SELF-ORGANIZED CRITICALITY IN SPATIAL EVOLUTIONARY GAME SIMULATION

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Abstract

This paper is a written report of the final project of the Math 636-mathematical modeling class. The objective of the project is to understand what self-organized criticality (SOC) is and its relationship with the power law distribution. Literature review helps us to achieve a further and deeper understanding of SOC. It also provides a theoretical framework for us to analyze the spatial version of the Prisoners' Dilemma. This paper will try to give a generalized definition of SOC and its properties/behaviors. We simulated the Prisoners' Dilemma through the evolutionary games using Python 2.0. We investigated the spatial version of the Prisoners' Dilemma and are trying to see whether it has a power law distribution. Furthermore, we examine whether the Prisoner's Dilemma system could display SOC.

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1 Introduction

1.1 Historic Background of SOC

Self-organized criticality (SOC) can be understood in the context of physics, biology, and other fields. There are many examples of SOC systems in those areas. This concept is relatively new. It was first proposed by Bak, Tang, and Wiesenfeld (BTW) in their paper in 1987 in Physical Review Letters. From then on, the concept has been widely applied to other fields outside the principle of physics. We attempt to lay down some possible universal SOC behaviors from a theoretical perspective. Continuation in this direction in the future could derive a possible set of general rules for determining if a given system or algorithm displays SOC.

According to BTW paper, flicker noise is a structure of states which are minimally stable within a complex system. It can be explained by SOC and modeled by the power law. These states are to be understood and visualized as dynamic spatial clusters over which a small local perturbation will propagate and spread. The complex system will evolve into this critical stage by self-organization and reach a critical point. When it reaches this critical stage, it has interesting dynamics and has the following properties:

- Its critical point is an attractor.
- It can be modeled by the power law distribution which shows scale invariance behavior.
- The system can reach this state free from initial sensitivity and without fine setting of the parameter value.

The system will become stable when the network of these barely stable states broken down to a level that the noise signal can no longer communicate through infinite distances. It is similar to the critical point at a percolation transition where a connected path first appears between the two sides.

1.2 General Definition of SOC

The BTW paper has provided a good framework of SOC and it has been applied to many other complex systems such as earthquake, evolutionary biology, stock market, solar corona, the brain, etc. Many believe SOC could also be applied to the size of wars. It can be understood as follow:

The theory of SOC assumes a system is open and out of equilibrium. SOC is a phenomenon that can be observed in a complex system with many interacting or interconnected components with extended spatial degrees of freedom. The dynamics of the system is independent from the physical nature of those components. There exists a slow driving force–stress acting upon the system and an event–avalanche occurs. An avalanche takes place for a quick release of energy due to the stress of the system. Since the components are well connected, it has a domino effect throughout the system and avalanches occur where local perturbations propagate throughout the system. The system will continuously evolve under these circumstances in either time or space, until it reaches a critical point and self organized around the point.

This critical point is usually right before the system behavior stabilizes or turns chaotic and it is an attractor. SOC usually occurs in a transient or critical stage, in between predictable and chaotic behaviors of the system, or in between two phases of the system. Some regard SOC as phase transition. Better understanding of the elements and the behaviors of SOC can lead to better understanding of the complex system and its phase transition. The phase transition is cause by the "avalanches," not the "stress" that is acting on the system. The statistical nature of these avalanches has a power law distribution. This happens at every length scale of the system and therefore also demonstrates scale-invariance behavior, fractal like structure. The complex system can reach this transient stage free from initial sensitivity and without fine setting of the parameter value

Percolation[1] theory is used to postulate when exactly the phase change would occur. Although it is not the scope of this paper, we wish to mention that the qualitative analysis of the SOC, from dynamical system perspective, could be used to understand the attractor of the system. Furthermore, one can postulate when this phase change takes place. Similar to percolation, the phase change occurs when percolation threshold first appeared.

Now, we could generalize that the following necessary elements for SOC to display:

- A complex system, with many interacting components.
- A slow driving force (stress), acting upon on the system.
- An avalanche (event), on set by the stress and causes a domino effect throughout the system.
- A critical point as an attractor.

Some of the properties of SOC are:

- The system evolves in time or space and self organizes around its critical point. It (maintains a stable form or) shows transient phenomena-avalanches.
- The statistical nature of the avalanches during the transient stage of the system has a power law distribution, which also gives scale invariance behavior.
- The system can evolve to this state free from initial sensitivity and without fine setting of the parameter value.
- It is a series of avalanches causes the system change phase, not the slow driving force acting upon the whole system.

1.3 Examples of systems displayed SOC and other application of SOC

1.3.1 Forest fire model

In the forest fire model, we visualize a cellular grid where each cell can contain a tree or not. We can let a black cell represent a cell with a tree, and a white cell to represent an empty spot with no tree. The trees are the interacting components and are programmed to appear randomly at some predetermined rate. We can then add a slow driving force to the system (such as lightning) at a random spot on the lattice. If it hits a spot with a tree, an event will

[1] Percolation is a simple probabilistic model. Imagine that we have a n by n grid with no path connected the top and bottom sides. It asks the question: what is the probability that an open path exists between the two sides in a random medium. Percolation threshold is when the open path first occurs and connected both sides.

occur (fire), which burns the tree and its neighbors, turning them into white empty cells. The cluster size is how many cells the fire wipes out, which we can measure. The components can only interact with other components it is connected to in the cluster. The fire is mainly localized because the clusters are small. The entire forest network is still evolving. However, after a while, a critical point is reached where clusters of trees percolate through the entire network. This means that if one of the trees is stressed with the lightning, the resulting fire will travel through the forest network in a domino effect. After this transient period, a minimally stable state is reached and the cycle repeats with new trees growing until it reaches that critical point again.

These properties of an SOC system generate a power law distribution. There are many small clusters of trees that start off in the beginning. They exist in high probability and get wiped out by fire more often. It takes time for the system to grow into a connected network, and adding the perturbation of lightning kills off smaller clusters of trees that are growing, before they can even form into bigger clusters. This leads to high frequencies of small clusters getting on fire. The system hasn't stabilized yet, and if you wait long enough, a highly connected cluster can form. This is the critical point that occurs when the system/network is able to form long range connectivity, and creates available paths across through the entire system. It takes time for highly linked clusters to form. When the system is now stressed (lightning occurs), we get the rare occurrence of large events. After this critical point, the system stabilizes, and quickly goes back into this cycle. [**?**]

1.3.2 Vanadium Oxide

Vanadium oxide is a thin film measured in nanometers. It changes from insulator to metallic and therefore conduct electricity at a critical temperature around 340 K. The transition is temperature driven through a series of resistance jumps. This complex system has many interacting parts, VO2 molecules. The slow driving force is the temperature in this case. It stresses the system and causes these resistance jumps (avalanches).

In between the temperatures of 335 K and 339 K, there is a significant decrease of resistance. It shows various sizes of resistance jumps. There are more small jumps but few large jumps. It demonstrates a power law distribution of these jumps. Further statistical analysis shows that there is this particular jump causes the system changes from insulator to metallic phase regardless of the temperature that was given to stress the system. Therefore, the phase change is due to the avalanches, but not at a particular temperature.

1.4 How SOC relates to power law distribution

The simplifying aspect of complex systems with SOC state is that they can be described by a power law distribution. The distribution tells us that the probability of an event of size, s, occuring is $P(s) \sim s - a$. The power law distribution is a way to measure if a system exhibits SOC. However, not all observed power law behavior is an effect of dynamical systems self-organizing into a critical state [?] . There are other factors involved, thus, we conjecture that the power law distribution of the avalanches is a necessary condition but not sufficient enough to determine whether a system displays SOC. We believe in the absence of the power law distribution, the system is most likely not a SOC system.

1.4.1 Important characteristics of the system

We expect SOC behavior and its corresponding power law distribution to be present in evolving systems consisting of many interacting components. The system may also be slowly driven by an external force. There is a tendency for the system to self-organize into a critical scale-invariant state. The state is critical in the sense that local perturbations will propagate throughout the system, creating emergent phenomenon that cannot be exhibited by individual components that make up the system. This behavior does not depend on the details of each system. The state is scale-invariant in the sense that no characteristic event size exists. The same perturbation applied at different times or/and to different regions can lead to a response of any size. The process building up to the critical state takes place over a very long period, but the relaxation/event occurs in a relatively short period. There is thus a series of minimally stable states exhibited by the system before it reaches the threshold. This is also known as separation of timescales. [?]

Before criticality is reached, the effect of perturbation will be localized. When criticality is reached, the effect of perturbation on a localized region gets distributed throughout the system. Thus, cooperative behavior seems to arise from the repeated internal interaction of each component in the building up stage. Cooperative behavior in complex critical system operate through percolation networks, and evolution of this network is the mechanism responsible for generating the power law distribution shared by this class of dynamical system.

1.4.2 Percolation network

Percolation occurs during a phase transition, where clusters that span the system are percolating through the system. This allows a connected path for localized perturbation to travel across the system and create a larger effect. In complex systems, percolating structure is exhibited in slowly evolving systems with many interacting components. The long time period and slow driving force allow the interacting components time to evolve into many minimally stable states. Components that are able to affect each other cluster together in a sort of network of connected paths. The system eventually reaches a critical state where it is in some interconnected configuration. When the system is perturbed at this marginally stable state, it relaxes and an event occurs that can be measured.

The size of the event is proportional to the size of the effected cluster, which represents how many components the system can distribute its applied perturbation to. The cluster sizes are continually modified based on the internal interaction and the external driving force. The clustering network starts off sparse and eventually grows into percolation geometry. Since the system will initially have more time to form networks with small clusters rather than large ones, the external perturbation has a higher probability affecting a small cluster. In addition, the network may disconnect after the system relaxes, allowing small network of clusters to form again. Thus, small activity has an even more frequent chance of occurring. It takes time for highly interconnected networks to form. Larger event rarely happen because a path through the entire system hasn't yet had a chance to form. This scale-invariant behavior gives rise to the power law distribution.

The exponent of the power law can tell us the connectedness of the system. A flat slope represents a system that forms a large network with many connections throughout the entire system. A steep slope has many independent small clusters, and is not highly interconnected overall.

2 The Prisoner's Dilemma

2.1 The Canonical Prisoner's Dilemma (PD)

The Prisoner's Dilemma is a mathematical game which encapsulates the inherent contradiction of doing what's best for you to the detriment of those around you – it's a great strategy as long as you are in the minority, but if everyone acted as you act, then everyone would be worse off than if they had all done the right thing to begin with. In its simplest incarnation, the Prisoner's Dilemma is played by only two players, and each player has only two strategies to choose from: Cooperate or Defect. The payoff matrix for the two-player

	Cooperate	Defect
Cooperate	R, R	S, T
Defect	T, S	P, P

Table 1: The canonical Prisoner's Dilemma where $T > R > P > S$.

Prisoner's Dilemma is given in Table 1. Since $T > R$ and $P > S$, regardless of which strategy the other player adopts, it is always more advantageous to Defect. This rationale holds for both players, since the game is symmetric. The rational 'solution' to the game, then, is for both players to Defect. Herein lies the crux of the dilemma. It is more advantageous for both players to Cooperate, since each would then receive a pyaoff of R which is strictly greater than P – the payoff for mutual Defection. Thus the individually 'rational' solution is not optimal for either of the players.

Much attention has been paid to this 'classic' dilemma. Not least among the reasons being that cooperation does in fact exist within societies, even though the temptation for individual members to defect is ever-present.

2.2 The Iterated Prisoner's Dilemma (IPD)

One extension of the classic two-player Prisoner's Dilemma that has undergone much study and research is the Iterated Prisoner's Dilemma (IPD), where the two players play against each other many times in a row[2]. Playing repeatedly gives rise to many interesting strategies contingent upon what strategy the other player has played in the past. One of the most simple, yet surprisingly robust strategies is called Tit-For-Tat (`TFT`). `TFT` cooperates on the first move and merely mimicks the strategy of the other player tehrerafter. This approach has the interesting behavior of punishing the other player by Defecting if the other player Defected in the previous round, but rewarding the other player by Cooperating if the other player Cooperated in the previous round. `TFT` embodies the charastics of being *nice* by opening with the goodwill to Cooperate, but willing to quickly *retaliate* if wronged, but also willing to quickly *forgive*. Moreover, `TFT` is *non-envious* of the other player, by not trying to always outscore him or her.

2.3 The Evolutionary and Saptial Prisoner's Dilemma

Imagine now that there is an entire population of players playing the Prisoner's Dilemma with one another. Assuming that the members of the population eventually die off and are replaced according to the proportion of their strategy (genotype) in the population, strict Cooperation does not succeed in a society containing even one single Defector. Defectors quickly overtake the entire population. What if, instead of having the members of such a population interact with one another entirely at random, they all resided at some fixed location and interacted only with their immediate neighbors? Could this modification help to foster the evolution of Cooperation within the society? Researchers Martin A. Nowak and Robert M. May addressed this question in their 1992 article in Nature, *Evolutionary Games and Spatial Chaos.* They chose to analyze players on a square lattice, each playing the Prisoner's Dilemma with their neighbors. Each player was assigned an initial strategy (either Cooperate or Defect) and after each round of pay compared her total payoff to those of her neighbors. If a neighbor with a different strategy had a higher payoff, then she would switch strategies in the next round of play. The parameter of interest Nowak and May chose to investigate was the ratio T/R, that is the ratio of the 'Temptation to Defect' to 'Reward for mutual Cooperation'. High values of T/R the system would not surprisingly quickly settle down to one of all Defectors. For values of T/R close to 1 however, the majority of the board would be covered with Cooperators. So it turns out that imposing some spatial restrictions on large

[2]Robert Axelrod gives an excellent analysis of the IPD in his seminal book, *The Evolution of Cooperation.*

populations of players playing the Prisoner's Dilemma can be beneficial to the evolution of cooperation.

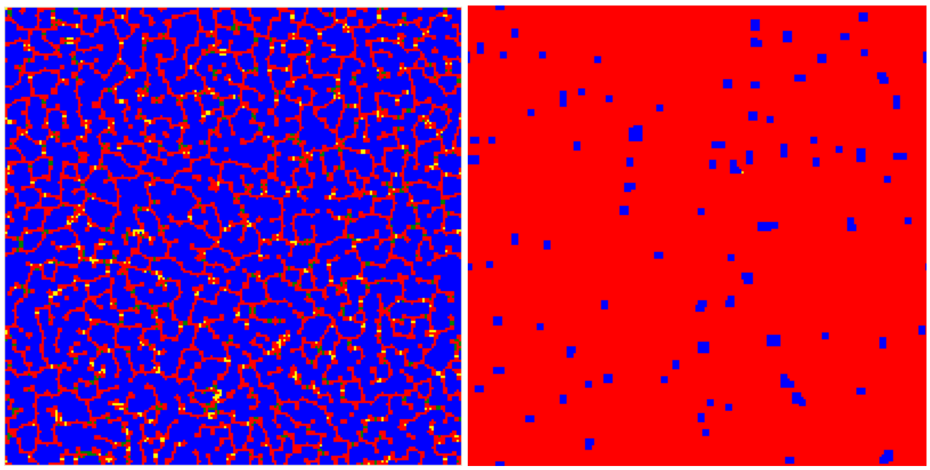

Figure 1: Both figures show the stable states of the Prisoner's Dilemma played on a square lattice with the Moore neighborhood and the Imitate Best Neighbor update rule. The image on the left corresponds to $T/R = 1.76$. Although Defectors (red) still have an advantage over Cooperators (blue) in this region, the spatial connectivity allows large clusters of Cooperators to prevail. The image on the right corresponds to $T/R = 2.1$. Already by this point the Defectors have a clear advantage. Only a few clusters of Cooperators were able to survive. (Note: Here we have used the payoff values from Nowak and May (1992): $R = 1, P = S = 0$.

There exists an interesting region between the regimes where neither Cooperators nor Defectors dominate. This regime exhibits properties of spatial chaos. The dynamics never settle down to a steady state, but continually evolve. A snapshot of such a system is shown in Figure 2.3.

We were curious what sort of distribution the size of the clusters of Cooperators actually have in the chaotic regime. Figure 2.3 is a histogram plot of Cooperator cluster sizes for the Prisoner's Dilemma on a latice with the von Neuman neighborhood.

We attempted to fit the data to a power law, but unfortunately with little success.

2.4 Simulation

Spatial games, such as the one described in the previous section, are too complex to study analytically however they can be simulated using a computer. We wrote a computer program to simulate and analyze a spatial prisoners' dilemma. The simulation of this model was created in Python and is freely

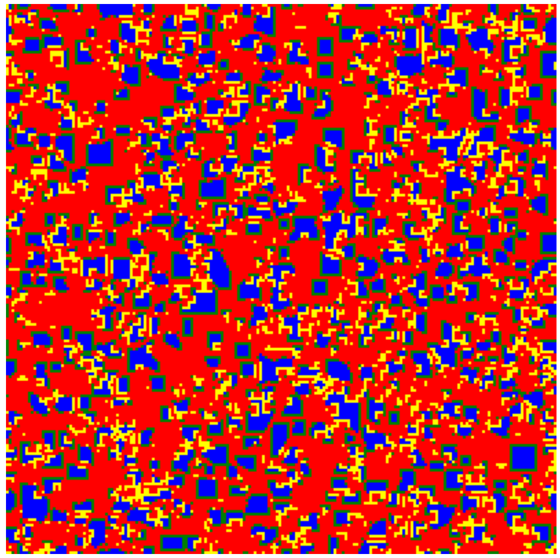

Figure 2: The chaotic regime. The green and yellow squares represent the change of the strategies from one time step to the next. Green squares are Cooperators who were Defectors in the previous round, while yellow squares are Defectors who were Cooperators in the previous round.

available online [3]

The simulation allowed us to define the dimensions of the grid both in the vertical and horizontal direction. This would set up the system with $N \times M$ cells each randomly assigned to be either defectors or cooperators. Then we could evolve the system forward in time which would calculate the values of each cell according to the payoff matrix. After each step every cell would update to it's new strategy as described in 2.3 and the new state of the grid is displayed.

The values of R, S, T, P are adjustable parameters in the simulation. To make things simpler we fixed the parameters $R = 1$ and $S = P = 0$ so that only T may be varied making it a one parameter system. This simplified the system and we were able to study how the system changed for different values of T. Adjusting this parameter we found that the system has four different ranges of T in which the system has different behavior.

The first region when $T < 1.\bar{3}$ evolves to a stable configuration. In this setup cluster of cooperators score well and expand filling most of the space. However some defectors remain as individual defectors surrounded by cooperators will remain defectors. This leaves strips of defectors while the gird is otherwise cooperators.

The next region exists where $1.\bar{3} < T < 1.5$. In this region system never

[3] http://github.com/f4hy/math636-soc-project

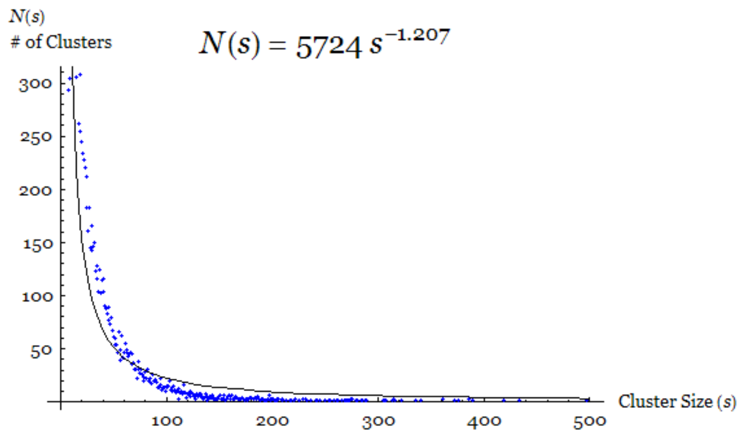

Figure 3: The distribution of Cooperation cluster sizes for the Prisoner's Dilemma on a 100 $\times$ 100 lattice with the von Neuman neighborhood. The data were taken over ten time steps each for ten completely random (50% Cooperators, 50% Defectors) initial conditions, after 25 generations of relaxation. The payoffs were $T = 1.4, R = 1, P = S = 0$.

settles down to any stable configuration. This is a chaotic region in which the grid continually fluctuates. Next the region of $1.5 < T < 2.0$ has cycles. In this cyclic region the randomness settles down into configurations which cycle between a few configurations. The last region of $T > 2.0$ is uninteresting as the defectors always score better than cooperators and thus dominate the entire grid.

To help analyze the system qualitative the simulation would output statistical data about the current state. After each time step the program would write a log displaying how many steps had been taken, the ratio of cooperators to defectors. Also since the system organizes itself into clusters of one type we wanted to analyze the properties of the clusters. We defined a cluster to be an area in which a group of cooperators were continuously connected via neighbors of cooperators. The simulation also output the number of clusters after each step and a the frequency of how often a cluster of a given size occurred.

2.5 Analysis

Since exhibiting power law behavior is sometimes necessary condition for characterizing a system as an SOC, we attempted to find a power law relating cluster size and cluster frequency. [4] Cluster size is the number of adjacent

[4] If cluster size *gamma* and cluster frequency $f(\gamma)$ obey a power law relation, then we must have $f(\gamma) = C\gamma^{\alpha} + b$. To analyze the data, we first we plot all data points $(\gamma, f(\gamma))$ for a given lattice size. Since it is not necessary that α be an integer, it is unlikely that we will be

cooperators in a select time step and cluster frequency is the number of times a given cluster size appears in a given time step. Killingback and Doebeli [?] found that the Hawk-Dove spatial evolutionary game exhibits self-organized criticality over a particular range of values of one parameter. However, they assert that the simulation of the Prisoner's Dilemma by Nowak and May [?] does not exhibit SOC. Nowak and May simulated the Prisoner's Dilemma with parameter values $R = 1$, $T = b(b > 1)$, and $S = P = 0$. They determined-using the von Neumann neighborhood rule-that chaotic behavior occurs for $\frac{4}{3} < b < \frac{3}{2}$ Killingback and Doebelsi [?] noted that for different values of b, only static, periodic and chaotic behavior occurs. Thus this particular PD parameter setup does not exhibit SOC behavior.

If the simulation of Nowak and May does not obey a power law relation, then it does not exhibit SOC. Thus we need only show that this particular PD setup with the above parameter values does not obey a power law relation. We ran our PD simulation in taking $b = 4$. Thus according to Nowak and May [?], our simulation should generate class III (chaotic) cellular automata. Next we set the initial proportion of cooperators to defectors as 0.50 with a random setup of cooperators to defectors. For several different square lattice sizes, we ran the simulation over 50 times steps before analyzing cluster size and cluster frequency and obtained the following results:

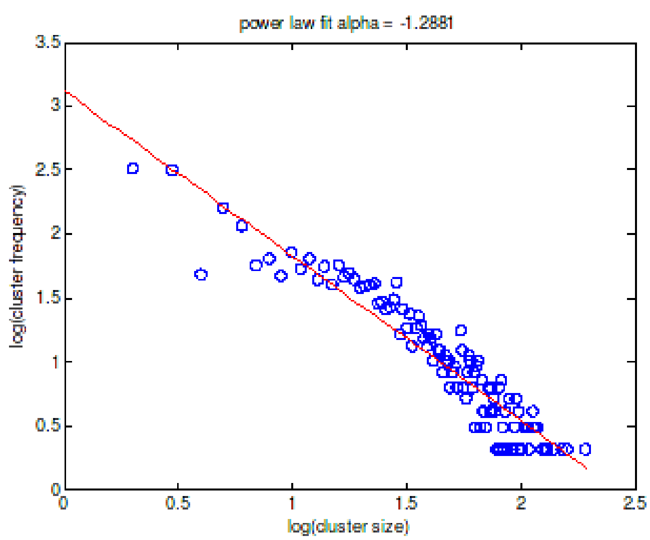

Figure 4: 399×399 matrix

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able to determine an appropriate α be visual inspection of the scatter plot. Thus we utilize a log-log plot since $f(\gamma) = C\gamma^{\alpha} + b \rightarrow \log f(\gamma) = \alpha \log \gamma + k$, where $k = b + \log C$. Thus we can easily compute the exponent α by applying a least squares fit to the data in the log-log plot.

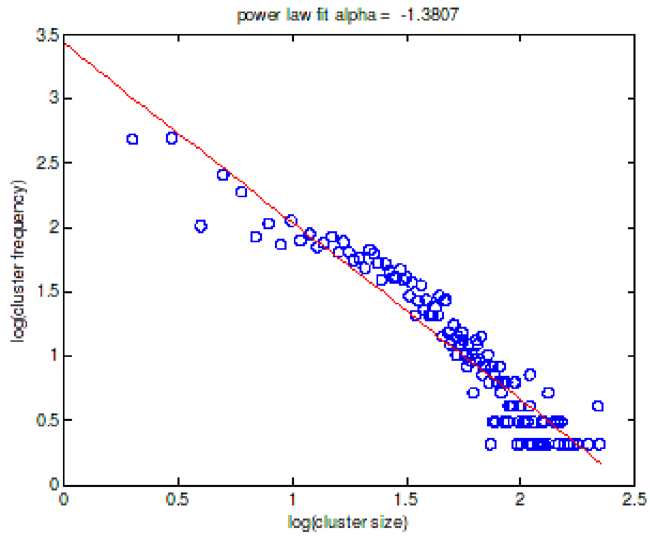

Figure 5: 502 × 502 matrix

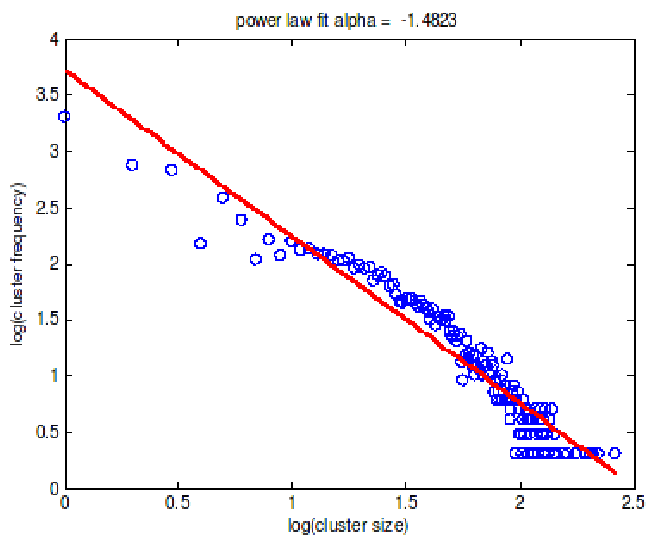

Figure 6: 605 × 605 matrix

It is evident from Table 2 that for increasing lattice size, the sum-squared error of the power law fit is monotonically increasing. This does not seem unusual since the lattice size is increasing. Thus it seems that this particular PD

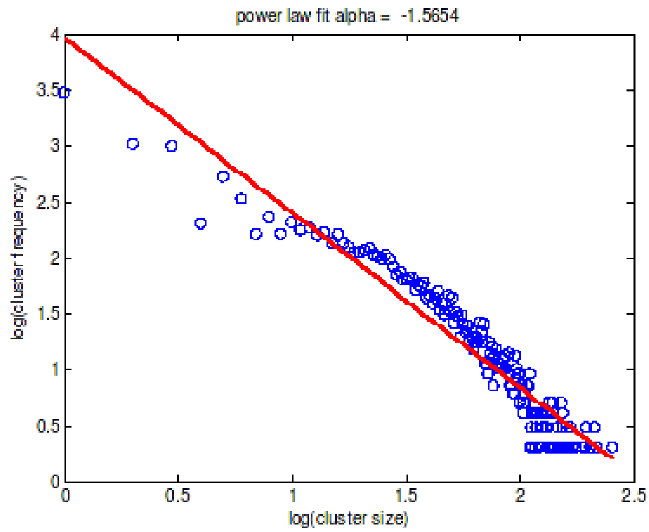

Figure 7: 707×707 matrix

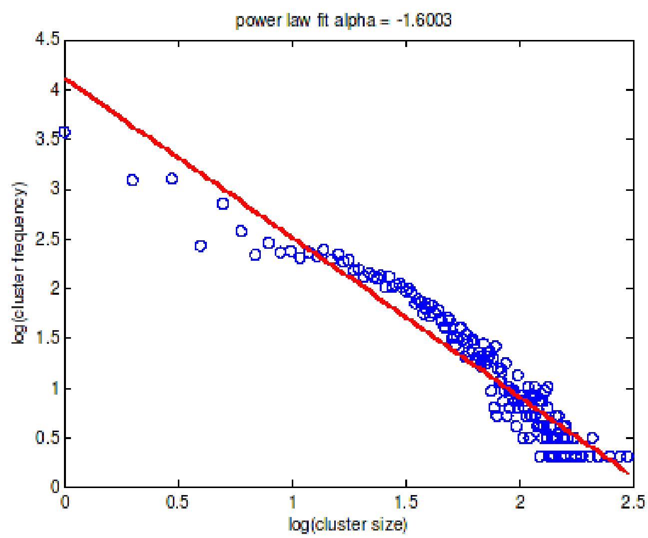

Figure 8: 795×795 matrix

setup does not obey a power law relation. Indeed when we ran the simulation the automata behaved chaotically. Thus we would classify the cellular automata as Class III cellular automata. Simply based on the class of the cellular automata

N x N	Number of Clusters	Avg. Cluster Size	α	Sum Squared Error
399 x 399	3507	17.83	-1.29	3.39
502 x 502	5482	18.14	-1.38	4.44
605 x 605	8044	18.02	-1.48	5.4
707 x 707	11121	17.67	-1.57	6.54
795 x 795	13926	17.81	-1.6	7.63

Table 2: Power Law Fit Data Analysis For T=1.4, R=1, P=S=0

we see that the spatial PD does not exhibit SOC for this particular set of parameters since only class IV cellular automata can exhibit SOC

Fort and Viola [**?**] note that the PD with the above parameter values does not obey the following strict Prisoner's Dilemma inequalities:

$$T > R > P > S \quad \text{and} \quad 2R > S + T \tag{1}$$

They parameterized the PD payoff matrix with a single parameter $\tau = \frac{T}{R}$ as:

$$M = \begin{bmatrix} (1,1) & (-\tau,\tau) \\ (\tau,-\tau) & (-1,-1) \end{bmatrix} \tag{2}$$

With this parameterization and several other similar parameterizations, Fort and Viola concluded that cluster size vs. cluster frequency is a power law relation for $\tau < 3$. They used the log-log plot method described above to measure the exponent of the proposed power law relation. They found for a 400×400 lattice the exponent was $-1.79 \pm .02$ for 150 "lattice sweeps." Thus it remains possible that if we run our simulation again but with parameter values that obey the strict PD inequalities given above, we might find that the cluster size vs. cluster frequency relation is a power law.

We reran the simulation with $T = 1.3, R = 1.1, P = 0.4$, and $S = 0$ and noticed immediately that the cellular automata stabilize in under 10 time steps. Thus these automata are class I (stable) cellular automata and cannot exhibit SOC. Nevertheless we continued to analyze the data as previously. Again we began with an initial proportion of cooperators to defectors 0.50 and a random set up and ran 50 time steps for several different lattice sizes.

N x N	Number of Clusters	Avg. Cluster Size	α	Sum Squared Error
399 x 399	1462	7.39	-3.03	1.61
502 x 502	2348	7.26	-3.24	1.03
605 x 605	3383	7.24	-3.52	1.03
707 x 707	4711	7.22	-3.69	1.15
795 x 795	6042	7.15	-3.95	1.57

Table 3: Power Law Fit Data Analysis For T=1.4, R=1, P=S=0

Note that we have more data points in the second simulation than in the first. We are unsure whether this accounts for the low SSE calculated in Table

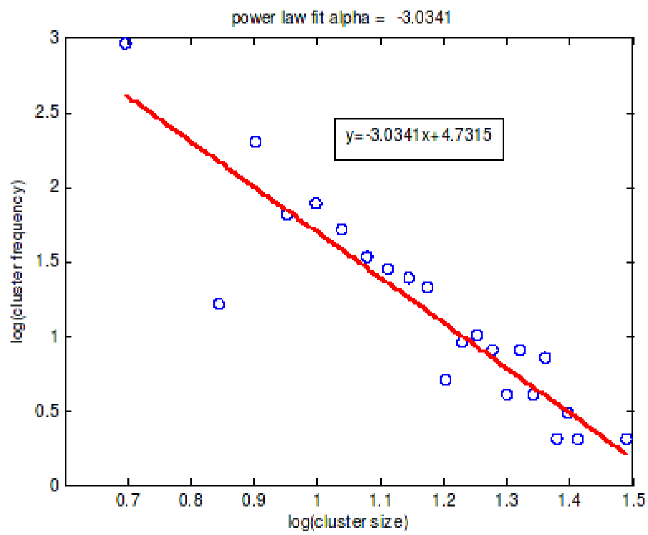

Figure 9: 399 × 399 matrix

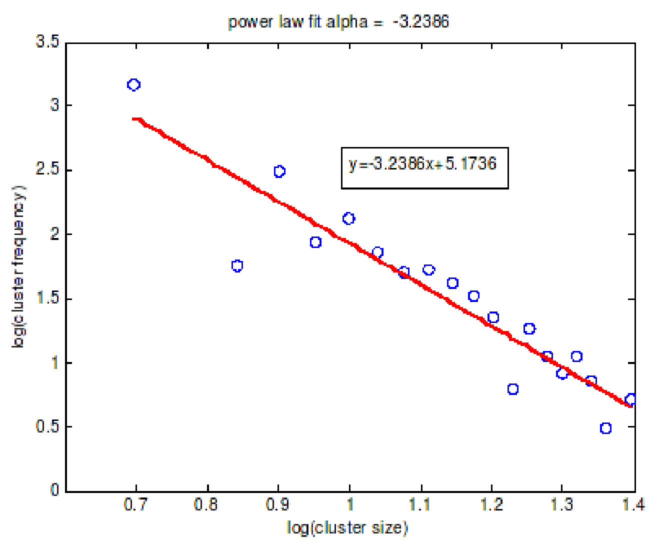

Figure 10: 502 × 502 matrix

3. Since the SSE of α given in Table 3 does not monotonically increase but sometimes decreases for increasing lattice sizes, it is possible that we have a power law relation. But it is impossible that the spatial PD with our particular

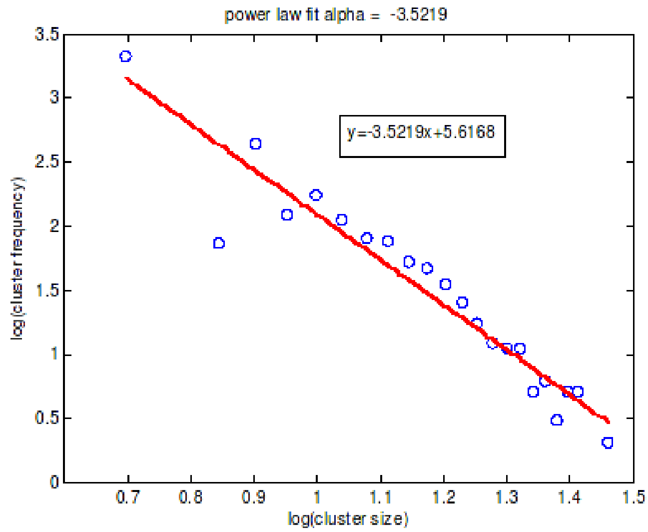

Figure 11: 605×605 matrix

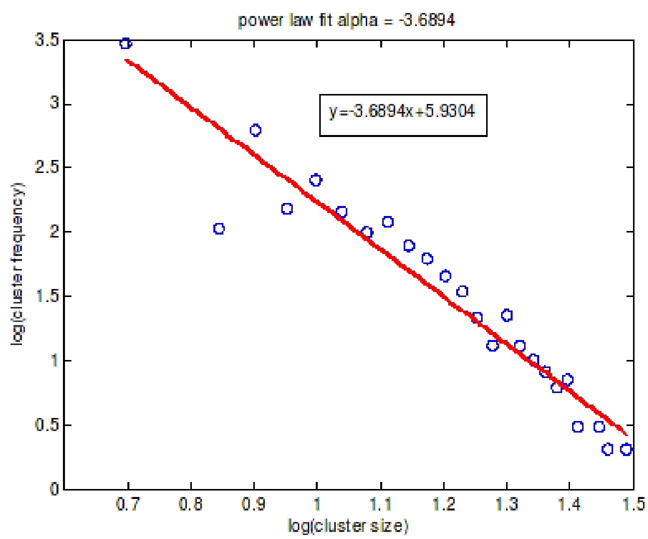

Figure 12: 707×707 matrix

selection of parameters exhibits self-organized criticality because the parameters used generate Class I (stable) cellular automata, as noted above.

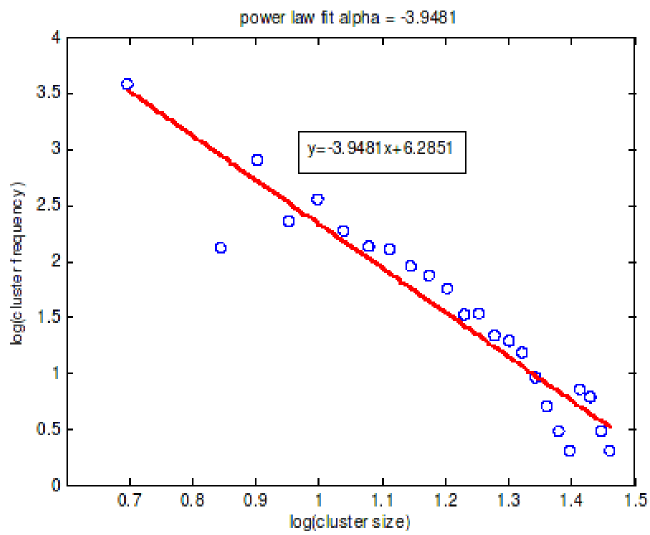

Figure 13: 795 × 795 matrix

3 Conclusion

3.1 Spatial Prisoners Dilemma and SOC

In the end we wanted to know if the spatial prisoners dilemma system that we chose to analyze can be classified as a self organizing critical system. After studying the system we can conclude that the specific system we studied does not exhibit the properties of an SOC system.

An SOC system must self organize to a critical state while our system can self organize to a stable point this point is not a critical state. Looking at the three regions the first region described in 2.4 stabilizes fairly quickly to strips of defectors with mostly cooperators. This state is very stable and does not create avalanches or long transients when perturbed. A perturbation of this stable state will either immediately return to the stable state or remain local just affecting a few neighbors and become a new stable state that is only slightly altered form the original. No perturbations create avalanches which distribute to other regions of the grid so the system is not in a critical state.

The second state of chaos never self organizes to any configuration. This state is in constant change and any small perturbation is unnoticed as the system remains in chaos.

The final cyclic state is similar to the first state in that all perturbations remain local. Any perturbations either destroys the local cycles of cooperators and becomes all defectors or has no lasting affect and the system retains its same cycle.

It is interesting to note that although our system does not exhibit SOC it is scale invariant. We were able to fit power laws to the data and clusters formed at all different sizes. While our particular spatial prisoners' dilemma did not have SOC properties similar systems might. There are many variations to this by having mixed strategies where each node does not act the same way to all of it's neighbors.

4 A bit about Cellular Automata

While performing a thorough study of a specific subset of 1-dimensional cellular automata, Stephen Wolfram [?] devised a classification system for cellular automata [5]. Wolfram noted four distinct classes:

1. Class I: evolution leads to a homogeneous state
2. Class II: evolution leads to simple or periodic structures
3. Class III: evolution leads to a chaotic state
4. Class IV: evolution leads to complex global structures.

As Figure 4 illustrates, it is easy to visualize these class distinctions for 1D cellular automata [6].

5 The Spatial Hawk-Dove Game

	Dove	Hawk
Dove	$1, 1$	$0, 2$
Hawk	$2, 0$	$1-\beta, 1-\beta$

Table 4: A single parameter Hawk-Dove game.

For the parameterization of the Hawk-Dove game given in Table 4, Killingback and Doebeli observe two regions of 'interesting' dynamics. One region they claim is merely chaotic, while the other exhibits "complex, self-organized dynamics". Although we have not yet had the chance to do a thorough comparison between the Killingback-Doebeli Hawk-Dove (HD) game and the parameterization of the Prisoner's Dilemma (PD) that we have investigated, we believe (with some some hesitation) that the PD has only one 'interesting' region and that this region is 'merely' chaotic. One indication which Killingback and Doebeli cite as evidence that the 'critical' region is in fact critical, is the fact that after

[5] See also Jean-Philippe Rennard's website *Introduction to Cellular Automata*: `http://www.rennard.org/alife/english/acintrogb05.html\#nac22`

[6] The image in Figure is taken from `http://www.stephenwolfram.com/publications/articles/general/84-computer/`

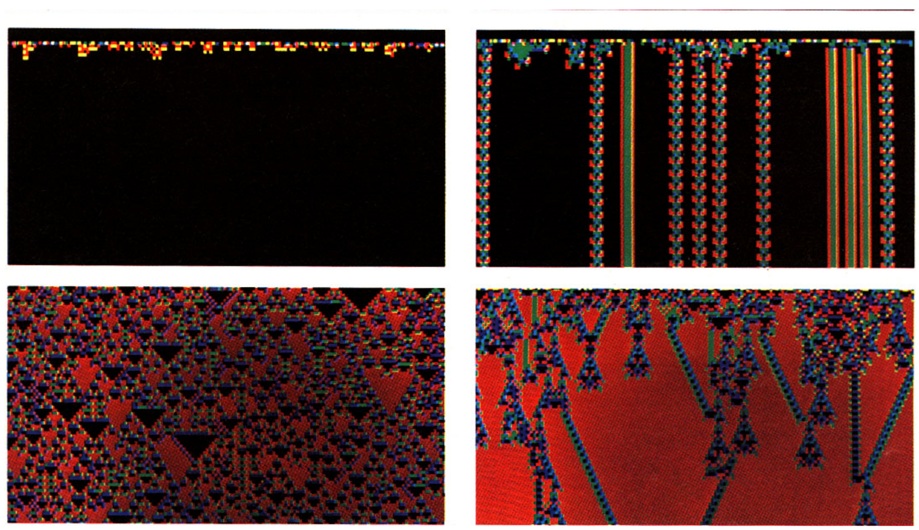

Figure 14: Image taken from Wolfram [Sept84] [**?**]. "The patterns in the [...] four photographs [shown here] begin with disordered states. Even though the values of the cells in these initial states are chosen at random, the evolution of the cellular automata gives rise to structures of four basic classes. In the two classes shown in the [top] row of photographs the long-term behavior of the cellular automata is comparatively simple; in the two classes shown in the bottom row it can be highly complex. The behavior of many natural systems may well conform to this classification."(ibid)

an extremely long transient, the HD settles into a cyclic state. For example, on a mere 35×35 lattice, the transient lasted approximately 70,000 iterations. Moreover, they claim that the transient time increases "at least exponentially" with the size of the lattice. We have not been nearly patient enough to observe whether or not the PD eventually stabilizes. (We have been observing the PD on lattices at least 50×50 and as large as 700×700.) Killingback and Doebeli analyze further aspects of the HD beyond the long transient. Among them are the power spectrum of the frequency-of-hawks time series during the transient phase, the average mutual information of the system from one generation to another for varying beta, and the time taken for avalanches to propagate through the critically stable system. In the future, we would like to probe the PD for these same pieces of information and compare the results to those of the HD game in order to determine more conclusively that the PD does not exhibit critical dynamics.

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Our group has split into three teams. Each team is concentrated (but not limited) on a different part of the project.

Rita and Jennifer have formed the theoretical framework behind the SOC, trying to generalize the definition, its properties/behaviors, and its relationship with the power law distribution.

Brendan and Brent have created the computer program to analyze the prisoner dilemma using the cellular automaton. They further developed a way to count the cluster sizes and collected data for further analysis. Also did theoretical analysis of prisoners' dilemma and other spatial games.

Daniel and Mike have fitted the power law distribution to the data that Brendan and Brent have collected. They further reviewed literatures and studied example of dynamical systems that display SOC.

All members of the group contributed significantly on the understanding of SOC.

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