Non-Uniform Fourier Transform: A Tutorial

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1 Introduction

The need of analysis of irregularly sampled data arises from many scientific disciplines, such as astronomy and astrophysics, geoscience and seismics, oceanography, telecommunications, remote sensing and medical imaging. The analysis of this kind of data is more complicated than the one of regularly sampled data. Therefore a common approach is to resample the irregular data on a regular grid.

It is however of interest to explore the possibility of analytical tools capable of dealing directly with irregularly sampled data.

In linear Image Processing, an ubiquitous operation is that of convolution of an input signal $p(t)$ with a filter $g(t)$, such that the output signal $y(t)$ produced by the convolution $y(t) = p(t) * g(t)$ highlights particular characteristics of the original input signal. The convolution can be simplified to a multiplication $Y(\omega) = P(\omega)G(\omega)$ using the Fourier transform. The investigation of the Fourier transform in the case of irregularly sampled input signal is therefore of great interest, since the knowledge of the Fourier transform in the irregular sampling case allows one to perform convolution in the irregular case and therefore opens the possibility of performing linear Image Processing on irregularly sampled signals.

2 From Discrete Fourier Transform to Non-Uniform Fourier Transform

2.1 Definition of the Discrete Fourier Transform (DFT)

Let us take into consideration the definition of Fourier transform in the continuous domain first: Under certain conditions upon the function $p(t)$ the Fourier transform of this function exists and can be defined as

$$P(\omega) = \int_{-\infty}^{+\infty} p(t)e^{-j\omega t}dt \tag{1}$$

where $\omega = 2\pi f$ and f is a temporal frequency. With the inverse Fourier transform, the original signal is given by:

$$p(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} P(\omega)e^{j\omega t}d\omega \tag{2}$$

Let us take into consideration now the case of the discrete Fourier transform (DFT). In this case we have a finite number N of samples of the signal $p(t)$ taken at regular intervals of duration T_S (which can be considered a sampling interval). In practical cases the signal $p(t)$ has not an infinite duration, but its total duration is $T = NT_S$ and we have a set $\{p_n\}$ of samples of the signal $p(t)$ taken at regular intervals. We can define $p_n \doteq p(t_n)$, where $t_n = nT_S$, for $n = 0, \ldots, N-1$, is the sampling coordinate.

In the case of the discrete Fourier transform, not only we want the signal to be discrete and not continuous, but we also want the Fourier transform, which is a function of the temporal frequency, to be defined only at regular points of the frequency domain. That is to say that the function $P(\omega)$ is not defined for every value of ω but only for certain values ω_m. We want the samples $P(\omega_m)$ to be regularly spaced as well, so that all the samples ω_m are multiples of a dominant frequency $\frac{1}{T}$, that is to say $\omega_m = m\left(\frac{2\pi}{T}\right)$, for $m = 0, \ldots, N-1$. Let us note now that T is equal to the finite duration of the signal $p(x)$ from which we want to define its discrete Fourier transform. Note also that we assume the number of samples in frequency to be equal to the number of samples in the temporal domain, that is N. This is not a necessary condition, but it simplifies the notation.

All that said, the direct extension of Definition (1) to the discrete domain is:

$$P(\omega_m) = \sum_{n=0}^{N-1} p(t_n)e^{-j\omega_m t_n} \tag{3}$$

Considering that ω_m can assume only discrete values $m\left(\frac{2\pi}{T}\right)$ and x_n can assume only discrete values nT_S, it is possible to rewrite Definition (3) as:

$$P(\omega_m) = \sum_{n=0}^{N-1} p(t_n)e^{-j\left(m\frac{2\pi}{T}\right)\left(nT_S\right)} = \sum_{n=0}^{N-1} p(t_n)e^{-j\left(m\frac{2\pi}{NT_S}\right)\left(nT_S\right)} \tag{4}$$

It is now possible to simplify and express the dependence on ω_m only in terms of m and the dependence on x_n only in terms of n. In that way the final definition of the DFT is:

$$P(m) = \sum_{n=0}^{N-1} p_n e^{-j\frac{2\pi}{N}mn} \tag{5}$$

And the inverse of the discrete Fourier transform (IDFT) as:

$$p_n = \frac{1}{N}\sum_{m=0}^{N-1} P(m) e^{-j\frac{2\pi}{N}mn} \tag{6}$$

2.2 Definition of Non-uniform Discrete Fourier Transform (NDFT)

Now we want to generalize the definition and the computation of the Fourier transform from the regular sampling to the irregular sampling domain. In the general case, the definition of the Nonuniform Discrete Fourier Transform (NDFT) is the same as the one given by Equation 3, taking into consideration that the samples can the taken at irregular intervals both in time (t_n) and/or in frequency (ω_m).

However, in practice, we want to take into consideration a more restricted case, which is the case where the samples are irregularly taken in the time domain t but regularly taken in the frequency domain. That is to say that the samples $P(m)$ of the irregular Fourier transform are taken at multiples of a quantity Δk, which is a fixed quantity in the Fourier domain. The fixed quantity Δk in the regular case corresponds to $\frac{2\pi}{T}$. The extension from regular to irregular sampling, therefore, depends on the duration of the signal $p(t)$ and not on the fact that the samples t_n are taken at regular or irregular intervals.

The definition of the nonuniform discrete Fourier transform (NDFT) is as follows:

$$P(m) = \sum_{n=0}^{N-1} p_n e^{-jm\Delta k t_n} \tag{7}$$

It is common practice to set $\Delta k = \frac{2\pi}{T}$ where T is the range of extension for the samples t_n. In that case the formulation of the NDFT is very similar to the one of the DFT except of the presence of the spatial coordinates t_n instead of the index n. In this case, the NDFT is defined as:

$$P(m) = \sum_{n=0}^{N-1} p_n e^{-j\frac{2\pi}{T}mt_n} \tag{8}$$

From a computational point of view, two differences have to be noticed between DFT and NDFT. The first difference is that samples in frequency are taken at intervals $\frac{2\pi}{T}$ in the irregular case instead of $\frac{2\pi}{N}$ in the regular case (T being the duration of the signal $p(t)$, with $t \in [0, T]$, and N is the number of samples of the signal $p(t)$). The second difference is that, instead of the integer index n in the regular case, in the irregular case the irregular sampling coordinate t_n appears in the exponent.

3 Signal Reconstruction by using the Fourier transform

The irregular sampling problem is concerned with the problem of dealing with signals and images which may be represented by samples on an irregular grid. One of the most frequent problems to solve is the one of reconstruction of the signal from its samples. The NDFT can be used directly for reconstruction. By computing the Fourier coefficients at all required discrete regular frequencies and then Fourier transforming back with the use of the inverse DFT, a kind of interpolation of the irregularly sampled signal can be obtained.

In order to explain this application, let us first examine the problem of reconstruction of a signal from its regular samples with the use of the inverse Fourier transform.

3.1 Signal reconstruction from regularly sampled data

Let us consider the following one-dimensional signal $\mathbf{p}(t)$. It assumes values $\mathbf{p}(t) = [0; 1; 2; 3; 4; 2; 0; 2; 4; 2; 0; 2; 4; 3; 2; 1; 0]$, the plot of which is shown in Figure 1 a. The corresponding Fourier transform is function $P(f)$ visualized in Figure 1 b. The total duration of the signal is $T = 17$.

Let us consider a regular subsampling of the original signal $p(t)$, $\mathbf{p}_r(t)$. It assumes values $\mathbf{p}_r(t) = [0; 2; 4; 0; 4; 0; 4; 2; 0]$. The regular sampling pattern used is $\mathbf{t}_s$, the values of which are $\mathbf{t}_s = [-8; -6; -4; -2; 0; 2; 4; 6; 8]$.

The sampled values are shown as magenta circles in Figure 2 a. The total number of regular samples is $N = 9$. The DFT of the regularly sampled signal is plotted in Figure 2 b in magenta superimposed to the Fourier transform of the original signal in blue. It is possible to notice that the Fourier transform of the regularly sampled signal has the same shape of the original Fourier

(a) (b)

Figure 1: Original signal $p(t)$ (a) and its Fourier transform $P(f)$ (b).

(a) (b)

Figure 2: Regularly sampled signal $\mathbf{p}_r(t)$ superimposed on the original signal $p(t)$ (a) and its Discrete Fourier transform (DFT) $\mathbf{P}_r(f)$ (in magenta) superimposed on the Fourier transform of the original signal (in blue)(b).

transform, the peaks are slightly lower (the signal loses some energy) and the higher frequencies are missing, in fact only half of the Fourier coefficients are present.

If we Fourier transform the $\mathbf{P}_r(f)$ back, which is to say we use the inverse DFT, we obtain the original signal $p(t)$ if the regular sampling frequency is above the Nyquist limit of twice the highest frequency present in the original signal. A smoothed (low-pass) approximation of the original signal is obtained, if the Nyquist condition is not met. In the case of $N = 9$ the Nyquist condition is respected and the signal is reconstructed perfectly as shown in Figure 3. In the case the number of samples is lowered and therefore the sampling interval increases, the reconstruction is imperfect as shown from

Figure 3: Reconstructed signal using the inverse Fourier Transform (IDFT) using $N = 9$ regularly spaced samples.

the examples in Figure 4 a for $N = 8$ and Figure 4 b for $N = 6$.

3.2 Signal reconstruction from irregularly sampled data

Now we want to reconstruct the signal $p(t)$ from a collection of samples taken irregularly using the inverse Fourier transform. Let us take into consideration an irregular sampling $\mathbf{p}_i(t) = [0; 1; 4; 0; 2; 2; 3; 2; 0]$ of the original signal $p(t)$. The number of samples taken into consideration is the same as in the case of the regular sampling, $N = 9$.

The irregular sampling sequence is $\mathbf{t}_i = [-8; -7; -4; -2; -1; 1; 5; 6; 8]$, so these are the irregular sampling coordinates that have to be used in the computation of the NDFT. The irregular samples are shown as green circles in Figure 5 a and the corresponding NFT is shown as the green curve in Figure 5 b.

Now it is possible to calculate the regular inverse DFT of the Fourier transform $\mathbf{P}_i(f)$, which provides a reconstruction $\mathbf{p}_{ir}(t)$ of the original signal $p(t)$. This reconstruction is shown in green in Figure 6, superimposed to the original signal in blue.

The reconstructed signal has the same shape of the original signal and provides a good approximation, however it is not an interpolation of the original signal, as the values of the original signal at the sampling points are

(a)

(b)

Figure 4: Reconstruction of the original signal (in blue) using $N = 8$ regularly spaced samples (a) and $N = 6$ regularly spaced samples (b). The reconstruction is not perfect because the number of samples is reduced.

(a) (b)

Figure 5: Irregularly sampled signal $\mathbf{p}_i(t)$ superimposed on the original signal $p(t)$ (a) and its Nonuniform Discrete Fourier transform (NDFT) $\mathbf{P}_i(f)$ (in green) superimposed to the Fourier transform of the original signal, $P(f)$ (in blue) and the DFT of the regularly sampled signal, $\mathbf{P}_r(f)$ (in magenta) (b).

Figure 6: Reconstructed signal using the inverse Fourier Transform (IDFT) of the NDFT $\mathbf{P}_i(f)$.

not recovered.

Let us make a comparison now between the computation of the DFT and the computation of the NDFT. For the computation of the DFT it is necessary to calculate the matrix:

$$F_N = \left(e^{-j2\pi \mathbf{m}^T \mathbf{m}/N}\right) \tag{9}$$

where $\mathbf{m}^T$ is the transpose of vector $\mathbf{m} = [-4; -3; -2; -1; 0; 1; 2; 3; 4]$ which contains the $N = 9$ indices of the Fourier coefficients. The coefficients obtained by the multiplication $\mathbf{m}^T\mathbf{m}/N$ are shown in Figure 7 a.

Using a matrix formulation the calculation of the DFT $\mathbf{P}_r$ can be expressed as:

$$\mathbf{P}_r = \mathbf{p}_r F_N \tag{10}$$

In order to compute the NDFT it is necessary to calculate the matrix:

$$A_N = \left(e^{-j2\pi \mathbf{t}_i^T \mathbf{m}/T}\right) \tag{11}$$

where $\mathbf{t}_i^T$ is the transpose of vector $\mathbf{t}_i$ which contains the $N = 9$ coordinates of the irregular sampling sequence. Once again, we remind that $T = 17$ is the total length of the signal $p(t)$. The coefficients obtained by the multiplication $\mathbf{t}_i^T\mathbf{m}/T$ are shown in Figure 7 b.

Using a matrix formulation the calculation of the NDFT $\mathbf{P}_i$ can be expressed as:

$$\mathbf{P}_i = \mathbf{p}_i A_N \tag{12}$$

Inspection of Figure 7 helps visualize the difference in the two matrices used for the regular and irregular calculation of the Fourier transform. Each line in the diagrams on Figure 7 corresponds to a matrix row. The regular circles in Figure 7 a correspond to the column values as column index assumes the $N = 9$ values from -4 to 4. In Figure 7 b the $N = 9$ cross values are placed irregularly and their position is dependent on the sampling sequence represented by vector $\mathbf{t}_i$.

By using these coefficients as exponents of the exponential Fourier kernel, the complex values shown in Figure 8 are obtained. The values of the Fourier exponential functions in the regular matrix are shown as magenta circles on the complex plane, while the corresponding functions for the irregular matrix are shown as green crosses in the complex plane. Since matrices F_N and A_N are square matrices of dimension N, one would expect to see $N \times N$ values. Since the values shown are much fewer, this means that some values are assumed by elements of the matrices more than once.

(a)

(b)

Figure 7: Regular matrix F_N coefficients obtained from the vector multiplication $\mathbf{m}^T\mathbf{m}/N$ (a) and irregular matrix A_N coefficients obtained from the vector multiplication $\mathbf{t}_i^T\mathbf{m}/T$ (b) as row and colum values of a $N \times N$ matrix.

Figure 8: Values assumed by the elements of the regular matrix F_N in magenta and of the irregular matrix A_N in green, in the complex plane.

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