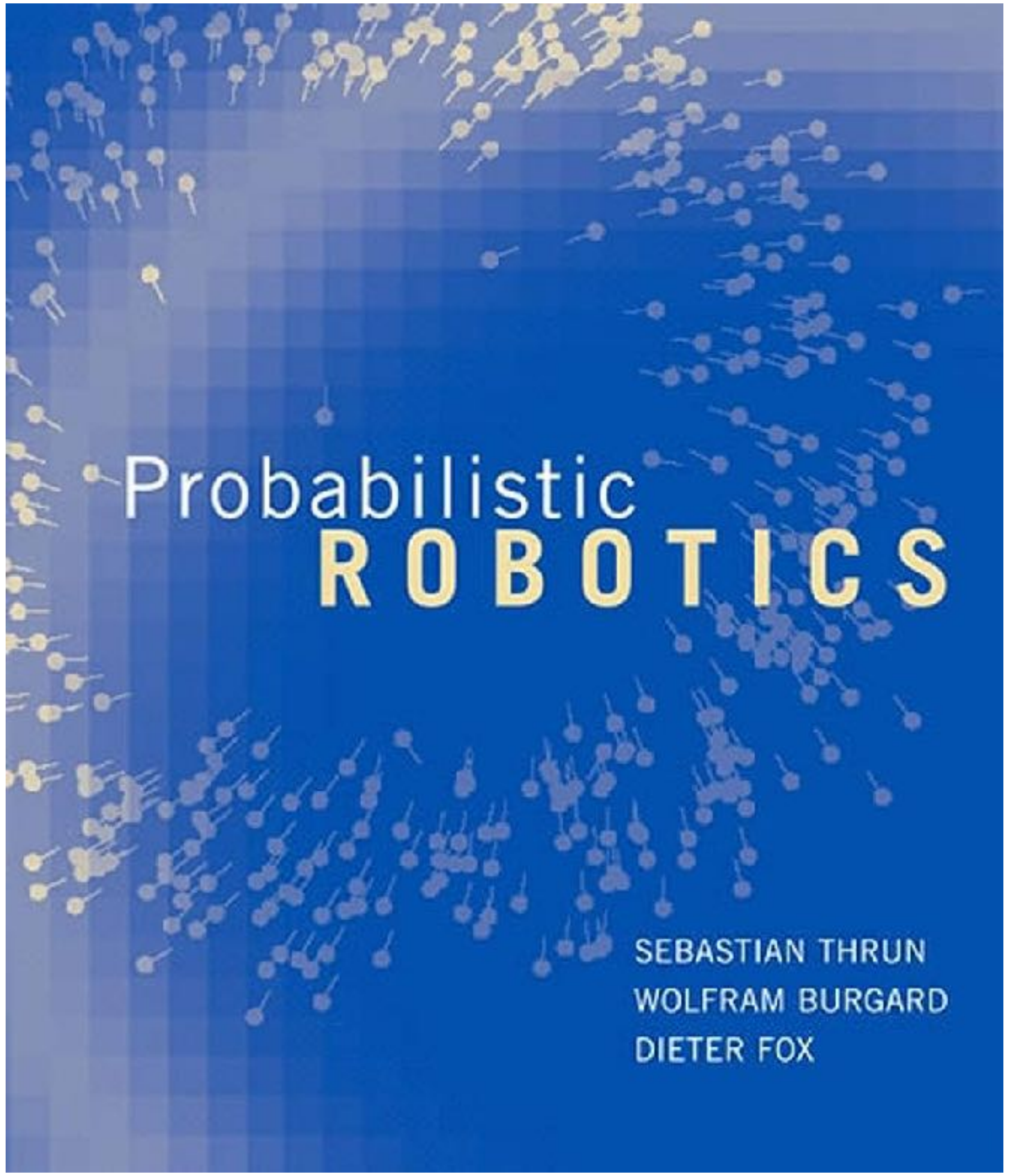
Lecture 17
Mobile Robotics - I - Probability



Course logistics

- **Quiz 14-15 was posted yesterday and was due before the lecture.**
- Quizzes will be posted on Tuesday every week from now.
- Project 4 is posted on 10/30 and will be due 11/13
  - Start early!
- Details on the Final Project will be announced later next week.
- **No lectures on 11/06. No OH on 11/06.**
- Karthik will do OH on 11/08 between 0830-1000 instead.
- Chahyon's OH for next week will be on Zoom.

Probabilistic Robotics

Key idea: Explicit representation of uncertainty

(using the calculus of probability theory)

- Perception = state estimation
- Action = utility optimization

Discrete Random Variables

- X denotes a random variable.
- X can take on a countable number of values in $\{x_1, x_2, \ldots, x_n\}$.
- $P(X = x_i)$, or $P(x_i)$, is the probability that the random variable X takes on value x_i.
- $P(\,.\,)$ is called probability mass function.
- E.g. $P(\text{room}) = < 0.7, 0.2, 0.08, 0.02 >$

Joint and Conditional Probability

- $P(X = x \text{ and } Y = y) = P(x, y)$

- $P(x \mid y)$ is the probability of x given y

$$P(x \mid y) = \frac{P(x, y)}{P(y)}$$

$$P(x, y) = P(x \mid y)P(y)$$

- If X and Y are independent then

$$P(x, y) = P(x)P(y)$$

- If X and Y are independent then

$$P(x \mid y) = P(x)$$

Law of Total Probability, Marginals

Discrete Case

$$\sum_x P(x) = 1$$

$$P(x) = \sum_y P(x, y)$$

$$P(x) = \sum_y P(x \mid y) P(y)$$

Continuous Case

$$\int p(x)dx = 1$$

$$p(x) = \int p(x, y)dy$$

$$p(x) = \int p(x \mid y)p(y)dy$$

Events

- P(+x, +y) ?
- P(+x) ?
- P(-y OR +x) ?
- Independent?

$$P(X, Y)$$

X	Y	P
+x	+y	0.2
+x	-y	0.3
-x	+y	0.4
-x	-y	0.1

Marginal Distributions

$$P(X, Y)$$

X	Y	P
+x	+y	0.2
+x	-y	0.3
-x	+y	0.4
-x	-y	0.1

$$P(x) = \sum_y P(x, y)$$

$$P(X)$$

X	P
+x	
-x	

$$P(y) = \sum_x P(x, y)$$

$$P(Y)$$

Y	P
+y	
-y	

Conditional Probabilities

$$P(X, Y)$$

X	Y	P
+x	+y	0.2
+x	-y	0.3
-x	+y	0.4
-x	-y	0.1

- P(+x | +y) ?
- P(-x | +y) ?
- P(-y | +x) ?

Bayes Formula

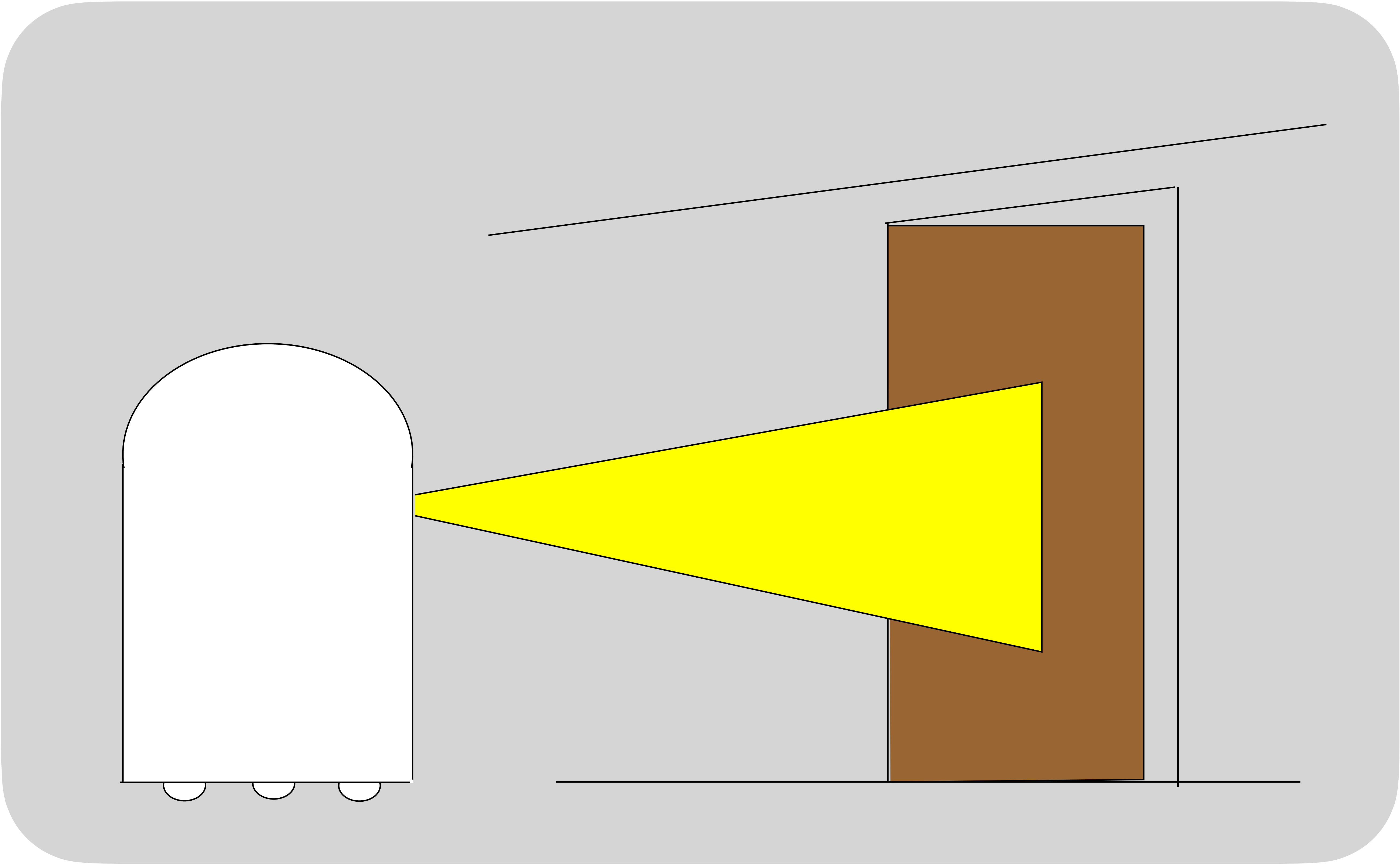
$$P(x,y) = P(x \mid y)P(y) = P(y \mid x)P(x)$$

$$\boxed{P(x \mid y) = \frac{P(y \mid x)P(x)}{P(y)} = \frac{\text{likelihood} \times \text{prior}}{\text{evidence}}}$$

- Often causal knowledge is easier to obtain than diagnostic knowledge.
- Bayes rule allows us to use causal knowledge.

Simple Example of State Estimation

- Suppose a robot obtains measurement z
- What is $P(open \mid z)$?



Example

$$P(z \mid \text{open}) = 0.6 \qquad P(z \mid \neg\text{open}) = 0.3$$

$$P(\text{open}) = 0.5 \qquad P(\neg\text{open}) = 0.5$$

$$P(\text{open} \mid z) =$$

$$P(\text{open} \mid z) = \frac{0.6 \times 0.5}{0.6 \times 0.5 + 0.3 \times 0.5} = \frac{2}{3} = 0.67$$

- z raises the probability that the door is open.

Normalization

$$P(x \mid y) = \frac{P(y \mid x)P(x)}{P(y)} = \eta P(y \mid x)P(x)$$

$$\eta = P(y)^{-1} = \frac{1}{\sum_{x'} P(y \mid x')P(x')}$$

Conditioning

- Bayes rule and background knowledge:

$$P(x \mid y, z) = \frac{P(y \mid x, z) P(x \mid z)}{P(y \mid z)}$$

$$P(x \mid y) \stackrel{?}{=} \int P(x \mid y, z) P(z) dz$$

$$\stackrel{?}{=} \int P(x \mid y, z) P(z \mid y) dz$$

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$$P(x \mid y) = \int P(x \mid y, z) P(z \mid y) dz$$

Conditional Independence

$$P(x, y \mid z) = P(x \mid z) P(y \mid z)$$

- Equivalent to

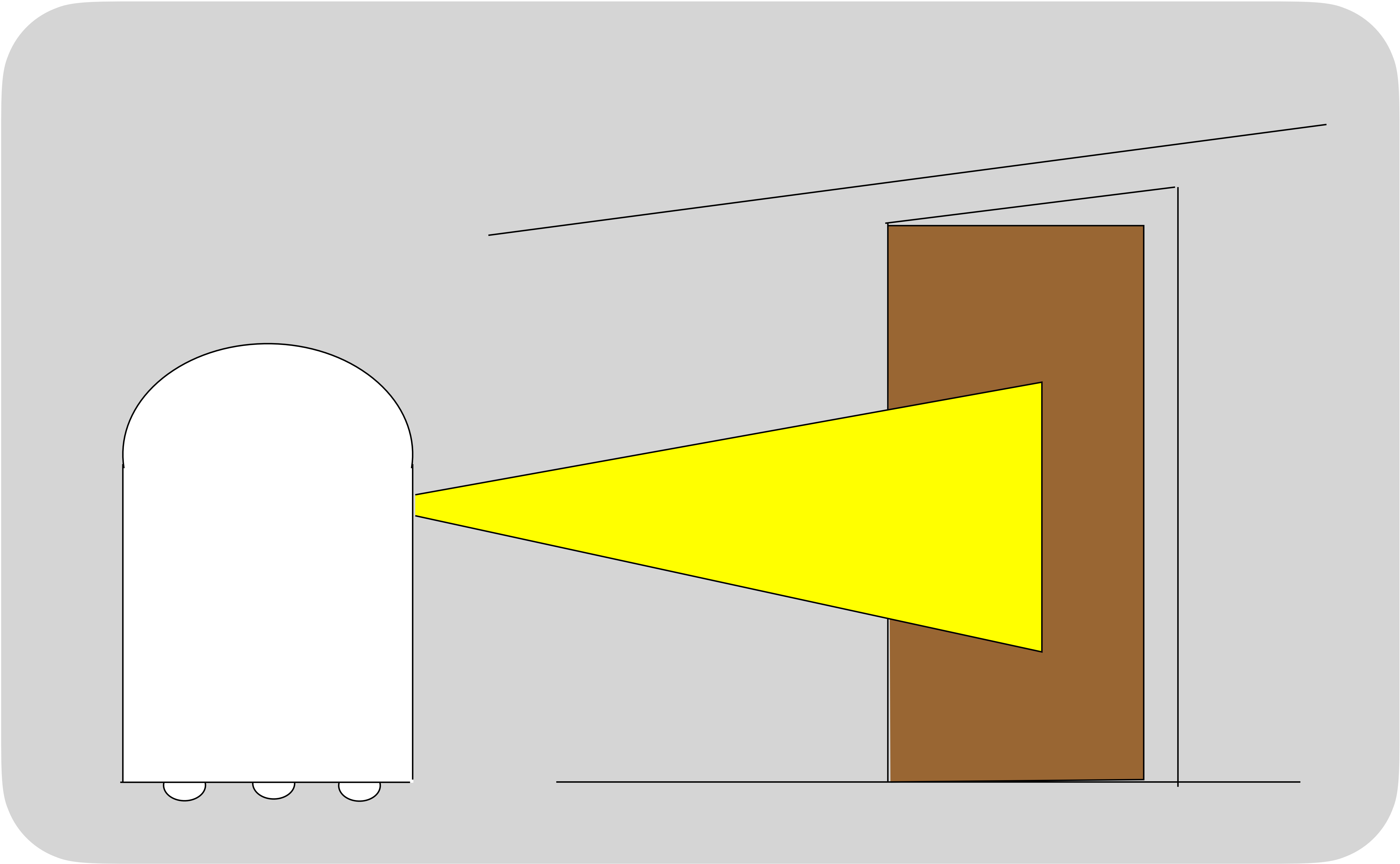
$$P(x \mid z) = P(x \mid z, y)$$

and

$$P(y \mid z) = P(y \mid z, x)$$

Simple Example of State Estimation

- Suppose our robot obtains another observation z_2.
- What is $P(open \mid z_1, z_2)$?



Recursive Bayesian Updating

$$P(x \mid z_1, \ldots z_n) = \frac{P(z_n \mid x, z_1, \ldots z_{n-1}) P(x \mid z_1, \ldots z_{n-1})}{P(z_n \mid z_1, \ldots z_{n-1})}$$

Markov assumption: z_n is conditionally independent of $z_1, \ldots, z_{n-1}$ given x.

$$P(x \mid z_1, \ldots z_n) = \frac{P(z_n \mid x) P(x \mid z_1, \ldots z_{n-1})}{P(z_n \mid z_1, \ldots z_{n-1})}$$

$$= \eta P(z_n \mid x) P(x \mid z_1, \ldots z_{n-1})$$

$$= \eta_{1..n} \prod_{i=1...n} P(z_i \mid x) P(x)$$

Example: Second Measurement

$$P(z_2 \mid \text{open}) = 0.5 \quad P(z_2 \mid \neg\text{open}) = 0.6$$

$$P(\text{open} \mid z_1) = 2/3 \quad P(\neg\text{open} \mid z_1) = 1/3$$

$$P(\text{open} \mid z_2, z_1) = \cdot$$

$$= \frac{1/2 \times 2/3}{1/2 \times 2/3 + 3/5 \times 1/3} = \frac{5}{8} = 0.625$$

- z_2 lowers the probability that the door is open.

Bayes Filters: Framework

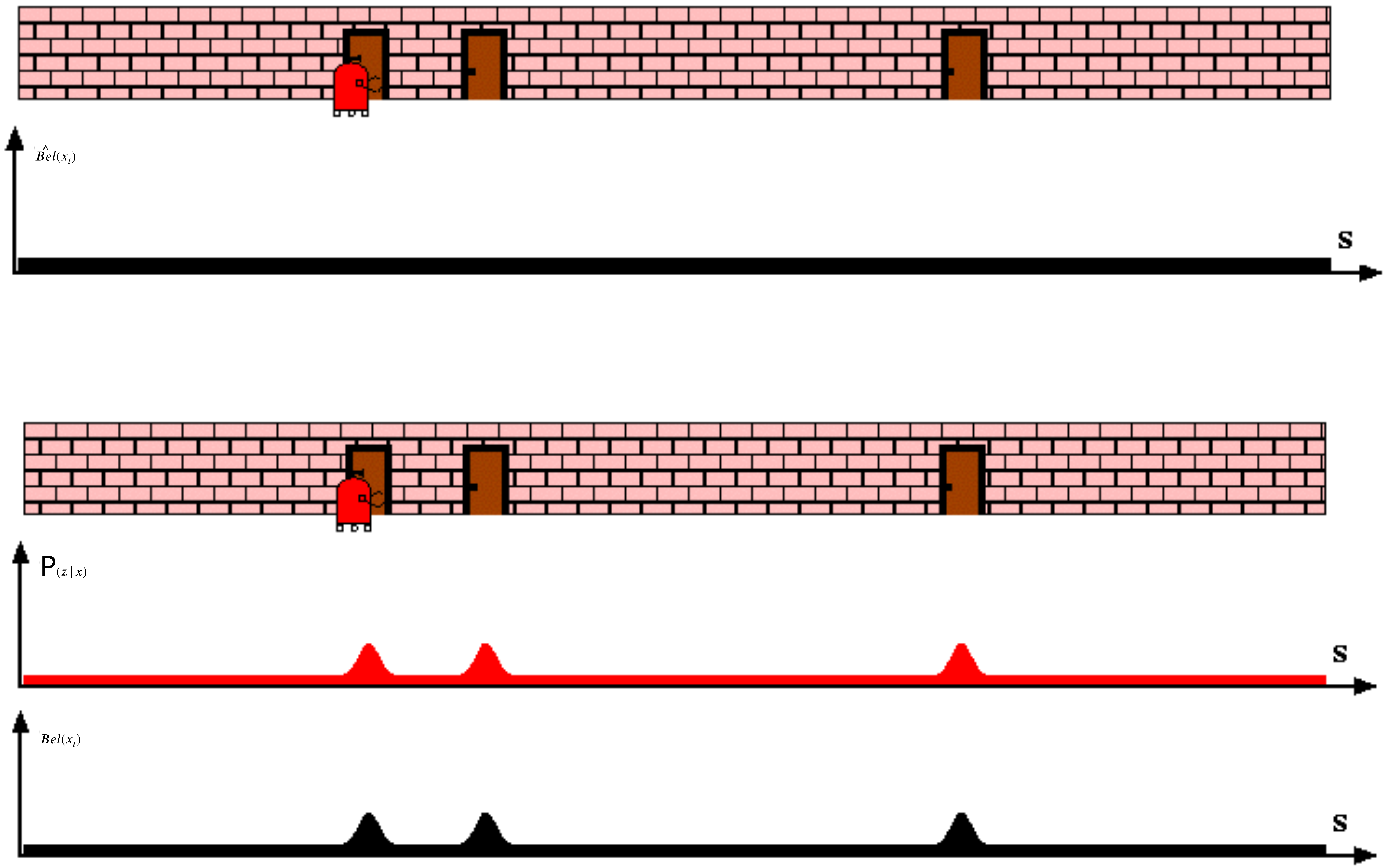
- **Given:**
  - Stream of observations z and action data u:

$$d_t = \{u_1, z_2, \ldots, u_{t-1}, z_t\}$$

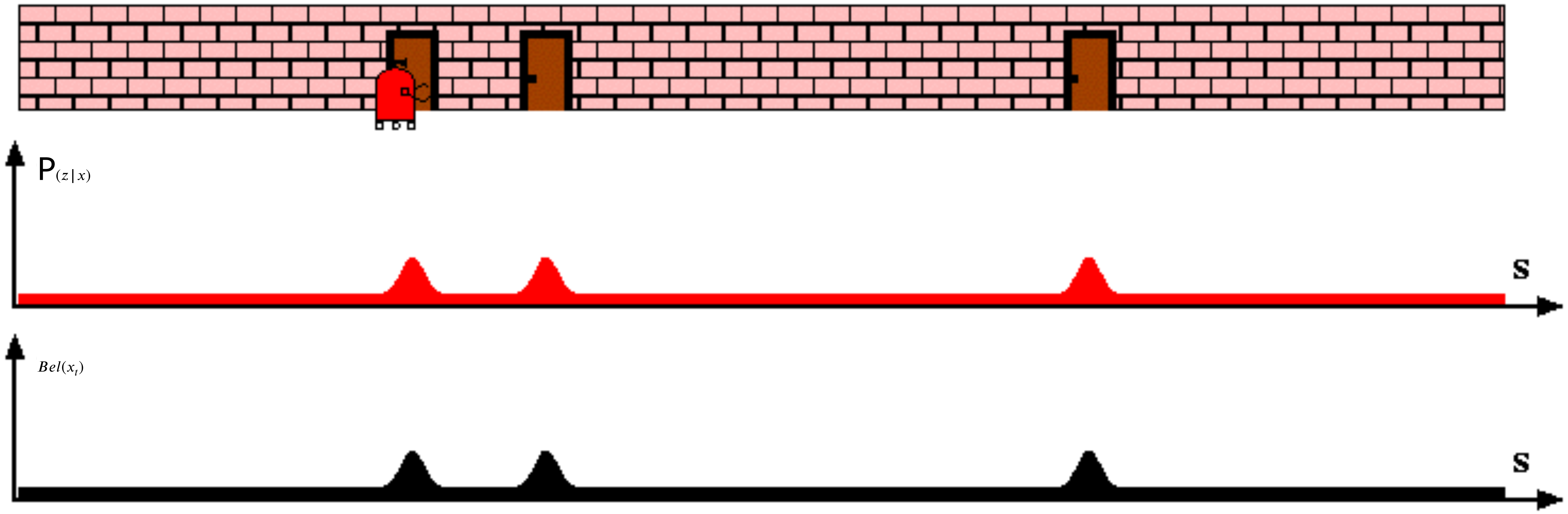
  - Sensor model $P(z \mid x)$.
  - Action model $P(x \mid u, x')$.
  - Prior probability of the system state $P(x)$.
- **Wanted:**
  - Estimate of the state X of a dynamical system.
  - The posterior of the state is also called **Belief**:

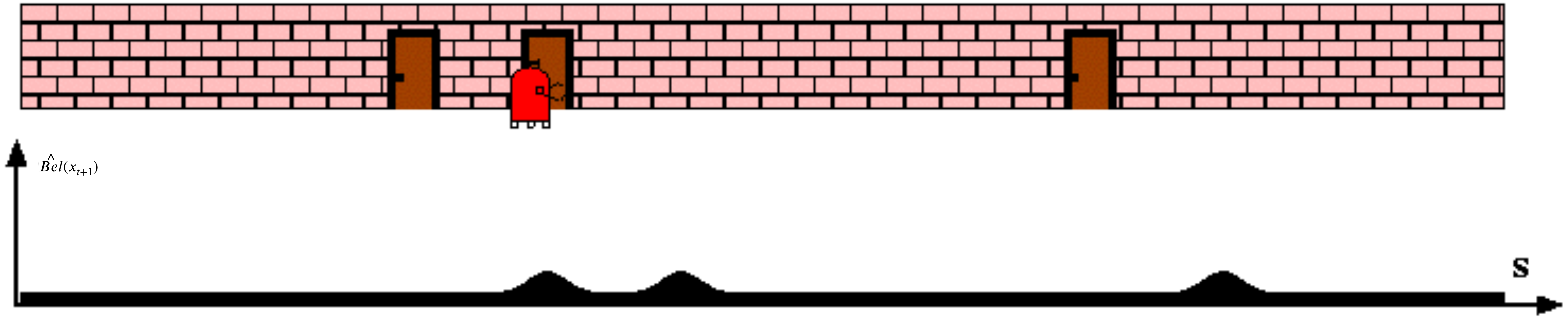
$$Bel(x_t) = P(x_t \mid u_1, z_2, \ldots, u_{t-1}, z_t)$$

Bayes Filters for Robot Localization

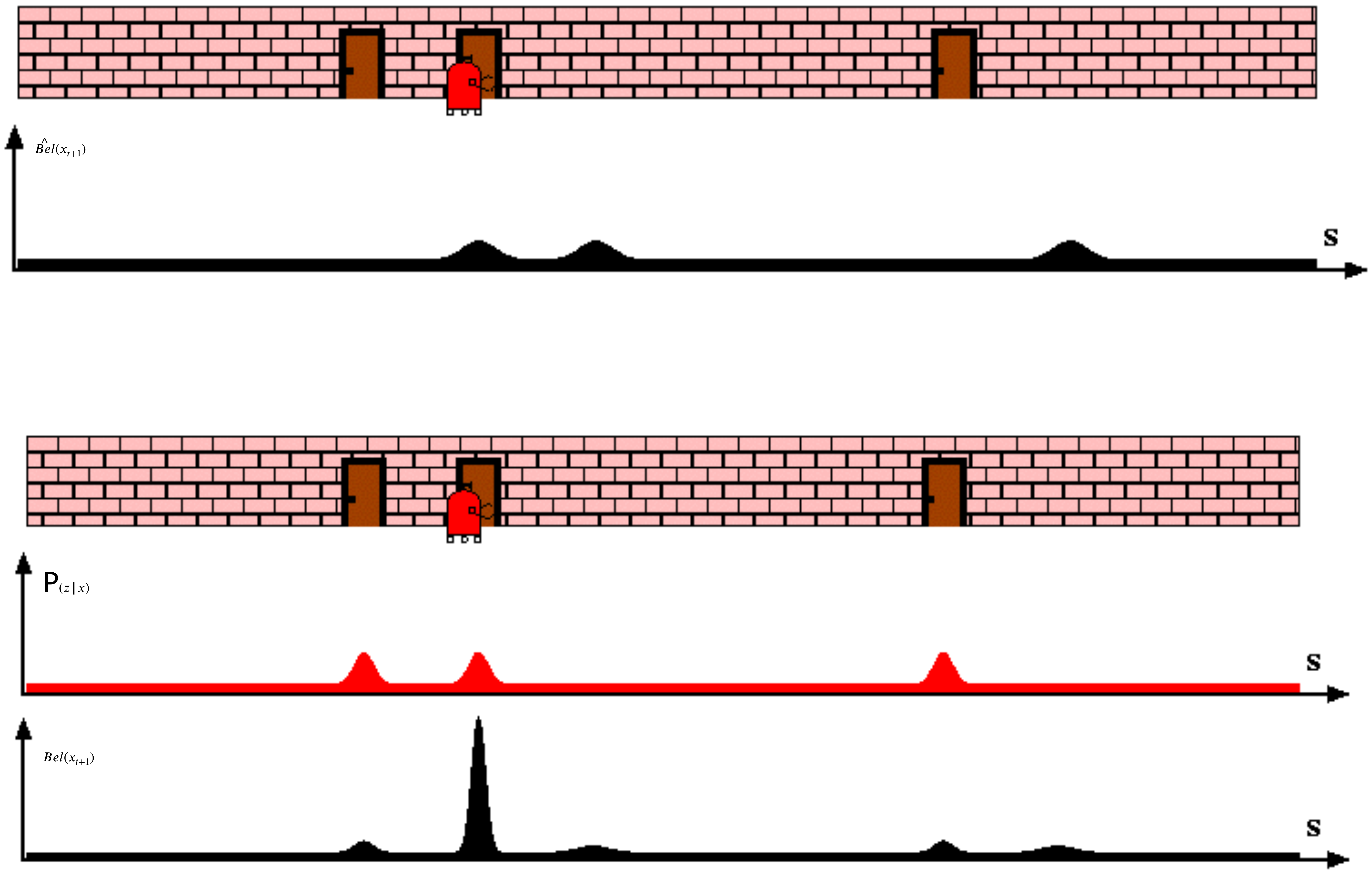


Bayes Filters for Robot Localization

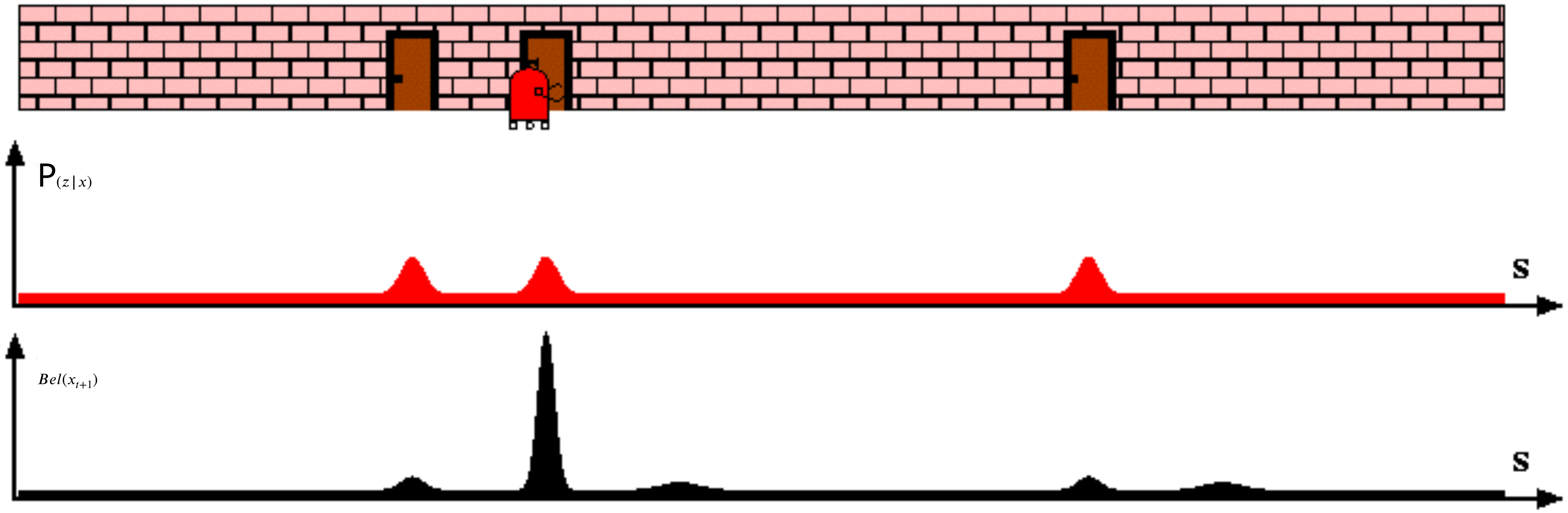


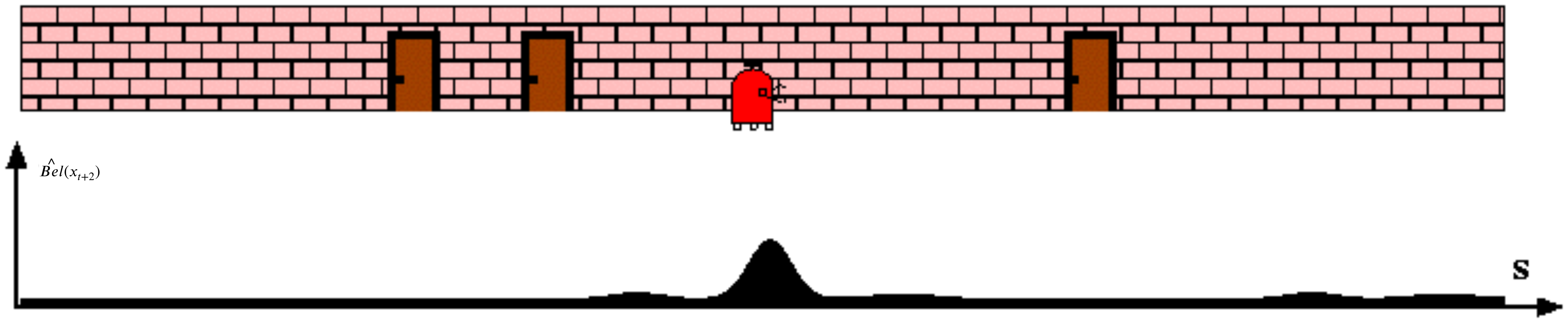


Bayes Filters for Robot Localization



Bayes Filters for Robot Localization





Bayes Filters

z = observation
u = action
x = state

$$\boxed{Bel(x_t)} = P(x_t \mid u_1, z_1 \ldots, u_t, z_t)$$

Bayes $$= \eta\ P(z_t \mid x_t, u_1, z_1, \ldots, u_t)\ P(x_t \mid u_1, z_1, \ldots, u_t)$$

Markov $$= \eta\ P(z_t \mid x_t)\ P(x_t \mid u_1, z_1, \ldots, u_t)$$

Total prob.
$$= \eta\ P(z_t \mid x_t) \int P(x_t \mid u_1, z_1, \ldots, u_t, x_{t-1}) P(x_{t-1} \mid u_1, z_1, \ldots, u_t)\, dx_{t-1}$$

Markov $$= \eta\ P(z_t \mid x_t) \int P(x_t \mid u_t, x_{t-1})\ P(x_{t-1} \mid u_1, z_1, \ldots, u_t)\, dx_{t-1}$$

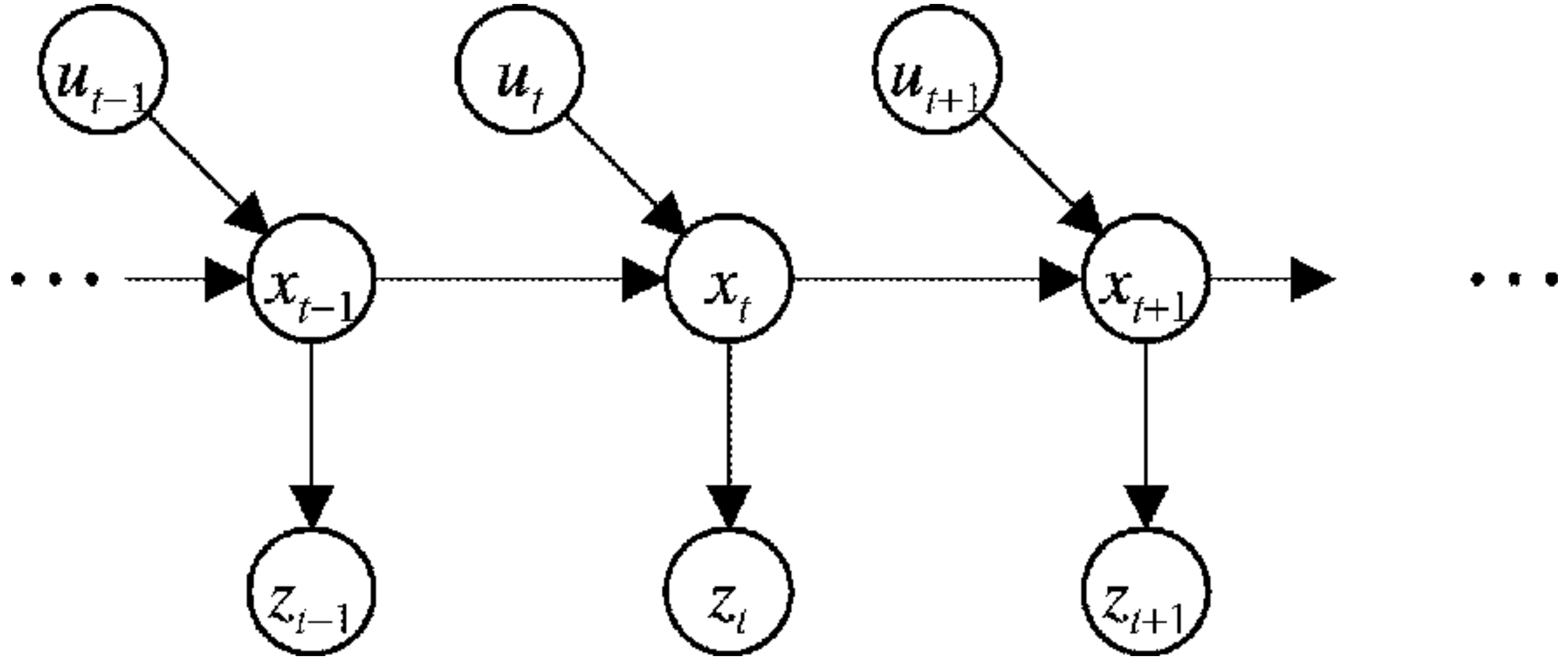
$$\boxed{= \eta\ P(z_t \mid x_t) \int P(x_t \mid u_t, x_{t-1})\ Bel(x_{t-1})\, dx_{t-1}}$$

Slide borrowed from Dieter Fox

$$Bel(x_t) = \eta \; P(z_t \mid x_t) \int P(x_t \mid u_t, x_{t-1}) \; Bel(x_{t-1}) \; dx_{t-1}$$

1. Algorithm **Bayes_filter**(*Bel(x),d*):
2. *n*=0
3. If *d* is a perceptual data item *z* then
4. For all *x* do
5. $Bel'(x) = P(z \mid x) Bel(x)$
6. $\eta = \eta + Bel'(x)$
7. For all *x* do
8. $Bel'(x) = \eta^{-1} Bel'(x)$
9. Else if *d* is an action data item *u* then
10. For all *x* do
11. $Bel'(x) = \int P(x \mid u, x') \; Bel(x') \; dx'$
12. Return *Bel'(x)*

Markov Assumption



$$P(z_t \mid x_{0:t}, z_{1:t-1}, u_{1:t}) = p(z_t \mid x_t)$$

$$P(x_t \mid x_{1:t-1}, z_{1:t-1}, u_{1:t}) = p(x_t \mid x_{t-1}, u_t)$$

Underlying Assumptions

- Static world
- Independent noise
- Perfect model, no approximation errors

Bayes Filters are Familiar!

$$Bel(x_t) = \eta P(z_t \mid x_t) \int P(x_t \mid x_{t-1} u_t) Bel(x_{t-1}) dx_{t-1}$$

- Kalman filters
- Particle filters
- Hidden Markov models
- Dynamic Bayesian networks
- Partially Observable Markov Decision Processes (POMDPs)

Summary

- Bayes rule allows us to compute probabilities that are hard to assess otherwise.
- Under the Markov assumption, recursive Bayesian updating can be used to efficiently combine evidence.
- Bayes filters are a probabilistic tool for estimating the state of dynamic systems.

Next Lecture

Mobile Robotics - II - Motion & Sensor Models