notes on
multivariable calculus with analytic geometry

by robert picardo
december 2021–may 2022

1 Overview

Multivariable calculus centers

2 Euclidean vector space $\mathbb{R}^n$

In Calculus III, we will deal with the **vectors** that are elements of the $\mathbb{R}^n$ vector space. A vector $\mathbf{v} = \vec{v}$ is represented by an arrow with **magnitude** and **direction**.

The **zero vector** $\mathbf{0}$ is a point; it has zero magnitude and direction.

$\mathbf{u} = \mathbf{v}$ if and only if they have the same direction and magnitude.

Parallel vectors are vectors with the same direction. **Anti-parallel vectors** are vectors with exact opposite directions.

Point-vector correspondence: Each point $(x, y, z, \cdots)$ in $\mathbb{R}^n$ corresponds to a vector $\langle x, y, z, \cdots \rangle$ that can be constructed by the tail at the origin and head at the point.

A **unit vector** $\hat{\mathbf{v}}$ is a vector with magnitude $\|\hat{\mathbf{v}}\| = 1$.

In the $\mathbb{R}^2$ plane, we have the **standard basis vectors** $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$, so that

$$\hat{\mathbf{v}} = \langle a, b \rangle = a\hat{\mathbf{i}} + b\hat{\mathbf{j}}$$

2.1 Vector addition

$$\mathbf{a} + \mathbf{b} = \langle a_1, a_2, \cdots a_n \rangle + \langle b_1, b_2, \cdots b_n \rangle = \langle a_1 + b_1, \ldots, a_n + b_n \rangle$$

2.2 Scalar multiplication

The **scalar product** $a\mathbf{v}$ of a and $\mathbf{v}$ is a vector with magnitude

$$||a\mathbf{v}|| = |a| \cdot ||\mathbf{v}||.$$

Its direction will either be the same as $\mathbf{v}$ if $a > 0$ or the opposite from $\mathbf{v}$ if $a < 0$. If $a = 0$, $a\mathbf{v} = \mathbf{0}$.

2.3 Dot product $\cdot$

The **dot product** $\cdot : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, also called the standard **inner product**, is defined

$$\begin{aligned}\mathbf{a} \cdot \mathbf{b} &= \langle a_1, a_2, \cdots \rangle \cdot \langle b_1, b_2, \cdots \rangle \\ &= a_1 b_1 + a_2 b_2 + \cdots a_n b_n \\ &= ||\mathbf{a}||\, ||\mathbf{b}|| \cos\theta\end{aligned}$$

Proof. The **law of cosines** implies that

$$||\mathbf{a} - \mathbf{b}||^2 = ||\mathbf{a}||^2 + ||\mathbf{b}||^2 - 2||\mathbf{a}||||\mathbf{b}|| \cos\theta$$

$$\begin{aligned}||\mathbf{a} - \mathbf{b}||^2 &= (\mathbf{a} - \mathbf{b}) \cdot (\mathbf{a} - \mathbf{b}) \\ &= \mathbf{a} \cdot (\mathbf{a} - \mathbf{b}) - \mathbf{b} \cdot (\mathbf{a} - \mathbf{b}) \\ &= \mathbf{a} \cdot \mathbf{a} - \mathbf{a} \cdot \mathbf{b} - \mathbf{a} \cdot \mathbf{b} + \mathbf{b} \cdot \mathbf{b} \\ &= ||\mathbf{a}||^2 + ||\mathbf{b}||^2 - 2\mathbf{a} \cdot \mathbf{b} \\ \therefore ||\mathbf{a}||^2 + ||\mathbf{b}||^2 - 2\mathbf{a} \cdot \mathbf{b} &= ||\mathbf{a}||^2 + ||\mathbf{b}||^2 - 2||\mathbf{a}||||\mathbf{b}|| \cos\theta \\ \therefore \mathbf{a} \cdot \mathbf{b} &= ||\mathbf{a}||||\mathbf{b}|| \cos\theta\end{aligned}$$

■

f and g are **orthogonal** iff their **inner product** is 0.

Vectors $\mathbf{v}$ and $\mathbf{u}$ are said to be **orthogonal** iff $\mathbf{v} \cdot \mathbf{u} = \mathbf{0}$.

If $\mathbf{v}, \mathbf{u}$ are non-zero vectors, then $\mathbf{v}, \mathbf{u}$ are orthogonal implies $\mathbf{v}, \mathbf{u}$ are **perpendicular**. The zero vector $\mathbf{0}$ is orthogonal to every vector:

$$\mathbf{0} \cdot \mathbf{v} = 0 \qquad \mathbf{v} \cdot \mathbf{0} = 0$$

Let $\mathbf{u}, \mathbf{v}$ be non-zero vectors. The **projection** of $\mathbf{u}$ onto $\mathbf{v}$, denoted $\operatorname{proj}_{\mathbf{v}} \mathbf{u}$, is defined

$$\operatorname{proj}_{\mathbf{v}} \mathbf{u} = \frac{\mathbf{u} \cdot \mathbf{v}}{\mathbf{v} \cdot \mathbf{v}} \mathbf{v}$$

2.4 Cross product $\times$

$$\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ a_i & a_j & a_k \\ b_i & b_j & b_k \end{vmatrix}$$
$$= (a_j b_k - a_k b_j)\hat{\mathbf{i}} - (a_i b_k - a_k b_i)\hat{\mathbf{j}} + (a_i b_j - a_j b_i)\hat{\mathbf{k}}$$

The cross product is only for 3-**dimensional vectors**. To generalize to higher dimensions, we can use the **wedge product**.

$$\begin{aligned} \mathbf{v}\times\mathbf{u} &= -\mathbf{u}\times\mathbf{v} && \text{(anti-commutative)} \\ \mathbf{u}\times(\mathbf{v}+\mathbf{w}) &= \mathbf{u}\times\mathbf{v}+\mathbf{u}\times\mathbf{w} && \text{(distributive)} \\ (\mathbf{u}+\mathbf{v})\times\mathbf{w} &= \mathbf{u}\times\mathbf{w}+\mathbf{v}\times\mathbf{w} && \text{(distributive)} \\ (\mathbf{u}\times\mathbf{v})\cdot\mathbf{u} = 0 \quad & (\mathbf{u}\times\mathbf{v})\cdot\mathbf{v} = 0 && \text{(normality to } \mathbf{u},\mathbf{v}) \end{aligned}$$

$$\mathbf{u}\times(\mathbf{v}\times\mathbf{w}) = (\mathbf{u}\cdot\mathbf{w})\mathbf{v} - (\mathbf{u}\cdot\mathbf{v})\mathbf{w}$$

$$\text{volume of a parallelepiped} = \mathbf{u}\cdot(\mathbf{v}\times\mathbf{w})$$

Cauchy-Schwarz inequality:
$$|\mathbf{v}\cdot\mathbf{w}| \le ||\mathbf{v}||||\mathbf{w}||$$

Proof. $|\mathbf{v}\cdot\mathbf{w}| = ||\mathbf{v}||\,||\mathbf{w}||\,|\cos\theta|$. Since $|\cos\theta| \le 1$, $||\mathbf{v}||\,||\mathbf{w}||\,|\cos\theta| \le ||\mathbf{v}||\,||\mathbf{w}|| \implies |\mathbf{v}\cdot\mathbf{w}| \le ||\mathbf{v}||\,||\mathbf{w}||$.

Triangle inequalities:

$$||\mathbf{v}+\mathbf{w}|| \le ||\mathbf{v}|| + ||\mathbf{w}||$$
$$||\mathbf{v}|| - ||\mathbf{w}|| \ge ||\mathbf{v}-\mathbf{w}||$$

Jacobi identity:
$$\mathbf{u}\times(\mathbf{v}\times\mathbf{w}) + \mathbf{v}\times(\mathbf{w}\times\mathbf{u}) + \mathbf{w}\times(\mathbf{u}\times\mathbf{v}) = \mathbf{0}$$

3 Lines and planes

Let L be a **line** and $\mathbf{v}$ be any non-$\mathbf{0}$ vector parallel to L. Let $\mathbf{r}$ be a vector defined by the displacement from origin to some chosen point on L.

The set of all points (x_0, y_0, z_0) of the line L corresponds to the set of vectors of the form

$$\mathbf{f}(t) = \mathbf{r} + t\mathbf{v} \quad (t \in \mathbb{R})$$

Solving for t for each of the $\hat{\mathbf{\imath}}, \hat{\mathbf{\jmath}}$, and $\hat{\mathbf{k}}$ components of $\mathbf{f}(t)$ gets you a symmetric equivalence.

Symmetric representation for a line:

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

Plane: A plane that goes through (x_0, y_0, z_0) such that the vector $\mathbf{n} = \langle a, b, c \rangle$ is *normal* to the plane,

$$\langle a, b, c \rangle \cdot (x - x_0, y - y_0, z - z_0) = 0$$

4 Surfaces

A **surface embeddable in three dimensions** is, roughly, the solution set of

$$F(x, y, z) = 0$$

for some continuous function $F : \mathbb{R}^3 \to \mathbb{R}$.

A **plane** is the solution set to $ax + by + cz + d = 0$.

A **sphere** is a set of points that are a constant distance r, called the **radius**, from a point (z_0, y_0, z_0), called the **center**. It follows that a sphere is defined as the solution set to

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 - r^2 = 0$$

Alternatively, for any point (x, y, z), let $\mathbf{x} = x\hat{\mathbf{\imath}} + y\hat{\mathbf{\jmath}} + z\hat{\mathbf{k}}$ and $\mathbf{x_0} = x_0\hat{\mathbf{\imath}} + y_0\hat{\mathbf{\jmath}} + z_0\hat{\mathbf{k}}$. Then,

$$||\mathbf{x} - \mathbf{x_0}|| = r$$

is the **vector** solution set that forms a **sphere**.

Quadratic surfaces are of the form

$$ax^2 + by^2 + cz^2 + dxy + exz + fyz + gx + hy + ij + k = 0$$

where $a, b, \cdots k$ are constants, and $a, b, \cdots f$ are zero.

Examples of 2-D surfaces **non-embeddable** in 3-D space include the **Klein bottle**.

4.1 Paraboloids

Ellipsoid:

$$\frac{(x-x_0)^2}{a^2}+\frac{(y-y_0)^2}{b^2}+\frac{(z-z_0)^2}{c^2}=1$$

a, b, c are called the **principal axes** of the ellipsoid (a, b are called the *semi-minor* and *semi-major axis* for an ellipse).

Hyperboloid of one sheet:

$$\frac{(x-x_0)^2}{a^2}+\frac{(y-y_0)^2}{b^2}-\frac{(z-z_0)^2}{c^2}=1$$

Hyperboloid of two sheets:

$$-\frac{(x-x_0)^2}{a^2}-\frac{(y-y_0)^2}{b^2}+\frac{(z-z_0)^2}{c^2}=1$$

Elliptic paraboloid:

$$\frac{(x-x_0)^2}{a^2}+\frac{(y-y_0)^2}{b^2}=z-z_0$$

Hyperbolic paraboloid (the **Pringles chip**):

$$\frac{(x-x_0)^2}{a^2}-\frac{(y-y_0)^2}{b^2}=z-z_0$$

Elliptic cone:

$$\frac{(x-x_0)^2}{a^2}+\frac{(y-y_0)^2}{b^2}=\frac{(z-z_0)^2}{c^2}$$

5 Coordinates

5.1 Cylindrical coordinates (r, θ, z)

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases} \qquad \tan\theta = \frac{y}{z} \qquad r = \sqrt{x^2+y^2}$$

5.2 Spherical coordinates (ρ, θ, ϕ)

$\rho =$ **radial distance** from a point P to the origin $(p \geq 0)$
$\theta =$ **angle** from positive x-axis to radius $(0 \leq \theta \leq 2\pi)$
$\phi =$ **angle** from positive z-axis to radius $(0 \leq \phi \leq \pi)$

$$x^2 + y^2 + x^2 = \rho^2 \qquad x^2 + y^2 = r^2$$

$$\begin{aligned} x &= r\cos\theta = \rho\sin\phi\cos\theta \\ y &= r\sin\theta = \rho\cos\phi\sin\theta \\ z &= \rho\cos\phi \end{aligned}$$

A **sphere** centered at the origin with radius ρ is defined $\rho = k \in \mathbb{R}_{\geq 0}$.

6 Vector-valued functions

6.1 Derivative

$$\mathbf{v}'(t) = \langle f'(t), g'(t) \rangle$$

Properties:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\,\mathbf{c} &= \mathbf{0} \\ \frac{\mathrm{d}}{\mathrm{d}t}\,k\mathbf{f} &= k\,\frac{\mathrm{d}\mathbf{f}}{\mathrm{d}t} \\ \frac{\mathrm{d}}{\mathrm{d}t}\,(\mathbf{f} \pm \mathbf{g}) &= \frac{\mathrm{d}\mathbf{f}}{\mathrm{d}t} + \frac{\mathrm{d}\mathbf{g}}{\mathrm{d}t} \\ \frac{\mathrm{d}}{\mathrm{d}t}\,[u(t)\mathbf{f}] &= \frac{\mathrm{d}u}{\mathrm{d}t}\,\mathbf{f} + u(t)\,\frac{\mathrm{d}\mathbf{f}}{\mathrm{d}t} \\ \frac{\mathrm{d}}{\mathrm{d}t}\,(\mathbf{f} \cdot \mathbf{g}) &= \frac{\mathrm{d}\mathbf{f}}{\mathrm{d}t} \cdot \mathbf{g} + \mathbf{f} \cdot \frac{\mathrm{d}\mathbf{g}}{\mathrm{d}t} \\ \frac{\mathrm{d}}{\mathrm{d}t}\,(\mathbf{f} \times \mathbf{g}) &= \frac{\mathrm{d}\mathbf{f}}{\mathrm{d}t} \times \mathbf{g} + \mathbf{f} \times \frac{\mathrm{d}\mathbf{g}}{\mathrm{d}t} \\ \frac{\mathrm{d}\mathbf{f}}{\mathrm{d}s} &= \frac{\mathrm{d}\mathbf{f}}{\mathrm{d}t}\,\frac{\mathrm{d}t}{\mathrm{d}s} \end{aligned}$$

6.2 Arc length

$$s = \int_a^b \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\,\mathrm{d}t = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2}\,\mathrm{d}t$$

7 Partial derivative ∂ and gradient ∇

The **partial derivatives** of a function $f : \mathbb{R}^2 \to \mathbb{R}$ are

$$\begin{aligned} \frac{\partial f}{\partial x}(x,y) &= \partial_x f = \frac{\mathrm{d}}{\mathrm{d}t} f(x+t, y) \\ \frac{\partial f}{\partial y}(x,y) &= \partial_y f = \frac{\mathrm{d}}{\mathrm{d}t} f(x, y+t) \end{aligned}$$

$$\begin{aligned} \frac{\partial^2 f}{\partial y \partial x}(x,y) &= \frac{\partial}{\partial y}\left[\frac{\partial f}{\partial x}(x,y)\right] & \frac{\partial^2 f}{\partial x^2}(x,y) &= \frac{\partial}{\partial x}\left[\frac{\partial f}{\partial x}(x,y)\right] \\ \frac{\partial^2 f}{\partial x \partial y}(x,y) &= \frac{\partial}{\partial x}\left[\frac{\partial f}{\partial y}(x,y)\right] & \frac{\partial^2 f}{\partial x^2}(x,y) &= \frac{\partial}{\partial x}\left[\frac{\partial f}{\partial x}(x,y)\right] \end{aligned}$$

Clairaut's theorem on mixed partials:
$\partial_{xy} f(x,y) = \partial_{yx} f(x,y)$ if they are continuous.

The **gradient** of f is

$$\nabla f(x,y) = \left\langle \frac{\partial f}{\partial x}(x,y), \frac{\partial f}{\partial y}(x,y) \right\rangle$$

- in the direction of greatest increase
- normal to the level curve of f

The **directional derivative** at $\mathbf{x} = (x_0, y_0)$ in the $\hat{\mathbf{v}}$-direction is

$$\begin{aligned} \partial_{\hat{\mathbf{v}}} f(\mathbf{x}) &= \lim_{h \to 0} \frac{f(\mathbf{x} + h\hat{\mathbf{v}})) - f(\mathbf{x})}{h} \\ &= \nabla f(\mathbf{x}) \cdot \hat{\mathbf{v}} \end{aligned}$$

8 Extrema and optimization

8.1 Critical points

A local extremum of a differentiable $f : \mathbb{R}^2 \to \mathbb{R}$ is located at any point where

$$\nabla f(x, y) = \mathbf{0}$$

8.2 Second partial derivative test

Let $D(x, y)$ be the determinant of the Hessian matrix of f:

$$D(x, y) = \begin{vmatrix} \partial_{xx} f & \partial_{xy} f \\ \partial_{yx} f & \partial_{yy} f \end{vmatrix} = (\partial_{xx} f) \left(\partial_{yy} f\right) - \left(\partial_{xy} f\right)^2$$

$D(x, y) > 0, \quad \partial_{xx} f < 0 \implies$ local minimum
$D(x, y) > 0, \quad \partial_{xx} f > 0 \implies$ local maximum
$D(x, y) > 0 \implies$ saddle point

8.3 Constrained optimization and the Lagrange multiplier λ

Let $f, g : \mathbb{R}^2 \to \mathbb{R}$ be continuous functions such that g forms a non-empty closed bounded set and $\nabla g(x, y) \neq \mathbf{0}$ for all $g(x, y) = c \in \mathbb{R}$. To optimize f under the constraint of $g(x, y) = c$, solve

$$\nabla f(x, y) = \lambda \nabla g(x, y)$$

$\lambda \in \mathbb{R}$ is called the **Lagrange multiplier**.

9 Multiple integrals

The **double integral** over a region $R \subseteq \mathbb{R}^2$ is

$$\iint\limits_R f(x, y) \, \mathrm{d}A = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y) \, \mathrm{d}y \, \mathrm{d}x = \int_c^d \int_{h_1(x)}^{h_2(x)} f(x, y) \, \mathrm{d}x \, \mathrm{d}y$$

The **triple integral** over a solid $V \subseteq \mathbb{R}^3$ is

$$\iiint\limits_V f(x, y, z) \, \mathrm{d}V = \iiint\limits_V f(x, y, z) \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z$$

9.1 Substitution for multiple variables

The **Jacobian determinant** of a transformation with continuous partial derivatives $(u, v) \mapsto (x(u, v), y(u, v))$ is

$$\left|\frac{\partial(x, y)}{\partial(u, v)}\right| = \begin{vmatrix} \partial_u x & \partial_u y \\ \partial_v x & \partial_v y \end{vmatrix} = (\partial_u x)(\partial_v y) - (\partial_v y)(\partial_u x)$$

Let $\phi : R \to R' : (u, v) \mapsto (x(u, v), y(u, v))$ be an injective transformation with continuous partial derivatives, where $\left|\frac{\partial(x,y)}{\partial(u,v)}\right| \neq 0$ for all $u, v \in R'$. If f is continuous, then

$$\iint\limits_R f(x, y)\,\mathrm{d}A = \iint\limits_{R'} f(x(u, v), y(u, v)) \left|\frac{\partial(x, y)}{\partial(u, v)}\right| \mathrm{d}u\,\mathrm{d}v$$

$$\left|\frac{\partial(x, y)}{\partial(r, \theta)}\right| = r \qquad \mathrm{d}x\,\mathrm{d}y = r\,\mathrm{d}r\,\mathrm{d}\theta$$

$$\left|\frac{\partial(x, y, z)}{\partial(\rho, \theta, \phi)}\right| = \rho^2 \sin\phi \qquad \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \rho^2 \sin\phi\,\mathrm{d}\rho\,\mathrm{d}\theta\,\mathrm{d}\phi$$

10 Line integrals

10.1 Line integral for scalar fields

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a surface, and x and y be functions of t. The **line integral** of f along a curve C is

$$\int_C f(x, y)\,\mathrm{d}s = \int_a^b f(x, y)\sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\,\mathrm{d}t$$

10.2 Line integral for vector fields

Let $\mathbf{F} = \langle p(x, y), q(x, y)\rangle$ be a vector field, and $\mathbf{r} = \langle x, y\rangle$. The **line integral** of $\mathbf{F}$ along C is

$$\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_C p(x, y)\,\mathrm{d}x + q(x, y)\,\mathrm{d}y$$

Letting C be a curve in $\mathbf{F}$ with smooth parametrization $x(t), y(t)$ for $a \leq t \leq b$,

$$\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_a^b \mathbf{F}\left(x(t), y(t)\right) \cdot \mathbf{r}'(t)\,\mathrm{d}t$$

10.3 Conservative vector fields

$$\underset{\text{conservative}}{\mathbf{F} = (A, B)} \iff \oint_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = 0 \iff \exists \phi : \nabla\phi = \mathbf{F} \iff \frac{\partial A}{\partial y} = \frac{\partial B}{\partial x} \iff \underset{\text{path-independent}}{\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r}}$$

11 Gradient theorem

Given a path C from $\mathbf{A}$ to $\mathbf{B}$,

$$\int_C \nabla\phi \cdot \mathrm{d}\mathbf{r} = \phi(\mathbf{B}) - \phi(\mathbf{A})$$

12 Green's theorem

Let $U \subset \mathbb{R}^2$ be a simply connected region with positively oriented boundary ∂U, and A, B be functions defined on U with continuous partial derivatives. Then,

$$\oint\limits_{\partial U} A\,\mathrm{d}x + B\,\mathrm{d}y = \iint\limits_U \left(\frac{\partial B}{\partial x} - \frac{\partial A}{\partial y}\right) \mathrm{d}x\,\mathrm{d}y$$

12.1 Multiply connected regions

Given a multiply connected region—enclosed by C_1 with a hole enclosed by C_2 in the middle, for example—Green's theorem still applies.

$$\begin{aligned}\oint\limits_{C_1 \cup C_2} \mathbf{F} \cdot \mathrm{d}\mathbf{r} &= \oint\limits_{\partial R_1} \mathbf{F} \cdot \mathrm{d}\mathbf{r} + \oint\limits_{\partial R_2} \mathbf{F} \cdot \mathrm{d}\mathbf{r} \\ &= \iint\limits_R \left(\frac{\partial B}{\partial x} - \frac{\partial A}{\partial y}\right) \mathrm{d}x\,\mathrm{d}y\end{aligned}$$

13 Surface integrals

13.1 Surface integral for scalar fields

The **surface integral** of $f : \mathbb{R}^3 \to \mathbb{R}$ over the surface $\Sigma \subset \mathbb{R}^3$, parametrized by $x(u,v), y(u,v), z(u,v)$, is

$$\iint_\Sigma f(x,y,z)\,\mathrm{d}\sigma = \iint_\Sigma f(x,y,z) \left\| \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right\| \mathrm{d}u\,\mathrm{d}v$$

13.2 Surface integral for vector fields

The **surface integral** of vector field $\mathbf{F}$ over the surface $\Sigma \subset \mathbb{R}^3$, parametrized by $x(s,t), y(s,t), z(s,t)$, is

$$\iint_\Sigma \mathbf{F} \cdot \mathrm{d}\boldsymbol{\sigma} = \iint_{\Sigma'} \mathbf{F}(x,y,z) \left(\frac{\partial \mathbf{r}}{\partial s} \times \frac{\partial \mathbf{r}}{\partial t} \right) \mathrm{d}s\,\mathrm{d}t$$

14 Divergence theorem

Let $V \subset \mathbb{R}^3$ be a compact solid with smooth boundary ∂V, and $\mathbf{F}$ be a continuously differentiable vector field. Then,

$$\oiint_{\partial V} \mathbf{F} \cdot d\boldsymbol{\sigma} = \iiint_V \nabla \cdot \mathbf{F}\,\mathrm{d}V$$

$$\operatorname{div} \mathbf{F} = \nabla \cdot \mathbf{F} = \left\langle \partial_x, \partial_y, \partial_z \right\rangle \cdot \langle F_1, F_2, F_3 \rangle$$

15 Stokes' theorem

Let Σ be an orientable smooth surface in $\mathbb{R}^3$ with closed boundary $\partial\Sigma$, and $\mathbf{F}$ be a continuously differentiable vector field. Then,

$$\oint_{\partial\Sigma} \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \iint_\Sigma \nabla \times \mathbf{F} \cdot \mathrm{d}\boldsymbol{\sigma}$$

$$\operatorname{curl} \mathbf{F} = \operatorname{rot} \mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \partial_x & \partial_y & \partial_z \\ F_1 & F_2 & F_3 \end{vmatrix}$$

An **orientable surface** is a surface with a possible smooth normal $\hat{\mathbf{n}}$.

16 Generalized Stokes' theorem

Let ω be a smooth $(n-1)$-differential form with compact support on a smooth n-dimensional oriented manifold χ. Then,

$$\int_{\partial\chi} \omega = \int_{\chi} \mathrm{d}\omega$$

where $\mathrm{d}\omega$ is the exterior derivative of ω.

17 Potentials

ϕ is a scalar potential for $\mathbf{F}$ if

$$\nabla\phi = \mathbf{F}$$

ϕ exists if

$$\nabla \times \mathbf{F} = \mathbf{0}$$

$\mathbf{A}$ is a vector potential for $\mathbf{F}$ if

$$\nabla \times \mathbf{A} = \mathbf{F}$$

$\mathbf{A}$ exists if

$$\nabla \cdot \mathbf{F} = 0$$

18 Laplacian

$$\Delta = \nabla^2 = \nabla \cdot \nabla$$

19 ∇ in non-rectangular coordinates

19.1 ∇ in cylindrical coordinates

Given ϕ, scalar potential of $\mathbf{F} = F_r\hat{\mathbf{e}}_r + F_\theta\hat{\mathbf{e}}_\theta + F_z\hat{\mathbf{e}}_z$,

$$
\begin{aligned}
\nabla\phi &= (\partial_r\phi)\,\hat{\mathbf{e}}_r + \left(\frac{\partial_\theta\phi}{r}\right)\hat{\mathbf{e}}_\theta + (\partial_z\phi)\,\hat{\mathbf{e}}_z \\
\nabla\cdot\mathbf{F} &= \frac{1}{r}\partial_r(rF_r) + \frac{1}{r}\partial_\theta F_\theta + \partial_z F_z \\
\nabla\times\mathbf{F} &= \left(\frac{1}{r}\partial_\theta F_z - \partial_z F_\theta\right)\hat{\mathbf{e}}_r + (\partial_z F_\theta - \partial_r F_z)\,\hat{\mathbf{e}}_\theta \\
&\quad + \frac{1}{r}\left(\partial_r(rF_\theta) - \partial_\theta F_r\right)\hat{\mathbf{e}}_z \\
\nabla^2\phi &= \frac{1}{r}\partial_r\left(r\partial_r\phi\right) + \frac{1}{r^2}\partial^2_{\theta\theta}\phi + \partial_{zz}\phi
\end{aligned}
$$

19.2 ∇ in spherical coordinates

Given Φ, scalar potential of $\mathbf{F} = F_\rho\hat{\mathbf{e}}_\rho + F_\theta\hat{\mathbf{e}}_\theta + F_\phi\hat{\mathbf{e}}_\phi$,

$$
\begin{aligned}
\nabla\Phi &= \left(\partial_\rho\Phi\right)\hat{\mathbf{e}}_\rho + \left(\frac{\partial_\theta\Phi}{\rho\sin\phi}\right)\hat{\mathbf{e}}_\theta + \left(\frac{\partial_\phi\Phi}{\rho}\right)\hat{\mathbf{e}}_\phi \\
\nabla\cdot\mathbf{F} &= \frac{1}{\rho^2}\partial_\rho(\rho^2F_\rho) + \frac{1}{\rho\sin\phi}\partial_\theta F_\theta + \frac{1}{\rho\sin\phi}\partial_\phi\left(\sin\phi F_\phi\right) \\
\nabla\times\mathbf{F} &= \frac{1}{\rho\sin\phi}\Big(\partial_\phi\left(\sin\phi F_\phi\right) - \partial_\theta F_\theta\Big)\hat{\mathbf{e}}_\rho + \frac{1}{\rho}\left(\partial_\rho\left(\rho F_\theta\right) - \partial_\phi F_\rho\right)\hat{\mathbf{e}}_\theta \\
&\quad + \left(\frac{1}{\rho\sin\phi}\partial_\theta F_\rho - \frac{1}{\rho}\partial_\rho\left(\rho F_\theta\right)\right)\hat{\mathbf{e}}_\phi \\
\nabla^2\phi &= \frac{1}{\rho^2}\partial_\rho(\rho^2\partial_\rho\Phi) + \frac{1}{\rho^2\sin^2\phi}\partial_{\theta\theta}\Phi + \frac{1}{\rho^2\sin^2\phi}\partial_\phi\left(\sin\phi\partial_\phi\Phi\right)
\end{aligned}
$$

19.3 Transformation matrices

From cylindrical to Cartesian,

$$
\mathbf{T}_{\varepsilon_3\leftarrow\mathrm{cyc}} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}
$$

From Cartesian to cylindrical,

$$\mathbf{T}_{\mathrm{cyc}\leftarrow\varepsilon_3} = \mathbf{T}^{-1}_{\varepsilon_3\leftarrow\mathrm{cyc}} == \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

These notes, taken from December 2021 to 30 May 2022, were written under the following courses:

- Eugene L. Berg. *Advanced Placement Calculus AB*, Jefferson High School, The College Board (see the Course and Exam Description here).
- Sean-Giacomo Nguyen. `MATH 252` *Calculus with Analytic Geometry II*, Skyline College.
- Shawn M. Westmoreland. `MATH 253` *Calculus with Analytic Geometry III*, College of San Mateo. Based on *Vector Calculus* by Michael Corral.