MAT1341 Notes - By Eric Hua

Contents

Pre-knowledge

Set: A set is an unordered collection of objects.

- $\mathbb{Z}$ = the set of all integers.
- $\mathbb{R}$ = the set of all real numbers.
- Empty set $\emptyset$ or $\{\ \ \}$.

For example, the set of integers greater than 1 and less than 5: $\{2, 3, 4\}$, or $\{2, 4, 3\}$, or ...

Elements of a set: If S is a set and x is an object in the set S, we write $x \in S$. If x is not in S, then we write $x \notin S$. We refer to the objects in a set as its elements.

Subset: A Subset T of a set S is another set which contains some of elements of the set S, we write $T \subset S$.

For example, let
$M = \{$MAT1341 students$\}$,
$A = \{$MAT1341 students with grade A+$\}$.
Then $A \subset M$.

General logic:

- If you say the statement may be false, you must give an explicit counterexample.
- If you say the statement is always true, then you CANNOT use an example to justify your response. You must give a clear explanation that works in all cases.

True/False: The line $L_1: \ x + y = 2$ and the line $L_2: \ x + y = 3$ are different.

True/False: The line $L_1: \ x + y = 2$ and the line $L_2: \ x + y = 3$ are same.

Chapter 2. Vector Geometry

Vectors in $\mathbb{R}^n$

$$\mathbb{R}^n = \{(x_1, x_2, ..., x_n) | x_i \in \mathbb{R}\}.$$

If u or $\vec{u} \in \mathbb{R}^n$, then

$$\vec{u} = (u_1, u_2, ..., u_n) = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} = [u_1 \quad u_2 \quad \cdots \quad u_n]^T,$$

here T means transpose. The magnitude (or length, or norm)

$$||(u_1, u_2, ..., u_n)|| = \sqrt{u_1^2 + \cdots + u_n^2}.$$

Special vectors:

- Zero vector $\vec{0} = (0, 0, ..., 0)$.
- unit vector: $||\vec{u}|| = 1$.
- In $\mathbb{R}^2$: $\vec{i} = (1, 0)$, $\vec{j} = (0, 1)$; In $\mathbb{R}^3$: $\vec{i} = (1, 0, 0)$, $\vec{j} = (0, 1, 0)$, $\vec{k} = (0, 0, 1)$.

Manipulation of vectors in $\mathbb{R}^n$: Let $\vec{u} = (u_1, ..., u_n)$, $\vec{v} = (v_1, ..., v_n)$.

- Addition: $\vec{u} + \vec{v} = (u_1 + v_1, ..., u_n + v_n)$.
- Scalar multiple: Let c be a scalar, then $c\vec{u} = (cu_1, cu_2, ..., cu_n)$.
- $\vec{u} = \vec{v}$ if and only if $u_1 = v_1, ..., u_n = v_n$).
- $\vec{u} // \vec{v}$ if and only if $\vec{u} = c\vec{v}$ for a constant c.

Properties: Let c, d be scalars.

- $\vec{u} + \vec{v} = \vec{v} + \vec{u}$
- $(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$
- $\vec{u} + \vec{0} = \vec{u}, \quad \vec{u} + (-\vec{u}) = \vec{0}$
- $(cd)\vec{u} = c(d\vec{u})$

- $(c+d)\vec{u} = c\vec{u} + d\vec{u}$
- $c(\vec{u}+\vec{v}) = c\vec{u} + c\vec{v}$
- $1\vec{u} = \vec{u}, \quad (-1)\vec{u} = -\vec{u}, \quad 0\vec{u} = \vec{0}$
- $\vec{u}//\vec{v} \Leftrightarrow \vec{v} = c\vec{u}$

Linear combination: Let $\vec{u}_1, ..., \vec{u}_n$ be n vectors, k_1,..., k_n be n scalars, then

$$k_1\vec{u}_1 + ... + k_n\vec{u}_n$$

is called a linear combination of the n vectors.

Example 1. *Is $\vec{w}$ a linear combination of $\vec{u}$ and $\vec{v}$?*
(a) $\vec{w} = (2,3)$, $\vec{u} = (1,0)$, $\vec{v} = (0,1)$.
(b) $\vec{w} = \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix}$, $\vec{u} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$.
(c) $\vec{w} = (1,2,3,4)$, $\vec{u} = (1,0,0,0)$, $\vec{v} = (0,0,1,0)$.

Solution: (a): Yes, $\vec{w} = 2\vec{u} + 3\vec{v}$.

(b): No.

(c): No. If $\vec{w} = a\vec{u} + b\vec{v}$, then

$$(1,2,3,4) = a(1,0,0,0) + b(0,0,1,0)$$

$$= (a,0,0,0) + (0,0,b,0)$$

$$= (a,0,b,0).$$

Thus $a = 1$, $2 = 0$, $3 = b$, $4 = 0$, a contradiction.

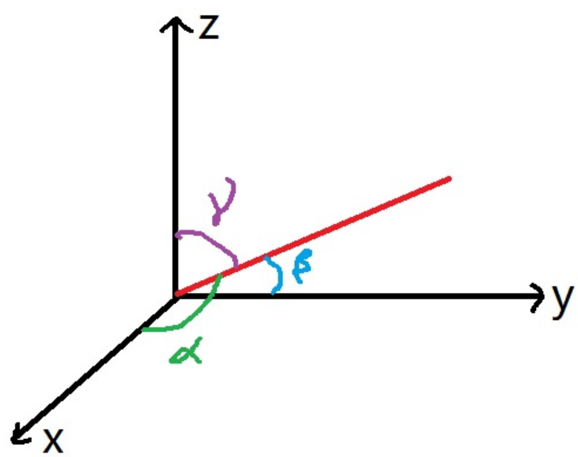
Dot product and applications

- Dot product (inner product): Let $\vec{u} = (u_1, ..., u_n)$, $\vec{v} = (v_1, ..., v_n)$, then $\vec{u} \cdot \vec{v} = u_1 v_1 + \cdots + u_n v_n$.

- Angle: Let θ be the angle between $\vec{u}$ and $\vec{v}$ which satisfies $0 \le \theta \le \pi$, then $\cos\theta = \frac{\vec{u}\cdot\vec{v}}{||\vec{u}||\,||\vec{v}||}$.

- Orthogonal: $\vec{u} \perp \vec{v}$ if $\vec{u} \cdot \vec{v} = 0$.

- Direction angles to the three axis and direction cosines of vectors:



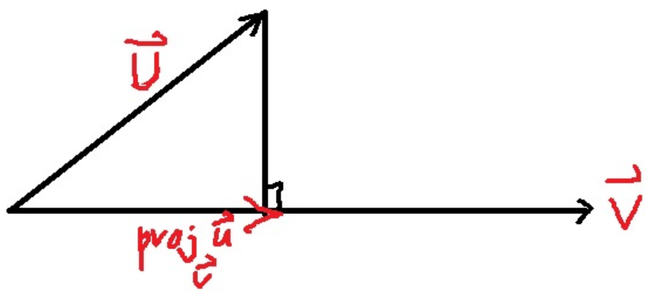
$$\cos\alpha = \frac{\vec{u}\cdot\vec{i}}{||\vec{u}||\,||\vec{i}||}, \quad \cos\beta = \frac{\vec{u}\cdot\vec{j}}{||\vec{u}||\,||\vec{j}||}, \quad \cos\gamma = \frac{\vec{u}\cdot\vec{k}}{||\vec{u}||\,||\vec{k}||}.$$

They satisfy

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1, \qquad \frac{\vec{u}}{||\vec{u}||} = (\cos\alpha, \cos\beta, \cos\gamma).$$

- Projection: The projection of $\vec{u}$ onto (along) $\vec{v}$ is

$$proj_{\vec{v}}\vec{u} = \left(\frac{\vec{u}\cdot\vec{v}}{||\vec{v}||^2}\right)\vec{v}, \quad comp_{\vec{v}}\vec{u} = \frac{\vec{u}\cdot\vec{v}}{||\vec{v}||}.$$



Example 2. *Let $\vec{u} = (1, 2)$ and $\vec{v} = (3, -1)$. Find $proj_{\vec{v}}\vec{u}$ and $proj_{\vec{u}}\vec{v}$.*

Solution: $proj_{\vec{v}}\vec{u} = \frac{1}{10}(3, -1)$ and $proj_{\vec{u}}\vec{v} = \frac{1}{5}(1, 2)$.

Example 3. *Let $\vec{u} = (1, 2, -2)$, $\vec{v} = (-2, -2, 1)$. Write $\vec{u} = \vec{u_1} + \vec{u_2}$ s.t. $\vec{u_1} \perp \vec{v}$, $\vec{u_2} // \vec{v}$.*

Solution:

$$\vec{u}_2 = proj_{\vec{v}}\vec{u} = \left(\frac{\vec{u} \cdot \vec{v}}{||\vec{v}||^2}\right)\vec{v} = \frac{-8}{9}(-2, -2, 1) = \left(\frac{16}{9}, \frac{16}{9}, -\frac{8}{9}\right).$$

$$\vec{u}_1 = \vec{u} - \vec{u}_2 = (1, 2, -2) - \left(\frac{16}{9}, \frac{16}{9}, -\frac{8}{9}\right) = \left(-\frac{7}{9}, \frac{2}{9}, -\frac{10}{9}\right).$$

Example 4. *Let $\vec{u} = (1, 2, -2)$, $\vec{v} = (-2, -2, 1)$, Find the cosine of the angle between $\vec{u}$ and $\vec{v}$.*

Solution:

$$\cos\theta = \frac{\vec{u} \cdot \vec{v}}{||\vec{u}||\,||\vec{v}||} = \frac{-8}{9}.$$

Properties: Let c be a scalar.

- $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$
- $\vec{w} \cdot (\vec{u} + \vec{v}) = \vec{w} \cdot \vec{u} + \vec{w} \cdot \vec{v}$
- $c(\vec{u} \cdot \vec{v}) = (c\vec{u}) \cdot \vec{v} = \vec{u} \cdot (c\vec{v})$
- $\vec{u} \cdot \vec{0} = 0$
- $\vec{u} \cdot \vec{u} = ||\vec{u}||^2$.

Chapter 3. Lines and Planes

Lines

Line L: A line is determined by a point $\vec{p}$ and a vector (direction vector) $\vec{d}$ parallel to the line. It can be described as

$$L = \{\vec{p} + t\vec{d}\,|t \in \mathbb{R}\},$$

t is a parameter.

Line L in $\mathbb{R}^2$:

- Slope-intercept form: $y = mx + b$.
- vector form: Let $\vec{p} = (p_1, p_2)$ be a point on L (a position vector). Let $\vec{v} = (v_1, v_2)$ be a vector parallel to the line L (a direction vector). Then the line is:

  $$L = \{\vec{p} + t\vec{v}|t \in \mathbb{R}\}$$

  or

  $$(x, y) = \vec{p} + t\vec{v},\ t \in \mathbb{R}.$$

Example 5. *Find the line through points $P(1, 2)$ and $Q(3, -2)$.*

Solution: A direction vector $\vec{v} = Q - P = \begin{bmatrix} 2 \\ -4 \end{bmatrix}$. Thus

$$L = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} + t \begin{bmatrix} 2 \\ -4 \end{bmatrix} |t \in \mathbb{R} \right\} = \{(1, 2) + t(2, -4)|t \in \mathbb{R}\} = \{(1, 2) + t(1, -2)|t \in \mathbb{R}\}.$$

Line L in $\mathbb{R}^3$:

Let $P(p_1, p_2, p_3)$ be a point on the line L. Let $\vec{v}$ be a nonzero vector which is parallel L.

- vector form:

  $$L = \{\vec{p} + t\vec{v}|t \in \mathbb{R}\}, \quad \vec{p} = (p_1, p_2, p_3)$$

  or

  $$(x, y, z) = \vec{p} + t\vec{v},\ t \in \mathbb{R}.$$

- Parametric form: $x = p_1 + tv_1$, $y = p_2 + tv_2$, $z = p_3 + tv_3$.

- Symmetric form: $\frac{x-p_1}{v_1} = \frac{y-p_2}{v_2} = \frac{z-p_3}{v_3}$.

Example 6. *If a line goes through two points P and Q, then $(x, y, z) = \vec{p} + t(\vec{q} - \vec{p})$, where $\vec{q}$, $\vec{p}$ are the position vectors of Q, P.*

Example 7. *Find the parametric equation of the line through $P(1, 2, 3)$ and $Q(3, 1, 1)$.*

Solution:

$$x = 1 + 2t, y = 2 - t, z = 3 - 2t.$$

Remark. Relation between two lines L_1 and L_2

- parallel
- intersected
- skewed

The Cross Product in $\mathbb{R}^3$ and Applications

- Cross product: Let $\vec{u} = (u_1, u_2, u_3)$, $\vec{v} = (v_1, v_2, v_3)$, then

$$\vec{u} \times \vec{v} = (u_2 v_3 - u_3 v_2, +u_3 v_1 - u_1 v_3, u_1 v_2 - u_2 v_1).$$

- Orthogonal: $\vec{u} \times \vec{v} \perp \vec{u}$, $\vec{u} \times \vec{v} \perp \vec{v}$.

Example 8. *Find a vector that is orthogonal to both* $\vec{u} = (1, 2, -1)$, $\vec{v} = (0, 2, 3)$.

Solution: Any scalar multiple of $\vec{u} \times \vec{v} = (8, -3, 2)$.

Properties: Let c be a scalar.

- $\vec{u} \times \vec{v} = -\vec{v} \times \vec{u}$
- $\vec{w} \times (\vec{u} + \vec{v}) = \vec{w} \times \vec{u} + \vec{w} \times \vec{v}$
- $(\vec{u} + \vec{v}) \times \vec{w} = \vec{u} \times \vec{w} + \vec{v} \times \vec{w}$
- $c(\vec{u} \times \vec{v}) = (c\vec{u}) \times \vec{v} = \vec{u} \times (c\vec{v})$
- $\vec{u} \times \vec{0} = \vec{0}$
- $\vec{u} \times \vec{u} = \vec{0}$
- $\vec{u} \times (\vec{v} \times \vec{w}) = (\vec{u} \cdot \vec{w})\vec{v} - (\vec{u} \cdot \vec{v})\vec{w}$.
- $||\vec{u} \times \vec{v}|| = ||\vec{u}||\,||\vec{u}|| \sin\theta$, where θ is the angle between $\vec{u}$ and $\vec{v}$
- $||\vec{u} \times \vec{v}||$ is the area of the parallelogram determined by $\vec{u}$ and $\vec{v}$.
- $||\vec{u} \cdot (\vec{v} \times \vec{w})||$ is the volume of the parallelepiped spanned by $\vec{u}, \vec{v}, \vec{w}$. If it is 0, then these three vectors are in the same plane.

Example 9. *Find the area of the parallelogram determined by* $\vec{u} = (1, 2, -1)$, $\vec{v} = (0, 2, 3)$.

Solution: $A = ||\vec{u} \times \vec{v}|| = ||(8, -3, 2)|| = \sqrt{77}$.

Example 10. *Find the area of the triangle with vertices* $P(1, 2, 3)$, $Q(-3, 2, 1)$, *and* $R(2, 4, 5)$.

Solution: $\vec{PQ} = Q - P = (-4, 0, -2)$, $\vec{PR} = R - P = (1, 2, 2)$.

$$A = \frac{1}{2}||\vec{PQ} \times \vec{PR}|| = ||(4, 6, -8)|| = \sqrt{29}.$$

Example 11. *Find the volume of the parallelepiped spanned by* $\vec{u} = (1, 2, 3), \vec{v} = (1, 3, 2), \vec{w} = (1, 2, 2)$.

Solution: Volume $= ||\vec{u} \cdot (\vec{v} \times \vec{w})|| = 1$.

Plane in $\mathbb{R}^3$

A plane Π is determined by a point and a normal vector $\vec{n}$ which is perpendicular to the plane.

Let p be a point on the plane. Let $\vec{n} = (a, b, c)$ be a nonzero vector which is perpendicular to the plane.

- Point-normal form: $\Pi = \{\vec{x} \in \mathbb{R}^3 | (\vec{x} - \vec{p}) \cdot \vec{n} = 0\}$, $\vec{x} = (x, y, z)$.
- Parametric form: $(x, y, z) = p + t\vec{v}_1 + s\vec{v}_2$, where $\vec{v}_1$ and $\vec{v}_2$ are two non-parallel non-zero vectors, parallel to the plane.
- Standard form (or scalar equation, or linear equation): $ax + by + cz = d$ or

$$\Pi = \left\{(x, y, z) \in \mathbb{R}^3 | ax + by + cz = d\right\}.$$

Example 12. *Find the scalar equation of the plane through three points $P(1, 2, 3)$, $Q(-3, 2, 1)$, and $R(2, 4, 5)$.*

Solution: $\vec{PQ} = Q - P = (-4, 0, -2)$, $\vec{PR} = R - P = (1, 2, 2)$.
$\vec{n} = \vec{PQ} \times \vec{PR} = (4, 6, -8)$.
Thus

$$4(x - 1) + 6(y - 2) - 8(z - 3) = 0, \Rightarrow 4x + 6y - 8z = -8.$$

Example 13. *Find the scalar equation of the plane through two lines:*
$L_1: \ x = -1 + 2s, y = 3 - s, z = 1 + 2s, \ s \in \mathbb{R}$, *and*
$L_2: \ x = 2 + t, y = 1 - t, z = 4 + t, \ t \in \mathbb{R}$.

Solution: 1. Find the intersection: (x,y,z)=(1,2,3).
2. Find a normal vector:

$$n = (2, -1, 2) \times (1, -1, 1) = (1, 0, -1).$$

Thus

$$x - z = -2.$$

Chapter 11 and 12. Solving systems of linear equations

Linear systems

Example 14. *David inherited \$50,000 and invested part of it in a money market account which pays 6% annually, and part in a mutual fund which pays 5% annually. After one year, he received a total of \$2,700 in simple interest from the two investments. Find the amount David invested in each category.*

Solution: Let
$x =$ The amount of money invested in the money market account.
$y =$ The amount of money invested in a mutual fund.
Then

$$\begin{aligned} x + y &= 50000 \\ 0.06x + 0.05y &= 2700 \end{aligned}$$

The solution is $x = 20,000$, $y = 30,000$.

Definition 1. *A* ***linear equation*** *in variables $x_1, x_2, \ldots, x_n$ has the form*

$$a_1x_1 + a_2x_2 + a_3x_3 + \cdots + a_nx_n = b$$

where the numbers $a_1, \ldots, a_n \in \mathbb{R}$ are the equation's coefficients and $d \in \mathbb{R}$ is the constant. An n-tuple $(s_1, s_2, \ldots, s_n) \in \mathbb{R}^n$ is a ***solution*** *of, or satisfies, that equation if substituting the numbers $s_1, \ldots, s_n$ for the variables gives a true statement: $a_1s_1 + a_2s_2 + \ldots + a_ns_n = d$.*

A ***system of linear equations (or linear system)***

$$\begin{array}{ccccccccc} a_{1,1}x_1 & + & a_{1,2}x_2 & + & \cdots & + & a_{1,n}x_n & = & b_1 \\ a_{2,1}x_1 & + & a_{2,2}x_2 & + & \cdots & + & a_{2,n}x_n & = & b_2 \\ & & & & & & & \vdots & \\ a_{m,1}x_1 & + & a_{m,2}x_2 & + & \cdots & + & a_{m,n}x_n & = & b_m \end{array}$$

has the solution $(s_1, s_2, \ldots, s_n)$ if that n-tuple is a solution of all of the equations in the system.
A linear system is called inconsistent if it has no solution. Otherwise it is called consistent.

Remark. A linear equation in which all the coefficients are zero is called degenerate.

Theorem 1. *Any linear system has,*

1. *no solution*
2. *one solution*

3. infinitely many solutions.

In case 1, the linear system is called inconsistent. In cases 2 and 3, the linear system is called consistent.

Augmented matrix and row operations

Finding the set of all solutions is solving the system.

Definition 2. *If we have two linear systems and they have the same solution set then the two linear systems are called* ***equivalent****.*

Elementary operations: There are three types of elementary operations to a system of linear equations.

1. Replacement: Replace an equation by the sum of itself and the multiple of another equation.

2. Interchange: Interchange two equations.

3. Scaling: Multiply an equation by a non-zero constant.

Theorem 2. *Elementary operations will result in an equivalent system.*

Matrices:

Definition 3. *An $m \times n$ (m by n)* matrix *A with m rows and n columns with entries in $\mathbb{R}$ is a rectangular array of the form*

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix},$$

where $\forall (i,j) \in \{1,2,\ldots,m\} \times \{1,2,\ldots,n\}, \quad a_{ij} \in \mathbb{R}$.

As a shortcut, we often use the notation $A = [a_{ij}]$ to denote the matrix A with entries a_{ij}. Notice that when we refer to the matrix we put parentheses—as in "$[a_{ij}]$," and when we refer to a specific entry we do not use the surrounding parentheses—as in "a_{ij}."

- Diagonal matrix: Except for entries on diagonal (main diagonal), all other entries are 0,

- Zero matrix: all entries are 0,
- Identity matrix I_n: all entries on diagonal are 1, other entries are 0.

Example 15. $A = \begin{bmatrix} 0 & -1 & 1 \\ 1 & 2 & 3 \end{bmatrix}$ *is a* 2×3 *matrix and*

$B = \begin{bmatrix} -2 & 1 \\ 1 & 2 \\ 0 & 3 \end{bmatrix}$ *is a* 3×2 *matrix.*

$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Coefficient Matrix and Augmented Matrix:

To solving linear systems, we put all the coefficients of each variable aligned in columns to get the **coefficient matrix**. By adding an additional column to the coefficient matrix consisting of the values on the right hand side of the equal sign to give the **augmented matrix**.

Example 16. *Find the coefficient matrix and augmented matrix of the linear system*

$$\begin{array}{rcrcrcl} & & & & 3x_3 & = & 9 \\ x_1 & + & 5x_2 & - & 2x_3 & = & 2 \\ \frac{1}{3}x_1 & + & 2x_2 & & & = & 3 \end{array}$$

Solution:

$$\text{coefficient matrix} = \begin{bmatrix} 0 & 0 & 3 \\ 1 & 5 & -2 \\ 1/3 & 2 & 0 \end{bmatrix}, \quad \text{augmented matrix} = \left[\begin{array}{ccc|c} 0 & 0 & 3 & 9 \\ 1 & 5 & -2 & 2 \\ 1/3 & 2 & 0 & 3 \end{array}\right].$$

Elementary row operations: There are three types of elementary row operations.

1. Replacement: Replace one row by the sum of itself and the multiple of another: $R_i + cR_j \to R_i$.
2. Interchange: Interchange two rows: $R_i \leftrightarrow R_j$.
3. Scaling: Multiply all entries in a row by a non zero constant: $cR_i \to R_i$.

Definition 4. *Two matrices are row equivalent if one matrix can be transformed into another matrix by a sequence of elementary row operations.*

REF and RREF

A **leading entry of a row**: is the leftmost, nonzero entry in the row (nonzero row).

Row echelon form (REF): A rectangular matrix is in row-reduced echelon form if it satisfies the following three properties.

1. All non zero rows are above any zero rows.
2. The leading entry in each nonzero row is 1, which is called leading 1.
3. Each leading 1 is in a column to the right of the leading 1 above.

Reduced row echelon form (RREF): A rectangular matrix is in reduced row echelon form if it satisfies the 4th condition:

4. Each leading 1 is the only nonzero entry in its column.

Example 17. $\begin{bmatrix} 1 & 3 & 0 & 9 \\ 0 & 0 & 1 & 2 \\ 0 & k & 0 & 0 \end{bmatrix}$ *is not RREF if* $k \neq 0$*; it is RREF if* $k = 0$.

We are interested in performing row operations until one of these two matrix structures arises.

Uniqueness of RREF: Each matrix is row equivalent to one and only one reduced echelon matrix.

Properties:

- The augmented matrix of a inconsistent linear system is row equivalent to a matrix with the last non-zero row
$$\begin{bmatrix} 0 & \cdots & 0 & * \end{bmatrix}.$$

Remark: The variables correspond to leading 1's, are called **leading variables (basic variables)**.

Gaussian elimination

1. Carry the augmented matrix to RREF;
2. The system is inconsistent if a row is [0 ... 0 1];
3. Otherwise, assign parameters to non-leading variables, and use the equations corresponding to the reduced row-echelon matrix to solve for the leading variables (the unknowns) in terms of parameters.

Example 18. *Solve the following system by using elementary row operations (row reduction):*

$$\begin{array}{ccccccc} x & + & y & - & 2z & = & -2 \\ & & y & + & 3z & = & 7 \\ x & & & - & 5z & = & -9 \end{array}$$

Solution: We can start by going to echelon form:

$$\text{augmented matrix } = \begin{bmatrix} 1 & 1 & -2 & -2 \\ 0 & 1 & 3 & 7 \\ 1 & 0 & -5 & -9 \end{bmatrix} \xrightarrow{R_3 \to R_3 - R_1} \begin{bmatrix} 1 & 1 & -2 & -2 \\ 0 & 1 & 3 & 7 \\ 0 & -1 & -3 & -7 \end{bmatrix}$$

$$\xrightarrow{R_3 \to R_3 + R_2} \begin{bmatrix} 1 & 1 & -2 & -2 \\ 0 & 1 & 3 & 7 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_1 \to R_1 - R_2} \begin{bmatrix} 1 & 0 & -5 & -9 \\ 0 & 1 & 3 & 7 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

We write this back to a new linear system:

$$\begin{cases} x & = & -9 + 5z \\ y & = & 7 - 3z \\ z & = & free \end{cases}$$

Set $z = t$ (parameter), then the general solution of the system is

$$S = \{(-9 + 5t, 7 - 3t, t) |\ t \in \mathbb{R}\}.$$

Chapter 13 Applications and examples of linear system

Rank

Definition 5. *The rank of a matrix A = the number of leading 1s in RREF = the number of nonzero rows in RREF.*

For example, $rank \begin{bmatrix} 1 & -3 & 2 & 5 & 3 \\ 0 & 0 & 4 & 7 & 4 \\ 0 & 0 & -8 & -14 & -8 \end{bmatrix} = 2.$

Theorem 3. *For a linear system with coefficient matrix A, where A is $m \times n$, let p be the rank of the coefficient matrix, let q be the rank of the augmented matrix. Then the system has*

1. *no solution, if $p < q$;*
2. *only one solution if $p = q = n$;*
3. *infinitely many solutions if $p = q < n$.*

Example 19. *Consider the following system of linear equations*

$$\begin{aligned} x_1 + 4x_2 - 8x_3 &= 3 \\ 2x_1 + 5x_2 - 7x_3 &= 0 \\ -3x_1 - 7x_2 + kx_3 &= c - 1 \end{aligned}$$

(i) Find value(s) of k and c such that the system has no solution.
(ii) Find value(s) of k and c such that the system has only one solution.
(iii) For the value(s) of k and c such that the system has infinitely many solutions.

Solution: (i)

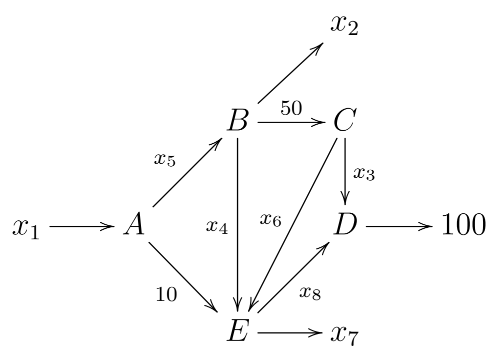
$$\text{augmented matrix} = \begin{bmatrix} 1 & 4 & -8 & 3 \\ 2 & 5 & -7 & 0 \\ -3 & -7 & k & c-1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 4 & -5 \\ 0 & 1 & -3 & 2 \\ 0 & 0 & k-9 & c-2 \end{bmatrix}$$

(i) If $k = 9$, $c \neq 2$, then no solution.
(ii) For $k \neq 9$, the system has only one solution.
(iii) When $k = 9$, $c = 2$, infinitely many solutions.

An Application to Network Flow

Junction Rule: At each junction, total flow in = total flow out.

Example 20. *Consider the traffic flow described by the following diagram. The letters A through E label intersections. The arrows indicate the direction of flow (all roads are one-way) and their labels indicate flow in cars per minute.*



(a) Construct a linear system describing the traffic flow , including all constraints on the variables $x_i, i = 1, \ldots, 8$.

Solution: Set "flow in" = "flow out" at each intersection:

$$\begin{array}{llrl} A & & x_1 & = x_5 + 10 \\ B & & x_5 & = x_2 + x_4 + 50 \\ C & & 50 & = x_3 + x_6 \\ D & & x_3 + x_8 & = 100 \\ E & & 10 + x_4 + x_6 & = x_7 + x_8 \end{array}$$

Constraints: x_i $(i = 1, ..., 8)$ are non-negative integers.

Remark. In some other textbooks, they add a row for "total flow in=total flow out". Actually that is not necessary.

(b) Carry the augmented matrix to the reduced row echelon form, and find the general flow pattern (i.e., the general solution of the linear system with more constraints).

Solution:
$$\left[\begin{array}{cccccccc|c} 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 10 \\ 0 & 1 & 0 & 0 & -1 & 0 & 1 & 0 & -90 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 100 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 40 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & -50 \end{array}\right]$$

$$\begin{cases} x_1 = x_5 + 10 \\ x_2 = x_5 - x_7 - 90 \\ x_3 = -x_8 + 100 \\ x_4 = x_7 + 40 \\ x_5 = \text{free} \\ x_6 = x_8 - 50 \\ x_7 = \text{free} \\ x_8 = \text{free} \end{cases}$$

i.e.,

$$\begin{cases} x_1 = s + 10 \\ x_2 = s - t - 90 \\ x_3 = -u + 100 \\ x_4 = t + 40 \\ x_5 = s \\ x_6 = u - 50 \\ x_7 = t \\ x_8 = u \end{cases},$$

where $s, t, u \in \mathbb{R}$, or

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \end{bmatrix} = \begin{bmatrix} 10 \\ -90 \\ 100 \\ 40 \\ 0 \\ -50 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + u \begin{bmatrix} 0 \\ 0 \\ -1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} \quad s, t, u \in \mathbb{R}.$$

Constraints: $s - t \geq 90$, $50 \leq u \leq 100$.
Remark: For a consistent system, if it has free variables, it has infinite solutions; if no free variable, then only one solution.

(c) What is the minimum and maximum number of traffic along the road ED?

Solution: Since $x_6 = x_8 - 50 \geq 0$, $x_8 \geq 50$. From $x_3 = -x_8 + 100 \geq 0$ we imply that $x_8 \leq 100$. Thus $50 \leq x_8 \leq 100$.

(d) Suppose that due to road work, the flow along ED is limited to a maximum of 70 cars per minute. What is the maximum possible flow along CE?

Solution: Since $x_6 = x_8 - 50$, from $x_8 \leq 70$ it follows that $x_6 \leq 20$, i.e., the maximum number of cars along CE is 20 cars per minutes.

Chapter 14 Matrix multiplications

Matrix multiplication

Let $A = [a_{ij}]_{m\times r}$ and $B = [b_{ij}]_{r\times n}$. Then

$$AB = [c_{ij}]_{m\times n},$$

where

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{ir}b_{rj} = \begin{bmatrix} a_{i1} & a_{i2} & \cdots & a_{in} \end{bmatrix} \begin{bmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{nj} \end{bmatrix}.$$

Example 21.

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 7 & 8 \\ 9 & 10 \\ 11 & 12 \end{bmatrix} = \begin{bmatrix} 1(7)+2(9)+3(11) & 1(8)+2(10)+3(12) \\ 4(7)+5(9)+6(11) & 4(8)+5(10)+6(12) \end{bmatrix} = \begin{bmatrix} 58 & 64 \\ 139 & 154 \end{bmatrix}$$

Remark. In order to have the product AB of two matrices A and B, the number of columns of A must equal the number of rows of B. So, if A is an $m \times r$ and B is an $s \times n$ matrix, in order to have the product AB, we need r = s. The resulting matrix AB will be an $m \times n$ matrix.

Properties of matrix multiplication: Let A, B, C be matrices for which sums and products are defined.

1. A(BC) = (AB)C (associativity)
2. A(B+C)= AB + AC (Left distributivity)
3. (B+C)A = BA + CA (Right distributivity)
4. r(AB) = (rA)B = A(rB)
5. Let A be $m \times n$, then $I_m A = A = AI_n$.
6. $AB \neq BA$.
7. $(AB)^T = B^T A^T$.
8. $(A + B)^T = A^T + B^T$.

9. $(kA)^T = k(A^T)$ for a scalar k.
10. $(A^T)^T = A$.
11. $AB = AC$ can not imply $B = C$.

Example. Let $A = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 4 & 4 \\ 1 & 3 \end{bmatrix}$, $C = \begin{bmatrix} 2 & 4 \\ 2 & 3 \end{bmatrix}$. Then $AB = AC$, but $B \neq C$.

Example. Let $A = \begin{bmatrix} 1 & -1 & 2 \\ 3 & 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}$. Calculate AB and AB^T.

Solution: AB not possible.

$$AB^T = \begin{bmatrix} 1 & -1 & 2 \\ 3 & 1 & 0 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}^T = \begin{bmatrix} 1 & -1 & 2 \\ 3 & 1 & 0 \end{bmatrix} \begin{bmatrix} a & d \\ b & e \\ c & f \end{bmatrix} = \begin{bmatrix} a-b+2c & d-e+2f \\ 3a+b & 3d+e \end{bmatrix}.$$

Powers of a matrix $A^k = AAA \cdots A$.

Example. Expand $(A-B)(A+B)$.

Solution: $(A-B)(A+B) = A^2 + AB - BA + B^2$.

Example 22. *Let* $A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$. *Compute* A^2, A^3, A^4, *then predict* A^{2012}.

Solution:

$$\begin{aligned}
A^2 &= AA = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & -2 & 0 & 1 \end{bmatrix}, \\
A^3 &= A^2A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & -3 & 0 & 1 \end{bmatrix}, \\
A^4 &= A^3A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 4 & 1 & 0 \\ 0 & -4 & 0 & 1 \end{bmatrix}
\end{aligned}$$

Therefore $A^{2018} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 2018 & 1 & 0 \\ 0 & -2018 & 0 & 1 \end{bmatrix}$.

Vector equations and matrix equations

Definition 6. *$A\vec{x} = \vec{b}$ is called matrix equation, where A is the coefficient matrix. Let $A =$ $[\vec{a}_1 \dots \vec{a}_n]$, $\quad \vec{a}_1 = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}, \dots, \vec{a}_n = \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}$, $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$. Then $A\vec{x} = x_1\vec{a}_1 + \dots + x_n\vec{a}_n$. The equation*

$$x_1\vec{a}_1 + \dots + x_n\vec{a}_n = \vec{b}$$

is called vector equation.

For example. The linear system :

$$\begin{aligned}
x_1 - x_4 &= 1 \\
x_1 - 2x_3 &= 2 \\
x_1 + 2x_2 + 3x_3 &= 3 \\
x_2 + 5x_3 &= 4
\end{aligned}$$

$$\Leftrightarrow$$

The vector equation:

$$x_1 \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 0 \\ 2 \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ -2 \\ 3 \\ 5 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$$

$$\Leftrightarrow$$

The matrix equation:

$$\begin{bmatrix} 1 & 0 & 0 & -1 \\ 1 & 0 & -2 & 0 \\ 1 & 2 & 3 & 0 \\ 0 & 1 & 5 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}.$$

Homogeneous System: The linear system $A\vec{x} = \vec{b}$ is called homogeneous if $\vec{b} = \vec{0}$. Otherwise, it is non-homogeneous. Zero vector $\vec{0}$ is always a solution of $A\vec{x} = \vec{0}$, which is called a trivial solution; any non-zero solution is called a non-trivial solution.

Example 23. *Consider the following system of linear equations*

$$\begin{aligned} x_1 + 4x_2 - 8x_3 &= 0 \\ 2x_1 + 5x_2 - 7x_3 &= 0 \\ -3x_1 - 7x_2 + kx_3 &= 0 \end{aligned}$$

(i) Find value(s) of k such that the system has only trivial solution.
(ii) Find value(s) of k such that the system has non-trivial solutions.

Solution: (i)

$$\text{augmented matrix} = \begin{bmatrix} 1 & 4 & -8 & 0 \\ 2 & 5 & -7 & 0 \\ -3 & -7 & k & 0 \end{bmatrix} \xrightarrow[R_3 \to R_3 + 3R_1]{R_2 \to R_2 - 2R_1} \begin{bmatrix} 1 & 4 & -8 & 0 \\ 0 & -3 & 9 & 0 \\ 0 & 5 & k-24 & 0 \end{bmatrix}$$

$$\xrightarrow{R_2 \to -\frac{1}{3}R_2} \begin{bmatrix} 1 & 4 & -8 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 5 & k-24 & 0 \end{bmatrix} \xrightarrow{R_3 \to R_3 - 5R_2} \begin{bmatrix} 1 & 4 & -8 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & k-9 & 0 \end{bmatrix}$$

Hence, for $k \neq 9$, the system has only trivial solution.

(ii) For $k = 9$, the system has non-trivial solution. When $k = 9$, from the discussion in (i), we have

$$\text{augmented matrix} = \begin{bmatrix} 1 & 4 & -8 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_1 \to R_1 - 4R_2} \begin{bmatrix} 1 & 0 & 4 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Thus

$$\begin{aligned} x_1 + 4x_3 &= 0 \\ x_2 - 3x_3 &= 0 \end{aligned}$$

i.e.,

$$\begin{aligned} x_1 &= -4t \\ x_2 &= 3t \\ x_3 &= t \ (free) \end{aligned}$$

Then the general solution is: $(x_1, x_2, x_3) = t(-4, 3, 1)$, $t \in \mathbb{R}$.

Chapter 4 Vector Spaces

Vector Spaces.

An example of vector space: Search engines, which convert documents into word frequency vectors in determining content.

Definition 7. *Let V be a set of elements. The set V is called a vector space, if the operations addition $+$ and scalar multiplication $\cdot$ are defined on V satisfying the following 10 axioms:*

(1). If $\vec{u}, \vec{v} \in V$, then $\vec{u} + \vec{v} \in V$.

(2). If $\vec{u}, \vec{v} \in V$, then $\vec{u} + \vec{v} = \vec{v} + \vec{u}$.

(3). If $\vec{u}, \vec{v}, \vec{w} \in V$, then $\vec{u} + (\vec{v} + \vec{w}) = (\vec{u} + \vec{v}) + \vec{w}$.

(4). There exists an element, denoted by $\vec{0}$, in V, such that $\vec{0} + \vec{u} = \vec{u}$ for every $\vec{u}$.

(5). For every $\vec{u} \in V$, there exists an element, denoted by $-\vec{u}$, in V such that $\vec{u} + (-\vec{u}) = \vec{0}$.

(6). Operation scalar multiplication is defined for every number c and every $\vec{u}$ in V, and $c \cdot \vec{u} \in V$.

(7). Operation scalar multiplication satisfies the distributive law: $c \cdot (\vec{u} + \vec{v}) = c \cdot \vec{u} + c \cdot \vec{v}$.

(8). Operation scalar multiplication satisfies the second distributive law: $(c + d) \cdot \vec{u} = c \cdot \vec{u} + d \cdot \vec{u}$.

(9). Operation scalar multiplication satisfies the associative law: $(cd) \cdot \vec{u} = c \cdot (d \cdot \vec{u})$.

(10). For every element $u \in V$, $1 \cdot \vec{u} = \vec{u}$.

Conditions we usually need here:

- Closure:
  1. Closed under addition: If $\vec{u}$, $\vec{v} \in V$, then $\vec{u} + \vec{v} \in V$.
  2. Closed under scalar multiplication: If $\vec{u} \in V$, $k \in \mathbb{R}$, then $k\vec{u} \in V$.
- Existence:
  1. There exists zero vector $\vec{0}$ such that: If $\vec{u} \in V$, then $\vec{u} + \vec{0} = \vec{0} + \vec{u} = \vec{u}$.
  2. There exists negative vector $-\vec{u}$, for any $\vec{u} \in V$, such that $\vec{u} + (-\vec{u}) = \vec{0}$.

Remark. Usually, we use $\bigoplus$ for general addition, $\bigodot$ or $\bigotimes$ for general multiplication.

Example 24. *1. $\mathbb{R}^n$ with the usual operations is a vector space.*

2. $\mathbb{C}^n$ *with the usual operations is a vector space.*

3. P*=the set of all polynomials with the usual operations are vector space.*

4. $F(\mathbb{R})$ *of all real-valued functions on* $\mathbb{R}$ *with the usual operations is a vector space.*

5. $F[a, b]$ *of all real-valued functions on* $[a, b]$ *with the usual operations is a vector space.*

Example 25. *(Space of equations) Let* $\mathbb{E} = \{a_1x_1 + ... + a_nx_n = b | a_1, ..., a_n, b \in \mathbb{R}\}$ *under usual addition and scalar multiplication is a vector space.*

Solution:

$$\vec{0}: \quad 0x_1 + ... + 0x_n = 0.$$

Example 26. $M_{22} = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} | a, b, c, d \in \mathbb{R} \right\}$ *is a vector space under addition*

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} a+e & b+f \\ c+g & d+h \end{bmatrix}$$

and scalar multiplication

$$k \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} ka & kb \\ kc & kd \end{bmatrix}.$$

Solution:

$$\vec{0} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad -\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} -a & -b \\ -c & -d \end{bmatrix}.$$

Proposition 1. *(i)* $\vec{w} + \vec{v} = \vec{u} + \vec{v}$ *implies* $\vec{w} = \vec{u}$.
(ii) $0\vec{v} = \vec{0}$.
(iii) $c\vec{0} = \vec{0}$.
(iv) $a\vec{v} = \vec{0}$ *implies* $a = 0$ *or* $\vec{v} = \vec{0}$.

Example 27. *On* $\mathbb{R}^2$ *we define (irregular) addition and scalar multiplication as*

$$\begin{bmatrix} x \\ y \end{bmatrix} \bigoplus \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} x+u-2 \\ y+v \end{bmatrix}, \quad k \bigodot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ky \\ kx \end{bmatrix}.$$

(a) Find the zero vector and negative vectors. (b) Is it a vector space?

Solution: (a). The zero vector is $\begin{bmatrix} 2 \\ 0 \end{bmatrix}$, since $\begin{bmatrix} x \\ y \end{bmatrix} \bigoplus \begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} x+2-2 \\ y+0 \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$.

The negative of $\begin{bmatrix} x \\ y \end{bmatrix}$ is $\begin{bmatrix} -x+4 \\ -y \end{bmatrix}$:

$$\begin{bmatrix} x \\ y \end{bmatrix} \bigoplus \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} x+u-2 \\ y+v \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \quad u = -x+4, v = -y.$$

(b). Not a vector space. For example,

$$3 \bigodot (2 \bigodot \begin{bmatrix} 4 \\ 5 \end{bmatrix}) = 3 \bigodot \begin{bmatrix} 10 \\ 8 \end{bmatrix} = \begin{bmatrix} 24 \\ 30 \end{bmatrix},$$

$$(3 \cdot 2) \bigodot \begin{bmatrix} 4 \\ 5 \end{bmatrix} = 6 \bigodot \begin{bmatrix} 4 \\ 5 \end{bmatrix} = \begin{bmatrix} 30 \\ 24 \end{bmatrix} \neq 3 \bigodot (2 \bigodot \begin{bmatrix} 4 \\ 5 \end{bmatrix}).$$

Example 28. *Determine whether each of the following is a vector space. (a) The integer set $\mathbb{Z}$ with the usual operations.*

(b) $\mathbb{R}^2$ with $c \cdot (x, y) = (cx, 0)$.

(c) $\mathbb{R}^3$ with the scalar multiplication $k \cdot (x, y, z) = (kx, yz, ky)$.

Solution: (a) Does not satisfies (6). For example, $\sqrt{2}(3)$ is not integer.

(b) Does not satisfies (6). For example, $1 \cdot (2, 3) = (2, 0) \neq (2, 3)$.

(c) It does not satisfies Axioms (9) and (10).

Chapter 5 Subspaces

Subspaces

Definition 8. *A set U is a subspace of a vector space V if U is a vector space with respect to the operations of V.*

Theorem 4. *(**Subspace Test**): U is a subspace of V if*

(i) the zero vector is in U;

(ii) U is closed under scalar multiplication: if x is in U, then ax is in U for any scalar a,

(iii) U is closed under addition: if x, y are in U, then $x + y$ is in U.

Example 29. *Determine whether each of the following is a subspace or not.*

- $S = \{(x, y, z) \mid x + z = y, x, y, z \in \mathbb{R}\}$. *Give a complete geometric description of* S.

  Solution:

  – *Notice that* $(0, 0, 0) \in S$ *since* $0 + 0 = 0$.

  – *If* (a, b, c) *and* (a', b', c') *are in* S, *by definition we have*

  $$a + c = b \qquad \text{and} \qquad a' + c' = b'$$

  which implies that

  $$(a + a') + (c + c') = (a + c) + (a' + c') = b + b'.$$

  Thus $(a, b, c) + (a', b', c') = (a + a', b + b', c + c') \in S$, *so* S *is closed under addition.*

  – *If* (a, b, c) *is in* S *and* k *is any scalar, we have that* $k(a, b, c) = (ka, kb, kc)$, $ka + kc = k(a + c) = kb$. *This implies that* S *is closed under scalar multiplication.*

  Thus, S *is a subspace of* $\mathbb{R}^3$.

  S *is a plane through origin with normal vector* $(1, -1, 1)$.

- $T = \{(x, y, 1) | x, y \in \mathbb{R}\}$ *with the usual operations.*

  Solution: *NO.*

- $U = \{(x, y, 0) | x, y \in \mathbb{R}\}$ *with the usual operations.*

  Solution: *YES.*

- $V = \{(a - 1, b, b) | a, b, c \in \mathbb{R}\}$.

  Solution: *Yes.*

- $W = \{(x, y, z) | xz = y\}$ *with the usual operations.*

  Solution: *NOT a vector space. For example,* $(1, 3, 3), (2, 6, 3) \in W$, *but* $(1, 3, 3) + (2, 6, 3) = (3, 9, 6) \notin W$, $2(1, 3, 3) = (2, 6, 6) \notin W$.

- $X = \{(a, a+1, a-1) | a \in \mathbb{R}\}$;
  Solution: *X is not a subspace of $\mathbb{R}^3$.*

  (1) $\vec{0} \notin X$. In fact, if $(a, a+1, a-1) = \vec{0}$, then $a = 0$, $a - 1 = 0$, $a + 1 = 0$, a contradiction.
  (2) X is not closed under addition: for example, $(1, 2, 0) + (2, 3, 1) \notin X$.
  (3) X is not closed under scalar multiplication: for example, $3(1, 2, 0) \notin X$.

Example 30. • *$\{\vec{0}\}$ is a subspace. A subspace that is not $\{\vec{0}\}$ is a proper subspace.*

- *Any line through the origin is a subspace.*

- *Any plane through the origin in $\mathbb{R}^3$ is a subspace. For example, $v = \{w \in \mathbb{R}^3 | \; w \cdot (2, -1, 1) = 0\}$ is a subspace.*

- $S = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} | a = 3b + 2c, a, b, c \in R \right\}$ *is a subspace of M_{22}.*

- *P_n, the set of all polynomials with degree $\leq n$, is a subspace of P.*

- $S = \{p \in P_2 | p(1) = 0\}$ *is a subspace of P.*

- $S = \{p \in P_2 | p(1) = 3\}$ *is not a subspace of P.*

- *$C(\mathbb{R})$ of all real-valued continuous functions on $\mathbb{R}$ with the usual operations is a subspace of $F(\mathbb{R})$.*

Example 31. *Let $A = \begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix}$. Then the set of matrices in M_{22} that are commutative with A is a subspace of M_{22}.*

Solution: Let $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be commutable with A, i.e.,

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \Rightarrow \begin{bmatrix} a+3b & 2a-b \\ c+3d & 2c-d \end{bmatrix} = \begin{bmatrix} a+2c & b+2d \\ 3a-c & 3b-d \end{bmatrix},$$

$$\Rightarrow 3b = 2c, a = b + d.$$

Let $c = 3s, d = t$. Then $b = 2s$, $a = 2s + t$. Then $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = s \begin{bmatrix} 2 & 2 \\ 3 & 0 \end{bmatrix} + t \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

Chapter 6 Spanning Sets

Given a set of vectors $\vec{v}_1, \cdots, \vec{v}_m$ in the vector space V,

$$Span\{\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_m\} = \{c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_m\vec{v}_m| \quad c_1, c_2, ..., c_m \in \mathbb{R}\}.$$

Proposition 2. *(i) If $v \in S$, then $v \in spanS$.*

(ii) If a subspace W contains every vector in S, then W contains span S.

(iii) If $\vec{b}$ is a linear combination of $v_1, v_2, \cdots, v_k$, then $span\{\vec{b}, v_1, v_2, \cdots, v_k\} = span\{v_1, v_2, \cdots, v_k\}$.

(iv) ***Any span is a subspace.***

Example 32. *(1)* $(1, 2, 0, 1)$ *is in span*$\{(2, 1, 2, 0), (0, -3, 2, -2)\}$.

(2) $span\{u + v, u, v\} = span\{u, v\}$.

(3) $R^n = span\{e_1, e_2, ..., e_n\}$, where $e_1 = (1, 0, ..., 0)$, ..., $e_n = (0, 0, ..., 0, 1)$.

(4) $M_{22} = span\{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\}$.

(5) Verify that the set of vectors $S = \{(1, 0, 0), (1, 0, 1), (1, 1, 1)\}$ spans $\mathbb{R}^3$.

(6) Find k such that $(2k, 3k, 5k, k^2)$ is in span$\{(1, 0, 1, 1), (0, 1, 1, -1)\}$.

(7) $W = \{(x, y, z)|y = 2x + 3z, x, z \in \mathbb{R}\} = span\{(1, 2, 0), (0, 3, 1)\}$.

(8) Let $A = \begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix}$. Then the set of matrices in M_{22} that are commutative with A is a subspace of M_{22}, which is spanned by $\left\{ \begin{bmatrix} 2 & 2 \\ 3 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right\}$.

(9) Let $f(x) = \sin x$, $g(x) = \cos x$, $k(x) = 1$ for any x. Then $k(x) \notin span\{f, g\}$, but $k(x) \in span\{f^2, g^2\}$.

Solution: (6) Let $(2k, 3k, 5k, k^2) = a(1, 0, 1, 1) + b(0, 1, 1, -1)$. Then $a = 2k$, $b = 3k$, $a + b = 5k$, $a - b = k^2$. We imply that $-k = k^2$, $k = 0, -1$.

Example 33. $P_2 = span\,\{1, x, x^2\} = span\,\{2, 3, x, 3x, 1 + x + x^2, 1 - x + 3x^2, 1 + 3x - x^2\}$.

Solution: Each member in the set $\{2, 3, x, 3x, 1 + x + x^2, 1 - x + 3x^2, 1 + 3x - x^2\}$ is a linear combination of the members in $\{1, x, x^2\}$.

Example 34. $P_2 \neq span\,\{1, x^2\}$.

Solution: For example, $x \in P_2$, but x is not in the span.

Chapters 7-8 Linear Independence

Example 35. *Let* $u = (1,2,3)$, $v = (2,4,6)$, $w = (3,5,6)$. *Then vectors* u *and* v *are colinear,* u *and* w *are non-colinear.*

Solution:

$$2u - v = 0$$

If

$$au + bw = 0$$

then

$$(a+3b, 2a+5b, 3a+6b) = 0, \Rightarrow a = -3b, a = -2.5b, a = -2b.$$

Thus $b = 0$, $a = 0$.

Definition 9. *A set of vectors* $\{\vec{v}_1, \cdots, \vec{v}_m\}$ *in a vector space* V *is linearly independent* ***LI*** *if the vector equation*

$$x_1\vec{v}_1 + x_2\vec{v}_2 + \cdots + x_m\vec{v}_m = \vec{0}$$

implies that $x_1, x_2, \cdots, x_m = 0$. *The set is said to be linearly dependent* ***LD*** *if there is a non-trivial solution to the vector equation.*

Method to show LI:

- Construct the linear equation.
- Solve the equation.

Example 36. *Given* $\vec{v_1} = (1,2,3), \vec{v_2} = (3,5,8), \vec{v_3} = (1,1,2)$. *Show that* $\{\vec{v_1}, \vec{v_2}, \vec{v_3}\}$ *is linearly dependent and find the linear combination.*

Solution: Let

$$x_1\vec{v_1} + x_2\vec{v_2} + x_3\vec{v_3} = 0.$$

$$\left[\begin{array}{ccc|c} 1 & 3 & 1 & 0 \\ 2 & 5 & 1 & 0 \\ 3 & 8 & 2 & 0 \end{array}\right] \xrightarrow{R_2 - 2R_1, R_3 - 3R_1} \left[\begin{array}{ccc|c} 1 & 3 & 1 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & -1 & -1 & 0 \end{array}\right] \xrightarrow{R_3 - R_2} \left[\begin{array}{ccc|c} 1 & 3 & 1 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{R_1 + 3R_2} \left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{R_1 + 3R_2} \left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Thus $x_1 = 2x_3$, $x_2 = -x_3$. Take $x_3 = 1$, we have

$$2\vec{v_1} - \vec{v_2} + \vec{v_3} = \vec{0}.$$

Example 37. 1. *In $F(\mathbb{R})$, the set $\left\{\sin^2 x, \cos^2 x, 2\right\}$ is linearly dependent.*

2. *In P_2, the set $\{1, x, x^2\}$ is linearly independent.*

3. *In P_2, the set $\{1 + x + x^2, 1 - x + 3x^2, 1 + 3x - x^2\}$ is linearly dependent.*

Example 38. *Let $\{\vec{v_1}, \vec{v_2}, \vec{v_3}, \vec{v_4}\}$ be linearly independent. Determine if the following set is linearly independent or dependent: $S = \{\vec{v_1} - \vec{v_2}, \vec{v_2} - \vec{v_3}, \vec{v_3} - \vec{v_4}, \vec{v_4} - \vec{v_1}\}$.*

Solution: We need to set up the equation

$$c_1(\vec{v_1} - \vec{v_2}) + c_2(\vec{v_2} - \vec{v_3}) + c_3(\vec{v_3} - \vec{v_4}) + c_4(\vec{v_4} - \vec{v_1}) = \vec{0}, \Rightarrow$$

$$(c_1 - c_4)\vec{v_1} + (-c_1 + c_2)\vec{v_2} + (-c_2 + c_3)\vec{v_3} + (-c_3 + c_4)\vec{v_4} = \vec{0}, \Rightarrow$$

$$c_1 = c_2 = c_3 = c_4. \quad e.g., c_1 = c_2 = c_3 = c_4 = 1.$$

Thus dependent.

Theorem 5. 1. *A set of two vectors is linearly dependent if and only if one of the vectors is a multiple of the other.*

2. *A set of two or more vectors is linearly dependent if and only if at least one vector may be written as a linearly combination of the others.*

3. *If the zero vector is in a set of vectors, then the set of vectors is linearly dependent.*

4. *Let $W = \{v_1, ..., v_k\}$ be linearly independent. Then Let $\{u, v_1, ..., v_k\}$ is linearly independent iff $u \notin span\{v_1, ..., v_k\}$.*

Fundamental Theorem: Let U be a vector space spanned by m vectors, and U contains k linearly independent vectors, then $k \leq m$.

Example 39. *Let $W = \{v_1, ..., v_k\} \subset \mathbb{R}^n$. If $k > n$, then the set W is linearly dependent.*

Chapters 9-10 Basis and dimension

Basis

Definition 10. • *A basis for a vector space V is a linearly independent set of vectors that spans V.*

- *If $\{w_1, ..., w_k\}$ is a basis of the space V, then k is called the dimension of V and is denoted by* $\dim V$.

- *If $\{w_1, ..., w_k\}$ is a basis of the space V, then for any $v \in V$, we have*

$$v = c_1 w_1 + ... + c_k w_k,$$

  where $c_1, ..., c_k$ are called ***coordinates, or Fourier coefficients.***

Invariant Theorem: Let U be a vector space. If $\{g_1, ..., g_k\}$ and $\{f_1, ..., f_m\}$ are two bases of U, then $k = m$.

Example 40. *1. $\{e_1, ..., e_n\}$ is the standard basis og $\mathbb{R}^n$.*

2. The set $\{1, x, x^2, ..., x^n\}$ is the standard basis of P_n.

3. Let E_{ij} be the matrix of $m \times n$ where the (i, j) entry is 1, all other entries are 0. Then the set $\{E_{11}, ..., E_{mn}\}$ is the standard basis of M_{mn}. For example,

$$\{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\}$$

is the standard basis for M_{22}.

Method to find a basis of a subspace:

- Step 1: Find a spanning set.

- Find a maximum LI subset.

Example 41. *Find a basis to each of the following subspaces:*

$$\begin{aligned} U &= \{(a + 2b + 3c, a + c, b + a + 2c, a - b) | a, b, c \in \mathbb{R}\} \\ V &= \{(a, b, c, d) | a + 2b = c, 3b - 2c = d; a, b, c, d \in \mathbb{R}\} \end{aligned}$$

Solution:

- $U = \{a(1, 1, 1, 1) + b(2, 0, 1, -1) + c(3, 1, 2, 0) | a, b, c \in \mathbb{R}\}$

  $= Span\{(1, 1, 1, 1), (2, 0, 1, -1), (3, -1, 0, 0)\}$.

Next we show that the set of three vectors $S = \{(1,1,1,1),(2,0,1,-1),(3,-1,0,0)\}$ is linearly dependent. If

$$a(1,1,1,1) + b(2,0,1,-1) + c(3,-1,0,0) = \vec{0},$$

then

$$\begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & 1 \\ 1 & 1 & 2 \\ 1 & -1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Thus a basis is $\{(1,1,1,1),(2,0,1,-1)\}$, vectors containing leading 1's.

- For the subspace V, from $a+2b=c, 3b-2c=d$ we imply that $d=-2a-b$. Thus $V = \{(a,b,a+2b,-2a-b)|a,b \in \mathbb{R}\} = \{a(1,0,1,-2)+b(0,1,2,-1)|a,b \in \mathbb{R}\} = Span\{(1,0,1,-2),(0,1,2,-1)\}$.

  Similarly we can show that the set of two vectors $T = \{(1,0,1,-2),(0,1,2,-1)\}$ is linearly independent. Thus T is a basis of V.

Example 42. *Find a basis of the subspace of* $W = \{p \in P_2 | p(2) = 0\}$.

Solution: Let

$$p(x) = ax^2 + bx + c$$

Then $p(2) = 4a + 2b + c = 0, \Rightarrow c = -4a - 2b$. Thus

$$p(x) = ax^2 + bx + -4a - 2b = a(x^2 - 4) + b(x - 2).$$

$$W = span\{x^2 - 4, x - 2\}.$$

Note that

$$a(x^2 - 4) + b(x - 2) = 0 \Rightarrow a = 0, b = 0.$$

Thus the set $B = \{x^2 - 4, x - 2\}$ is LI, which is a basis of W.

Example 43. *Find a basis to the following subspace of* M_{22}*:*

$$W = \left\{ \begin{bmatrix} 2x & 3x \\ y+z & y+z-x \end{bmatrix} \middle| x, y, z \in \mathbb{R} \right\}$$

Solution:

$$\begin{aligned} W &= \left\{ x \begin{bmatrix} 2 & 3 \\ 0 & -1 \end{bmatrix} + y \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} + z \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} | x, y, z \in \mathbb{R} \right\} \\ &= Span \left\{ \begin{bmatrix} 2 & 3 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \right\} = Span \left\{ \begin{bmatrix} 2 & 3 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \right\}. \end{aligned}$$

Next we show that the set of two vectors $S = \left\{ \begin{bmatrix} 2 & 3 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \right\}$ is linearly independent. Set up the equation

$$a \begin{bmatrix} 2 & 3 \\ 0 & -1 \end{bmatrix} + b \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} = \vec{0}.$$

then

$$\begin{bmatrix} 2a & 3a \\ b & b-a \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

$$\Rightarrow 2a = 0, 3a = 0, b = 0, b - a = 0,$$

$$\Rightarrow a = 0, b = 0.$$

Thus S is independent, which is a basis of W.

Dimension

Definition 11. *The dimension of a vector space V is defined to be the number of vectors in a basis. We write it as $\dim V$.*

Example 44. *1.* $\dim\{\vec{0}\} = 0$.

2. $\dim P_n = n + 1$.

3. $\dim M_{mn} = mn$.

Theorem 6. *Let V be a subspace space with $\dim V = k$. Then*

1. *Any set of more than k vectors in V is linearly dependent.*

2. *Any set of less than k vectors in V can not span V.*

3. *Every basis for V has exactly k vectors.*

Example 45. *Find a basis for* $V = \{v \in \mathbb{R}^4 | v \cdot (1,2,3,4) = 0\}$.

Solution: Let $v = (a,b,c,d)$. Then $a + 2b + 3c + 4d = 0$, $a = -2b - 3c - 4d$.

$$v = (-2b - 3c - 4d, b, c, d) = (-2b, b, 0, 0) + (-3c, 0, c, 0) + (-4d, 0, 0, d)$$

$$= b(-2,1,0,0) + c(-3,0,1,0) + d(-4,0,0,1).$$

Let $S = \{(-2,1,0,0), (-3,0,1,0), (-4,0,0,1)\}$. Then $V = spanS$. From

$$b(-2,1,0,0) + c(-3,0,1,0) + d(-4,0,0,1) = 0, \Rightarrow b = c = d = 0.$$

Thus S is LI. Hence S is a basis of V.

Example 46. *Let* $V = \{p(x) \in P_2 | p(4) = 0\}$, $W = \{p(x) \in P_2 | p(3) = 0\}$. *Then* V *and* W *is a subspace of* P_2.

(1) Find a basis and the dimension of W.

(2) Extend the basis in (1) to a basis of P_2.

(3) Find $W \bigcap V$.

Solution: (1). Let $p(x) \in W$, then $p(x) = (x-3)(ax+b)$. Taking $a = 1, b = 0$ and $a = 0, b = 1$, we get a basis

$$B = \{x - 3, x(x-3)\}.$$

$dimW = 2$.

(2) For any $p(x) \in P_2$, if $p(3) \neq 0$, then $p(x) \notin spanB = W$. For example, $p(x) = 1$. Thus $\{1, x-3, x(x-3)\}$ is a basis of P_2.

(3). For any $p(x) \in W \bigcap V$, $p(3) = 0, p(4) = 0$. Thus $p(x) = a(x-3)(x-4)$. Thus $W \bigcap V = span\{(x-3)(x-4)\}$.

Example 47. *Extend the set* $\{(1,2,1,0), (1,1,2,1)\}$ *to a basis of* $\mathbb{R}^4$.

Solution:

Example 48. *Let* $V = spanB$, *where* $B = \{(1,2,1), (1,1,2), (1,1,1), (1,0,3), (0,1,1)\}$. *Cut down the set* B *to a basis of* V.

Solution:

Example 49. *Let* $W = spanS$, *where* $S = \{1 + x, x + x^2, 1 - x + 3x^2, 1 + 3x - x^2\}$.

(1) Cut down S *to get a basis* B *for* W.

(2) Is $W = P_2$*?*

(3) Express $p(x) = 2 - 5x - x^2$ *as a linear combination of the vectors in* B.

Solution:

(1) By row operations we can derive that $B = \{1 + x, x + x^2, 1 - x + 3x^2\}$.

(2) Yes, since dim W=3=dim P_2.

(3) Let

$$p(x) = a(1 + x) + b(x + x^2) + c(1 - x + 3x^2).$$

Then we get a system

$$\begin{cases} a + c &= 2 \\ a + b - c &= -5 \\ b - 3c &= -1. \end{cases}$$

$a = 8$, $b = -19$, $c = -6$. Thus

$$p(x) = 8(1 + x) - 19(x + x^2) - 6(1 - x + 3x^2).$$

Example 50. *Let* $U = \{p(x) \in P_3 | P(1) = 0, p(2) = 0\}$, $V = \{p(x) \in P_3 | P(1) = 0, p(3) = 0\}$. *We define*

$$U + V = \{u + v | u \in U, v \in V\}.$$

Find $\dim(U + V)$, *and a basis of* $U + V$.

Solution:

Chapters 15-17 Column space, row space, null space

Definition 12. *Let* $A = [\vec{a}_1 \dots \vec{a_n}]_{m \times n}$.

- *The columns of A span (or generate) a subspace of* $\mathbb{R}^m$ *called the column space of A, and is denoted by col(A):*

$$col(A) = Span\{\vec{a}_1, \dots, \vec{a_n}\}.$$

- *The rows of A span (or generate) a subspace of* $\mathbb{R}^n$ *called the row space of A, and is denoted by row(A).*

- *The null space (kernel space) of A:*

$$null(A) = ker(A) = \{v \in \mathbb{R}^n | Av = 0\}.$$

  It is a subspace of $\mathbb{R}^n$*.*

- *The image space of A:*

$$im\ A = \{w \in \mathbb{R}^m | w = Av, v \in \mathbb{R}^n\}.$$

  It is a subspace of $\mathbb{R}^m$ *.*

Example 51. *Let* A *be* $n \times n$*. Let* λ *be a number. Define*

$$E_\lambda(A) = \{v \in \mathbb{R}^n | Av = \lambda v\},$$

which is a subspace of $\mathbb{R}^n$*.*

Proof. $E_\lambda(A) = null(\lambda I - A)$.

Theorem 7. *Let* A *be* $m \times n$ *with the rank* r*. Then*

- *If* $A \to B$ *by elementary row operations, then rowA=rowB.*
- $n - r$ *basic solutions of the equation* $Ax = 0$ *form a basis for* $null(A)$*.*
- *If* A *is row reduced to a row-echelon matrix* B*, then the nonzero rows of* B *form a basis of row*A*. Moreover, if the columns* $j_1, \ldots, j_k$ *in* B *contain leading 1's, then the corresponding columns in* A *form a basis of col*A*.*
- ***The Rank Theorem:*** $dim(col\ A) = dim(row\ A) = rank\ A = n - dim(null A)$*.*

Proof. If we multiply row j by k then add to row i, then
rowA $= span\{R_1, ..., R_i, ..., R_j, ..., R_n\} = span\{R_1, ..., R_i + kR_j, ..., R_j, ..., R_n\} =$ rowB.

Proposition 3. *Let* A *be* $m \times n$*. Then im A=col A.*

Proof. Let $A = [c_1\ ...\ c_n]$, and let $\{e_1, ..., e_n\}$ be the standard basis of $\mathbb{R}^n$. Then $Ae_i = c_i$. Thus $col\ A \subset im\ A$.

Conversely, any $Ax = x_1c_1 + ... + x_nc_n \subset col\ A$.

Example 52. *Let* $A = \begin{bmatrix} 1 & -3 & 1 & 12 & 7 \\ 0 & 0 & 1 & 7 & 4 \\ -2 & 6 & 0 & -10 & -6 \end{bmatrix}$.

Find a basis for $rowA$, $colA$ *and* $nullA$.

Solution:

$A \sim \begin{bmatrix} 1 & -3 & 0 & 5 & 3 \\ 0 & 0 & 1 & 7 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$. Thus $\{(1,-3,0,5,3),[(0,0,1,7,4)\}$ is a basis of $rowA$;

$\left\{ \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\}$ is a basis of $colA$.

To find a basis for $nullA$, consider $A \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = 0$.

$$Augmented\ matrix \sim \begin{bmatrix} 1 & -3 & 0 & 5 & 3 \\ 0 & 0 & 1 & 7 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Thus

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 3x_2 - 5x_4 - 3x_5 \\ x_2 \\ -7x_4 - 4x_5 \\ x_4 \\ x_5 \end{bmatrix} = x_2 \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -5 \\ 0 \\ -7 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -3 \\ 0 \\ -4 \\ 0 \\ 1 \end{bmatrix}.$$

Therefore, a basis of $nullA$ is

$$\left\{ \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ 0 \\ -7 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ -4 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

Example 53. *Let* $X = span\{(1,-1,1,0),(0,1,1,1),(1,2,4,3),(1,0,2,2)\}$.
a) Find any basis for X, *and hence find* $\dim X$.

b) Find a basis for X which is a ***<u>subset</u>*** *of the given spanning set above.*
c) Extend your basis for X in part (b) to a basis of $\mathbb{R}^4$.
d) If X were the row space of a 4×4 matrix A, how many parameters would there be in the general solution to $Ax = 0$?

Solution: a) Let $v_1 = (1, -1, 1, 0),\ v_2 = (0, 1, 1, 1),\ v_3 = (1, 2, 4, 3),\ v_4 = (1, 0, 2, 2)$.

Method1: Consider as Column Space:

$$[v_1\ v_2\ v_3\ v_4] = \begin{bmatrix} 1 & 0 & 1 & 1 \\ -1 & 1 & 2 & 0 \\ 1 & 1 & 4 & 2 \\ 0 & 1 & 3 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

A basis is $\{v_1, v_2, v_4\}$, and $\dim X = 3$.

Method2: Consider as Row Space:

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 2 & 4 & 3 \\ 1 & 0 & 2 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

A basis is $\{(1, 0, 2, 0), (0, 1, 1, 0), (0, 0, 0, 1)\}$.

b) In a).

c) Method1: Consider as Column Space:

$$[v_1\ v_2\ v_4\ e_1\ e_2\ e_3\ e_4] \sim \begin{bmatrix} 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & -1 & 0 \end{bmatrix}.$$ Thus A basis is $\{v_1, v_2, v_4, e_1\}$.

Method2: Consider as Row Space:

$$\begin{bmatrix} v_1 \\ v_2 \\ v_4 \\ e_3 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 2 & 2 \\ 0 & 0 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

d) 1 parameter.

Theorem 8. *Let A be $m \times n$ matrix. Then the following statements are equivalent.*

1. $rankA = n$
2. *The rows of A spans $\mathbb{R}^n$.*
3. *The columns of A are linearly independent.*
4. *$A^T A$ is invertible.*
5. *There exists $n \times m$ matrix C such that $CA = I_n$.*
6. *The equation $AX = 0$ has only the trivial solution.*

Theorem 9. *Let A be $m \times n$ matrix. Then the following statements are equivalent.*

1. $rankA = m$
2. *The columns of A spans $\mathbb{R}^m$.*
3. *The rows of A are linearly independent.*
4. *AA^T is invertible.*
5. *There exists $n \times m$ matrix C such that $AC = I_m$.*
6. *The equation $AX = b$ is consistent for every $b \in \mathbb{R}^m$.*

Chapter 18 Matrix Inverses

Definition 13. *Given an $n \times n$ matrix A, the inverse of A is an $n \times n$ matrix B such that*

$$BA = AB = I,$$

where I is the $n \times n$ identity. The inverse of A is denoted by A^{-1}.

Example 54. *The inverse of 2×2 matrix: If $ad - bc \neq 0$, then*

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

Theorem 10. *If A is an invertible $n \times n$ matrix, then for each $\vec{b} \in \Re^n$, the equation $A\vec{x} = \vec{b}$ has the unique solution $\vec{x} = A^{-1}\vec{b}$.*

Properties of inverses:

1. If A is invertible then A^{-1} is invertible and $(A^{-1})^{-1} = A$
2. If A and B are invertible then AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$
3. If A is invertible then A^T is invertible and $(A^T)^{-1} = (A^{-1})^T$.
4. If A is invertible then A^k is invertible and $(A^k)^{-1} = (A^{-1})^k$.
5. If A is invertible then cA is invertible and $(cA)^{-1} = (1/c)A^{-1}$.
6. $I^{-1} = I$.
7. A triangular matrix is invertible if and only if no entry on the main diagonal is zero.
8. The inverse of an upper (lower) triangular matrix is also an upper (lower) triangular matrix.

True/False Let A, B and C denote 2×2 matrices, where A is invertible. If $AB = AC$, then $B = C$.

Solution: T.

Matrix Inversion Algorithm:

$$[A|I] \xrightarrow{\text{elementary row operations}} [I|A^{-1}].$$

Example 55. *Find A^{-1}, where*

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 6 \\ 3 & 6 & 10 \end{bmatrix}.$$

Solution:

$$[A|I] = \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 5 & 6 & 0 & 1 & 0 \\ 3 & 6 & 10 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_3 \to R_3 - 3R_1]{R_2 \to R_2 - 2R_1} \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & -3 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_1 \to R_1 - 2R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 5 & -2 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & -3 & 0 & 1 \end{array}\right] \xrightarrow{R_1 \to R_1 - 3R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 14 & -2 & -3 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & -3 & 0 & 1 \end{array}\right].$$

Thus

$$A^{-1} = \begin{bmatrix} 14 & -2 & -3 \\ -2 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}.$$

Example 56. *Given the matrix equation*

$$A(B^{-1} + DX)C^T = I,$$

where A, B, C, D and X are $n \times n$ invertible matrices. Solve for X in terms of A, B, C, D.

Solution:

$$X = D^{-1}[A^{-1}(C^T)^{-1} - B^{-1}].$$

Conditions for invertibility

Theorem 11. *(The Invertible Matrix Theorem) Let A be a square $n \times n$ matrix. Then the following statements are equivalent.*

1. *A is an invertible matrix.*
2. *A is row equivalent to the identity matrix.*
3. *The rank of A is n.*
4. *The equation $A\vec{x} = \vec{0}$ has only the trivial solution.*
5. *The equation $A\vec{x} = \vec{b}$ has a solution for each $\vec{b} \in \mathbb{R}^n$.*
6. *There is an $n \times n$ matrix D such that $AD = I$.*
7. *A^T is invertible.*

Proposition 4. *Let A be $n \times n$, $V = \{v_1, ..., v_k\} \subset \mathbb{R}^n$. Then V is linearly independent if and only if $AV = \{Av_1, ..., Av_k\}$ is linearly independent.*

Chapter 19-20 Orthogonality

Orthogonal Sets and Expansion Theorem

Definition 14. *(Orthogonality) Let $\vec{u}, \vec{v}$ be two vectors in $\mathbb{R}^n$. They are said to be orthogonal if $\vec{u} \cdot \vec{v} = 0$.*

Theorem 12. *(The Pythagorean Theorem) Two vectors $\vec{u}, \vec{v}$ are orthogonal if and only if*

$$||\vec{u} + \vec{v}||^2 = ||\vec{u}||^2 + ||\vec{v}||^2.$$

Definition 15. *If each pair of distinct vectors in a set is orthogonal then the set is called an orthogonal set.*

Example 57. *Let* $\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ 3 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 2 \\ -1 \\ 0 \\ -1 \end{bmatrix}.$

(a) Show that the set $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \}$ is an orthogonal set.

(b) Let $\vec{x} = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$. *Find a, b, c, d such that the set $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{x}\}$ is orthogonal.*

Solution: (a) We need to check

$$\begin{aligned} \vec{v_1} \cdot \vec{v_2} &= 0 + 0 + (-2) + 2 = 0 \\ \vec{v_1} \cdot \vec{v_3} &= 0 + (-5) + 4 + 1 = 0 \\ \vec{v_2} \cdot \vec{v_3} &= 0 + 0 + (-2) + 2 = 0. \end{aligned}$$

Thus $\vec{v_1} \perp \vec{v_2}$, $\vec{v_1} \perp \vec{v_3}$, $\vec{v_2} \perp \vec{v_3}$, the set is orthogonal.

(b) We have a system

$$\begin{aligned} \vec{x} \cdot \vec{v}_1 &= \ = 0 \\ \vec{x} \cdot \vec{v}_2 &= 0 \\ \vec{x} \cdot \vec{v}_3 &= 0 \end{aligned}$$

i.e.,

$$\begin{aligned} a + 2b + c &= \ = 0 \\ a - b + c + 3d &= 0 \\ 2a - b - d &= 0 \end{aligned}$$

The augmented matrix

$$\left[\begin{array}{ccccc} 1 & 2 & 1 & 0 & 0 \\ 1 & -1 & 1 & 3 & 0 \\ 2 & -1 & 0 & -1 & 0 \end{array}\right] \sim \left[\begin{array}{ccccc} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 3 & 0 \end{array}\right]$$

$$a = d, b = d, c = -3d.$$

Thus $\vec{x} = d \begin{bmatrix} 1 \\ 1 \\ -3 \\ 1 \end{bmatrix}$.

Theorem 13. *If a set of non-zero vectors is orthogonal, then the set is linearly independent.*

Definition 16. *(Orthogonal Basis) If $S = \{\vec{v_1}, \vec{v_2}, ..., \vec{v_m}\}$ is an orthogonal set of non-zero vectors, then it is called an orthogonal basis for the subspace $W = Span\{\vec{v_1}, \vec{v_2}, ..., \vec{v_m}\}$.*

Example 58. *Let $v_1 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}, \vec{v_2} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \vec{v_3} = \begin{bmatrix} -5 \\ -2 \\ 1 \end{bmatrix}$.*

Show that $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is an orthogonal basis for $\mathbb{R}^3$.

Proof.

$$\begin{aligned} \vec{v_1} \cdot \vec{v_2} &= 0 + (-2) + 2 = 0 \\ \vec{v_1} \cdot \vec{v_3} &= (-5) + 4 + 1 = 0 \\ \vec{v_2} \cdot \vec{v_3} &= 0 + (-2) + 2 = 0. \end{aligned}$$

Thus $\vec{v_1} \perp \vec{v_2}$, $\vec{v_1} \perp \vec{v_3}$, $\vec{v_2} \perp \vec{v_3}$. The set $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is orthogonal of non-zero vectors, so they are linearly independent. Three such vectors automatically form a basis for $\mathbb{R}^3$.

Definition 17. *If $S = \{\vec{u_1}, \vec{u_2}, ..., \vec{u_m}\}$ is an orthogonal set of unit vectors, then it is called an orthonormal set. If an orthonormal set S spans some subspace W, then S is called an orthonormal basis for W. (S is an orthogonal set that spans W, so is linearly independent and thus a basis for W.)*

Example 59. *The set $\{\vec{e_1}, \vec{e_2}, ..., \vec{e_n}\}$ is an orthonormal set that spans $\mathbb{R}^n$, thus an orthonormal basis for $\mathbb{R}^n$, which is called the standard basis for $\mathbb{R}^n$.*

Theorem 14. *(Expansion Theorem) If $\vec{y}$ is a vector in a subspace W which has orthogonal basis $S = \{\vec{v_1}, \vec{v_2}, ..., \vec{v_m}\}$, then it may be written uniquely as a linear combination of the vectors in S,*

$$\vec{y} = c_1\vec{v_1} + c_2\vec{v_2} + ... + c_m\vec{v}_m, \quad c_k = \frac{\vec{y} \cdot \vec{v}_k}{\vec{v}_k \cdot \vec{v}_k}, k = 1, ..., m.$$

Proof.

$$\begin{aligned}\vec{y}\cdot\vec{v_1} &= (c_1\vec{v_1}+c_2\vec{v_2}+...+c_m\vec{v_m})\cdot\vec{v_1}\\ &= c_1\vec{v_1}\cdot\vec{v_1}+c_2\vec{v_2}\cdot\vec{v_1}+...+c_m\vec{v_m}\cdot\vec{v_1}\\ &= c_1\vec{v_1}\cdot\vec{v_1}+0.\end{aligned}$$

Example 60. *Let* $\vec{v}_1 = \begin{bmatrix}1\\2\\1\\0\end{bmatrix}, \vec{v}_2 = \begin{bmatrix}1\\-1\\1\\3\end{bmatrix}, \vec{v}_3 = \begin{bmatrix}2\\-1\\0\\-1\end{bmatrix}. \ \vec{x} = \begin{bmatrix}-3\\4\\-1\\-4\end{bmatrix}$. *Represent* $\vec{x}$ *as a linear combination of* $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$.

Solution: Note that

$$\begin{aligned}\vec{v_1}\cdot\vec{v}_2 &= 1-2+1+0=0\\ \vec{v_1}\cdot\vec{v}_3 &= 2-2+0+0=0\\ \vec{v_2}\cdot\vec{v}_3 &= 2+1+0-3=0.\end{aligned}$$

Thus the set $\{\vec{v_1}, \vec{v}_2, \vec{v}_3\}$ is orthogonal. Let

$$\vec{x} = c_1\vec{v_1}+c_2\vec{v_2}+c_3\vec{v}_3.$$

Then

$$c_1 = \frac{\vec{x}\cdot\vec{v}_1}{\vec{v}_1\cdot\vec{v}_1} = \frac{-3+8-1+0}{1+4+1+0} = \frac{2}{3}.$$

$$c_2 = \frac{\vec{x}\cdot\vec{v}_2}{\vec{v}_2\cdot\vec{v}_2} = \frac{-3-4-1-12}{1+1+1+9} = -\frac{5}{3}.$$

$$c_3 = \frac{\vec{x}\cdot\vec{v}_3}{\vec{v}_3\cdot\vec{v}_3} = \frac{-6-4+0+4}{4+1+0+1} = -1.$$

Hence

$$\vec{x} = \frac{2}{3}\vec{v_1} - \frac{5}{3}\vec{v_2} - \vec{v}_3.$$

Orthogonal Complements

Definition 18. *(Orthogonal Complement) Let* W *be a subspace of* $\mathbb{R}^n$*. The orthogonal complement of* W *is defined as*

$$W^{\perp} = \{v \in \mathbb{R}^n | v\cdot w = 0, \forall w \in W\}.$$

Example 61. *Let W be a plane and L a line intersecting W. At the point of intersection of the line L to the plane W, L is orthogonal to W. We call L the orthogonal complement of W and denote it by $L = W^{\perp}$. Similarly, we may think of W as being perpendicular to L and so may be called the orthogonal complement of L and is denoted by, $W = L^{\perp}$.*

Properties of Orthogonal Complement:

- $W^{\perp}$ is a subspace.
- $W \bigcap W^{\perp} = \{0\}$.
- If $W = span\{w_1, ..., w_k\}$, then $v \in W^{\perp}$ if and only if $v \cdot w_i = 0$ for all i.
- $(rowA)^{\perp} = nullA$, $(colA)^{\perp} = nullA^T$.

Example 62. *Let*

$$W = span\left\{ \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} \right\}, \quad \vec{y} = \begin{bmatrix} 1 \\ 4 \\ -7 \end{bmatrix}.$$

1) Show that $\vec{y} \in W^{\perp}$.
2) Find other vectors in $W^{\perp}$.

Solution: 1)

$$\begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} \cdot \vec{y} = 0, \quad \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} \cdot \vec{y} = 0.$$

2) Let $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in W^{\perp}$. Then

$$\begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} \cdot \vec{x} = 0, \quad \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} \cdot \vec{x} = 0 \Rightarrow$$

$$A\vec{x} = 0, \quad A = \begin{bmatrix} -1 & 2 & 1 \\ 3 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1/7 \\ 0 & 1 & 4/7 \end{bmatrix}.$$

The solution of this is: $x_1 = (-1/7)x_3, x_2 = (-4/7)x_3$, i.e.,

$$\vec{x} = s \begin{bmatrix} 1 \\ 4 \\ -7 \end{bmatrix} = s\vec{y}.$$

Projection

Definition 19. *(Orthogonal projection) Let W be a subspace of $\mathbb{R}^n$. Let $S = \{\vec{v_1}, \vec{v_2}, ..., \vec{v_m}\}$ be an orthogonal basis of W. Then the orthogonal projection of $\vec{y}$ onto W is defined as*

$$\operatorname{proj}_W \vec{y} = c_1\vec{v_1} + c_2\vec{v_2} + ... + c_m\vec{v}_m, \quad c_k = \frac{\vec{y} \cdot \vec{v}_k}{\vec{v}_k \cdot \vec{v}_k}, k = 1, ..., m.$$

The complement of $\vec{y}$ orthogonal to W is

$$perp_W\vec{y} = \vec{y} - \operatorname{proj}_W \vec{y}.$$

Theorem 15. *(The orthogonal Decomposition Theorem) Let W be a subspace of $\mathbb{R}^n$ with orthogonal basis $\{\vec{v_1}, \vec{v_2}, ..., \vec{v_m}\}$. Then each $\vec{y}$ in $\mathbb{R}^n$ can be written uniquely in the form*

$$\vec{y} = \vec{y_1} + \vec{y_2}, \quad \vec{y_1} \in W, \vec{y_2} \in W^{\perp},$$

where

$$\vec{y_1} = \operatorname{proj}_W \vec{y} = \frac{\vec{y} \cdot \vec{v}_1}{\vec{v_1} \cdot \vec{v_1}}\vec{v_1} + \cdots + \frac{\vec{y} \cdot \vec{v}_m}{\vec{v}_m \cdot \vec{v}_m}\vec{v_m}, \quad \vec{y_2} = \vec{y} - \vec{y_1}.$$

Example 63. *Let* $\vec{y} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$, $\vec{v_1} = \begin{bmatrix} 0 \\ 1 \\ -2 \\ 1 \end{bmatrix}$, $\vec{v_2} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 2 \end{bmatrix}$, $\vec{v_3} = \begin{bmatrix} 0 \\ -5 \\ -2 \\ 1 \end{bmatrix}$. *Let $W = Span\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$. Find $\vec{y_1} \in W, \vec{y_2} \in W^{\perp}$ such that $\vec{y} = \vec{y_1} + \vec{y_2}$.*

Solution: Since

$$\begin{aligned} \vec{v_1} \cdot \vec{v_2} &= 0 + 0 + (-2) + 2 = 0 \\ \vec{v_1} \cdot \vec{v_3} &= 0 + (-5) + 4 + 1 = 0 \\ \vec{v_2} \cdot \vec{v_3} &= 0 + 0 + (-2) + 2 = 0. \end{aligned}$$

Thus $\vec{v_1} \perp \vec{v_2}$, $\vec{v_1} \perp \vec{v_3}$, $\vec{v_2} \perp \vec{v_3}$, the set $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is orthogonal basis of W.

$$\vec{y_1} = \operatorname{proj}_W \vec{y} = \frac{\vec{y} \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1}\vec{v_1} + \frac{\vec{y} \cdot \vec{v}_2}{\vec{v}_2 \cdot \vec{v}_2}\vec{v_2} + \frac{\vec{y} \cdot \vec{v}_3}{\vec{v}_3 \cdot \vec{v}_3}\vec{v_3} = 0\begin{bmatrix} 0 \\ 1 \\ -2 \\ 1 \end{bmatrix} + \frac{3}{5}\begin{bmatrix} 0 \\ 0 \\ 1 \\ 2 \end{bmatrix} + \frac{-6}{30}\begin{bmatrix} 0 \\ -5 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 6 \\ 3 \\ 0 \end{bmatrix}.$$

$$\vec{y}_2 = \vec{y} - \vec{y}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 0 \\ 6 \\ 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -5 \\ -2 \\ 1 \end{bmatrix}.$$

Property: If $\vec{y} \in W$, then $\operatorname{proj}_W \vec{y} = \vec{y}$.

Proof. Let $\{\vec{v_1}, \vec{v_2}, ..., \vec{v_m}\}$ be an orthogonal basis for W. Since $\vec{y} \in W$,

$$\vec{y} = d_1\vec{v_1} + d_2\vec{v_2} + ... + d_m\vec{v_m}. \Rightarrow$$

$$\vec{y} \cdot \vec{v_1} = d_1\vec{v_1} \cdot \vec{v_1}, ..., \vec{y} \cdot \vec{v_m} = d_m\vec{v_m} \cdot \vec{v_m}.$$

Thus

$$\begin{aligned} \operatorname{proj}_W \vec{y} &= \frac{\vec{y} \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1}\vec{v_1} + \cdots + \frac{\vec{y} \cdot \vec{v}_m}{\vec{v}_m \cdot \vec{v}_m}\vec{v_m} \\ &= \frac{d_1\vec{v_1} \cdot \vec{v_1}}{\vec{v}_1 \cdot \vec{v}_1}\vec{v_1} + \cdots + \frac{d_m\vec{v_m} \cdot \vec{v_m}}{\vec{v}_m \cdot \vec{v}_m}\vec{v_m} \\ &= d_1\vec{v_1} + d_2\vec{v_2} + ... + d_m\vec{v_m} \\ &= \vec{y}. \end{aligned}$$

Theorem 16. *Let W be a subspace of $\mathbb{R}^n$ with orthonormal basis $\{\vec{u_1}, \vec{u_2}, ..., \vec{u_m}\}$. Then for each $\vec{y}$ in $\mathbb{R}^n$,*

$$\operatorname{proj}_W \vec{y} = (\vec{y} \cdot \vec{u}_1)\vec{u_1} + \cdots + (\vec{y} \cdot \vec{u}_m)\vec{u_m}.$$

Theorem 17. *If W is a subspace of $\mathbb{R}^n$, then*

$$\dim W + \dim W^{\perp} = n.$$

Proof. (1) A basis of W and a basis of $W^{\perp}$ form a linearly independent set; (2) By decomposition theorem, any vector in $\mathbb{R}^n$ can be written as a linear combination of the set.

Approximation

Theorem 18. *(The Best Approximation Theorem) Let W be a subspace of $\mathbb{R}^n$, $\vec{y}$ in $\mathbb{R}^n$ and $\hat{\vec{y}}$ be the orthogonal projection of $\vec{y}$ onto W. Then $\hat{\vec{y}}$ is the closest point in W to $\vec{y}$ in the sense that*

$$||\vec{y} - \hat{\vec{y}}|| < ||\vec{y} - \vec{v}||$$

for all $\vec{v} \in W$ distinct from $\hat{\vec{y}}$.

Example 64. *Let* $\vec{y} = \begin{bmatrix} 3 \\ -1 \\ 1 \\ 13 \end{bmatrix}$, $\vec{v}_1 = \begin{bmatrix} 1 \\ -2 \\ -1 \\ 2 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} -4 \\ 1 \\ 0 \\ 3 \end{bmatrix}$. *Let* $W = Span\{\vec{v}_1, \vec{v}_2\}$. *Find the distance from* $\vec{y}$ *to W.*

Solution: The closest point in W to $\vec{y}$ is $proj_W\vec{y}$. So the distance is $||\vec{y} - proj_W\vec{y}||$. Note that $\vec{v}_1 \cdot \vec{v}_2 = 0$, $\{\vec{v}_1, \vec{v}_2\}$ is orthogonal basis of W,

$$proj_W\vec{y} = \frac{\vec{y}\cdot\vec{v}_1}{\vec{v}_1\cdot\vec{v}_1}\vec{v}_1 + \frac{\vec{y}\cdot\vec{v}_2}{\vec{v}_2\cdot\vec{v}_2}\vec{v}_2 = \begin{bmatrix} -1 \\ -5 \\ -3 \\ 9 \end{bmatrix}, \Rightarrow \vec{y} - proj_W\vec{y} = \begin{bmatrix} 4 \\ 4 \\ 4 \\ 4 \end{bmatrix}, \Rightarrow ||\vec{y} - proj_W\vec{y}|| = 8.$$

The Gram-Schmidt Algorithm

Let $S = \{\vec{v}_1,\ \vec{v}_2, \cdots,\ \vec{v}_k\}$ be a set of vectors which are linearly independent, and let

$$\begin{aligned} \vec{w}_1 &= \vec{v}_1 \\ \vec{w}_2 &= \vec{v}_2 - \frac{\vec{v}_2\cdot\vec{w}_1}{||\vec{w}_1||^2}\vec{w}_1 \\ &\cdots \\ \vec{w}_k &= \vec{v}_k - \frac{\vec{v}_k\cdot\vec{w}_1}{||\vec{w}_1||^2}\vec{w}_1 - \frac{\vec{v}_k\cdot\vec{w}_2}{||\vec{w}_2||^2}\vec{w}_2 - \cdots - -\frac{\vec{v}_k\cdot\vec{w}_{k-1}}{||\vec{w}_{k-1}||^2}\vec{w}_{k-1}. \end{aligned}$$

Then $\{\vec{w}_1,\ \vec{w}_2, \cdots,\ \vec{w}_k\}$ is an orthogonal set.

Example 65. *Consider the following independent set* $S = \{\vec{v}_1,\ \vec{v}_2,\ \vec{v}_3\}$ *of vectors from* $\mathbb{R}^4$*:*

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 6 \\ 0 \\ 0 \\ 2 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} -1 \\ -1 \\ 2 \\ 4 \end{bmatrix}.$$

Use the Gram-Schmidt algorithm to convert the set $S = \{\vec{v}_1,\ \vec{v}_2,\ \vec{v}_3\}$ *into an orthogonal set* $B = \{\vec{w}_1,\ \vec{w}_2,\ \vec{w}_3\}$.

Solution: Let $\vec{w}_1 = \vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$. Then

$$\vec{w}_2 = \vec{v}_2 - \frac{\vec{v}_2 \cdot \vec{w}_1}{||\vec{w}_1||^2}\vec{w}_1 = \begin{bmatrix} 6 \\ 0 \\ 0 \\ 2 \end{bmatrix} - \frac{8}{4}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ -2 \\ -2 \\ 0 \end{bmatrix}.$$

$$\vec{w}_3 = \vec{v}_3 - \frac{\vec{v}_3 \cdot \vec{w}_1}{||\vec{w}_1||^2}\vec{w}_1 - \frac{\vec{v}_3 \cdot \vec{w}_2}{||\vec{w}_2||^2}\vec{w}_2 = \begin{bmatrix} -1 \\ -1 \\ 2 \\ 4 \end{bmatrix} - \frac{4}{4}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \frac{-6}{24}\begin{bmatrix} 4 \\ -2 \\ -2 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ -2.5 \\ 0.5 \\ 3 \end{bmatrix}.$$

Chapter 1 Complex Numbers

Complex number is

$$z = a + bi, \quad a \in \mathbb{R}, b \in \mathbb{R}, i^2 = -1.$$

Addition and multiplication:

$$(a + bi) + (c + di) = (a + c) + (b + d)i,$$

$$(a + bi)(c + di) = (ac - bd) + (ad + bc)i.$$

The conjugate of $z = a + bi$ is: $\bar{z} = a - bi$. We have

$$z\bar{z} = a^2 + b^2.$$

The modulus of z is defined by

$$|z| = r = \sqrt{a^2 + b^2}.$$

Polar form: $z = a + bi = re^{i\theta} = r(\cos\theta + i\sin\theta)$, where $\cos\theta = \frac{a}{r}$, $\sin\theta = \frac{b}{r}$.

Properties

1. $\bar{z} = z$ if and only if z is real,
2. $\overline{z + w} = \bar{z} + \bar{w}$,
3. $\overline{zw} = \bar{z}\bar{w}$,
4. $|wz| = |w||z|$,
5. $|z + w| \le |z| + |w|$ (triangle inequality),
6. let $z = r(\cos\theta + i\sin\theta)$, then $z^n = r^n(\cos n\theta + i\sin n\theta)$ (De Moivre's Theorem).
7. let $z = r(\cos\theta + i\sin\theta)$, then $\sqrt[n]{z} = \sqrt[n]{r}\left(\cos\frac{\theta+2k\pi}{n} + i\sin\frac{\theta+2k\pi}{n}\right)$, where $k = 0, ..., n-1$.

Example 66. *Simplify* $\frac{3+4i}{4-5i}$.

Solution: $\frac{3+4i}{4-5i} = \frac{(3+4i)(4+5i)}{(4-5i)(4+5i)} = \frac{-8+31i}{41} = -\frac{8}{41} + \frac{31}{41}i$.

Example 67. *Find the polar form:* $\frac{\sqrt{2}-\sqrt{2}i}{-2+2\sqrt{3}i}$.

Solution: We change the numerator and the denominator to polar form.

For the numerator $\sqrt{2}-\sqrt{2}i$: $r=2$, $\cos\theta=\frac{\sqrt{2}}{2}$, $\sin\theta=\frac{-\sqrt{2}}{2}$. $\Rightarrow$ $\theta=\frac{7\pi}{4}$. Thus $\sqrt{2}-\sqrt{2}i=2e^{\frac{7\pi}{4}i}$.

Similarly, the denominator $-2+2\sqrt{3}i=4e^{\frac{2\pi}{3}i}$.

$$\frac{\sqrt{2}-\sqrt{2}i}{-2+2\sqrt{3}i}=\frac{2}{4}e^{\frac{7\pi}{4}i-\frac{2\pi}{3}i}=\frac{1}{2}e^{\frac{13\pi}{12}i}.$$

Example 68. *Let $z=\sqrt{3}+i$. Calculate z^6.*

Solution: $r=2$, $\theta=\frac{\pi}{6}$. Thus

$$z=\sqrt{3}+i=2e^{i\pi/6}.$$

$$z^6=2^6e^{6i\pi/6}=2^6e^{\pi i}=64(-1+0i)=-64.$$

Example 69. *Solve the following equation $x^2-x+4=0$.*

Solution:

$$x=\frac{1\pm\sqrt{15}i}{2}.$$

Example 70. *Construct a polynomial with real coefficients such that $3-2i$ is a root.*

Solution: Since $3-2i$ is a root, $3+2i$ should be a root. Thus the polynomial $p(x)$ has the factor

$$(x-3+2i)(x-3-2i)=(x-3)^2+4=x^2-6x+13.$$

Chapter 21 Determinants

Cofactor Expansion

Definition 20. *Let* $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. *The determinant of* A *is defined as*

$$\det A = |A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

For a $n \times n$ *matrix* A*, let* A_{ij} *be the matrix obtained from* A *by deleting the* i*-th row and* j*-th column. The* $(i,j)^{th}$ *cofactor of* A *is the number,*

$$c_{ij} = (-1)^{i+j} \det A_{ij}.$$

$$\det A = a_{i1}c_{i1} + a_{i2}c_{i2} + ... + a_{in}c_{in},$$

which is called a cofactor expansion across the i*-th row. Similarly,*

$$\det A = a_{1j}c_{1j} + a_{2j}c_{2j} + ... + a_{nj}c_{nj},$$

which is called a cofactor expansion across the j*-th column.*

Example 71. *Calculate* $\det A$*, where*

$$A = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 1 & 1 \\ 3 & 4 & 2 \end{bmatrix}.$$

Solution: We do cofactor expansion across the 2nd row.

$$\begin{aligned} \det A &= a_{21}c_{21} + a_{22}c_{22} + a_{23}c_{23} \\ &= 2(-1)^{2+1} \det \begin{bmatrix} 3 & 5 \\ 4 & 2 \end{bmatrix} + (-1)^{2+2} \det \begin{bmatrix} 1 & 5 \\ 3 & 2 \end{bmatrix} + (-1)^{2+3} \det \begin{bmatrix} 1 & 3 \\ 3 & 4 \end{bmatrix} \\ &= 2(14) + (-13) + 5 = 20. \end{aligned}$$

Geometric Interpretation:

If A is 2×2, then $|A|$ represents the area of the parallelogram formed by the two row vectors. If A is 3×3, then $|A|$ represents the volume of the parallelepiped formed by the three row vectors.

Definition 21. *A triangular matrix is a matrix that is all zeros either above or below the diagonal. An upper triangular matrix means all entries below the main diagonal are zero; an lower triangular matrix means all entries above the main diagonal are zero.*

Theorem 19. *If A is a triangular matrix then det$A$ is the product of the entries on the main diagonal of A.*

Example 72. *Calculate* $\det A$*, where*

$$A = \begin{bmatrix} 5 & 3 & 5 & 7 \\ 0 & 1 & 1 & 9 \\ 0 & 0 & 2 & 12 \\ 0 & 0 & 0 & 12 \end{bmatrix}.$$

Solution: A is an upper triangular matrix. $\det A = 5(1)(2)(12) = 120$.

Theorem 20. *If A is a 3×3 matrix, $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, then*

$$\det A = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{31}a_{22}a_{13} - a_{32}a_{23}a_{11} - a_{33}a_{21}a_{12}.$$

Remark. This comes from the three main diagonals and three other diagonals by repeating the first two columns.

Elementary Operations and Determinants

1. If $A \xrightarrow{R_i \to R_i + kR_j} B$, then $\det B = \det A$.

2. If $A \xrightarrow{R_i \leftrightarrow R_j} B$, then $\det B = -\det A$.

3. If $A \xrightarrow{R_i \to kR_i} B$, then $\det B = k \det A$.

Example 73. *Let*

$$A = \begin{bmatrix} 1 & -3 & 2 & -4 \\ -4 & 12 & -4 & 5 \\ 2 & -5 & 4 & -3 \\ -3 & 10 & -1 & 7 \end{bmatrix}.$$

(a) Calculate $\det A$ *by using row reduction. (b) Find C_{23}, the (2,3)-cofactor of A.*

Solution:

$$\det A = \begin{vmatrix} 1 & -3 & 2 & -4 \\ -4 & 12 & -4 & 5 \\ 2 & -5 & 4 & -3 \\ -3 & 10 & -1 & 7 \end{vmatrix} \overset{\substack{R_2\to R_2+4R_1 \\ R_3\to R_3-2R_1 \\ R_4\to R_4+3R_1}}{=\!=\!=\!=\!=\!=\!=} \begin{vmatrix} 1 & -3 & 2 & -4 \\ 0 & 0 & 4 & -11 \\ 0 & 1 & 0 & 5 \\ 0 & 1 & 5 & -5 \end{vmatrix}$$

$$\overset{R_2\leftrightarrow R_4}{=\!=\!=\!=\!=\!=} - \begin{vmatrix} 1 & -3 & 2 & -4 \\ 0 & 1 & 5 & -5 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 4 & -11 \end{vmatrix} \overset{R_3\to R_3-R_2}{=\!=\!=\!=\!=\!=} - \begin{vmatrix} 1 & -3 & 2 & -4 \\ 0 & 1 & 5 & -5 \\ 0 & 0 & -5 & 10 \\ 0 & 0 & 4 & -11 \end{vmatrix}$$

$$\overset{R_3\to -\frac{1}{5}R_3}{=\!=\!=\!=\!=\!=} - (-5) \begin{vmatrix} 1 & -3 & 2 & -4 \\ 0 & 1 & 5 & -5 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 4 & -11 \end{vmatrix} \overset{R_4\to R_4-4R_3}{=\!=\!=\!=\!=} 5 \begin{vmatrix} 1 & -3 & 2 & -4 \\ 0 & 1 & 5 & -5 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & -3 \end{vmatrix}$$

$$= 5(1)(1)(-3) = -15.$$

(b)

$$C_{23} = (-1)^{2+3} \det A_{23} = -\det \begin{bmatrix} 1 & -3 & -4 \\ 2 & -5 & -3 \\ -3 & 10 & 7 \end{bmatrix}$$

$$\overset{\substack{R_2\to R_2-2R_1 \\ R_3\to R_3+3R_1}}{=} - \begin{vmatrix} 1 & -3 & -4 \\ 0 & 1 & 5 \\ 0 & 1 & -5 \end{vmatrix} \overset{R_3\to R_3-R_2}{=} - \begin{vmatrix} 1 & -3 & -4 \\ 0 & 1 & 5 \\ 0 & 0 & -10 \end{vmatrix} = -1(1)(-10) = 10.$$

Example 74. *Given that* $\det A = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 2$, *what is* $\begin{vmatrix} a-2g & b-2h & c-2i \\ 3g & 3h & 3i \\ 5d & 5e & 5f \end{vmatrix}$*?*

Solution: $\begin{vmatrix} a-2g & b-2h & c-2i \\ 3g & 3h & 3i \\ 5d & 5e & 5f \end{vmatrix} = \text{-15} \begin{vmatrix} a-2g & b-2h & c-2i \\ d & e & f \\ g & h & i \end{vmatrix}$

$= \text{-15} \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = -15(2) = -30.$

Example 75. *(Vandermonde determinant)*

$$det \begin{bmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^{n-1} \end{bmatrix} = \prod_{i>j;\ i,j=1}^{n} (x_i - x_j).$$

Properties of determinants:

1. If A is a square matrix then $\det A^T = \det A$.
2. If A has two identity rows (or columns), then $\det A = 0$.
3. If A is an $n \times n$ matrix and k a scalar then $\det(kA) = k^n \det A$.

4. $|AB| = |A| \cdot |B|$.

Properties of determinants:

5. If A is invertible then $det(A^{-1}) = \frac{1}{\det A}$.

6. A square matrix A is invertible $\Leftrightarrow \det A \neq 0$.

Example 76. *Let A, B and C be 3×3 invertible matrices, $det(A) = 3$, $det(B) = 5$, $det(C) = 6$. Calculate $det(A^{-1}C^2(-2B^T))$.*

Solution:

$$\frac{1}{3}6^2(-2)^3 5 = -480.$$

7. Block upper (lower) triangular matrices:

$$\det \begin{bmatrix} A & X \\ 0 & B \end{bmatrix} = \det A \det B, \quad \det \begin{bmatrix} A & 0 \\ X & B \end{bmatrix} = \det A \det B.$$

8. **Inverse Formula:** If A is $n \times n$ and invertible, then

$$A^{-1} = \frac{1}{\det A} adj A, \quad adj A = \begin{bmatrix} c_{11} & c_{21} & \dots & c_{n1} \\ c_{12} & c_{22} & \dots & c_{n2} \\ \vdots & \vdots & \vdots & \vdots \\ c_{1n} & c_{2n} & \dots & c_{nn} \end{bmatrix},$$

where $adj A$ is called adjugate of A.

Proposition 5. *Let A be $n \times n$. Then* $\det(adj A) = (\det A)^{n-1}$.

A square matrix A is called **orthogonal** if $A^{-1} = A^T$.

Example 77. $A = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$.

Chapter 22 Eigenvalues and eigenvectors

Eigenvalues and Eigenvectors

Definition 22. *An eigenvector of an $n \times n$ matrix A is a nonzero vector $\vec{x}$ such that $A\vec{x} = \lambda\vec{x}$ for some scalar λ. A scalar λ is called an eigenvalue of A, $\vec{x}$ is called the eigenvector corresponding to λ.*

To determine whether a given value λ is an eigenvalue of a matrix A we need to find a non-zero vector $\vec{x}$ such that $A\vec{x} = \lambda\vec{x}$. This is the same as determining whether the matrix equation

$$(\lambda I - A)\vec{x} = 0$$

has a non-trivial solution.

Characteristic equation: $c_A(\lambda) = \det(\lambda I - A)$ is called the characteristic polynomial of A and

$$c_A(\lambda) = \det(\lambda I - A) = 0$$

is called the characteristic equation.

Theorem 21. *The solutions of the characteristic equation are the eigenvalues of A. The eigenvectors $\vec{x}$ corresponding to λ are the nonzero solutions of $(\lambda I - A)\vec{x} = 0$.*

Basic eigenvectors: Any basic solution of $(\lambda I - A)\vec{x} = 0$ is called a basic eigenvector (or basic-λ-eigenvector). Any set of non-zero multiples of a basic eigenvector of $(\lambda I - A)\vec{x} = 0$ will be called a set of basic eigenvectors corresponding to λ. We treat a set of basic eigenvectors as only one basic eigenvector.

Eigenspace: $E_\lambda(A) = \{\vec{x} | (\lambda I - A)\vec{x} = 0\}$ is called eigenspace corresponding to λ.

Example 78. *Let* $A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$. *Find all eigenvalues.*

Solution: Eigenvalues are 5, 1. Corresponding eigenvectors are $\vec{x} = s\begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\vec{x} = t\begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

Example 79. *Let* $A = \begin{bmatrix} 1 & -2 \\ 1 & 3 \end{bmatrix}$. *Find all eigenvalues.*

Solution: Eigenvalues are $2 \pm i$.

Example 80. *Let* $A = \begin{bmatrix} 3 & -4 & 2 \\ 1 & -2 & 2 \\ 1 & -5 & 5 \end{bmatrix}$. *Find the eigenvalues and eigenvectors, and basis to each eigenspace.*

Solution:

$$\det(A - \lambda I) = \begin{vmatrix} 3-\lambda & -4 & 2 \\ 1 & -2-\lambda & 2 \\ 1 & -5 & 5-\lambda \end{vmatrix} \overset{R_3 \to R_3 - R_2}{=\!=\!=} \begin{vmatrix} 3-\lambda & -4 & 2 \\ 1 & -2-\lambda & 2 \\ 0 & -3+\lambda & 3-\lambda \end{vmatrix}$$

$$= (3-\lambda) \begin{vmatrix} 3-\lambda & -4 & 2 \\ 1 & -2-\lambda & 2 \\ 0 & -1 & 1 \end{vmatrix} = (3-\lambda) \left\{ (3-\lambda) \begin{vmatrix} -2-\lambda & 2 \\ -1 & 1 \end{vmatrix} - 1 \begin{vmatrix} -4 & 2 \\ -1 & 1 \end{vmatrix} \right\}$$

$$= (3-\lambda)\left\{(3-\lambda)(-\lambda) + 2\right\} = (3-\lambda)(\lambda-1)(\lambda-2).$$

Thus the eigenvalues are 1, 2, 3 .

When $\lambda = 3$,

$$A - 3I = \begin{bmatrix} 0 & -4 & 2 \\ 1 & -5 & 2 \\ 1 & -5 & 2 \end{bmatrix} \xrightarrow{R_3 \to R_3 - R_2} \begin{bmatrix} 0 & -4 & 2 \\ 1 & -5 & 2 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1/2 \\ 0 & 1 & -1/2 \\ 0 & 0 & 0 \end{bmatrix}.$$

Thus $(A - 3I)\vec{x} = 0$ has the solution

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \frac{1}{2} x_3 \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}.$$

We take $\vec{x}_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$ as a basic eigenvector.

$$E_3 = \{c(1,1,2) | c \in \mathbb{R}\}.$$

A basis of E_3 is: $\{(1,1,2)\}$.

When $\lambda = 1$,

$$A - I = \begin{bmatrix} 2 & -4 & 2 \\ 1 & -3 & 2 \\ 1 & -5 & 4 \end{bmatrix} \xrightarrow{R_1 \to \frac{1}{2}R_1} \begin{bmatrix} 1 & -2 & 1 \\ 1 & -3 & 2 \\ 1 & -5 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Thus $(A - I)\vec{x} = 0$ has the solution

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

We take $\vec{x}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ as a basic eigenvector.

$$E_1 = \{c(1, 1, 1) | c \in \mathbb{R}\}.$$

A basis of E_1 is: $\{(1, 1, 1)\}$.

When $\lambda = 2$,

$$A - 2I = \begin{bmatrix} 1 & -4 & 2 \\ 1 & -4 & 2 \\ 1 & -5 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Thus $(A - 2I)\vec{x} = 0$ has the solution

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}.$$

We take $\vec{x}_1 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$ as a basic eigenvector.

$$E_2 = \{c(2, 1, 1) | c \in \mathbb{R}\}.$$

A basis of E_3 is: $\{(2, 1, 1)\}$.

Example 81. *Let* $A = \begin{bmatrix} 4 & 5 \\ -1 & 0 \end{bmatrix}$. *Find all eigenvalues and corresponding eigenvectors.*

Solution: $\det(A - \lambda I) = \lambda^2 - 4\lambda + 5 \Rightarrow \lambda = 2 \pm i$.

When $\lambda = 2 + i$,

$$\left[\begin{array}{c|c} A-(2+i)I & 0 \end{array}\right] = \left[\begin{array}{cc|c} 2-i & 5 & 0 \\ -1 & -2-i & 0 \end{array}\right] \xrightarrow{\text{row reduce}} \left[\begin{array}{cc|c} 5 & 5(2+i) & 0 \\ -1 & -2-i & 0 \end{array}\right]$$

$$\xrightarrow{\text{row reduce}} \left[\begin{array}{cc|c} 1 & 2+i & 0 \\ -1 & -2-i & 0 \end{array}\right] \xrightarrow{\text{row reduce}} \left[\begin{array}{cc|c} 1 & 2+i & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Thus, $x_1 + (2+i)x_2 = 0$, $x_1 = -(2+i)x_2$, x_2 free.

$$E_{2+i} = \{c\begin{bmatrix} -2-i \\ 1 \end{bmatrix} \mid c \in \mathbb{R}\}.$$

Therefore, a basis of

$$E_{2+i} = \{\begin{bmatrix} -2-i \\ 1 \end{bmatrix}\}.$$

When $\lambda = 2 + -$,

$$\left[\begin{array}{c|c} A-(2-i)I & 0 \end{array}\right] = \left[\begin{array}{cc|c} 2+i & 5 & 0 \\ -1 & -2+i & 0 \end{array}\right] \xrightarrow{\text{row reduce}} \left[\begin{array}{cc|c} 5 & 5(2-i) & 0 \\ -1 & -2+i & 0 \end{array}\right]$$

$$\xrightarrow{\text{row reduce}} \left[\begin{array}{cc|c} 1 & 2-i & 0 \\ -1 & -2+i & 0 \end{array}\right] \xrightarrow{\text{row reduce}} \left[\begin{array}{cc|c} 1 & 2-i & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Thus, $x_1 + (2-i)x_2 = 0$, $x_1 = -(2-i)x_2$, x_2 free.

$$E_{2-i} = \{c\begin{bmatrix} -2+i \\ 1 \end{bmatrix} \mid c \in \mathbb{R}\}.$$

Therefore, a basis of

$$E_{2+i} = \{\begin{bmatrix} -2+i \\ 1 \end{bmatrix}\}.$$

Properties:

(1) Let $\vec{x}$ be an eigenvector of the matrix A corresponding to the eigenvalue a. For any positive integer n, a^n is an eigenvalue of A^n with corresponding eigenvector $\vec{x}$.

(2) Let $\vec{x}$ be an eigenvector of the matrix A corresponding to the eigenvalue a. If A is invertible, then $\frac{1}{a}$ is an eigenvalue of A^{-1} with corresponding eigenvector $\vec{x}$.

(3) Let $\vec{x}$ be an eigenvector of the matrix A corresponding to the eigenvalue a. If A is invertible, then for any integer n, a^{-n} is an eigenvalue of A^{-n} with corresponding eigenvector $\vec{x}$.

(4) Let $\vec{x}$ be an eigenvector of both the matrices A and B associated with eigenvalues a and b. $\vec{x}$ is an eigenvector of $A+B$ associated with eigenvalue $a+b$.

(5) Let $\vec{x}$ be an eigenvector of both the matrices A and B associated with eigenvalues a and b. $\vec{x}$ is an eigenvector of AB associated with eigenvalue ab.

(6) A is invertible if and only if 0 s not an eigenvalue.

Solution: Proof.

(1) $A^n\vec{x} = a^n\vec{x}$.

(2) $A\vec{x} = a\vec{x} \Rightarrow \vec{x} = A^{-1}(a\vec{x}) = aA^{-1}\vec{x}$. Thus $A^{-1}\vec{x} = \frac{1}{a}\vec{x}$.

(3) By (1) and (2), let $B = A^n$. Then

$$B\vec{x} = a^n\vec{x}.$$

$$A^{-n}\vec{x} = B^{-1}\vec{x} = \frac{1}{a^n}\vec{x}.$$

(4) $(A+B)\vec{x} = A\vec{x} + B\vec{x} = a\vec{x} + b\vec{x} = (a+b)\vec{x}$.

(5) $(AB)\vec{x} = Ab\vec{x} = bA\vec{x} = ba\vec{x}$.

(6) Assume A is not invertible. Then Ax = 0 = 0x has non-trivial solution by invertible matrix theorem. So by definition, 0 is an eigenvalue.

Assume 0 is an eigenvalue. Thus there is some nontrivial solution to Ax = 0x = 0. By the invertible matrix theorem, A is not invertible.

Example 82. *Let* $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 3 & 2 & 4 \\ 0 & 0 & 5 & -1 \\ 0 & 0 & 0 & -3 \end{bmatrix}$. *Find eigenvalues of* A, A^3, A^{-3}.

Solution:

A: 1, 3, 5, -3.

A^3: $1^3, 3^3, 5^3, (-3)^3$, i.e., 1, 27, 125, -27.

A^{-3}: 1, 1/27, 1/125, -1/27.

Chapter 23 Diagonalization

Multiplicity:

- The multiplicity of an eigenvalue is equal to the number of times it is a root of the characteristic equation.

Similar matrices: Two matrices A and B are similar if there is an invertible matrix P such that,

$$A = PBP^{-1}.$$

Theorem 22. *If $n \times n$ matrices A and B are similar, then they have the same characteristics polynomial and hence the same eigenvalues (with the same multiplicities).*

Proof.

$$\begin{aligned}
\det(A - \lambda I) &= \det(PBP^{-1} - \lambda PP^{-1}) \\
&= \det[P(B - \lambda I)P^{-1}) \\
&= \det(P)\det(B - \lambda I)\det(P^{-1}) \\
&= \det(P)\det(B - \lambda I)\frac{1}{\det(P)} \\
&= \det(B - \lambda I).
\end{aligned}$$

A diagonal matrix is a matrix with only zeros on its off diagonal entries.

Definition 23. *If an $n \times n$ matrix A is similar to a diagonal matrix D then A is said to be diagonalizable.*

Theorem 23. *(Diagonalization Theorem) Let A be an $n \times n$ matrix.*

- *A is diagonalizable if and only if A has n eigenvectors $\vec{x}_1, ..., \vec{x}_n$ such that $P = [\vec{x}_1 \cdots \vec{x}_n]$ is invertible. When this is the case, $P^{-1}AP = diag(\lambda_1, ..., \lambda_n)$, where each λ_i is the eigenvalue corresponding to $\vec{x}_i$.*

- *If A has n distinct eigenvalues, then A is diagonalizable.*

- *A is diagonalizable if and only if the multiplicity of each eigenvalue λ equals the number of basic eigenvectors corresponding to λ. That is equivalent to: A is diagonalizable if and only if for each eigenvalue λ with the multiplicity m, $(A - \lambda I)x = 0$ has exactly m parameters.*

Example 83. $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 3 & 2 & 4 \\ 0 & 0 & 5 & -1 \\ 0 & 0 & 0 & 7 \end{bmatrix}$ *is diagonalizable.*

Solution: 4 distinct eigenvalues.

Example 84. $B = \begin{bmatrix} 3 & -1 \\ 1 & 5 \end{bmatrix}$ *is not diagonalizable.*

Solution: $\lambda = 4$, one eigenvector $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

Example 85. *Let* $A = \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix}$.

1) Find P and D such that $A = PDP^{-1}$.

2) Calculate A^4.

Solution: 1)

$$P = \begin{bmatrix} 1 & -3 \\ 1 & 4 \end{bmatrix}, D = \begin{bmatrix} 5 & 0 \\ 0 & -2 \end{bmatrix}; \ or \ P = \begin{bmatrix} -3 & 1 \\ 4 & 1 \end{bmatrix}, D = \begin{bmatrix} -2 & 0 \\ 0 & 5 \end{bmatrix}.$$

2) Let $P = \begin{bmatrix} 1 & -3 \\ 1 & 4 \end{bmatrix}$, then $P^{-1} = \frac{1}{7}\begin{bmatrix} 4 & 3 \\ -1 & 1 \end{bmatrix}$.

$$A^4 = \left\{PDP^{-1}\right\}^4 = PD^4P^{-1} = \begin{bmatrix} 1 & -3 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} 5 & 0 \\ 0 & -2 \end{bmatrix}^4 \frac{1}{7} \begin{bmatrix} 4 & 3 \\ -1 & 1 \end{bmatrix}$$

$$= \frac{1}{7} \begin{bmatrix} 1 & -3 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} 625 & 0 \\ 0 & 16 \end{bmatrix} \begin{bmatrix} 4 & 3 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 364 & 261 \\ 348 & 277 \end{bmatrix}.$$

Example 86. *Diagonalize the matrix* $A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$

Solution: Step 1: Find all eigenvalues: The characteristic polynomial is $-\lambda^3 + 6\lambda^2 - 9\lambda + 4$. So $\lambda = 1, 4$.

Step 2: Basic eigenvectors corresponding to $\lambda = 1$: $\left\{ \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \right\}$.

Basic eigenvectors corresponding to $\lambda = 4$: $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$.

Step 3: Construct P and D:

$$P = \begin{bmatrix} -1 & -1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}, D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix}.$$

Invertible $\not\leftrightarrow$ diagonalizable: Let $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. Then A is invertible but not diagonalizable, B is diagonalizable but not invertible.

Chapter 24 Linear Transformation

Transformations

Definition 24. *A transformation, (function or mapping), T from vector space V to vector space W is a rule that assigns a vector $T(\vec{x}) \in W$ for each vector $\vec{x} \in V$.*

$$T : V \to W$$

V is called the domain and W the codomain of T. The vector $T(\vec{x})$ is called the image of $\vec{x}$. The set of all images of T is called the range of T.

Example 87. *$T(x, y) = (x + y, x - y, 2x + 3y)$ is a transformation from $\mathbb{R}^2 \to \mathbb{R}^3$.*

Example 88. *$T(a, b, c) = ax^2 + bx + c$ is a transformation from $\mathbb{R}^3 \to P_2$.*

Definition 25. *(Matrix transformation induced by A): Let A be $m \times n$. Then $T(x) = Ax$ is a transformation from $\mathbb{R}^n \to \mathbb{R}^m$.*

Definition 26. *A transformation T is **linear** if*

1. *$T(\vec{u} + \vec{v}) = T(\vec{u}) + T(\vec{v})$ for all $\vec{u}, \vec{v}$ in the domain of T.*
2. *$T(c\vec{u}) = cT(\vec{u})$ for all $\vec{u}$ and scalars c.*

Properties: If T is linear, then

- $T(\vec{0}) = \vec{0}$.
- $T(c\vec{u} + d\vec{v}) = cT(\vec{u}) + dT(\vec{v})$ for all $\vec{u}, \vec{v}$ in the domain of T and all scalars c, d.

Example 89. *T_A is linear for any matrix A.*

Matrix of a linear transformation:

Theorem 24. *Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then there exists a unique matrix A such that $T(\vec{x}) = A\vec{x}$, for all $\vec{x} \in \mathbb{R}^n$. A is an $m \times n$ matrix:*

$$A = [T(\vec{e}_1)...T(\vec{e}_n)].$$

*A is called the **matrix associated with** T (or the standard matrix for the linear transformation T).*

Example 90. *Let* $T(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}) = \begin{bmatrix} x_1 - 2x_2 \\ -x_1 + 3x_2 \\ 3x_1 - 2x_2 \end{bmatrix}$ *or* $T(x_1, x_2) = (x_1 - 2x_2, -x_1 + 3x_2, 3x_1 - 2x_2)$.

1) Find the matrix associated with T.

2) Solve $T(\vec{x}) = (0, 1, 4)$, *or find* $\vec{x} = (x_1, x_2)$ *such that* $T(x_1, x_2) = (0, 1, 4)$.

Solution: 1) $T(\vec{x}) = \begin{bmatrix} x_1 - 2x_2 \\ -x_1 + 3x_2 \\ 3x_1 - 2x_2 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ -1 & 3 \\ 3 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$. Thus $A = \begin{bmatrix} 1 & -2 \\ -1 & 3 \\ 3 & -2 \end{bmatrix}$.

Method 2: Note that $T(1, 0) = (1, -1, 3)$, $T(0, 1) = (-2, 3, -2)$. Then we can get A.

2)

$$\text{augmented matrix} = \begin{bmatrix} 1 & -2 & 0 \\ -1 & 3 & 1 \\ 3 & -2 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Thus $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

Example 91. *Let* $T : \mathbb{R}^2 \to P_3$ *by* $T(\begin{bmatrix} 2 \\ 1 \end{bmatrix}) = 1 - x - x^2$, *and* $T(\begin{bmatrix} 1 \\ 2 \end{bmatrix}) = 2x + x^3$. *Find* $T(\begin{bmatrix} 5 \\ 4 \end{bmatrix})$ *and* $T(\begin{bmatrix} a \\ b \end{bmatrix})$.

Solution: Note that $\begin{bmatrix} 5 \\ 4 \end{bmatrix} = 2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. Thus

$$T(\begin{bmatrix} 5 \\ 4 \end{bmatrix}) = 2(1 - x - x^2) + 1(2x + x^3) = 1 - 2x^2 + x^3.$$

Kernel and Image

Definition 27. *Let* $T : V \to W$ *be a linear transformation. The kernel (or nullspace) of* T*:*

$$ker(T) = \{\vec{v} \in V | T(\vec{v}) = \vec{0}\};$$

The image (or range) of T*:*

$$im(T) = \{T(\vec{v}) | \vec{v} \in V\}.$$

Theorem 25. *If $T\vec{x} = A\vec{x}$, then $ker(T) = null(A)$, $im(T) = im(A) = col(A)$.*

Example 92. *Let $T(x, y, z) = (x + 2z, y + 3z, -2x - 4z, -y - 3z)$.*

1) Find a basis, dimension, geometric meaning for $ker(T)$.

2) Find a basis, dimension, geometric meaning for $im(T)$.

Solution:

$\ker T = \{s(-2, -3, 1) |\ s \in \mathbb{R}\}$.

$\mathrm{im} T = \{s(1, 0, -2, 0) + t(0, 1, 0, -1) |\ s, t \in \mathbb{R}\}$.

Dimension

Dimension Theorem (Rank Theorem): Let $T : V \to W$ be a linear transformation. Then

$$\dim(kerT) + \dim(imT) = \dim V.$$

Proof. Let $\dim V = n$, and let $\{v_1, \cdots, v_k\}$ be a basis for $ker(T)$. Then we can extend it to a basis of V: $\{v_1, \cdots, v_k, v_{k+1}, \cdots v_n\}$. We only need to prove that $\{T(v_{k+1}), \cdots T(v_n)\}$ is a basis for $range(T)$.

Example 93. *Let $T : \mathbb{R}^2 \to M_{22}$ and $S : M_{22} \to P_3$ be two linear transformations defined by:*

$$T(\begin{bmatrix} a \\ b \end{bmatrix}) = \begin{bmatrix} a & b \\ a+b & a-b \end{bmatrix}, \quad S(\begin{bmatrix} a & b \\ c & d \end{bmatrix}) = a - b + (b + c)x + (a - d)x^2 + (a + c)x^3.$$

Find $\dim(imT), \dim(imS), \dim(kerT), \dim(kerS)$.