Program PDSTRIP: Public Domain Strip Method

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1 Purpose of PDSTRIP

PDSTRIP (for public-domain strip) computes the seakeeping of ships and other floating bodies according to the 'strip method' originally proposed by *Korvin-Kroukowski and Jacobs (1957)*. Here, the slightly different method of *Söding (1969)* is applied for motions, and the procedure of *Hachmann (1991)* for the pressure.

Responses in regular waves are given as complex transfer functions, i.e. as ratio between the complex amplitude of a response and the complex amplitude of waves of various frequencies and directions causing that response. The absolute value of the complex transfer function is called response amplitude operator. For so-called linear responses this ratio is independent from wave amplitude. PDSTRIP is mainly confined to such linear responses; however, it takes into account a few nonlinear effects.

Responses in natural seaways are given as significant amplitudes. These are defined as the average of the one-third largest positive maxima of the response, neglecting the 2/3 smaller positive maxima. The significant amplitude is twice the standard deviation (from the average value zero) of the response.

The following responses are computed:

- Translations in directions x, y, z of the ship-fixed coordinate origin
- Rotations around the three coordinate axes
- The translation of specified points on the body in 3 coordinate directions
- The relative translation between these points and the water. Here the water is assumed to be disturbed by the incident waves, but not by the ship[1]
- The acceleration at these points; if required, after weighing with a function of motion frequency (encounter frequency)
- The pressure at a specified number of points on each offset section
- Sectional force (3 components) and moment (3 components) in cross sections (x = constant) of the body
- Longitudinal and transverse drift force and yaw drift moment on the body
- Water drift velocity in a specified height

PDSTRIP can handle unsymmetrical bodies including heeled ships. Forces on fins or sails can be taken into account. The water may be deep or shallow, but the water depth must be constant in space and time. PDSTRIP cannot deal with multi-hulls like catamarans etc.

2 Coordinate system used in describing the theory

An inertial coordinate system ξ, η, ζ is used here to formulate the waves and the motion equation. Its three axes are directed forward, to starboard and downward respectively. The system moves forward with the average ship velocity, but does not follow the periodical ship motions. In the time average, the coordinate origin is, unless specified otherwise, located at the intersection of midship plane, midship section and base-line (keel).

Coordinates used in the data input and ship-fixed coordinates are described later.

[1] Diffraction/radiation waves are neglected.

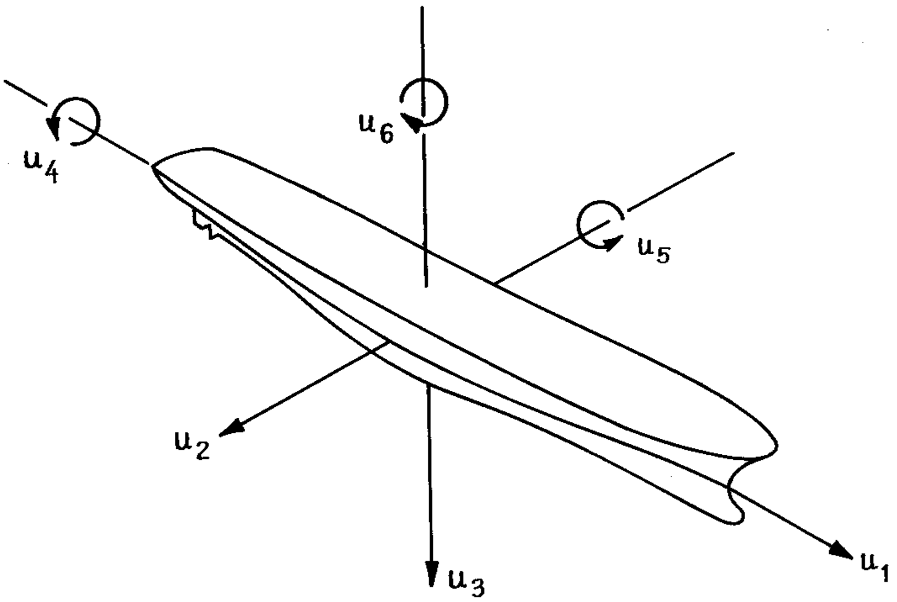


Fig. 1: Direction of coordinate axes; origin may be at keel and not as shown in waterline plane

3 Computation of the added mass, damping and excitation matrices of ship sections

3.1 Fundamental equations and boundary conditions for added mass and damping

To determine the added mass and damping of ship sections, the 2-dimensional potential flow of an incompressible fluid around symmetrical (to the plane $y = 0$) or asymmetrical bodies is to be computed. In this chapter, the coordinate origin is assumed in the undisturbed water surface. The flow is presupposed to oscillate sinusoidally over time with circular frequency ω, so that the flow potential may be expressed as

$$\phi(y, z, t) = \text{Re } [\hat{\phi}(y, z)e^{i\omega t}] \tag{1}$$

For a ship with forward speed v, the sections oscillate with encounter frequency

$$\omega_e = \omega - kv \cos \mu \tag{2}$$

μ is the wave direction (0 for stern waves) amd k the wave number. Nonetheless in this chapter we use the symbol ω to designate the circular frequency of motion.

Eq.(1) excludes any stationary flow. The flow is assumed to be excited alone by sinusoidal translations of the body in y and z direction, and by a sinusoidal rotation about the coordinate origin.

The potential has to satisfy the following conditions:

1. Laplace equation within the fluid region, to enforce the condition of incompressibility:

$$\phi_{yy} + \phi_{zz} = 0 \quad \text{for} \quad z > 0 \quad \text{outside of the body} \tag{3}$$

   (Indices y and z designate partial derivatives with respect to y, z.)

2. Bottom condition:

$$\text{For shallow water:} \quad \phi_z = 0 \qquad \text{at } z = H \tag{4}$$

$$\text{For deep water:} \quad \lim_{z\to\infty} \phi_z = 0 \tag{5}$$

   H denotes the water depth.

3. Free-surface condition:

   At the undisturbed free surface, a condition combining the conditions of constant pressure and no flow through the real (wavy) surface, linearized with respect to wave steepness, yields:

$$\phi_{tt} - g\phi_z = 0 \quad \text{at} \quad z = 0 \tag{6}$$

4. Hull boundary condition:

   There is no flow through the (submerged part of the) hull contour:

$$\nabla\phi \cdot \vec{n} = \vec{v} \cdot \vec{n} \quad \text{along the contour} \tag{7}$$

   $\vec{v}$ is the motion velocity of the body at the respective contour point, and $\vec{n}$ is the (inward) unit normal on the contour.

5. Radiation condition:

   Waves created by the hull propagate away from the hull. To formulate this as a boundary condition, the formula for linear shallow-water (Airy) waves is applied:

$$\phi = \text{Re}\left[\frac{-ic\hat{\zeta}}{\sinh(kH)}\cosh(k[z-H])e^{i(\omega t \mp ky)}\right] \tag{8}$$

   In the argument of the exponential function the $-$ sign holds for waves running in $+y$ direction, and the $+$ sign for waves running in opposite direction. Thus the two signs hold for the sides $y > 0$ and $y < 0$ of the body, respectively.

Eq.(1) yields then for the complex amplitude of the potential $\hat{\phi}$:

$$\begin{array}{rcll}
\text{Laplace} & \hat{\phi}_{yy} + \hat{\phi}_{zz} = 0 & \text{for } z > 0 \text{ outside of the body} & (9)\\
\text{Bottom} & \hat{\phi}_z = 0 & \text{at } z = H \quad \text{for shallow water} & (10)\\
 & \lim\limits_{z\to\infty} \hat{\phi}_z = 0 & \text{for deep water} & (11)\\
\text{Free surface} & \frac{\omega^2}{g}\hat{\phi} + \hat{\phi}_z = 0 & \text{at } z = 0 & (12)\\
\text{Hull} & \nabla\hat{\phi}\cdot\vec{n} = \hat{\vec{v}}\cdot\vec{n} & \text{along the contour} & (13)\\
\text{Radiation} & \hat{\phi}_y = \mp ik\hat{\phi} & & (14)
\end{array}$$

$\hat{\vec{v}}$ is the complex amplitude of the body motion velocity corresponding to $\vec{v} = \text{Re}\ (\hat{\vec{v}}e^{i\omega t})$. We use three different motion vectors: $\hat{\vec{v}} = i\omega\,\{1;0\}$ for sway, $\hat{\vec{v}} = i\omega\,\{0;1\}$ for heave, $\hat{\vec{v}} = i\omega\,\{-z;y\}$ for roll. The radiation condition (14) was derived applying partial derivative with respect to y to Eq.(8).

3.2 Solution method

The numerical solution follows a "patch method", *Söding (1993), Bertram (2000)*, which computes the forces more accurately than a traditional panel method. The patch method approximates the potential $\hat{\phi}$ as a superposition of point sources

$$\hat{\phi}(y,z) = \sum_{i=1}^{n} q_i\,\frac{1}{2}\ln[(y-y_i)^2 + (z-z_i)^2] \tag{15}$$

where q_i are the source strengths of the n sources at locations (y_i, z_i). This satisfies the Laplace equation (9) everywhere except at the location of the sources (y_i, z_i) which are therefore located within the section contour or above the line $z = 0$.

The section contour is defined by given offset points. For each contour segment between adjacent offset points, one source is generated near to the midpoint between the two offset points, however shifted from the midpoint to the interior of the section by 1/20 of the segment length. Along the average water surface $z = 0$ grid points are generated automatically. Near to the body, their distance is equal to 1.5 of the offset point distance on the contour at the waterline. Farther to the sides, the distance increases by a factor of 1.5 from one segment to the next, until a maximum distance of 1/12 of a wavelength (of the waves generated by the body oscillations) is attained. Source points are again located above the mid-points of each free-surface segment, here however at a distance of one segment length. The number of free-surface grid points used is 55 for a symmetrical body of which only one half needs to be discretized, and $2 \cdot 55$ for asymmetrical bodies where the water surface to both sides of the section must be discretized.

Whereas in the panel method the boundary conditions are, usually, satisfied at a 'collocation point' in the middle of each segment, in the patch method the integral of the boundary condition over each segment has to be used. For the body boundary condition (13) this is simple: $\int_A^B \nabla\hat{\phi} \cdot \vec{n}\; ds$, i.e. the flux induced by a source at S through a segment between points A and B, is equal to the source strength times the angle ASB divided by 2π. The total flux is the sum of the fluxes coming from all sources. This method is used also for the second term in the free-surface condition (12) which – after integration over a segment – is also the flux through that segment. For the integral over the first term of (12) the approximation

$$\int_A^B \hat{\phi}\; ds \approx \frac{1}{2}\{\phi(A + 0.316[B - A]) + \phi(A + 0.684[B - A])\} \cdot |B - A| \tag{16}$$

is used. The constants 0.316 and 0.684 were determined such that the integral is approximated correctly for a source which is located near to the midpoint of the segment AB, whereas for sources farther off from the segment the errors are small anyway.

The bottom condition (10) is satisfied exactly by using mirror images of all sources below the bottom at the point $(y_i,\ 2H - z_i)$ for a source at $(y_i,\ z_i)$. For deep water the condition (11) is satisfied automatically by the approach (15); however, for vertical motion the accuracy is improved by adding another source and specifying the additional condition that the sum of all source strengths is zero. The location of the additional source is at $y = 0$ (above the body) at a distance above the waterline of 1/2 the distance to the farthest free-surface grid point.

Also the radiation condition (14) is integrated over a panel between points A (nearer to the body) and B (farther out). Using again the approximation (16) results in

$$\phi_B - \phi_A = \mp\frac{ik}{2}(\phi(A + 0.316[B - A]) + \phi(A + 0.684[B - A])(y_B - y_A) \tag{17}$$

from which follows

$$i(\phi_B - \phi_A) - \frac{k}{2}|y_B - y_A|(\phi(A + 0.316[B - A]) + \phi(A + 0.684[B - A]) = 0 \tag{18}$$

This condition is applied in the outer range of the free surface, for asymmetrical bodies on both sides.

The details of satisfying the radiation condition are of great importance for the accuracy of the method and for the necessary length of the discretized part of the free surface. This length, on the other hand, influences significantly the computer time. Therefore a number of improvements have been made in the treatment of the radiation condition.

A complex wave number

$$k = k_r + ik_i \tag{19}$$

with negative k_i, when used e.g. in (14), generates waves with decreasing amplitude in the direction of wave propagation. A damping region at some distance from the body will decrease the effects of truncating the infinitely long free surface; thus a complex k with negative imaginary part will be advantageous in the outer range of the free surface. In the program this is done implicitly: The imaginary part of k introduces into (18) a term

$$ik_i|y_B - y_A|\phi_{\text{average}} \tag{20}$$

Applying the y derivative of the radiation condition (14), i.e.

$$\phi_{yy} = -k^2\phi \tag{21}$$

to the term (20) transforms it to

$$-i\frac{k_i}{k^2}|y_B - y_A|\phi_{yy,\,\text{average}} \tag{22}$$

This term may be interpreted as a Taylor expansion of the first term in (18) for potentials ϕ shifted by δy from the original points A and B:

$$\begin{aligned} i[\phi(y_B+\delta y) - \phi(y_A+\delta y)] &\approx i[\phi(y_B) + \delta y\phi_y(y_B) - \phi(y_A) - \delta y\phi_y(y_A)] \quad (23)\\ &\approx i[\phi(y_B) - \phi(y_A)] + i\delta y(y_B - y_A)\phi_{yy,\,\text{average}} \quad (24)\end{aligned}$$

The second (Taylor expansion) term corresponds to (22) if

$$\mp\frac{k_i}{k^2} \approx \mp\frac{k_i}{k_r^2} = \delta y \tag{25}$$

Thus, instead of directly applying an imaginary k, the potentials ϕ_A and ϕ_B on the left-hand side of (18) are determined not at y_A and y_B, but at $y_A \mp k_i/k_r^2$ and $y_B \mp k_i/k_r^2$, respectively. Based on numerical tests a quadratic increase of this outward shift from zero at the beginning of the damping region to one panel length at the outer end(s) of the free-surface discretization was selected.

However, in test computations there remained a small influence of the coordinate y_D where the radiation condition and the damping region started: The results oscillated, only slowly fading out for large y_D, with a wavelength of half the length of the waves radiated from the body. Therefore, for each body and frequency two cases are computed with y_D values differing by 1/4 wavelength of the radiated waves. The results found in both cases are averaged. This removes the oscillations of results over y_D and results in high accuracy for a very broad range of frequencies with the selected moderate number of free-surface panels: 55 on each side of the body, of which 25 (in one case) and 28 (in the other case) apply the free-surface condition and the rest the radiation condition.

The linear equation system resulting from the boundary conditions is solved for the complex amplitudes of all source strengths. The flow potential follows then from (15). According to Bernoulli's equation, the complex amplitude of the pressure is

$$p = -\rho\phi_t \tag{26}$$

This pressure amplitude is integrated over the section contour to give the complex amplitudes of horizontal force, vertical force and x (roll) moment, each for horizontal, vertical and rolling motion of the section with unit amplitude. Considering one of the three force terms $\hat{f}$ due to one of the three motion amplitudes $\hat{u}$, we have the proportionality

$$\hat{f} = \text{constant} \cdot \hat{u} \tag{27}$$

If we write this proportionality in the form

$$\hat{f} = -\hat{m}(-\omega^2\hat{u}) \tag{28}$$

we can interpret $\hat{m}$ as one element of the complex added mass matrix because $-\omega^2\hat{u}$ is the complex amplitude of the acceleration if the sinusoidal motion occurs with circular frequency ω. Instead, we may consider the force amplitude $\hat{f}$ as the sum of a damping and a mass term:

$$\hat{f} = -m(-\omega^2\hat{u}) - d(i\omega\hat{u}) \tag{29}$$

Here $i\omega\hat{u}$ is the complex amplitude of the motion velocity. Both equations (28) and (29) are compatible with each other if

$$\hat{m} = m - \frac{id}{\omega} \tag{30}$$

This illustrates the relation between complex added mass, real added mass and damping. A corresponding relation holds for the complex added mass matrix, the real added mass matrix and the real damping matrix, which result from combining the 3 forces due to 3 motions within a 3×3 matrix. The matrix depends on the motion frequency.

3.3 Computation of wave excitation force of ship sections

Wave excitation forces and moments are the sum of a so-called Froude-Kriloff and a diffraction contribution. The former is produced by the pressure distribution in the wave minus hydrostatic pressure, acting on the (non-moving) section, neglecting the influence of the section on the pressure; the latter is generated by the change of wave-induced pressure due to the presence of the (non-moving) section.

The flow potential of a wave of wave number k, complex amplitude $\hat{\zeta}$ and direction μ is

$$\hat{\phi} = \mathrm{Re}\left[\frac{-ic\hat{\zeta}}{\sinh(kH)}\cosh k(z-H)e^{i(\omega_e t - k\{x\cos\mu - y\sin\mu\})}\right]. \tag{31}$$

Because of the assumption of linearity with respect to wave height, we consider here only waves of amplitude ($= \frac{1}{2}\cdot$ height) $=1$ and omit this factor 1.

The complex amplitude of the wave pressure (without disturbance by the body) follows from (31) and from (26) if we consider the wave in a coordinate system which is fixed to the earth, i.e. if $\omega_e = \omega$:

$$\hat{p} = \rho\frac{-\omega^2 e^{iky\sin\mu}}{2k\sinh kH}\left(e^{k(z-H)} + e^{-k(z-H)}\right) \tag{32}$$

The cosh function in (31) was substituted here by two exponential functions. This is advantageous in the following, especially if both the case of deep water and shallow water must be considered.

The Froude-Kriloff forces are computed as pressure integrals over the section contour. These are summed from integrals over each contour segment between adjacent offset points $\vec{x}_1 = (y_1, z_1)$ and $\vec{x}_2 = (y_2, z_2)$. Using (32) one finds:

$$\int_{\vec{x}_1}^{\vec{x}_2}\hat{p}\,ds = -\rho g\frac{e^{iky_1\sin\mu}}{2\cosh kH}\left(e^{kz_1}e^{-kH}\frac{e^{k(\Delta z + i\Delta y\sin\mu)} - 1}{k(\Delta z + i\Delta y\sin\mu)} + e^{-kz_1}e^{kH}\frac{e^{k(-\Delta z + i\Delta y\sin\mu)} - 1}{k(-\Delta z + i\Delta y\sin\mu)}\right)|\vec{x}_2 - \vec{x}_1| \tag{33}$$

where $\Delta y = y_2 - y_1$ and $\Delta z = z_2 - z_1$.

The computation of the wave diffraction force and moment differ from that of the added mass force and moment only with respect to the inhomogeneous terms in the body boundary condition. Here these terms are the fluxes which would occur between panel endpoints $\vec{x}_1$ and $\vec{x}_2$ if there were no change of the wave-induced flow by the presence of the body. In the diffraction calculation, this wave flux has to be cancelled by the source-induced flow.

From the flow speed vector

$$\begin{Bmatrix}\hat{\phi}_y\\ \hat{\phi}_z\end{Bmatrix} = \frac{-i\omega e^{iky\sin\mu}}{2k\sinh kH}\left(\begin{Bmatrix}ik\sin\mu\\ k\end{Bmatrix}e^{kz}e^{-kH} + \begin{Bmatrix}ik\sin\mu\\ -k\end{Bmatrix}e^{-kz}e^{kH}\right) \tag{34}$$

the wave flux between $\hat{x}_1$ and $\hat{x}_2$ follows:

$$\begin{aligned}\int_{\vec{x}_1}^{\vec{x}_2}(\hat{\phi}_y, \hat{\phi}_z)\,d\vec{x} &= \frac{-i\omega e^{iky_1\sin\mu}}{2\sinh kH}\left(e^{kz_1}e^{-kH}\frac{e^{k(\Delta z + i\Delta y\sin\mu)} - 1}{k(\Delta z + i\Delta y\sin\mu)}(\Delta y - i\Delta z\sin\mu)\right.\\ &\left.+e^{-kz_1}e^{kH}\frac{e^{k(-\Delta z + i\Delta y\sin\mu)} - 1}{k(-\Delta z + i\Delta y\sin\mu)}(\Delta y + i\Delta z\sin\mu)\right)|\vec{x}_2 - \vec{x}_1| \end{aligned} \tag{35}$$

Using the dispersion relation

$$\omega^2 = gk \tanh kH \tag{36}$$

the first fraction in (35) can be transformed to the expression used in the program:

$$\frac{-igke^{iky_1 \sin\mu}}{2\omega \cosh kH} \tag{37}$$

3.4 Details of the code and its application

The direct evaluation of the expressions $[\exp(\alpha) - 1]/\alpha$ in Eqs.(33) and (35) is inaccurate for small α. Therefore for $|\alpha| < 0.01$ they are substituted by the expression $1 - \alpha/2$ which follows from a Taylor expansion of the exponential function.

For reasons of shortness of code and computation time, the determination of the complex added-mass matrix and of the wave excitation is combined in one subroutine `sectionhydrodynamics`. To avoid unnecessary computer time for symmetrical sections, this subroutine combines two separate routines: one for asymmetrical sections, the other for symmetrical ones. For symmetrical sections, the method of mirror images of sources on both sides of the symmetry plane $y = 0$ is used. That is possible also for oblique waves where the actual potential is the sum of a symmetrical and an anti-symmetrical part. These routines handle both deep and shallow water; in the latter case, mirror sources below the water bottom are used to satisfy the bottom boundary condition.

The program code (subroutine `sectiondata.f`) is suitable also for fully submerged sections and for sections consisting of several unconnected parts. However, only one of these parts may intersect the water surface; thus intersections of both hulls of a catamaran cannot be handled by this feature.

Fig. 2 demonstrates how the offset points given to describe the section contour should be arranged. The offset points determine the discretization for the hydrodynamic calculation. Thus the computed results would be inaccurate if, e.g. a broad flat bottom of a section were described by only two offset points at its ends. It is recommended to use approximately equal distance between offset points on a section. The recommended number of offset points is about 10 per half section if the section shape is relatively smooth, and some more otherwize. To gain experience, one may test how much results change if more points per section are used.

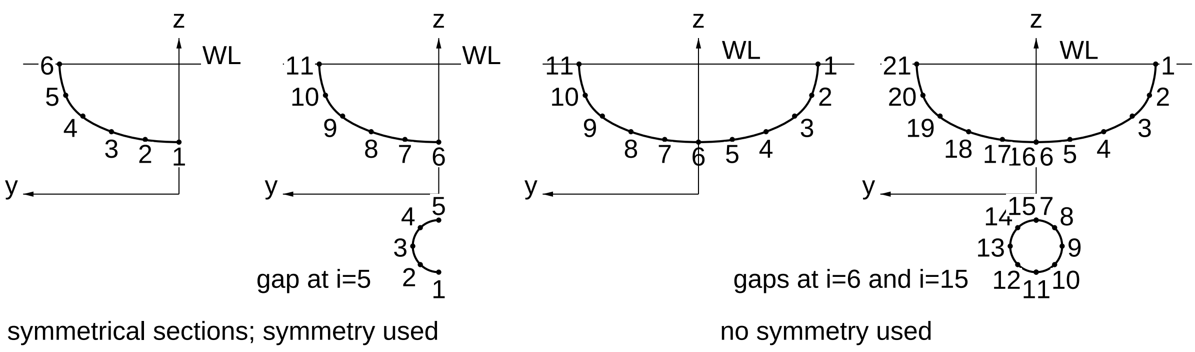


Fig. 2: Arrangement of offset points on sections consisting of one or two separate parts. Coordinate axes are those of the input coordinate system.

4 Determination of complex transfer functions

4.1 Coordinate systems

Besides the inertial coordinate system ξ, η, ζ defined in chapter 2, we introduce here ship-fixed coordinates x, y, z pointing forward, to port side and downward respectively. Averaged over time both coordinate origins coincide, but at any time instant the origin of the ship-fixed system differs from that of the inertial system by the ship translational motions surge (in ξ direction), sway (in η direction) and heave (in ζ direction).

A steady trim of the ship is considered in the ship hull geometry description in the xyz-System and the mass data (center of gravity, mass moments of inertia).

We assume in the following motions to be small. Thus terms being proportional to second or even higher powers of motion amplitudes are neglected without further mentioning. This yields the transformation between the two coordinate systems:

$$\left\{ \begin{array}{c} \xi \\ \eta \\ \zeta \end{array} \right\} = \left[\begin{array}{ccc} 1 & -u_6 & u_5 \\ u_6 & 1 & -u_4 \\ -u_5 & u_4 & 1 \end{array} \right] \cdot \left\{ \begin{array}{c} x \\ y \\ z \end{array} \right\} + \left\{ \begin{array}{c} u_1 \\ u_2 \\ u_3 \end{array} \right\} \tag{38}$$

u_1	surge motion	(positive forward)
u_2	sway motion	(positive to starboard)
u_3	heave motion	(positive down)
u_4	roll angle	(positive down on starboard)
u_5	pitch angle	(positive bow up)
u_6	yaw angle	(positive bow to starboard

In the following we will consider complex amplitudes of ship responses in waves. Complex amplitudes are denoted by the hat symbol. The absolute value of the complex amplitude is the (real) amplitude, also called response amplitude operator RAO. The ratio between imaginary and real part of the complex amplitude describes the phase shift between the response and the exciting wave.

4.2 Equation of motion

We want to compute the complex amplitude of the generalised motion vector $\hat{\vec{u}} = \{\hat{u}_1, \hat{u}_2, \hat{u}_3, \hat{u}_4, \hat{u}_5, \hat{u}_6\}^T$. Newton's equations for accelerated bodies yield for harmonic motions:

$$-\omega_e M \hat{\vec{u}} = \hat{\vec{F}} \tag{39}$$

Here $\hat{\vec{F}}$ is the amplitude of the generalised force vector (including 3 forces and 3 moments). M is the mass matrix:

$$M = \left[\begin{array}{cccccc} m & 0 & 0 & 0 & mz_G & -my_G \\ 0 & m & 0 & -mz_G & 0 & mx_G \\ 0 & 0 & m & my_G & -mx_G & 0 \\ 0 & -mz_G & my_G & \theta_{xx} & -\theta_{xy} & -\theta_{xz} \\ mz_G & 0 & -mx_G & -\theta_{xy} & \theta_{yy} & -\theta_{yz} \\ -my_G & mx_G & 0 & -\theta_{xz} & -\theta_{yz} & \theta_{zz} \end{array} \right] \tag{40}$$

with:

m	ship's mass
x_G, y_G, z_G	ship's center of gravity
θ_{xx}, ...	mass moments of inertia

The mass moments of inertia are defined with respect to the coordinate origin, e.g.:

$$\theta_{xx} = \int (y^2 + z^2)\, dm \tag{41}$$

$$\theta_{xy} = \int xy\, dm \tag{42}$$

For symmetric mass distribution, $\theta_{xy} = \theta_{yz} = 0$.

$\hat{\vec{F}}$ is the amplitude of the force exerted by the water on the ship. It consists of a hydrostatic part and hydrodynamic parts due to the ship motions, the incident wave and its diffraction. This yields the fundamental equation of motion

$$(-\omega_e^2 M - B + S)\hat{\vec{u}} = \hat{\vec{F}}_e \tag{43}$$

M is a real mass matrix, B is a complex matrix due to the ship motions, S a real matrix due to hydrostatics, and $\hat{\vec{F}}_e$ contains the amplitudes of the forces due to the incident wave and its diffraction.

4.3 Restoring forces – Hydrostatics matrix S

The restoring matrix S is given without derivation for the general case where the ship is not symmetric in y:

$$S = \begin{bmatrix} 0 & 0 & \rho g A_{tr} & \rho g y_{tr} A_{tr} & \begin{matrix} -\rho g x_{tr} A_{tr} + gm \\ -\rho g \int A \, dx \end{matrix} & 0 \\ 0 & 0 & 0 & \rho g \int A \, dx - gm & 0 & 0 \\ 0 & 0 & \rho g \int B \, dx & \rho g \int y_W B \, dx & -\rho g \int xB \, dx & 0 \\ 0 & 0 & \rho g \int y_W B \, dx & \begin{matrix} \rho g \int y_W^2 B \, dx + gmz_G \\ -\rho g \int A z_S \, dx \end{matrix} & -\rho g \int x y_W B \, dx & 0 \\ 0 & 0 & \begin{matrix} \rho g z_{tr} A_{tr} \\ -\rho g \int xB \, dx \end{matrix} & \begin{matrix} \rho g y_{tr} z_{tr} A_{tr} \\ -\rho g \int x y_W B \, dx \end{matrix} & \begin{matrix} -\rho g \int A z_S \, dx + gmz_G \\ +\rho g \int x^2 B \, dx \end{matrix} & 0 \\ 0 & 0 & -\rho g y_{tr} A_{tr} & \begin{matrix} \rho g \int xA \, dx - gmx_G \\ -\rho g y_{tr}^2 A_{tr}^2 \end{matrix} & -gmy_G + \rho g \int y_S A \, dx & 0 \end{bmatrix} \tag{44}$$

The integration is extended over the ship length. A is the area of the section (below the average water surface), B is waterline breadth, $y_W(x)$ the mean transverse coordinate of the waterline at the section, and (y_S, z_S) are the coordinates of the section area center; all for the transverse section at longitudinal coordinate x. A_{tr} is the section area of the dry transom stern below the waterline. It is zero if the ship has no transom stern or if the transom stern is wetted. y_{tr}, z_{tr} are the coordinates of the center of gravity of the dry transom area.

4.4 Radiation forces

Let $\hat{\vec{f}}_x$ denote the complex amplitude of the force and moment exerted by the water on the cross section at x in 2-dimensional flow. The vector contains 3 components: force in y-direction, force in z direction and moment about x-axis. $\hat{\vec{f}}_x$ is proportional to the 3-component motion amplitude vector $\hat{\vec{u}}_x = \{\hat{u}_2, \hat{u}_3, \hat{u}_4\}$:

$$\hat{\vec{f}}_x = A\omega_e^2 \hat{\vec{u}}_x = \begin{bmatrix} \bar{m}_{22} & \bar{m}_{23} & \bar{m}_{24} \\ \bar{m}_{32} & \bar{m}_{33} & \bar{m}_{34} \\ \bar{m}_{42} & \bar{m}_{43} & \bar{m}_{44} \end{bmatrix} \omega_e^2 \hat{\vec{u}}_x \tag{45}$$

The elements of the complex added mass matrix can be interpreted as real-value added mass and damping, e.g. for the first element:

$$\bar{m}_{22} = m_{22} + \frac{n_{22}}{i\omega_e} \tag{46}$$

In all elements of A, the first index refers to the force component, the second to the motion causing this force. Motion and moments refer to the origin of the cross section at $y = z = 0$, not to its center.

The complex added mass and the exciting forces are computed in the 2-d section module, see the corresponding chapter. We can re-write the above equation in the form:

$$\hat{\vec{f}}_x = (-i\omega_e) A (i\omega_e \hat{\vec{u}}_x) \tag{47}$$

This form can be interpreted as the time derivative of the momentum (mass times velocity). For ships (non-cylindrical) with forward speed v, we take a substantial derivative instead of the partial time derivative, obtaining:

$$\hat{\vec{f}}_x = \left(-i\omega_e + v\frac{d}{dx}\right) A(i\omega_e \hat{\vec{u}}_x) \tag{48}$$

The 3-component cross-section velocity $\hat{\vec{u}}_x$ follows from the global 6-component ship motion:

$$i\omega_e \hat{\vec{u}}_x = W\hat{\vec{u}} \tag{49}$$

with

$$W = \begin{bmatrix} 0 & i\omega_e & 0 & 0 & 0 & i\omega_e x - v \\ 0 & 0 & i\omega_e & 0 & -i\omega_e x + v & 0 \\ 0 & 0 & 0 & i\omega_e & 0 & 0 \end{bmatrix} \tag{50}$$

The transformation between 3-component section force $\hat{\vec{f}}_x$ and 6-component global force/length $\hat{\vec{f}}$ follows from:

$$V = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -x & 0 \\ x & 0 & 0 \end{bmatrix} \tag{51}$$

Integration over the ship yields the 6-d radiation force:

$$B\hat{\vec{u}} = \int\limits_L V \left(-i\omega_e + v\frac{d}{dx}\right) AW\ dx \cdot \hat{\vec{u}} \tag{52}$$

Thus we have:

$$B = \int\limits_L V \left(-i\omega_e + v\frac{d}{dx}\right) AW\ dx \tag{53}$$

The integration extends over the whole ship length. However, the term vd/dx must be omitted at those section where the flow separates off the hull. This is the case at the trailing edge of a deadwood and at an immersed dry transom stern.

The hydrodynamic added mass in surge cannot be computed following the 2-d strip computation. Instead, we use an empirical formula:

$$m_{11} = \frac{m}{\pi\sqrt{\rho L_{pp}^3/m} - 14} \tag{54}$$

This yields the longitudinal force exerted from the water on the ship:

$$\hat{f}_1 = \omega_e^2 m_{11} \hat{u}_1 \tag{55}$$

Let this force act at (y_0, z_0) (approximately at the center of the hull). Then this induces a generalized force vector with respect to the origin of the hull:

$$\hat{\vec{F}}_1 = \vec{U}_1 \hat{f}_1 \tag{56}$$

$$\vec{U}_1 = \begin{Bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ z_0 \\ -y_0 \end{Bmatrix} \tag{57}$$

Correspondingly we couple the surge motion $\hat{u}_1$ of the hull with the global hull motions:

$$\hat{\vec{F}}_1 = \vec{U}_1 \hat{f}_1 \tag{58}$$

$$\vec{u}_1 = \vec{U}_1^T \hat{u} \tag{59}$$

In sum, we then get:

$$\hat{\vec{F}}_1 = \omega_e^2 m_{11} \vec{U}_1 \vec{U}_1^T \hat{\vec{u}} \tag{60}$$

Thus we have a 6×6 addition to the matrix B:

$$\Delta B = \omega_e^2 m_{11} \vec{U}_1 \vec{U}_1^T \tag{61}$$

4.5 Exciting force

Let $\hat{\vec{f}}_{e,x}$ be the complex amplitude of the 3-dimensional force (including moment around x axis; per length) on a cross section in 2-dimensional flow. The force is proportional to the complex amplitude $\hat{\zeta}_x$ of the wave at $x,\ y = 0$. The force consists of a Froude-Krilov part (index 0) and a diffraction part (index 7):

$$\hat{\vec{f}}_{e,x} = (\hat{\vec{f}}_{e,x0} + \hat{\vec{f}}_{e,x7})\hat{\zeta}_x \tag{62}$$

The compex 3-component vectors $\hat{\vec{f}}_{e,x0}$ and $\hat{\vec{f}}_{e,x7}$ are computed in the 2-d section hydrodynamics module.

For a ship with forward speed, for reasons explained at eq. (48) we replace $\hat{\vec{f}}_{e,x7}$ by

$$\hat{\vec{f}}_{e,x7} - \frac{v}{i\omega}\frac{d\hat{\vec{f}}_{e,x7}}{dx} \tag{63}$$

except at places with flow separation (deadwood, transom stern).

The complex amplitude of the wave elevation ζ is for a regular Airy wave with direction μ (0 for following waves, $\pi/2$ for waves coming from starboard):

$$\hat{\zeta} = \hat{\zeta}_0 e^{-ik(x\cos\mu - y\sin\mu)} \tag{64}$$

Here $\hat{\zeta}_0$ is the complex amplitude at origin. $k = 2\pi/\lambda$ is the wave number, λ the wave length. Thus we have:

$$\hat{\zeta}_x = \hat{\zeta}_0 e^{-ikx\cos\mu} \tag{65}$$

$\hat{\vec{f}}_{e,x}$ is converted into a 6-component force vector in the global coordinate system by left multiplication with transformation matrix V according to Eq(51). We then get the first contribution to the exciting force vector:

$$\hat{\vec{f}}_{e,1} = \int_L V\left(\hat{f}_{e,x0} + \hat{f}_{e,x7} + \frac{iv}{\omega}\frac{d\hat{f}_{e,x7}}{dx}\right) e^{-ikx\cos\mu} dx \cdot \hat{\zeta}_0 \tag{66}$$

For surge motion and more accurate computation of yaw and pitch motions, we need to consider in addition the longitudinal forces exerted from the wave on the ship. *Blume (1976)* showed that the diffraction part is negligibly small in this case. We then consider only the Froude-Krilov contribution. The longitudinal force on the ship is then:

$$\int_L \hat{p}\frac{dA}{dx}dx + \hat{p}_{tr}A_{tr} \tag{67}$$

Here $\hat{p}$ is the pressure amplitude at a point fixed in space.. Instead of using the average of $\hat{p}$ over a cross section, we use here the value at the center of the cross section, having the coordinates (x, y_x, z_x).

The index tr denotes values at the transom stern. These are zero if there is no transom stern or if the transom stern is wetted (i.e. for low or zero speed).

For the section center, we get the pressure $\hat{p}$:

$$\hat{p} = -\rho g e^{-k(z_x+T)} e^{iky_x \sin\mu} e^{-kx\cos\mu} \hat{\zeta}_0 = \alpha \hat{\zeta}_x \tag{68}$$

with $\alpha = -\rho g e^{-k(z_x+T)} e^{iky_x \sin\mu}$

The longitudinal forces/length $\hat{p}\ dA/dx$ induce also contributions to the pitch moment. Thus we have as second contribution to the 6-component exciting force vector:

$$\hat{\vec{f}}_{e,2} = \left(\int_L \begin{Bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ z_x \\ -y_x \end{Bmatrix} \alpha \frac{d\ A}{dx} e^{-ikx\cos\mu}\ dx + \begin{Bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ z_x \\ -y_x \end{Bmatrix} \alpha A_{tr} e^{-ikx_{tr}\cos\mu} \right) \hat{\zeta}_0 \tag{69}$$

5 Fin forces

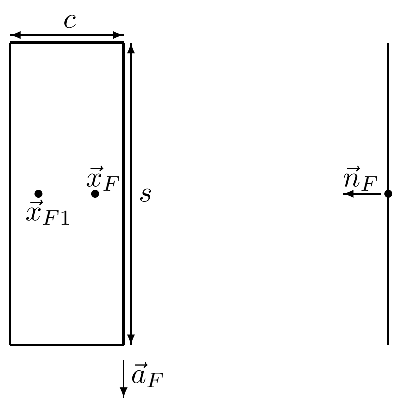
This section describes how rudders and other fixed and movable submerged fins are handled. Their shape is characterized by a mean chord length c (in mean flow direction) and a span s measured in the y, z-plane. Fins having an aspect ratio $\Lambda = s/c \ll 1$, and fins which continue the hull to the rear (like skegs or rudders) may be treated as parts of the hull. Generally, movable fins included already in the hull may be treated additionally as fins, but these fins should produce only the forces due to the rotation of the fin (especially a rudder) because the forces on the fin in its average position are contained in the hull forces already. This is achieved by specifying the parameters C_M and C_r (see later) as zero.

Fins having a span s which is in the same size range or larger than the distance between fin center and the "roll axis" (approximately parallel to the x axis through the center of gravity of ship's mass) will be treated more accurately if they are subdivided into two or more "part fins" so that the sum of their spans s is equal to that of the total fin. In that case, the lift gradient $dC_L/d\alpha$ (see later) of the part fins has to be determined using the aspect ratio Λ of the total fin, not that of the part fins. Generally, it may be recommended to test whether subdivision of a large fin into part fins gives substantially different results.

The program takes account only of the oscillating forces normal to the (average) center plane of the fin; in linear approximation, this force is the lift. Resistance forces and lift forces depending nonlinearly on the angle of attack are neglected. Especially, the stall angle is not taken into account.

A fin generates waves and vortices. Their influence on other fins is neglected. That may be important if one fin operates behind another one, e.g. a rudder behind the keel fin of a sailing yacht.

The following scetch shows a fin seen from starboard (left) and from behind.



The position of a fin is characterized by its pressure center $\vec{x}_F$. Normally a good guess of the pressure center is: at half span, 1/4 c behind the profile nose at this profile. To estimate the forces on a fin an angle of attack is determined at a reference point $\vec{x}_{F1}$ which is located $0.5c$ in $-x$ direction behind $\vec{x}_F$. To determine the angle of attack, ship and wave orbital motions are taken into account, but radiation and diffraction waves are neglected.

The direction of the axis of a movable fin is characterized by the vector $\vec{a}_F = (a_x, a_y, a_z)$. In the following $\vec{a}_F$ is assumed to have length one; in the input its length is arbitrary. Example: For a rudder $\vec{a}_F = \{0, 0, 1\}$. Then the rudder angle is taken positive to port[2].

The angle α_{S3} of movable fins specifies the right-hand rotation of a fin around axis $\vec{a}_F$. A linear relationship between ship motions and control mechanism is presupposed:

$$\hat{\alpha}_{S3} = \hat{\delta}_{F1}\hat{u}_1 + ... + \hat{\delta}_{F6}\hat{u}_6 = \hat{\delta}_F \hat{u} \tag{70}$$

$\hat{\delta}_F$ is a user-specified complex row vector specifying the active control of the fin. Depending on the position of the fin, the direction of $\vec{a}_F$ and the sign of the real parts (corresponding to restoring forces) and the imaginary parts (corresponding to damping forces) of the components of $\hat{\delta}_F$ a fin can increase or decrease the motions. Thus it is recommended to test whether the correct signs have been selected by comparing the motions with and without fin control.

Let $\vec{n}_F$ be a unit vector in the y, z-plane and normal to the fin center plane:

$$\vec{n}_F = \frac{(1,0,0) \times \vec{a}_F}{|(1,0,0) \times \vec{a}_F|} = (0, n_{F2}, n_{F3}) \tag{71}$$

The periodical fin force depends on the component of the flow in direction $\vec{n}_F$.

The undisturbed incident wave of complex amplitude $\hat{\zeta}$ for water depth D produces a flow speed in direction $\vec{n}_F$ of amplitude

$$\hat{v}_{FW}\hat{\zeta} = \frac{\omega\hat{\zeta}}{\sinh(kD)} e^{i(-kx_{F1}\cos\mu + ky_F \sin\mu)} \vec{n}_F \left\{ \begin{array}{c} 0 \\ \sin\mu \cosh(k(z_F - z_B)) \\ -i\sinh(k(z_F - z_B)) \end{array} \right\} \tag{72}$$

where z_B is the vertical coordinate of the bottom. For deep water the corresponding formula for $z_B \to \infty$ is taken.

The complex amplitude of the fin velocity in direction $\vec{n}_F$ relative to the inertial coordinate system follows from the ship motions:

$$\hat{v}_{FS} = \vec{n}_F \left\{ \begin{array}{c} \hat{\dot{\xi}} \\ \hat{\dot{\eta}} \\ \hat{\dot{\zeta}} \end{array} \right\} = i\omega_e \vec{n}_F \left(\begin{bmatrix} 0 & -\hat{u}_6 & \hat{u}_5 \\ \hat{u}_6 & 0 & -\hat{u}_4 \\ -\hat{u}_5 & \hat{u}_4 & 0 \end{bmatrix} \left\{ \begin{array}{c} x_{F1} \\ y_F \\ z_F \end{array} \right\} + \left\{ \begin{array}{c} \hat{u}_1 \\ \hat{u}_2 \\ \hat{u}_3 \end{array} \right\} \right) \tag{73}$$

With $\vec{n}_F = (0, n_{F2}, n_{F3})$ this can be transformed to the more convenient form

$$\hat{v}_{FS} = i\omega_e \cdot W_F \cdot \hat{u} \tag{74}$$

with

$$W_F = \{0,\ n_{F2},\ n_{F3},\ -n_{F2}z_F + n_{F3}y_F,\ -n_{F3}x_{F1},\ n_{F2}x_{F1}\} \tag{75}$$

The presence of the hull is assumed to change the orbital velocity by a factor C_r to

$$C_r \cdot \hat{v}_{FW} \cdot \hat{\zeta} \tag{76}$$

[2]Remember: z points down.

C_r is to be given as input. It is about 1 for fins at a sufficient distance from the hull; it is > 1 for fins attached to the hull and pointing to the sides or downward, and it is nearly zero for fins attached at the aft end of the hull, if they do not reach out of the hull's wake. The same factor C_r is applied to the velocity $\hat{v}_{FS}$ due to the ship motions to take account of the hull influence on $\hat{v}_{FS}$.

The added mass of a fin for accelerations in direction $\vec{n}_F$ is determined as

$$m_F = C_M \cdot \frac{\pi}{4} \rho c^2 s \tag{77}$$

C_M is a user-specified input. $C_M \approx 1$ for $s/c \gg 1$, $C_M \approx \Lambda$ for $s/c \ll 1$. The aspect ratio is defined here as:

$$\Lambda = \begin{cases} 2s/c & \text{if the fin is attached to the hull without gap for water to flow through} \\ s/c & \text{if the flow around both ends of the span is not restricted} \end{cases} \tag{78}$$

Use Fig. 3 to estimate C_M as a function of $1/\Lambda$ given on the abscissa.

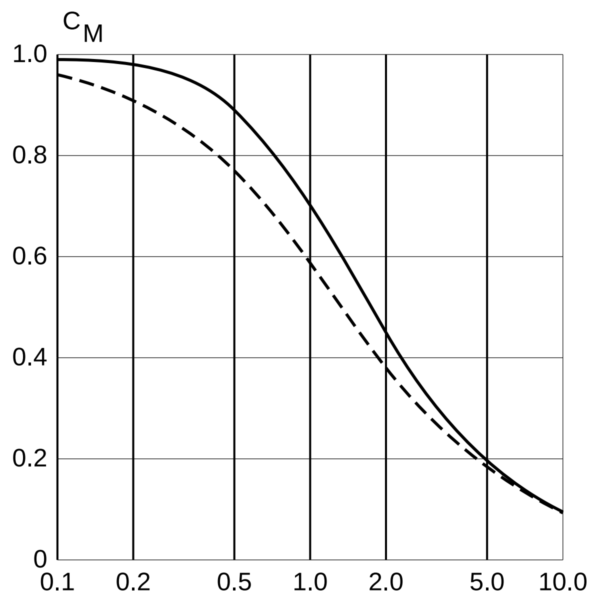


Fig. 3: Added mass coefficient C_M depending on the inverse aspect ratio Λ. Full line for elliptical, broken line for rectangular planform.

The complex amplitude of the acceleration due to orbital motion and ship motion is $i\omega \cdot C_r \cdot \hat{v}_{FW} \cdot \hat{\zeta}$ and $i\omega_e \cdot C_r \cdot \hat{v}_{FS}$, respectively. Thus the added-mass force at the fin due to the wave is

$$\hat{F}_{FW} = m_F \cdot i\omega \cdot C_r \cdot \hat{v}_{FW} \cdot \hat{\zeta} \tag{79}$$

and due to the ship motion

$$\hat{F}_{FS} = m_F \cdot \omega_e^2 \cdot C_r \cdot W_F \cdot \hat{u} \tag{80}$$

The complex amplitude of the force in direction $\vec{n}_F$ due to the time-varying angle of attack is

$$\hat{L} = \frac{\rho}{2} \cdot v^2 \cdot s \cdot c \cdot C_L = v^2 \cdot b_F \cdot \hat{\alpha} \tag{81}$$

v is the (average) speed ahead, $\hat{\alpha}$ the complex amplitude of the angle of attack, and

$$b_F = \frac{\rho}{2} \cdot s \cdot c \cdot \frac{dC_L}{d\alpha} \tag{82}$$

if the lift coefficient is assumed proportional to the angle of attack. To improve the accuracy for larger wave amplitudes, the lift coefficient is approximated, instead, as

$$C_L = \frac{dC_L}{d\alpha}\alpha + C_d\alpha|\alpha| \tag{83}$$

C_d is a coefficient of fin resistance in transverse flow. For fins with sharp ends of the span $C_d = 1$ is a good choice. As the second term depends nonlinearly on the angle of attack, an equivalent linearization based on the given wave height for quadratical effects is used for it.

Also $dC_L/d\alpha$ is a user-specified input: the linear change of the lift coefficient with attack angle for $\alpha = 0$. For fins in free flow it can be approximated as

$$\frac{dC_L}{d\alpha} = 2\pi\frac{\Lambda(\Lambda + 0.7)}{(\Lambda + 1.7)^2} \tag{84}$$

However, $\frac{dC_L}{d\alpha}$ may be influenced substantially by various effects:

- Propeller slipstream. Because in the formula for L the ship speed v is used instead of the flow speed at the fin, the lift coefficient gradient according to (84) should be multiplied by the square of the ratio actual inflow speed over ship speed.
- Ship's wake. To be taken into account by the ratio (inflow speed to the fin $/v)^2$.
- Boundary layer along the hull
- Non-viscous interaction between hull and fin. Increases the lift gradient.
- Exceeding the stall angle would decrease the lift.
- Gaps between different parts of the fin decrease the lift.
- Boundary layer at the fin (effective only for small fins in model experiments; decreases the lift)

The influence of these effects is discussed extensively by *Bohlmann 1990.* Some of the effects are discussed also by Söding (1991). Both papers are in German.

The complex amplitude of the angle of attack $\hat{\alpha}$ is the sum of the following contributions:

- Due to the wave, for $v \neq 0$:

$$\hat{\alpha}_W = \frac{C_r \cdot \hat{v}_{FW}}{v} \cdot \hat{\zeta} \tag{85}$$

  For $v = 0$ the expression is not evaluated because $\hat{L} = 0$ then.

- Due to the fin moving with the ship in direction $\vec{n}_F$:

$$\hat{\alpha}_{S1} = -C_r \cdot \frac{\hat{v}_{FS}}{v} = -i\omega_e \cdot C_r \cdot \frac{W_F}{v} \cdot \hat{u} \tag{86}$$

- Due to the rotation of the fin with the ship:

$$\hat{\alpha}_{S2} = C_r\tilde{a}_F \begin{Bmatrix} \hat{u}_4 \\ \hat{u}_5 \\ \hat{u}_6 \end{Bmatrix} \tag{87}$$

  Here $\tilde{a}_F$ is the unit vector in the y, z plane in the foil's span direction:

$$\tilde{a}_F = \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix} \times \vec{n}_F = \begin{Bmatrix} 0 \\ -n_{F3} \\ n_{F2} \end{Bmatrix} \tag{88}$$

$\hat{\alpha}_{S2}$ may be expressed in the form

$$\hat{\alpha}_{S2} = C_r \cdot c_F \cdot \hat{u} \tag{89}$$

with

$$c_F = \{0, 0, 0, 0, -n_{F3}, n_{F2}\} \tag{90}$$

The above angles were defined positive if they turn right around $\tilde{a}_F$. $\tilde{a}_F$ is – except for a possible inclination of the fin axis agains the section plane $x =$ constant – opposite to $\vec{a}_F$. Thus

– the angle of attack, right around $\tilde{a}_F$, due to fin control is

$$\hat{\alpha}_{S3} = -C_\delta \cdot \hat{\delta}_F \cdot \hat{u} \cdot \sqrt{a_y^2 + a_z^2} \tag{91}$$

C_δ is a user-specified factor:
$C_\delta = 1$ for fins turning as a single body (e.g. spade rudder)
$C_\delta < 1$ if the front part of the fin is fixed and only the aft part rotated (semi-balanced rudder)
$C_\delta > 1$ if the front part moves by the angle $\hat{\delta}_F \hat{u}$ and an aft flap at larger angle (Becker rudder)

The factor $\sqrt{a_y^2 + a_z^2} \leq 1$ takes account of a possible inclination of the fin axis against the yz plane.

Eqs.(79) to (91) can be combined to the total complex amplitude of the fin force in direction $\vec{n}_F$:

$$\begin{aligned} \hat{F}_F &= \left[\omega_e^2 \cdot C_r \cdot m_F \cdot W_F - i\omega_e \cdot b_F \cdot C_r \cdot v \cdot W_F + b_F \cdot v^2 \cdot C_r \cdot c_F - b_F \cdot v^2 \cdot C_\delta \cdot \hat{\delta}_F \sqrt{a_y^2 + a_z^2}\,\right] \hat{u} \\ &\quad + (i\omega \cdot C_r \cdot m_F \cdot \hat{v}_{FW} + b_F \cdot C_r \cdot v \cdot \hat{v}_{FW})\,\hat{\zeta} \end{aligned} \tag{92}$$

F_F acts at point $\vec{x}_F$. It is transformed to a 6-component generalised force vector acting at the coordinate origin by left multiplication with V_F:

$$V_F = \left\{ \begin{array}{c} \vec{n}_F \\ \vec{x}_F \times \vec{n}_F \end{array} \right\} \tag{93}$$

In the expression $V_F \hat{F}_F$ the part of (92) between brackets gives a contribution to the hydrodynamic matrix B, the term in parentheses in the second line a contribution to the exciting vector $\vec{F}_e$.

The time-averaged drift force due to fins is elaborated in chapter 17.

6 Sail forces

Periodical forces on sails are considered because they influence ship motions. Especially the damping effect on roll motion is often dominant. With some precautions the method to take account of sails may also be applied to wind forces on superstructures of ships without sails, or on deck containers etc. Like for fins, it may be necessary for accurate results to subdivide a large sail horizontally into several part sails.

At first damping forces are considered. The steady force on a sail, acting normal to the "sail plane", is approximated as

$$F_S = \frac{1}{2} \rho_{\text{air}} |\vec{s}| |\vec{u}_W - \vec{u}_S|^2 C_N(\alpha) \tag{94}$$

ρ_{air} is the density of air. PDSTRIP assumes $\rho_{\text{air}} = \rho/800$. α is the angle of attack of the wind relative to the sail approximated as a plane. C_N is the normal force coefficient of the wind force on the sail. Only forces normal to the sail plane are considered. $\vec{s}$ is a vector normal to the sail plane, having the sail area as length. $\vec{u}_W$ is the "apparent" wind velocity, i.e. the wind velocity relative to the ship sailing with its average speed ahead as given. $\vec{u}_S$ is the oscillating velocity of the sail speed reference point x_{S1} due to ship motions. Its complex amplitude is

$$\hat{\vec{u}}_S = i\omega_e \left(\begin{bmatrix} 0 & -\hat{u}_5 & \hat{u}_6 \\ \hat{u}_5 & 0 & -\hat{u}_4 \\ -\hat{u}_6 & \hat{u}_6 & 0 \end{bmatrix} \begin{Bmatrix} x_{S1} \\ y_{S1} \\ z_{S1} \end{Bmatrix} + \begin{pmatrix} \hat{u}_1 \\ \hat{u}_2 \\ \hat{u}_3 \end{pmatrix} \right) = i\omega_e \cdot W_S \cdot \hat{u} \tag{95}$$

$$W_S = \begin{bmatrix} 1 & 0 & 0 & 0 & z_{S1} & -y_{S1} \\ 0 & 1 & 0 & -z_{S1} & 0 & x_{S1} \\ 0 & 0 & 1 & y_{S1} & -x_{S1} & 0 \end{bmatrix} \tag{96}$$

$\vec{x}_{S1}$ is located 0.5 times average chord length behind the sail force reference point $\vec{x}_S$:

$$\vec{x}_{S1} = \vec{x}_S + 0.5\vec{c} \tag{97}$$

$\vec{c}$ is the "chord vector". Its length is the average chord length of the sail, and the direction is from the forward to the backward edge of the sail. $\vec{c}$ and the "mast vector" $\vec{m}_S$ (points upward; length = height of the sail) determine the sail area vector $\vec{s}$:

$$\vec{s} = \vec{c} \times \vec{m}_S \tag{98}$$

The sail force reference point $\vec{x}_S$ may be assumed to lie between the sail area center of gravity and half the height of the sail with respect to height, and about 1/4c behind the front end of the sail at that height.

The normal force coefficient C_N is approximated as

$$C_N = 2\pi \frac{\Lambda(\Lambda + 0.7)}{(\Lambda + 1.7)^2} \alpha \tag{99}$$

Λ is the aspect ratio, i.e. height / average chord length. If wind flow under the lower edge of the sail is severely restricted, Λ should be increased, for full suppression of the flow below the lower edge of the sail by a factor of 2. Further, C_N is limited in the program by ± 1.5 to take account of the stall angle.

In (94), due to the ship motions the quantities $\vec{u}_S$ and α change. Linearizing this change, the complex amplitude of $|\vec{u}_W - \vec{u}_S|^2$ is

$$-2\vec{u}_W \hat{\vec{u}}_S \tag{100}$$

and the complex amplitude of α is approximately

$$\hat{\alpha} = \frac{-\hat{\vec{u}}_S \vec{s}}{|\vec{u}_W||\vec{s}|} \tag{101}$$

For large absolute value of $\hat{\alpha}$ the expression for $\hat{\alpha}$ is inaccurate, but in that case the term containing $\hat{\alpha}$ is small anyway.

This gives the complex amplitude of the sail force as

$$\hat{F}_S = \frac{1}{2} \rho_{\text{air}} |\vec{s}| \left(-2\vec{u}_W \hat{\vec{u}}_S C_N(\bar{\alpha}) + |\vec{u}_W|^2 \frac{\partial C_N}{\partial \alpha} \hat{\alpha} \right) \tag{102}$$

The second term is zero for stall, i.e. if $C_N = \pm 1.5$. $\bar{\alpha}$ is the average angle of attack (positive, if u_W has a positive component in direction $\vec{s}$):

$$\bar{\alpha} = \arcsin \frac{\vec{u}_W \cdot \vec{s}}{|\vec{u}_W||\vec{s}|} \tag{103}$$

To apply $\hat{F}_S$ in the computation of the motion transfer functions, the ship motion $\hat{u}$ has to be separated:

$$\hat{F}_S = -\frac{1}{2}\rho_{\text{air}}|\vec{s}| \left(2\vec{u}_W C_N(\bar{\alpha}) + |\vec{u}_W|\frac{\partial C_N}{\partial \alpha}\frac{\vec{s}}{|\vec{s}|}\right) i\omega_e \cdot W_S \cdot \hat{u} \tag{104}$$

The force and moment acting on the ship due to the sail force is $V_S\hat{F}_S$ with the 6 by 1 matrix

$$V_S = \left\{ \begin{array}{c} \vec{s}/|\vec{s}| \\ \vec{x}_F \times \vec{s}/|\vec{s}| \end{array} \right\} \tag{105}$$

For light ships with large sails, the added air mass of the sails may contribute considerably to the moment of inertia around the longitudinal axis. Therefore also added-mass forces due to the sails are taken into account. Their complex amplitude is

$$\hat{F}_{Sa} = -c_M \frac{\pi\rho_{\text{air}}\vec{s}|\vec{c}|}{4}(i\omega_e)^2 \cdot W_S\hat{u} \tag{106}$$

c_M can be estimated as described for fins. The effect of the sail added-mass force on the ship force and moment is again determined by left-multiplication of $\hat{F}_{Sa}$ with V_S.

The contribution of sails to the matrix B is the sum of the damping and mass force without the right-hand factor $\hat{u}$:

$$B_S = -\sum_{\text{all sails}} \rho_{\text{air}} V_S \left(\vec{u}_W C_N(\bar{\alpha})|\vec{s}| + \frac{1}{2}|\vec{u}_W|\frac{\partial C_N}{\partial \alpha}\vec{s} + \frac{\pi}{4}c_M i\omega_e|\vec{c}|\vec{s}\right) i\omega_e W_S \tag{107}$$

For aftward wind, the point of attack of the wind force $\vec{x}_S$ and the velocity reference point $\vec{x}_{S1}$ exchange their positions. This is of minor importance and is neglected in the program.

7 Motion-dependent forces

The program allows to specify motion-dependent forces produced by thrusters and used, e.g., to dampen the roll motion. One example are Voith-Schneider propellers in crane barges at zero forward speed.

Here it is assumed that such a force acts:

- at the location (x_P, y_P, z_P) of the propulsor,
- oscillates harmonically over time with complex amplitude $\hat{P}$
- acts in direction (a_{xP}, a_{yP}, a_{zP})

(a_{xP}, a_{yP}, a_{zP}) can be input with arbitrary absolute value. It will be normalized to absolute value 1 by the program and is assumed normalized in the following.

$\hat{P}$ is set proportional to the motion amplitudes:

$$\hat{P} = \{f_1, f_2, f_3, f_4, f_5, f_6\} \cdot \hat{u} \tag{108}$$

Here the rotations u_4, u_5, u_6 are assumed to be expressed in radians, not degrees. The complex factors f_i must be chosen such that neither the maximum propulsor force P_{max}, nor the maximum rate of its change $\dot{P}_{max}$ are exceeeded.

$\hat{P}$ induces in the ξ, η, ζ system a generalized force (6 components):

$$\underbrace{\begin{Bmatrix} a_{xP} \\ a_{yP} \\ a_{zP} \\ y_P a_{zP} - z_P a_{yP} \\ z_P a_{xP} - x_P a_{zP} \\ x_P a_{yP} - y_P a_{xP} \end{Bmatrix}}_{V} \hat{P} = V \cdot \{f_1, f_2, f_3, f_4, f_5, f_6\} \cdot \hat{u} \tag{109}$$

The following 6×6 matrix has to be added to the matrix B:

$$B_P = V \cdot \{f_1, f_2, f_3, f_4, f_5, f_6\} \tag{110}$$

8 Suspended load

A mass m hangs on a rope of length l. The upper end of the rope is attached to a ship-fixed point (x_l, y_l, z_l). For simplification, only the motions of the loads in y-direction are considered, because these are more important than those in x-direction. Damping of the motion of the load e.g. due to air resistance is neglected. Thus the only damping mechanism is the coupling of the load motion to the ship motions.

In the fundamental motion equation

$$[S - B - \omega_e^2 M]\hat{\vec{u}} = \hat{\vec{F}}_e$$

S describes the ship with load fixed (not free to move) in $(x_l, y_l, z_l + l)$.

The motion of the load is described in the ship fixed coordinate system by the complex motion amplitude $\hat{y}_m$. The pendular motion of the load induces an excitation in addition to $\hat{\vec{F}}_e$:

$$\hat{l} = \begin{Bmatrix} 0 \\ \omega^2 \\ 0 \\ g - \omega^2(z_l + l) \\ 0 \\ \omega^2 x_l \end{Bmatrix} m \cdot \hat{y}_m \tag{111}$$

The motion of the load is described in the inertial system by the linearized motion equation:

$$m \cdot \ddot{\eta}_m = \frac{\eta_l - \eta_m}{l} \cdot mg \tag{112}$$

We obtain correspondingly for the complex amplitude:

$$-\omega^2 \cdot m \cdot \hat{\eta}_m = \frac{\hat{\eta}_l - \hat{\eta}_m}{l} \cdot mg \tag{113}$$

The transformation equation between inertial and ship-fixed coordinates (38) yields the complex amplitudes $\hat{\eta}_l$ and $\hat{\eta}_m$:

$$\hat{\eta}_l = \hat{u}_6 \cdot x_l - \hat{u}_4 \cdot z_l + \hat{u}_2 \tag{114}$$

$$\hat{\eta}_m = \hat{u}_6 x_l - \hat{u}_4(z_l + l) + \hat{u}_2 + \hat{y}_m \tag{115}$$

Inserting these terms in (113) yields:

$$\hat{y}_m = \frac{\hat{u}_4(z_l + l - g/\omega^2) - \hat{u}_6 x_m - \hat{u}_2}{1 - g/(l\omega^2)} \tag{116}$$

We consider three cases as verification for this equation:

1. For $\omega \to 0$ we obtain as expected the static deflection of the load for heel $\hat{y}_m = l\hat{u}_4$.
2. For $\omega = \sqrt{g/l}$ (resonance) we obtain as expected $\hat{y}_m = \infty$.
3. For $\omega \to \infty$ we obtain $\hat{y}_m = \hat{u}_4(z_l + l) - \hat{u}_6 x_m - \hat{u}_2$. This means according to (115): $\hat{\eta}_m = 0$, i.e. the load remains unmoved in the inertial system as expected.

The system of equation for the calculation of the ship motions must then be extended for the influence of a load performing a pendular motion in y-direction:

$$\begin{bmatrix} \begin{matrix} S - B - \omega_e^2 M \\ (6 \times 6) \end{matrix} & \begin{matrix} 0 \\ -\omega_e^2 m \\ 0 \\ -[g - \omega_e^2(z_l + l)]m \\ 0 \\ -\omega_e^2 x_l m \end{matrix} \\ 0 \quad 1 \quad 0 \quad -(z_l + l - g/\omega_e^2) \quad 0 \quad x_m & 1 - g/(l\omega_e^2) \end{bmatrix} \cdot \begin{Bmatrix} \hat{u} \\ (6 \times 1) \\ \\ \hat{y}_m \end{Bmatrix} = \begin{pmatrix} \hat{\vec{F}}_e \\ (6 \times 1) \\ \\ 0 \end{pmatrix} \tag{117}$$

If a transverse force depends on the shift of the load in the ship-fixed system rather than the ship motions, deduct in (117) $\alpha_7 V$ from the 7^{th} column in the l.h.s. matrix in rows 1 to 6. α_7 is the ratio between complex amplitudes of transverse force and load shift. V is the transformation matrix as defined in (109).

9 Surf-riding

Symbol v designates the average forward speed of the ship. Its component in the direction of wave propagation is $v \cos \mu$. The total time-varying horizontal velocity of the ship in the direction of wave propagation is

$$v \cos \mu + \zeta_A \ \mathrm{Re}(i\omega_e[\hat{u}_1 \cos \mu + \hat{u}_2 \sin \mu]e^{i\omega_e t}) \tag{118}$$

where ζ_A is the wave amplitude. The maximum over time of this velocity is

$$v_{max} = v \cos \mu + \zeta_A |\omega_e| \cdot |\hat{u}_1 \cos \mu + \hat{u}_2 \sin \mu| \tag{119}$$

and the minimum is

$$v_{min} = v \cos \mu - \zeta_A |\omega_e| \cdot |\hat{u}_1 \cos \mu + \hat{u}_2 \sin \mu| \tag{120}$$

If, for positive ω_e, v_{max} exceeds the phase speed $c = \omega/k$ of the wave, the linearization used in the program is inappropriate because the ship will surf-ride on the wave once it has attained wave celerity c. Actually, due to nonlinear effects surf-riding will occur even if v_{max} (computed by a linear calculation as is done here) is a little smaller than c; but that will be neglected.

Negative encounter frequency ω_e means that the ship overtakes the waves from behind. In this case, surf-riding will occur if v_{min} is smaller than c, or even for a small positive difference $v_{min} - c$, which, however, is neglected as before.

If one of these sufficient conditions for surf-riding is found, an appropriate warning is given in the output list, and the transfer functions of these cases are not used for determining the behaviour in natural seaways. This is necessary because in this region of small absolute values of the encounter frequency ω_e many transfer functions have sharp peaks with excessive maxima which would produce unreasonable results for several reasons: In case of surf-riding, the linearity assumption is not valid; the motion is no longer oscillatory; the ship may broach-to; and, due to its dangers, in practice one tries to avoid surf-riding.

The wave amplitude ζ_A is determined from the wave steepness given in the input data. If *steep* is given as zero or too small, incorrect results may be determined in natural waves, e.g. for the added resistance in waves. For wind waves, *steep* = 0.1 may be recommended.

10 Quadratic drag forces

Equantions for maneuvering simulations include a cross-flow contribution to the hull forces. This transverse drag force depends quadratically on the transverse velocity v_q at the sections:

$$F_q = \int\limits_L \frac{\rho}{2} v_q |v_q| T C_d \, dx \tag{121}$$

Here T is the section draft, B its breadth, and C_d is the drag coefficient; usually they are functions of x. This equation cannot be applied directly for various reasons:

1. Horizontal and vertical relative motions appear simultaneously.
2. Both section and water move.
3. Despite the nonlinear force components, we still want to use a linear system of equations to determine the motions.
4. The transverse velocity of the section is not known initially. It depends on the drag forces.

We discuss these points individually:

1. Different directions of motion

   Consider a circular, unmoved cross section of diameter d in a deep water wave, i.e. in a flow where the absolute value of velocity is $|\vec{v}|$. Then the drag force/length in y-direction is

   $$f_y = \frac{\rho}{2} |\vec{v}| v_y T C_d \tag{122}$$

   v_y is the y component of $\vec{v}$. The equation is here generalized for two harmonically osciallating velocity components v_y and v_z of different amplitude and arbitrary phase difference, and for cross sections of arbitrary shape with draft T and width B:

   $$f_y = \frac{\rho}{2} \sqrt{v_y^2 + v_z^2} v_y T C_y \tag{123}$$
   $$f_z = \frac{\rho}{2} \sqrt{v_y^2 + v_z^2} v_z B C_z \tag{124}$$

   These equations should be good approximations for all ship cross sections.

2. Simultaneous motion of section and water

   The above formulae can be applied if we interpret v_y and v_z as relative velocities between water and ship. We then set

   $$v_y = w_y - u_y \tag{125}$$

   Here w_y is the y component of the velocity of the water, u_y that of the section. Correspondingly for direction z.

3. Linearization

   The principle of equivalent linearization defines the equivalent linear drag force such that it performs the same work (energy) as the nonlinear force over one motion period $\mathcal{T}$. For the transverse force, this yields:

   $$\int\limits_0^{\mathcal{T}} \frac{\rho}{2} \sqrt{v_y^2 + v_z^2} v_y u_y T C_y \, dt = \int\limits_0^{\mathcal{T}} \frac{\rho}{2} \bar{v} v_y u_y T C_y \, dt \tag{126}$$

$\bar{v}$ on the right-hand side denotes an equivalent linearized velocity, constant in time. From this equation $\bar{v}$ follows:

$$\bar{v} = \frac{\overline{\sqrt{v_y^2 + v_z^2} \cdot v_y \cdot u_y}}{\overline{v_y \cdot u_y}} \tag{127}$$

The overlines denote time averages over a period. The time average in the nominator cannot be given in closed form. One possibility is to integrate numerically for known values of $\hat{v}_y$, $\hat{v}_z$ and $\hat{u}_y$. Because the drag coefficients and wave amplitudes are not known exactly either, we may as well use an approximation formula for the average, using the general mathematical relations for quantities a, b, and c oscillating harmonically with ω_e:

$$\overline{ab} = \frac{1}{2}\,\mathrm{Re}(\hat{a}\hat{b}^*) \tag{128}$$

$$\overline{|a|bc} = \frac{|\hat{a}|}{\pi}\,\mathrm{Re}\left(\frac{1}{3}\hat{b}\hat{c}\frac{\hat{a}^*}{\hat{a}} + \hat{b}^*\hat{c}\right) \tag{129}$$

The asterisk denotes the conjugate complex. The above time average $M = \overline{\sqrt{v_y^2 + v_z^2} \cdot v_y \cdot u_y}$ can be approximated for various special cases:

- For $|\hat{v}_z| \ll |\hat{v}_y|$:

$$M \approx \overline{|v_y|v_y u_y} = \frac{4}{3\pi}|\hat{v}_y|\,\mathrm{Re}(\hat{v}_y\hat{u}_y^*) \tag{130}$$

- For $|\hat{v}_z| \gg |\hat{v}_y|$:

$$M \approx \overline{|v_z|v_y u_y} = \frac{4}{3\pi}|\hat{v}_z|\,\mathrm{Re}\left(\frac{1}{4}\hat{v}_y\hat{u}_y\frac{\hat{v}_z^*}{\hat{v}_z} + \frac{3}{4}\hat{v}_y\hat{u}_y^*\right) \tag{131}$$

- For $|\hat{v}_z| = |\hat{v}_y|$ and 90° phase difference between the two velocity components:

$$M = |\hat{v}_y|\overline{v_y u_y} = \frac{1}{2}|\hat{v}_y|\,\mathrm{Re}(\hat{v}_y\hat{u}_y^*) = \frac{\sqrt{2}}{4}\sqrt{|\hat{v}_y|^2 + |\hat{v}_z|^2}\,\mathrm{Re}(\hat{v}_y\hat{u}_y^*) \tag{132}$$

- For v_z and v_y in phase, arbitrary ratio of amplitudes:

$$M = \frac{\sqrt{|\hat{v}_y|^2 + |\hat{v}_z|^2}}{|\hat{v}_y|}\overline{|v_y|v_y u_y} = \frac{4}{3\pi}\sqrt{|\hat{v}_y|^2 + |\hat{v}_z|^2}\,\mathrm{Re}(\hat{v}_y\hat{u}_y^*) \tag{133}$$

The formulae for the four cases show that the last formula approximates all four cases reasonably well.

The time-average in the denominator yields simply $\mathrm{Re}(\hat{v}_y\hat{u}_y^*)$. Thus we have:

$$\bar{v} = \frac{8}{3\pi}\sqrt{|\hat{v}_y|^2 + |\hat{v}_z|^2} \tag{134}$$

The equivalent linearized drag force/length on a cross section is then:

$$f_y = \frac{\rho}{2}\bar{v}v_y T C_y = \frac{\rho}{2}\bar{v}(w_y - u_y)TC_y \tag{135}$$

$$f_z = \frac{\rho}{2}\bar{v}v_z B C_z = \frac{\rho}{2}\bar{v}(w_z - u_z)BC_z \tag{136}$$

If there are no dedicated experimental values for C_y, it is recommended to use $C_y = 0.8$ for well rounded ship sections, and $C_y = 1.2$ for ship sections with bilge keels and sections with a knuckle at the keel or at the bilge. For C_z the following values may be recommended: 0.4 vor sections which intersect the waterline at maximum breadth (here an essential resistance occurs only during upward motion of the section), and 0.8 for fully submerged sections.

In above equations, the terms involving w_y, w_z contribute to the exciting force vector and the terms involving u_y, u_z to the complex added mass matrix. We consider first the terms contributing to the added mass matrix B. Due to the quadratic transverse drag force we have an additional contribution to B:

$$B_q \hat{u} = \int\limits_L V(-i\omega_e) A_q W \; dx \cdot \hat{u} \tag{137}$$

with

$$A_q = \frac{1}{i\omega_e} \begin{bmatrix} n_{q,22} & n_{q,23} & n_{q,24} \\ n_{q,32} & n_{q,33} & n_{q,34} \\ n_{q,42} & n_{q,43} & n_{q,44} \end{bmatrix} = \frac{1}{i\omega_e} \frac{\rho}{2} \bar{v} \begin{bmatrix} TC_y & 0 & 0 \\ 0 & BC_z & BC_z y \\ 0 & BC_z y & BC_z y^2 \end{bmatrix} \tag{138}$$

y is the transverse coordinate of the center of the section. The above equations can be simplified using the Matrix U:

$$B_q = \int\limits_L -V N_q W \; dx \tag{139}$$

$$N_q = \frac{\rho}{2} U^T \bar{v} \begin{bmatrix} TC_y & 0 \\ 0 & BC_z \end{bmatrix} U \tag{140}$$

$$U = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & y \end{bmatrix} \tag{141}$$

The contribution to the exciting forces is analoguously:

$$\hat{\vec{F}}_{e,q} = \int\limits_L V \hat{f}_{e,x} W \; dx \tag{142}$$

with

$$\hat{f}_{e,x} = \frac{\rho}{2} \bar{v} \left\{ \begin{array}{c} \hat{w}_y TC_y \\ \hat{w}_z BC_z \\ \hat{w}_z y BC_z \end{array} \right\} \tag{143}$$

With matrices U and W, see Eq.(50), and the generalized motion vector $\hat{\vec{u}}$, we can express the two transverse velocity components as follows:

$$\left\{ \begin{array}{c} \hat{u}_y \\ \hat{u}_z \end{array} \right\} = UW\hat{\vec{u}} \tag{144}$$

The complex amplitude of the water velocity is:

$$\left\{ \begin{array}{c} \hat{w}_y \\ \hat{w}_z \end{array} \right\} = \omega \hat{\zeta}_0 \cdot e^{-ik(x\cos\mu - y\sin\mu)} \cdot e^{-kT} \cdot \left\{ \begin{array}{c} \sin\mu \\ i \end{array} \right\} \tag{145}$$

Here we assumed that the section keel is the characteristic depth to base the drag force on.

4. Iteration

   The computation of B_q and $\hat{\vec{F}}_{e,q}$ require the motion vector $\hat{\vec{u}}$. This in turn requires a characteristic wave height. But even with given wave height, the motion vector can only be determined iteratively: As a first estimate, the motions $\hat{u}^{(1)}$ are determined without quadratic damping. Then estimates for B_q and $\hat{\vec{F}}_{e,q}$ can be obtained. These yield in turn a new estimate for the motion vector $\hat{u}^{(2)}$.

   In off-resonance cases, the iteration converges quickly, oscillating with rapidly decaying amplitude around the final solution.

Exactly at resonance, in the extreme case where linear damping is negligible compared to the quadratic damping, the motion is inversely proportional to the damping constant B_q. As the damping constant B_q in turn is proportional to the motion amplitude, we have:

$$\hat{u}_j^{(2)} \approx \frac{\text{factor}}{\hat{u}_j^{(1)}} \tag{146}$$

Thus, in this extreme case, the iteration would oscillate around the solution without convergence. To accelerate convergence and ensure convergence also for resonance, we employ a special iterative method described in the following. We compute a weighted average $\bar{u}$ of all previous estimates $\hat{u}^{(i)}$. $\bar{u}$ is used to compute B_q and $\hat{\vec{F}}_{e,q}$ which in turn yield a new estimate for $\hat{u}$, used to update the weighted average $\bar{u}$. The weights are $0.85^j/\epsilon$. j is the number of iterations between the considered value and the present step. Older iterations have thus a lower weight. ε is the nondimensional error, determined by the difference between $\bar{u}$ and the estimate $\hat{u}$ determined with this $\bar{u}$. Motions $\bar{u}$ with small error have thus a larger weight than motions resulting in larger errors. The number of iterations is limited to a maximum 200. If the remaining error exceeds still 1%, an error message is output and the program moves on to the next case. After the second iteration, we only compute the drag forces anew. The iteration requires thus very little extra computational time.

11 Negative encounter frequency

The encounter frequency is negative for following waves if the component of ship speed in the direction of wave propagation exceeds the phase velocity of the waves. However, the complex amplitudes of radiation forces are computed in the first part of PDSTRIP for positive frequencies only. The corresponding values for negative eoncounter frequency are the complex conjugate of those for positive encounter frequency of the same absolute value.

For negative encounter frequencies, the imaginary parts of the complex amplitudes indicate instantaneous values 1/4 period after the time when a wave trough is at the origin. For positive encounter frequencies (the more usual case), imaginary parts are instantaneous values 1/4 period before that time.

12 Small absolute value of encounter frequency

The strip method is sensitive to the behaviour of added mass and damping for small absolute values of the encounter frequency. The radiation force terms contain added mass terms without factor ω_e^2 or ω_e for ship speed $v \neq 0$. The damping terms even have terms containing $1/\omega_e$. This leads to infinite forces for $\omega_e = 0$ and unrealsitic phase shifts betwenn forces and pitch angle. Ultimately, the reason lies in an inaccurate boundary condition at the free surface, which is inherent in strip theory. The correct boundary condition for the radiation potentials would be:

$$g\phi_z = \left(\frac{\partial}{\partial t} - v\frac{\partial}{\partial x}\right)^2 \phi \tag{147}$$

Instead we use a condition that does not contain the term involving v. To reduce the associated errors, which are significant for small ω_e, the radiation and diffraction forces computed in the first part of PDSTRIP are interpolated introducing the following modifications:

- Only ω values above $2v/L$ are used in the interpolation.
- For $\omega_e < \omega_{\min}$, the smallest frequency used in the first part of PDSTRIP, the real parts of the radiation forces are taken as those for $\omega_{\min}$ times $(\omega_e/\omega_{\min})^2$, whereas the imaginary parts are multiplied with $(\omega_e/\omega_{\min})^4$.

– Diffraction forces are computed correspondingly, but using ω instead of ω_e.

If reasonably steep waves are specified in the input data, cases of small absolute value of encounter frequency are skipped anyway because of surf-riding (see the corresponding chapter).

13 Computation of motions at special points

According to the relation between inertial and ship-fixed coordinates, a ship-fixed point having coordinates x_b, y_b, z_b moves in the inertial system from its average position by the vector

$$\vec{\xi}_b = \left\{ \begin{array}{c} \xi_b \\ \eta_b \\ \zeta_b \end{array} \right\} = \begin{bmatrix} 0 & -\hat{u}_6 & \hat{u}_5 \\ \hat{u}_6 & 0 & -\hat{u}_4 \\ -\hat{u}_5 & \hat{u}_4 & 0 \end{bmatrix} \left\{ \begin{array}{c} x_b \\ y_b \\ z_b \end{array} \right\} + \left\{ \begin{array}{c} \hat{u}_1 \\ \hat{u}_2 \\ \hat{u}_3 \end{array} \right\} \tag{148}$$

Thus the complex amplitudes of the motion can be written in the more convenient form

$$\hat{\vec{\xi}}_b = \hat{\vec{u}} \begin{bmatrix} 1 & 0 & 0 & 0 & z_b & -y_b \\ 0 & 1 & 0 & -z_b & 0 & x_b \\ 0 & 0 & 1 & y_b & -x_b & 0 \end{bmatrix} = W_b \hat{\vec{u}} \tag{149}$$

14 Computation of sectional forces and moments

If required, sectional forces and moments (each of them having 3 components) are determined at transverse intersections of the ship. These intersections are located midway between the offset sections.

The relation between forces and accelerations of the part of the ship in front of any of these intersections corresponds to the respective relation for the total ship except for the sectional force and moment $\hat{\vec{F}}_{s,0}$ which acts on the forward ship part (and with opposite sign on the aft ship part):

$$[S_i(x) - B_i(x) - \omega_e^2 M_i(x)]\hat{u} - \hat{\vec{F}}_{e,i} = \hat{\vec{F}}_{s,0}(x) \tag{150}$$

Here $S_i, B_i, M_i, \hat{\vec{F}}_{e,i}$ are the restoring matrix, the complex added mass times ω_e^2 matrix, the mass matrix and the excitation, respectively, for the ship's part in front of the intersection.

Usually the intersection moments are specified referring to a point within the section, not referring to the global coordinate origin K. Here the reference point $(x_i, 0, 0)$ is chosen, where x_i is the x coordinate of the intersection. The conversion of $\hat{\vec{F}}_{s,0}$, which refers to K, to the "natural" section forces $\hat{\vec{F}}_s$ is performed by left multipication of $\hat{\vec{F}}_{s,0}$ with the matrix V_i:

$$V_i = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & x_i & 0 & 1 & 0 \\ 0 & -x_i & 0 & 0 & 0 & 1 \end{bmatrix} \tag{151}$$

The restoring matrix S_i is similar as for the whole ship, see Eq.(44):

$$S_i = \begin{bmatrix} 0 & 0 & \rho g A_x & \rho g y_x A_x & \begin{matrix} -\rho g x_s A_x + gm \\ -\rho g \int A\, dx \end{matrix} & 0 \\ 0 & 0 & 0 & \rho g \int A\, dx - gm & 0 & 0 \\ 0 & 0 & \rho g \int B\, dx & \rho g \int y_W B\, dx & -\rho g \int xB\, dx & 0 \\ 0 & 0 & \rho g \int y_W B\, dx & \begin{matrix} \rho g \int y_W^2 B\, dx + gmz_G \\ -\rho g \int Az_S\, dx \end{matrix} & -\rho g \int x y_W B\, dx & 0 \\ 0 & 0 & \begin{matrix} \rho g z_x A_x \\ -\rho g \int xB\, dx \end{matrix} & \begin{matrix} \rho g y_x z_x A_x \\ -\rho g \int x y_W B\, dx \end{matrix} & \begin{matrix} -\rho g \int A z_S\, dx + gmz_G \\ +\rho g \int x^2 B\, dx \end{matrix} & 0 \\ 0 & 0 & -\rho g y_x A_x & \begin{matrix} \rho g \int xA\, dx - gmx_G \\ -\rho g y_x^2 A_x^2 \end{matrix} & -gmy_G + \rho g \int y_S A\, dx & 0 \end{bmatrix} \tag{152}$$

The integration extends from x_s to the bow. A is the area of the section (below the average water surface), B its waterline breadth, $y_W(x)$ the mean transverse coordinate of the waterline at the section, and y_S, z_S are the coordinates of the section area center; all for the transverse section at longitudinal coordinate x. A_x is the underwater section area at the intersection (where sectional forces are determined), y_x, z_x are the coordinates of the section area center of A_x. The mass data m, x_G, y_G and z_G refer to the part of the vessel in front of the intersection.

15 Determination of pressure amplitudes

The pressure amplitudes are determined following the “pressure strip method” of *Hachmann (1991)*. The following text refers, however, to the derivation of this method in *Söding and Blume (1994)*.

The formula for the complex amplitude of the pressure (for unit wave amplitude) is, *Söding and Blume (1994)*:

$$\begin{aligned} \hat{p} &= \rho(-i\omega_e + v\frac{\partial}{\partial l}) \left[(\hat{\varphi}_2, \hat{\varphi}_3, \hat{\varphi}_4) i\omega_e \cdot W \cdot \hat{\vec{u}} + (\hat{\varphi}^w + \hat{\varphi}^d)\frac{i}{\omega} e^{-ikx\cos\mu} \right] \\ &\quad + \rho(\phi_y^0, \phi_z^0, y\phi_z^0 - z\phi_y^0) i\omega_e \cdot W \cdot \hat{\vec{u}} \end{aligned} \tag{153}$$

The symbols denote:

$\hat{p}$	complex amplitude of pressure
ρ	density
ω, ω_e	incident wave frequency, encounter wave frequency
v	(average) ship speed
$\partial/\partial l$	derivative along streamlines of the steady ship flow
$\hat{\varphi}_2$	2-d radiation potential for sway motion with unit amplitude
$\hat{\varphi}_3$	correspondingly for heave motion
$\hat{\varphi}_4$	correspondingly for roll motion
W	matrix for transformation of ship motions (6 components) into section motions (3 components)
$\hat{\vec{u}}$	vector of complex amplitude of ship motions (6 components)
$\hat{\varphi}^w$	complex amplitude of incident wave potential if the wave amplitude on the x-axis at the considered section is 1; $\hat{\varphi}^w = ge^{-kz}e^{iky\sin\mu}$
$\hat{\varphi}^d$	diffraction potential defined correspondingly
k	wave number
x, y, z	coordinates measured from keel point forward, to starboard, downwards
μ	encounter angle between ship and wave; 0° for following waves, 90° for waves from starboard
ϕ_0	potential of steady ship flow due to ship speed v. We assume a slender body for the partial derivatives of this potential, approximating:

$$\phi_y = -\frac{dy}{dx}v, \qquad \phi_z = -\frac{dz}{dx}v \tag{154}$$

dy/dx and dz/dx denote the derivatives along the streamlines of the steady ship flow.

The term in brackets in the pressure equation is denoted as 'potential' in the output of PDSTRIP. The term represents the complex amplitude of the flow potential due to time-harmonic flow around the ship, but does not contain the steady flow potential, although this contributes also to the pressures, as the second line of the pressure equation shows. An additional term to the above given hydrodynamic pressure amplitude $\hat{p}$ accounts for the periodic change of the hydrostatic pressure due to the vertical motions of the 'pressure points':

$$\rho g\{0, 0, 1, y, -x, 0\}\hat{\vec{u}} = \{\hat{u}_3 + y\hat{u}_4 - x\hat{u}_5\} \tag{155}$$

The pressure is computed at the offset sections given in the ship geometry input file. However, the pressure is not computed at the input offset points. Instead, the user has to specify the number N of 'pressure points' per section (per half-section in case of symmetry). PDSTRIP distributes the 'pressure points' at equidistant contour lengths along the submerged part of the section. The first pressure point of each section lies at the average waterline on port, the last point correspondingly on starboard side. The pressure points are regarded as approximation to the streamlines of the steady ship flow: We assume that the connection of the n-th pressure point on each strip ($1 \le n \le 2N$) forms such a streamline. This allows the numerical differentiation d/dx along streamlines. This may be grossly in error, e.g. at the end of a skeg where only one of two successive sections includes the skeg. In such a case inaccuracies of the pressure amplitudes will occur in case of nonzero speed.

The 2-d potentials $\hat{\varphi}_2, \hat{\varphi}_3, \hat{\varphi}_4$, and $\hat{\varphi}^d$ (radiation and diffraction potentials) are determined at first at points halfway between two offsets points on the section contour. For subsequent pressure computations, these potentials are linearly interpolated (at the free surface also extrapolated) at the pressure points.

The computation of deflections of and stresses in the ship structure requires to take account of the complex amplitudes of:

– the pressure at all pressure points (may be used directly or after interpolation to nodes of a finite-element model)

- fin forces and moments
- weight forces and moments[3]
- D'Alembert forces due to accelerated masses

In theory, the ship should then be in perfect equilibrium. In reality, there are rest forces and moments introduced into the structure at its assumed supports. Resons for these rest forces are:

- differences between the standard strip method and the Hachmann approach (these differences are 0 for zero ship speed and usually small otherwise)
- drag forces depending quadratically on wave height due to flow separation in the flow around the strips
- discretisation errors in the numerical interpolation
- discretisation errors in integrating over the ship length
- discretisation errors in approximating the differentiation d/dx.
- errors due to interpolation over the frequencies used in the first part of the program

Discretisation errors influence the pressure more than the global forces on the ship. An example: A 180m long ship, discretised using 20 offset sections, gave a pressure amplitude amidship near the waterline of 4.3 kPa for following waves of 40 m length (which is relatively short) and the usual 1 m amplitude. As the ship hardly moves in this wave, we expect, instead, a pressure amplitude close to $\rho g \zeta_w \approx 10$kPa. This value was only obtained after doubling the number of offset sections.

16 Significant amplitudes in natural seaways

The variance m_0 of a ship response in a natural seaway follows from the seaway spectrum S_ζ and the response amplitude operator Y:

$$m_0 = \int_0^\infty \int_0^{2\pi} S_\zeta(\omega, \mu) Y^2(\omega, \mu)\, d\mu\, d\omega \tag{156}$$

The seaway spectrum is described as a modified JONSWAP spectrum:

$$S_\zeta(\omega, \mu) = H_{1/3}^2 \cdot T_1 \cdot \frac{177.5 - 6.52\gamma}{(T_1\omega)^5} \cdot e^{-1.25(\omega_m/\omega)^4} \cdot \gamma^\Gamma \cdot \begin{cases} (\cos^n(\mu - \mu_0))/f & \text{for } |\mu - \mu_0| \le \pi/2 \\ 0 & \text{otherwise} \end{cases} \tag{157}$$

$$\Gamma = e^{-(\omega - \omega_m)^2/(2b^2\omega_m^2)} \tag{158}$$

with:

ω_m $= (4.65 + 0.182\gamma)/T_1$ circular frequency of the maximum of the spectrum

$H_{1/3}$ significant wave height of seaway

T_1 significant period of seaway (at 'centroid of spectrum')

[3]The weight force is constant in an earth-fixed, but not in a ship-fixed coordinate system.

μ_0 main direction of waves in spectrum (μ_0 means following waves, $\mu_0 = \pi/2$ menas waves from starboard)

γ peak enhancement factor; $\gamma = 1$ yields the Pierson-Moskowitz spectrum, $\gamma = 3.3$ an 'average' JONSWAP spectrum

f is the integral over the angular spreading function:

$$f = \int_{\mu_0-\pi/2}^{\mu_0+\pi/2} \cos^n(\mu - \mu_0) d\mu \tag{159}$$

This integral can be determined analytically for integer exponents n. In PDSTRIP, it is evaluated numerically, both for simplicity of the program and to allow non-integer exponents which may be used to fit the spectrum to measured data.

b $= 0.07$ for $\omega < \omega_m$, otherwise 0.09.

The significant amplitude $r_{1/3}$ of the response follows from the variance:

$$r_{1/3} = 2\sqrt{m_0} \tag{160}$$

17 Drift force

Drift forces are time-averaged forces on a body due to waves. The negative longitudinal drift force is the wave-induced additional resistance. For bodies with forward speed it is usually directed backward, thus constituting a resistance. There are also a vertical and a transverse drift force and drift moments. Of these, only the transverse drift force and the yaw drift moment are determined by PDSTRIP.

The drift forces are, approximately, proportional to the square of the wave amplitude. PDSTRIP determines only those parts of the ship motions, pressures etc. which depend linearly on the wave amplitude. From these linear responses the most important part of the drift forces can be derived easily, whereas other, usually much smaller components of the drift forces require much more involved computations. Basically, the approximation described by *Boese (1970)* is used here with modifications in several details. Additionally one effect neglected by Boese is taken into account: The stationary pressure term $\frac{1}{2}\rho v^2$ in Bernoulli's equation.

17.1 Effect of periodical hull pressure and ship rotation

The time-dependent pressure force (expressed in inertial coordinates) on a hull panel is

$$\vec{F}(t) = [p_0 + \mathrm{Re}(\hat{p}e^{i\omega_e t})][\vec{a}_0 + \mathrm{Re}(\hat{\vec{\alpha}}e^{i\omega_e t}) \times \vec{a}_0]. \tag{161}$$

Here p_0 is the average pressure, and $\vec{a}_0$ is the average normal vector, with length = panel area, pointing into the hull. $\hat{p}$ is the complex amplitude of the oscillatory pressure at ship-fixed points, and $\hat{\vec{\alpha}}$ the complex amplitude of the hull roation vector. In the second expression between brackets, the last term takes account of the rotation of the area vector (in inertial coordinates) due to ship rotation.

Expanding the product in (161) shows that $\vec{F}(t)$ contains a stationary term $p_0\vec{a}_0$, which is independent of the wave, and two terms oscillating with frequency ω_e. The time average of the latter terms is zero. Only the remaining term proportional to wave amplitude squared contributes to the drift force $\vec{f}$:

$$\vec{f} = \overline{\mathrm{Re}(\hat{p}e^{i\omega_e t})\mathrm{Re}(\hat{\vec{\alpha}}e^{i\omega_e t}) \times \vec{a}_0}. \tag{162}$$

Here the overbar indicates the time average. Using the formula

$$\mathrm{Re}(z) = \frac{1}{2}(z + z^*), \tag{163}$$

which is correct for any real number z if * designates the complex conjugate, from (162) follows

$$\vec{f} = \frac{1}{4}\overline{(\hat{p}e^{i\omega_e t} + \hat{p}^* e^{-i\omega_e t})(\hat{\vec{\alpha}}e^{i\omega_e t} + \hat{\vec{\alpha}}^* e^{-i\omega_e t})} \times \vec{a}_0 \tag{164}$$

Expanding the product under the overbar gives terms oscillating with $\pm 2\omega_e$, which do not contribute to the time average. Thus one obtains:

$$\vec{f} = \frac{1}{4}(\hat{p}^*\hat{\vec{\alpha}} + \hat{p}\hat{\vec{\alpha}}^*) \times \vec{a}_0 = \frac{1}{2}\mathrm{Re}(\hat{p}\hat{\vec{\alpha}}^*) \times \vec{a}_0 = -\frac{1}{2}\mathrm{Re}\left(\hat{p}\vec{a}_0 \times \hat{\vec{\alpha}}^*\right). \tag{165}$$

The sum over all hull panels gives the total drift force

$$\sum \vec{f} = -\frac{1}{2}\mathrm{Re}\left(\sum(\hat{p}\vec{a}_0) \times \hat{\vec{\alpha}}^*\right). \tag{166}$$

On the right-hand side, $\sum(\hat{p}\vec{a}_0)$ is the complex amplitude of the total pressure force on the hull in hull-bound coordinates. *Boese(1970)* substitues it by an expression involving the acceleration of the center of gravity of the hull. This substitution is not made in PDStrip.

17.2 Effect of the pressure due to squared velocity

The Bernoulli equation contains a term $\frac{1}{2}\rho\vec{v}^2$ contributing to the fluid pressure, where $\vec{v}$ is the fluid velocity. The latter consists of stationary and oscillatory contributions:

$$\vec{v} = \vec{v}_0 + \mathrm{Re}(\hat{\vec{v}}e^{i\omega_e t}). \tag{167}$$

Thus, the drift force on one panel due to the squared velocity is

$$\vec{F}(t) = -\frac{1}{2}\rho\left(\vec{v}_0 + \mathrm{Re}(\hat{\vec{v}}e^{i\omega_e t})\right)^2 \vec{a}_0. \tag{168}$$

Like in the previous paragraph, for the drift force terms independent from the wave and oscillatory terms having zero time average are omitted. Thus the panel drift force due to squared velocity is

$$\vec{f} = -\frac{1}{2}\rho\overline{\left(\mathrm{Re}(\hat{\vec{v}}e^{i\omega_e t})\right)^2}\vec{a}_0 = -\frac{1}{4}\rho|\hat{\vec{v}}|^2\vec{a}_0. \tag{169}$$

Again the total velocity-squared contribution to the drift force is found by summing over all hull panels:

$$\sum \vec{f} = -\frac{1}{4}\rho\sum|\hat{\vec{v}}|^2\vec{a}_0. \tag{170}$$

17.3 Effect of variable hull immersion

The previously discussed contributions to the drift force arise from summing over panels, or – in the limit for small panels – from integrating the pressure over the hull up to the time-averaged waterline. Another contribution stems from the pressure acting on the difference between the average and the actually immersed hull surface. To determine this relative-motion contribution to the drift force, a small segment of the average waterline is considered. The horizontal normal vector on this segment, directed into the body, is designated as $\vec{n}$; its length is assumed to be the segment length. The pressure at this segment (at the ship-fixed mean waterline) is designated as

$$p(t) = \mathrm{Re}(\hat{p}e^{i\omega_e t}). \tag{171}$$

If the actual water surface is below the average free surface, in reality the pressure at the average water surface is the air pressure, which is taken here as zero per definition. However, in the linearized pressure calculation this is not taken into account; instead a negative value $p(t)$ is calculated. Therefore, also in the correction for less than average immersion the negative value $p(t)$ has to be applied.

The instantaneous depth of the actual waterline below the average waterline is (up to order zero in wave amplitude, which is sufficient for the following)

$$z(t) = -\frac{p(t)}{\rho g}. \tag{172}$$

Thus the horizontal relative-motion drift force contribution of the segment becomes

$$\vec{f} = -\overline{\int_0^{z(t)} p(t,\zeta)\vec{n}dz}. \tag{173}$$

Here $p(t,\zeta)$ is the pressure at time t in height ζ below the average waterline. It changes linearly with ζ between $p(t)$ at $\zeta = 0$ and 0 at $\zeta = z$. Thus the integral over z is easily solved:

$$\vec{f} = -\frac{1}{2}\overline{p(t)z(t)}\vec{n}dz = \frac{1}{2\rho g}\overline{p(t)^2}\vec{n} = \frac{1}{4\rho g}|\hat{p}|^2\vec{n}. \tag{174}$$

Again the total contribution is attained by summing over all waterline segments:

$$\sum \vec{f} = \frac{1}{4\rho g}\sum |\hat{p}|^2\vec{n}. \tag{175}$$

17.4 Yaw drift moment

For each panel and each waterline segment, the contribution to the yaw drift moment is determined as the z component of

$$\vec{x} \times \vec{f}. \tag{176}$$

Because only the z (yaw) component of the moment is determined, it is sufficient to include in $\vec{f}$ only the horizontal components of the relative-motion drift force.

17.5 Drift force due to fins

The sinusoidal, first-order fin lift F_F (its complex amplitude is given in (92)) acts approximately in the direction of the vector $\vec{n}_F$. To determine its contribution to the second-order drift force, now we take into account that the lift, expressed in the intertial coordinate system, acts in the time-variable direction

$$\vec{n}_F + \mathrm{Re}[(\hat{\alpha}_W + \hat{\alpha}_{S1})e^{i\omega_e t}](1,0,0) + \vec{n}_F \times C_r \mathrm{Re}[(\hat{u}_4, \hat{u}_5, \hat{u}_6)e^{i\omega_e t}] \tag{177}$$

The second-order fin lift is the product of the first-order contribution to the direction times the first-order lift F_F:

$$\left\{\mathrm{Re}[(\hat{\alpha}_W + \hat{\alpha}_{S1})e^{i\omega_e t}](1,0,0) + \vec{n}_F \times C_r \mathrm{Re}[(\hat{u}_4, \hat{u}_5, \hat{u}_6)e^{i\omega_e t}]\right\} \mathrm{Re}[\hat{F}_F e^{i\omega_e t}]. \tag{178}$$

With (163) we obtain the following expression for the time-averaged drift force due to the fin lift:

$$\overline{\vec{F}_F} = \frac{1}{2}\mathrm{Re}[(\hat{\alpha}_W + \hat{\alpha}_{S1})\hat{F}_F^*](1,0,0) + \frac{1}{2}C_r\vec{n}_F \times \mathrm{Re}[(\hat{u}_4, \hat{u}_5, \hat{u}_6)\hat{F}_F^*] \tag{179}$$

Another second-order force due to fins is their induced resistance D. The corresponding resistance coefficient C_D is approximately

$$C_D = \frac{C_L^2}{\pi\Lambda_{eff}} \tag{180}$$

where Λ_{eff} is the effective aspect ratio of the fin. This expression holds exactly only for elliptical distribution of lift over the span, but the change in induced resistance for other lift distributions is small. Further, this expression holds only for stationary flow. To use it here as an approximation, we determine C_L from the fin force without added-mass terms. Thus we obtain:

$$-D = \frac{\frac{1}{2}\rho v^2 cs}{\pi\Lambda_{eff}}\left(\frac{dC_L}{d\alpha}\right)^2\left[\mathrm{Re}(\hat{\alpha}_W + \hat{\alpha}_{S1} + \hat{\alpha}_{S2} + \hat{\alpha}_{S3})e^{i\omega_e t}\right]^2(-1,0,0) \tag{181}$$

The time-averaged longitudinal drift force follows from this expression as

$$-\overline{D} = \frac{b_F v^2}{2\pi\Lambda_{eff}}\frac{dC_L}{d\alpha}\left|\hat{\alpha}_W + \hat{\alpha}_{S1} + \hat{\alpha}_{S2} + \hat{\alpha}_{S3}\right|^2(-1,0,0) \tag{182}$$

The variables occurring in this equation are explained in equations (82) and (85) to (91).

The effective aspect ratio may differ from the geometrical aspect ratio span/chord; for instance, for fins attached to a much larger body the effective aspect ratio is twice the geometrical one. However, in the fin input data the quantity $dC_L/d\alpha$ is given. This quantity depends on the effective aspect ratio. A good approximation is:

$$\frac{dC_l}{d\alpha} = 2\pi\frac{\Lambda_{eff}(\Lambda_{eff}+0.7)}{(\Lambda_{eff}+1.7)^2}. \tag{183}$$

Thus the given value $dC_l/d\alpha$ is used in the program to determine Λ_{eff}. From (183) follows:

$$\Lambda_{eff} = \frac{1}{2\pi - dC_L/d\alpha}\left(1.7dC_L/d\alpha - 0.7\pi \pm \sqrt{0.49\pi^2 + 3.4\pi dC_L/d\alpha}\right) \tag{184}$$

Here only the plus sign is appropriate.

17.6 Particle drift velocity

In this subchapter the coordinate origin is assumed to lie on the average water surface, and the x direction is the direction of wave propagation.

If the drift force on a floating body in waves is zero, the body simply follows the drift motion of the water particles. Thus to estimate the motion of the body, we need to know the water particle drift velocity. It is caused by the dependence of the orbital (fluid particle) velocity on depth: When the water particle is in the upper range of its orbit, it moves in the direction of wave propagation, whereas in the lower range of the orbit it moves in opposite direction. Because the motion is stronger in the upper part of the particle orbit, there is an average motion in the direction of wave propagation.

Up to the leading (second) order with respect to wave amplitude, the drift velocity can be determined from the first-order (Airy) wave potential, given here in shallow water (depth D) for a wave propagating in x direction with celerity c:

$$\phi = \mathrm{Re}\left(\frac{-ic\hat{\zeta}}{\sinh(kD)}\cosh(kz-kD)e^{i(\omega t - kx)}\right) \tag{185}$$

The particle velocity is the gradient of ϕ; thus

$$\left\{\begin{array}{c}\dot{x}\\ \dot{z}\end{array}\right\} = \left\{\begin{array}{c}\phi_x\\ \phi_z\end{array}\right\} = \mathrm{Re}\left(\frac{-ick\hat{\zeta}}{\sinh(kD)}\left\{\begin{array}{c}-i\cosh(kz-kD)\\ \sinh(kz-kD)\end{array}\right\}e^{i(\omega t-kx)}\right) \tag{186}$$

ck may be substituted by ω. For a particle on the orbit having its center at x_0, z_0, division of the complex amplitudes of $\dot{x}, \dot{z}$ by $i\omega$ gives the complex amplitudes of the coordinates x, z of the particle:

$$\left\{\begin{array}{c}x\\ z\end{array}\right\} = \mathrm{Re}\left(\frac{-\hat{\zeta}}{\sinh(kD)}\left\{\begin{array}{c}-i\cosh(kz-kD)\\ \sinh(kz-kD)\end{array}\right\}e^{i(\omega t-kx)}\right) + \left\{\begin{array}{c}x_0\\ z_0\end{array}\right\} \tag{187}$$

It is correct up to first order to substitute x, z by x_0, z_0 on the right-hand side.

To substitute x, z on the right-hand side of (186) by the time-invariant coordinates x_0, z_0, a Taylor expansion (truncated after the linear term) of the right-hand side around x_0, z_0 is made. This yields

$$\dot{x} = \mathrm{Re}\left[\frac{-\omega\hat{\zeta}}{\sinh(kD)}\left(\cosh(kz_0-kD) + k\sinh(kz_0-kD)\mathrm{Re}\left(\frac{-\hat{\zeta}\sinh(kz_0-kD)}{\sinh(kD)}e^{i(\omega t-kx_0)}\right)\right)\right.$$

$$\cdot\left(e^{i(\omega t-kx_0)}+(-ik)e^{i(\omega t-kx_0)}\cdot\mathrm{Re}\left(\frac{i\hat{\zeta}\cosh(kz_0-kD)}{\sinh(kD)}e^{i(\omega t-kx_0)}\right)\right)\Bigg] \tag{188}$$

The embedded Re functions in (188) are substituted (using again the symbol $*$ to designate the complex conjugate):

$$\mathrm{Re}\,\alpha=\frac{1}{2}(\alpha+\alpha^*) \tag{189}$$

E.g. the first of the embedded Re terms thus becomes

$$\mathrm{Re}\left(\frac{-\hat{\zeta}\sinh(kz_0-kD)}{\sinh(kD)}e^{i(\omega t-kx_0)}\right)=\frac{-\hat{\zeta}\sinh(kz_0-kD)}{2\sinh(kD)}e^{i(\omega t-kx_0)}+\frac{-\hat{\zeta}^*\sinh(kz_0-kD)}{2\sinh(kD)}e^{-i(\omega t-kx_0)} \tag{190}$$

Correspondingly the Re expression in the second line of Eq.(188) is the sum of two terms, one containing the factor $e^{i(\omega t-kx_0)}$, the other the factor $e^{-i(\omega t-kx_0)}$.

To derive the time-averaged drift motion velocity $\dot{x}_d$ one simply omits all terms in (188) which contain a factor $e^{i\cdots}$. This results in

$$\dot{x}_d=\frac{k\omega|\hat{\zeta}|^2}{2\sinh^2(kD)}\cosh[2k(z_0-D)] \tag{191}$$

This expression is directly applicable only for deep water. In shallow water the volume transport produced by the waves generates a current. Its size and direction depend on the shape of and the draft distribution within the total water basin, and on the wave field. Because this current cannot be determined here, it is reasonable to determine here only a 'reduced drift velocity'; that is the drift velocity (at a given reference height coordinate z_0) relative to the average (over height) current velocity in x direction. When determining the velocity of a ship over ground, one has to add the speed of the ship through the water, the reduced wave drift velocity of the water, and the current speed.

The reduced drift velocity is thus

$$\dot{x}_{d,r}=\dot{x}_d-\frac{1}{D}\int_0^D\dot{x}_d dz_0 = \frac{k\omega|\hat{\zeta}|^2}{2\sinh^2(kD)}\left(\cosh[2k(z_0-D)]-\frac{1}{D}\int_0^D\cosh[2k(z_0-D)]dz_0\right) \tag{192}$$

$$= \frac{k\omega|\hat{\zeta}|^2}{2\sinh^2(kD)}\left(\cosh[2k(z_0-D)]-\frac{1}{kD}\sinh(2kD)\right) \tag{193}$$

$$= \frac{1}{2}k\omega|\hat{\zeta}|^2\left(\frac{\cosh[2k(z0-D)]}{\sinh^2(kD)}-\frac{1}{kD\tanh(kD)}\right) \tag{194}$$

For large values of kD, the first term within parentheses cannot be determined directly because numerator and denominator may be too large; in that case the (practically) correct result is found by using in the first term of (194) a value of 30 for kD.

18 References

BERTRAM, V. (2000), *Practical Ship Hydrodynamics*, Butterworth+Heinemann, Oxford

BLUME, P. (1976), *Zur Frage der erregenden Längskraft in von achtern kommenden regelmässigen Wellen*, IfS report 334, Univ. Hamburg

BOESE, P. (1970), *Eine einfache Methode zur Berechnung der Widerstandserhöhung eines Schiffes im Seegang*, Schiffstechnik 17, pp.29-32

BOHLMANN, H.-J. (1990), *Berechnung hydrodynamischer Koeffizienten von Ubooten zur Vorhersage des Bewegungsverhaltens*, Bericht 513 des Instituts fuer Schiffbau der Universitaet Hamburg

HACHMANN, D. (1991), *Calculation of pressures on a ship's hull in waves*, Ship Technology Research 38, pp.111-133

KORVIN-KROUKOWSKI, B.V.; JACOBS, W.R. (1957), *Pitching and heaving motions of a ship in regular waves*, SNAME Transactions

SÖDING, H. (1969), *Eine Modifikation der Streifenmethode*, Schiffstechnik 16, pp.15-18

SÖDING, H. (1976), *Second-order forces on oscillating cylinders in waves*, Schiffstechnik 23, pp.205-209

SÖDING, H. (1988), *Berechnung der Bewegungen und Belastungen von SWATH-Schiffen und Katamaranen im Seegang (SEDOS)*, Report 483, Institut für Schiffbau, Hamburg

SÖDING, H. (1990), *Die Wirkung von Rolldämpfungsflossen abhängig von ihrem Einbauort*, internal report (to be obtained on request)

SÖDING, H. (1993), *A method for accurate force calculation in potential flow*, Ship Technology Research 40, pp.176-186

SÖDING, H.; BLUME, P. (1994), *Experimentelle Prüfung von Berechnungsmethoden für die Bewegungen von Katamaranen im Seegang*, Schrift Nr. 2412, Institut für Schiffbau, Hamburg

SUTULO, S.; GUEDES SOARES, C. (2004), *A boundary integral equations method for computing inertial and damping characteristics of arbitrary contours in deep fluid*, Ship Technology Research 51/2, pp.69-93

19 Files

The source text of the program is contained in the file `pdstrip.f90`.

The input data are read from the file `pdstrip.inp`. The input contains the name of another file containing the section offsets. Results are written into the output file `pdstrip.out`.

A computation for a new hull begins with determining the hydrodynamic data of the sections for various frequencies and wave encounter angles. These results are stored in a file `sectionresults`. If another computation is made using the same section data, the stored data may be used to reduce the CPU time.

Similarly, during the computation of transfer functions the files `responsefunctions`, `relativemotions` and `pressuretransfer` are generated. They can be used again if significant amplitudes or the mean added resistance is to be determined for the same loading case but different natural seaways.

A few scratch data files will also be generated by the program.

20 Coordinate systems used for input and output

Because naval architects and experts for ship motions use, traditionally, different coordinate directions, the program uses different coordinate systems for input and results: The input is specified in a coordinate system where the x axis is pointing forward, y to port, and z upward. Results, on the other hand, are expressed in coordinate directions designated by the numbers 1 for forward, 2 for starboard, and 3 for downward. This system was used here also in the description of the theory. Translations and forces are positive if they have these directions, while angles and moments are positive if they turn right-hand around these directions.

The coordinate origin K of the input and the output systems are the same. It is recommended to place K at the keel at midship section. The reason is: Surge, heave and sway motions computed by the program are those of the point K, and traditionally ship heave and sway motions are held to refer to the midship section.

Complex amplitudes $\hat{r}$ of all responses r are to be interpreted according to the formula

$$r(t) = \mathrm{Re}\,\left(\hat{r}e^{i\omega_e t}\right) \tag{195}$$

ω_e is the circular frequency of wave encounter. Thus, the real part of the complex amplitude is equal to the time function at the instant in which a wave trough is located vertically above or below point K, whereas the imaginary part is the time function one quarter of the encounter period earlier (for the more frequent case of positive encounter frequency) or later (for negative ω_e).

Results in natural seaways are, predominantly, significant amplitudes. These are defined as the average of the one third largest amplitudes, while the two thirds smaller amplitudes are disregarded. For the assumed Rayleigh distribution of amplitudes, this means that significant amplitudes are twice the standard deviation of the respective response from its time average.

21 Input data

All input data are read using the Fortran * format. Thus data values should be given as integer numbers for integer variables, or as real numbers with decimal point and possibly a power of 10 factor like in 1.0e6 (= one million). Complex input should be given as, e.g., (1.5, -0.5) to designate $1.5 - 0.5i$. Text input should be enclosed by apostrophs or quotes; these may only be omitted if the text does not involve special characters, especially blanks.

Data items are to be separated by at least one blank character. 3*0 ist the same as 0 0 0. In this description ==== means that the next input has to appear on a new line. The input 0/ means that only one item is specified as 0 whereas the following data items until ==== remain unspecified.

Physical quantities may be given in any system of units. However, output formats may be inappropriate if the numbers are too small or too large. The output formats have been designed for the units meter, second and (metric) ton. If other units are used, all quantities must use the same units, or units like tm^2 derived from the 3 base units by multiplication and powers. Thus if lengths are given in m and frequencies in Hertz, the speed must be given in m/s.

Results are output in the same units as those used for the input data, or units derived from these by multiplication and powers.

21.1 Input data in the file `pdstrip.inp`

n_p	$\pm$ number of points per section where the wave pressure should be determined. Use a negative number to suppress the output of pressure transfer functions in the list of results. 0 if no pressure and no drift forces or velocities should be determined. If $n_p \neq 0$ the ship's body must be specified as asymmetrical, and offsets on both sides of the sections must be given. Drift forces and water drift velocities are determined only if $n_p / = 0$. $n_p = \pm 1$ and ± 2 is inappropriate, and $\lvert n_p \rvert < 15$ results in inaccurate pressures and drift forces.
L_1	t, if added mass, damping and wave excitation of ship sections should be determined and stored in file `sectionresults`; f, if that may be skipped because the file has been generated in a previous run using the same section offsets.
L_2	t, if transfer functions should be determined and stored in file `responsefunctions`; f if this should be skipped because the file was generated already for the same data, or because only the file `sectionresults` should be generated.
L_3	t, if results in natural seaways should be determined; f otherwise.
====	
$text$	text of ≤ 80 characters to characterize the case
====	
g	gravity acceleration
ρ	water density
z_{wl}	z coordinate of the waterline (i.e. submergence of point K)
z_{bot}	z coordinate of the water bottom. Use –1e6 to indicate deep water.
z_0	z coordinate for which the water particle drift velocity is to be determined. Without influence if $n_p = 0$, because the drift velocity is determined only if $n_p > 0$. Recommended: Half hull draft if longitudinal speed is of main interest; z where transverse underwater force acts if transverse drift is of main interest.
====	
n_μ	number of encounter angles
$\mu(1 : n_\mu)$	encounter angles in degree (n_μ values), monotonically increasing. Use 0 for stern waves, 90 for waves from starboard, –90 for waves from port side. Use only values between –90 and +90. Transfer functions will be computed not only for these angles μ, but also for the angles $180° - \mu$, i.e. waves from forward directions.
====	
$filename$	name of the section offset file (see next chapter)
====	
	Only the above data are required if only part 1 of the program, i.e. determination and storage of section hydrodynamic data, should be run. Otherwise, i.e. if $L_2 = t$:
L_s	t, if section forces and moments at transverse sections through the ship's body should be determined; f otherwise. The position of these sections is in the middle between the given offset sections.
====	
m	mass of ship
x_G, y_G, z_G	coordinates of the mass center of gravity. Because the roll motion is sensitive to metacentric height, and because the metacentric height determined by this program may deviate from the exact value because of discretization errors, one should check the transverse metacentric height GM indicated in the output list and change z_G if necessary so that GM becomes correct.

r_x^2	square of the inertial radius relating to the axis through the center of gravity parallel to the x axis. The “roll moment of inertia” without added-mass effects around this axis is this value times m. r_x^2 may be computed as the mass-weighted average of $(y-y_G)^2+(z-z_G)^2$.
r_y^2	square of the inertial radius relating to the axis through the center of gravity parallel to the y axis; average of $(x-x_G)^2+(z-z_G)^2$.
r_z^2	square of the inertial radius relating to the axis through the center of gravity parallel to the z axis; average of $(x-x_G)^2+(y-y_G)^2$.
$\overline{(x-x_G)(y-y_G)}$	mass-weighted average of $(x-x_G)(y-y_G)$; zero for a mass distribution being symmetrical to the plane $y=0$. Used to determine the mixed moment of inertia $I_{xy}=m\overline{(x-x_G)(y-y_G)}$
$\overline{(y-y_G)(z-z_G)}$	corresponding value for I_{yz}
$\overline{(x-x_G)(z-z_G)}$	corresponding value for I_{xz}. Normally $\neq 0$ even for a symmetrical ship. Important for the coupling between yaw and roll motions.
====	
	If $L_s=t$ (intersection forces and moments to be determined), then the values m until $\overline{(x-x_G)(z-z_G)}$ have to be repeated for the part of the ship before each of these intersections, i.e. before the middle between the first and second offset section, second and third offset section, and so on. If $L_s=0$, only the data for the total ship are read.
====	
isep	“separation numbers” for each offset section. If n_s is the number of sections, n_s values are to be given which are either 0 or 1: 1 if the steady flow due to the ship’s forward speed separates between the respective section and the next (farther forward) one at least at a part of the section contour. 0 otherweise. Flow separation should be specified especially where the cross sections change discontinuously in draft, e.g. at the end of a skeg. In such a case there should be two offset sections arranged with a small distance; the aft one without skeg, the forward one with skeg, and flow separation should be specified at the end of the former (positioned aft) of these sections.
====	
steep	wave steepness, i.e. height (=double wave amplitude) over wave length.
maxh	maximum wave height
	These two data are used to determine the actual wave height $=\min(maxh, steep\cdot\lambda)$. This wave height is used to determine certain nonlinear effects on the transfer functions: resistive forces and moments on the body cross sections; nonlinear forces on fins; and the possibility of surf-riding. To compute significant amplitudes in a wind sea, *steep* $=0.1$ may be recommended.
===	
	Only if *steep* $\neq 0$, there follows a list of resistance coefficients:
C_y, C_z	resistance coefficients for horizontal and vertical flow, respectively; repeated for all offset sections. For sections having a well-rounded base, $C_y=0.8$ is recommended, whereas for sections being sharp at their base $C_y=1.0$ to 1.2 appears appropriate. For vertical flow $C_z=0.6$ is recommended.
====	
n_F	number of fins
====	
	If $n_F>0$, for each fin the following data have to be specified:
x_F, y_F, z_F	coordinates of the fin center

a_x, a_y, a_z	3 components of a vector specifying the fin's longitudinal axis (in span direction) ; for steered fins it specifies the axis of rotation. The length of the vector is arbitrary, the sign is important only for steered fins.
$\hat{\delta}_1, ..., \hat{\delta}_6$	$\hat{\delta}_1$ is the ratio between the complex amplitude of the steering angle in rad and that of the surge motion. $\hat{\delta}_2$ to $\hat{\delta}_6$ give corresponding ratios for the sway, heave, roll (in rad), pitch and yaw motions, respectively. If several of these ratios are non-zero, the actual angle is the sum of the contributions from the respective ship motions. Use 6*(0.,0.) to specify a fixed fin. The imaginary part of δ_i can produce a damping of motion i, while the real part may produce a restoring force or moment. Positive angles are right-hand rotated around the vector a_x, a_y, a_z
s	span = length of the fin (measured normal to the x axis)
c	chord = profile length of the fin (measured in x direction; average over fin length
C_M	added mass coefficient; for details see the report about the theory of the program
$dC_L/d\alpha$	gradient of lift coefficient; for details see the report about the theory of the program
C_r	factor for fin force change due to interaction with the hull; for details see the report about the theory of the program
C_δ	factor for change of inflow angle due to interaction with the hull; for details see the report about the theory of the program
C_d	cross-flow resistance coefficient of the fin; recommended values: 1 for rudder, 5 for bilge keels
====	
n_{sail}	number of sails
====	
	Only if $n_{sail} > 0$, the following data until c_M have to be specified:
u_w	'true' wind speed (relative to an earth-fixed point; not relative to ship)
μ_w	wind direction relative to the wave propagation direction. 0 if wind and waves propagate in the same direction; positive if a rotation from wave to wind direction points upward according to the right-hand rule.
====	
x_s, y_s, z_s	coordinates of the point of attack of the sail force (see chapter Sail forces for details of these and the following quantities)
c_x, c_y, c_z	components of the 'chord vector' (from front to aft end of sail if the apparent wind blows from forward directions; length = average chord length of the sail)
m_x, m_y, m_z	components of the 'mast vector' (from lower to upper end of the sail; length = sail extent in mast vector direction)
$\partial C_N/\partial\alpha$	change of the normal force coefficient with angle of attack
c_M	mass coefficient
====	
	The data from x_S to c_M must be repeated for n_{sail} sails.
n_{fo}	number of external forces depending on ship motions
====	
	If $n_{fo} > 0$, for each force the following data have to be specified. A new line is required (and comments allowed) only behind the data for the last force.
x_{fo}, y_{fo}, z_{fo}	coordinates of the point where the force acts on the ship
$a_{x,fo}, a_{y,fo}, a_{z,fo}$	vector giving the direction of the force. Vector length arbitrary
$\hat{f}_1, ..., \hat{f}_6$	6 complex numbers giving the ratio between the complex amplitudes of the force and the surge, sway, heave, roll, pitch and yaw motion, respectively
$\hat{f}_w$	ratio between complex amplitudes of the force and the transverse shift of a suspended weight from its average position
====	

m_w mass of a suspended weight. Give 5*0 for this and the following data to indicate that no suspended weight is present.

x_w, y_w, z_w coordinates of the suspension point (top of weight cable). y_w is read, but it does not influence the results.

l_w length of weight cable between suspension point and center of gravity of weight

====

n_m number of points where motions should be determined

====

If $n_m > 0$ for each of these points the following data must be specified:

x_m, y_m, z_m coordinates of the motion point

====

n_λ number of wave lengths for which transfer functions should be determined (≤ 200)

$\lambda_1, ..., \lambda_{n_\lambda}$ list of wave lengths

If significant amplitudes will be determined using these transfer functions, the number and the range of wave lengths should be sufficient to define the transfer functions accurately, especially near their peaks. Various RAOs have a sharp peak at the wave length for which the encounter frequency ω_e is near to the roll eigenfrequency. Others have peaks where ω_e is zero. These peaks must be resolved sufficiently to obtain accurate results for the significant amplitudes. Further one should keep in mind that, contrary to statements in the relevant literature, the results of the strip theory become inaccurate for wavelenghs below, say, 40% of the vessel length, whereas they are accurate – again contrary to literature statements – in long waves.

====

n_v number of different forward speeds

The following two data are repeated for each speed:

v average forward speed of the ship

$L_{transom}$ F if the transom is free from water due to forward speed; otherwise T.

====

No further data are required if $L3 = f$, i.e. if no significant amplitudes are to be determined. Otherwise:

n_{av} number of frequencies to specify the valuation function for accelerations

f_{av} frequency (not circular frequency) to define the valuation function

C_{av} valuation function for accelerations at frequency f_{av}

The data f_{av} and C_{av} must be repeated n_{av} times to specify the valuation function. Between the frequencies f_{av} the valuation function is interpolated linearly.

If no accelerations are to be determined, or if no frequency-dependent valuation of frequencies is required, use the data 2 0 1 100 1.

====

An arbitrary number of seaways may be specified. They characterize wind sea, not swell, by the following data:

$H_{1/3}$ significant height of the seaway

T_1 significant period of the seaway corresponding to the frequency where the area under the wave power spectrum has its centre of gravity. Expressed by the spectum moments m_i: $\omega_1 = m_1/m_0$ and $T_1 = 2\pi/\omega_1$. This period is significantly shorter than the peak frequency of the seaway.

μ_0 main direction of wave propagation (0 for stern sea, 90 for sea from starboard)

exp exponent used in the expression specifying the distribution of wave energy over directions: $\cos^{exp}(\mu - \mu_0)$. Recommended are values in the range 2 ... 4 for natural wind sea, or 1000 for long-crested irregular waves.

γ peak enhancement factor; 1 gives Pierson-Moskowith spectra, other values for Jonswap spectra. A typical value in wind seas is $\gamma = 3.3$.

====

To avoid an error stop due to missing further input data one may indicate the end of data by the input 0/.

21.2 Input data contained in the section offset file

The name of that file is given in the file `pdstrip.inp`.

The hull shape is described by $n_s \leq 100$ offset sections. The first of them is located at the transom or, if there is no immersed transom, at the aft end of the underwater hull; the last is at the bow. To obtain correct drift forces, sections of very small but positive area (e.g. 1 square centimeter) should be arranged at the bow, and at the stern if there is no immersed transom. Each section is described by $n_k \leq 100$ offset points.

For typical ship shapes, about 30 to 40 offset sections and about 10 offset points per half section are appropriate.

n_s	number of sections
sym	.true. if all ship sections are symmetrical to the plane $y = 0$ and only one half of the sections is described by offset points, beginning at the base ($y = 0$) and continuing up to the waterline on port side. Otherwise (for unsymmetrical or heeled sections, or for symmetrical sections both sides of which are described): .false. In the latter case the section is described by offset points beginning at the waterline on starboard side and ending at the waterline on port side. – The wave-induced pressure distribution and the wave drift force (including the added resistance) can be determined only if sym =.false.
T	reference draft. It is recommended to use the draft of the ship's body excluding fins
====	
	The following data until z_{n_k} must be repeated for each section (n_s times):
x_s	x coordinate of the section
n_k	number of offset points of the section
n_{gap}	number of gaps in the section contour; 0 for simply connected sections.
$gap(1 : n_{gap})$	index of contour points where a gap begins (only if $n_{gap} > 0$) For suitable numbers of contour points and their arrangement on the section compare chapter 3.4.
====	
$y_1 ... y_{n_k}$	list of y coordinates of all offset points of the section
$z_1 ... z_{n_k}$	list of z coordinates of all offset points of the section