notes on
single-variable calculus with analytic geometry

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1 Limits

1.1 Epsilon-delta definition

Let $(a, b) \subseteq \mathbb{R}$ be an interval, $c \in (a, b)$ be a limit point, and $f : (a, b) \setminus \{c\} \to \mathbb{R}$ be a function. Then, the **limit** of $f(x)$ as x approaches c is defined such that

$$\lim_{x\to c} f(x) = L \iff \forall\epsilon > \mathbb{R}_{>0} : \exists\delta > \mathbb{R}_{>0} : \forall x \in (a, b) : 0 < |x-c| < \delta \implies |f(x)-L| < \epsilon$$

$$\lim_{x\to c^-} f(x) = \lim_{x\to c^+} f(x) = L \iff \lim_{x\to c} f(x) = L$$

2 Continuous functions

f is **continuous** at $x = c$ if and only if the limit $\lim\limits_{x\to c} f(x)$ exists and $\lim\limits_{x\to c} f(x) = f(c)$.

2.1 Extreme value theorem

If f is continuous on $[a, b]$, then f has a minimum and maximum value on $[a, b]$.

2.2 Intermediate value theorem

If f is continuous on $[a, b]$, then

$$M \in [f(a), f(b)] \implies \text{there exists } c \in [a, b] \text{ such that } f(c) = M$$

3 Differentiation

3.1 Definition of derivative

$$f'(x) = \frac{\mathrm{d}f}{\mathrm{d}x} = \lim_{h\to 0} \frac{f(x + h) - f(x)}{h}$$

3.2 Properties

$$
\begin{aligned}
[cf(x)]' &= cf'(x) & \frac{\mathrm{d}}{\mathrm{d}x}(c) &= 0 \\
(fg)' &= f'g + fg' & \frac{\mathrm{d}}{\mathrm{d}x}x^n &= nx^{n-1} \\
\left(\frac{f}{g}\right)' &= \frac{f'g - fg'}{g^2} & \frac{\mathrm{d}}{\mathrm{d}x}f[g(x)] &= \frac{\mathrm{d}f}{\mathrm{d}g} \cdot \frac{\mathrm{d}g}{\mathrm{d}x}
\end{aligned}
$$

3.3 Common derivatives

$$
\begin{aligned}
(\sin x)' &= \cos x \\
(\cos x)' &= -\sin x \\
(\tan x)' &= \sec^2 x \\
(\sec x)' &= \sec x \tan x \\
(\csc x)' &= -\csc x \cot x \\
(\cot x)' &= -\csc^2 x
\end{aligned}
$$

$$
\begin{aligned}
(\arcsin x)' &= \frac{1}{\sqrt{1-x^2}} \\
(\arccos x)' &= -(\arcsin x)' \\
(\arctan x)' &= \frac{1}{1+x^2}
\end{aligned}
$$

$$
\begin{aligned}
(a^x)' &= a^x \ln a \\
(e^x)' &= e^x \\
(\ln x)' &= \frac{1}{x}, x > 0 \\
(\ln |x|)' &= \frac{1}{x}, x \neq 0 \\
(\log_a x)' &= \left(\frac{\ln x}{\ln a}\right)'
\end{aligned}
$$

3.4 Derivatives of inverse functions

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{\frac{\mathrm{d}x}{\mathrm{d}y}}$$

3.5 Higher-order derivatives

The **second derivative** is the derivative of the first derivative.

$$f''(x) = \frac{\mathrm{d}^2 f}{\mathrm{d}x^2} = [f'(x)]'$$

Likewise, the n^{th} **derivative** is the $(n-1)^{\text{th}}$ derivative of the first derivative.

$$f^{(n)}(x) = \frac{\mathrm{d}^n f}{\mathrm{d}x^n} = [f^{(n-1)}(x)]'$$

3.6 Differentiability

f is **differentiable** at $x = a$ iff $f'(a) = \dfrac{f(a+h) - a}{h}$ exists.

f is **differentiable at interval** I iff it is differentiable at every point $a \in I$.

4 Applications of differentiation

A **critical point** of f is any x where $f'(x) = 0$ or $f'(x)$ doesn't exist.

An **inflection point** of f is any c where its concavity, or the sign of $f''(x)$, changes through $x = c$.

$$\begin{aligned} f'(x) > 0 &\implies f \text{ increases} \\ f'(x) < 0 &\implies f \text{ decreases} \\ f'(x) = 0 &\implies f \text{ is constant} \end{aligned}$$

$$\begin{aligned} f'(x) > 0 &\implies f \text{ is concave up / convex} \\ f'(x) < 0 &\implies f \text{ is concave down / concave} \end{aligned}$$

Second derivative test: A critical point $x = c$ has either
a **relative maximum** if $f''(c) < 0$, or
a **relative minimum** if $f''(c) > 0$.

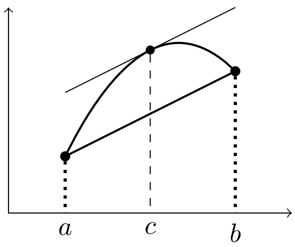
4.1 Mean value theorem

Given that a function f is

1. continuous over the closed interval $[a, b]$
2. differentiable over the open interval (a, b)

there is at least one value c in (a, b) where

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$



4.2 L'Hôpital's rule

Let a be a real number in some open interval I where f and g are differentiable, and that $g'(x) \neq 0$ in $I \setminus \{a\}$.
If $f(a) = g(a) = 0$, then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

5 Integration

5.1 Indefinite integrals

If $f'(x)$ is the derivative of $f(x)$, then $f(x) + C$ is called the **indefinite integral** or **antiderivative** of $f'(x)$.

$$\int f'(x)\,\mathrm{d}x = f(x) + C, \quad C \in \mathbb{R}$$

5.2 Integration by substitution

$$\int f(u)\,\mathrm{d}u = \int f(u)\frac{\mathrm{d}u}{\mathrm{d}x}\,\mathrm{d}x$$

5.3 Integration by parts

$$\int f'(x)g(x)\,\mathrm{d}x = f(x)g(x) - \int f(x)g'(x)\,\mathrm{d}x$$

5.4 Integration by trigonometric substitution

If the integrand contains:

$$\begin{aligned} a^2 - x^2 &\implies \text{substitute } x = a\sin\theta \\ a^2 + x^2 &\implies \text{substitute } x = a\cos\theta \\ x^2 - a^2 &\implies \text{substitute } x = a\sec\theta \end{aligned}$$

5.5 Integration by partial fractions

Given $\int \frac{P(x)}{Q(x)}\,\mathrm{d}x$ where $\deg(P) < \deg(Q)$,

$$\begin{aligned} \frac{P(x)}{(x-a_1)(x-a_2)\cdots(x-a_n)} &= \frac{A_1}{(x-a_1)} + \frac{A_2}{(x-a_2)} + \cdots \frac{A_n}{(x-a_n)} \\ \frac{P(x)}{(x-a)^n} &= \frac{A_1}{(x-a)} + \frac{A_2}{(x-a)^2} + \cdots \frac{A_n}{(x-a)^n} \\ \frac{P(x)}{(x-\alpha)(ax^2+bx^2+c)} &= \frac{A}{(x-\alpha)} + \frac{Bx+C}{ax^2+bx^2+C} \end{aligned}$$

5.6 Definite integrals

$$\int_a^b f(x)\,\mathrm{d}x = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$$

5.7 Numerical approximation

$$\int_a^b f(x)\,\mathrm{d}x \approx$$

$$L_n = \sum_{i=0}^{n-1} f(x_i)\Delta x = \Delta x \left[f(x_0) + \cdots + f(x_{n-1})\right]$$
$$R_n = \sum_{i=1}^{n} f(x_i)\Delta x = \Delta x \left[f(x_1) + \cdots + f(x_n)\right]$$
$$M_n = \sum_{i=1}^{n} f(\bar{x}_i)\Delta x = \Delta x \left[f(\bar{x}_1) + \cdots + f(\bar{x}_n)\right]$$

Trapezoidal rule:

$$\int_a^b f(x)\,\mathrm{d}x \approx$$

$$T_n = \frac{\Delta x}{2}\left[f(x_0) + 2f(x_1) + \cdots 2f(x_{n-1}) + f(x_n)\right]$$

where $\Delta x = \frac{b-a}{n}$ and $x_i = a + i\Delta x$

Simpson's rule:

$$\int_a^b f(x)\,\mathrm{d}x \approx$$

$$S_n = \frac{\Delta x}{3}\left[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \cdots 4f(x_{n-1}) + f(x_n)\right]$$

where n must be even and $\Delta x = \frac{b-a}{n}$

Error bounds for E_M, E_T, E_S

$$|E_M| \leq \frac{\max|f''(x)|(b-a)^3}{24n^2} \qquad |E_T| \leq \frac{\max|f''(x)|(b-a)^3}{12n^2}$$

$$|E_S| \leq \frac{\max|f^{(4)}(x)|(b-a)^5}{180n^4}$$

5.8 Fundamental theorem of calculus

$$f(x) = \int_a^x f'(t)\,\mathrm{d}t$$

$$\int_a^b f'(x)\,\mathrm{d}x = f(b) - f(a)$$

5.9 Improper integrals

$$\int_a^\infty f(x)\,\mathrm{d}x = \lim_{t\to\infty}\int_a^t f(x)\,\mathrm{d}x \quad \text{(type I)}$$

$$\lim_{x\to b^-} f(x) = \pm\infty \implies \int_a^b f(x)\,\mathrm{d}x = \lim_{t\to b^-}\int_a^t f(x)\,\mathrm{d}x \quad \text{(type II)}$$

$$\lim_{x\to a^+} f(x) = \pm\infty \implies \int_a^b f(x)\,\mathrm{d}x = \lim_{t\to a^+}\int_t^b f(x)\,\mathrm{d}x \quad \text{(type II)}$$

We say that an improper integral **converges** only if the limit exists, and that it **diverges** if the limit does not exist.

$$\int_{-\infty}^\infty f(x)\,\mathrm{d}x = \int_{-\infty}^c f(x)\,\mathrm{d}x + \int_c^\infty f(x)\,\mathrm{d}x$$

6 Applications of integration

6.1 Average value

$$f_{\text{avg}} = \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x$$

6.2 Mean value theorem for integrals

If f is continuous over $[a, b]$, then

$$\exists c : f(c) = f_{\text{avg}} = \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x$$

6.3 Volume by disk / washer method

Volume by **disk method**:

$$\pi \int_a^b [R(x)]^2 \, \mathrm{d}x \quad \text{or} \quad \pi \int_c^d [R(y)]^2 \, \mathrm{d}y$$

Volume by **washer method**:

$$\pi \int_a^b \left([R(x)]^2 - [r(x)]^2 \right) \mathrm{d}x \quad \text{or} \quad \pi \int_c^d \left([R(y)]^2 - [r(y)]^2 \right) \mathrm{d}y$$

Volume by **shell method**:

$$2\pi \int_a^b x f(x) \, \mathrm{d}x \quad \text{or} \quad 2\pi \int_c^d y f(y) \, \mathrm{d}y$$

6.4 Arc length

$$s = \int_a^b \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x} \right)^2} \, \mathrm{d}x \qquad s = \int_c^d \sqrt{1 + \left(\frac{\mathrm{d}x}{\mathrm{d}y} \right)^2} \, \mathrm{d}y$$

6.5 Area of surface of revolution

$$A = 2\pi \int_a^b r \, \mathrm{d}s$$

$$= 2\pi \int_a^b y \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x} \right)^2} \, \mathrm{d}x \qquad = 2\pi \int_c^d x \sqrt{1 + \left(\frac{\mathrm{d}x}{\mathrm{d}y} \right)^2} \, \mathrm{d}y$$

6.6 Work

The **work** required to move an object along an axis from $x = a$ to $x = b$ with magnitude $f(x)$ is

$$W = \lim_{n \to \infty} \sum_{i=1}^{n} W_i = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i) \Delta x = \int_a^b f(x) \, \mathrm{d}x$$

This is derived from the sum of **force** $\cdot$ **displacement** (the notion of work) over infinitely small displacements.

In a **lifting problem**, the object being lifted requires non-constant force, since different parts of the object are lifted different distances. The work W_i associated with the ith layer of the object of height h is

$$\begin{aligned} W_i &= F_i \cdot d_i \\ &= m \cdot g \cdot d_i \\ &= \rho V_i \cdot g \cdot (h - y_i) \\ &= \rho A(y_i) \Delta y \cdot g \cdot (h - y_i) \end{aligned}$$

$$W = \lim_{n\to\infty} \sum_{i=1}^{n} \rho g A(y_i)(h - y_i)\Delta y = \int_a^b \rho g A(y)(h - y)\,\mathrm{d}y$$

($g = 9.8\mathrm{m\,s^{-2}}$. Weight-density $\delta = \rho g$ may be given in Newtons.)

6.7 Hydrostatic pressure and force

The **pressure** p at depth d in a fluid of mass density ρ is $p = \rho g d = \delta d$.

We extend the notion of force ($F = p \cdot A = \rho g d \cdot A$) to surfaces with non-constant width. If we declare width of the ith strip as $L(y_i)$, then $A = L(y_i)\Delta y$, and its depth d is equal to y. The sum of all values of $F_i = \rho g y L(y_i)\Delta y$ gives:

The **fluid force** F on a flat side of **an object submerged vertically** in a fluid is

$$F = \lim_{n\to\infty} \sum_{i=1}^{n} \rho g y_i L(y_i)\Delta y = \int_a^b \rho g y L(y)\,\mathrm{d}y$$

6.8 Moments and center of mass

Many objects behave as if their masses are concentrated *at a single point*. This point is called the center of mass. Imagine a seesaw with two objects. Then its fulcrum must be at the center of mass, which is a point at which the (mass) · (distance) of each object is equal:

$$m_1 d_1 = m_2 d_2 \implies m_1(\bar{x} - x_1) = m_2(x_2 - \bar{x}) \implies \bar{x} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}$$

$$M_y = \sum_i m_i x_i \quad (\textbf{moment} \text{ about the } y\text{-axis})$$

$$M_x = \sum_i m_i y_i \quad (\textbf{moment} \text{ about the } x\text{-axis})$$

A system with constituent masses $m_1, m_2, \cdots m_n$, with total mass $M = \sum m_i$, has a **center of mass** $P(\bar{x}, \bar{y})$, where

$$\bar{x} = \frac{M_y}{M} = \frac{\sum m_i x_i}{\sum m_i}$$
$$\bar{y} = \frac{M_x}{M} = \frac{\sum m_i y_i}{\sum m_i}$$

6.9 Centroid of a lamina

A **lamina** (thin plate) with uniform area density ρ_A bounded by two curves $f \geq g$ has a **center of mass** $P(\bar{x}, \bar{y})$ where

$$\bar{x} = \frac{1}{A} \int_a^b x[f(x) - g(x)] \, \mathrm{d}x$$
$$\bar{y} = \frac{1}{2A} \int_a^b \left([f(x)]^2 - [g(x)]^2 \right) \mathrm{d}x$$
$$\text{where } A = \int_a^b [f(x) - g(x)] \, \mathrm{d}x$$

6.10 Pappus's theorem

Let $\mathcal{R}$ be the region of a plane with area A that we rotate about an axis disjoint from $\mathcal{R}$. Then the volume of the resulting solid is the product of A and the distance traveled by the centroid of $\mathcal{R}$.

7 Differential equations

7.1 Separable differential equations

Given a differential equation of the form

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x)g(y),$$

we **distribute the variables to opposite sides** and integrate.

$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} &= f(x)g(y) \\
\frac{1}{g(y)}\frac{\mathrm{d}y}{\mathrm{d}x} &= f(x) \\
\int \frac{1}{g(y)}\frac{\mathrm{d}y}{\mathrm{d}x}\,\mathrm{d}x &= \int f(x)\,\mathrm{d}x \qquad \text{(separate the variables)} \\
\int \frac{1}{g(y)}\,\mathrm{d}y &= \int f(x)\,\mathrm{d}x \qquad \text{(integration by } y\text{-substitution)}
\end{aligned}$$

7.2 Linear differential equations

A **linear differential equation** is defined by a linear polynomial in the unknown function and its derivative:

$$a_o(x)y + a_1(x)y' + a_2(x)y'' + \cdots + a_n(x)y^{(n)} + b(x) = 0$$

where $A_0(x), a_1(x), \cdots, a_n(x), b(x)$ are *differentiable* functions that don't need to be linear.

7.3 Logistic model

Given a population model with a **carrying capacity** M,

$$\frac{\mathrm{d}P}{\mathrm{d}t} = kP\left(1 - \frac{P}{M}\right) \implies P(t) = \frac{M}{1 + Ae^{-kt}} \qquad \left[A = \frac{M - P(0)}{P(0)}\right]$$

7.4 First-order linear ordinary differential equations

A **first-order linear ordinary differential equation** is a differential equation that involves the first derivative:

$$y' + P(x)y = Q(x)$$

$$y' + P(x)y = Q(x) \implies y = \frac{1}{\mu(x)} \int \mu(x)Q(x)\,\mathrm{d}x$$

where

$$\mu(x) = e^{\int P(x)\,\mathrm{d}x}$$

Proof. We multiply an **integrating factor** $\mu(x)$ such that the left side can be expressed as the derivative of another function through the *product rule*:

$$\mu(x)[y' + P(x)y] = \mu(x)Q(x)$$
$$\mu(x)y' + \mu(x)P(x)y = \mu(x)Q(x)$$

Let $\mu'(x) = \mu(x)P(x)$. Then,

$$\begin{aligned}\mu(x)y' + \mu'(x)y &= \mu(x)Q(x)\\ \big[\mu(x)y\big]' &= \mu(x)Q(x)\\ \mu(x)y &= \int \mu(x)Q(x)\,\mathrm{d}x\end{aligned}$$

$$y = \frac{1}{\mu(x)} \int \mu(x)Q(x)\,\mathrm{d}x$$

From our supposing that $\mu'(x) = \mu(x)P(x)$, we get

$$\frac{\mathrm{d}\mu}{\mathrm{d}x} = \mu(x)P(x) \implies \ln|\mu| = \int P(x)\,\mathrm{d}x \implies \mu(x) = \pm e^{\int P(x)\,\mathrm{d}x}$$

8 Parametric equations

A **line** with point (a, b) of slope $m = s/r$ is parametrized

$$x = a + rt, y = b + st, t \in \mathbb{R}$$

A **line** with point (a, b) and (c, d) is parametrized

$$x = a + t(c - a), y = b + t(d - b), t \in \mathbb{R}$$

A **circle of radius** r centered at (h, k) is parameterized

$$x = h + r\cos\theta, \qquad y = k + r\sin\theta$$

8.1 Slope of a tangent line

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\frac{\mathrm{d}y}{\mathrm{d}t}}{\frac{\mathrm{d}x}{\mathrm{d}t}}$$

8.2 Area under a parametric curve

$$A = \int_\alpha^\beta y \, \frac{\mathrm{d}x}{\mathrm{d}t} \, \mathrm{d}t = \int_\beta^\alpha y \, \frac{\mathrm{d}x}{\mathrm{d}t} \, \mathrm{d}t$$

8.3 Arc length

$$s = \int_\alpha^\beta \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2} \, \mathrm{d}t$$

8.4 Speed

$$\frac{\mathrm{d}s}{\mathrm{d}t} = \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}$$

8.5 Area of surface of revolution

The **area of the surface** obtained by rotating the curve about the x-axis is

$$S = 2\pi \int_\alpha^\beta r \, \mathrm{d}s = 2\pi \int_\alpha^\beta y \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2} \, \mathrm{d}t$$

9 Polar coordinates

To convert from polar to rectangular coordinates, use $x = r\cos\theta, y = r\sin\theta$. To convert from rectangular to polar coordinates, use $r^2 = x^2 + y^2$ and $\tan\theta = \frac{y}{x}$.

9.1 Slope of tangent line

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{(r\sin\theta)'}{(r\cos\theta)'}$$

9.2 Area in polar functions

An **area of region** $0 \leq g(\theta) \leq r \leq f(\theta)$ and $\alpha \leq \theta \leq \beta$ is

$$\frac{1}{2}\int_{\alpha}^{\beta} [f(\theta)]^2 - [g(\theta)]^2 \,\mathrm{d}\theta$$

9.3 Arc length

$$L = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\right)^2} \,\mathrm{d}\theta$$

10 Sequences

A **sequence**, denoted $\{a_n\}$, is an ordered collection of numbers defined by a function f with domain $\mathbb{N}$.

$$\{a_n\} : a_1, a_2, a_3, a_4, \cdots$$

Sequences can be defined by an **explicit formula** or **general term**: $a_n = f(n)$. However, some sequences can be defined **recursively**, *e.g.* the Fibonacci sequence:

$$F_1 = 1, \quad F_2 = 1, \quad F_n = F_{n-1} + F_{n-2}$$

We say that a sequence $\{a_n\}$ **converges to** L and write

$$\lim_{n\to\infty} a_n = L \quad \text{or} \quad a_n \to L$$

if $\forall \varepsilon > 0, \exists N \in \mathbb{N} : \forall n > N, |a_n - L| < \varepsilon$ (we can make the terms a_n as close to L as we like, and there will always be a sufficiently large n to satisfy this).

We say that a sequence $\{a_n\}$ **diverges to infinity**:

$$\lim_{n\to\infty} a_n = \infty \quad \text{or} \quad a_n \to \infty$$

if $\forall M, \exists N \in \mathbb{N} : \forall n > N, a_n > M$ (after some threshold N, all of the terms will be greater than *any* value of M we choose).

Squeeze theorem for sequences: Let $\{a_n\}, \{b_n\}, \{c_n\}$ be sequences such that for some number M:

$$a_n \leq b_n \leq c_n \quad (n > M) \quad \text{and} \quad \lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n.$$

Then,

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = \lim_{n\to\infty} c_n.$$

Continuous function theorem: If f is continuous and $\lim_{n\to\infty} a_n$ exists, then

$$\lim_{n\to\infty} f(a_n) = f\left(\lim_{n\to\infty} a_n\right)$$

10.1 Bounded monotonic sequences

A sequence is **bounded** if its bounded *from above and below.*

- **Bounded from above:** there exists an **upper bound** M such that $a_n \leq M$ for all n.
- **Bounded from below:** there exists an **lower bound** m such that $a_n \geq m$ for all n.

Convergent sequences are bounded.

Proof. Assume a convergent sequence $\{a_n\}$ where $a_n \to L$. By definition, for any $\varepsilon > 0$, there exists an N such that for all $n > N$, $|a_n - L| < \varepsilon$. Thus

$$|a_n - L| < \varepsilon \implies -\varepsilon < a_n - L < \varepsilon \implies L - \varepsilon < a_n < L + \varepsilon.$$

Therefore $\{a_n\}$ is bounded *from above* by $L + \varepsilon$ and *from below* by $L - \varepsilon$.

A sequence $\{a_n\}$ is **monotonic** if it is either:

- *increasing* if $\forall n, a_n < a_{n+1}$
- *decreasing* if $\forall n, a_n > a_{n+1}$

10.2 Monotone convergence theorem

If $\{a_n\}$ is a bounded sequence and there exists a positive integer N such that $\{a_n\}$ is monotone for all $n \geq N$, then $\{a_n\}$ converges.

Proof. Suppose the sequence $\{a_n\}$ is nondecreasing, *i.e.* $a_{n+1} \geq a_n$. Let $L = \sup\{a_n\}$ be the **least upper bound** (or **supremum**) of $\{a_n\}$.

For all $\varepsilon > 0$, there exists N such that $a_N > L - \varepsilon$. This is because otherwise $L - \varepsilon$ is an upper bound of $\{a_n\}$, which would contradict with the definition of L.

Since $\{a_n\}$ is nondecreasing,

$$a_n \geq a_N > L - \varepsilon \text{ for all } n \geq N.$$

Thus L is such that all terms with $n \geq N$ lie in the ε-neighborhood of L, *i.e.*

$$|a_n - L| < \varepsilon,$$

which is the definition of convergence.

10.3 Principle of mathematical induction

Let $P(n)$ be a statement applied over any natural number n.

If

1. $P(0)$ (the **base case**) is true, and
2. for any integer $k > 0$, $P(k) \implies P(k+1)$,

then $P(n)$ is **true for all** n.

Example. Show that the sequence defined recursively by $a_1 = 1, a_{n+1} = \sqrt{1 + a_n}$ converges.

Solution. The sequence is **bounded**. From the first few terms we see that the sequence is *bounded from below* by 1. We *claim* that the sequence is *bounded from above* by 2.

1. The **base case** $P(1) : a_1 = 1 < 2$ is true.
2. The **general case** $P(k)$:

$$a_{k+1} = \sqrt{1 + a_k} < \sqrt{1 + 2} = \sqrt{3} < 2.$$

   Therefore, $P(k) \implies P(k+1)$.

The sequence is bounded from above by the principle of mathematical induction.

The sequence is **monotone**.

1. The **base case** $P(1) : a_2 = \sqrt{1 + \sqrt{1}} > 1 = a_1$ is true.
2. The **general case** $P(k)$:

$$a_{k+2} = \sqrt{1 + a_{k+1}} > \sqrt{1 + a_k} = a_{k+1}$$

   Therefore, $P(k) \implies P(k+1)$.

The sequence is monotonically increasing by the principle of mathematical induction.
Thus the sequence is monotonic and bounded. By the **monotone convergence theorem**, the sequence is convergent, *i.e.*

$$\lim_{n\to\infty} a_n = L = \lim_{n\to\infty} a_{n+1}$$

$$\implies L = \sqrt{1+L} \implies L^2 - L - 1 = 0 \implies L = \frac{1+\sqrt{5}}{2} \quad (a_n > 0).$$

11 Infinite series

An **infinite series** is a sum of the terms in an infinite sequence. An infinite series converges to a sum s if

$$s = \lim_{n\to\infty} \sum_{k=1}^{n} a_k = \sum_{n=1}^{\infty} a_n$$

11.1 Geometric series

A **geometric series** with common **ratio** $r \neq 0$ is a sum of the terms of a **geometric sequence** ar^n $(a \neq 0)$:

$$s = \sum_{n=1}^{\infty} ar^{n-1}$$

$$\sum_{n=1}^{\infty} ar^{n-1} = \begin{cases} \frac{a}{1-r} & \text{if } |r| < 1 \\ \text{divergent} & \text{if} |r| > 1 \end{cases}$$

11.2 nth-term test for divergence

$$\lim_{n\to\infty} a_n \neq 0 \implies \sum a_n \text{ diverges}$$

$$\sum a_n \text{ converges} \implies \lim_{n\to\infty} a_n = 0$$

11.3 Integral test

If $f(n) = a_n$ is **positive and decreasing**, then

$$\int_1^{\infty} f(x)\,\mathrm{d}x \text{ and } \sum_{n=1}^{\infty} a_n \text{ converge or diverge together.}$$

Proof. Since f is positive and decreasing, we see visually that

$$a_2 + \cdots + a_N \leq \int_1^N f(x)\,\mathrm{d}x \leq \int_1^{\infty} f(x)\,\mathrm{d}x$$

If the integral on the right converges, then $a_2 + \cdots + a_N$ is bounded above. Thus $s_N = a_1 + \cdots + a_N$ is bounded above. By the *monotone convergence theorem*, $\{s_N\}$ is a bounded monotonic sequence which therefore converges.

Remainder estimate:

$$\int_{n+1}^{\infty} f(x)\,\mathrm{d}x \leq s - s_n \leq \int_n^{\infty} f(x)\,\mathrm{d}x$$

11.4 p-series test

$$\sum_{n=k}^{\infty} \frac{1}{n^p} \begin{cases} \text{converges if } p > 1 \\ \text{diverges if } p < 1 \qquad (k \in \mathbb{Z}_+) \end{cases}$$

$$\sum_{n=1}^{\infty} \frac{1}{n} \text{ is called the } \textbf{harmonic series.}$$

11.5 Direct comparison test

Let a_n, b_n be **positive sequences**.

$$a_n \le b_n \text{ and } \sum b_n \text{ converges} \implies \sum a_n \text{ converges}$$

$$b_n \le a_n \text{ and } \sum b_n \text{ diverges} \implies \sum a_n \text{ diverges}$$

11.6 Limit comparison test

Let $a_n, b_n > 0$ be **positive sequences**.

$$\lim_{n\to\infty} \frac{a_n}{b_n} > 0 \text{ exists} \implies \sum a_n, \sum b_n \text{ converge or diverge together}$$

Proof. Let $\sum b_n$ converge and $l = \lim_{n\to\infty} \frac{a_n}{b_n}$. Then, there exists $H > l$ such that $0 \le \frac{a_n}{b_n} \le H \implies a_n \le Hb_n$ $\sum b_n$ converges implies $\sum Hb_n$ converges. By the *direct comparison test*, $\sum a_n$ converges.

Let $\sum a_n$ converge. Then, $\lim_{n\to\infty} \frac{b_n}{a_n} = \frac{1}{l}$, and there exists $h > \frac{1}{l}$ such that $0 \le \frac{b_n}{a_n} \le h \implies b_n \le ha_n$ $\sum a_n$ converges implies $\sum Ha_n$ converges. By the *direct comparison test*, $\sum b_n$ converges.

11.7 Alternating series

An **alternating series** is a series whose terms are alternatively positive or negative (*e.g.* $(-1)^n b_n$, $(-1)^{n-1} b_n$, $\cos(n\pi) b_n$), where $b_n > 0$.

An alternating series $\sum (-1)^{n-1} b_n (b_n > 0)$ **converges** if it satisfies

$$b_{n+1} < b_n, \text{ for all } n \qquad \text{and} \qquad \lim_{n\to\infty} b_n = 0$$

Remainder:

$$|s - s_n| \le b_{n+1}$$

11.8 Absolute and conditional convergence

$$\sum |a_n| \text{ converges} \implies \sum a_n \textbf{ absolutely converges}$$
$$\sum |a_n| \text{ diverges and} \sum a_n \text{ converges} \implies \sum a_n \textbf{ conditionally converges}$$

11.9 Ratio test

Given a series $\sum a_n$,

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| \begin{cases} < 1 & \implies \sum a_n \text{ absolutely converges} \\ > 1 & \implies \sum a_n \text{ diverges} \end{cases}$$

11.10 Root test

Given a series $\sum a_n$,

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} \begin{cases} < 1 & \implies \sum a_n \text{ absolutely converges} \\ > 1 & \implies \sum a_n \text{ diverges} \end{cases}$$

12 Power series

A **power series** is a series of the form $\displaystyle\sum_{n=0}^{\infty} c_n(x-a)^n$.

12.1 Convergence theorem of power series

If $\displaystyle\sum_{n=0}^{\infty} a_n x^n$ converges at $x = c \neq 0$, then it converges absolutely on $|x| < |c|$

If $\displaystyle\sum_{n=0}^{\infty} a_n x^n$ diverges at $x = d$, then it diverges on $|x| < |d|$

12.2 Radius and interval of convergence

A power series $\sum_{n=0}^{\infty} c_n(x-a)^n$ converges either at

1. at $x = a \implies R = 0$ (**radius of convergence**)
2. at $x \in \mathbb{R} \implies R = \infty$
3. at $|x-a| < R$ and diverges at $|x-a| > R$

The **interval of convergence** is all x such that the power series converges. In case **3.**, the *endpoints* of the interval also need to be tested for convergence, so it has four possibilities:

$$(a-R, a+R) \quad (a-R, a+R] \quad [a-R, a+R) \quad [a-R, a+R]$$

12.3 Differentiation and integration of power series

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\sum_{n=0}^{\infty} c_n(x-a)^n\right] = \sum_{n=0}^{\infty} \frac{\mathrm{d}}{\mathrm{d}x}\left[c_n(x-a)^n\right]$$

$$\int \left[\sum_{n=0}^{\infty} c_n(x-a)^n\right] \mathrm{d}x = \sum_{n=0}^{\infty} \int \left[c_n(x-a)^n\right] \mathrm{d}x$$

where the interval of convergence $a - R < x < a + R$ is preserved (check endpoints).

13 Taylor and Maclaurin series

A **Taylor series approximation** centered at a is

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \qquad \text{for } |x-a| < R$$

A Taylor series centered at 0 is a **Maclaurin series**.

13.1 Taylor's theorem

Taylor's inequality / Lagrange error bound:

$$|R_n(x)| \leq \frac{M}{(n+1)!}|x-a|^{n+1} \text{ for } |x-a| \leq d$$

where $|f^{(n+1)}(x)| \leq M$ for $|x-a| \leq d$.

Taylor's theorem: Assume $f^{(n+1)}$ exists and is continuous.

$$R_n(x) = \frac{1}{n!}\int_a^x (x-t)^n f^{(n+1)}(t)\,\mathrm{d}t$$

13.2 Binomial series

$$(1+x)^n = \sum_{k=0}^{\infty} \binom{n}{k} x^k = \sum_{k=0}^{\infty} \frac{n!}{k!(n-k)!} x^k \qquad (k \in \mathbb{R}, |x| < 1)$$

These notes, taken from December 2021 to 30 May 2022, were written under the following courses:

- Eugene Berg. *Advanced Placement Calculus AB*, Jefferson High School, The College Board (see the Course and Exam Description here).
- Sean-Giacomo Nguyen. `MATH 252` *Calculus with Analytic Geometry II*, Skyline College.
- Shawn M. Westmoreland. `MATH 253` *Calculus with Analytic Geometry III*, College of San Mateo. Based on *Vector Calculus* by Michael Corral.