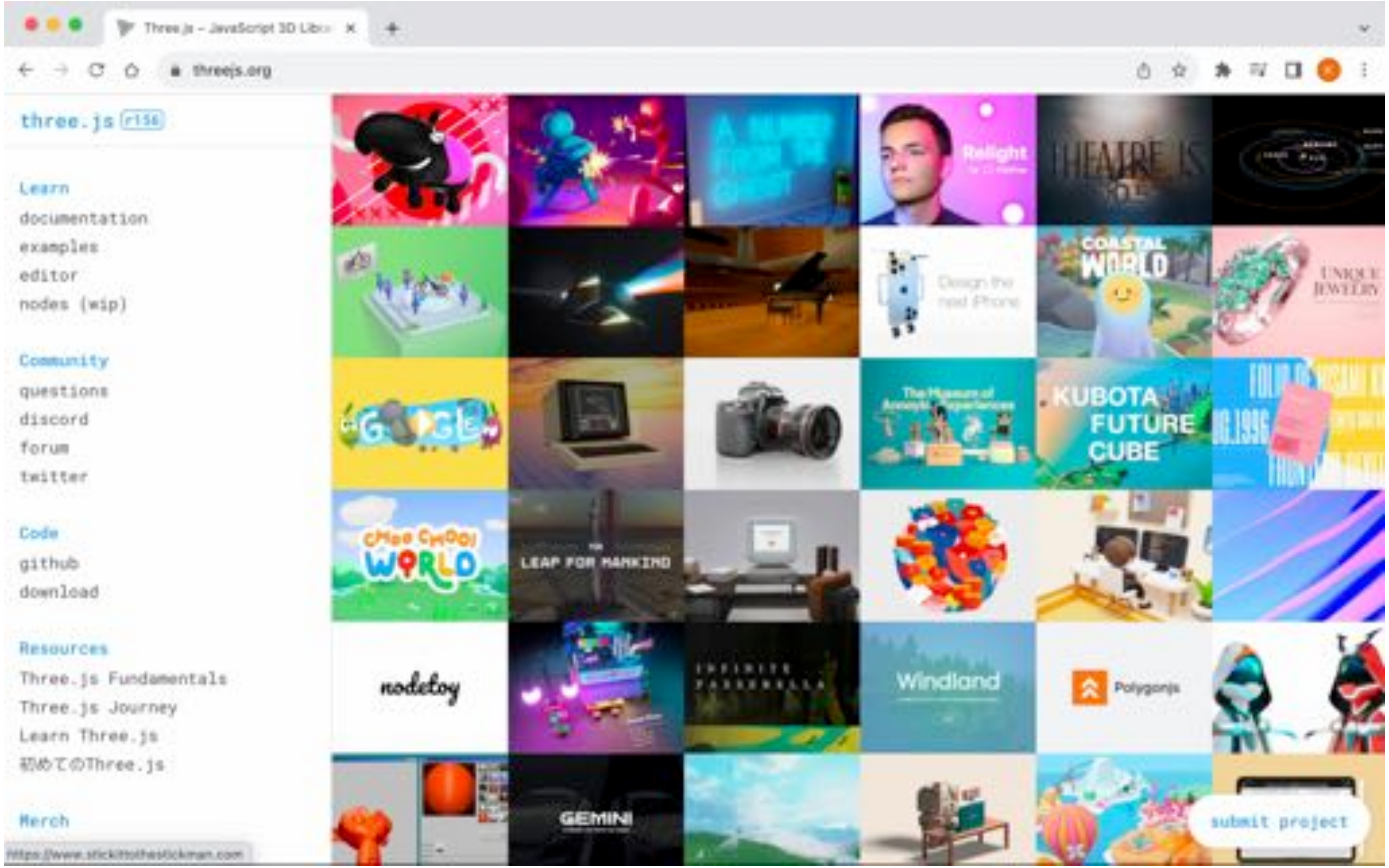
Lecture 03
Representations - I
Transformations

Course Logistics

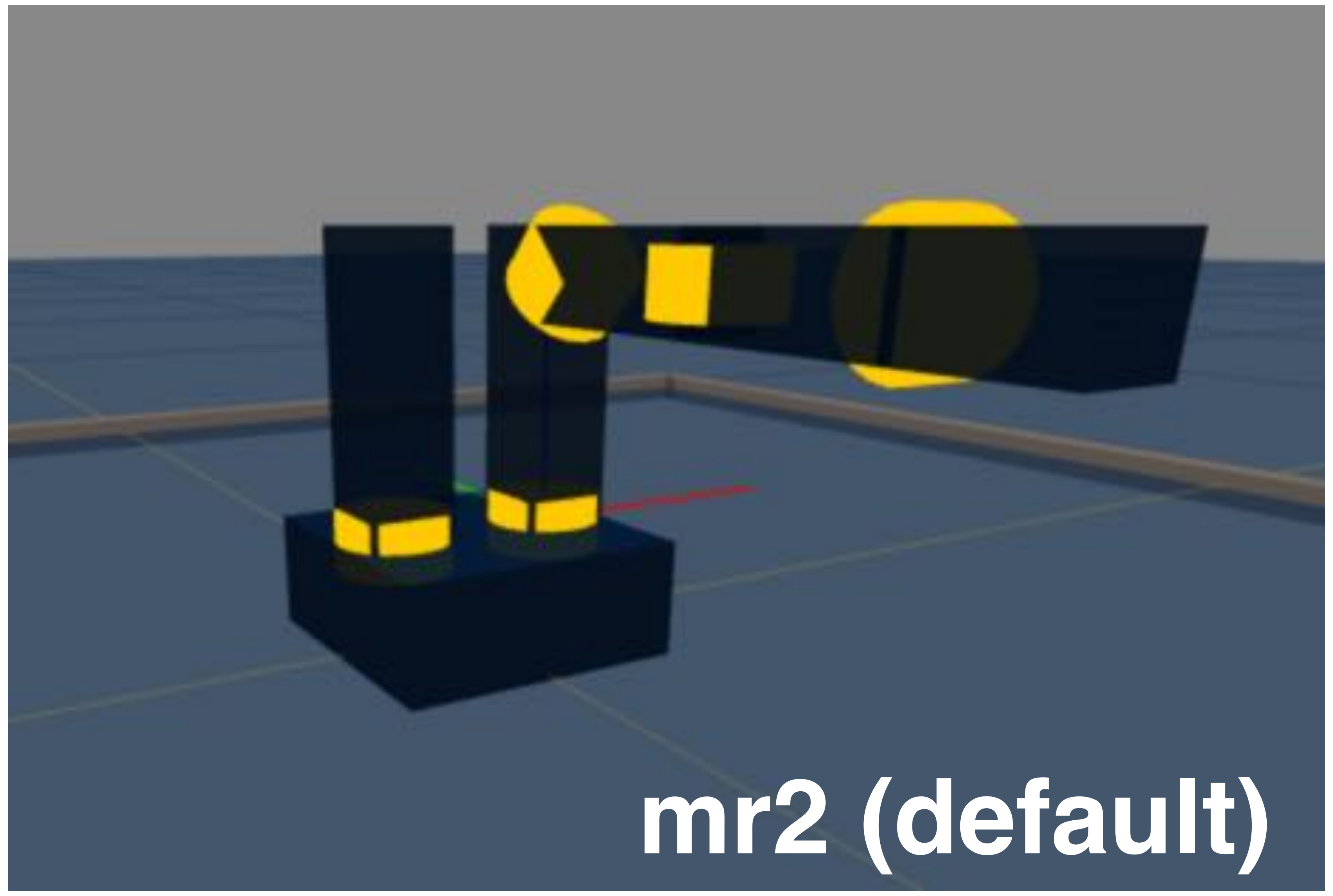
- Everyone should be on Ed discussion board now.
- Everyone should be on Gradescope now.
- **Quiz 1 posted today and was due before the lecture.**
- Quiz 2 will be posted in the same way on 09/18.
- Project 0 will be posted on 09/13 (**today**) and will be due 09/20.
- Autograder will be made available for you to test and submit your code.
- **Action items for you:**
  - Announcements will move from Canvas to Ed discussion posts starting today. So make sure you are getting email notifications from Ed.

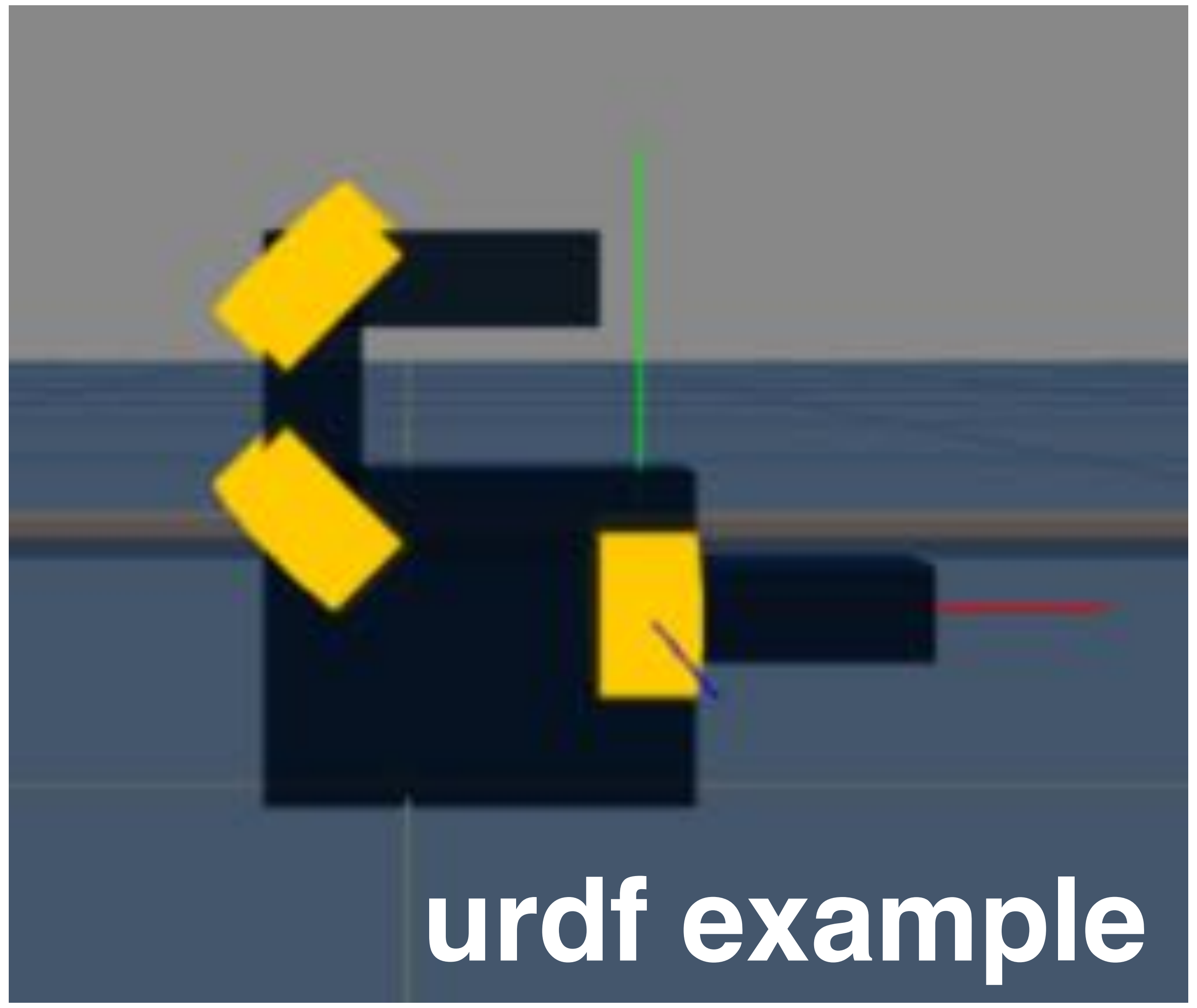
Why JavaScript?

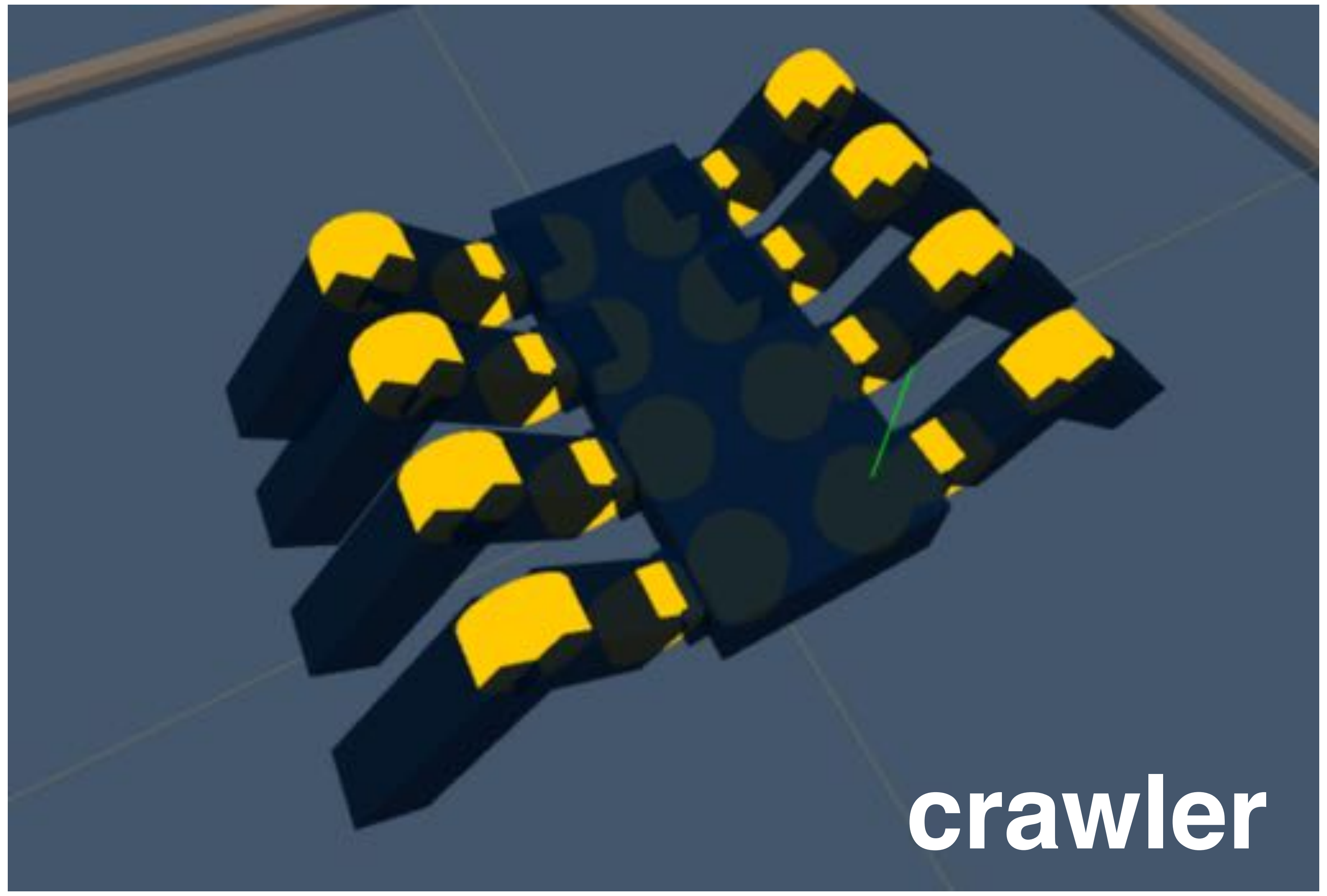
- ThreeJS



Why JavaScript?

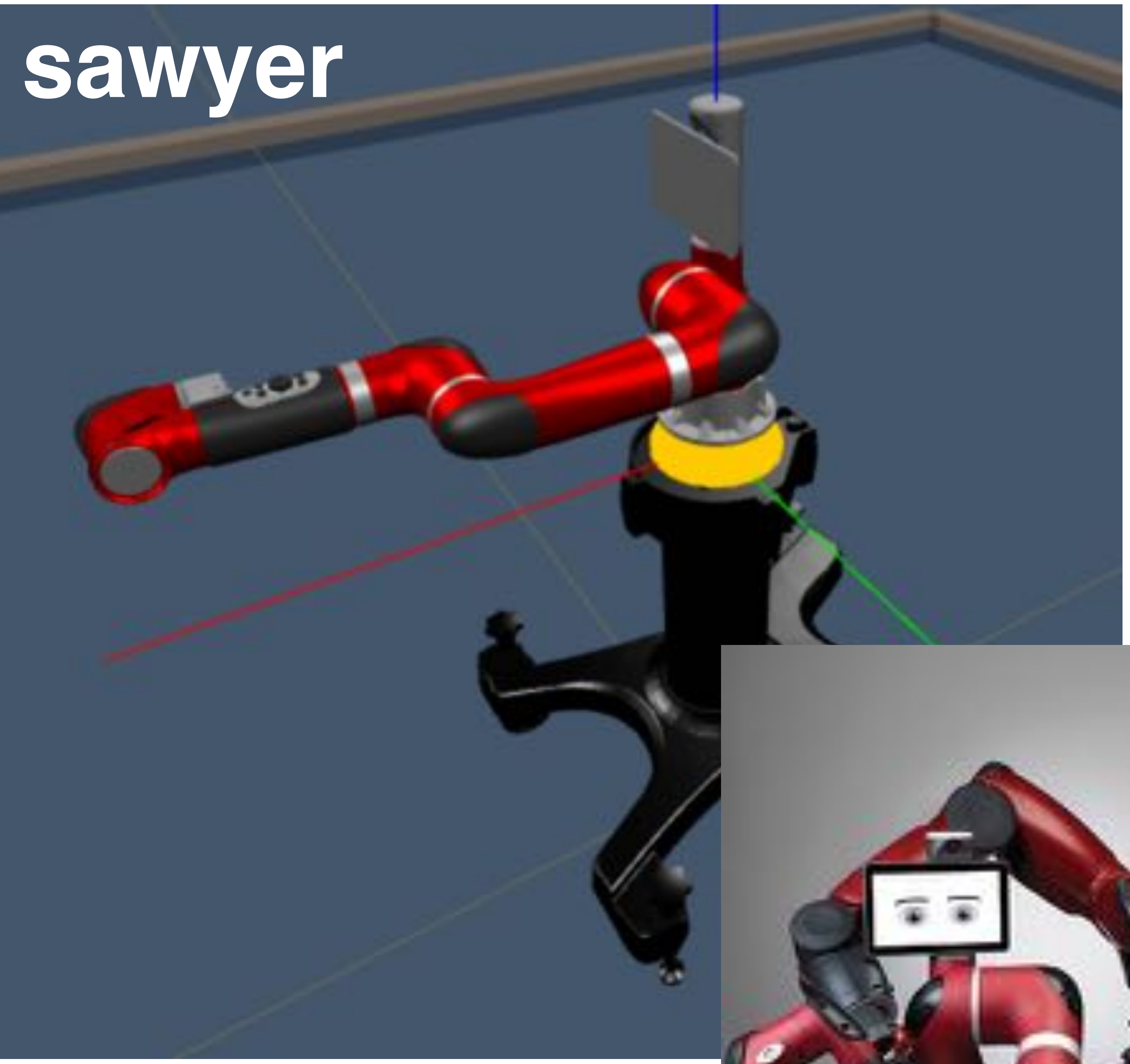


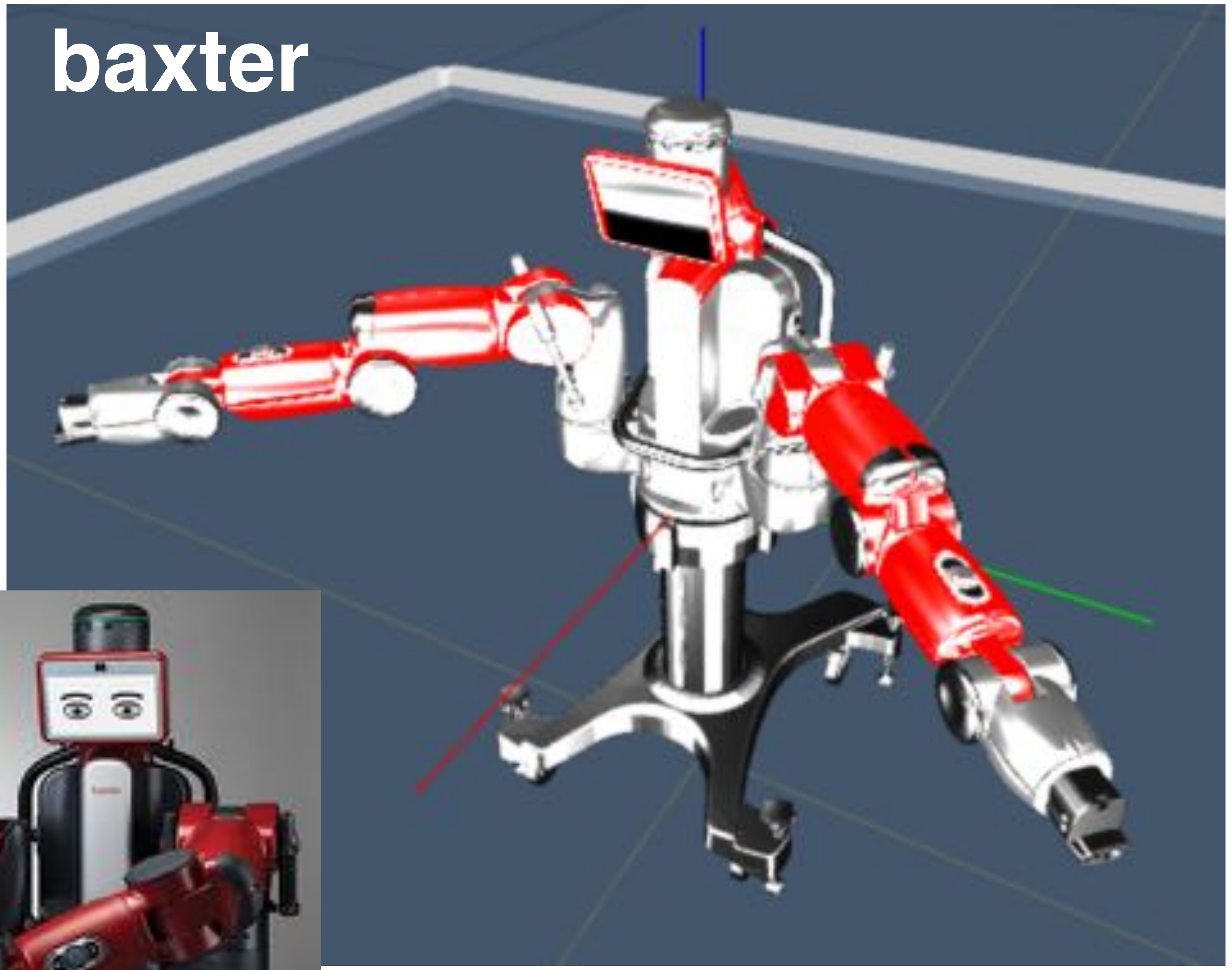


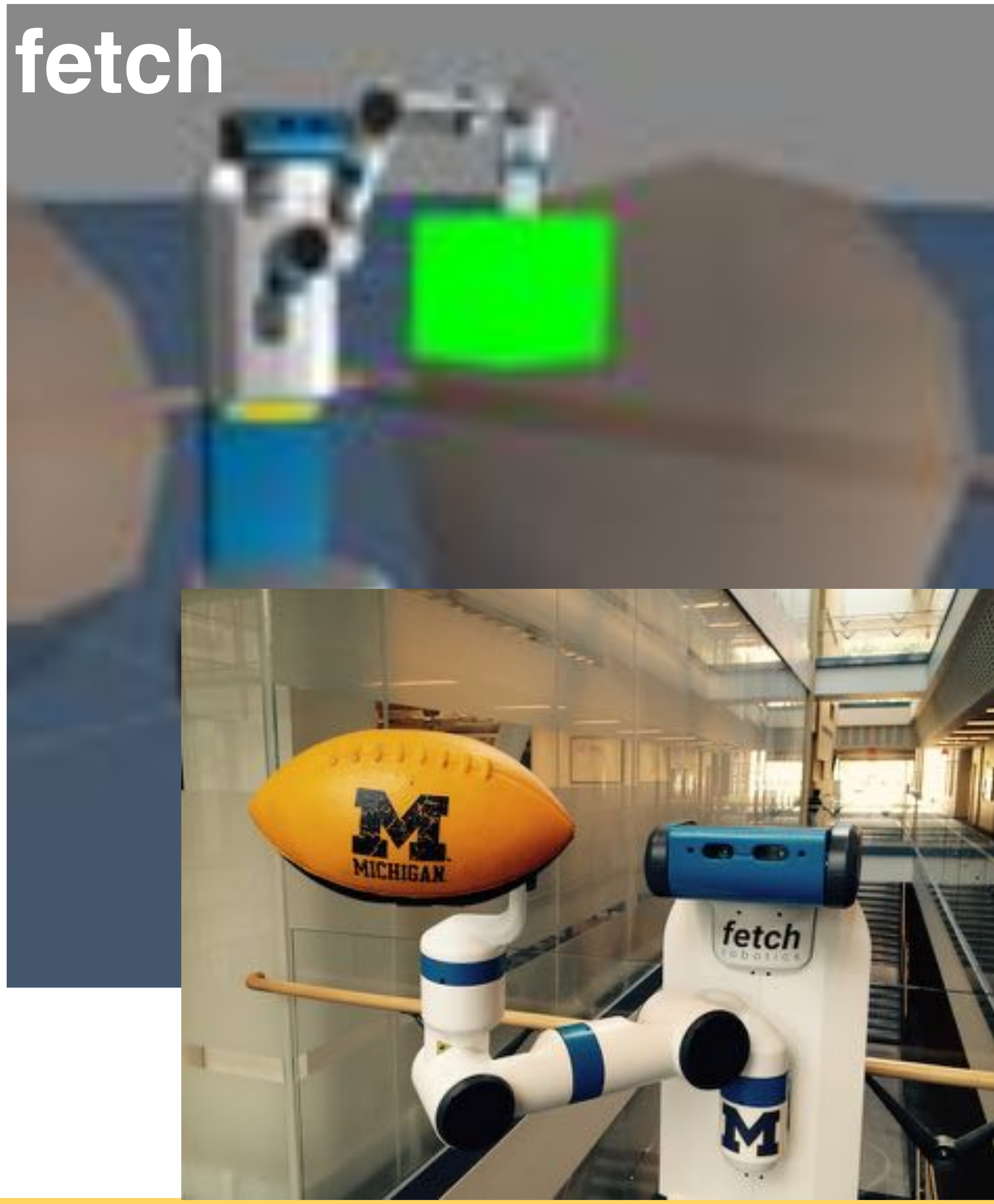


Why JavaScript?

You can load a URDF of a famous robot!

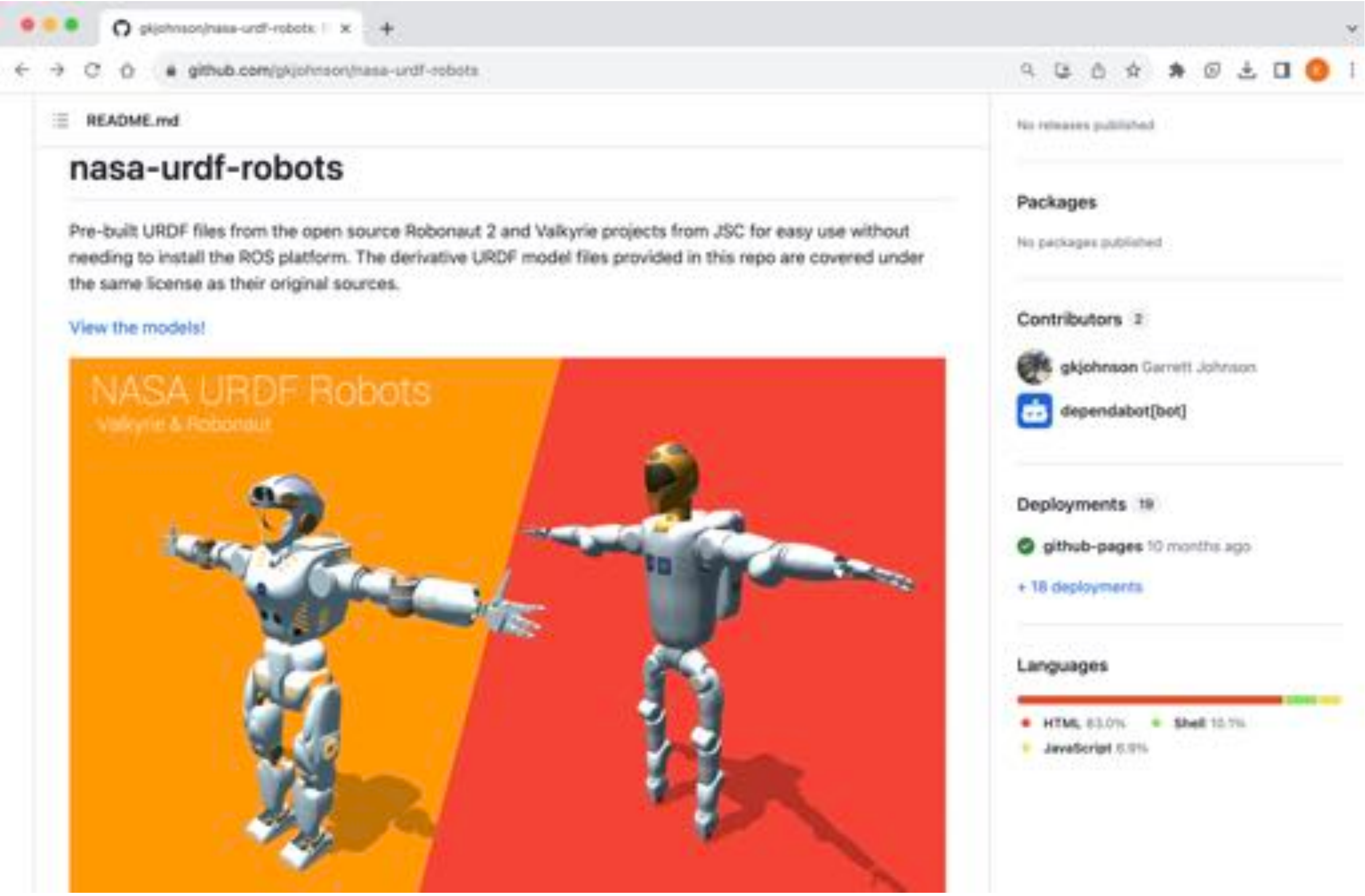


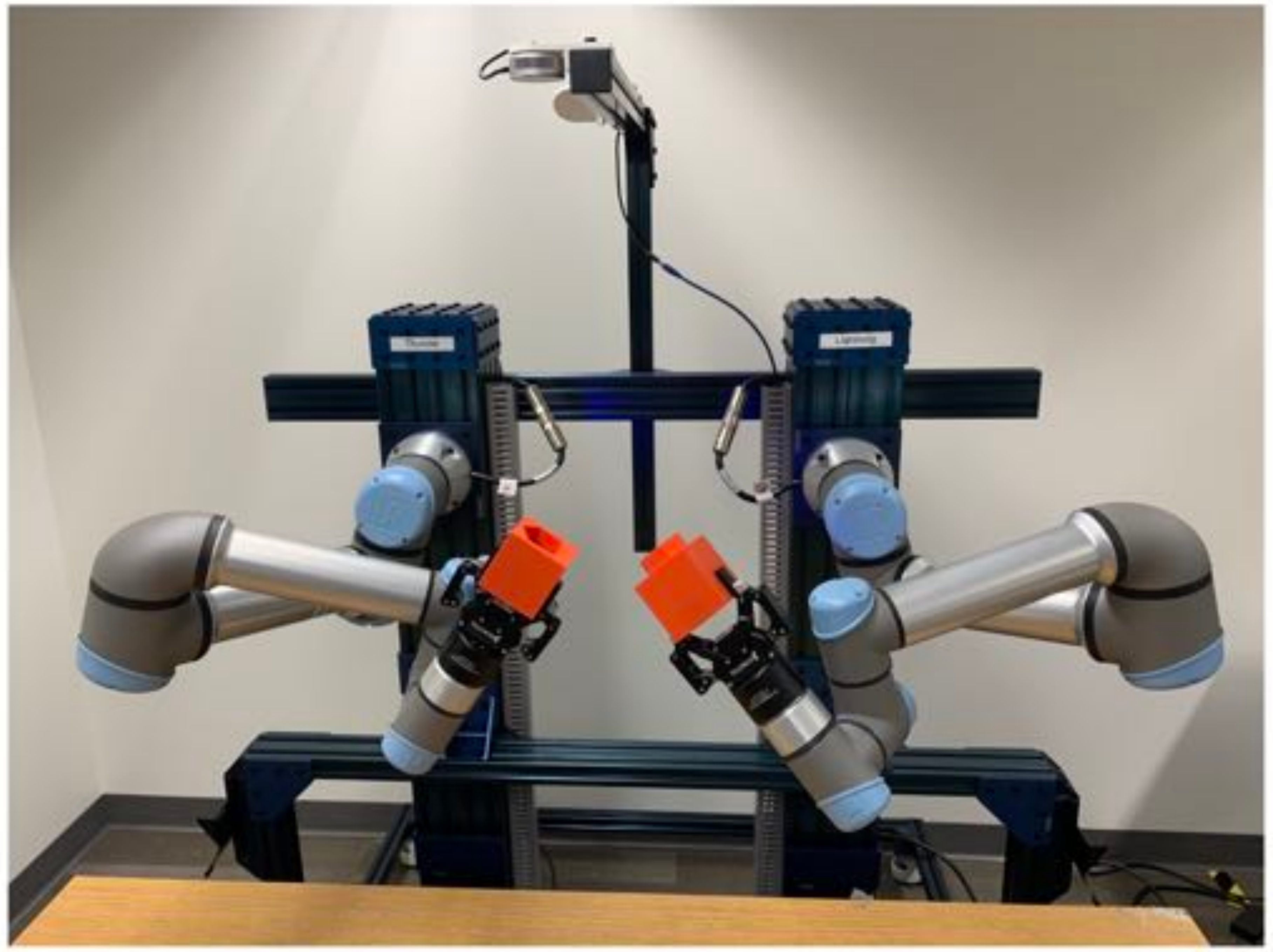




Why JavaScript?

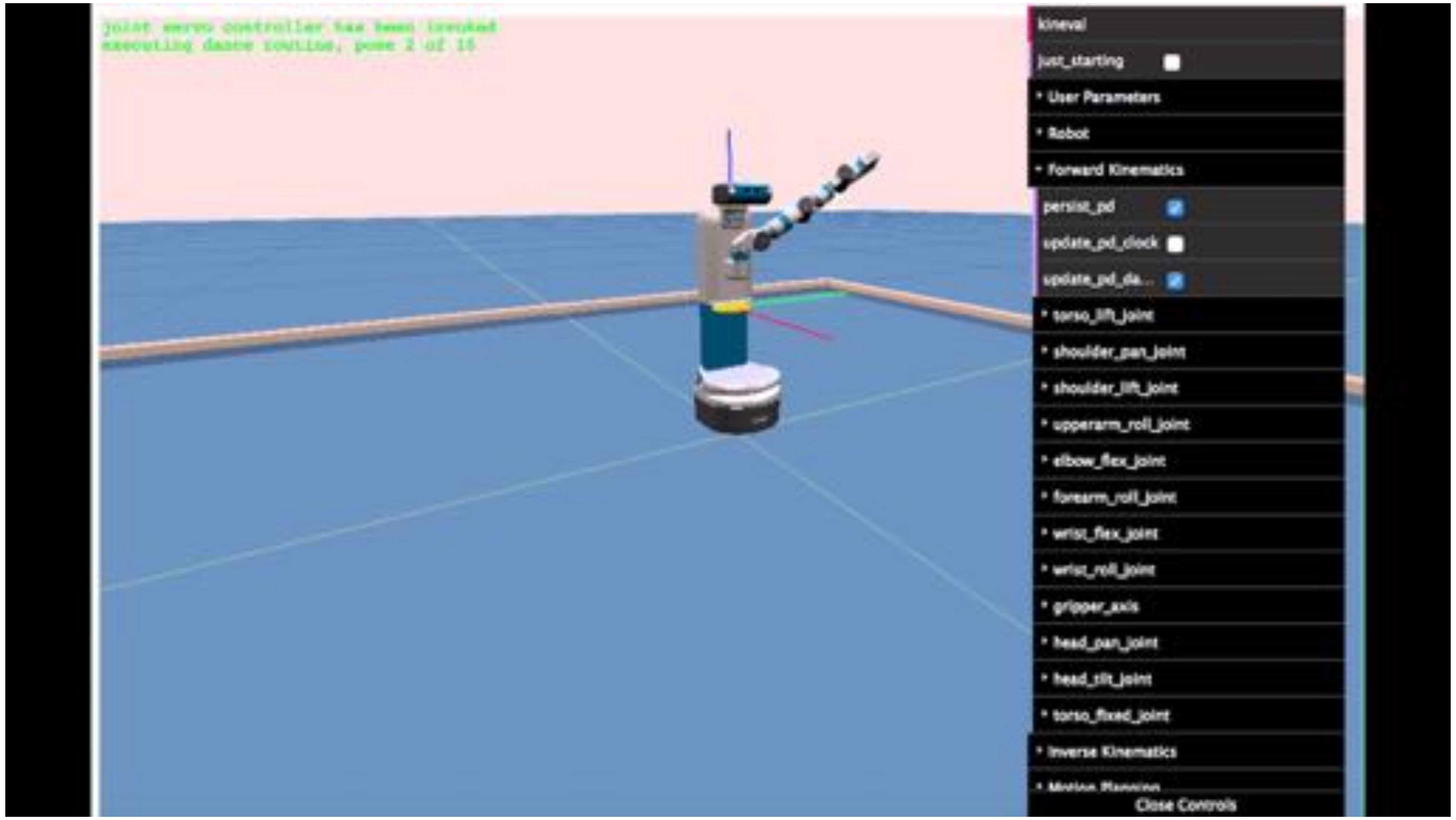
More robot models!!!

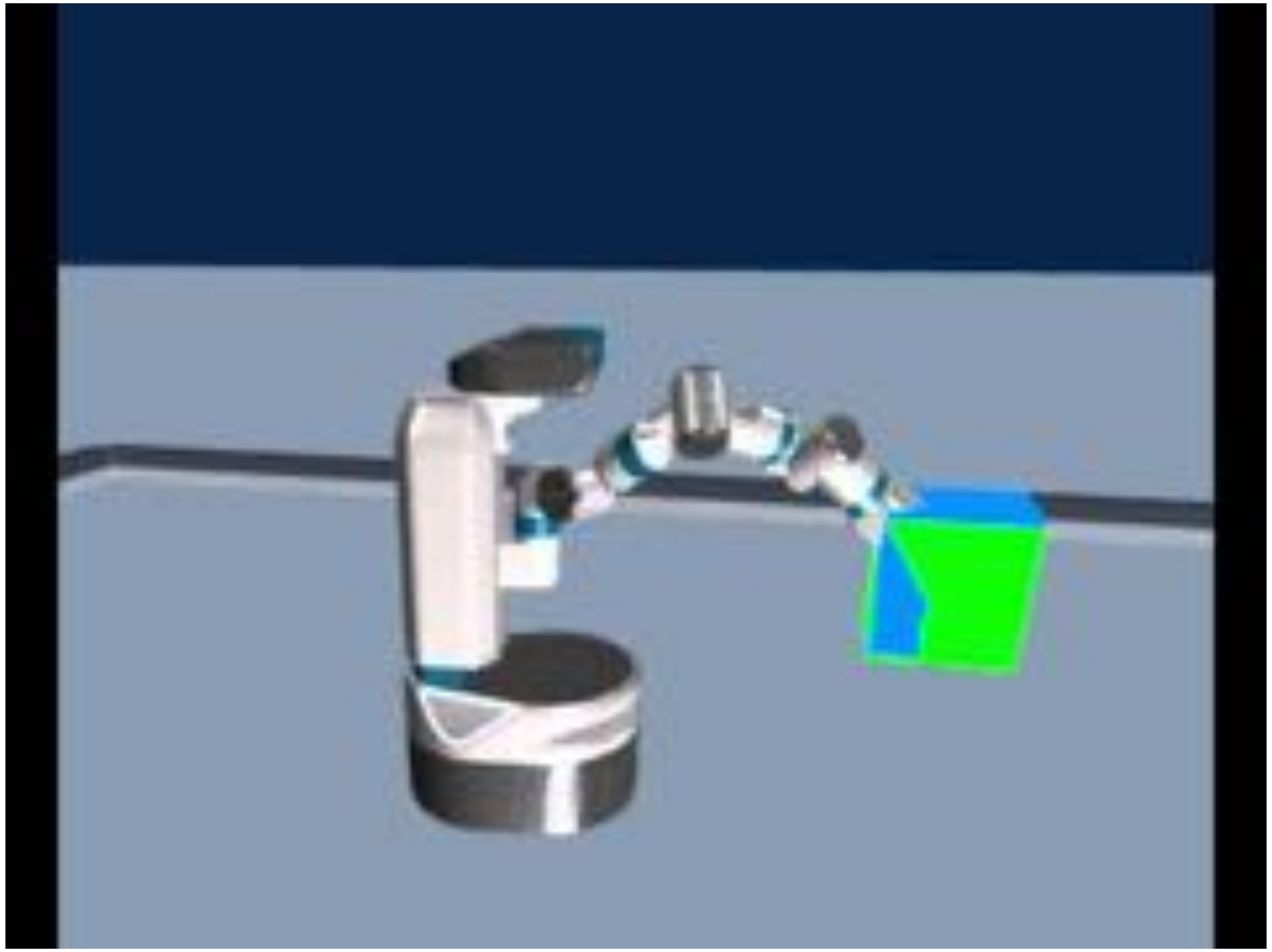




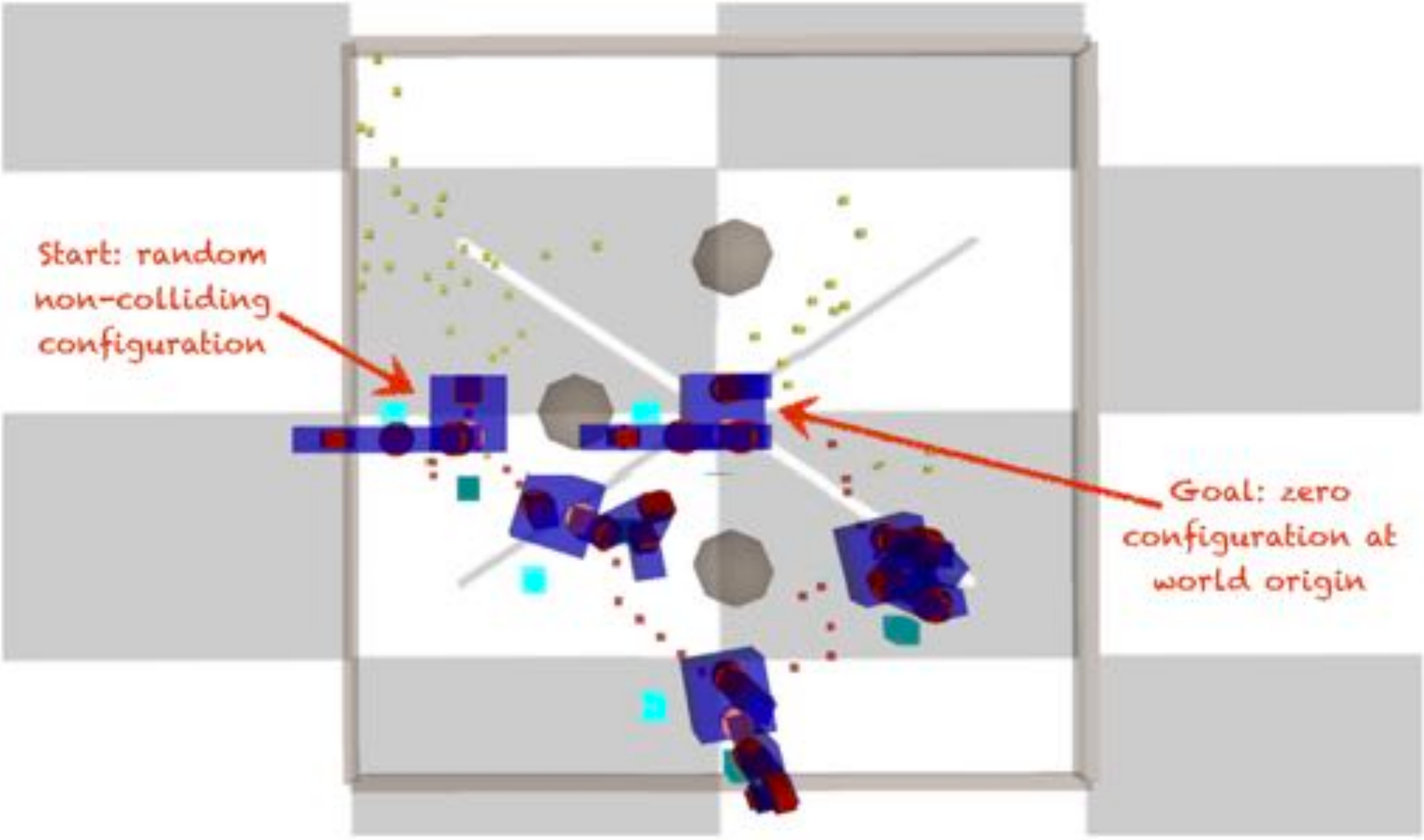
Robots that you can try to load URDFs off and work on your final project

How will the projects look like?

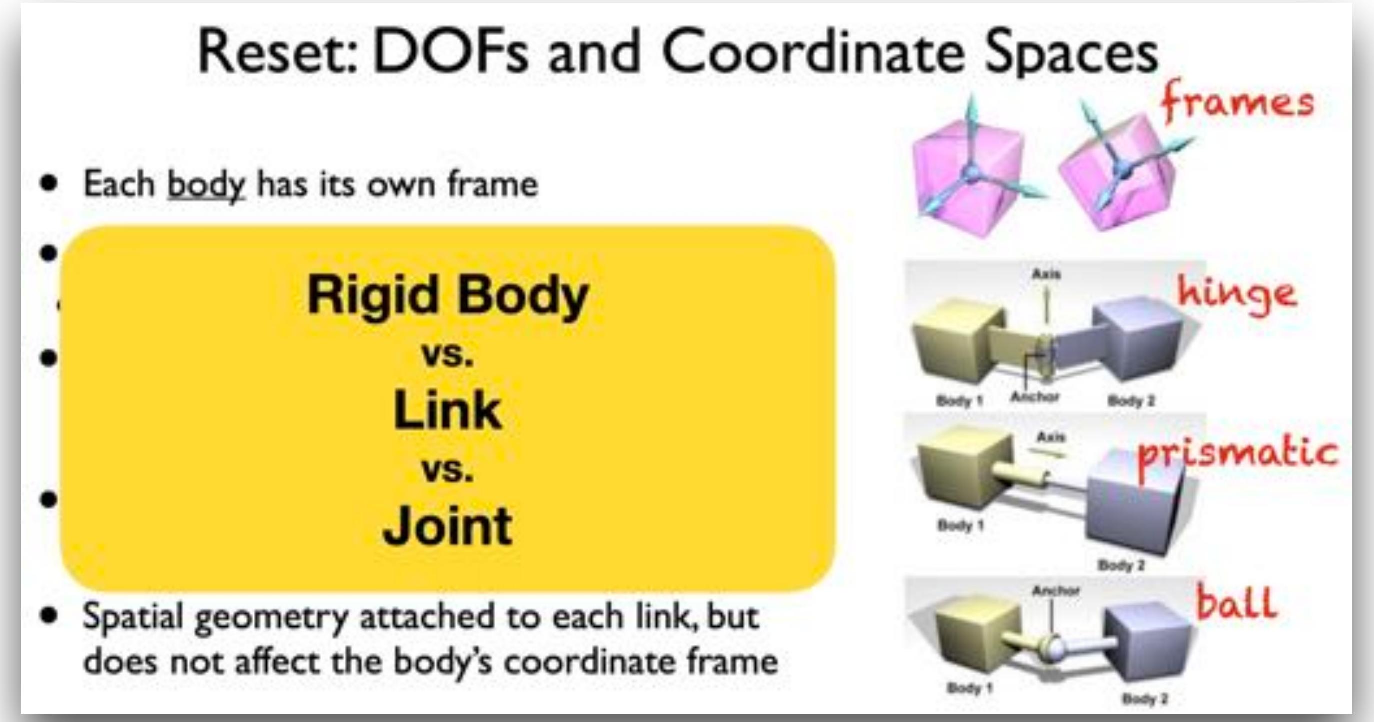




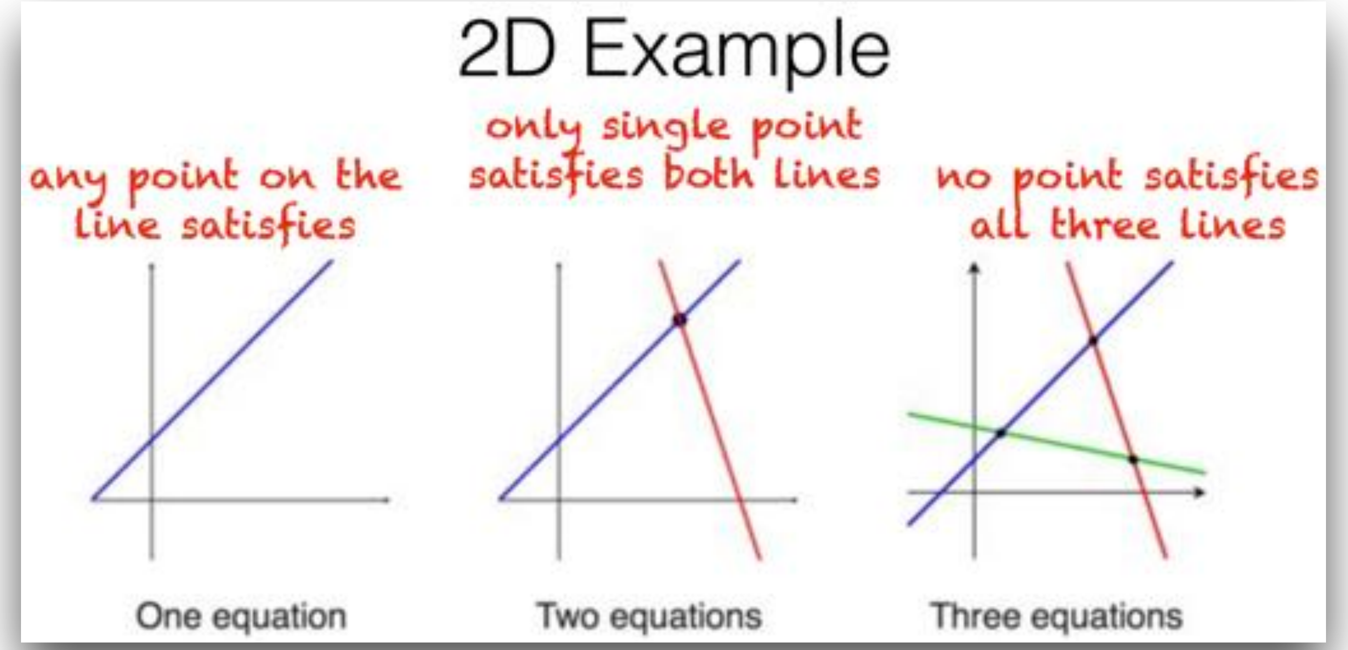
How will the projects look like?



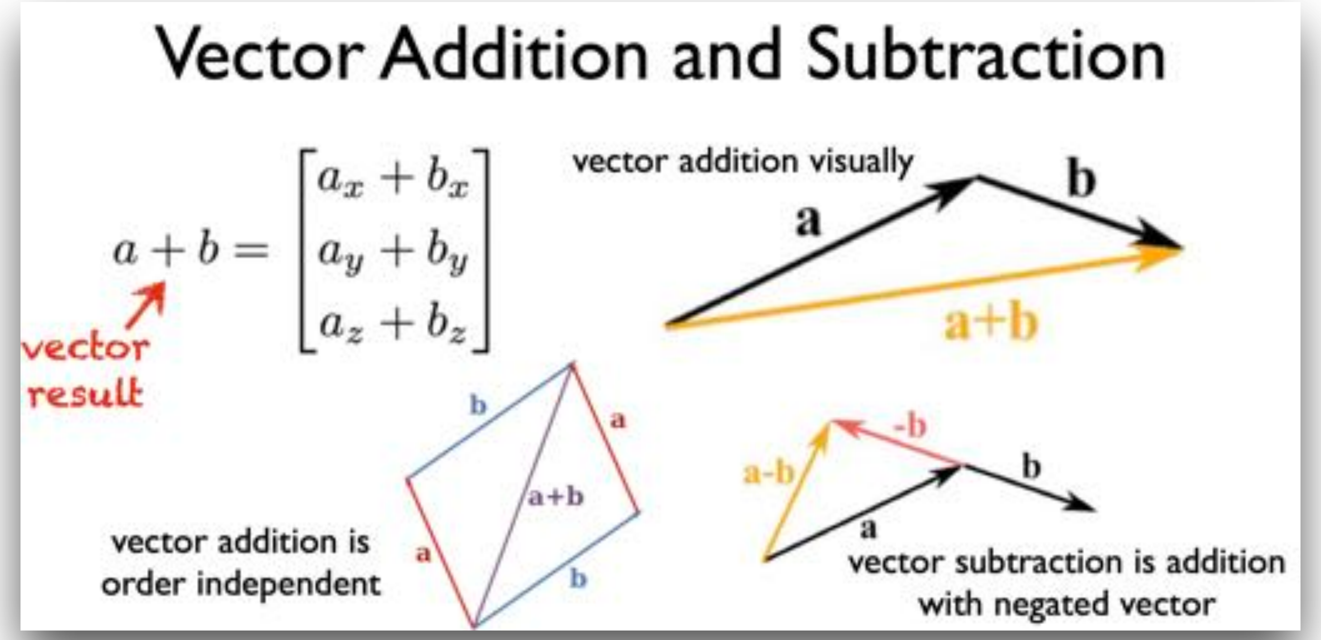
Previously



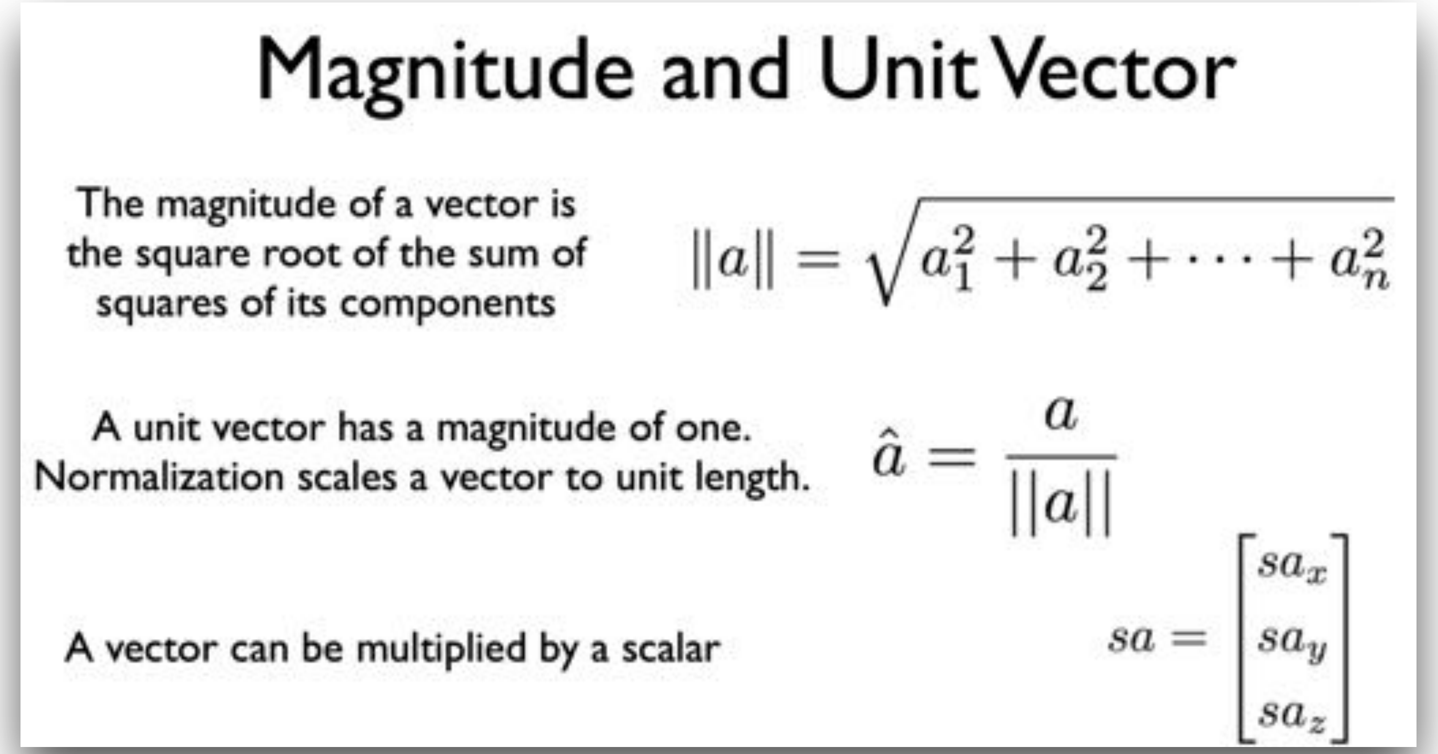
Reset: DOFs and Coordinate Spaces
Each body has its own frame
Rigid Body vs. Link vs. Joint
Spatial geometry attached to each link, but does not affect the body's coordinate frame
frames
hinge
prismatic
ball



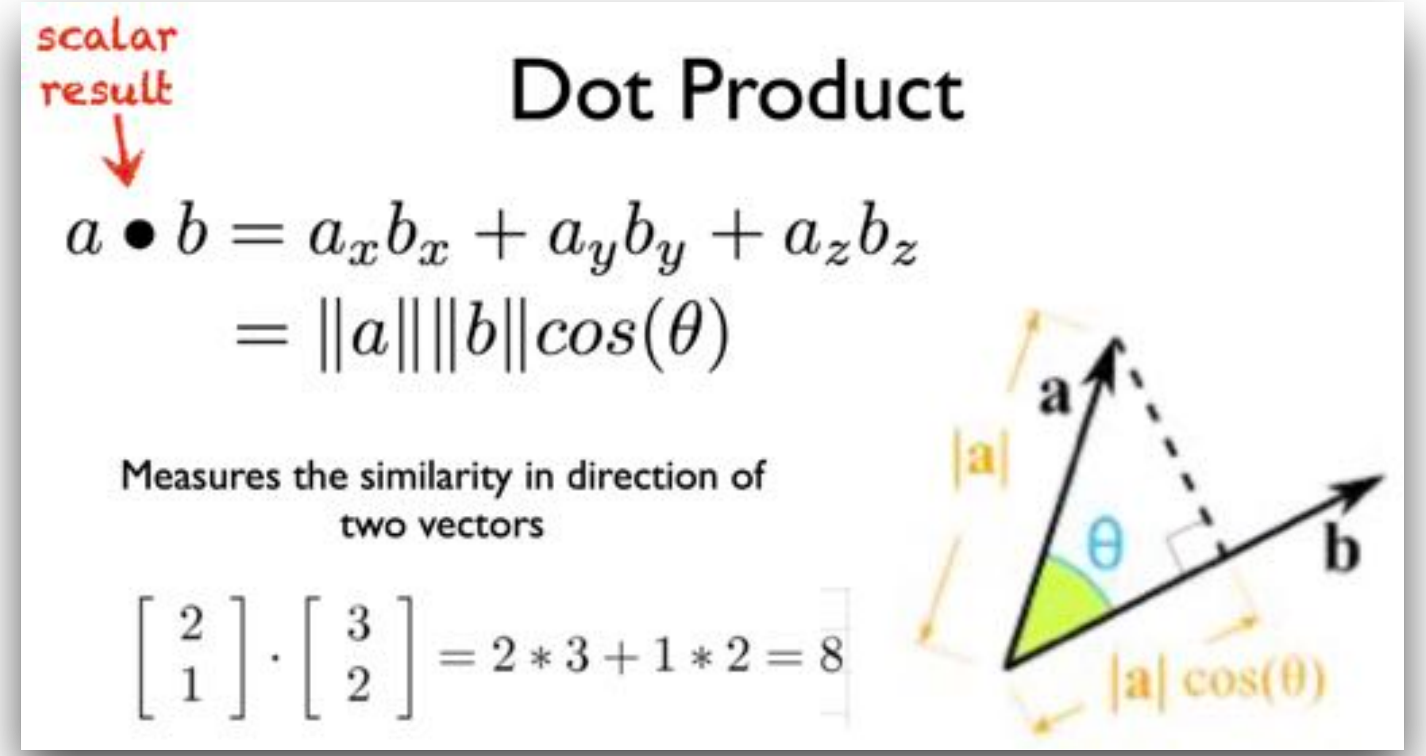
2D Example
any point on the line satisfies
only single point satisfies both lines
no point satisfies all three lines
One equation
Two equations
Three equations



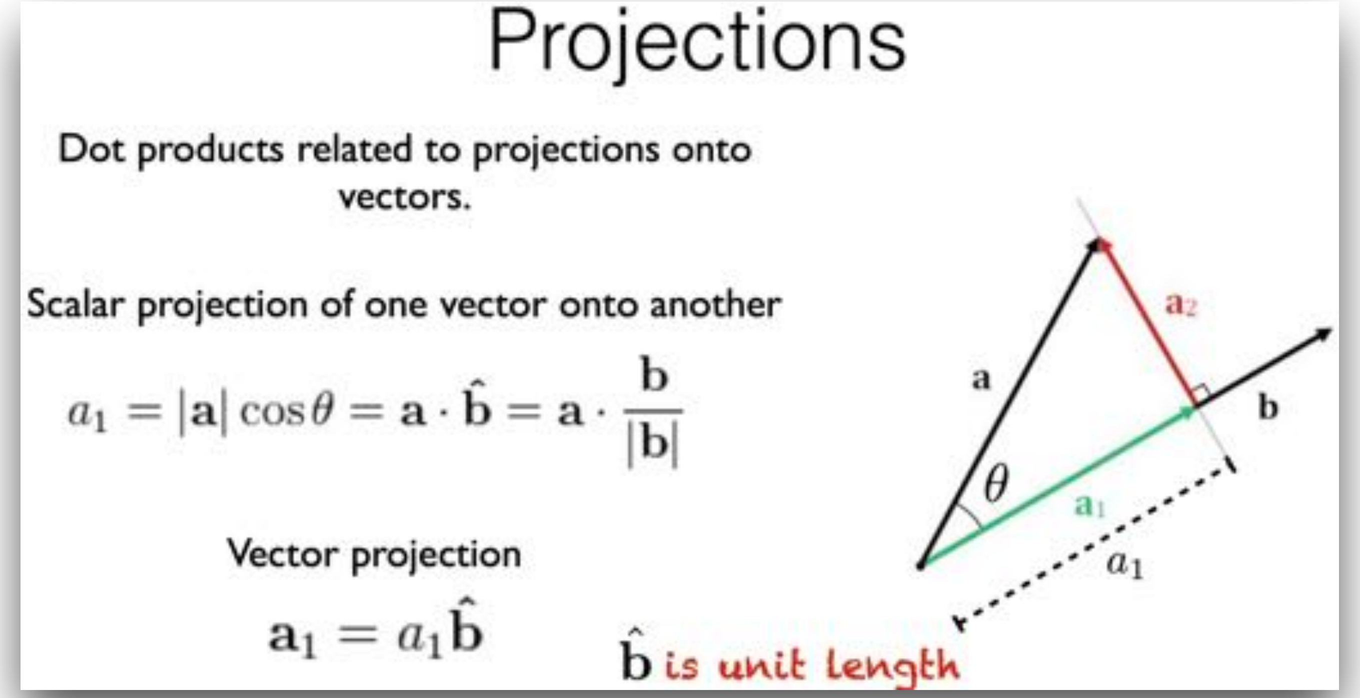
Vector Addition and Subtraction
$a + b = \begin{bmatrix} a_x + b_x \\ a_y + b_y \\ a_z + b_z \end{bmatrix}$
vector result
vector addition visually
a+b
vector addition is order independent
vector subtraction is addition with negated vector



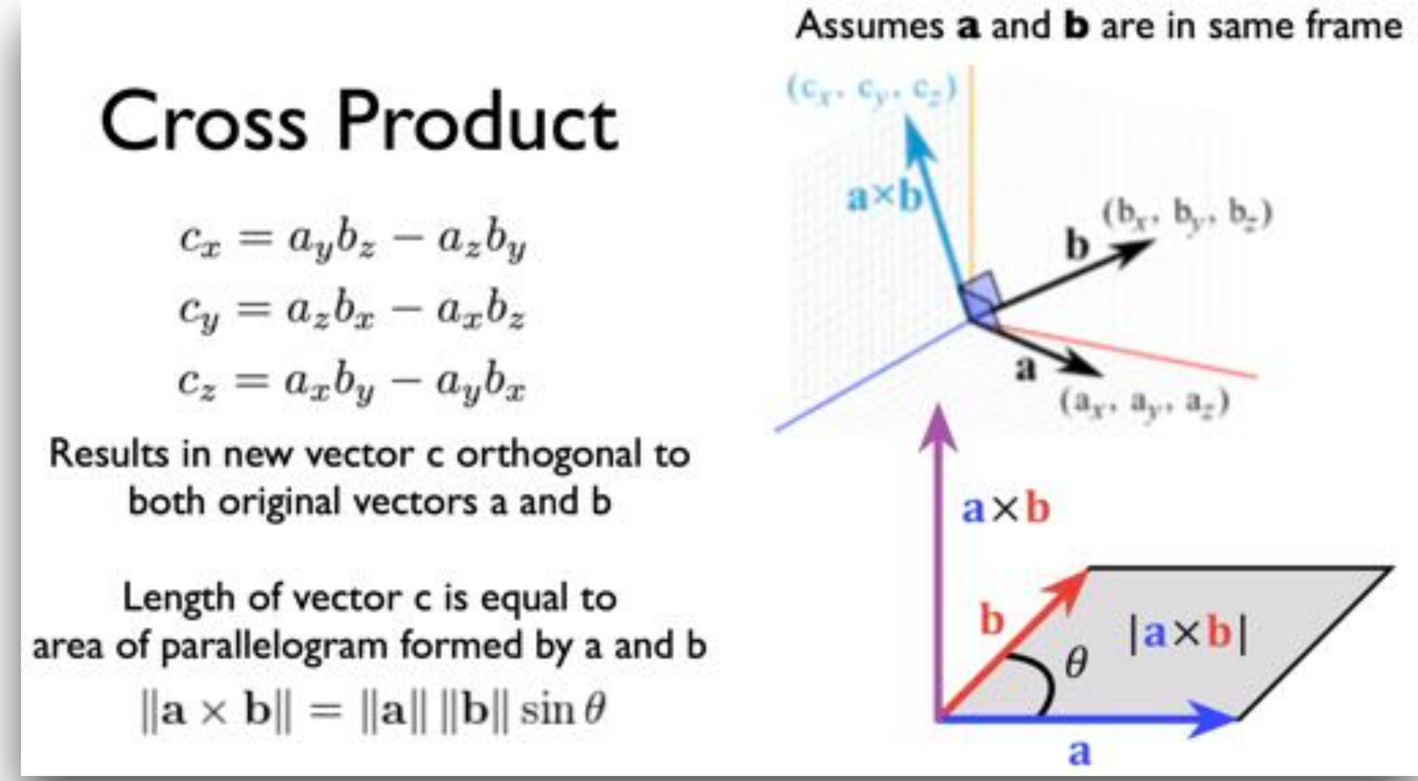
Magnitude and Unit Vector
The magnitude of a vector is the square root of the sum of squares of its components
$||a|| = \sqrt{a_1^2 + a_2^2 + \cdots + a_n^2}$
A unit vector has a magnitude of one. Normalization scales a vector to unit length.
$\hat{a} = \frac{a}{||a||}$
A vector can be multiplied by a scalar
$sa = \begin{bmatrix} sa_x \\ sa_y \\ sa_z \end{bmatrix}$



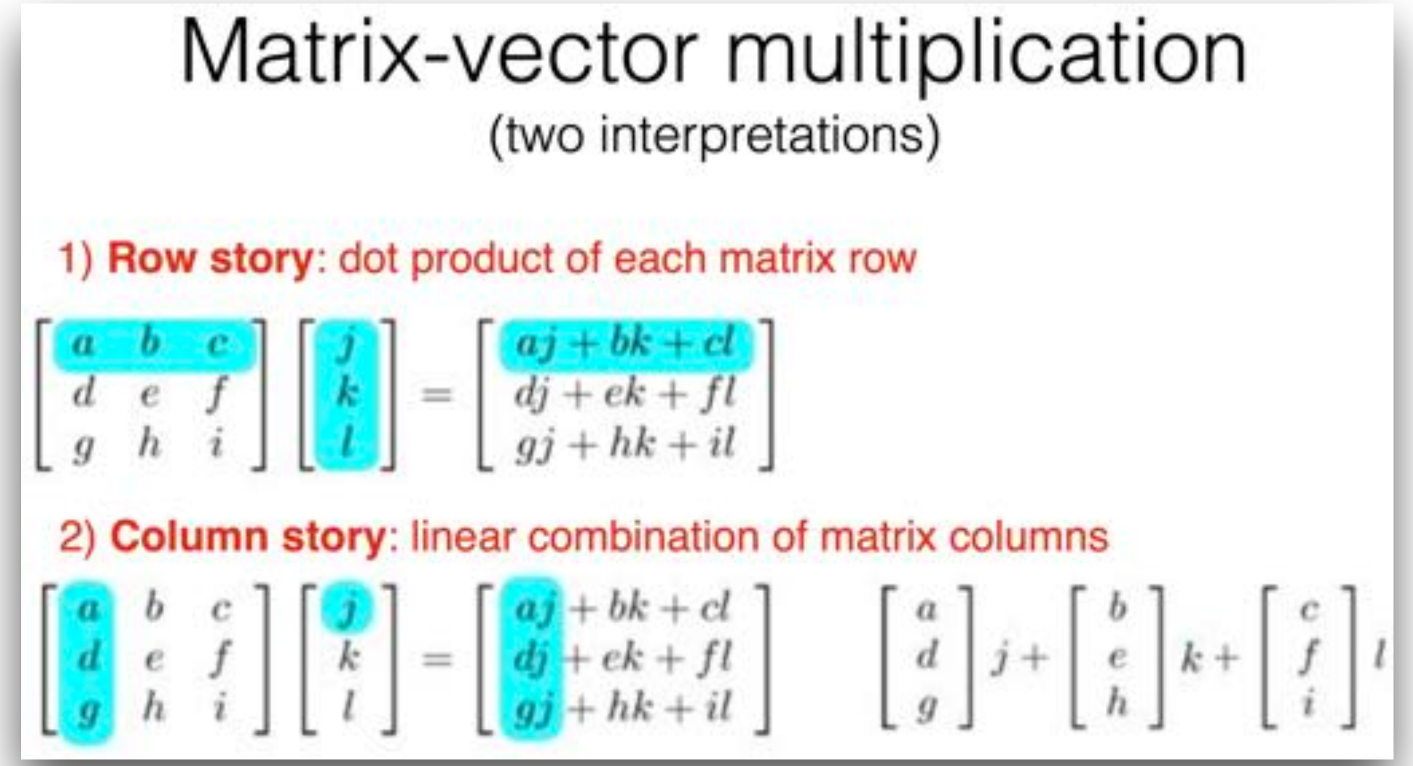
scalar result
Dot Product
$a \bullet b = a_x b_x + a_y b_y + a_z b_z = ||a|| ||b|| cos(\theta)$
Measures the similarity in direction of two vectors
$\begin{bmatrix} 2 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 2 \end{bmatrix} = 2 * 3 + 1 * 2 = 8$
|a|
|a| cos(θ)



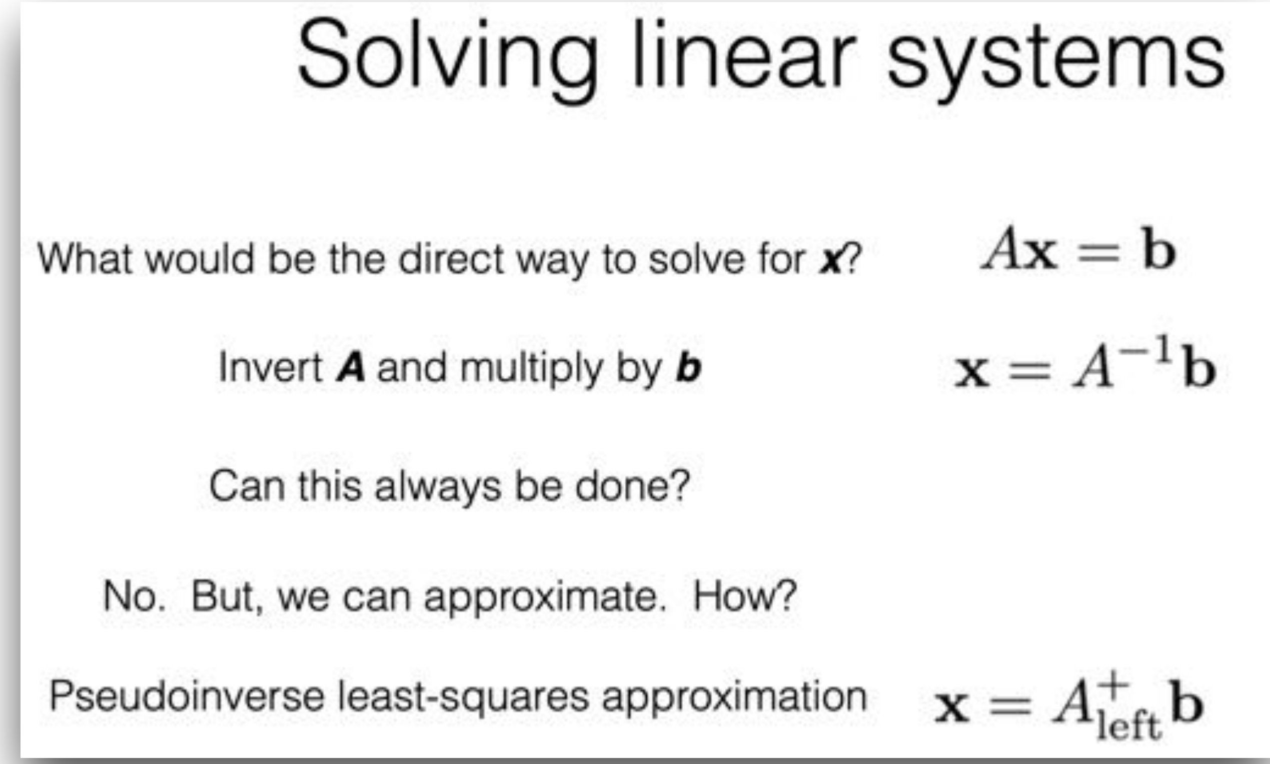
Projections
Dot products related to projections onto vectors.
Scalar projection of one vector onto another
$a_1 = |\mathbf{a}| \cos\theta = \mathbf{a} \cdot \hat{\mathbf{b}} = \mathbf{a} \cdot \frac{\mathbf{b}}{|\mathbf{b}|}$
Vector projection
$\mathbf{a}_1 = a_1 \hat{\mathbf{b}}$
$\hat{\mathbf{b}}$ is unit length



Cross Product
Assumes a and b are in same frame
$c_x = a_y b_z - a_z b_y$
$c_y = a_z b_x - a_x b_z$
$c_z = a_x b_y - a_y b_x$
Results in new vector c orthogonal to both original vectors a and b
Length of vector c is equal to area of parallelogram formed by a and b
$||\mathbf{a} \times \mathbf{b}|| = ||\mathbf{a}|| \, ||\mathbf{b}|| \sin\theta$
a×b
|a×b|



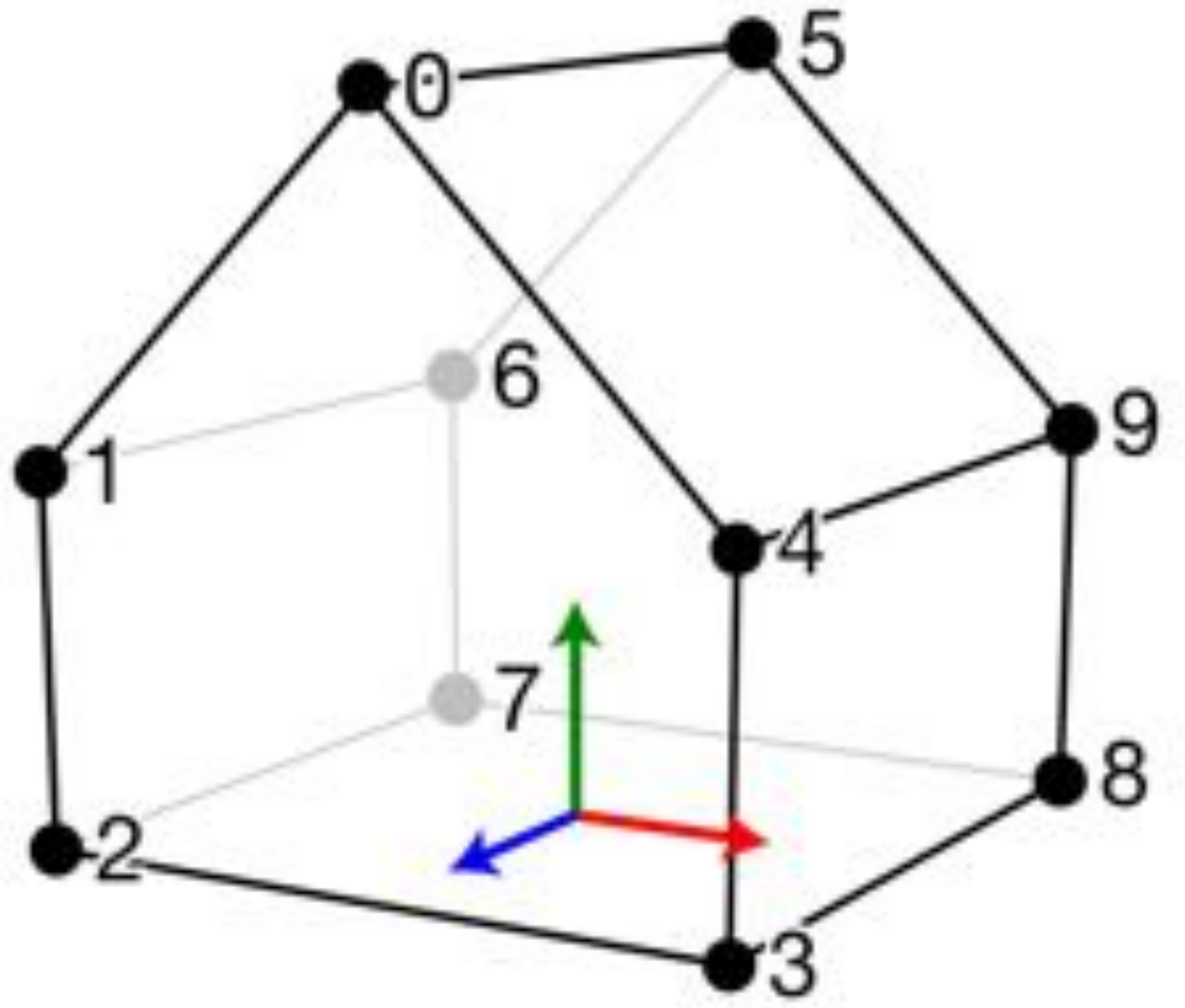
Matrix-vector multiplication
(two interpretations)
1) Row story: dot product of each matrix row
$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} j \\ k \\ l \end{bmatrix} = \begin{bmatrix} aj + bk + cl \\ dj + ek + fl \\ gj + hk + il \end{bmatrix}$
2) Column story: linear combination of matrix columns
$\begin{bmatrix} a \\ d \\ g \end{bmatrix} j + \begin{bmatrix} b \\ e \\ h \end{bmatrix} k + \begin{bmatrix} c \\ f \\ i \end{bmatrix} l$



Solving linear systems
What would be the direct way to solve for x?
$A\mathbf{x} = \mathbf{b}$
Invert A and multiply by b
$\mathbf{x} = A^{-1}\mathbf{b}$
Can this always be done?
No. But, we can approximate. How?
Pseudoinverse least-squares approximation
$\mathbf{x} = A^{+}_{\text{left}}\mathbf{b}$

How to define a Link Geometry

Link Geometry



http://csc.lsu.edu/~kooima/courses/csc4356/

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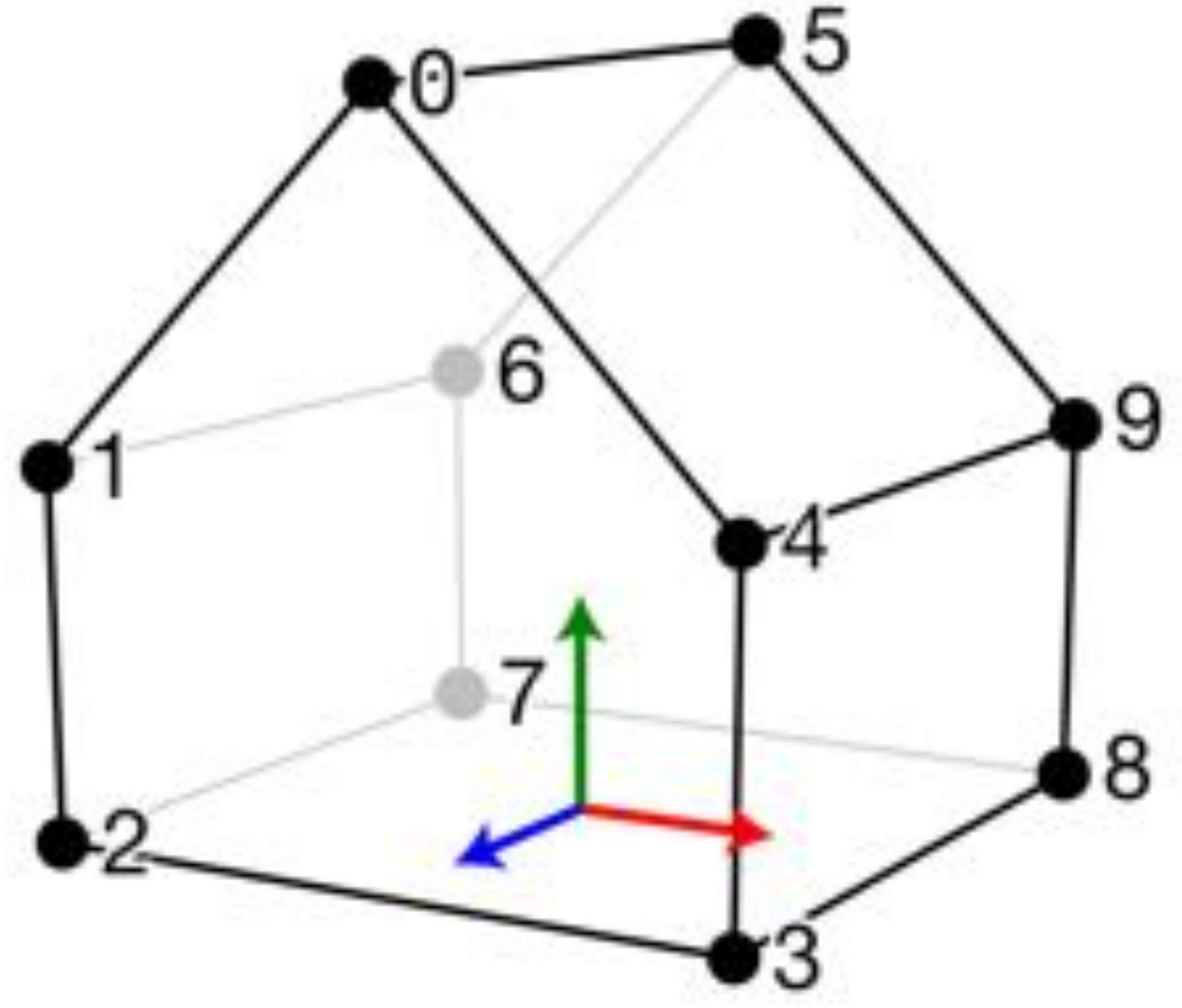
Link Geometry



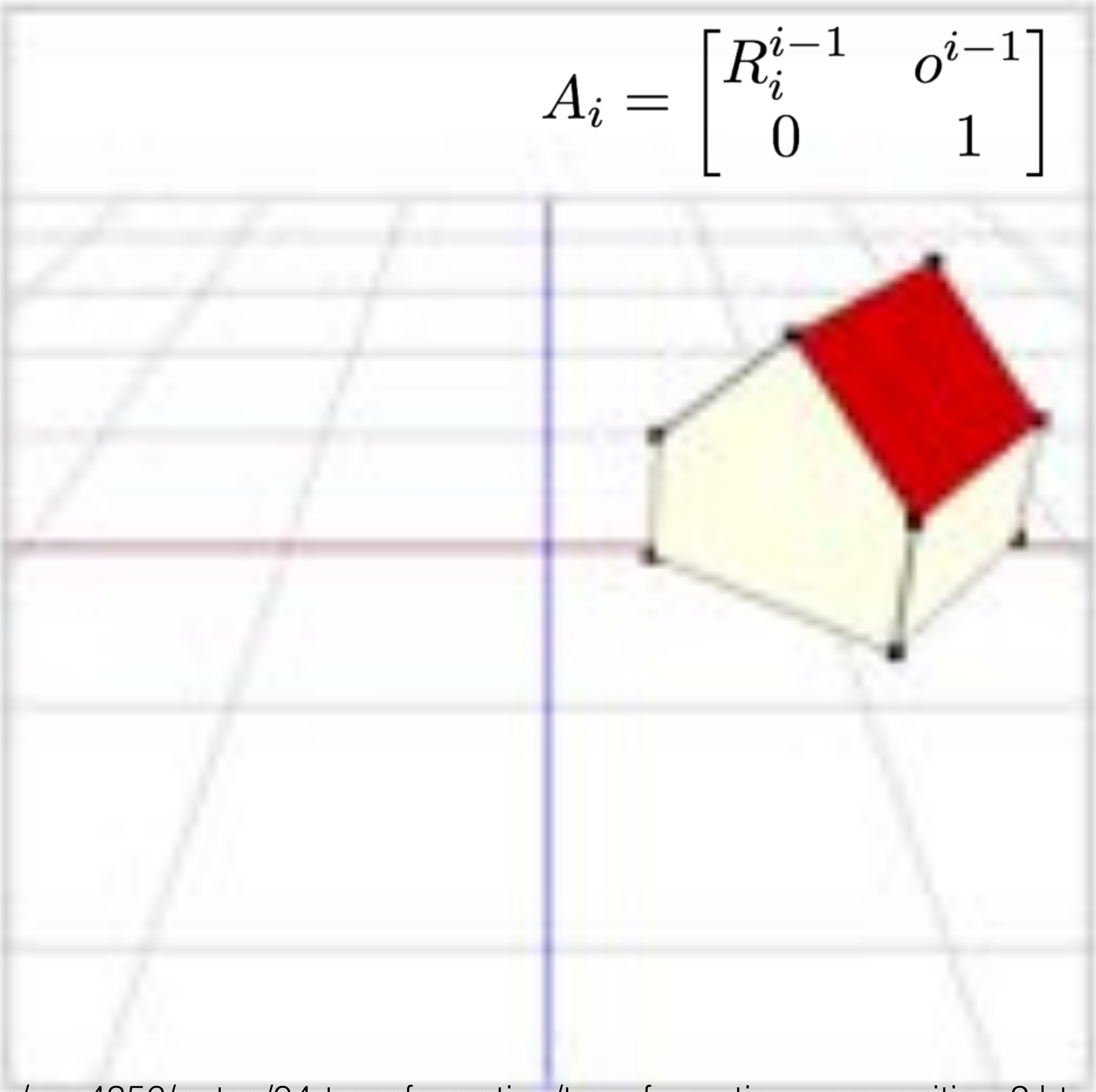
vertex index   vertex location

i	x	y	z
0	0.0	1.0	0.5
1	–0.5	0.5	0.5
2	–0.5	0.0	0.5
3	0.5	0.0	0.5
4	0.5	0.5	0.5
5	0.0	1.0	–0.5
6	–0.5	0.5	–0.5
7	–0.5	0.0	–0.5
8	0.5	0.0	–0.5
9	0.5	0.5	–0.5

Each robot link has a geometry specified as 3D vertices.
Vertices are connected into faces of the object's surface.
Vertices are defined wrt. the frame of the robots' link.

http://csc.lsu.edu/~kooima/courses/csc4356/

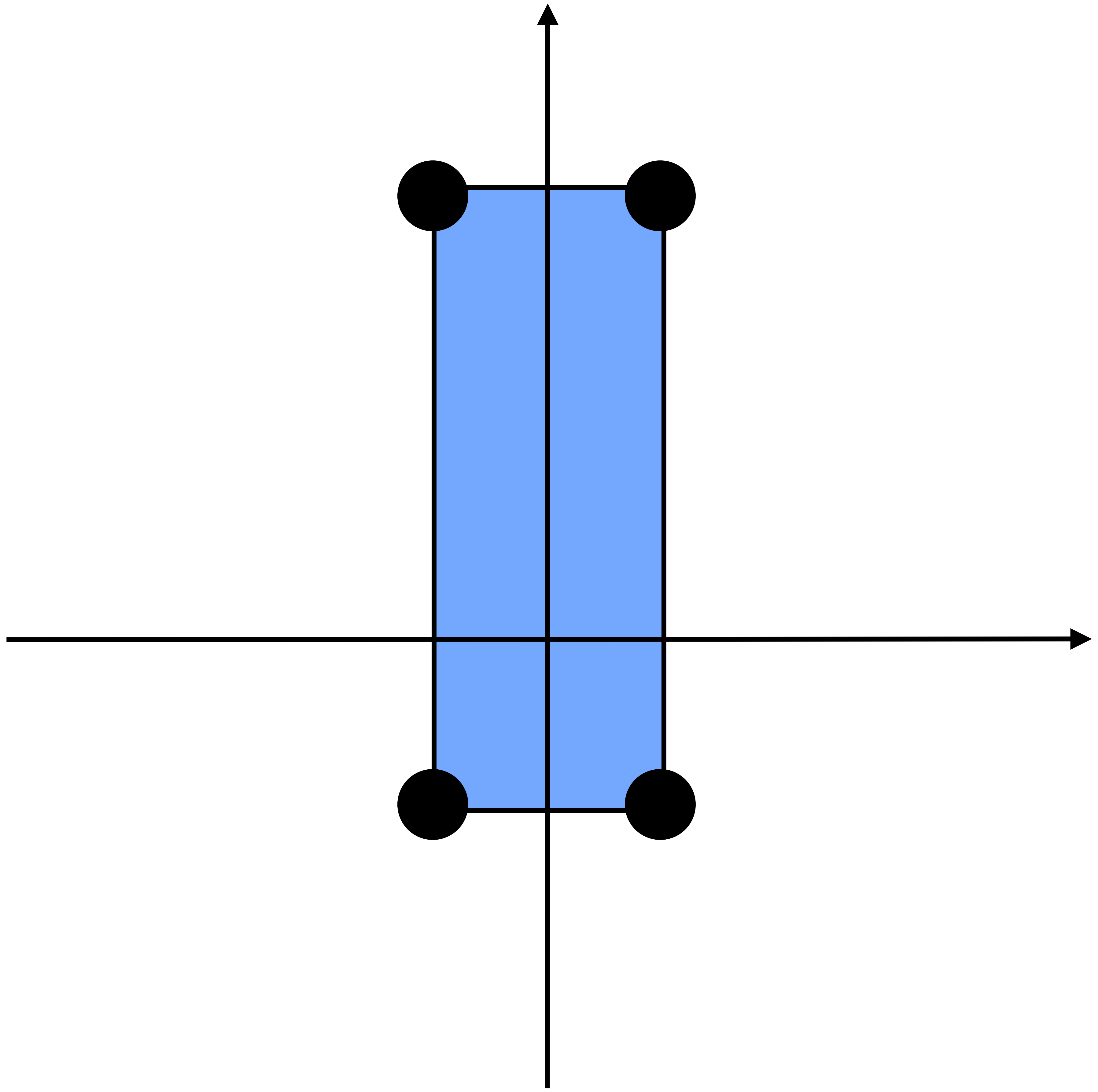
As the link frame moves, the geometry moves with it.



http://csc.lsu.edu/~kooima/courses/csc4356/notes/04-transformation/transformation-composition-2.html

2D Rotation

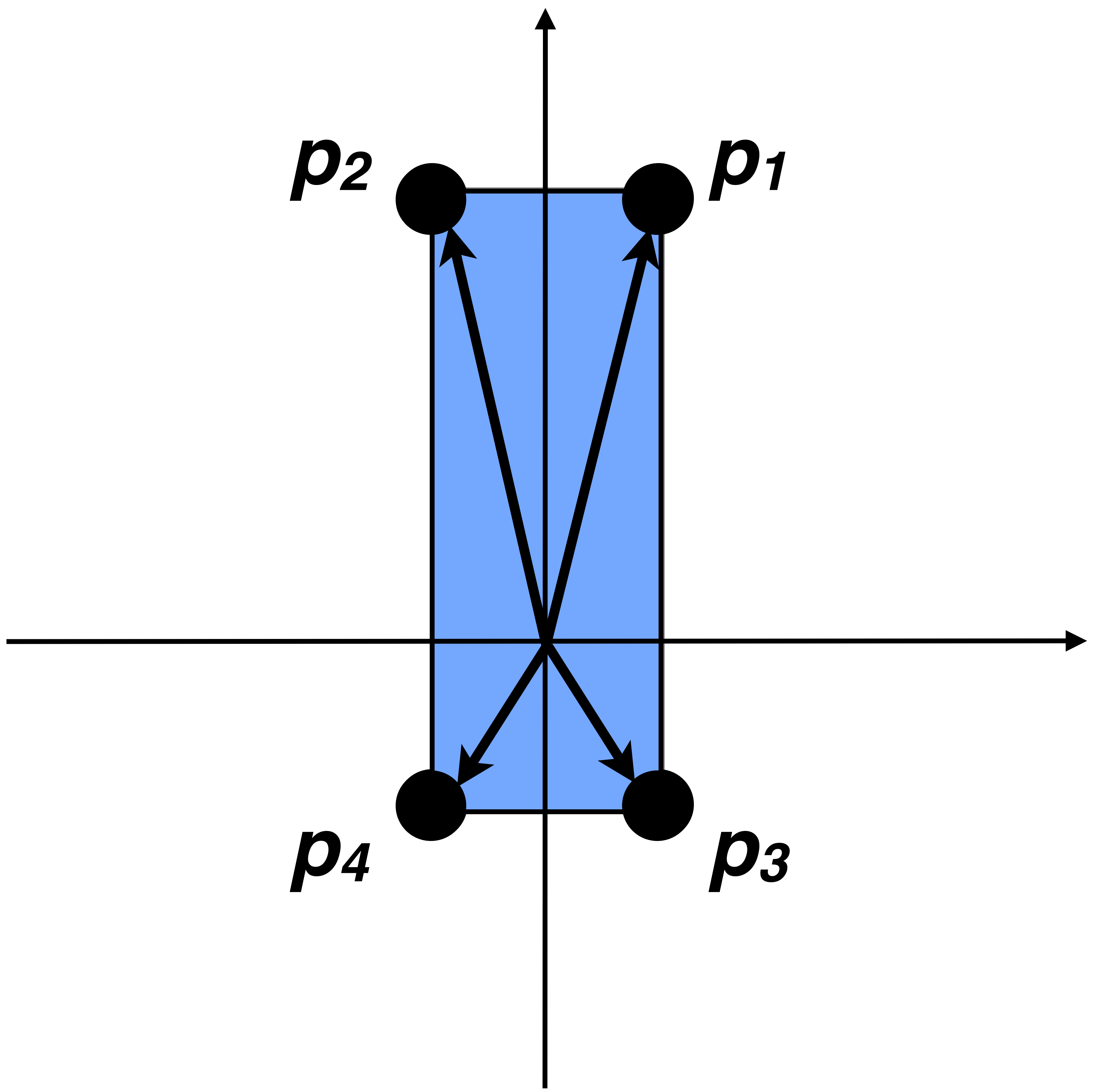
- Consider a link for a 2D robot with a box geometry of 4 vertices



2D Rotation

- Consider a link for a 2D robot with a box geometry of 4 vertices
- Vectors express position of vertices with respect to joint (at origin)

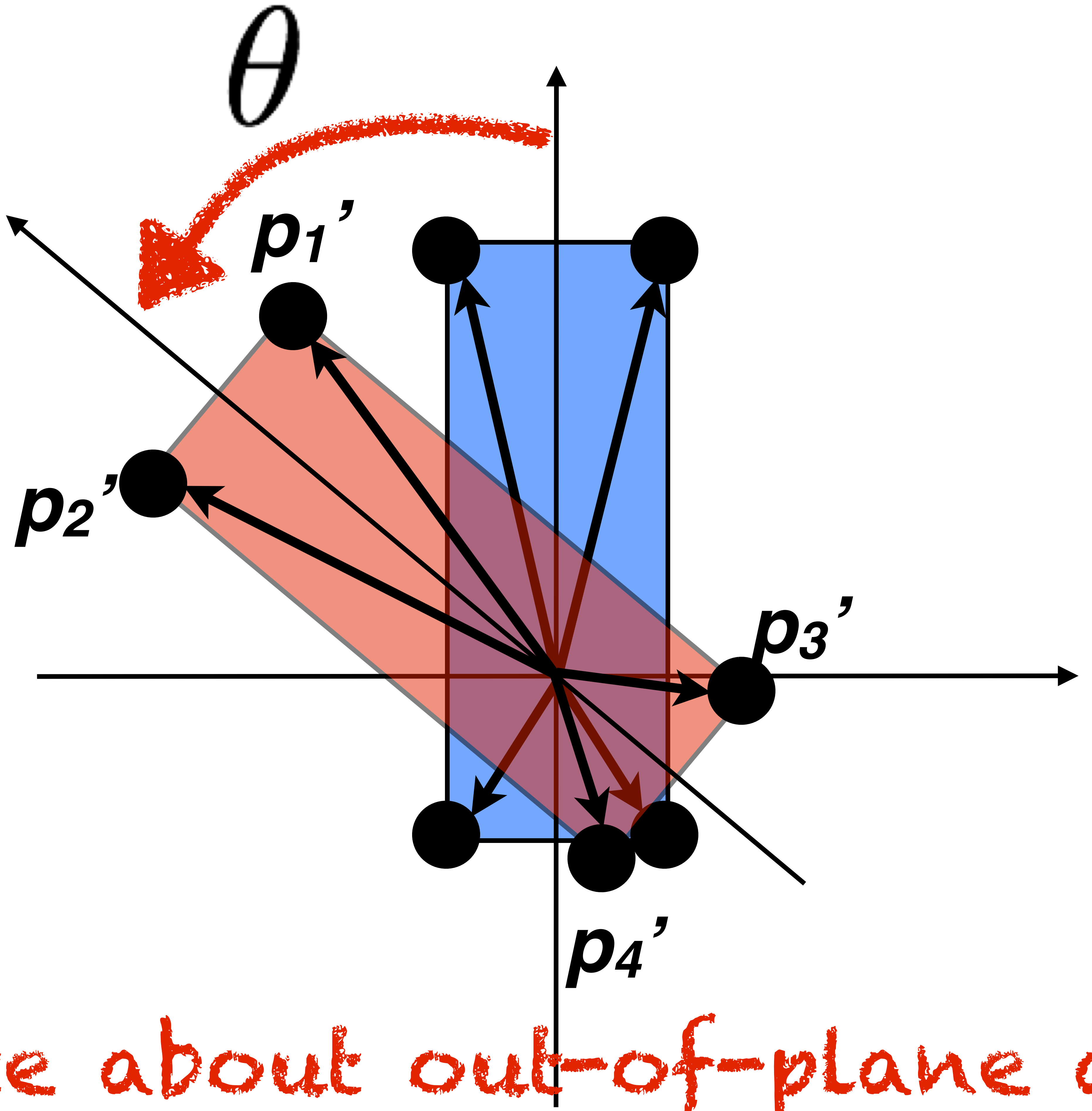
$$\boldsymbol{p_i} = [x_i, y_i]$$



2D Rotation

- Consider a link for a 2D robot with a box geometry of 4 vertices
- Vectors express position of vertices with respect to joint (at origin)
- How to rotate link geometry based on movement of the joint?

[redacted]

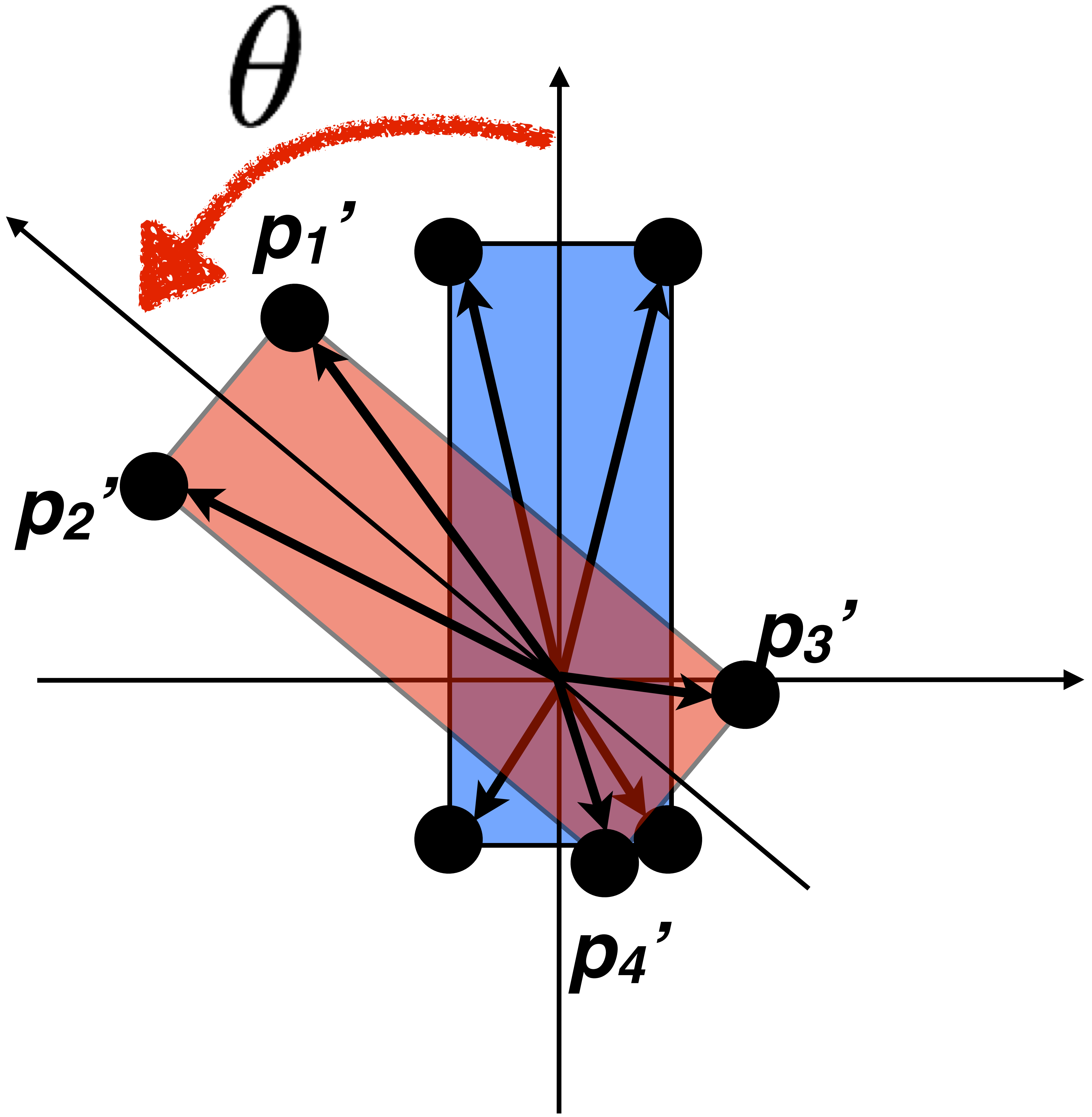


rotate about out-of-plane axis

2D Rotation

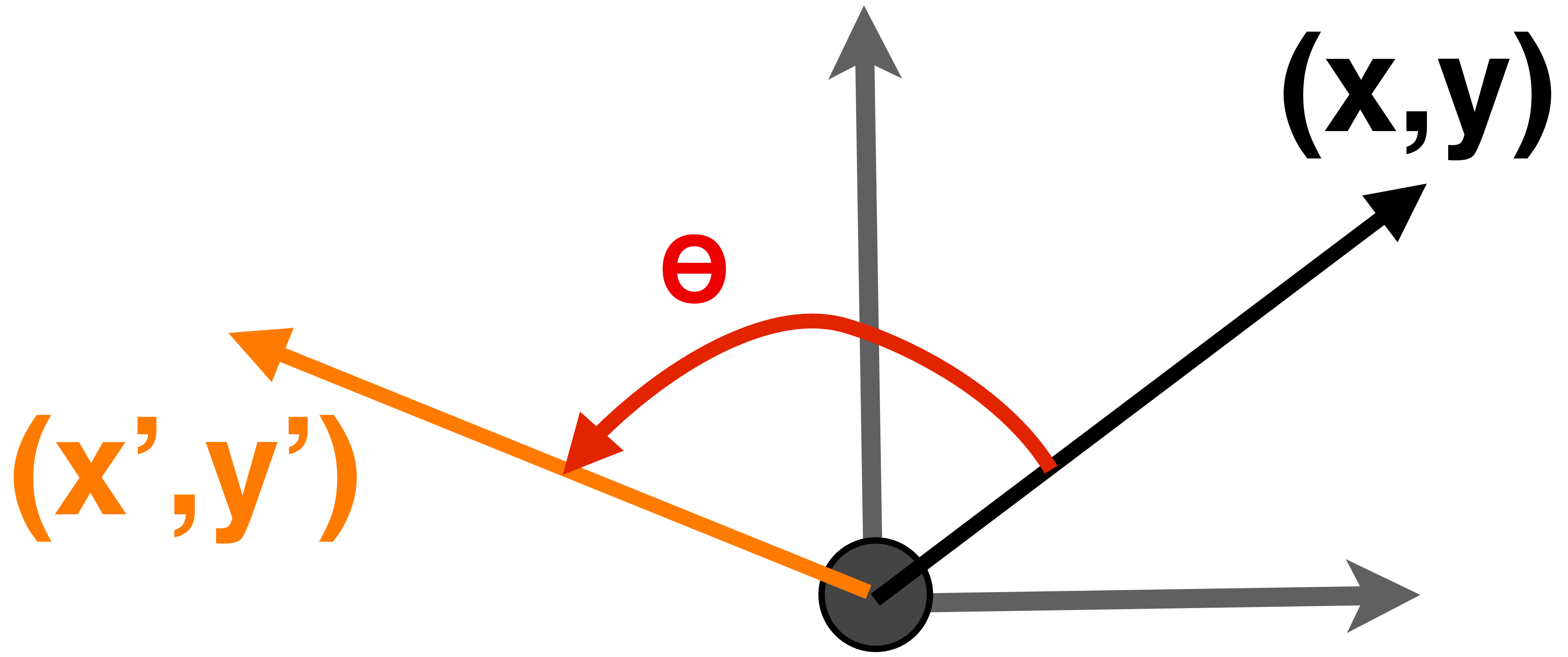
- Consider a link for a 2D robot with a box geometry of 4 vertices
- Vectors express position of vertices with respect to joint (at origin)
- How to rotate link geometry based on movement of the joint?

$$x' = x \cdot \cos(\theta) - y \cdot \sin(\theta)$$
$$y' = x \cdot \sin(\theta) + y \cdot \cos(\theta)$$



2D Rotation Matrix

(counterclockwise)

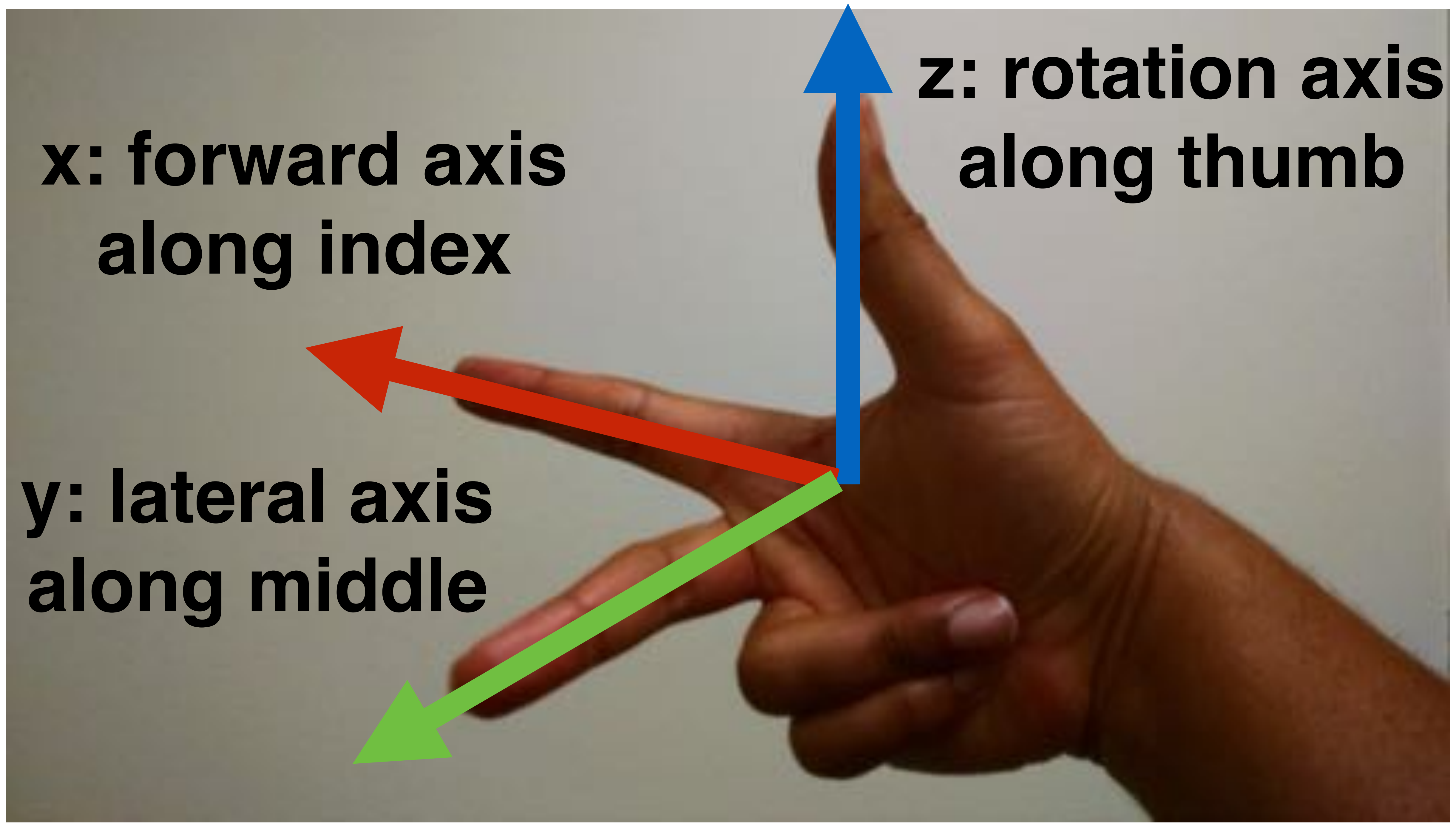


$$\begin{array}{c} \\ \\ \end{array}\qquad\qquad\qquad \color{red}{R(\theta)}$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

- Matrix multiply vector by 2D rotation matrix R
- Matrix parameterized by rotation angle θ
- Remember: this rotation is counterclockwise

Right-hand Rule



rotation occurs about axis from forward towards lateral,
or the "curl" of the fingers

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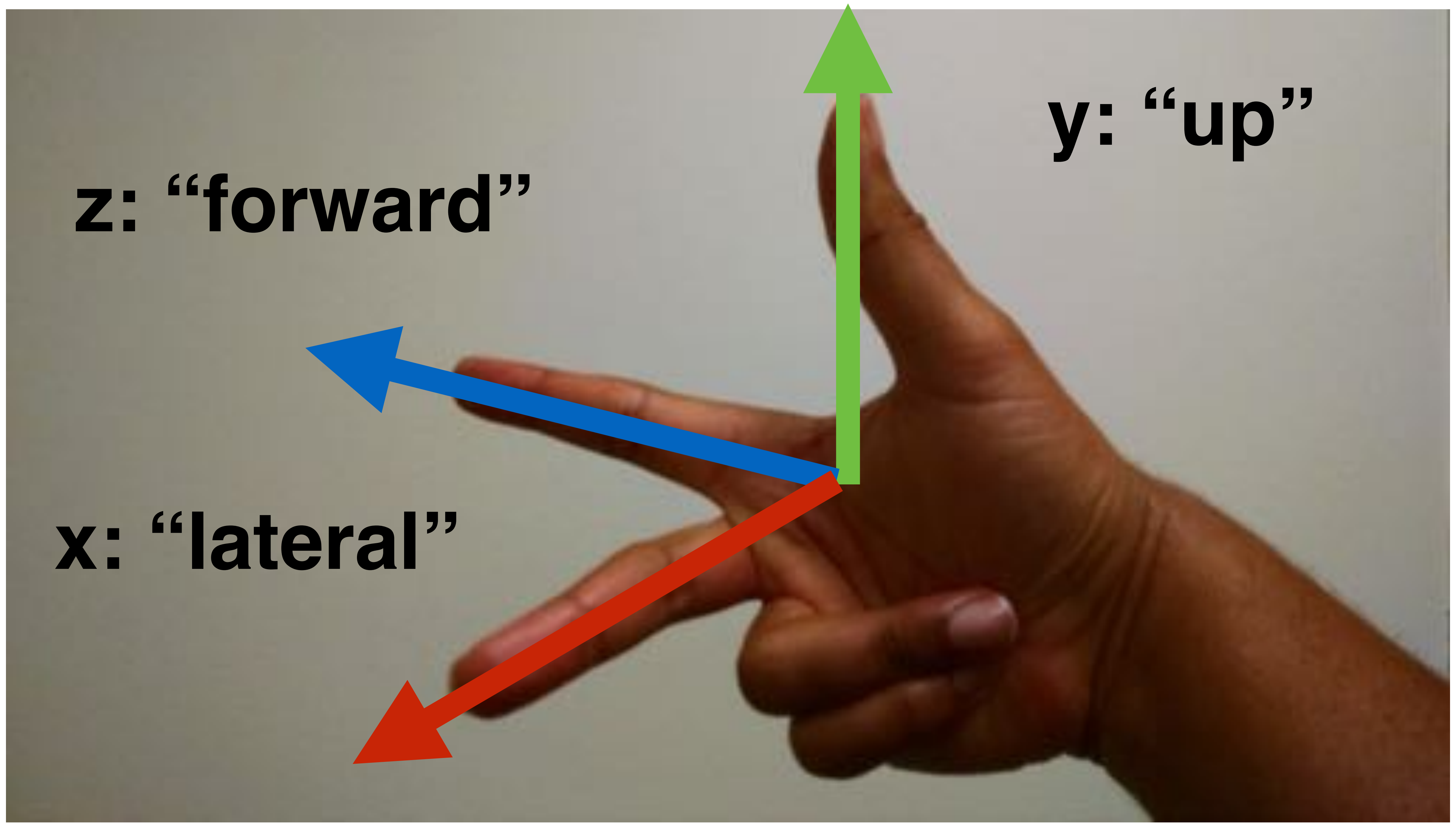
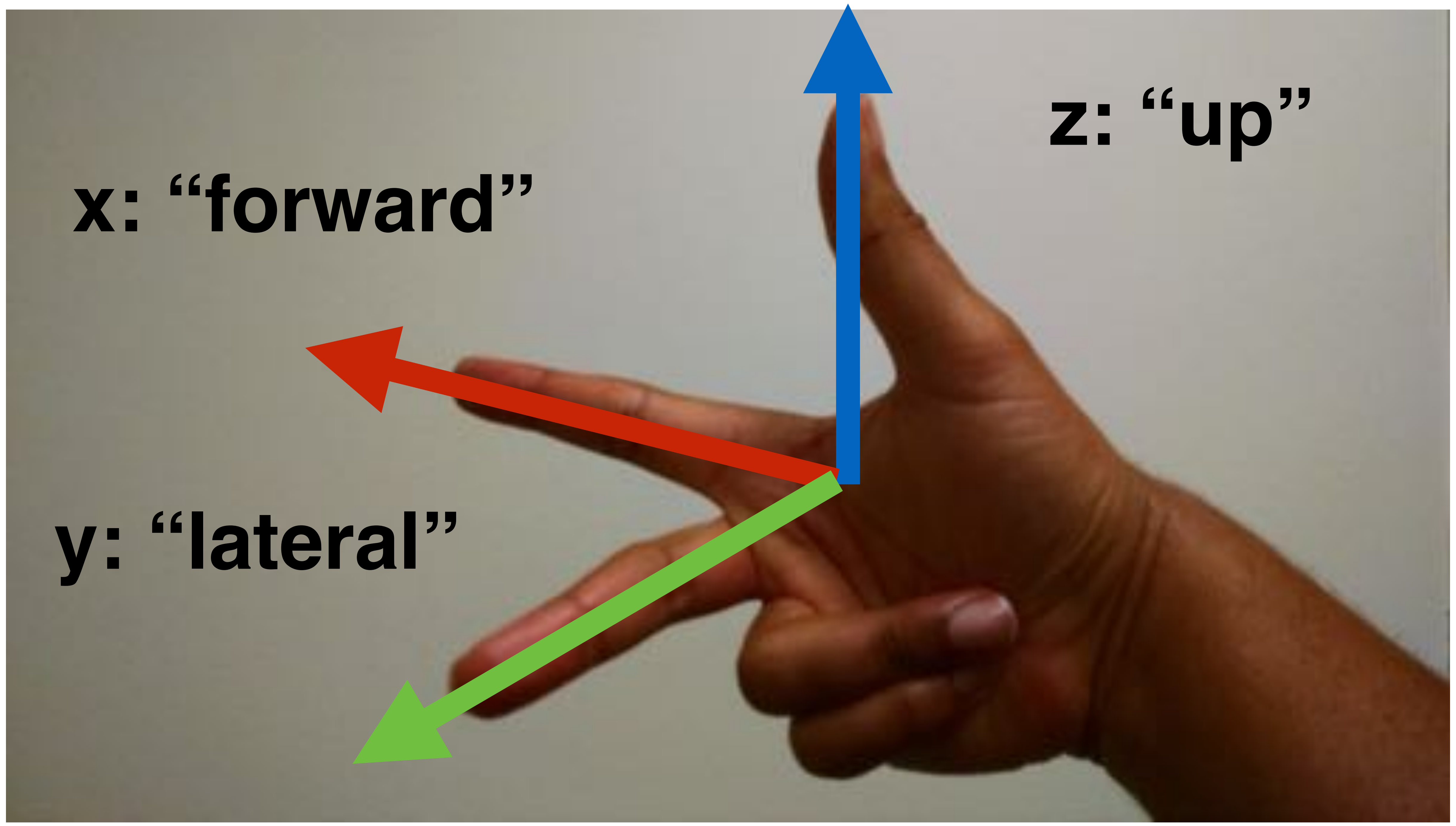
Coordinate conventions



threejs and KinEval
(used in the browser)



ROS and most of robotics
(used in URDF and rosbridge)

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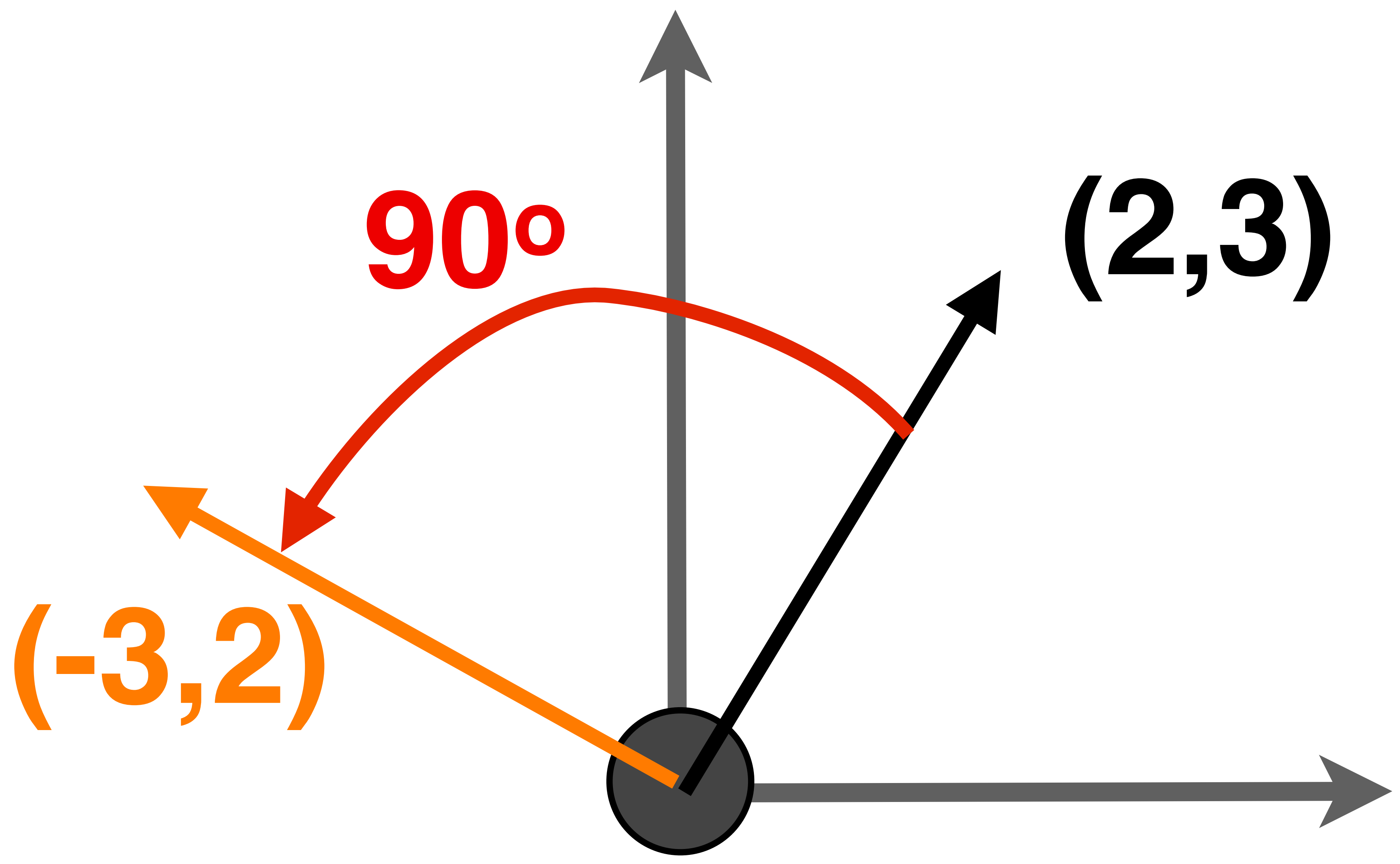
Checkpoint

- What is the 2D matrix for a rotation by 0 degrees?

- What is the 2D matrix for a rotation by 90 degrees?

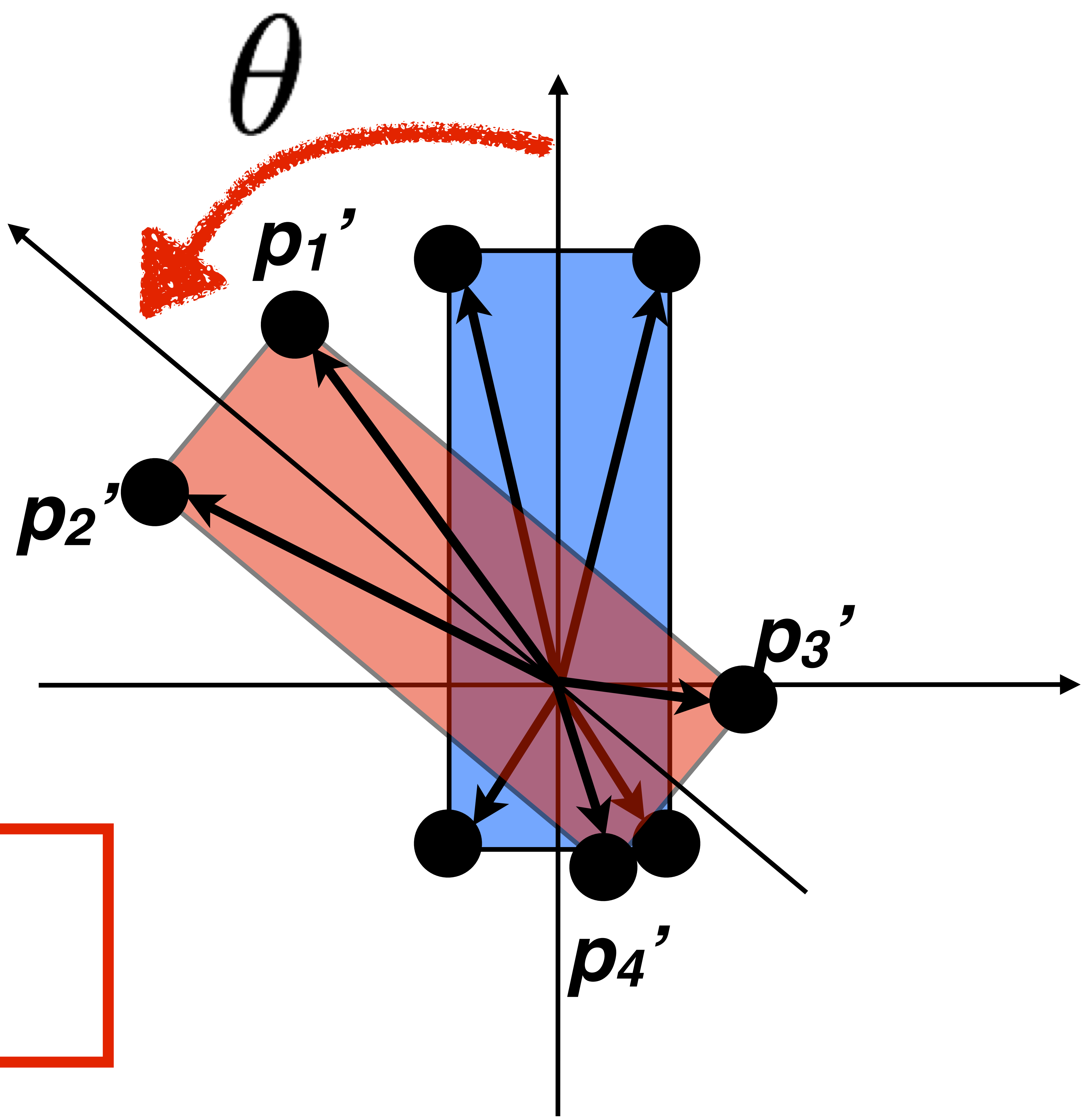
Example



cos(90°) = 0

sin(90°) = 1

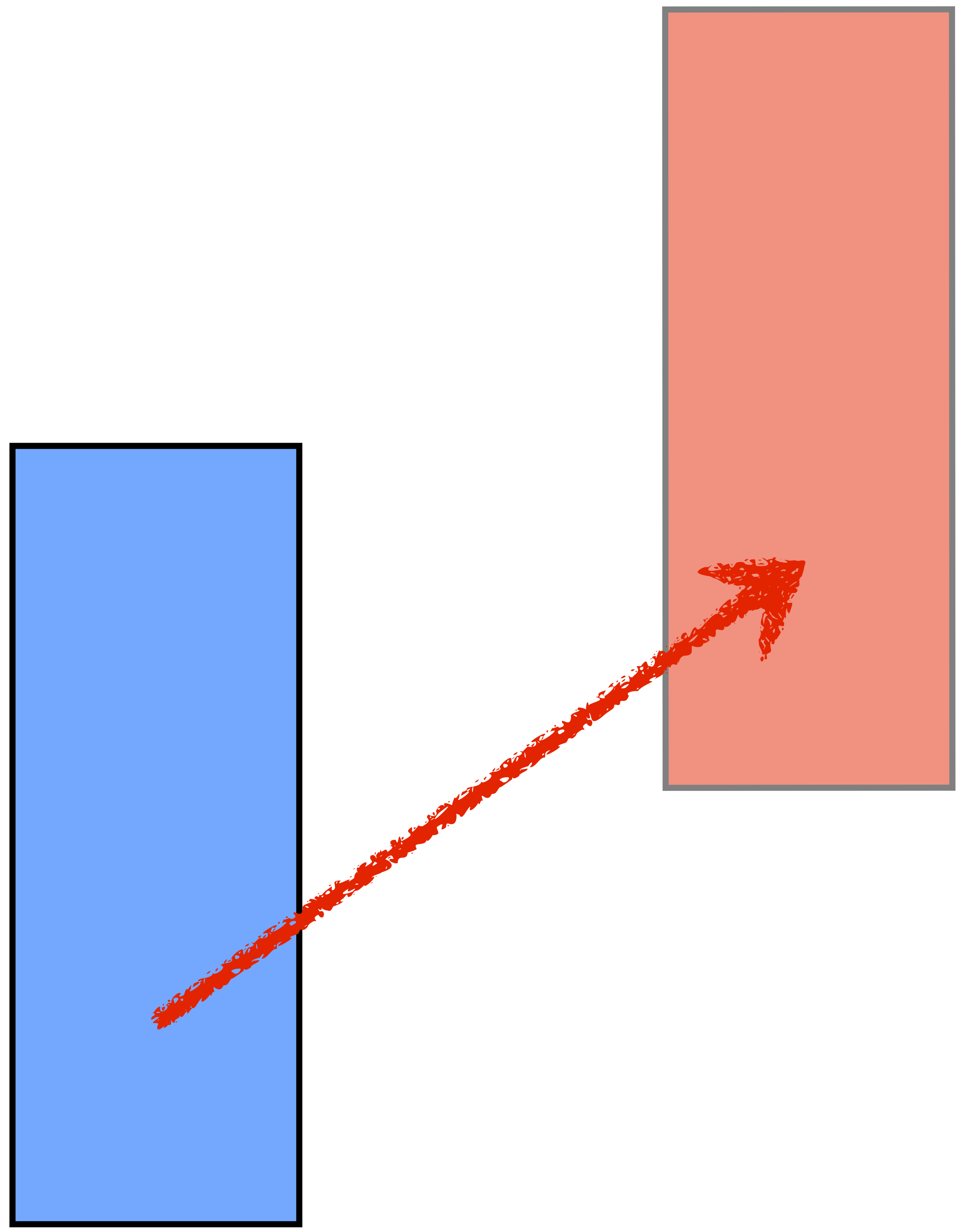
$$\begin{bmatrix} -3 \\ 2 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}}_{R(90^o)} \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$



Note: one matrix multiply can transform all vertices

$$\begin{bmatrix} p'_{1x} & p'_{2x} & p'_{3x} & p'_{4x} \\ p'_{1y} & p'_{2y} & p'_{3y} & p'_{4y} \end{bmatrix} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} p_{1x} & p_{2x} & p_{3x} & p_{4x} \\ p_{1y} & p_{2y} & p_{3y} & p_{4y} \end{bmatrix}$$

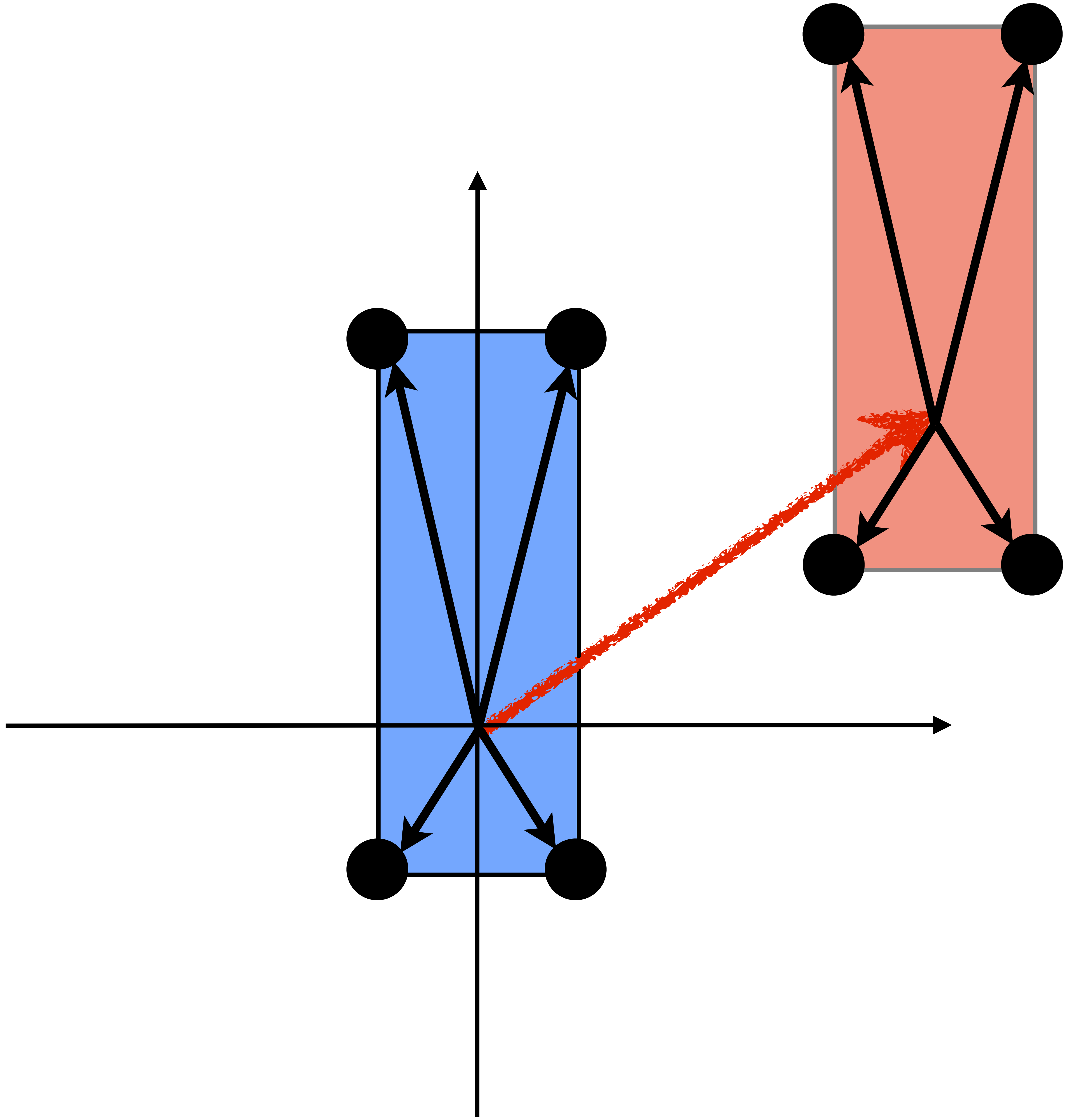
We can rotate.
Can we also translate?



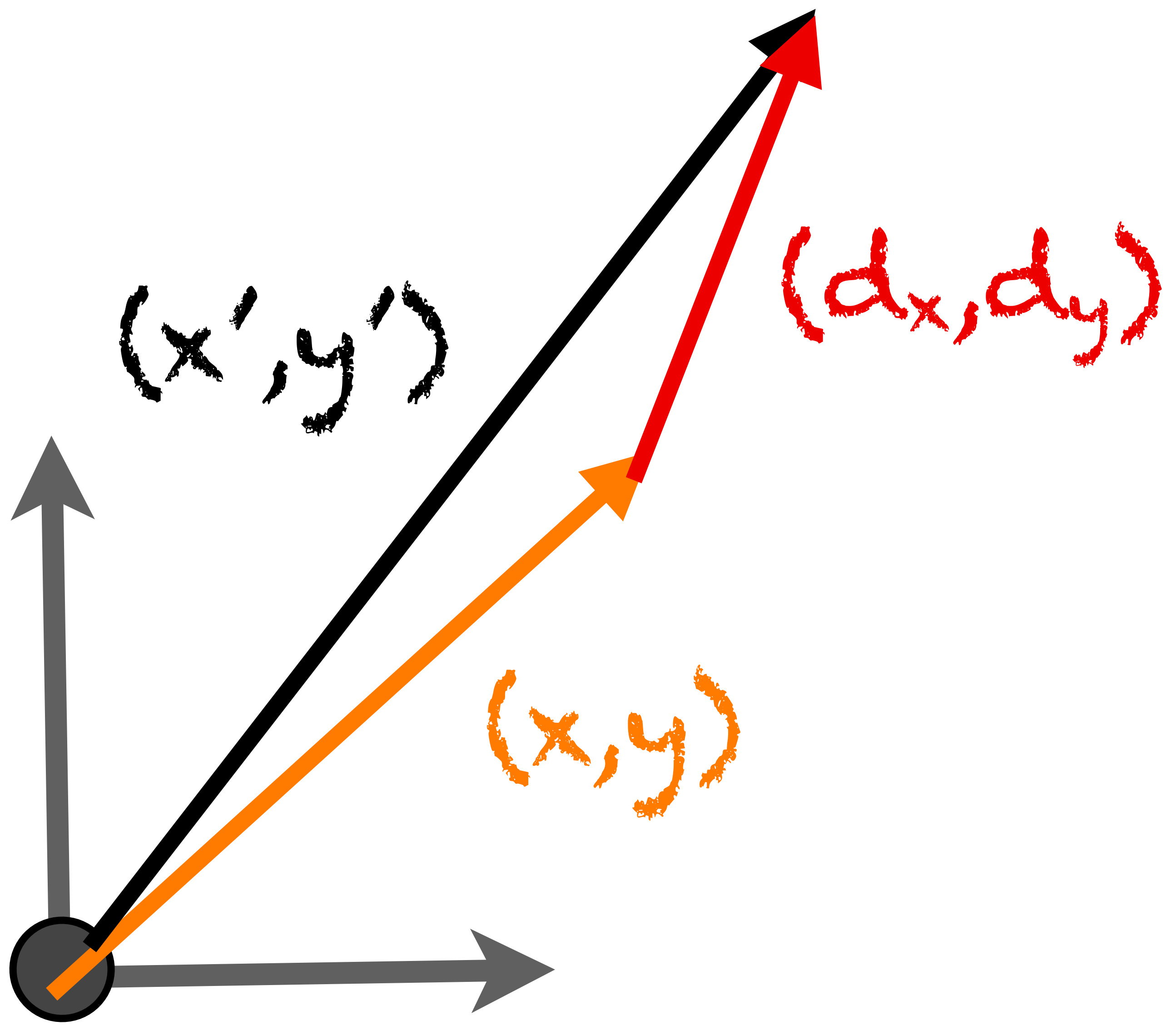
2D Translation

- Consider a link for a 2D robot with a box geometry of 4 vertices
- Vectors express position of vertices with respect to joint (at origin)
- How to translate link geometry to new location?

$$x' = x + d_x$$
$$y' = y + d_y$$



2D Translation Matrix



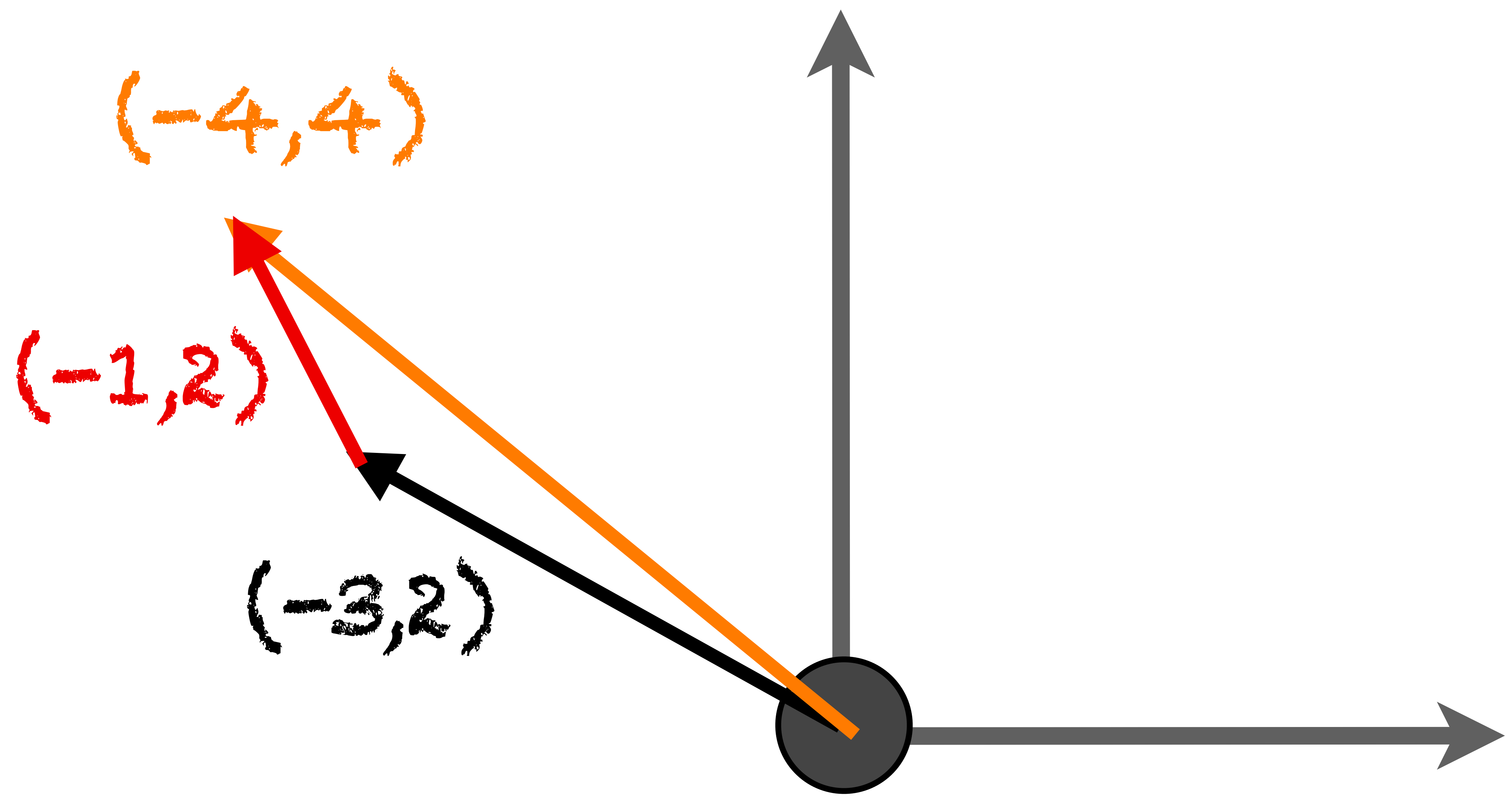
$$\underbrace{}_{}\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} x + d_x \\ y + d_y \\ 1 \end{bmatrix} = \overbrace{\begin{bmatrix} 1 & 0 & d_x \\ 0 & 1 & d_y \\ 0 & 0 & 1 \end{bmatrix}}^{D(d_x,d_y)} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

- Requires homogeneous coordinates
  - 3D vector of 2D position concatenated with a 1
  - A plane at *z=1* in a three dimensional space
- Matrix parameterized by horizontal and vertical displacement (d_x, d_y)

Checkpoint

- What is the 2D matrix for a translation by [-1,2]?

Example

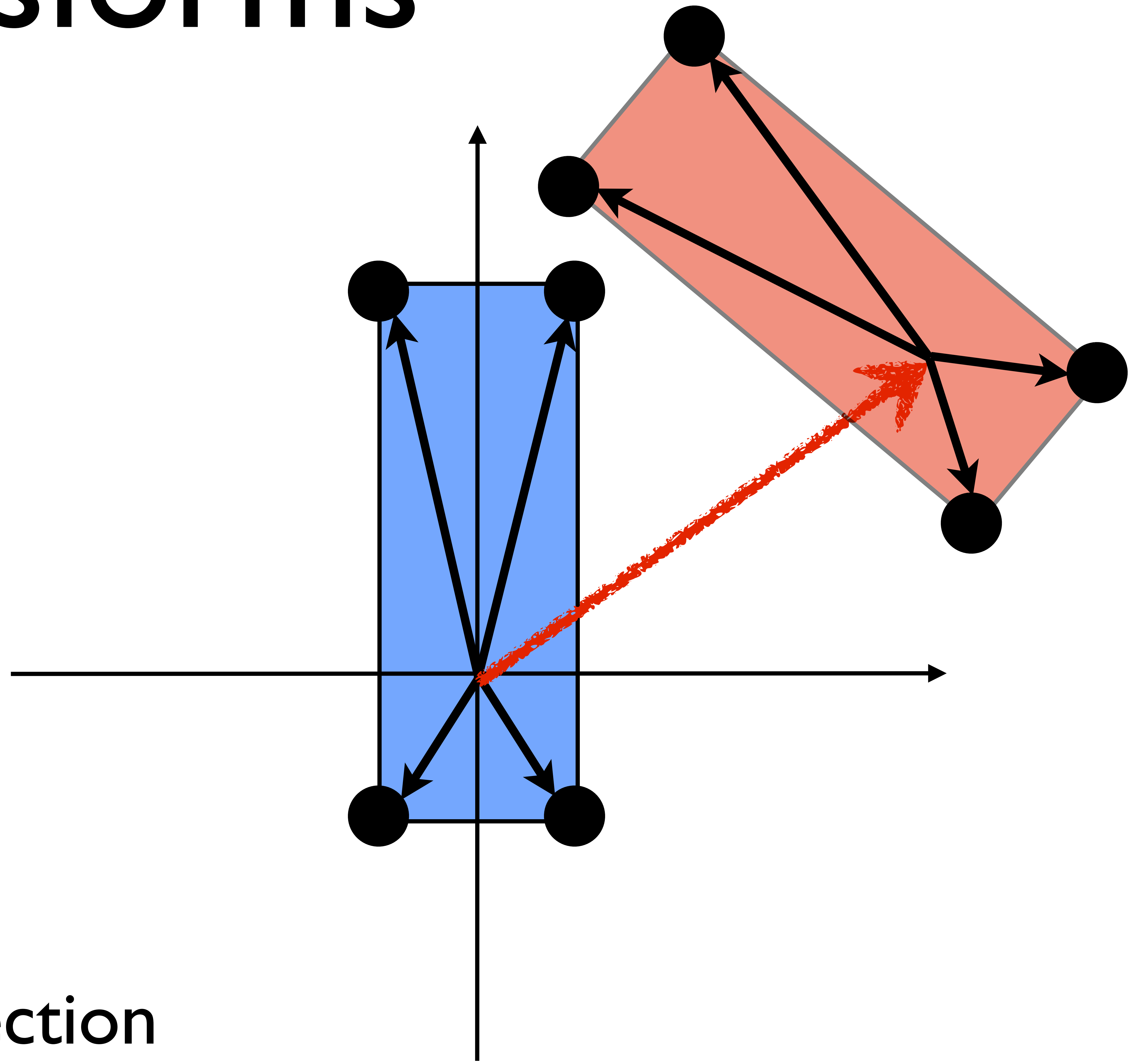


$$\begin{bmatrix} -4 \\ 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix}$$

D(-1,2)

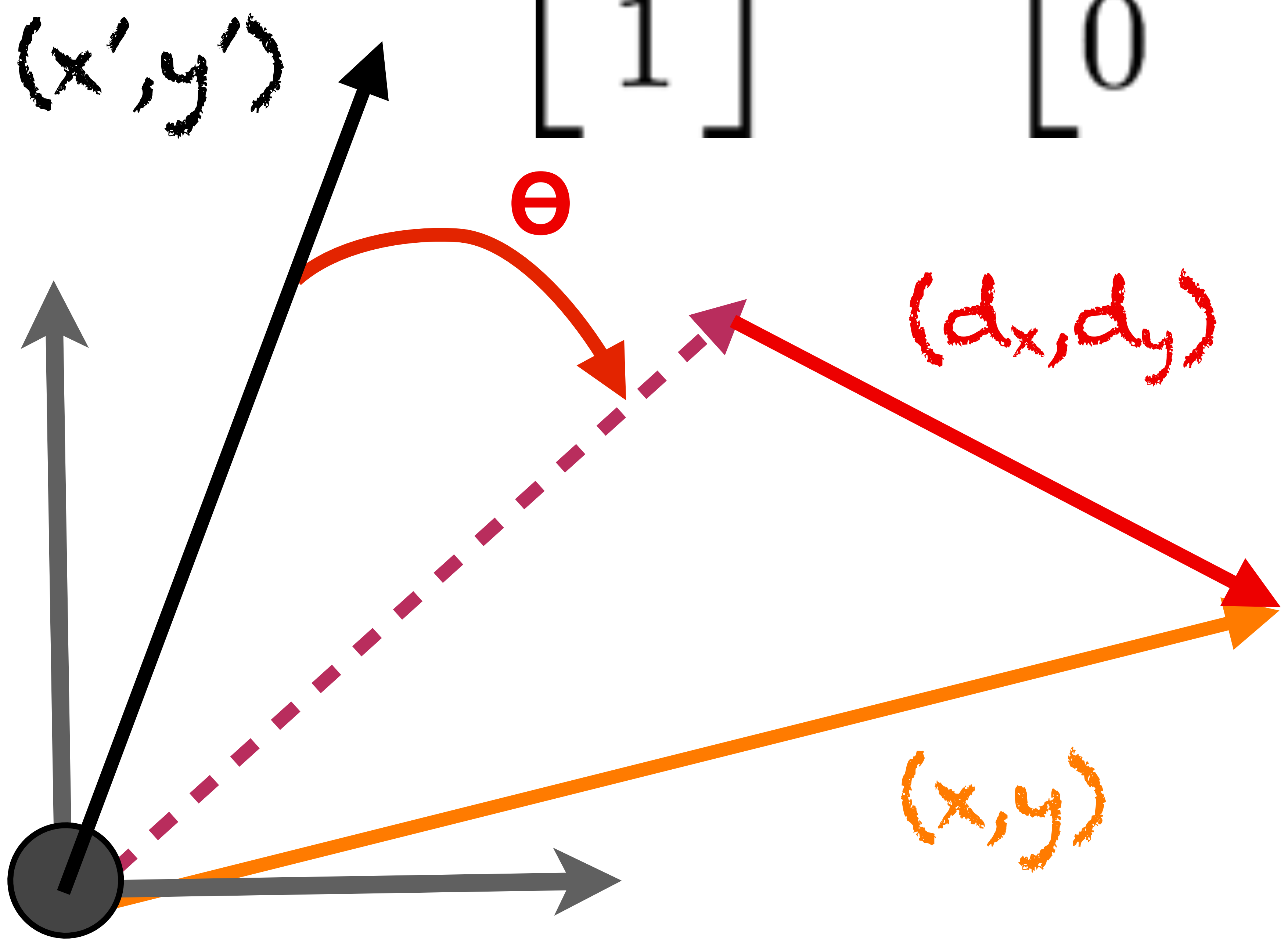
Rigid motions and Affine transforms

- Consider a link for a 2D robot with a box geometry of 4 vertices
- Vectors express position of vertices with respect to joint (at origin)
- How to both <u>rotate and translate</u> link geometry?
  - Rigid motion: rotate then translate
  - Affine transform: allows for rotation, translation, scaling, shearing, and reflection



Composition of Rotation and Translation

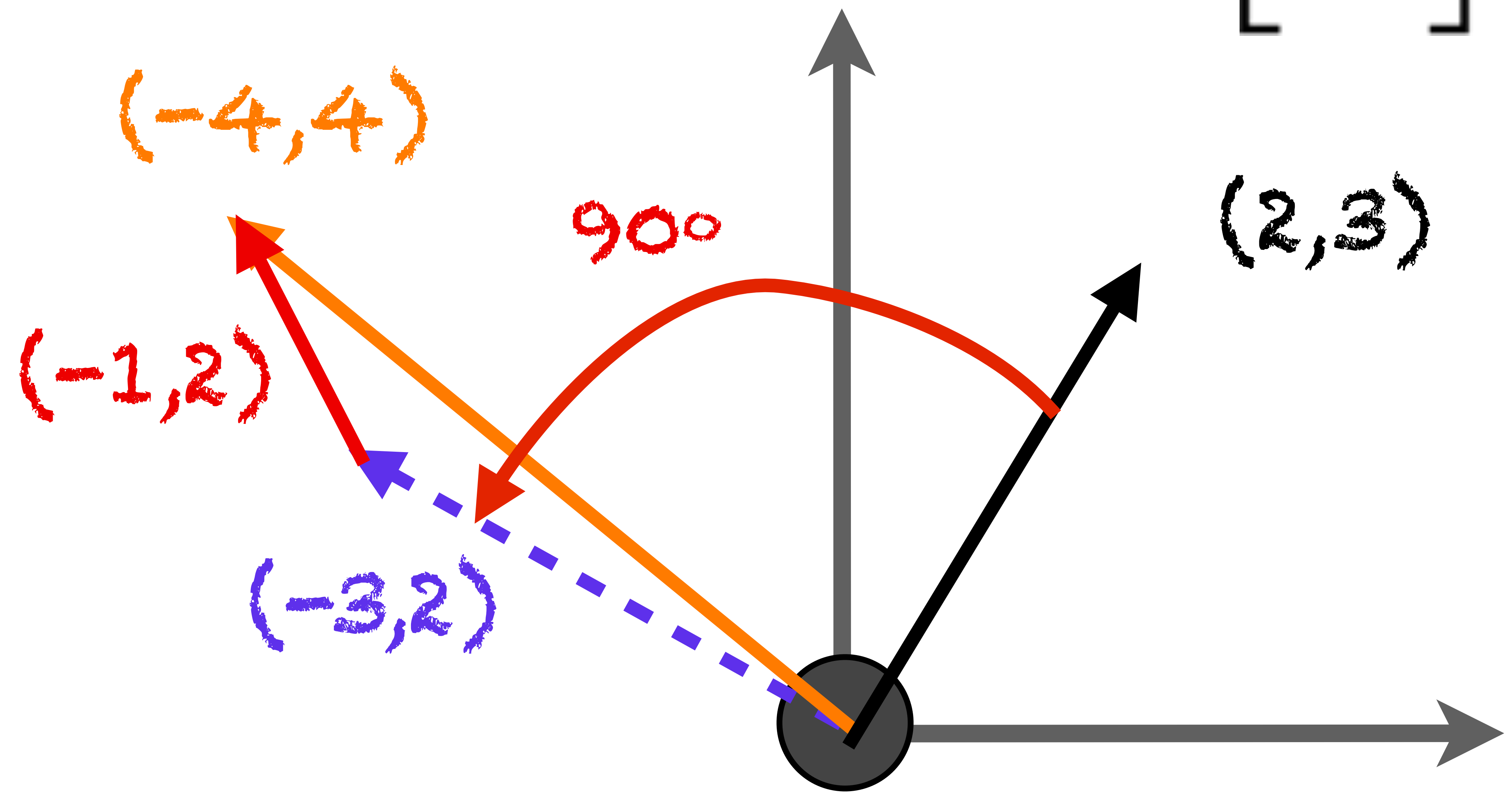
$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & d_x \\ 0 & 1 & d_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$



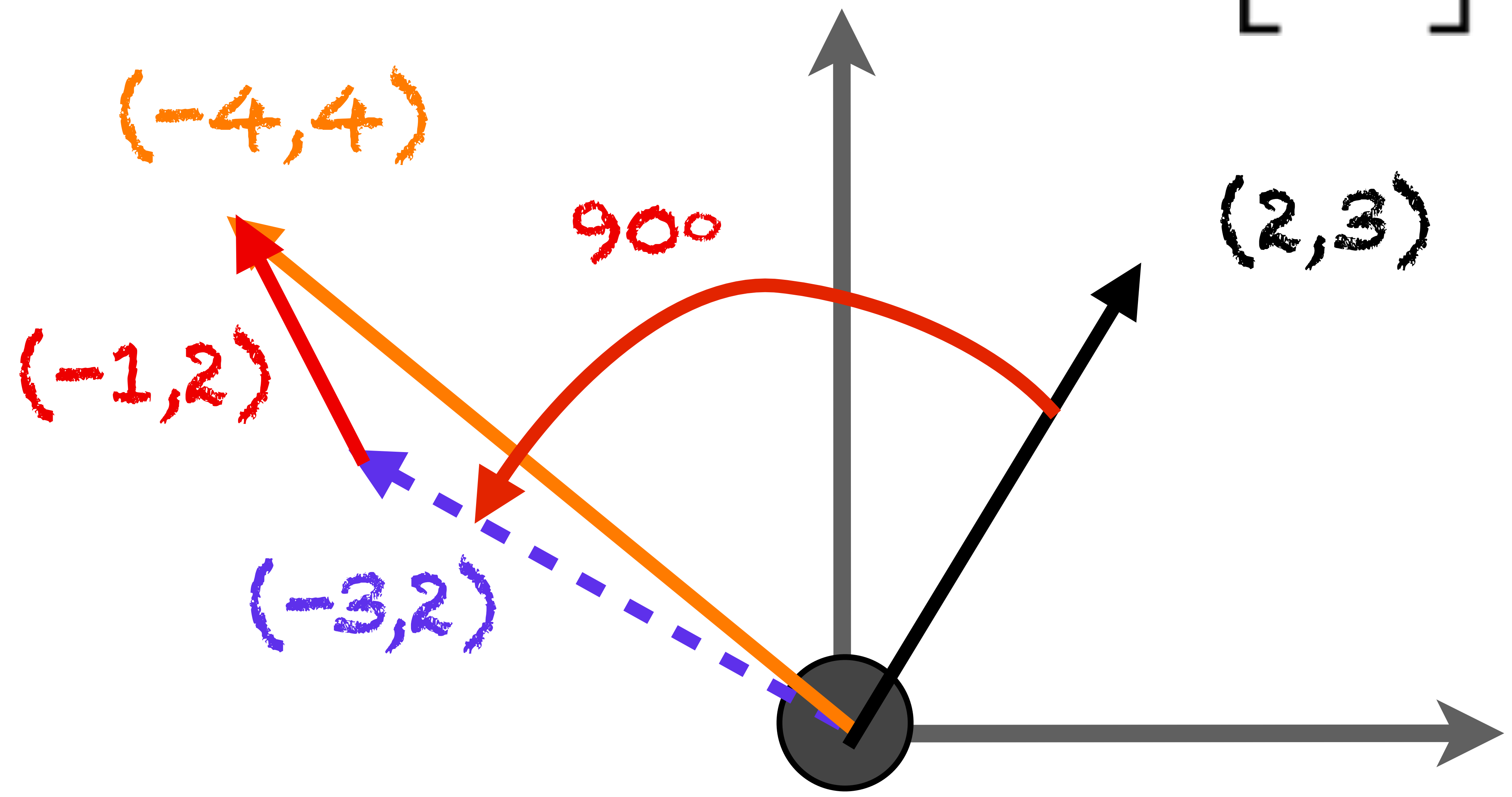
homogeneous rotation matrix

Example

$$\begin{bmatrix} -4 \\ 4 \\ 1 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}}_{D(-1,2)} \underbrace{\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{R(90^\circ)} \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$$



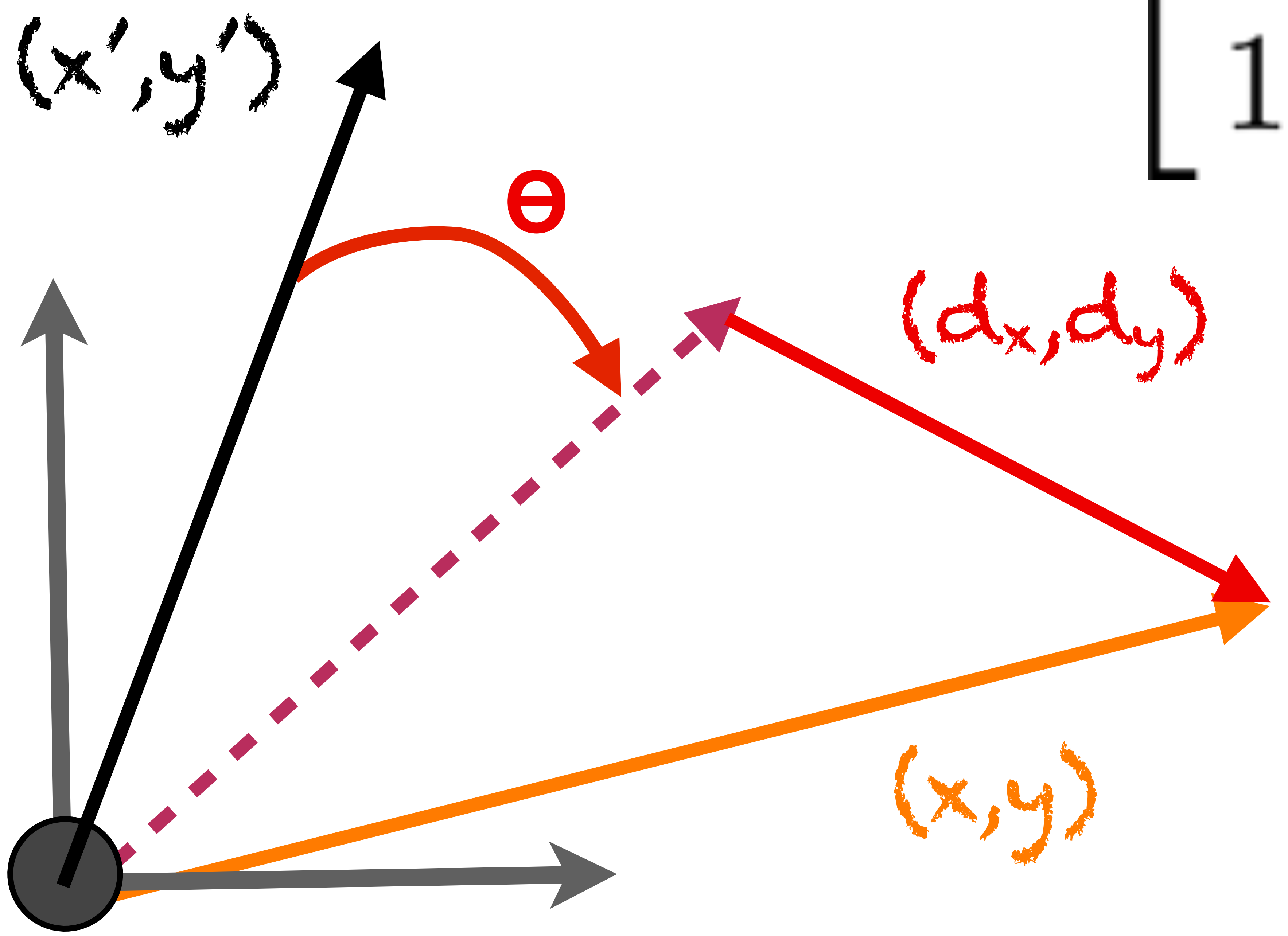
Example



$$\begin{bmatrix} -4 \\ 4 \\ 1 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}}_{D(-1,2)} \underbrace{\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{R(90°)} \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} -4 \\ 4 \\ 1 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & -1 & -1 \\ 1 & 0 & 2 \\ 0 & 0 & 1 \end{bmatrix}}_{D(-1,2)R(90°)} \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$$

Homogeneous Transform:
Composition of Rotation and Translation



$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & d_x \\ \sin(\theta) & \cos(\theta) & d_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$A_i = \begin{bmatrix} R_i^{i-1} & o^{i-1} \\ 0 & 1 \end{bmatrix}$$

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Homogeneous Transform

defines SE(2): Special Euclidean Group 2

$$H = \begin{bmatrix} R_{00} & R_{01} & d_x \\ R_{10} & R_{11} & d_y \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \mathbf{R}_{2\times 2} & \mathbf{d}_{2\times 1} \\ \mathbf{0}_{1\times 2} & 1 \end{bmatrix}$$

Homogeneous Transform

defines SE(2): Special Euclidean Group 2

$$H = \begin{bmatrix} R_{00} & R_{01} & d_x \\ R_{10} & R_{11} & d_y \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \mathbf{R}_{2\times 2} & \mathbf{d}_{2\times 1} \\ \mathbf{0}_{1\times 2} & 1 \end{bmatrix}$$

$$H \in SE(2)$$

Homogeneous Transform

defines SE(2): Special Euclidean Group 2

$$H = \begin{bmatrix} \boxed{\begin{matrix} R_{00} & R_{01} \\ R_{10} & R_{11} \end{matrix}} & \begin{matrix} d_x \\ d_y \end{matrix} \\ \begin{matrix} 0 & 0 \end{matrix} & 1 \end{bmatrix} = \begin{bmatrix} \mathbf{R}_{2\times 2} & \mathbf{d}_{2\times 1} \\ \mathbf{0}_{1\times 2} & 1 \end{bmatrix}$$

$H \in SE(2)$ $\quad \mathbf{R}_{2\times 2} \in SO(2)$

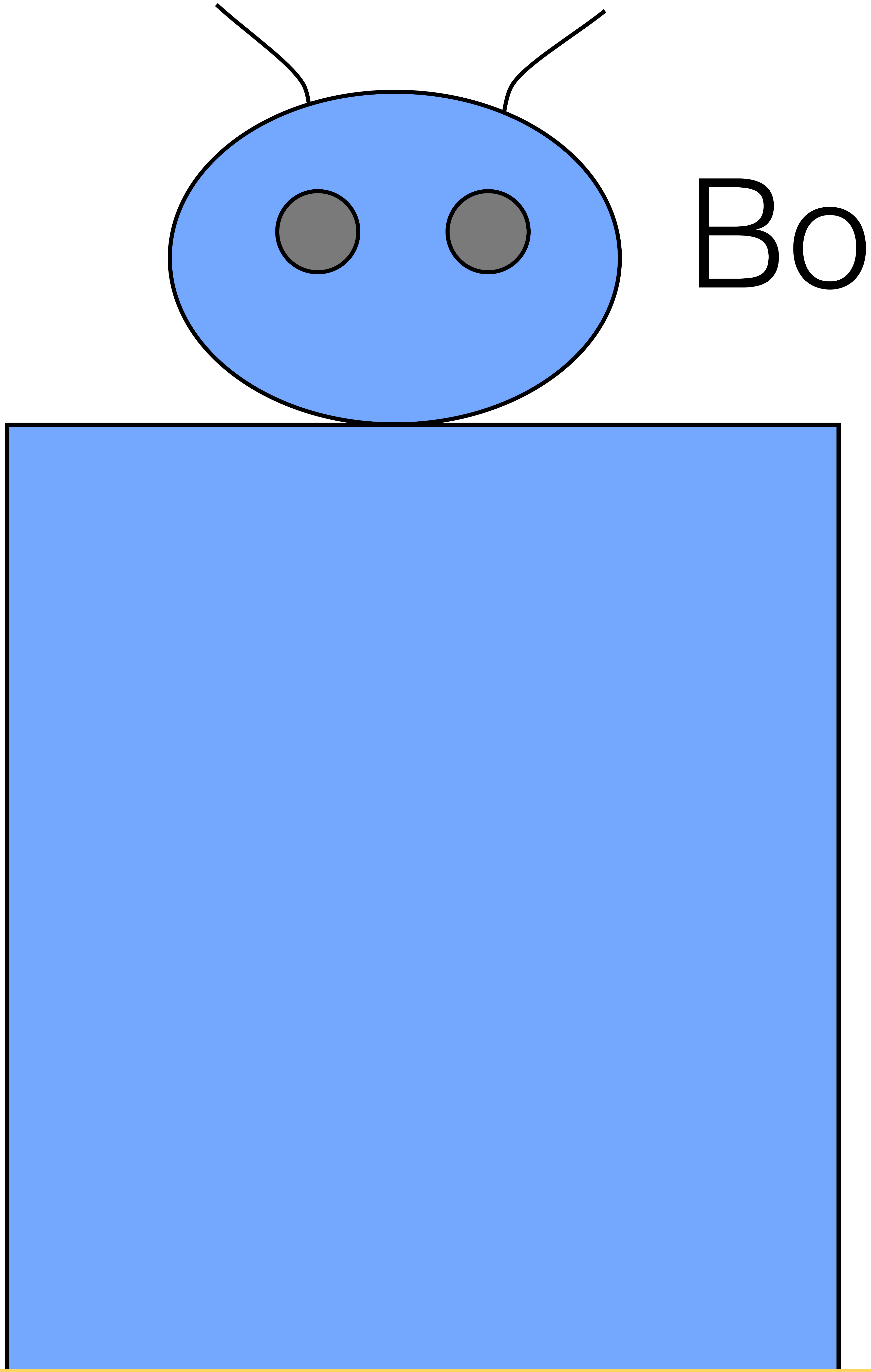
Homogeneous Transform

defines SE(2): Special Euclidean Group 2

$$H = \begin{bmatrix} \boxed{\begin{matrix} R_{00} & R_{01} \\ R_{10} & R_{11} \end{matrix}} & \boxed{\begin{matrix} d_x \\ d_y \end{matrix}} \\ \begin{matrix} 0 & 0 \end{matrix} & 1 \end{bmatrix} = \begin{bmatrix} \mathbf{R}_{2\times2} & \mathbf{d}_{2\times1} \\ \mathbf{0}_{1\times2} & 1 \end{bmatrix}$$

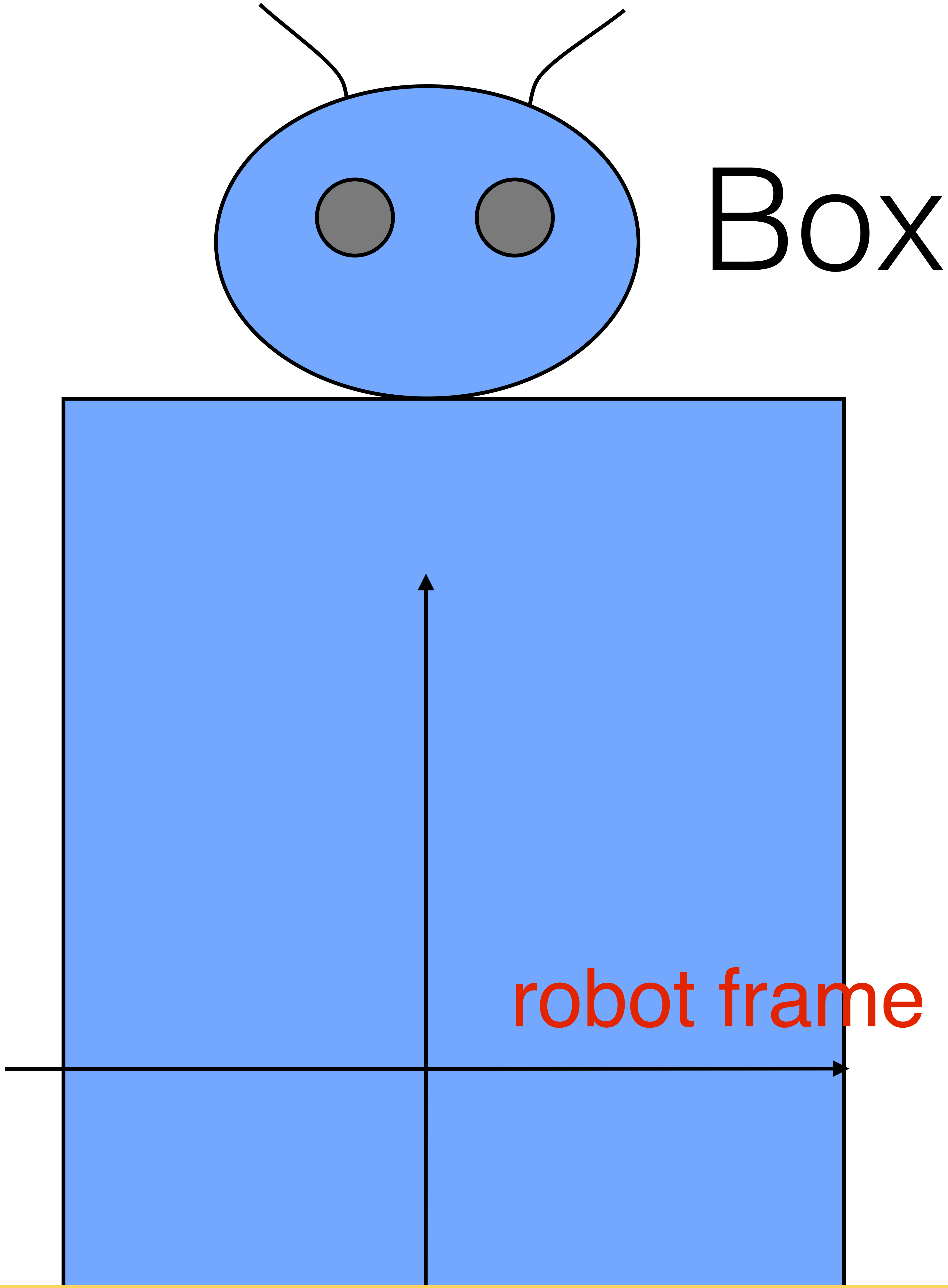
$$H \in SE(2) \quad \mathbf{R}_{2\times2} \in SO(2) \quad \mathbf{d}_{2\times1} \in \Re^2$$

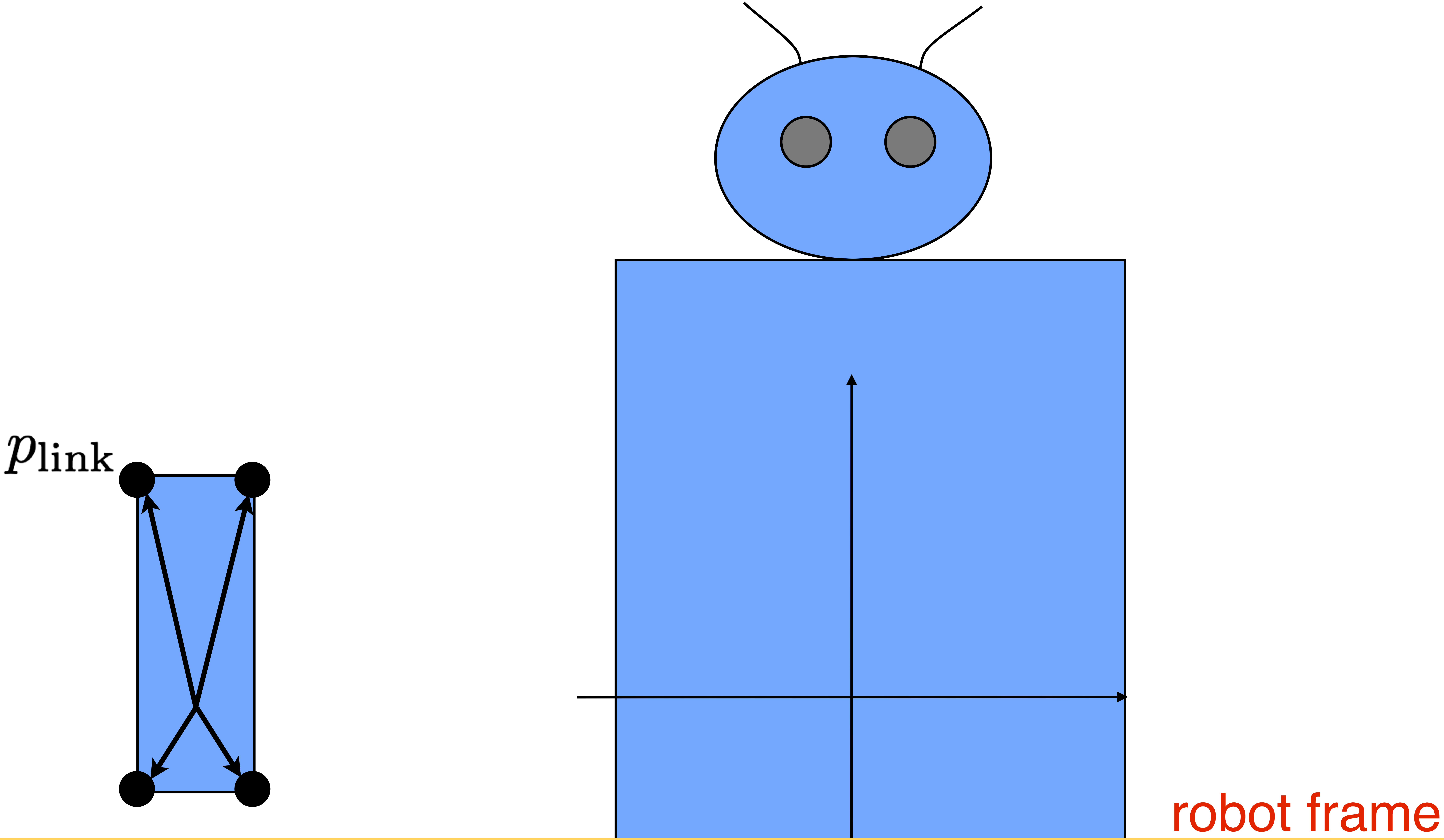
Example:
Let’s put an arm link on Boxy



Boxy the robot

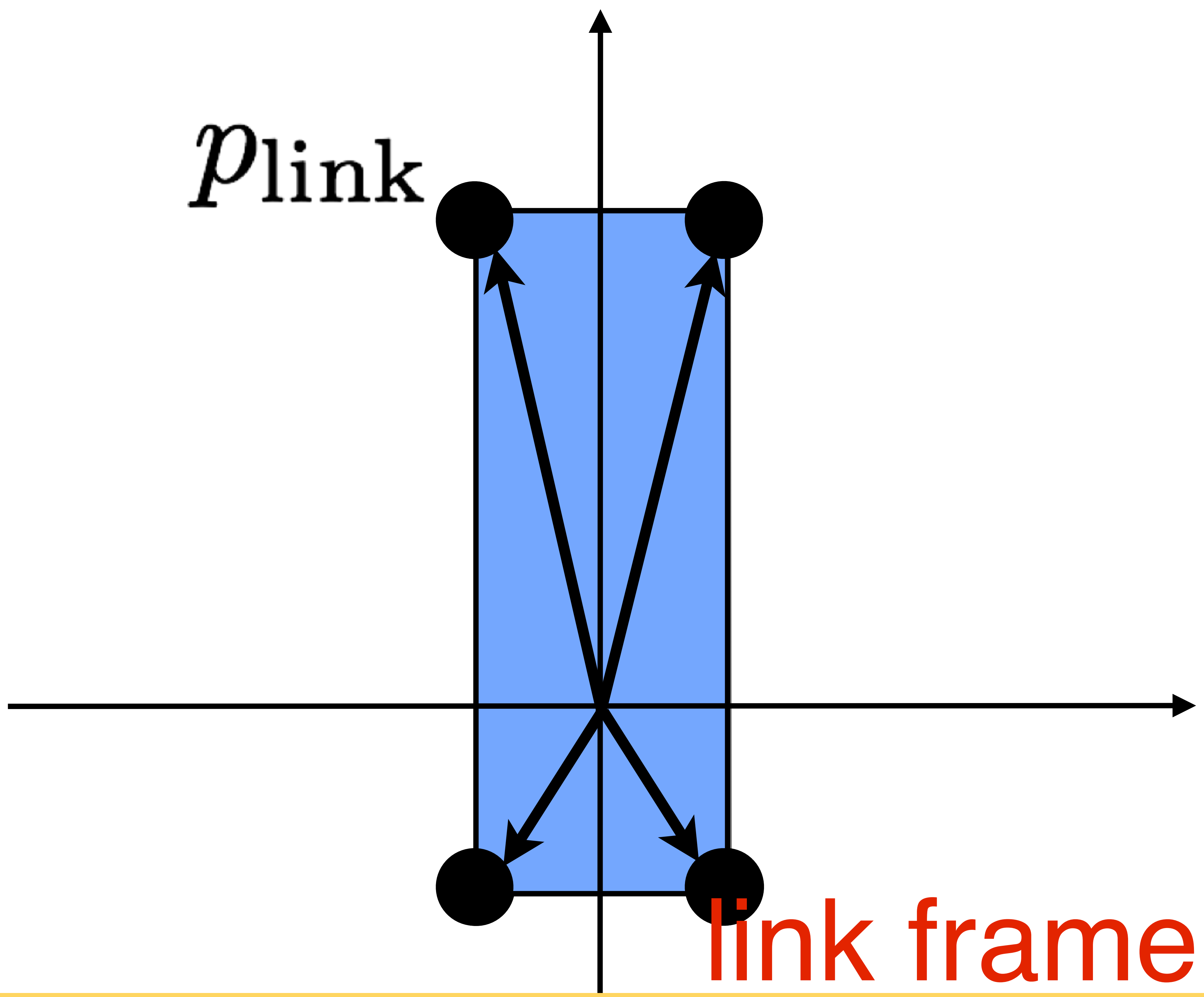
Boxy the robot

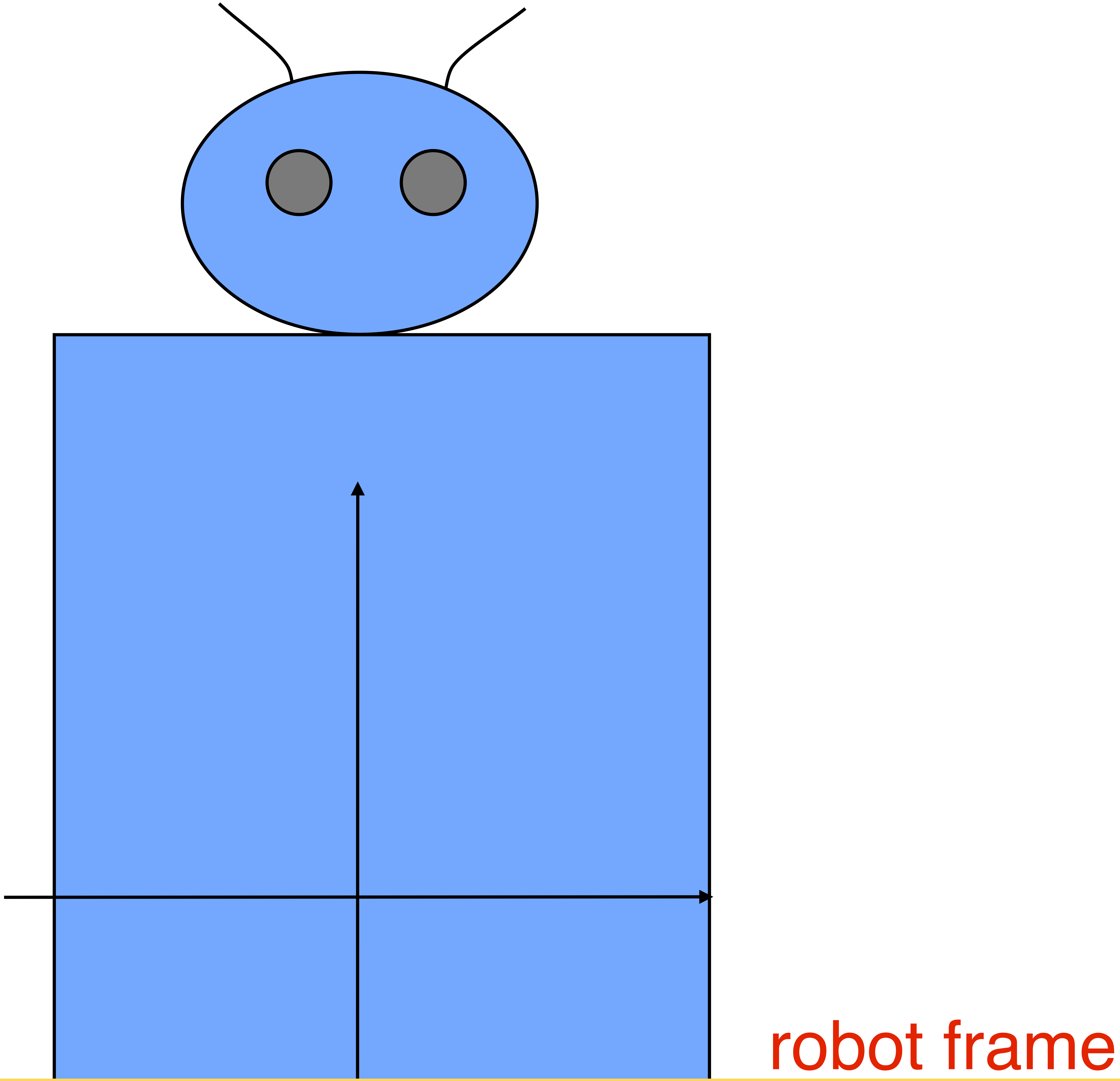



p_{link}
robot frame

Transform the link frame and its vertices into the robot frame

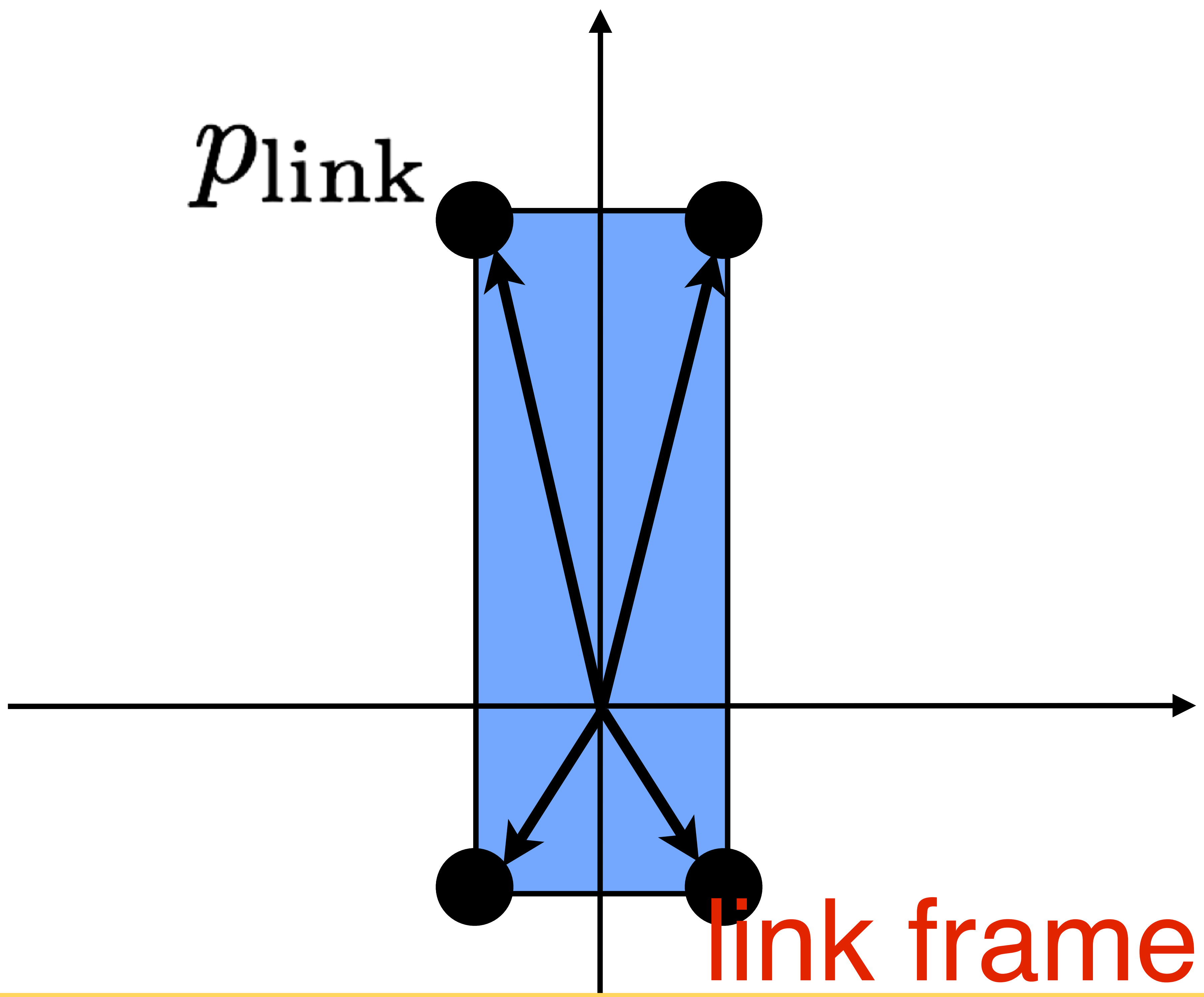
$$p_{\text{robot}} = T_{\text{link}}^{\text{robot}} p_{\text{link}}$$

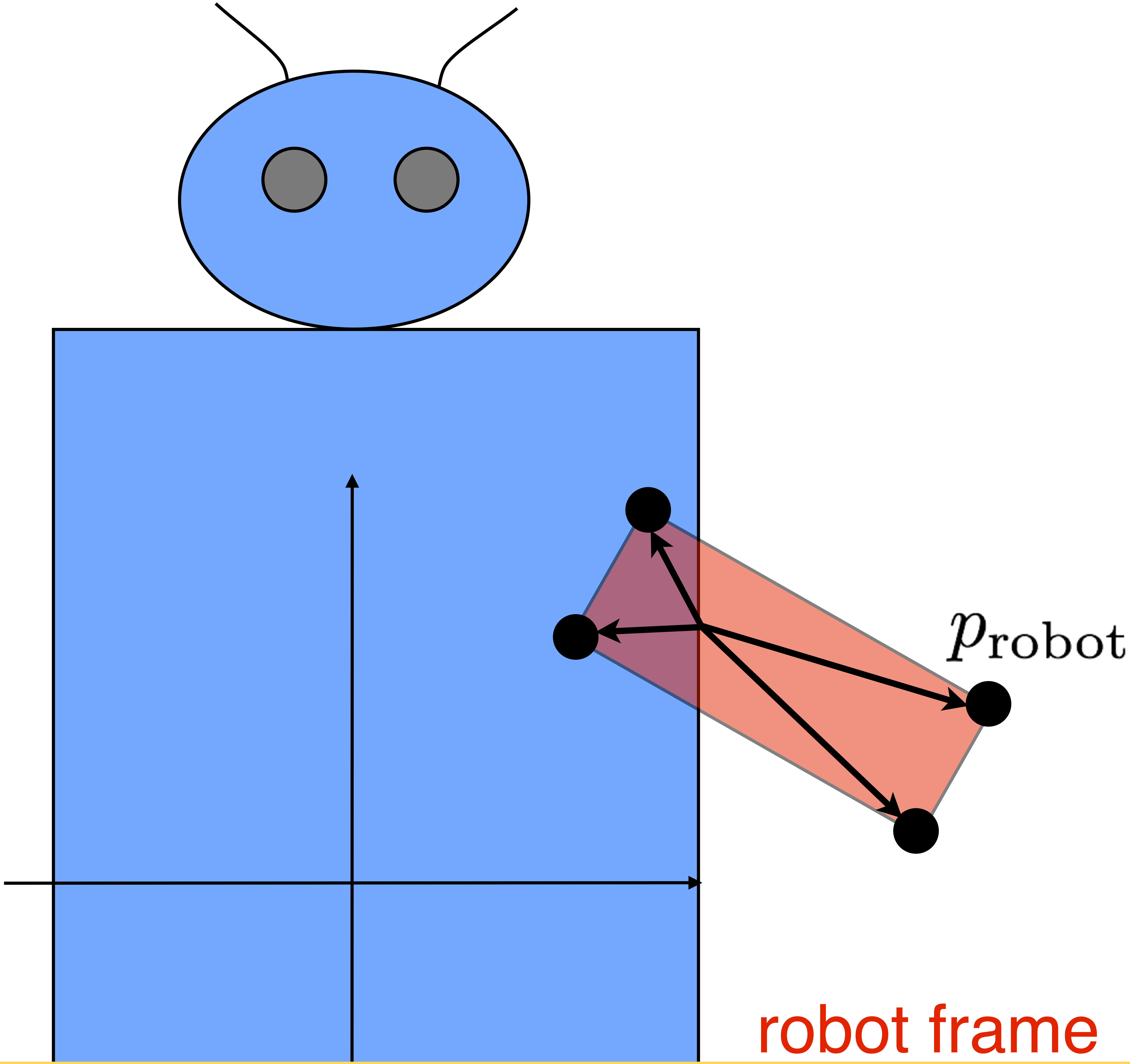




Transform the link frame and its vertices into the robot frame

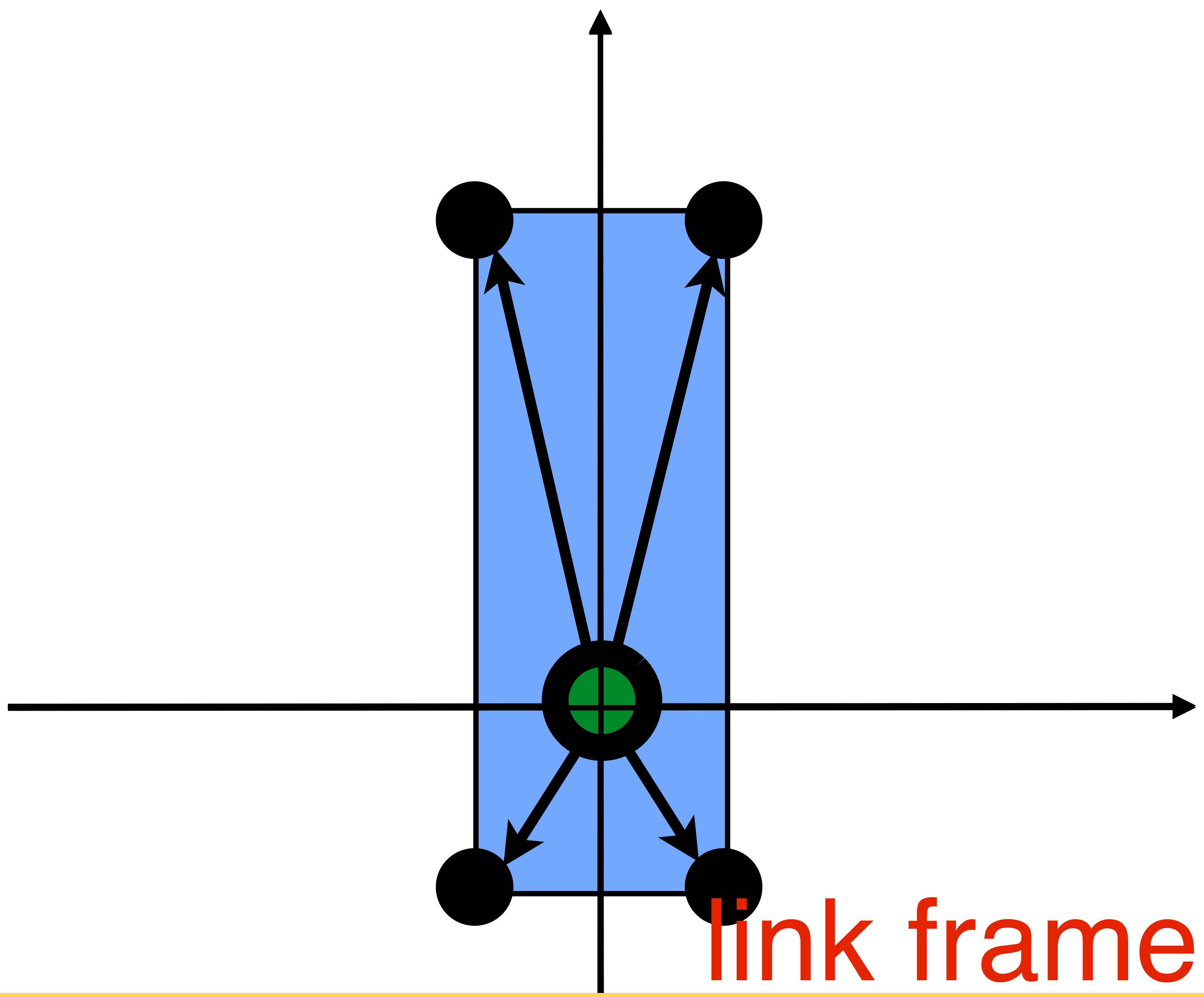
$$p_{\text{robot}} = T_{\text{link}}^{\text{robot}} p_{\text{link}}$$

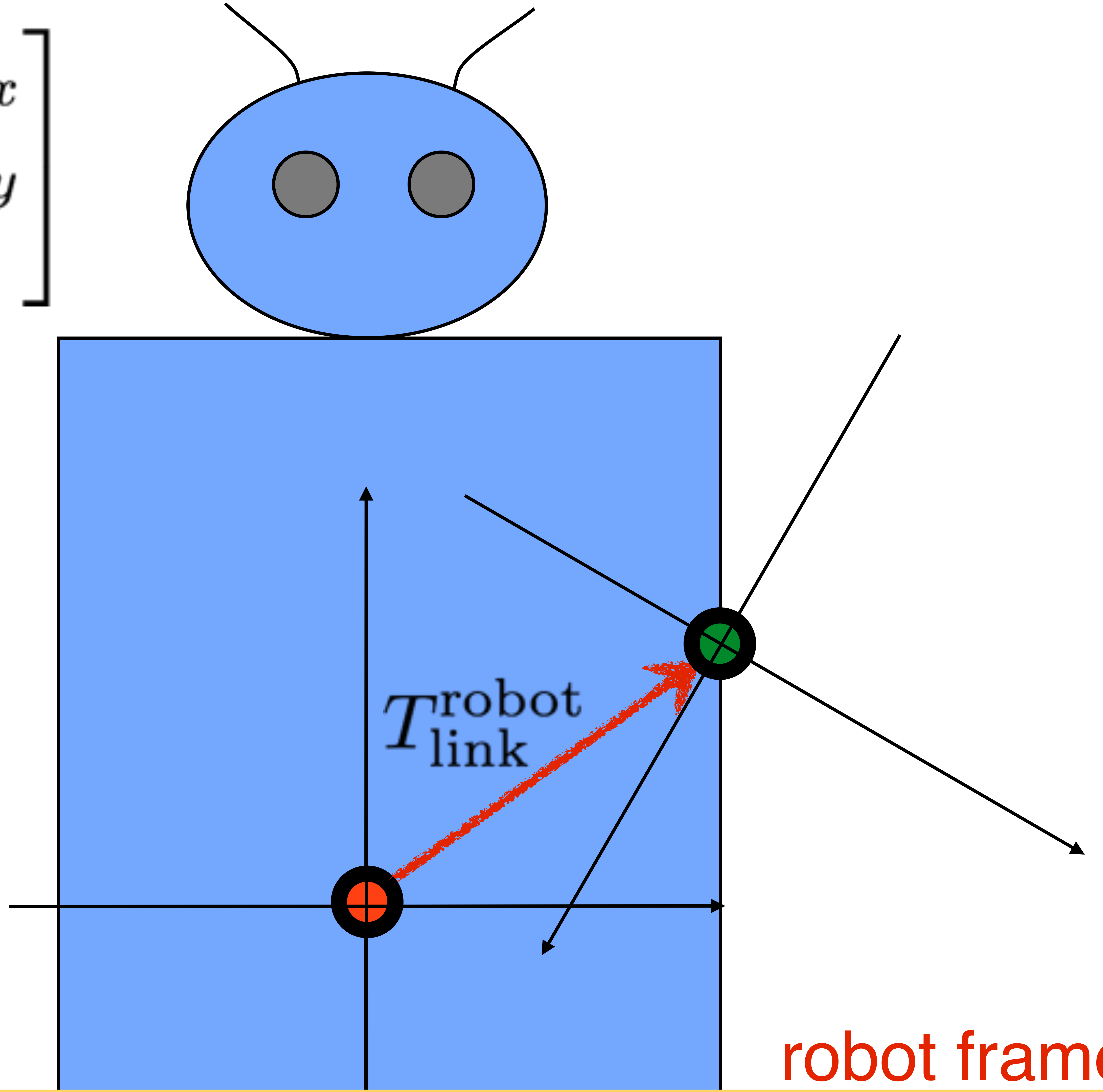




$$T_{\text{link}}^{\text{robot}} = \begin{bmatrix} R_{00} & R_{01} & d_x \\ R_{10} & R_{11} & d_y \\ 0 & 0 & 1 \end{bmatrix}$$

Can we think about this frame relation in steps?

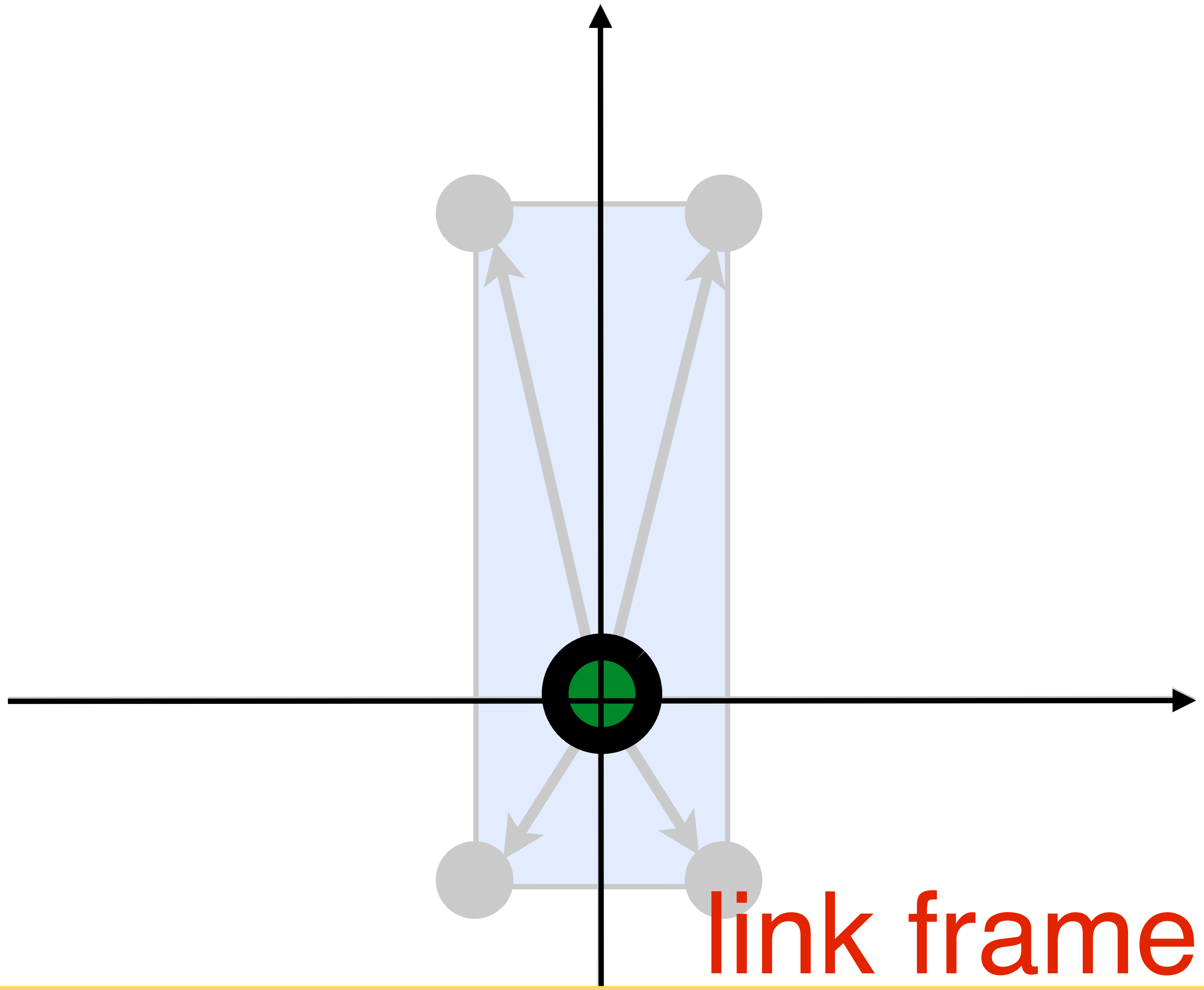


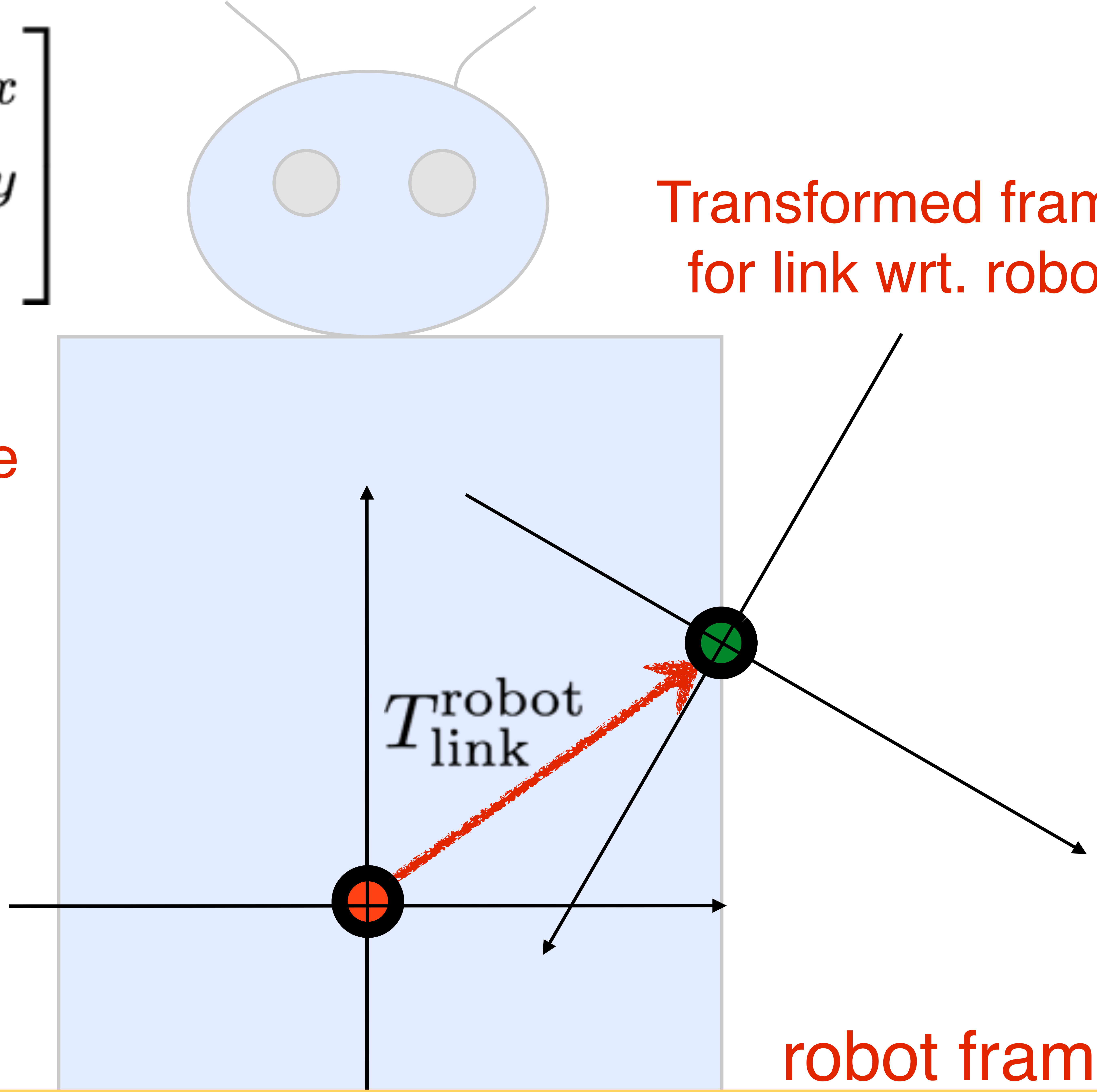


$$T_{\text{link}}^{\text{robot}} = \begin{bmatrix} R_{00} & R_{01} & d_x \\ R_{10} & R_{11} & d_y \\ 0 & 0 & 1 \end{bmatrix}$$

Transformed frame for link wrt. robot

First consider link in its own frame

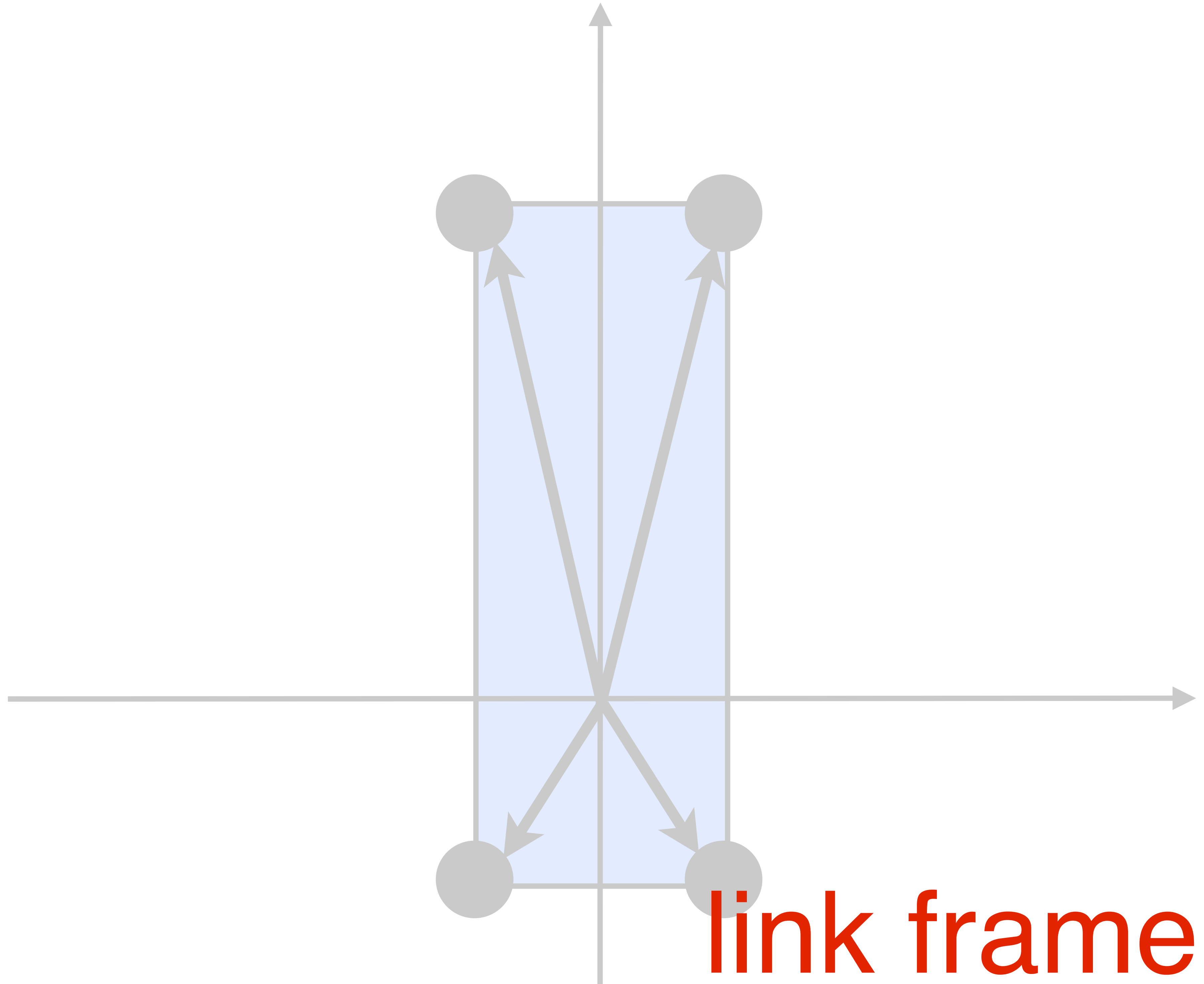




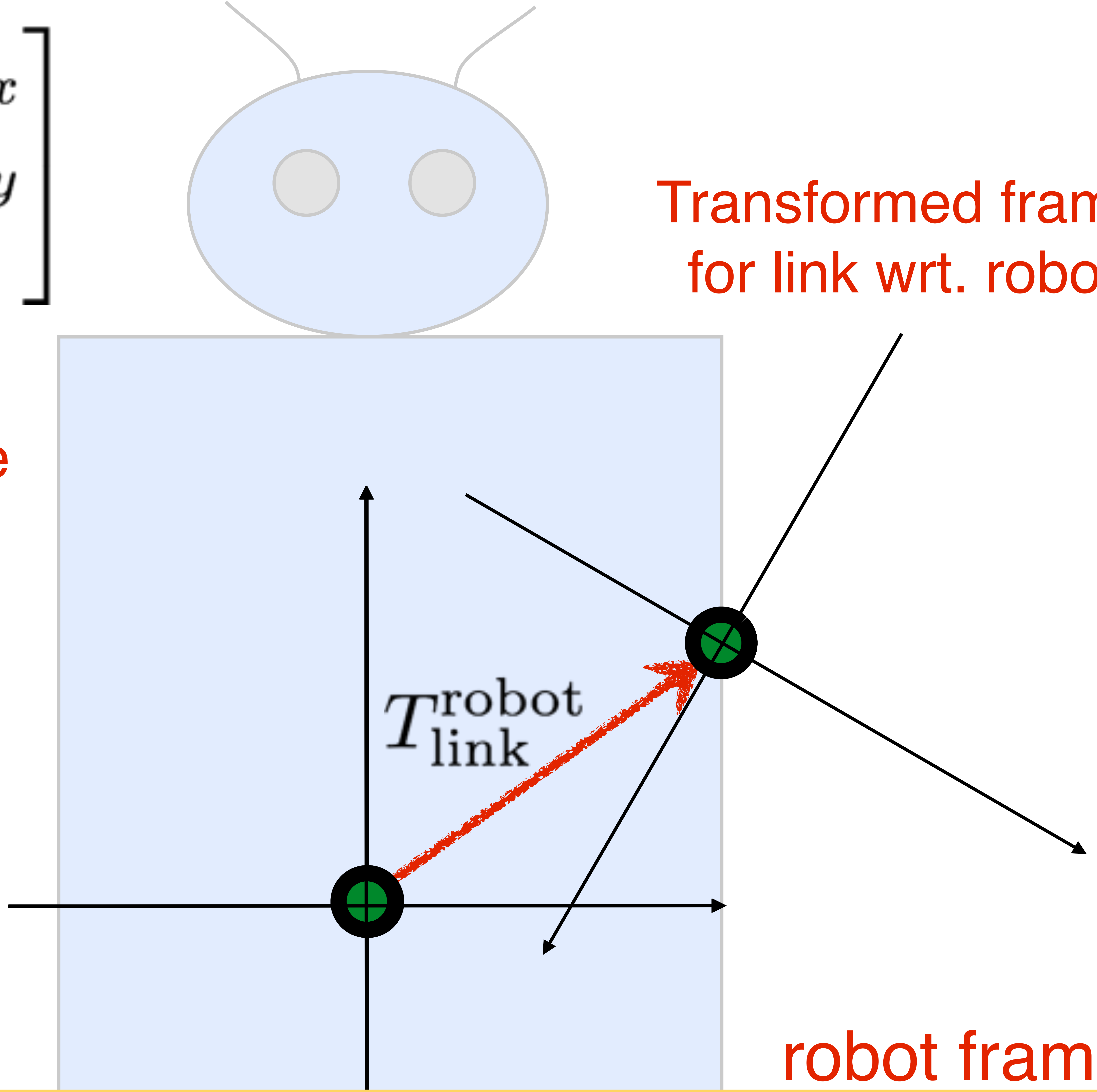
$$T^{\text{robot}}_{\text{link}} = \begin{bmatrix} R_{00} & R_{01} & d_x \\ R_{10} & R_{11} & d_y \\ 0 & 0 & 1 \end{bmatrix}$$

Transformed frame for link wrt. robot

as aligned with robot base frame



link frame

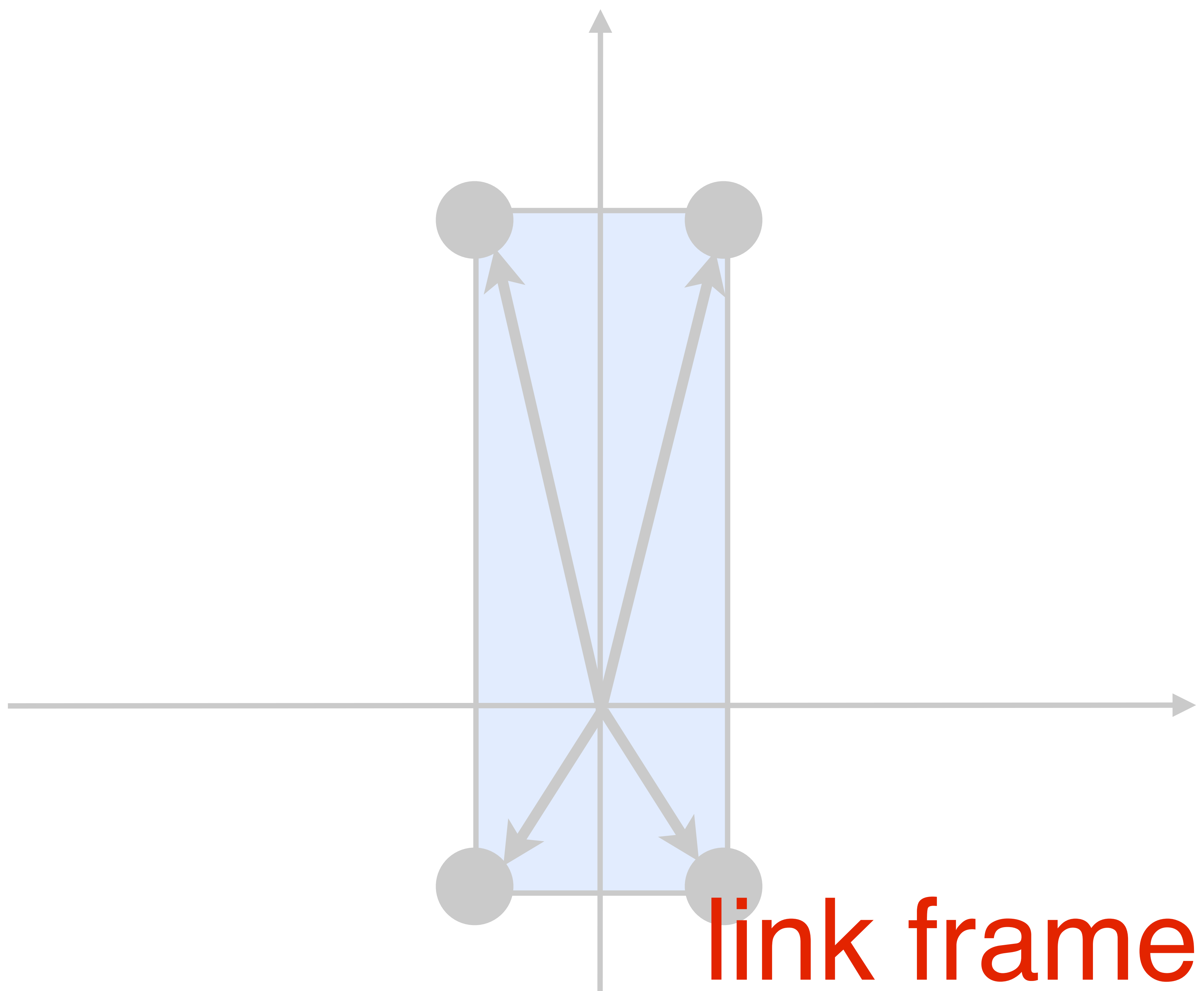


robot frame

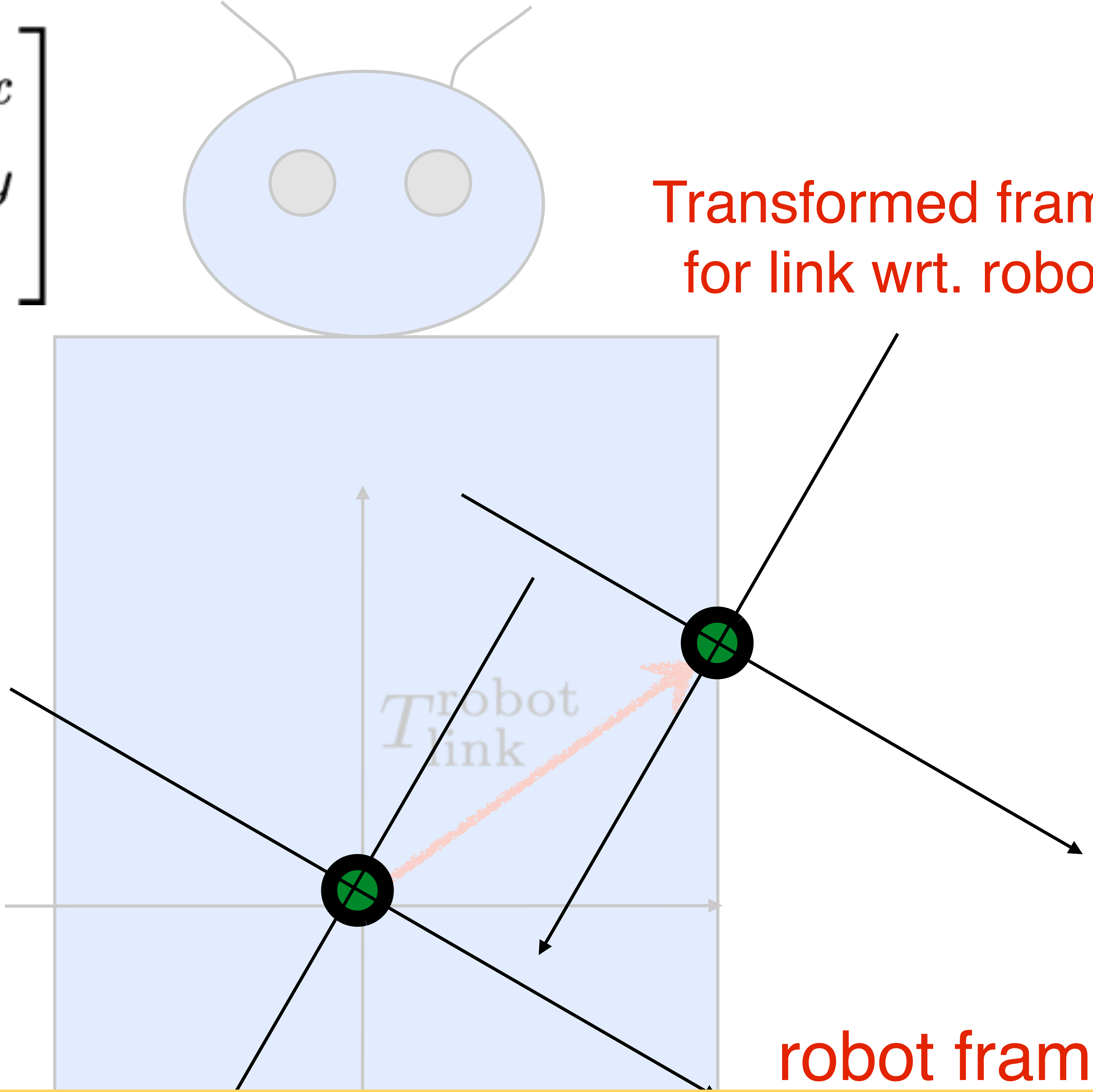
$$T_{\text{link}}^{\text{robot}} = \begin{bmatrix} R_{00} & R_{01} & d_x \\ R_{10} & R_{11} & d_y \\ 0 & 0 & 1 \end{bmatrix}$$

Transformed frame for link wrt. robot

Rotate link frame by *R*



link frame

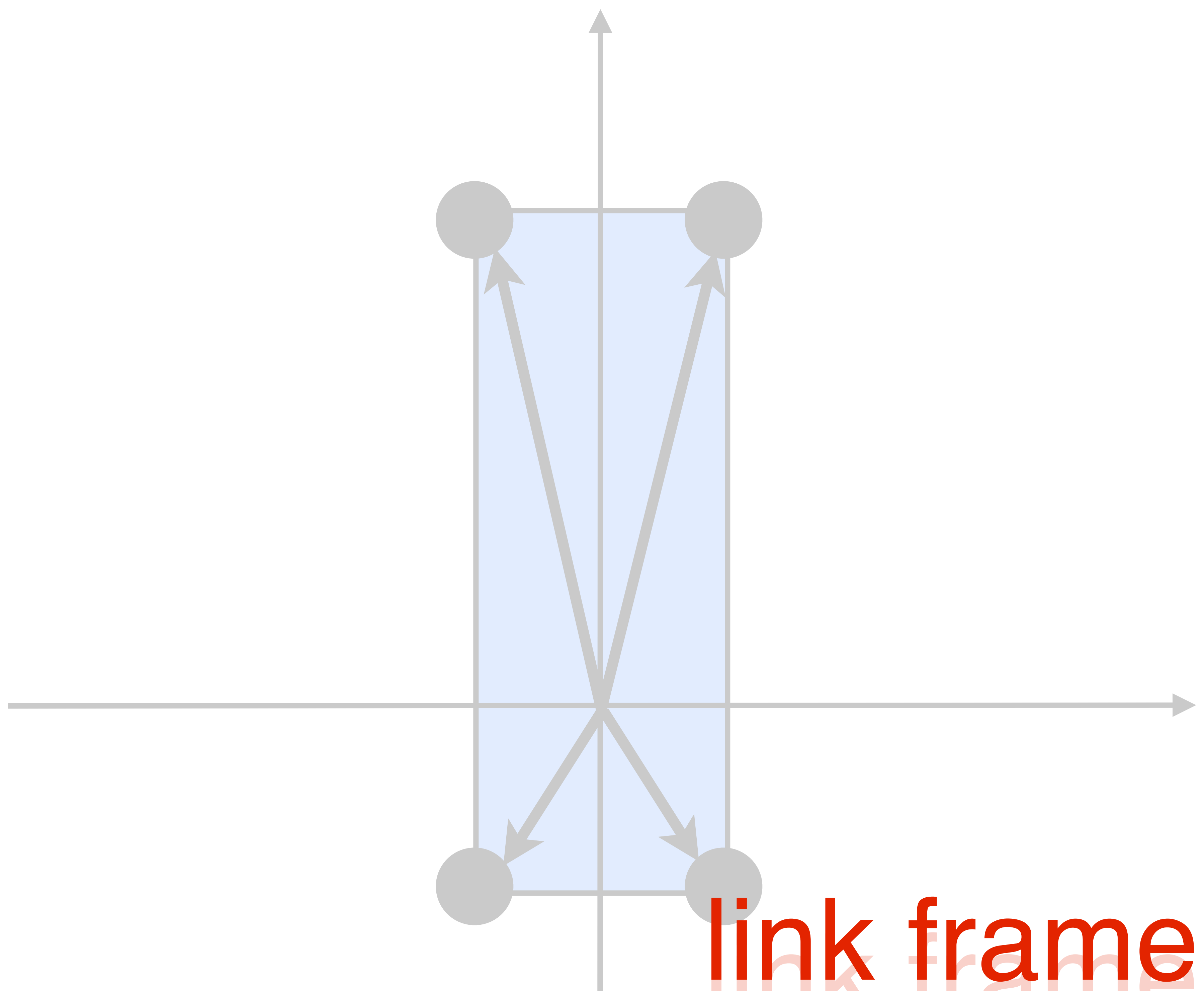


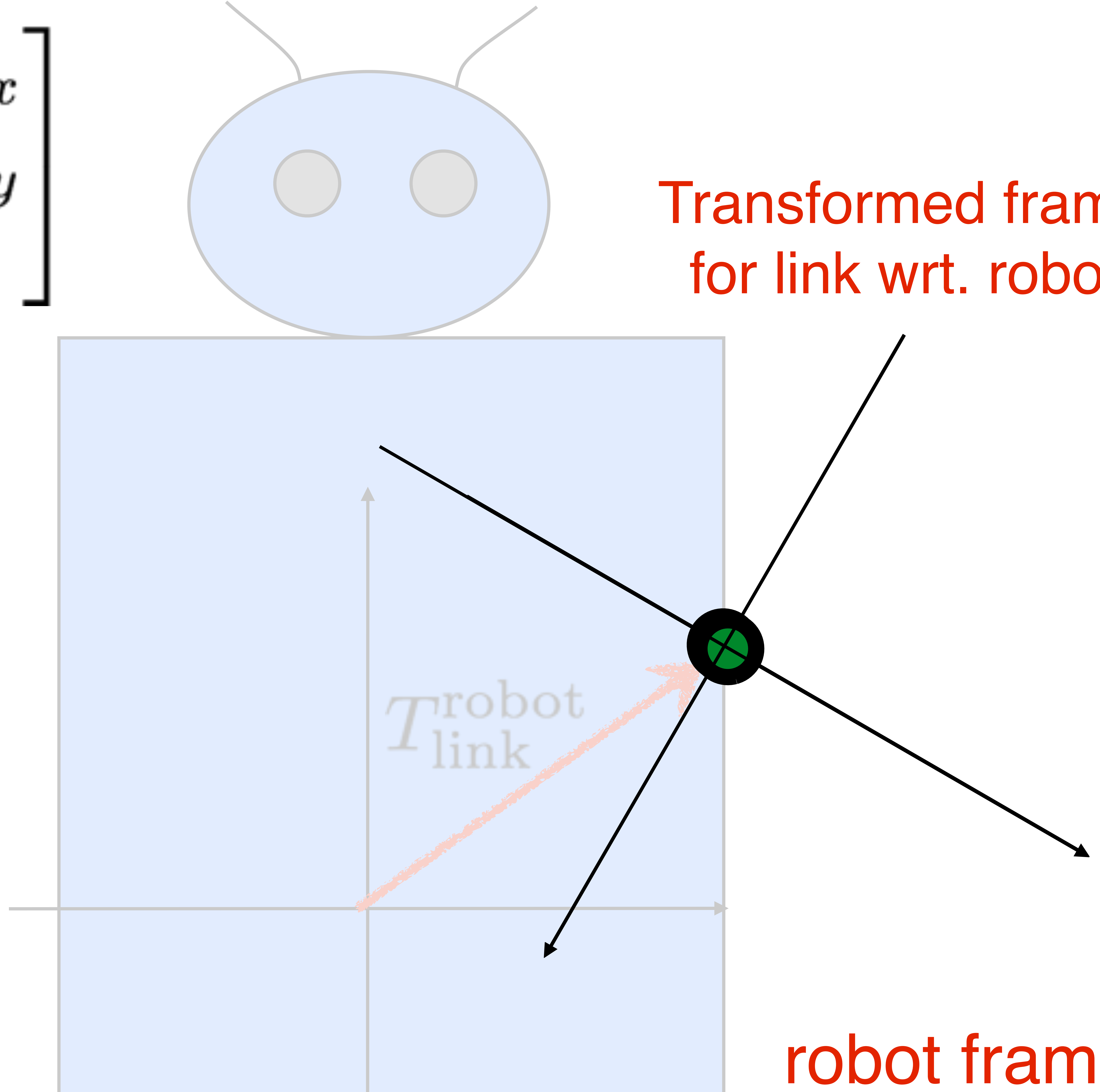
robot frame

$$T_{\text{link}}^{\text{robot}} = \begin{bmatrix} R_{00} & R_{01} & d_x \\ R_{10} & R_{11} & d_y \\ 0 & 0 & 1 \end{bmatrix}$$

Transformed frame for link wrt. robot

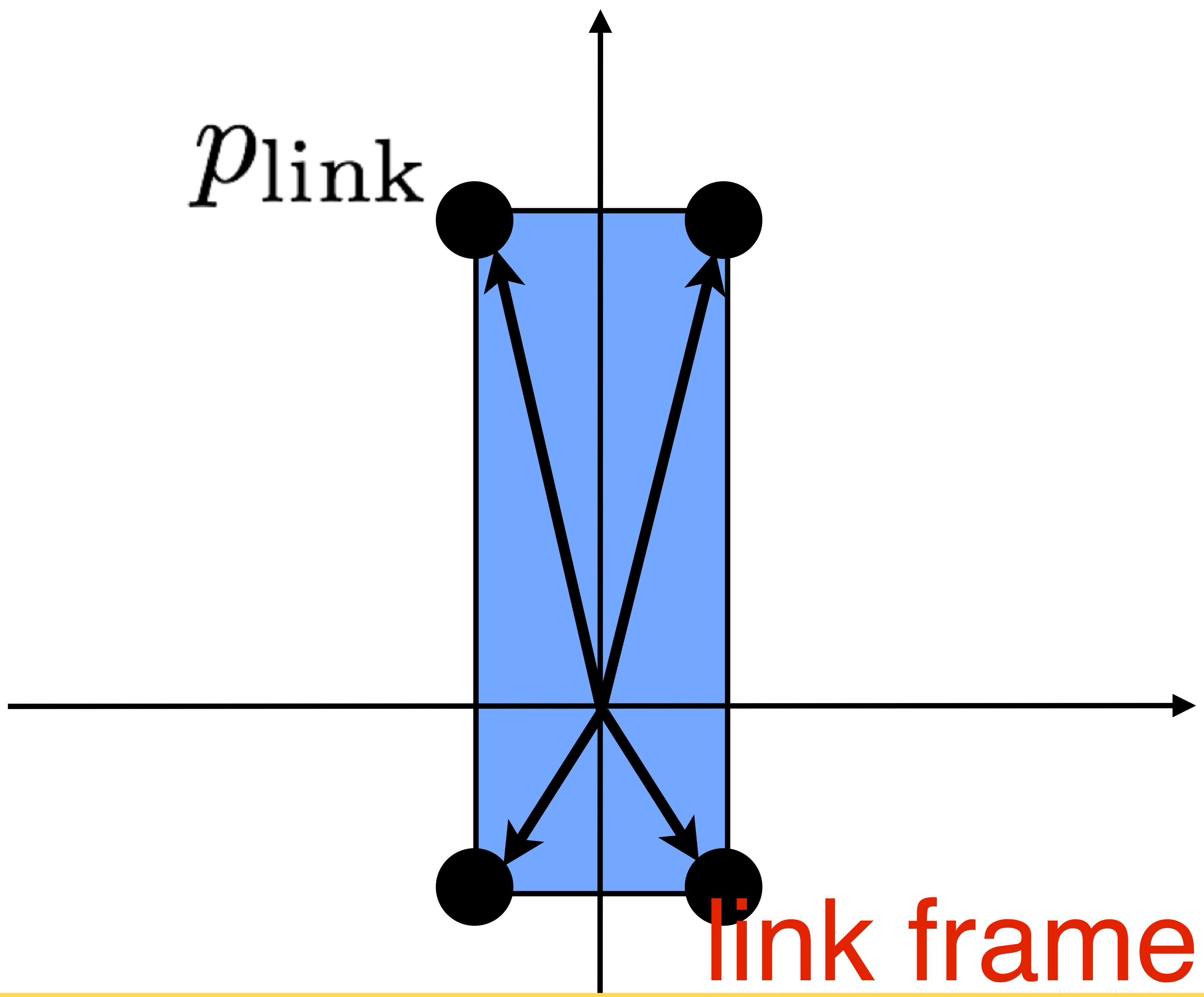
Translate link frame by *d*

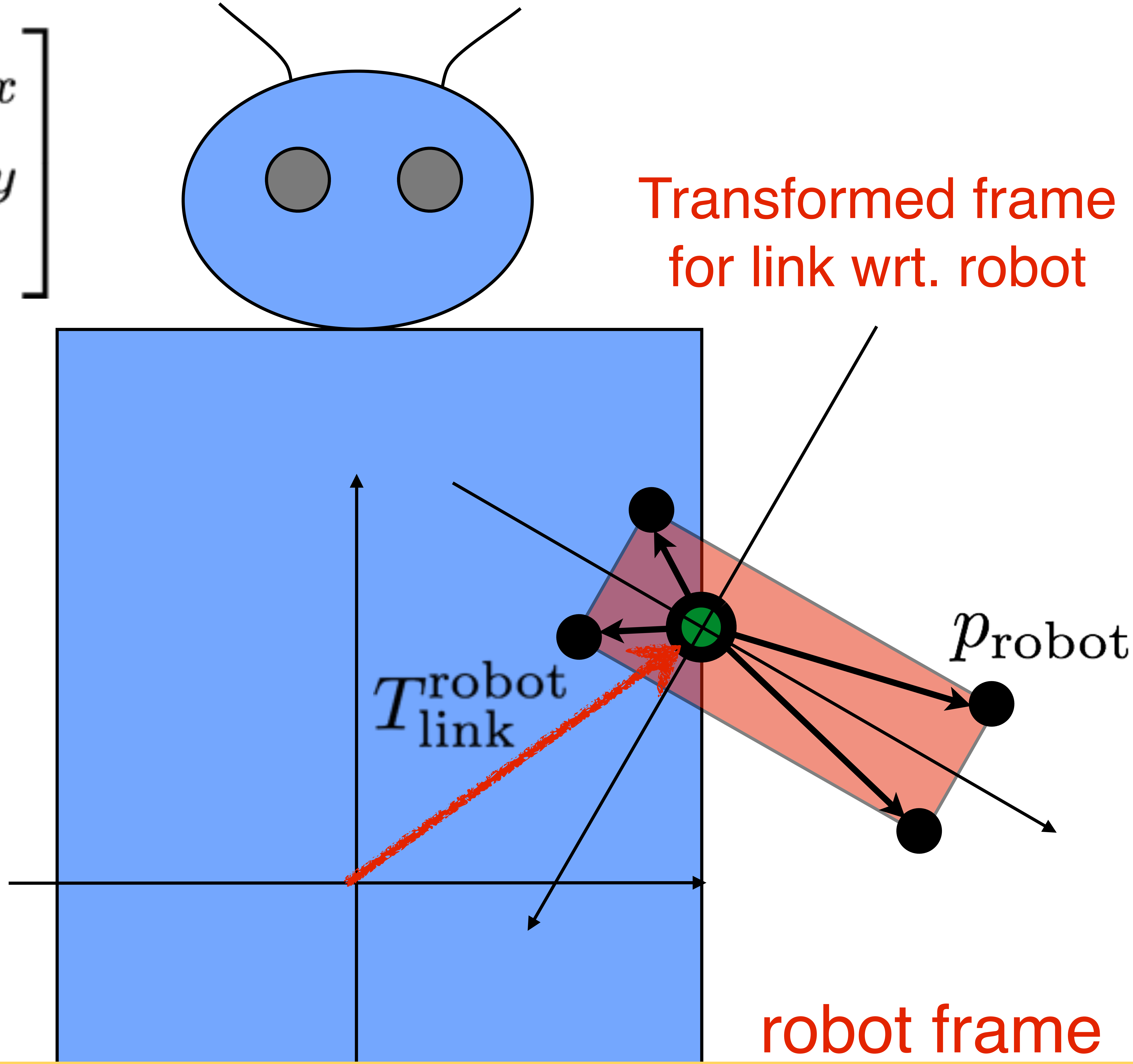




$$T_{\text{link}}^{\text{robot}} = \begin{bmatrix} R_{00} & R_{01} & d_x \\ R_{10} & R_{11} & d_y \\ 0 & 0 & 1 \end{bmatrix}$$

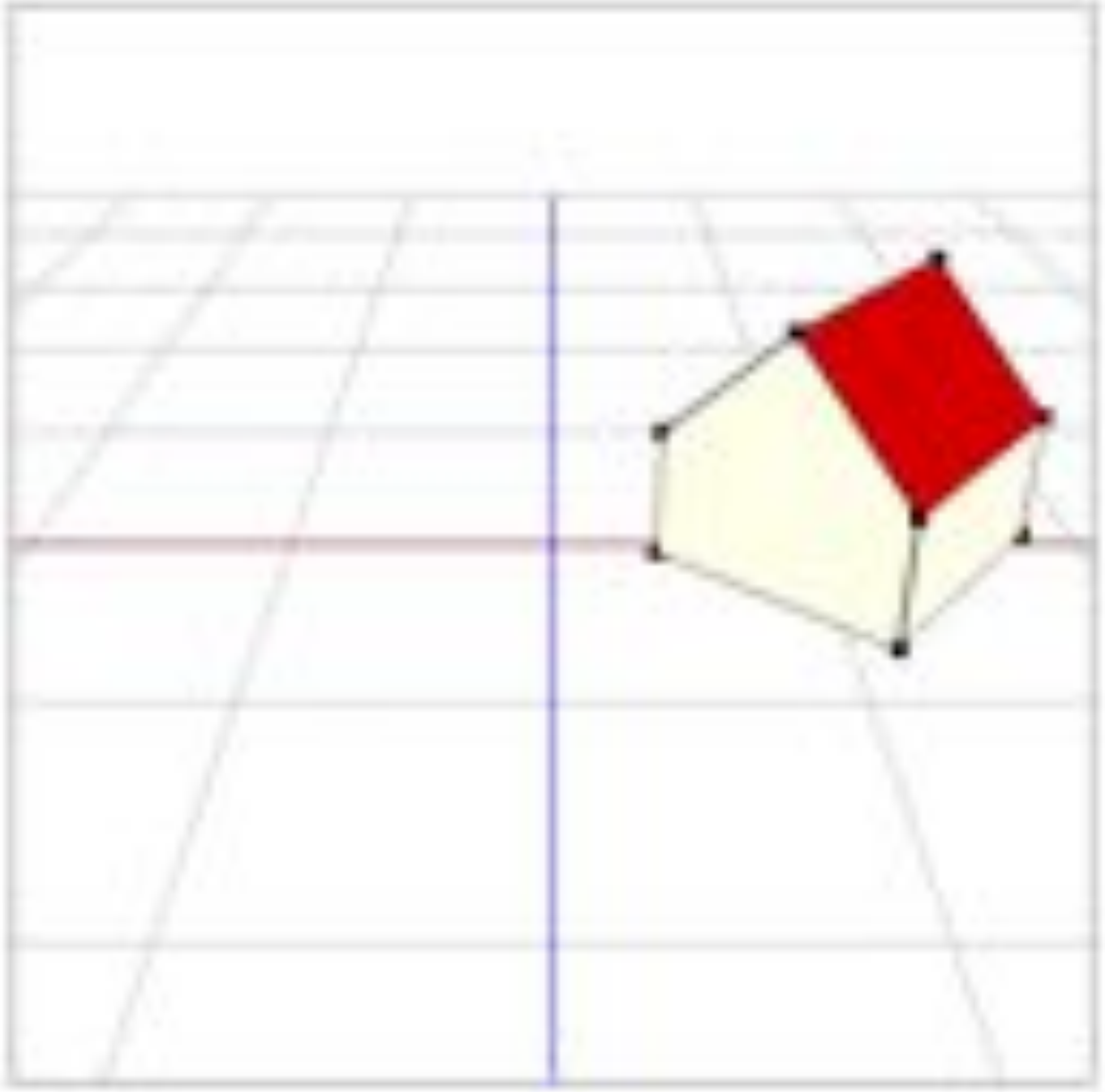
$$p_{\text{robot}} = T_{\text{link}}^{\text{robot}} p_{\text{link}}$$





Why not translate then rotate?

$M = R \cdot T$



$M = T \cdot R$

Note the difference in behavior.

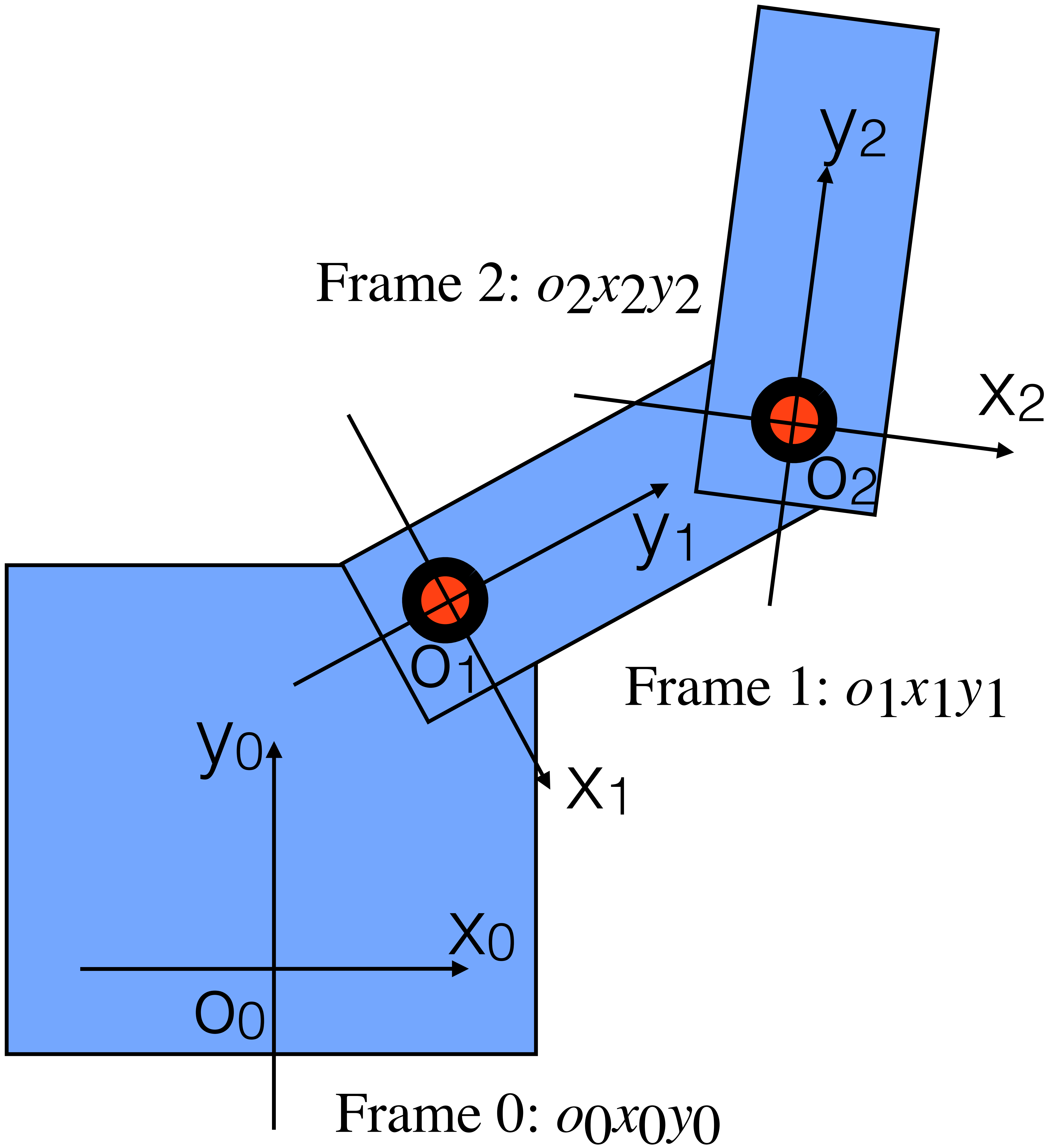
Translation along $x =$ 1.1

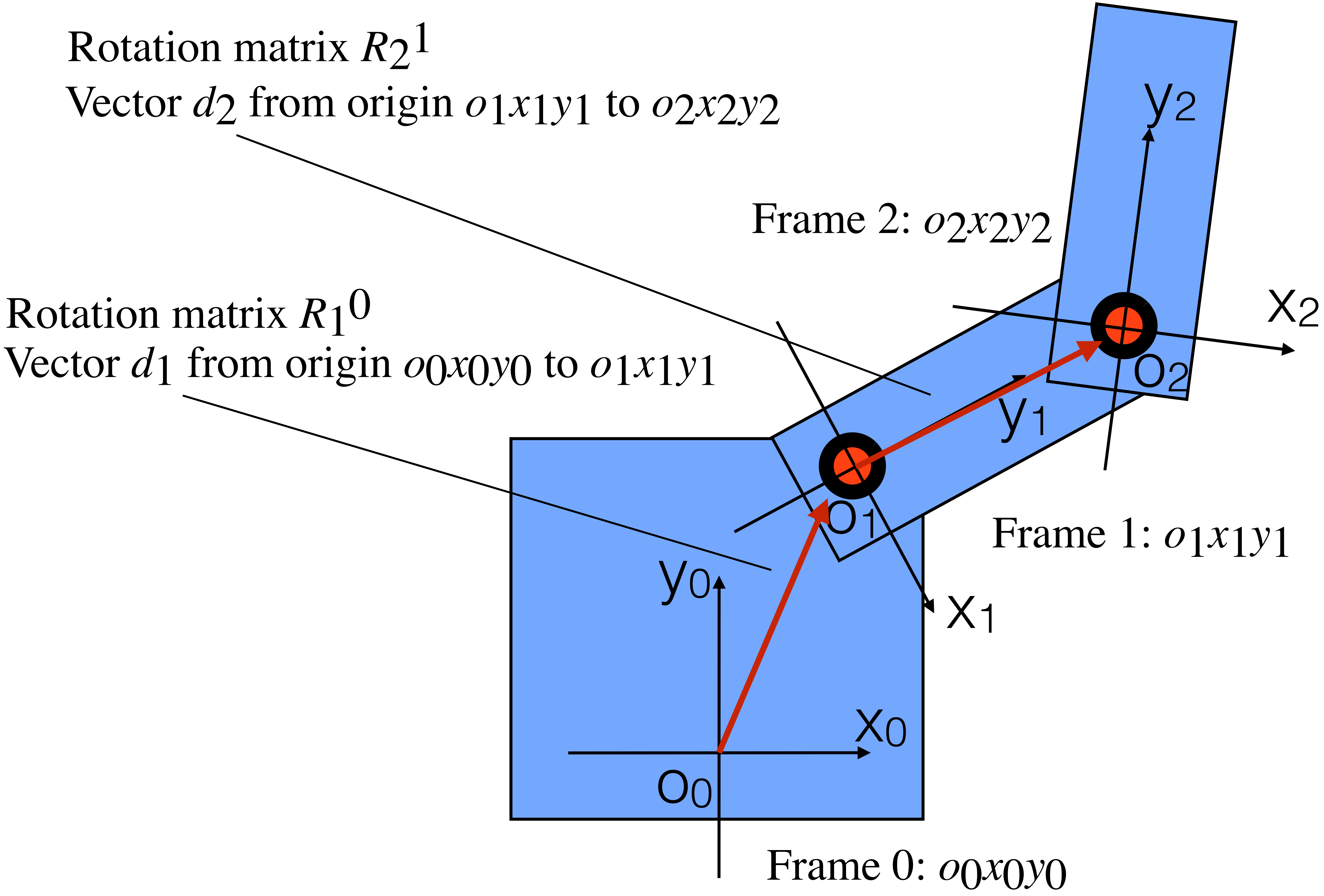
Rotation about $y =$ 140°

Can we compose multiple frame transforms?

Can we compose multiple frame transforms?

Consider the 3 frames of a planar 2-link robot


Frame 2: $o_2 x_2 y_2$
y_2
x_2
o_2
y_1
o_1
Frame 1: $o_1 x_1 y_1$
y_0
x_1
x_0
o_0
Frame 0: $o_0 x_0 y_0$


Rotation matrix R_2^1
Vector d_2 from origin $o_1x_1y_1$ to $o_2x_2y_2$
Rotation matrix R_1^0
Vector d_1 from origin $o_0x_0y_0$ to $o_1x_1y_1$
Frame 2: $o_2x_2y_2$
Frame 1: $o_1x_1y_1$
Frame 0: $o_0x_0y_0$
y_2
x_2
o_2
y_1
x_1
o_1
y_0
x_0
o_0

A point in frame 1 relates to a point in frame 0 by

$$p^0 = R_1^0 p^1 + d_1^0$$

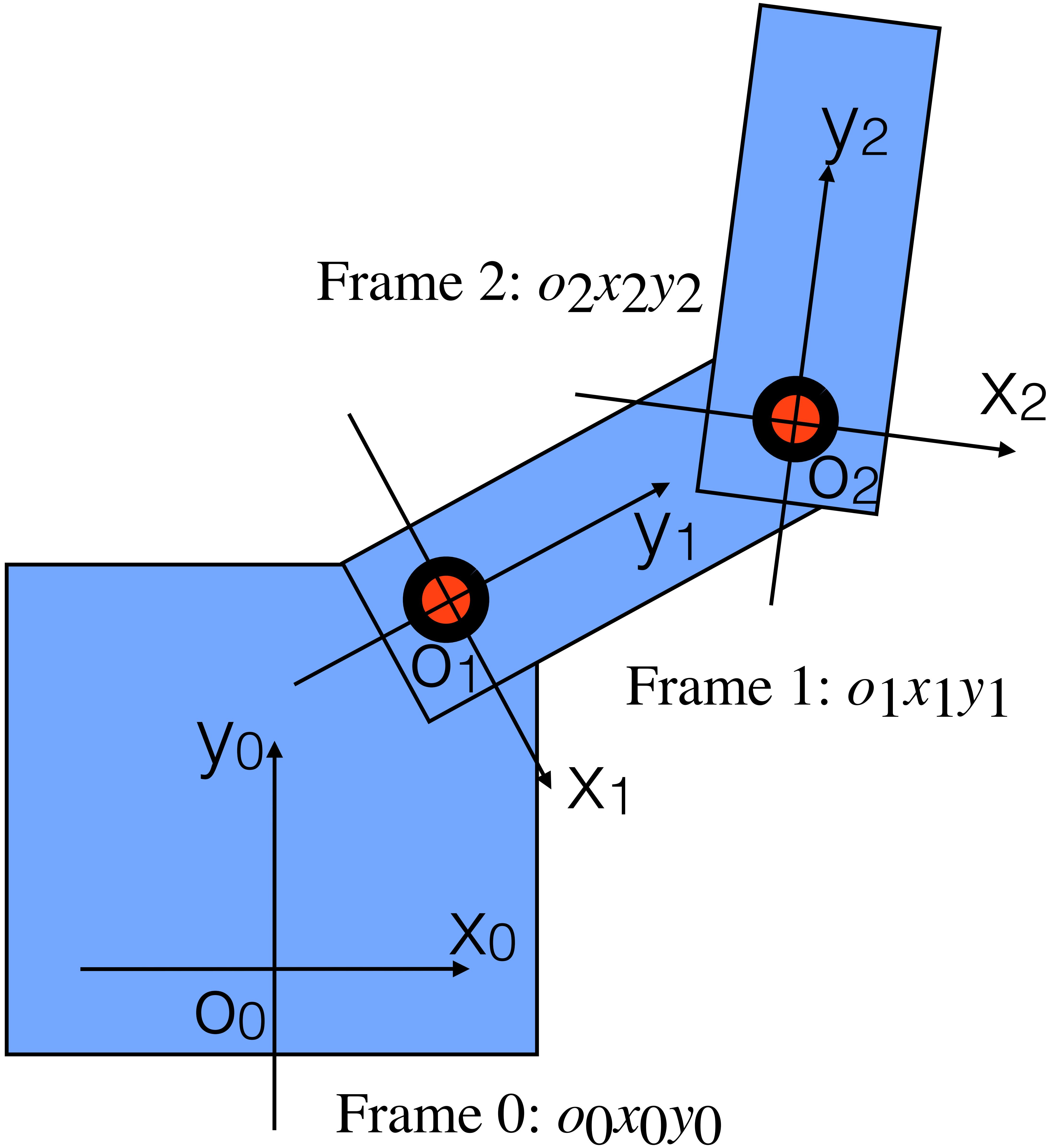
and point in frame 2 relates to point in frame 1 by

$$p^1 = R_2^1 p^2 + d_2^1$$

By substitution of p^1 into the expression for p^0,

a point in frame 2 relates to a point in frame 0 by

$$p^0 = \underbrace{R_1^0 R_2^1}_{R_2^0} p^2 + \underbrace{R_1^0 d_2^1 + d_1^0}_{d_2^0}$$



$$\begin{bmatrix} R_1^0 & d_1^0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} R_2^1 & d_2^1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} R_1^0 R_2^1 & R_1^0 d_2^1 + d_1^0 \\ 0 & 1 \end{bmatrix}$$

Alternatively, relat
transform from fra

$p^0 = R_2{}^0 p^2 + d$

where

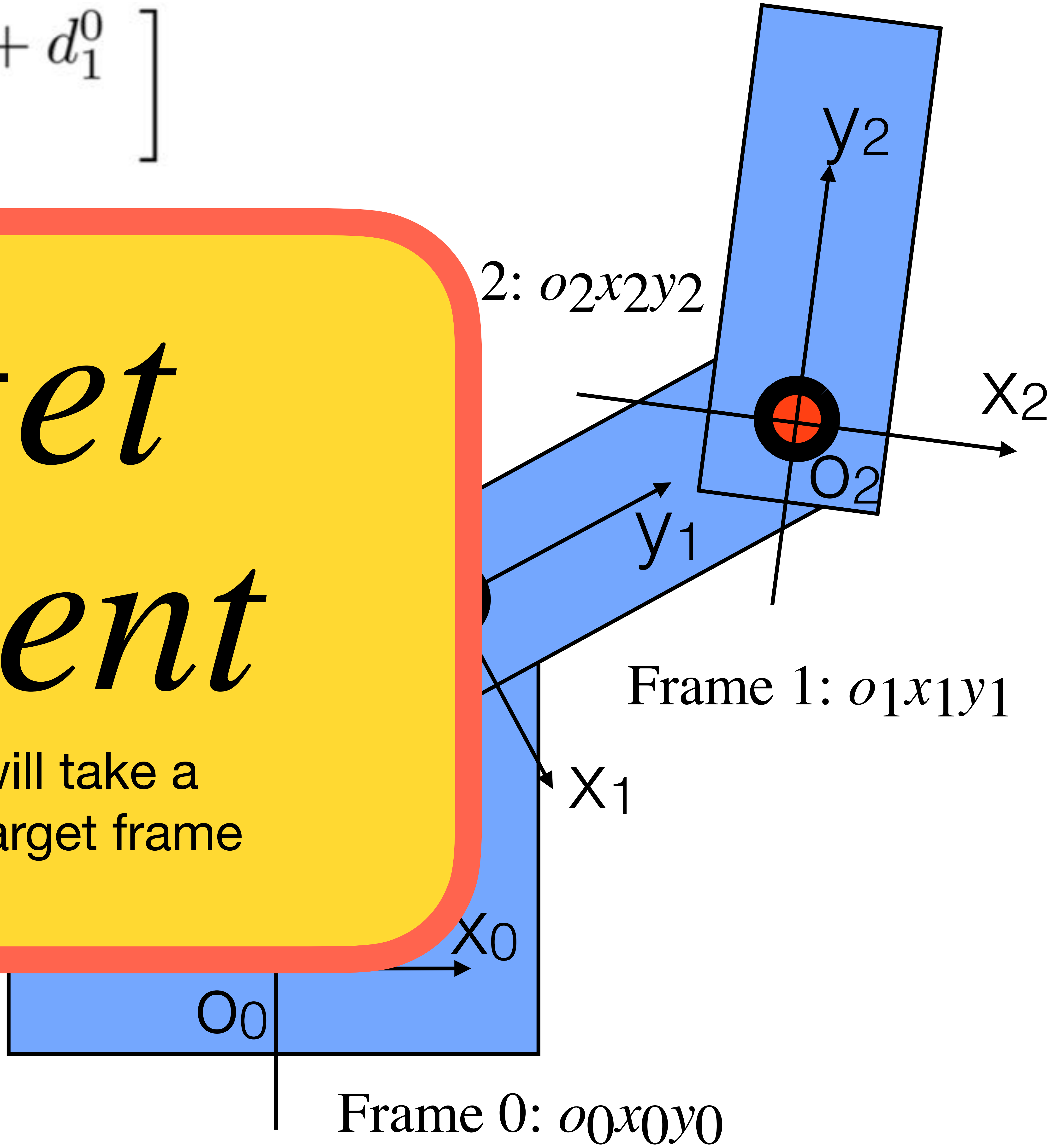
$R_2{}^0 = R_1{}^0 R_2{}^1$

$d_2{}^0 = R_1{}^0 d_2{}^1$

$$R^{target}_{current}$$

Rotation Matrix that will take a point in current to the target frame

which can be observed by block multiplying transforms



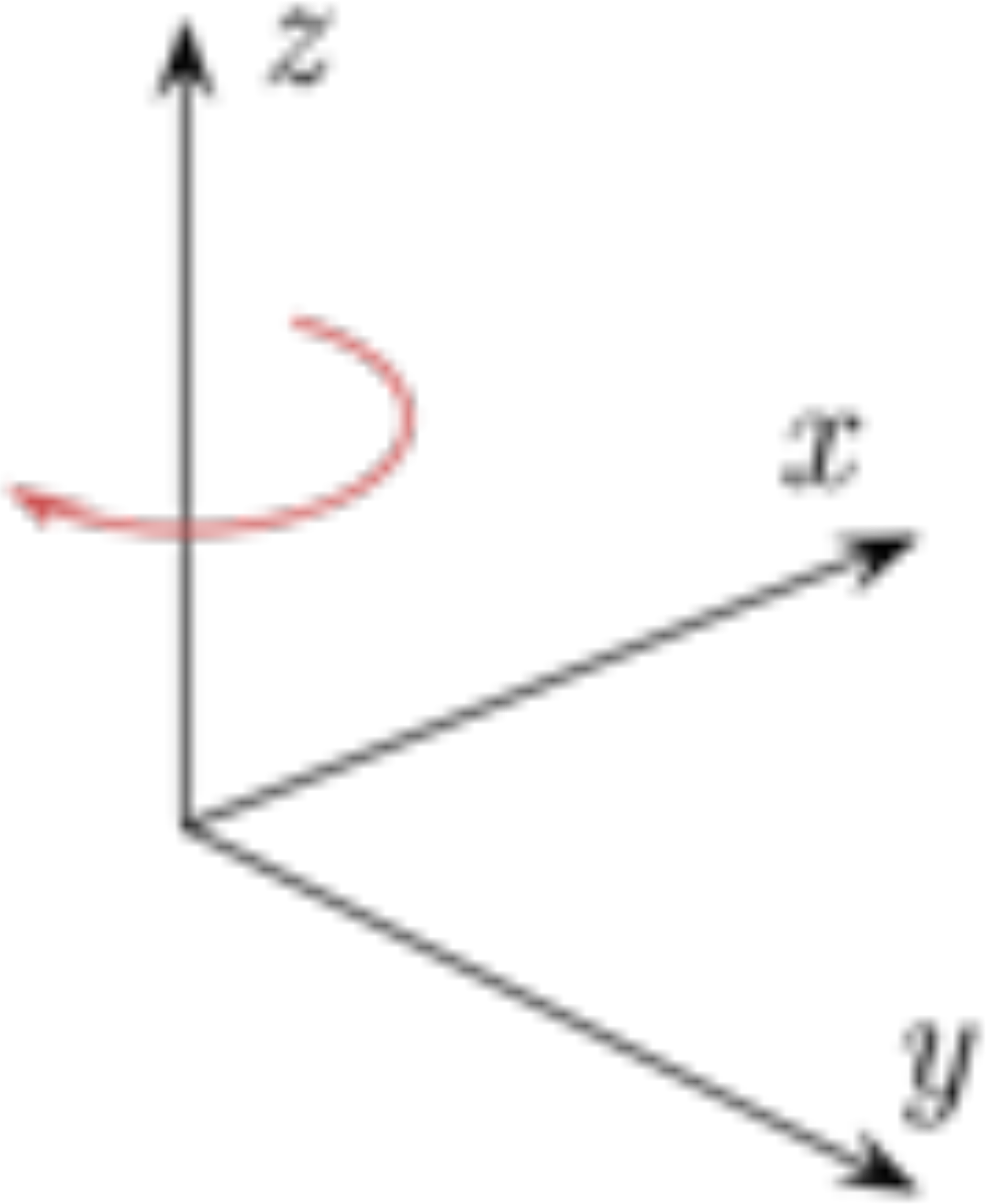
How do we extend this to 3D?

3D Translation and Rotation

$$D(d_x,d_y,d_z) \begin{bmatrix} 1 & 0 & 0 & d_x \\ 0 & 1 & 0 & d_y \\ 0 & 0 & 1 & d_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$R_z(\theta) \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 & 0 \\ \sin(\theta) & \cos(\theta) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

2D rotation in 3D is rotation about Z axis



Slide borrowed from Michigan Robotics autorob.org

3D Translation and Rotation

$$D(d_x, d_y, d_z) \begin{bmatrix} 1 & 0 & 0 & d_x \\ 0 & 1 & 0 & d_y \\ 0 & 0 & 1 & d_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$R_x(\theta) \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) & 0 \\ 0 & \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$R_z(\theta) \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 & 0 \\ \sin(\theta) & \cos(\theta) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$R_y(\theta) \begin{bmatrix} \cos(\theta) & 0 & \sin(\theta) & 0 \\ 0 & 1 & 0 & 0 \\ -\sin(\theta) & 0 & \cos(\theta) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

3D Homogeneous Transform

Rotate about each axis in order $R = R_x(\Theta_x)\, R_y(\Theta_y)\, R_z(\Theta_z)$

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 & d_x \\ 0 & 1 & 0 & d_y \\ 0 & 0 & 1 & d_z \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{D(d_x,d_y,d_z)} \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) & 0 \\ 0 & \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{R_x(\theta)} \underbrace{\begin{bmatrix} \cos(\theta) & 0 & \sin(\theta) & 0 \\ 0 & 1 & 0 & 0 \\ -\sin(\theta) & 0 & \cos(\theta) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{R_y(\theta)} \underbrace{\begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 & 0 \\ \sin(\theta) & \cos(\theta) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{R_z(\theta)}$$

3D Homogeneous Transform

$$\begin{bmatrix}1 & 0 & 0 & d_x\\ 0 & 1 & 0 & d_y\\ 0 & 0 & 1 & d_z\\ 0 & 0 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & \cos(\theta) & -\sin(\theta) & 0\\ 0 & \sin(\theta) & \cos(\theta) & 0\\ 0 & 0 & 0 & 1\end{bmatrix}\begin{bmatrix}\cos(\theta) & 0 & \sin(\theta) & 0\\ 0 & 1 & 0 & 0\\ -\sin(\theta) & 0 & \cos(\theta) & 0\\ 0 & 0 & 0 & 1\end{bmatrix}\begin{bmatrix}\cos(\theta) & -\sin(\theta) & 0 & 0\\ \sin(\theta) & \cos(\theta) & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}$$

$$= H_3 = \begin{bmatrix}R_{00} & R_{01} & R_{02} & d_x\\ R_{10} & R_{11} & R_{12} & d_y\\ R_{20} & R_{21} & R_{22} & d_z\\ 0 & 0 & 0 & 1\end{bmatrix} = \begin{bmatrix}\mathbf{R}_{3\times3} & \mathbf{d}_{3\times1}\\ \mathbf{0}_{1\times3} & 1\end{bmatrix}$$

$$\boxed{\begin{aligned} H_3 &\in SE(3)\\ \mathbf{R}_{3\times3} &\in SO(3)\\ \mathbf{d}_{3\times1} &\in \Re^3\end{aligned}}$$

3D Homogeneous Transform

$$H_3 = \begin{bmatrix} R_{00} & R_{01} & R_{02} & d_x \\ R_{10} & R_{11} & R_{12} & d_y \\ R_{20} & R_{21} & R_{22} & d_z \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \mathbf{R}_{3\times 3} & \mathbf{d}_{3\times 1} \\ \mathbf{0}_{1\times 3} & 1 \end{bmatrix} \in SE(3)$$

if $T_1^0 \in SE(3)$ and $T_2^1 \in SE(3)$ then composition holds:

$$\begin{bmatrix} R_1^0 & d_1^0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} R_2^1 & d_2^1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} R_1^0 R_2^1 & R_1^0 d_2^1 + d_1^0 \\ 0 & 1 \end{bmatrix}$$

such that points in Frame 2 can be expressed in Frame 0 by:

$$p^0 = T_1^0 T_2^1 p^2$$

Next lecture:
Representations II: Rotations & Quaternions

PR2 Fetches Sandwich from Subway 11 years ago!

Autonomous Subway sandwich delivery by a PR2 robot, from the University of Tokyo and TUM