Inverse Cascading

I. SYSTEM

We would like to study the following system describing 2D toroidal ITG turbulence (due to the corresponding instability) to see what causes the inverse cascade that is observed. The $(\kappa_n + \kappa_T)\,\partial_y \phi$ in the P equation and $-\kappa_B \partial_y P$ in the ϕ equation are responsible for the instability.

$$\begin{aligned}
\partial_t P + \{\phi, P\} + (\kappa_n + \kappa_T)\,\partial_y \phi &= \chi \nabla_\perp^2 P + D_P \nabla_\perp^4 P + \nu_P \nabla_\parallel^4 \tilde{P} \\
\partial_t \left(\tau \widetilde{\phi} - \nabla_\perp^2 \phi\right) + \left\{\phi, \left(\tau \widetilde{\phi} - \nabla_\perp^2 \phi\right)\right\} + \boldsymbol{\nabla}_\perp \cdot \{\boldsymbol{\nabla}_\perp \phi, P\} & \\
- (\kappa_B - \kappa_n)\,\partial_y \phi + (\kappa_n + \kappa_T)\,\partial_y \nabla_\perp^2 \phi - \kappa_B \partial_y P & \\
= -\chi \nabla_\perp^4 (a\phi - bP) + D_\phi \nabla_\perp^4 \left(\tau \tilde{\phi} - \nabla_\perp^2 \phi\right) + \nu_\phi \nabla_\parallel^4 \left(\tau \tilde{\phi} - \nabla_\perp^2 \tilde{\phi}\right) &
\end{aligned} \tag{1}$$

The inverse cascading means we need low-k damping for a stationary state to be achieved. [Nordman 1990] mentions that the inclusion of non-linear ion polarisation drift leads to inverse cascading. And [Nordman 1990], like a lot of papers, doesn't have $\boldsymbol{\nabla}_\perp \cdot \{\boldsymbol{\nabla}_\perp \phi, P\}$.

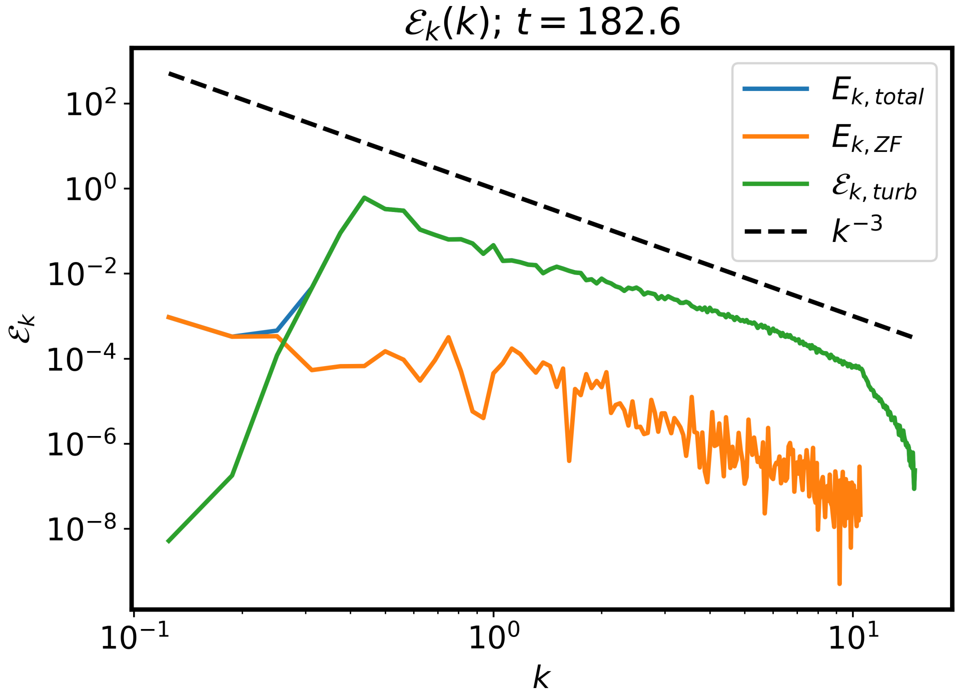


Figure 1: Energy spectrum, $\mathcal{E}_k$, of full system

A. Case 1: Does $\tau\widetilde{\phi}$ cause the inverse cascade

Omitting the $\tau\widetilde{\phi}$ fluctuations we get

$$\begin{aligned}\partial_t P + \{\phi, P\} + (\kappa_n + \kappa_T)\,\partial_y \phi = \chi \nabla_\perp^2 P + D_P \nabla_\perp^4 P + \nu_P \nabla_\parallel^4 \tilde{P} \\ \partial_t\left(-\nabla_\perp^2 \phi\right) + \left\{\phi, \left(-\nabla_\perp^2\phi\right)\right\} + \boldsymbol{\nabla}_\perp \cdot \{\boldsymbol{\nabla}_\perp \phi, P\} \\ - (\kappa_B - \kappa_n)\,\partial_y\phi + (\kappa_n + \kappa_T)\,\partial_y \nabla_\perp^2 \phi - \kappa_B \partial_y P \\ = -\chi \nabla_\perp^4 (a\phi - bP) + D_\phi \nabla_\perp^4 \left(-\nabla_\perp^2 \phi\right) + \nu_\phi \nabla_\parallel^4 \left(-\nabla_\perp^2 \tilde{\phi}\right)\end{aligned} \tag{2}$$

This is somewhat like the voriticity equation with terms like $-(\kappa_B - \kappa_n)\,\partial_y\phi$ that come from the density equation. For reference, the vorticity equation omitting $n_{pe} = \tau\widetilde{\phi}$ is

$$\begin{aligned}\partial_t \nabla_\perp^2 \phi + \left\{\phi, \nabla_\perp^2\phi\right\} - \boldsymbol{\nabla}_\perp \cdot \{\boldsymbol{\nabla}_\perp \phi, P\} \\ - (\kappa_n + \kappa_T)\,\partial_y \nabla_\perp^2 \phi + \kappa_B \partial_y P \\ = \chi \nabla_\perp^4 (a\phi - bP) + D_\phi \nabla_\perp^4 \nabla_\perp^2 \phi + \nu_\phi \nabla_\parallel^4 \left(\nabla_\perp^2 \tilde{\phi}\right)\end{aligned} \tag{3}$$

B. Case 2: Does $\tau\widetilde{\phi}$ in $\partial_t\left(\tau\widetilde{\phi} - \nabla_\perp^2\phi\right)$ cause the inverse cascade

Omitting $\tau\widetilde{\phi}$ in $\partial_t\left(\tau\widetilde{\phi} - \nabla_\perp^2\phi\right)$ we get

$$\begin{aligned}\partial_t P + \{\phi, P\} + (\kappa_n + \kappa_T)\,\partial_y \phi = \chi \nabla_\perp^2 P + D_P \nabla_\perp^4 P + \nu_P \nabla_\parallel^4 \tilde{P} \\ \partial_t\left(-\nabla_\perp^2 \phi\right) + \left\{\phi, \left(\tau\widetilde{\phi} - \nabla_\perp^2\phi\right)\right\} + \boldsymbol{\nabla}_\perp \cdot \{\boldsymbol{\nabla}_\perp \phi, P\} - (\kappa_B - \kappa_n)\,\partial_y\phi + (\kappa_n + \kappa_T)\,\partial_y \nabla_\perp^2 \phi - \kappa_B \partial_y P \\ = -\chi \nabla_\perp^4 (a\phi - bP) + D_\phi \nabla_\perp^4 \left(\tau\tilde{\phi} - \nabla_\perp^2 \phi\right) + \nu_\phi \nabla_\parallel^4 \left(\tau\tilde{\phi} - \nabla_\perp^2 \tilde{\phi}\right)\end{aligned} \tag{4}$$

C. Case 3: Does $\tau\widetilde{\phi}$ in $\left\{\phi, \left(\tau\widetilde{\phi} - \nabla_\perp^2\phi\right)\right\}$ cause the inverse cascade

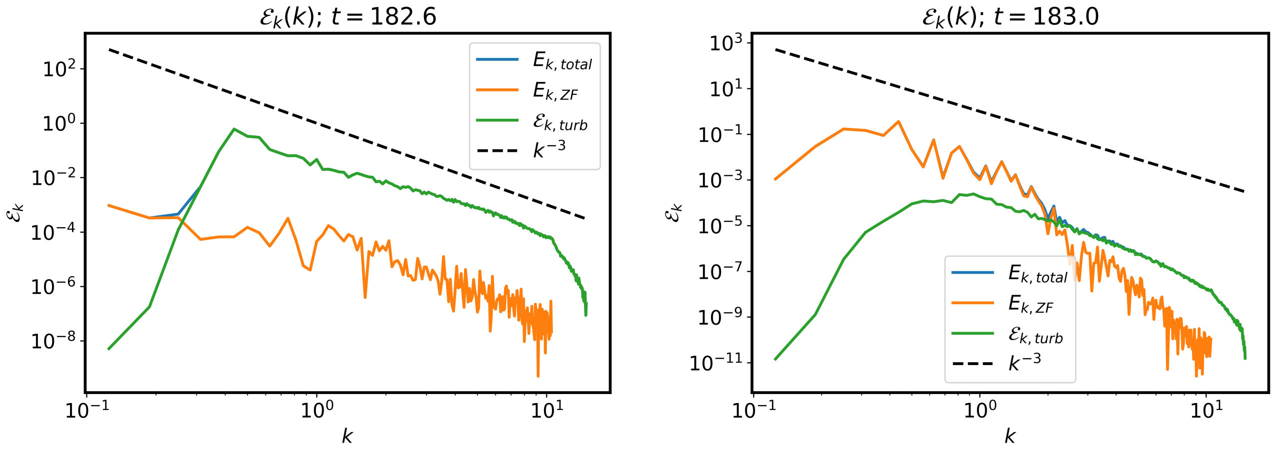
Omitting $\tau\widetilde{\phi}$ in the non-linear term $\left\{\phi, \left(\tau\widetilde{\phi} - \nabla_\perp^2\phi\right)\right\}$ we get

$$\begin{aligned}\partial_t P + \{\phi, P\} + (\kappa_n + \kappa_T)\,\partial_y \phi = \chi \nabla_\perp^2 P + D_P \nabla_\perp^4 P + \nu_P \nabla_\parallel^4 \tilde{P} \\ \partial_t\left(\tau\widetilde{\phi} - \nabla_\perp^2 \phi\right) + \left\{\phi, \left(-\nabla_\perp^2\phi\right)\right\} + \boldsymbol{\nabla}_\perp \cdot \{\boldsymbol{\nabla}_\perp \phi, P\} - (\kappa_B - \kappa_n)\,\partial_y\phi + (\kappa_n + \kappa_T)\,\partial_y \nabla_\perp^2 \phi - \kappa_B \partial_y P \\ = -\chi \nabla_\perp^4 (a\phi - bP) + D_\phi \nabla_\perp^4 \left(\tau\tilde{\phi} - \nabla_\perp^2 \phi\right) + \nu_\phi \nabla_\parallel^4 \left(\tau\tilde{\phi} - \nabla_\perp^2 \tilde{\phi}\right)\end{aligned} \tag{5}$$

D. Case 4: Does $\boldsymbol{\nabla}_\perp \cdot \{\boldsymbol{\nabla}_\perp \phi, P\}$ cause the inverse cascade

Omitting the non-linear term $\boldsymbol{\nabla}_\perp \cdot \{\boldsymbol{\nabla}_\perp \phi, P\}$ we get

$$\begin{aligned}\partial_t P + \{\phi, P\} + (\kappa_n + \kappa_T)\,\partial_y \phi = \chi \nabla_\perp^2 P + D_P \nabla_\perp^4 P + \nu_P \nabla_\parallel^4 \tilde{P} \\ \partial_t\left(\tau\widetilde{\phi} - \nabla_\perp^2 \phi\right) + \left\{\phi, \left(\tau\widetilde{\phi} - \nabla_\perp^2\phi\right)\right\} - (\kappa_B - \kappa_n)\,\partial_y\phi + (\kappa_n + \kappa_T)\,\partial_y \nabla_\perp^2 \phi - \kappa_B \partial_y P \\ = -\chi \nabla_\perp^4 (a\phi - bP) + D_\phi \nabla_\perp^4 \left(\tau\tilde{\phi} - \nabla_\perp^2 \phi\right) + \nu_\phi \nabla_\parallel^4 \left(\tau\tilde{\phi} - \nabla_\perp^2 \tilde{\phi}\right)\end{aligned} \tag{6}$$



(a) $\mathcal{E}_k$ of complete system

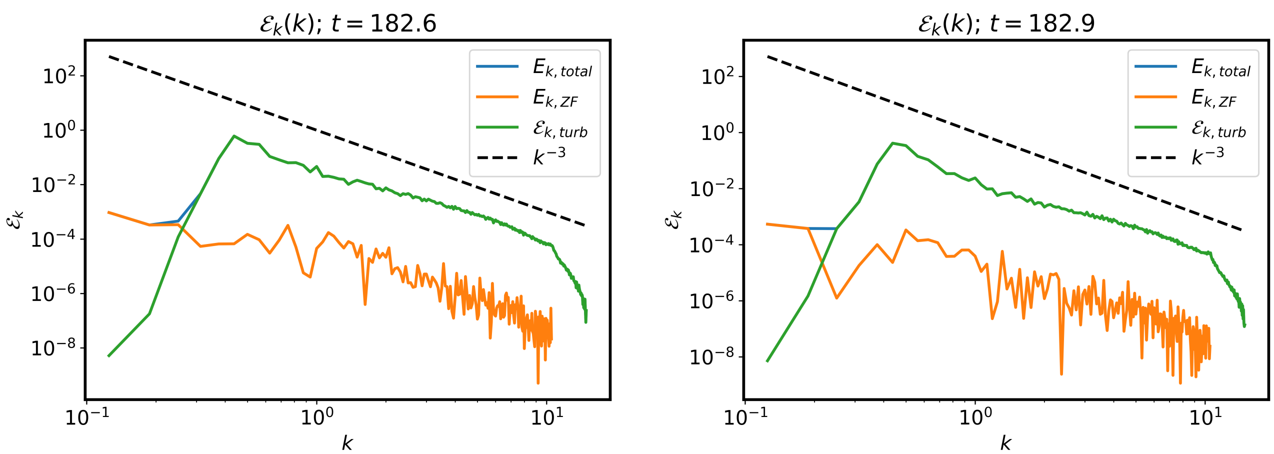
(b) $\mathcal{E}_k$ of system without diamagnetic stress, $\boldsymbol{\nabla}_\perp \cdot \{\boldsymbol{\nabla}_\perp \phi, P\}$

Figure 2: $\mathcal{E}_k$ comparison for Case 4

E. Case 5: Does $\left\{\phi, \left(\tau\widetilde{\phi} - \nabla_\perp^2 \phi\right)\right\}$ cause the inverse cascade

Omitting the non-linear term $\left\{\phi, \left(\tau\widetilde{\phi} - \nabla_\perp^2 \phi\right)\right\}$ we get

$$
\begin{aligned}
\partial_t P + \{\phi, P\} + (\kappa_n + \kappa_T)\,\partial_y \phi &= \chi \nabla_\perp^2 P + D_P \nabla_\perp^4 P + \nu_P \nabla_\parallel^4 \tilde{P} \\
\partial_t \left(\tau\widetilde{\phi} - \nabla_\perp^2 \phi\right) + \boldsymbol{\nabla}_\perp \cdot \{\boldsymbol{\nabla}_\perp \phi, P\} - (\kappa_B - \kappa_n)\,\partial_y \phi + (\kappa_n + \kappa_T)\,\partial_y \nabla_\perp^2 \phi - \kappa_B \partial_y P & \\
= -\chi \nabla_\perp^4 (a\phi - bP) + D_\phi \nabla_\perp^4 \left(\tau\tilde{\phi} - \nabla_\perp^2 \phi\right) + \nu_\phi \nabla_\parallel^4 \left(\tau\tilde{\phi} - \nabla_\perp^2 \tilde{\phi}\right) &
\end{aligned} \tag{7}
$$



(a) $\mathcal{E}_k$ of complete system

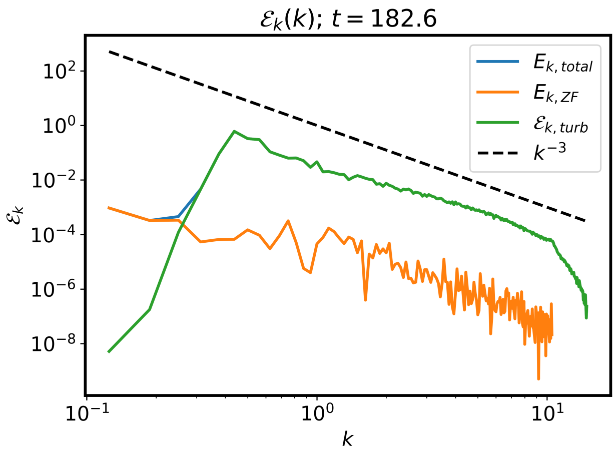
(b) $\mathcal{E}_k$ of system without $\left\{\phi, \left(\tau\widetilde{\phi} - \nabla_\perp^2 \phi\right)\right\}$

Figure 3: Case 5

F. Case 6: Role of $\boldsymbol{\nabla}_\perp \cdot \{\boldsymbol{\nabla}_\perp \phi, P\}$

Omitting the non-linear term $\{\phi, P\}$ in the P equation and $\left\{\phi, \left(\tau\widetilde{\phi} - \nabla_\perp^2 \phi\right)\right\}$in the ϕ equation, we get

$$
\begin{aligned}
\partial_t P + (\kappa_n + \kappa_T)\, \partial_y \phi &= \chi \nabla_\perp^2 P + D_P \nabla_\perp^4 P + \nu_P \nabla_\parallel^4 \tilde{P} \\
\partial_t \left(\tau\widetilde{\phi} - \nabla_\perp^2 \phi\right) + \boldsymbol{\nabla}_\perp \cdot \{\boldsymbol{\nabla}_\perp \phi, P\} - (\kappa_B - \kappa_n)\, \partial_y \phi + (\kappa_n + \kappa_T)\, \partial_y \nabla_\perp^2 \phi - \kappa_B \partial_y P & \\
= -\chi \nabla_\perp^4 (a\phi - bP) + D_\phi \nabla_\perp^4 \left(\tau\tilde{\phi} - \nabla_\perp^2 \phi\right) + \nu_\phi \nabla_\parallel^4 \left(\tau\tilde{\phi} - \nabla_\perp^2 \tilde{\phi}\right) &
\end{aligned} \tag{8}
$$



(a) $\mathcal{E}_k$ of complete system

data/energy_spectrum_kapt_1_2_chi_0_1_D_1_0_em3_H_1_0_em3_case6.

(b) $\mathcal{E}_k$ of system without $\{\phi, P\}$ and $\left\{\phi, \left(\tau\widetilde{\phi} - \nabla_\perp^2 \phi\right)\right\}$

Figure 4: Case 5