Problems in exact sequences of group homomorphisms. After §11.7 in Shahriar Shariari's *Algebra in Action.*

Preliminaries. **We say that the *sequence* of group (or ring, module, ...) homomorphisms**

$$\dots \to G_{i-1} \xrightarrow{\phi_{i-1}} G_i \xrightarrow{\phi_i} G_{i+1} \to \dots$$

is *exact* iff $\mathrm{im}\phi_{i-1} = \ker \phi_i$ **for all** i**.**

Also note that $\phi : G \to H$ **is...**

- surjective, iff $\mathrm{im}\phi = H$, iff $G \xrightarrow{\phi} H \xrightarrow{\mapsto e} \{e\}$ is exact. (Specifically, having the groups $G, H, \{e\}$ implies that the resulting sequence is exact. This does *not* the converse – that having $G \xrightarrow{\phi} H \xrightarrow{\psi} K$ exact gives $K = \{e_K\}$; just that $\ker\psi = \mathrm{im}\phi = H$, and so $\mathrm{im}\psi = \{e_K\}$.)
- injective, iff $\ker\phi = \{e\}$, iff $\{e\} \hookrightarrow G \xrightarrow{\phi} F$ is exact.

11.7.1 **Assume that** $V \xrightarrow{f} W \xrightarrow{g} U \xrightarrow{h} X$ **is exact. If** f **is surjective, is** h **necessarily injective?**

Yes. Since f is surjective, then

$$\begin{aligned} & \mathrm{im}\, f = W = \ker g && \text{(exactness)} \\ \Longrightarrow\ & \mathrm{im}\, g = \{e_U\} && (\ker g = W) \\ \Longrightarrow\ & \mathrm{im}\, g = \{e_U\} = \ker h && \text{(exactness)} \end{aligned}$$

and so h is injective.

11.7.2 **Assume that** $V \overset{f}{\twoheadrightarrow} W \xrightarrow{k} U \xrightarrow{h} X \overset{g}{\hookrightarrow} Y$ **is exact. Show how** $U = \{e\}$**.**

From f surjective, we deduce $\mathrm{im}k = \{e_U\}$:

$$\begin{aligned} \mathrm{im}\, f = W \ \Longrightarrow\ & \mathrm{im}\, f = W = \ker k && \text{(exactness)} \\ \Longrightarrow\ & \mathrm{im}\, k = \{e_U\} && (\ker k = W.) \end{aligned}$$

Simultaneously, from g injective we deduce $\mathrm{im}k = U$:

$$\begin{aligned} \ker g = \{e_x\} \ \Longrightarrow\ & \ker g = \{e_X\} = \mathrm{im}\, h && \text{(exactness)} \\ \Longrightarrow\ & \ker h = U && (\mathrm{im}\, h = \{e_X\}) \\ \Longrightarrow\ & \ker h = U = \mathrm{im}\, k && \text{(exactness.)} \end{aligned}$$

We thus conclude $U = \mathrm{im}k = \{e_U\}$

11.7.3 **Let** G **and** H **be groups, and let** $\phi : G \to H$ **be a group homomorphism. Check that the following is an exact sequence:**

$$\begin{array}{ccccccccc} \{e\} & \longrightarrow & \ker\phi & \hookrightarrow & G & \twoheadrightarrow & \phi(G) & \longmapsto & \{e\} \\ \\ e & \longmapsto & e & & g & \longmapsto & \phi(g) & & \\ \\ & & g & \longmapsto & g & & h & \longmapsto & e \end{array}$$

We compose the two exact sequences

- $\{e\} \to \ker\phi \hookrightarrow G$, which is exact because the inclusion map $\ker\phi \hookrightarrow G$ is an injection;
- $G \twoheadrightarrow \mathrm{im}\phi \to \{e\}$, which is exact because $\phi|_{\mathrm{im}\phi}$ is a surjection.

It remains to show that the sequence is exact at G. Here, $\ker\phi = \ker\phi \implies \mathrm{im}(i_{\ker\phi}) = \ker\phi$, as required.

11.7.4 **Let G, K, F be groups, and assume that the following sequence is exact:**

$$\{e\} \xrightarrow{\gamma} K \xrightarrow{\alpha} G \xrightarrow{\beta} F \xrightarrow{\delta} \{e\}$$

Show that $\ker\beta$ satisfies

(a) $K \cong \ker\beta$**, (b)** $G/\ker\beta \cong F$**.**

Immediately observe that, since the above sequence is exact, that α is injective and β is surjective.

As a moral exercise, we first transcribe all of these statements – include α injective, β surjective – into a Shahriari-style homomorphism diagram:

$$\begin{tikzcd}
 & G \arrow[r, "\beta"] \arrow[d, dash] & F \arrow[d, dash] \\
K \arrow[r, "\alpha"] \arrow[d, dash] & \ker\beta \arrow[r, "\beta"] \arrow[d, dash] & e \\
\{e\} \arrow[r, dash, "\alpha"] & \{e\} &
\end{tikzcd}$$

Note how exactness is encoded by the "stepped" diagram: $\alpha(K) = \ker\beta$.

For (a), we claim that $\alpha|_{\ker\beta}$ gives a bijection $K \to \ker\beta$. This map is injective, since α is injective, and restricting $\alpha|_{\ker\beta}$ does not change this. This map is also surjective, since exactness gives $\mathrm{im}\alpha = \ker\beta$, and $\alpha|_{\mathrm{im}\alpha} = \alpha|_{\ker\beta}$ is clearly surjective.

For (b), note that β is surjective and so $\mathrm{im}\beta = F$. The first homomorphism theorem then gives $G/\ker\beta \cong \mathrm{im}f \iff G/\ker\beta \cong F$.

11.7.5 **(Injectivity in the short five lemma; after Dummit and Foote.) Suppose that the following diagram commutes and has exact rows:**

$$\begin{tikzcd}
\{e\} \arrow[r] & A \arrow[r, "f"] \arrow[d, "\alpha"] & B \arrow[r, "g"] \arrow[d, "\beta"] & C \arrow[r] \arrow[d, "\gamma"] & \{e\} \\
\{e\} \arrow[r] & A' \arrow[r, "f'"'] & B' \arrow[r, "g'"'] & C' \arrow[r] & \{e\}
\end{tikzcd}$$

Show that α and γ injective imply β injective.

This is a diagram chase. Our strategy is to take a $b \in \ker\beta$, and show that $b = e$; hence, $\ker\beta = \{e\}$, and so β is injective.

Begin by observing that by exactness, f, f' are injective, and g, g' are surjective. All of the assumptions put together (and a few objects we later introduce) are put together in the following diagram:

$$\begin{tikzcd}
\{e\} \arrow[r] & a \in A \arrow[r, hook, "f"] \arrow[d, hook, "\alpha"] & b \in B \arrow[r, two heads, "g"] \arrow[d, "\beta"] & C \arrow[r] \arrow[d, hook, "\gamma"] & \{e\} \\
\{e\} \arrow[r] & A' \arrow[r, hook, "f'"'] & B' \arrow[r, two heads, "g'"'] & C' \arrow[r] & \{e\}
\end{tikzcd}$$

The chase begins.

Claim: $b \in \ker g$.

- $\beta(b) = e$,
- $g'(\beta(b)) = e$ (since g' a homomorphism)
- $\gamma(g(b)) = g'(\beta(b)) = e$ (by the diagram)
- $g(b) = e$ (by taking the left inverse of the injection γ.)

Since the rows in the above diagram are exact, $b \in \ker g = \operatorname{im} f$. So let $a \in A$ satisfy $f(a) = b$, $\beta(f(a)) = e$.

Claim: $a = e$.

- $\beta(f(a)) = e$,
- $f'(\alpha(a)) = \beta(f(a)) = e$ (by the diagram)
- $a = e$ (taking the left inverse of the composition of injections $f' \circ \alpha$.)

Hence, $f(a) = b \implies f(e) = b \implies e = b$, and we are done.

(Surjectivity in the short five lemma.) Similarly, show that α and γ surjective imply β surjective.

The following corresponding diagram will keep track of the diagram chase to come:

$$\begin{tikzcd}
\{e\} \arrow[r] & a \in A \arrow[r, hook, "f"] \arrow[d, two heads, "\alpha"] & b \in B \arrow[r, two heads, "g"] \arrow[d, "\beta"] & C \arrow[r] \arrow[d, two heads, "\gamma"] & \{e\} \\
\{e\} \arrow[r] & A' \arrow[r, hook, "f'"] & b' \in B' \arrow[r, two heads, "g'"] & C' \arrow[r, two heads] & \{e\}
\end{tikzcd}$$

Take $b' \in B'$. Through surjectivity, we will find $a \in A$ and $b \in B$ such that $b' = \beta(f(a)b)$, giving us $b \in \operatorname{im} \beta$.

Begin by considering $g'(b')$. Then

$$\begin{aligned}
g'(b') &= \gamma(c) && \text{(for some } c \in C \text{, since } \gamma \text{ surj.)} \\
&= g(\gamma(b)) && \text{(f.s. } b \in B \text{, since } g \text{ surj.)} \\
&= g'(\beta(b)) && \text{(by the diagram)} \\
\implies g'(b')\left[g'(\beta(b))\right]^{-1} &= e_{C'}. && \text{(recalling that these are group homomorphisms)} \\
g'(b'\,[\beta(b)]^{-1}) &=
\end{aligned}$$

So $b'\,[\beta(b)]^{-1} \in \ker g' = \operatorname{im} f'$: we finally use exactness, allowing us to move left in the diagram, and closer to B, β's domain.

$$\begin{aligned}
b'\,[\beta(b)]^{-1} &= f'(a') && \text{(f.s. } a' \in A' \text{, since the left hand side is in } \operatorname{im} f\text{)} \\
&= f'(\alpha(a)) && \text{(f.s. } a \in A \text{, since } \alpha \text{surj.)} \\
&= \beta(f(a)) && \text{(by the diagram)} \\
\implies b' &= \beta(f(a))\beta(b) && \text{(working again in groups)} \\
&= \beta(f(a)b).
\end{aligned}$$

So $b' = \beta(f(a)b)$, and indeed $b' \in \operatorname{im}\beta$.

11.7.6 **What are the maps that make the following sequence exact?**

$$\{I_{2\times 2}\} \to \mathrm{SL}(2,\mathbb{R}) \to \mathrm{GL}(2,\mathbb{R}) \to \mathbb{R}^{\times} \to \{1\}$$

Begin by observing that there is only one possible group homomorphism $\{I_{2\times 2}\} \to \mathrm{SL}(2,\mathbb{R})$: the injection map. Exactness at each group in the sequence allows us to deduce subsequent maps. We find the following:

$$\begin{tikzcd}
\{I_{2\times 2}\} \arrow[r] & \mathrm{SL}(2,\mathbb{R}) \arrow[r] & \mathrm{GL}(2,\mathbb{R}) \arrow[r] & \mathbb{R}^{\times} \arrow[r] & \{1\} \\
{} \arrow[r, hook, "i_{I_{2\times 2}}"] & {} \arrow[r, hook, "i_{\mathrm{SL}(2,\mathbb{R})}"] & {} \arrow[r, mapsto, "\det"] & {} \arrow[r, mapsto] & 1
\end{tikzcd}$$

All of these maps are group homomorphisms. Further check that the sequence is exact.