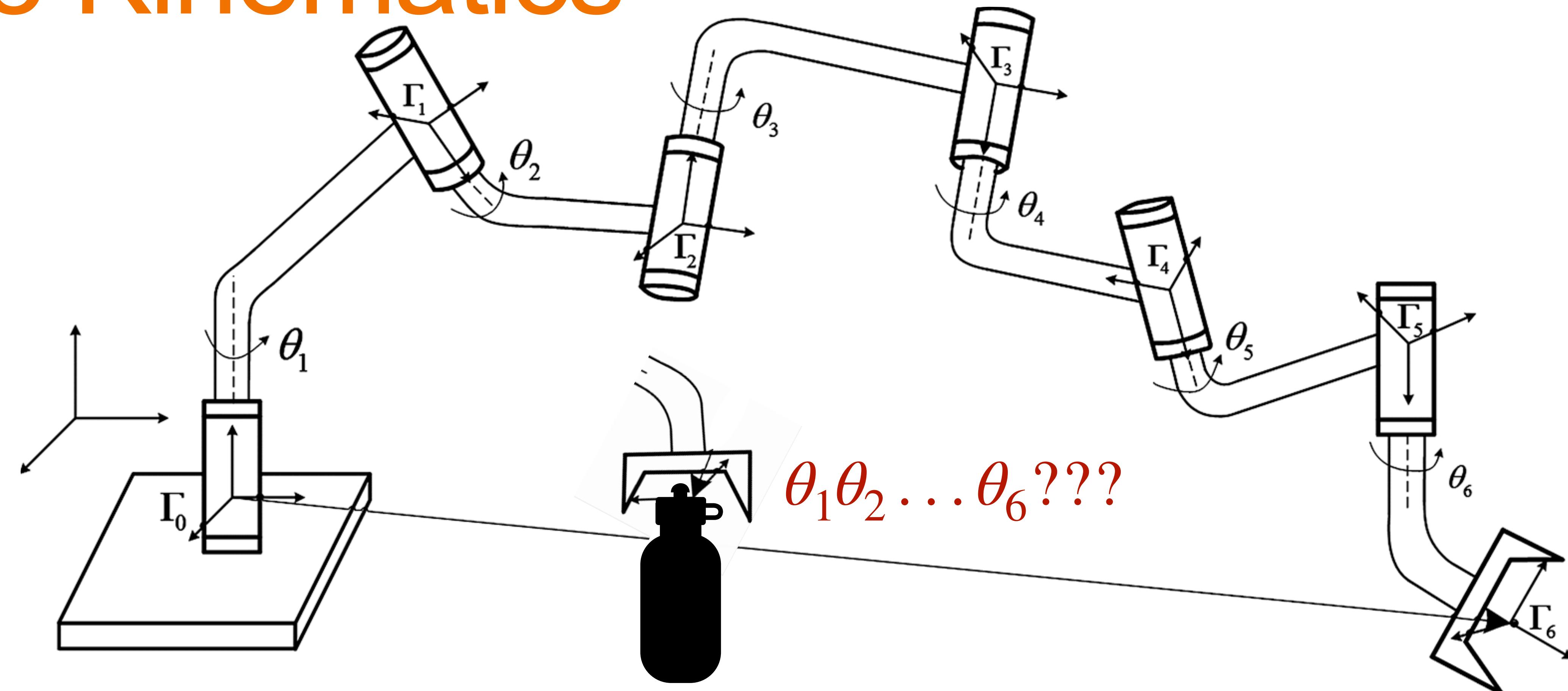


Lecture 06

Manipulation - II

Inverse Kinematics



Course Logistics

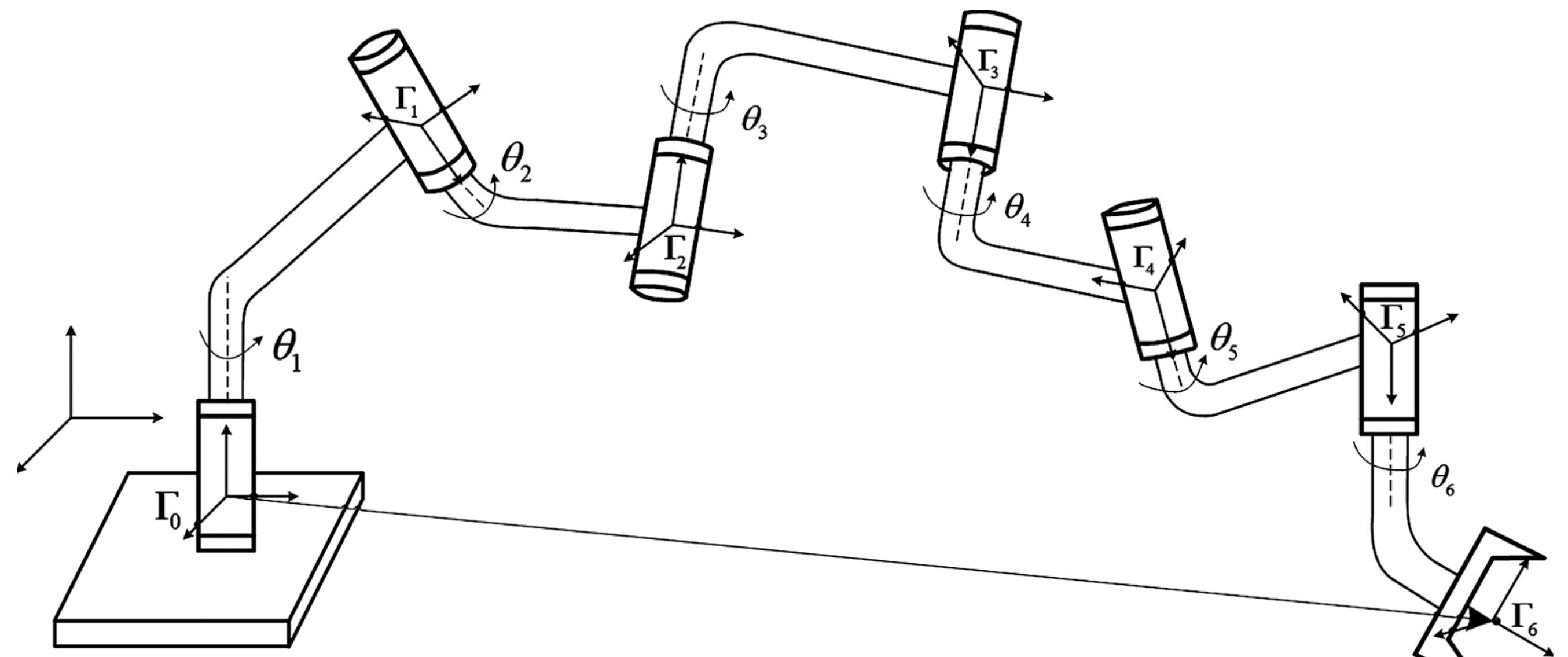
- Quiz 4 was posted today and was due before the lecture.
- Project 0 was posted on 09/13 and will be due **09/25 (today)**.
- Autograder is available for P0.
- Project 1 is posted on 09/20 and will be due 10/02.
- Autograder for P1 will be available by 09/26 (tomorrow).
 - *But you can start working on it!*



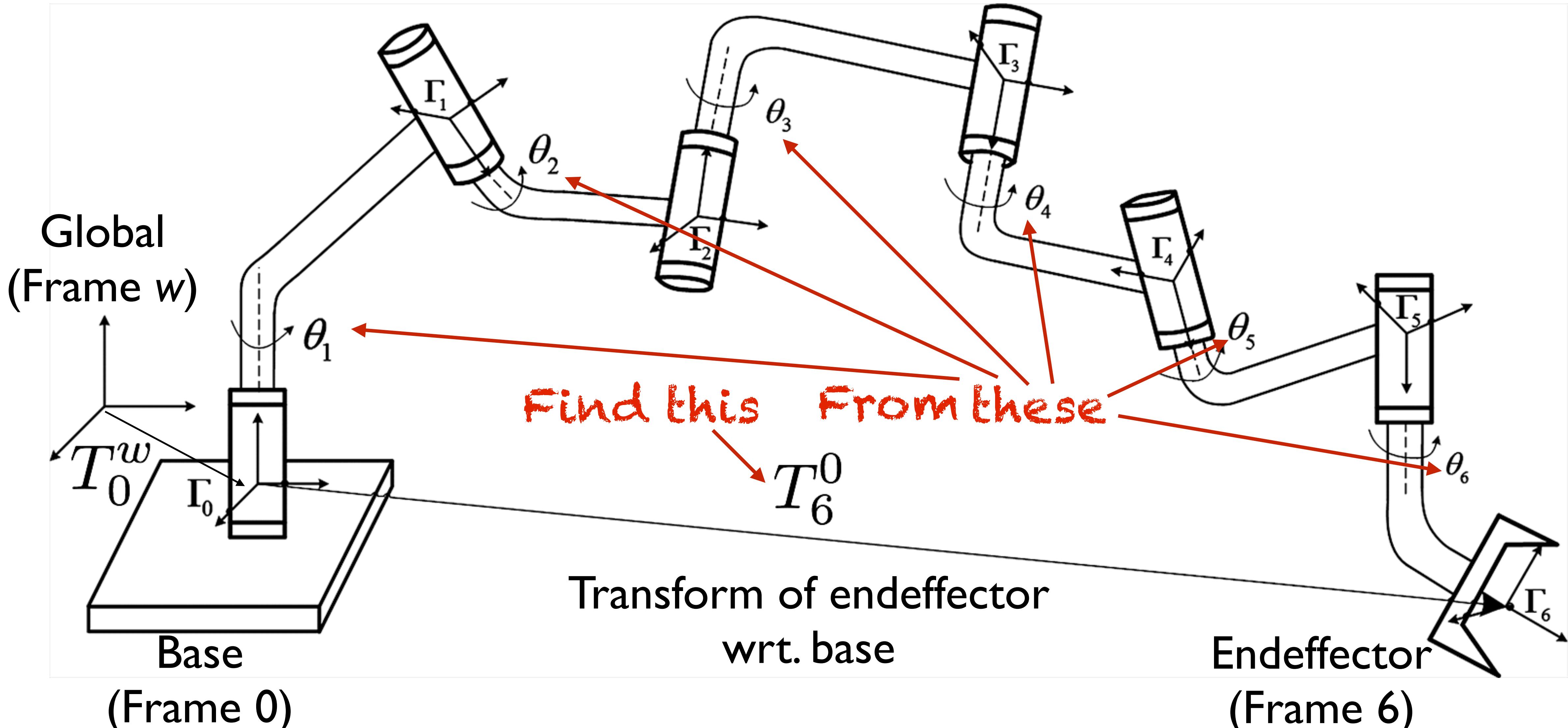
Robot Kinematics

Goal: Given the structure of a robot arm, compute

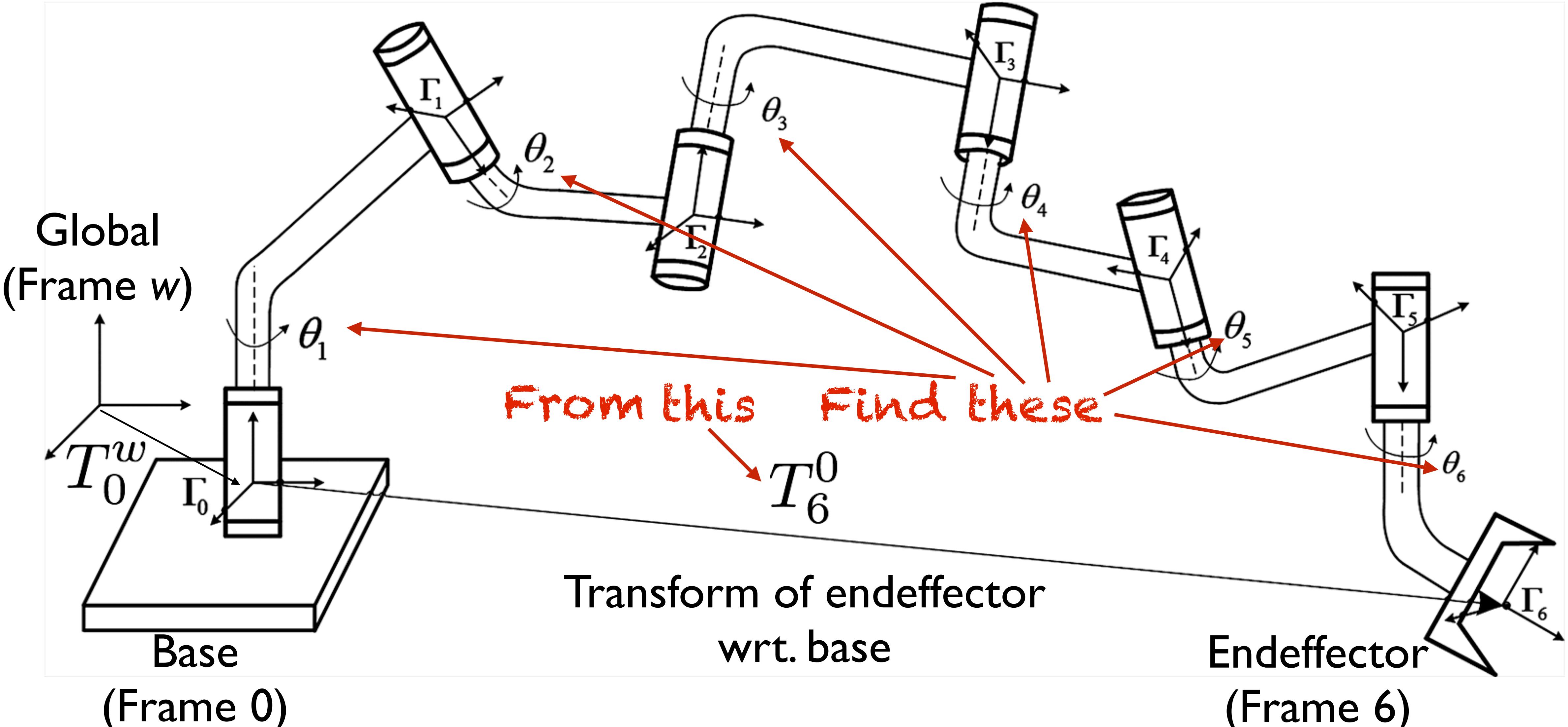
- **Forward kinematics:** infer the pose of the end-effector, given the state of each joint
- **Inverse kinematics:** inferring the joint states necessary to reach a desired end-effector pose.



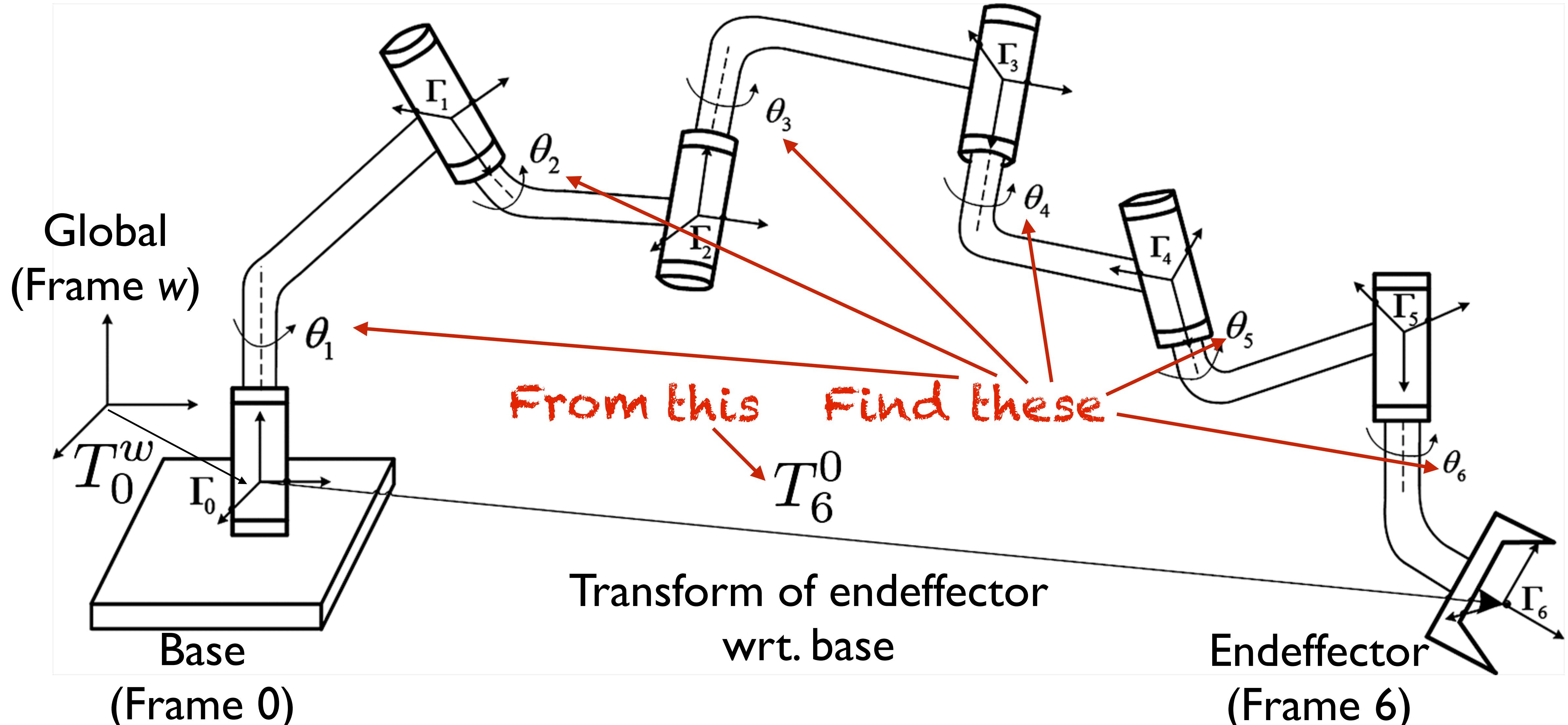
Forward kinematics: many-to-one mapping of robot configuration to reachable workspace endeffector poses



Inverse kinematics: one-to-many mapping of workspace endeffector pose to robot configuration



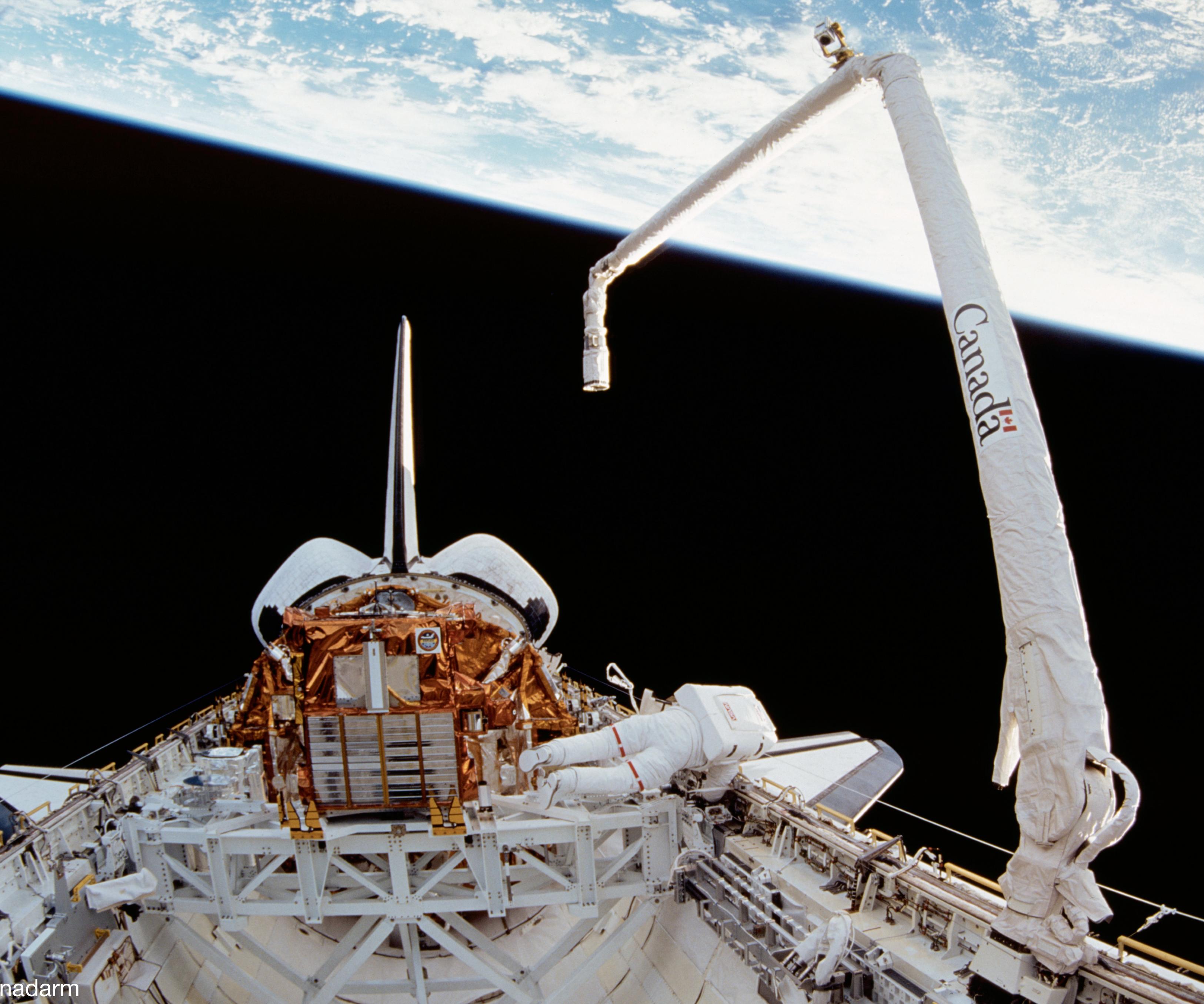
Inverse kinematics: how to solve for $q = \{\theta_1, \dots, \theta_N\}$ from T^0_N ?



Inverse Kinematics: 2 possibilities

- **Closed-form solution:** geometrically infer satisfying configuration
 - *Speed:* solution often computed in constant time
 - *Predictability:* solution is selected in a consistent manner
- **Solve by optimization:** minimize error of endeffector to desired pose
 - often some form of Gradient Descent (a la Jacobian Transpose)
 - *Generality:* same solver can be used for many different robots

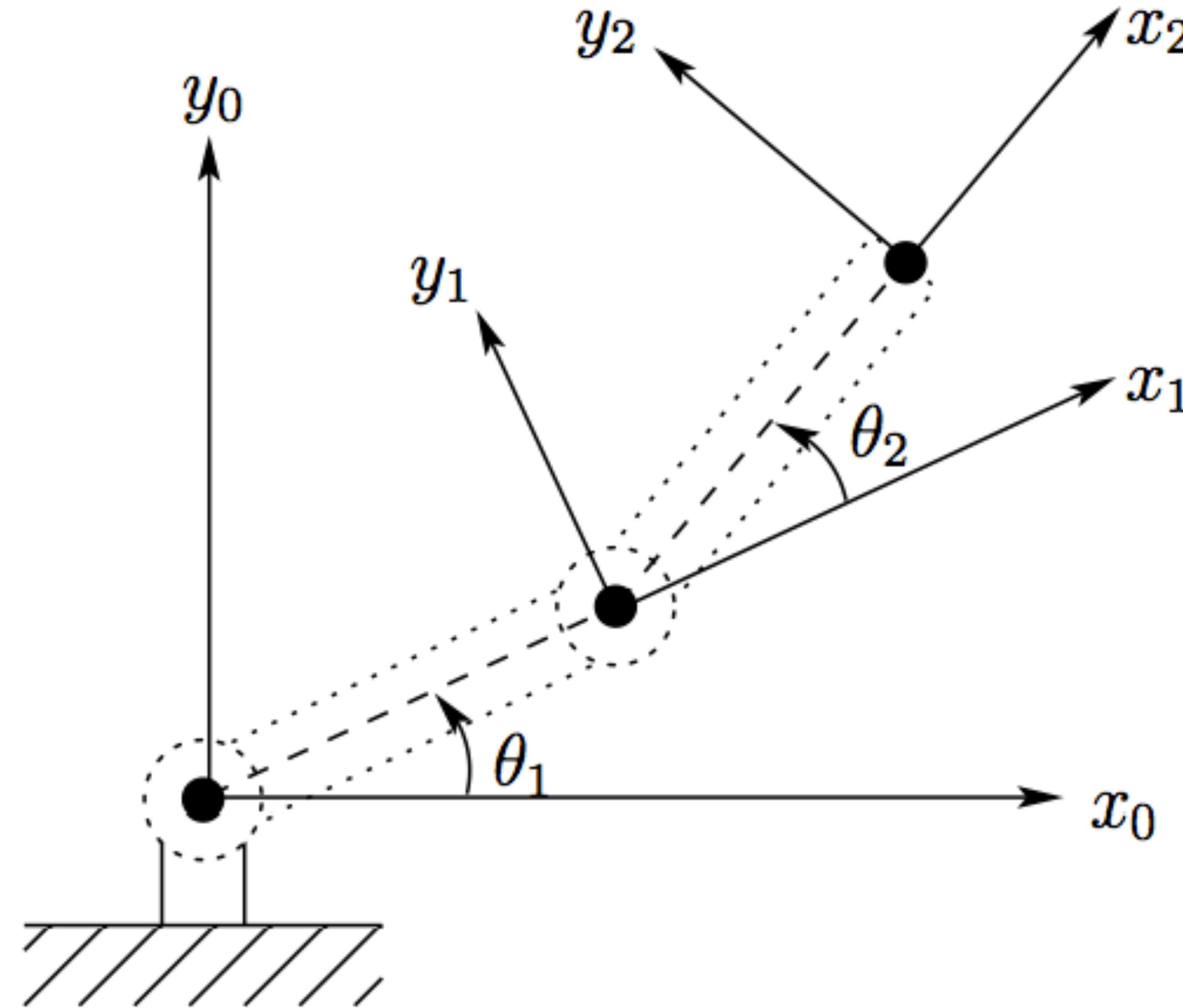
Let's define IK
starting from FK



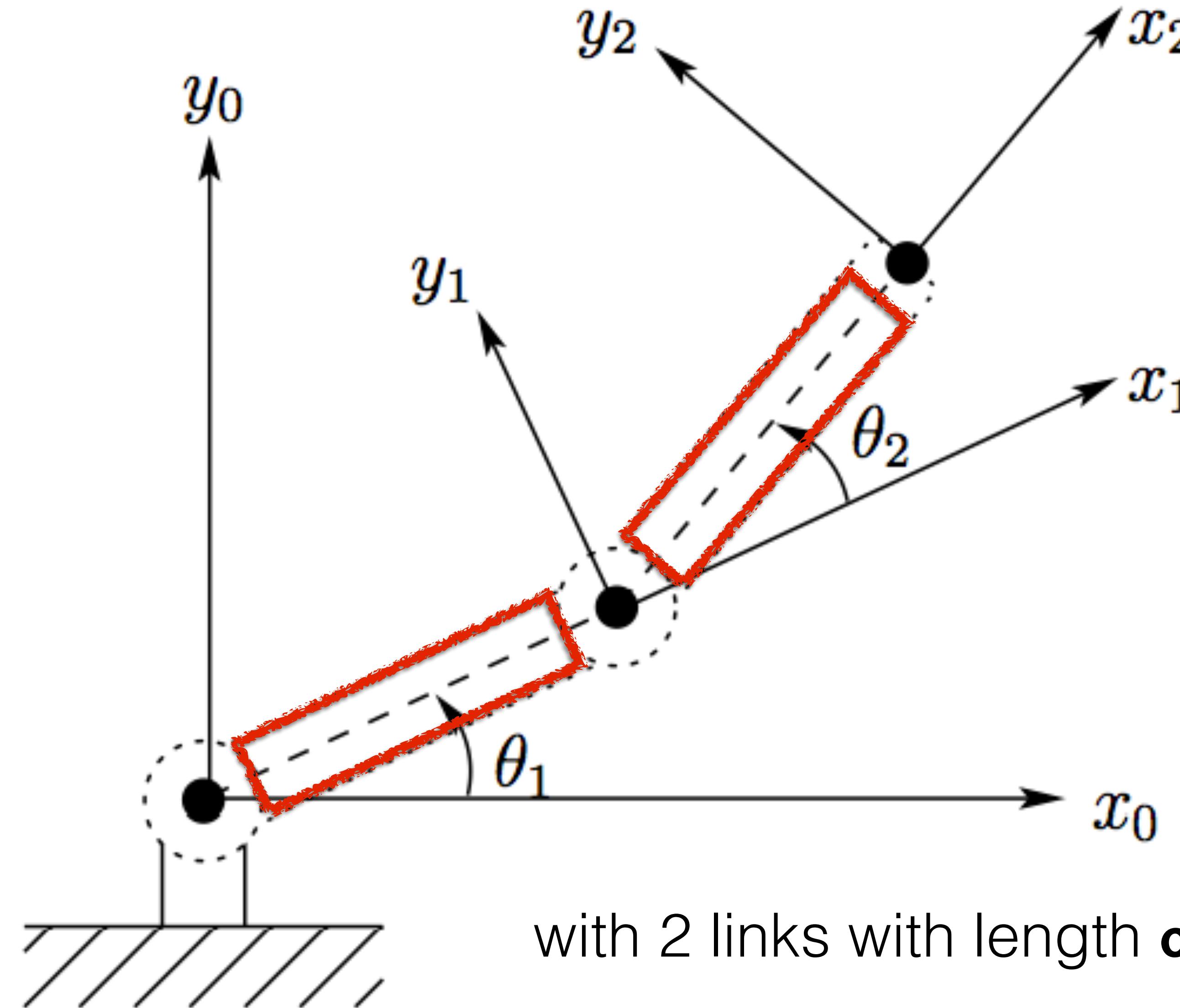
<https://en.wikipedia.org/wiki/Canadarm>



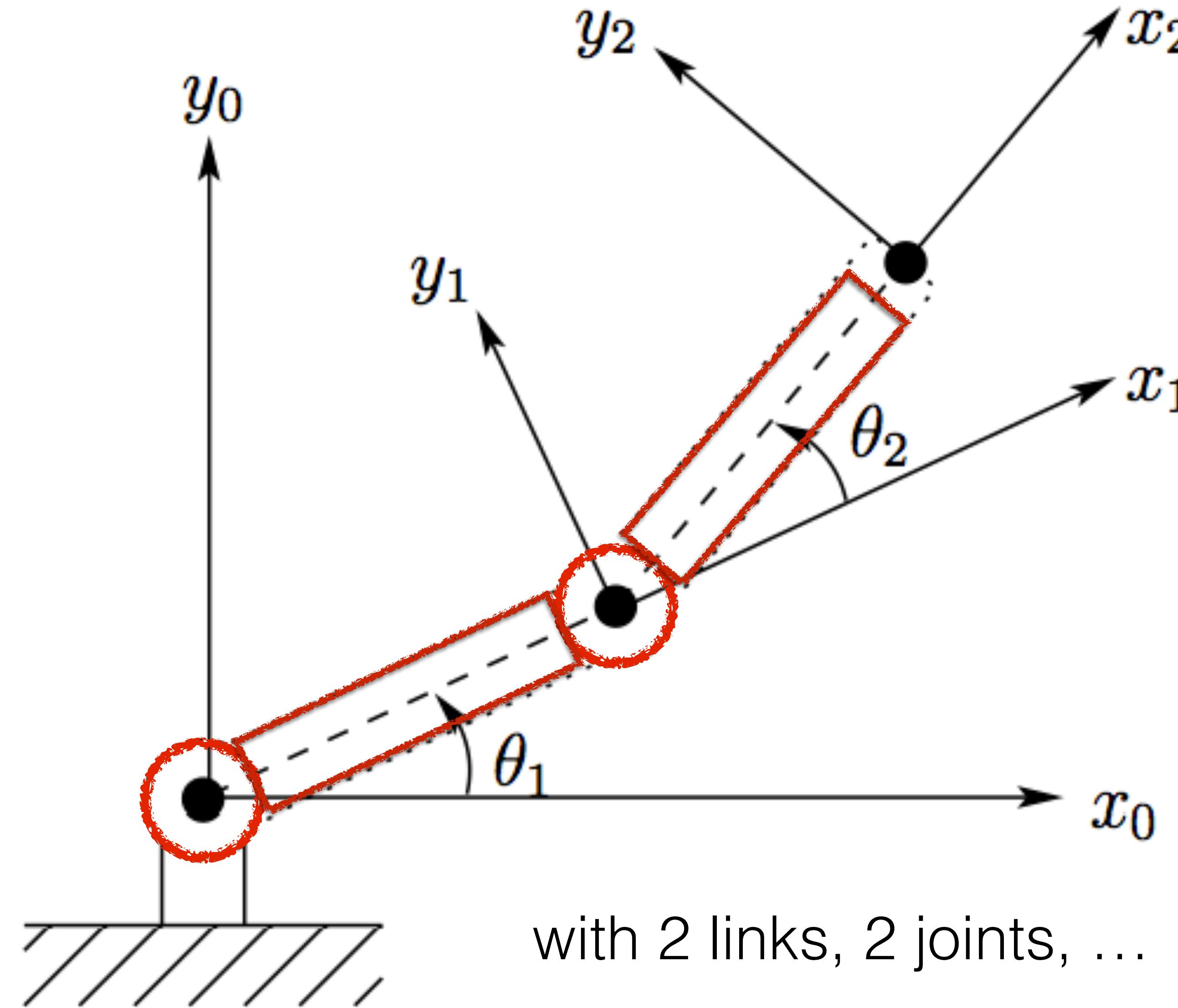
Consider a planar 2-link arm as an example



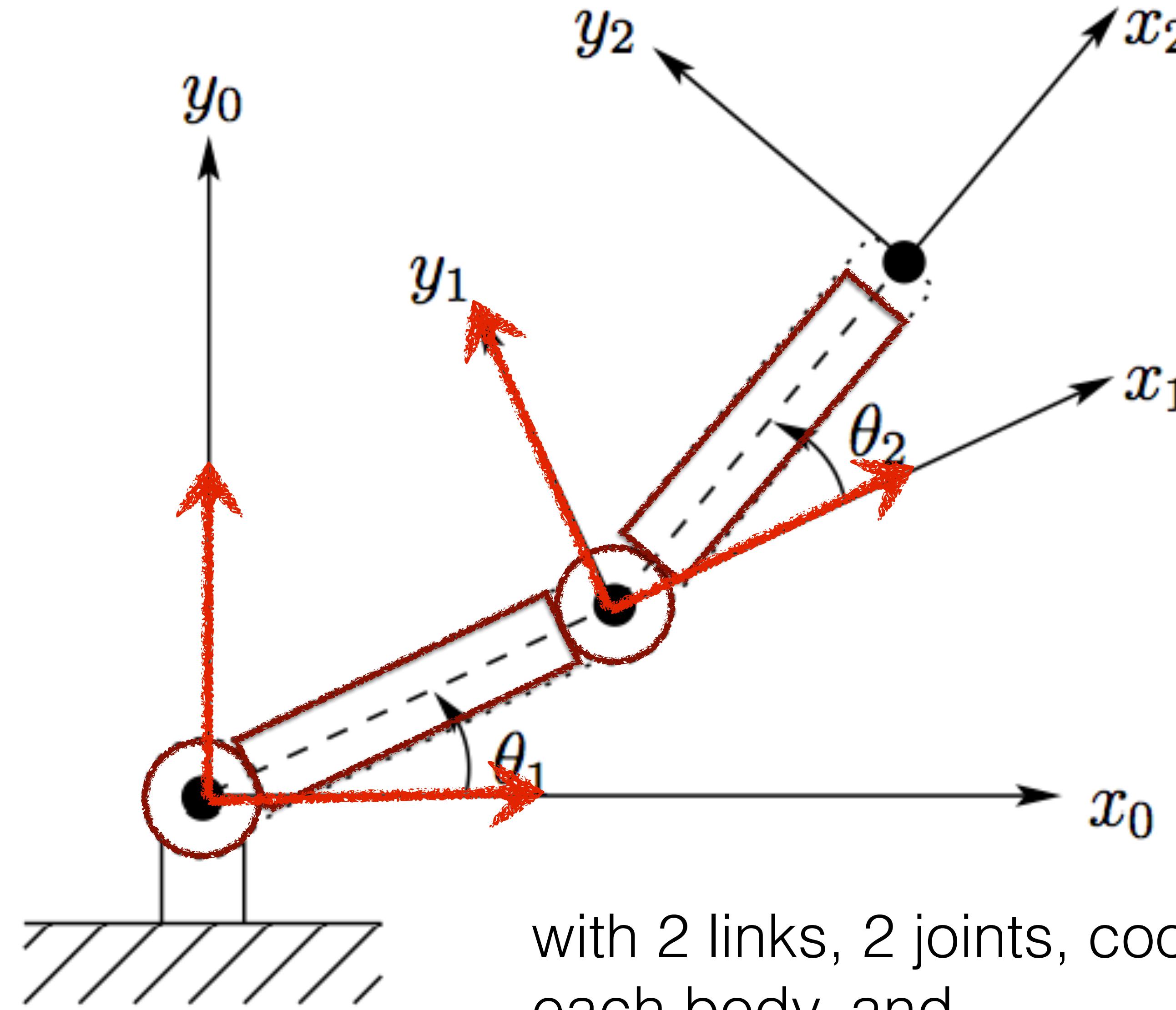
Consider a planar 2-link arm as an example



Consider a planar 2-link arm as an example

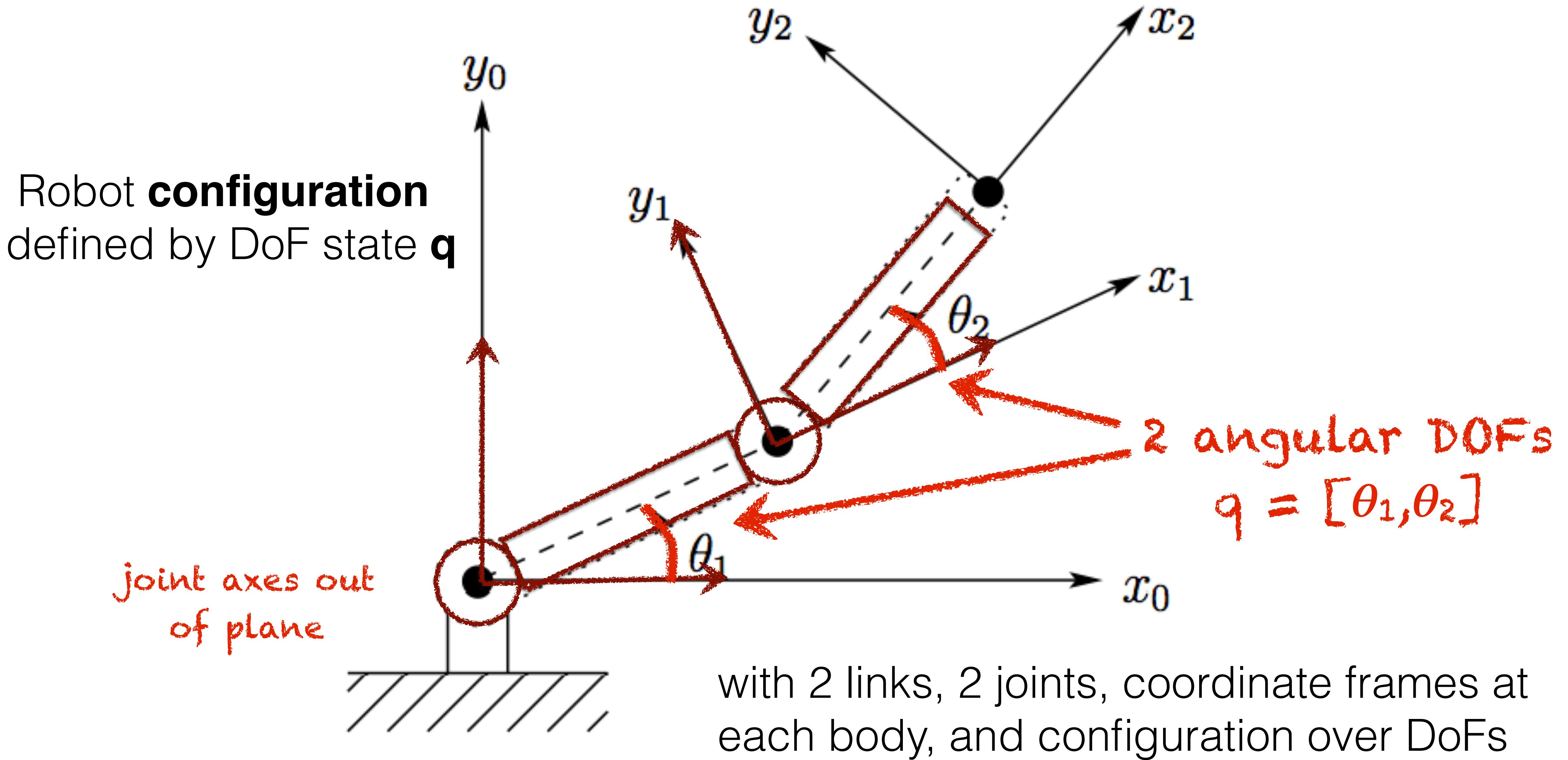


Consider a planar 2-link arm as an example

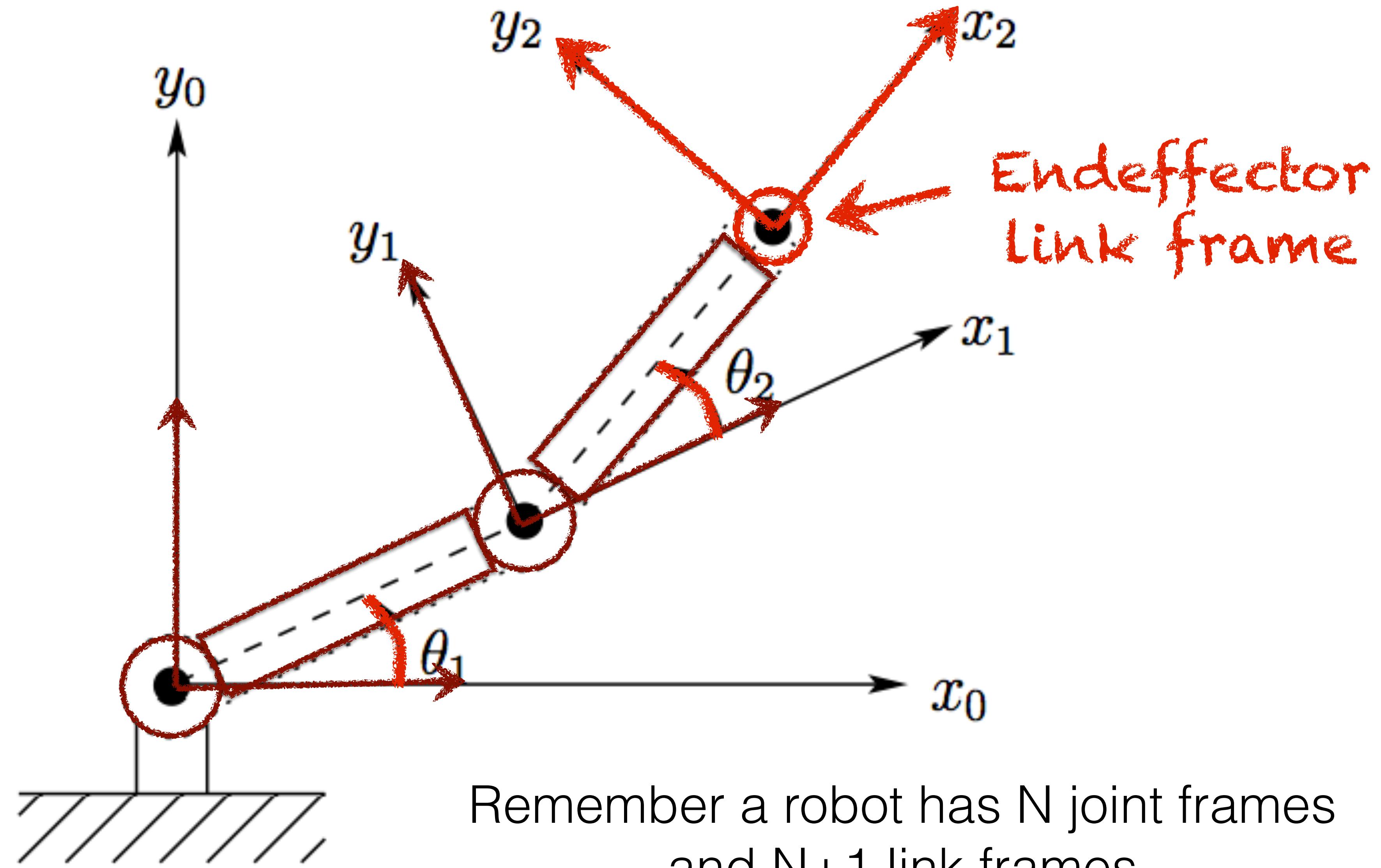


with 2 links, 2 joints, coordinate frames at each body, and ...

Consider a planar 2-link arm as an example



Consider a planar 2-link arm as an example



Remember a robot has N joint frames
and N+1 link frames

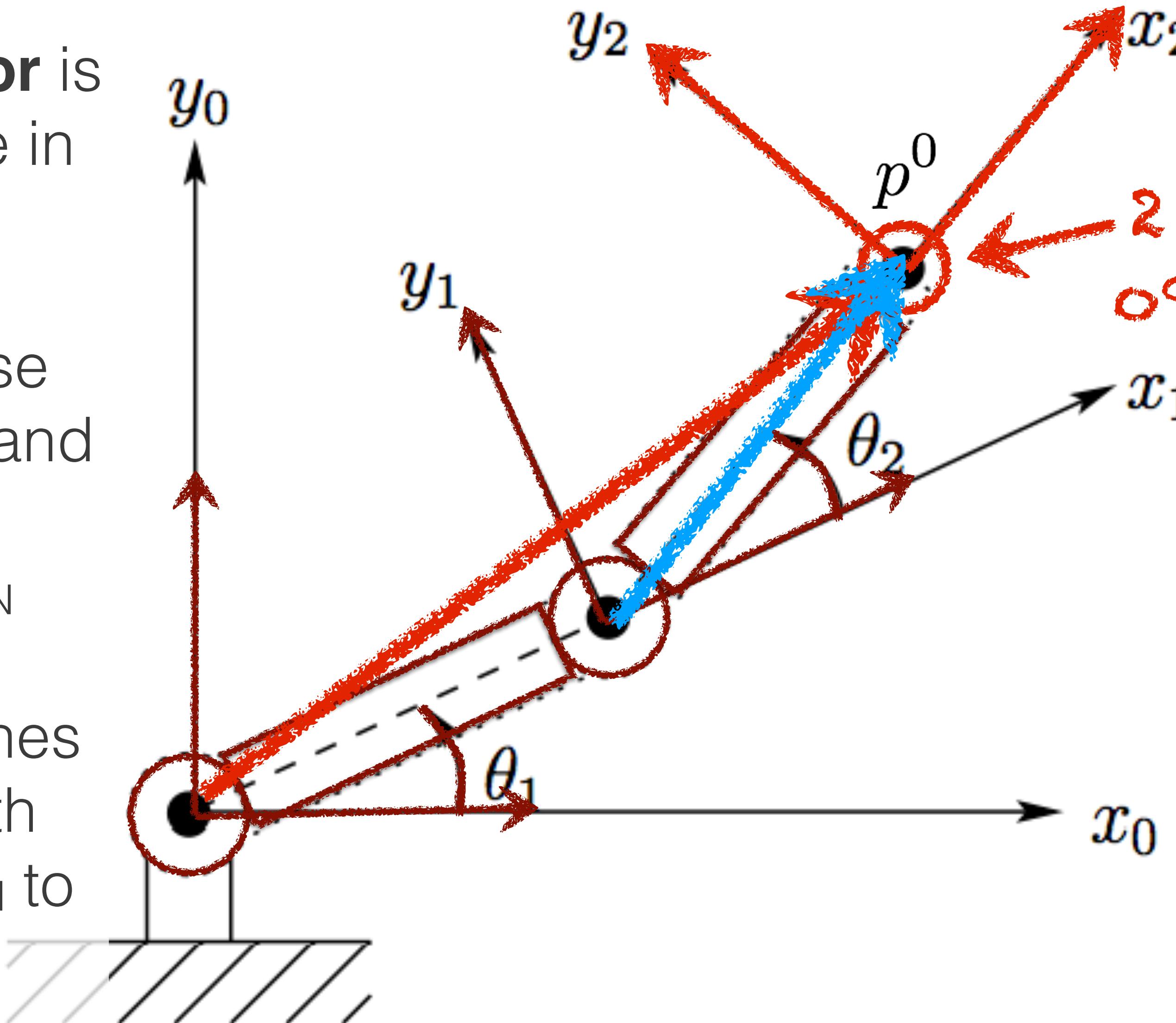
Consider a planar 2-link arm as an example

Frame 2 is the “tool frame”

Robot **endeffector** is
the gripper pose in
world frame

Endeffector pose
has position \mathbf{o}^0_N and
can consider
orientation \mathbf{R}^0_N

Endeffector defines
“tool frame” with
transform $\mathbf{H} = \mathbf{T}^0_N$ to
world frame

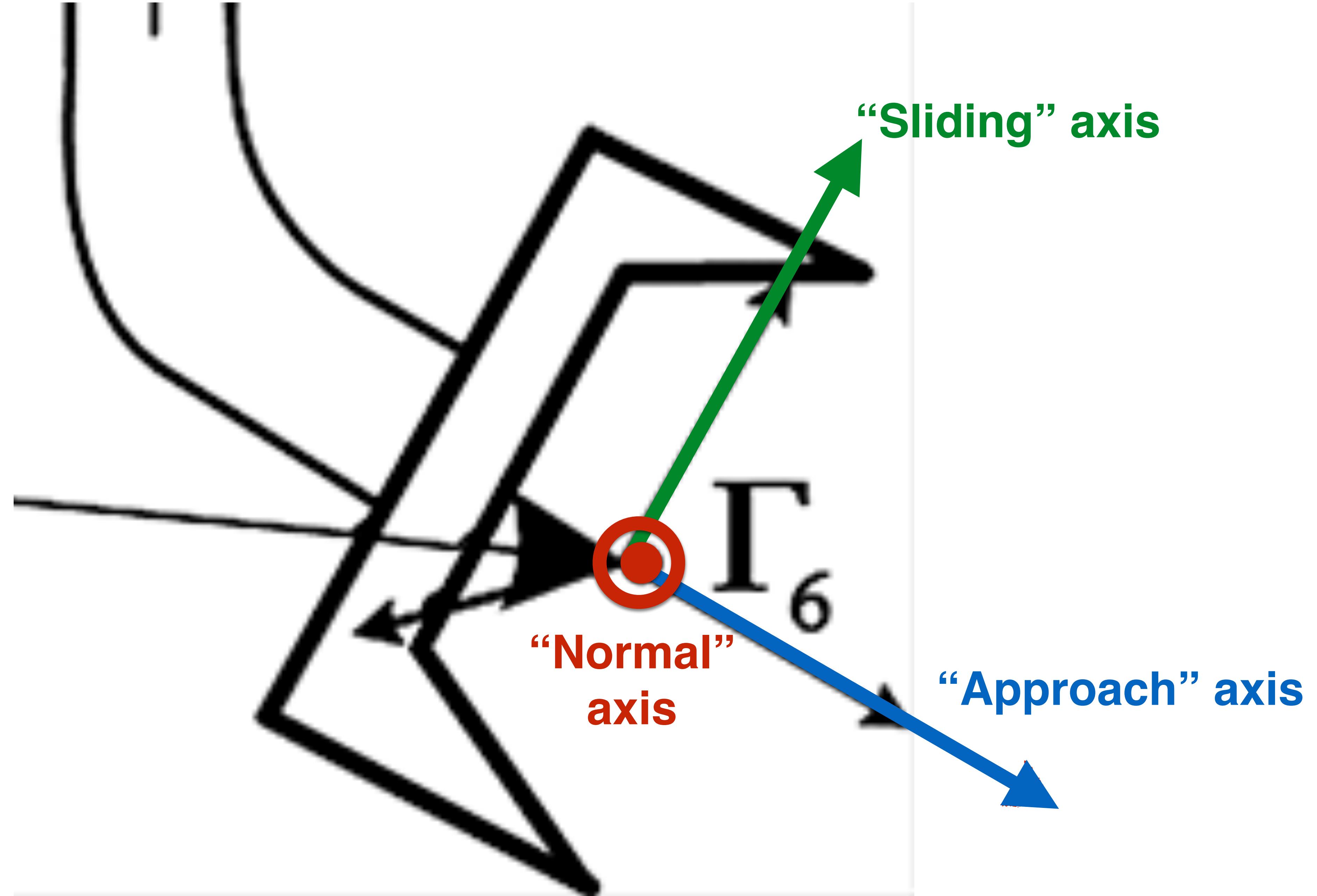


p^0 With respect to Frame 0

p^1 With respect to Frame 1

p^2 With respect to Frame 2 = (0, 0)

Endeffector axes

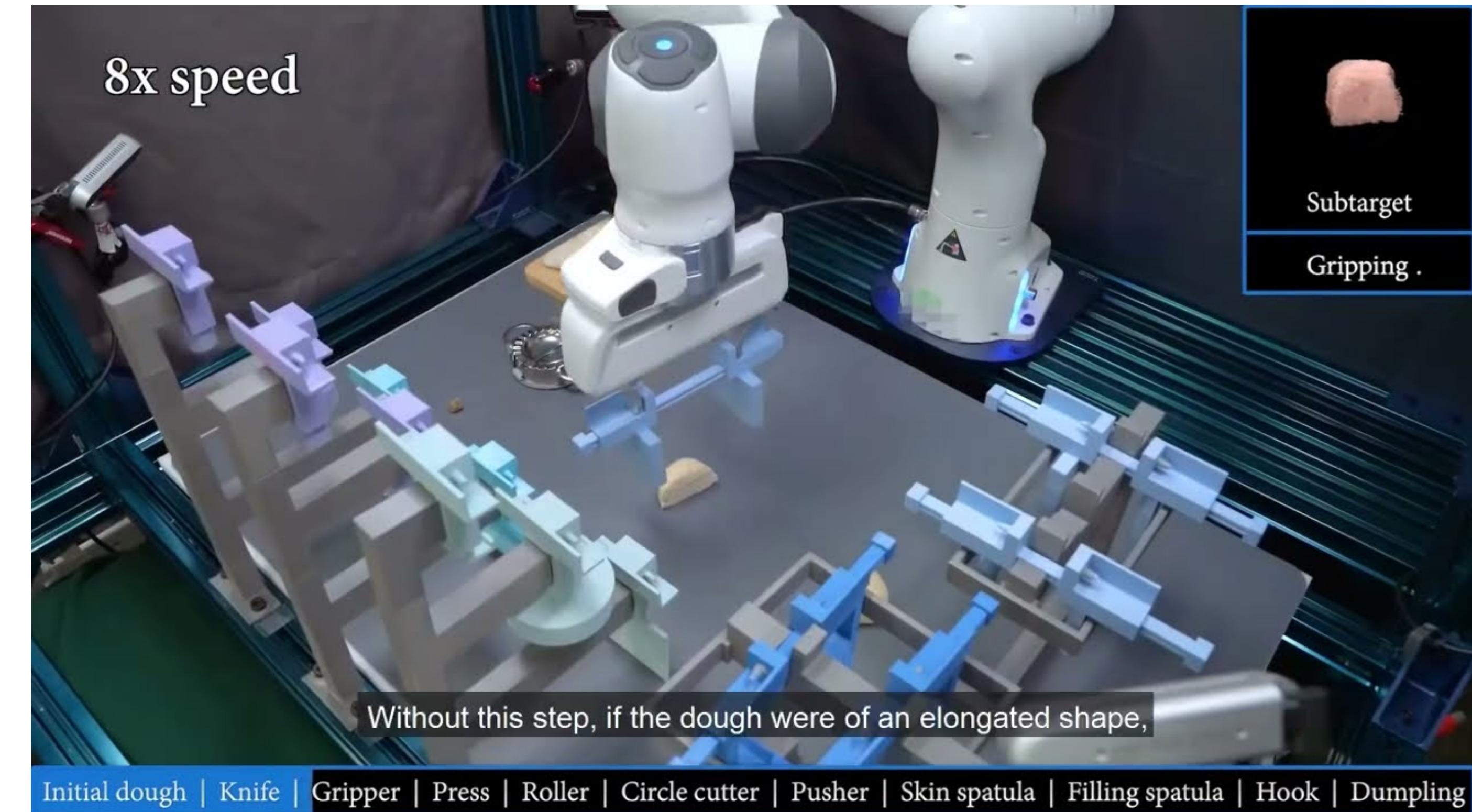


What are end-effectors?



<https://www.tthk.ee/inlearcs/7-robot-end-of-arm-tooling/>

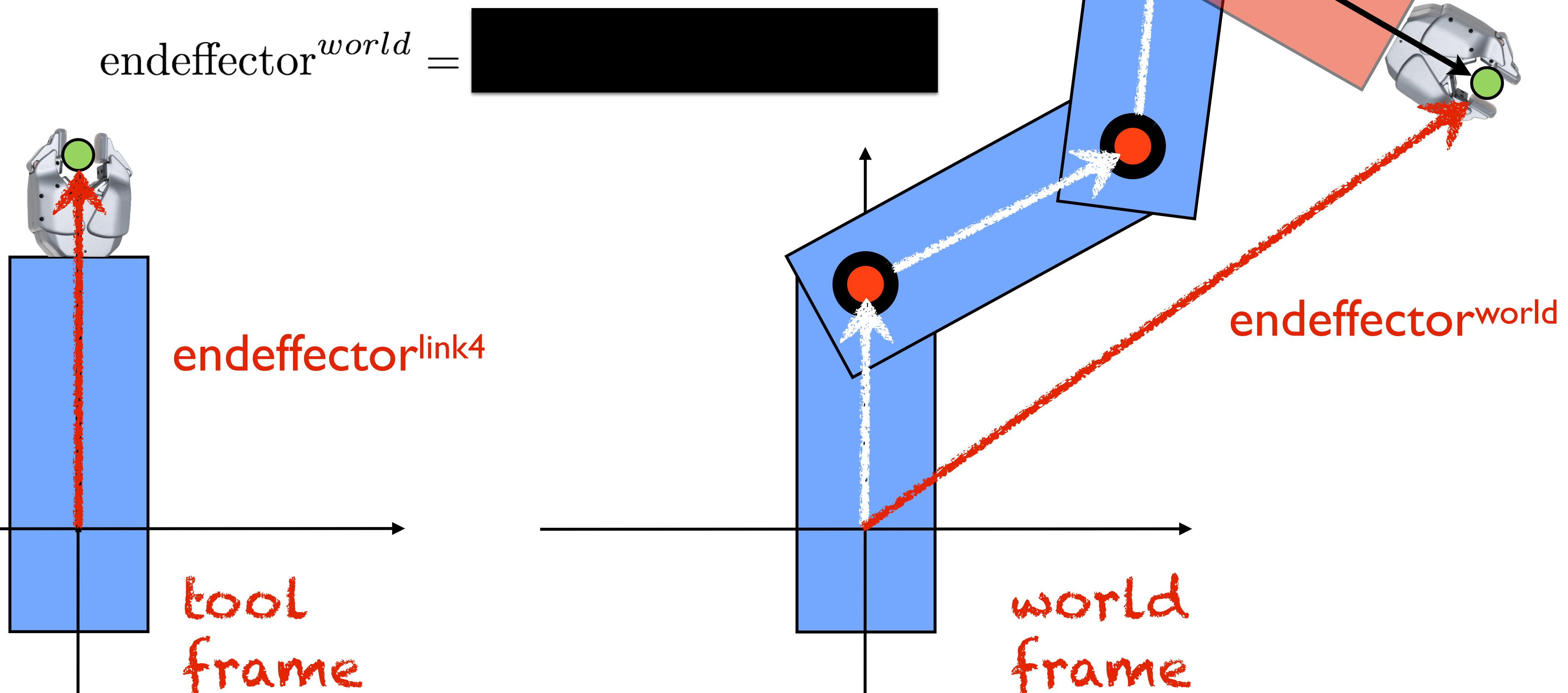
What are end-effectors?



Shi, Haochen, Huazhe Xu, Samuel Clarke, Yunzhu Li, and Jiajun Wu.
"RoboCook: Long-Horizon Elasto-Plastic Object Manipulation with Diverse Tools."
arXiv preprint arXiv:2306.14447 (2023).
<https://hshi74.github.io/robocook/>

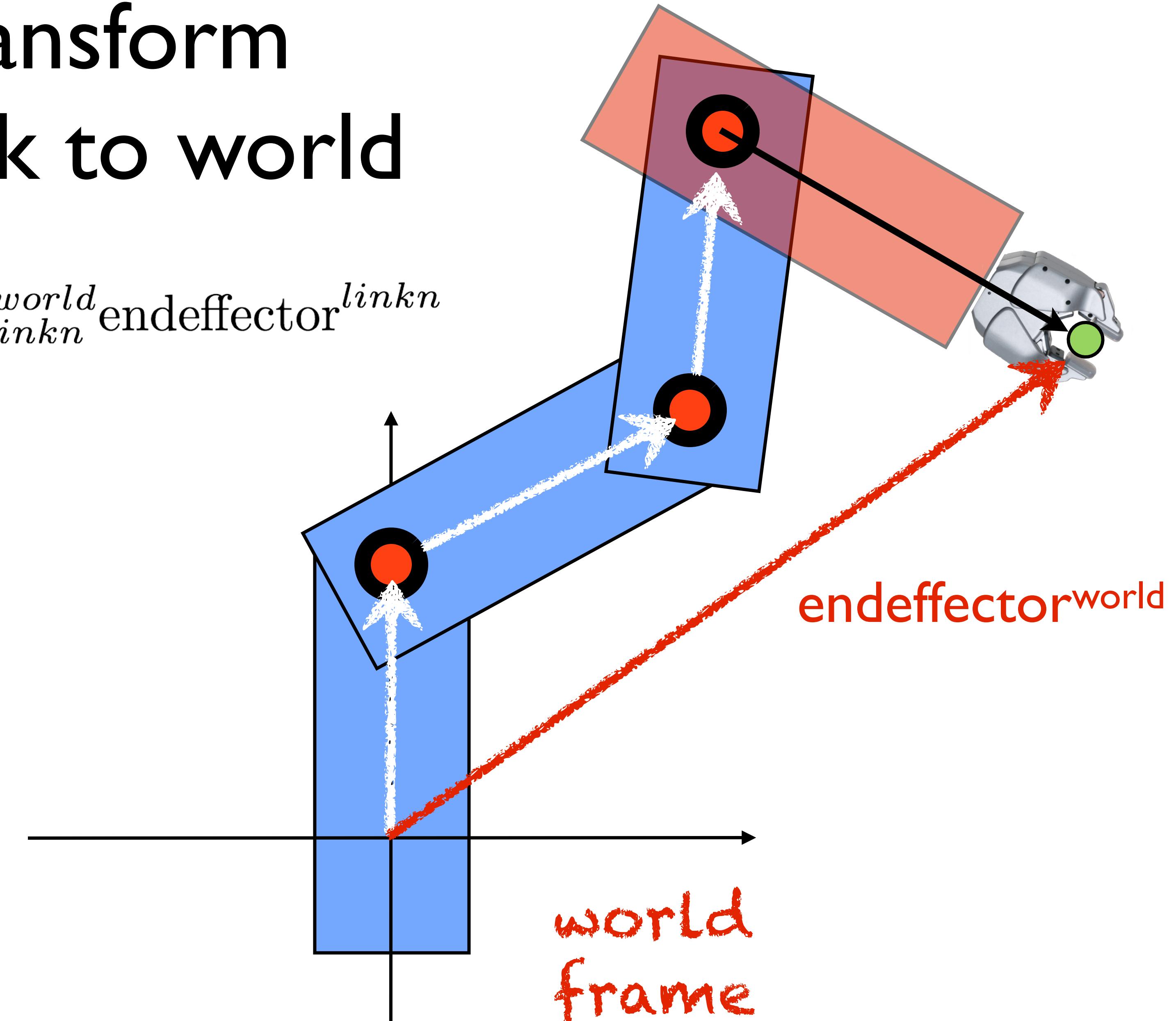
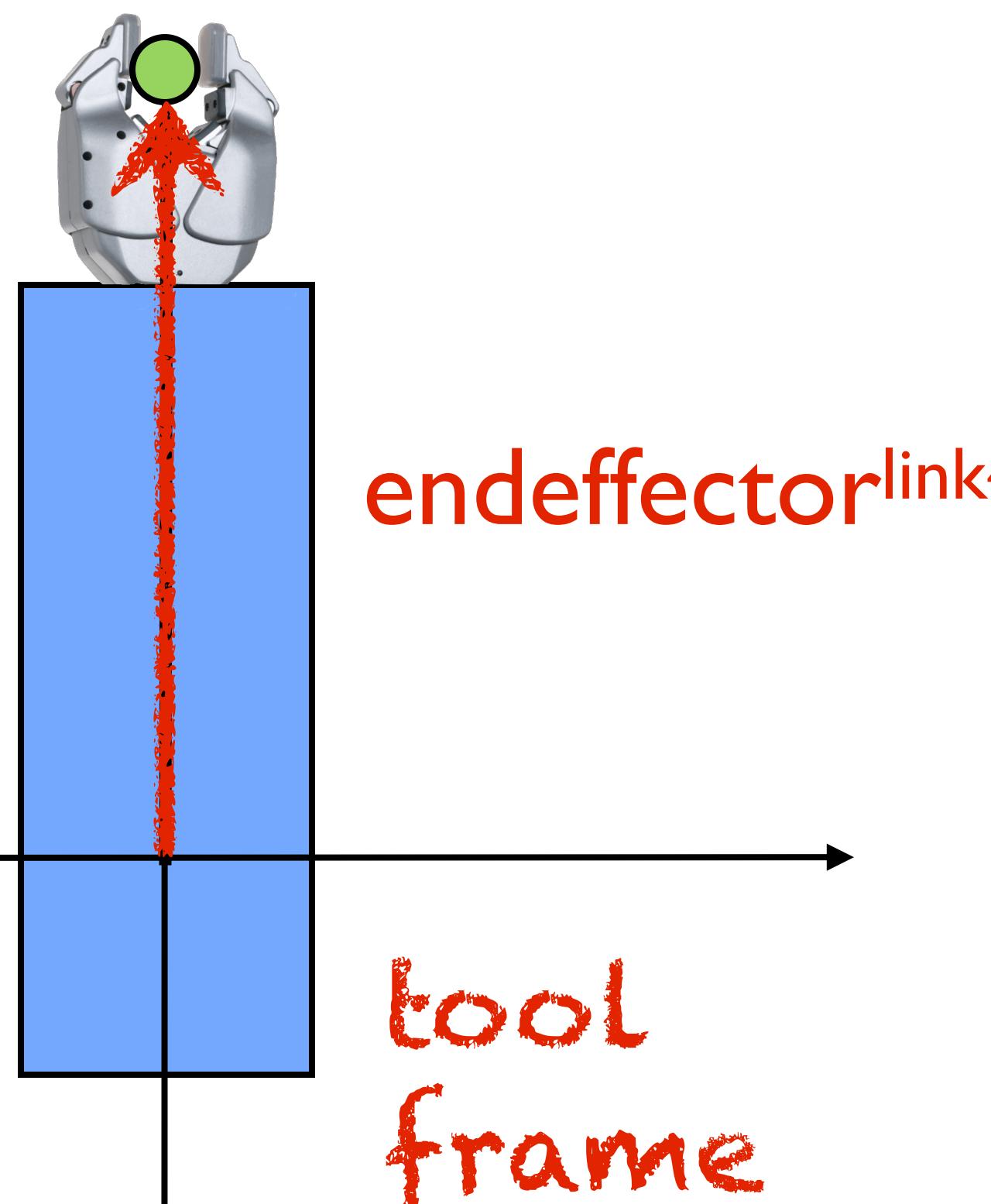
<https://www.tthk.ee/inlearcs/7-robot-end-of-arm-tooling/>

Checkpoint: Transform endeffector on link to world



Checkpoint: Transform endeffector on link to world

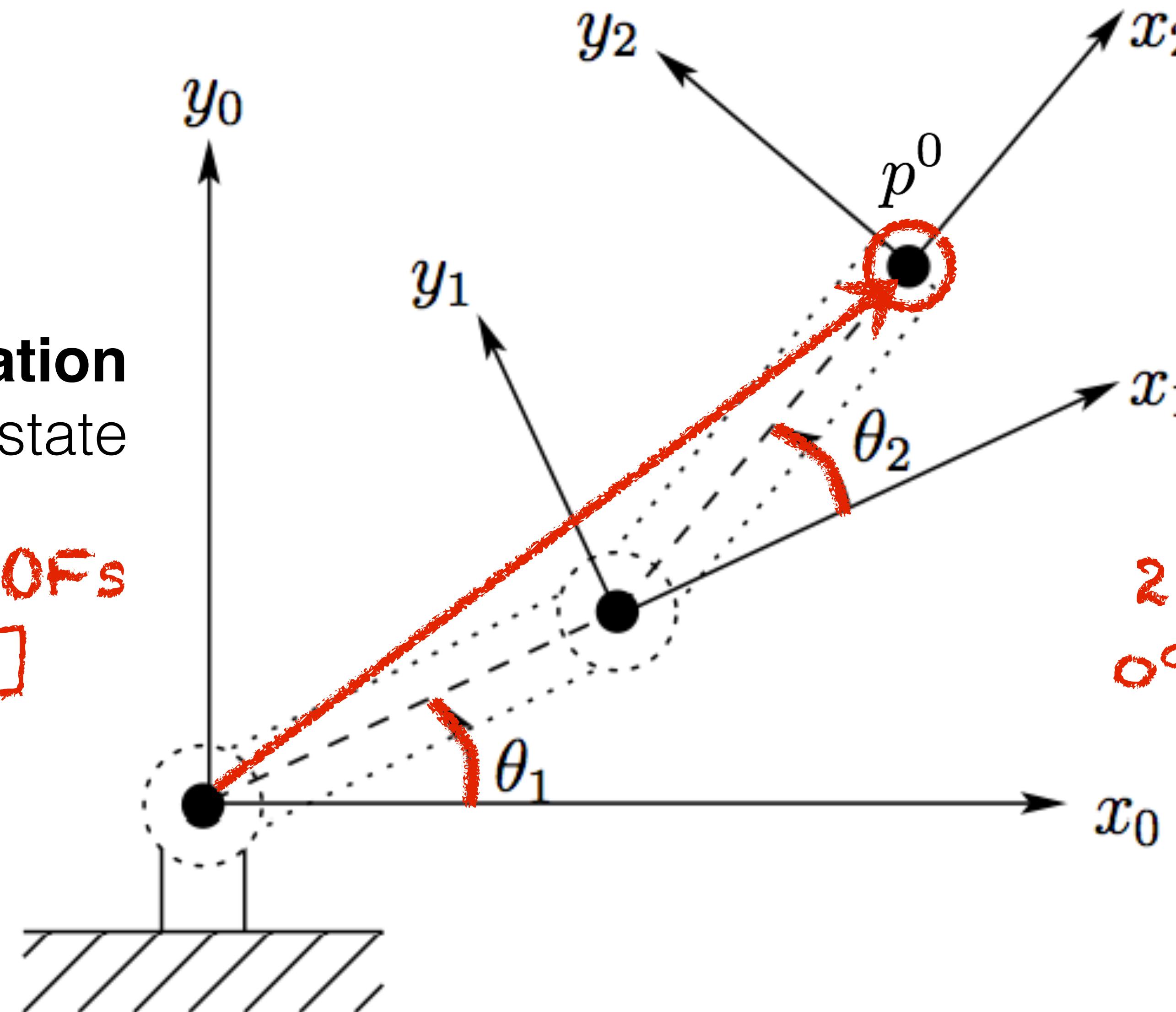
$$\text{endeffector}^{world} = T_{linkn}^{world} \text{endeffector}^{linkn}$$



Forward kinematics: “given configuration, compute endeffector”

Robot **configuration**
defined by DoF state

2 angular DOFs
 $q = [\theta_1, \theta_2]$



Robot **endeffector**
is the gripper pose
in world frame

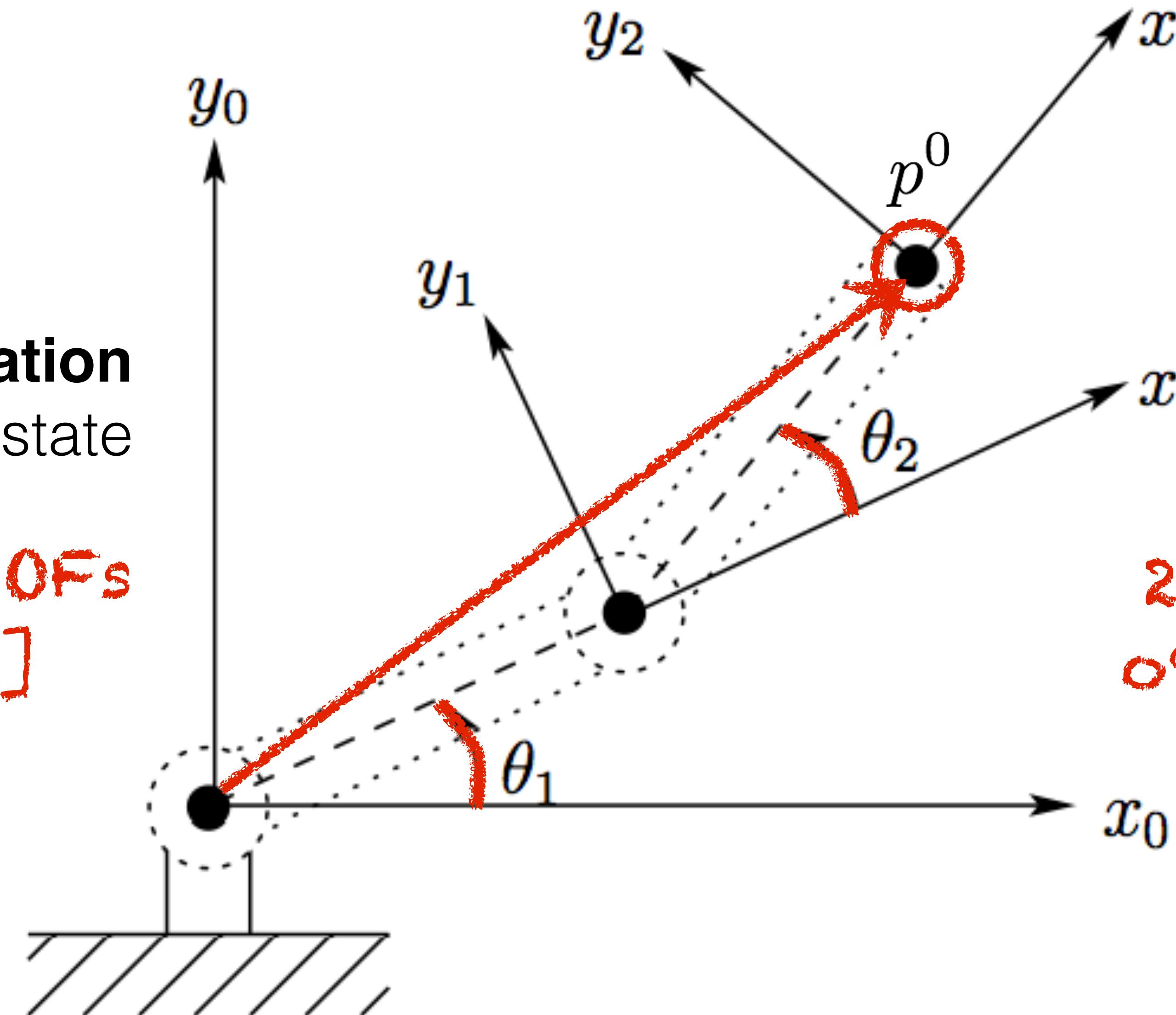
2 Cartesian DOFs
 $O^0_N = p^0 = (p_x^0, p_y^0)$

Forward kinematics: $[o^0_N, R^0_N] = f(q)$

$$p^o = f(\theta_1, \theta_2)$$

Robot **configuration**
defined by DoF state

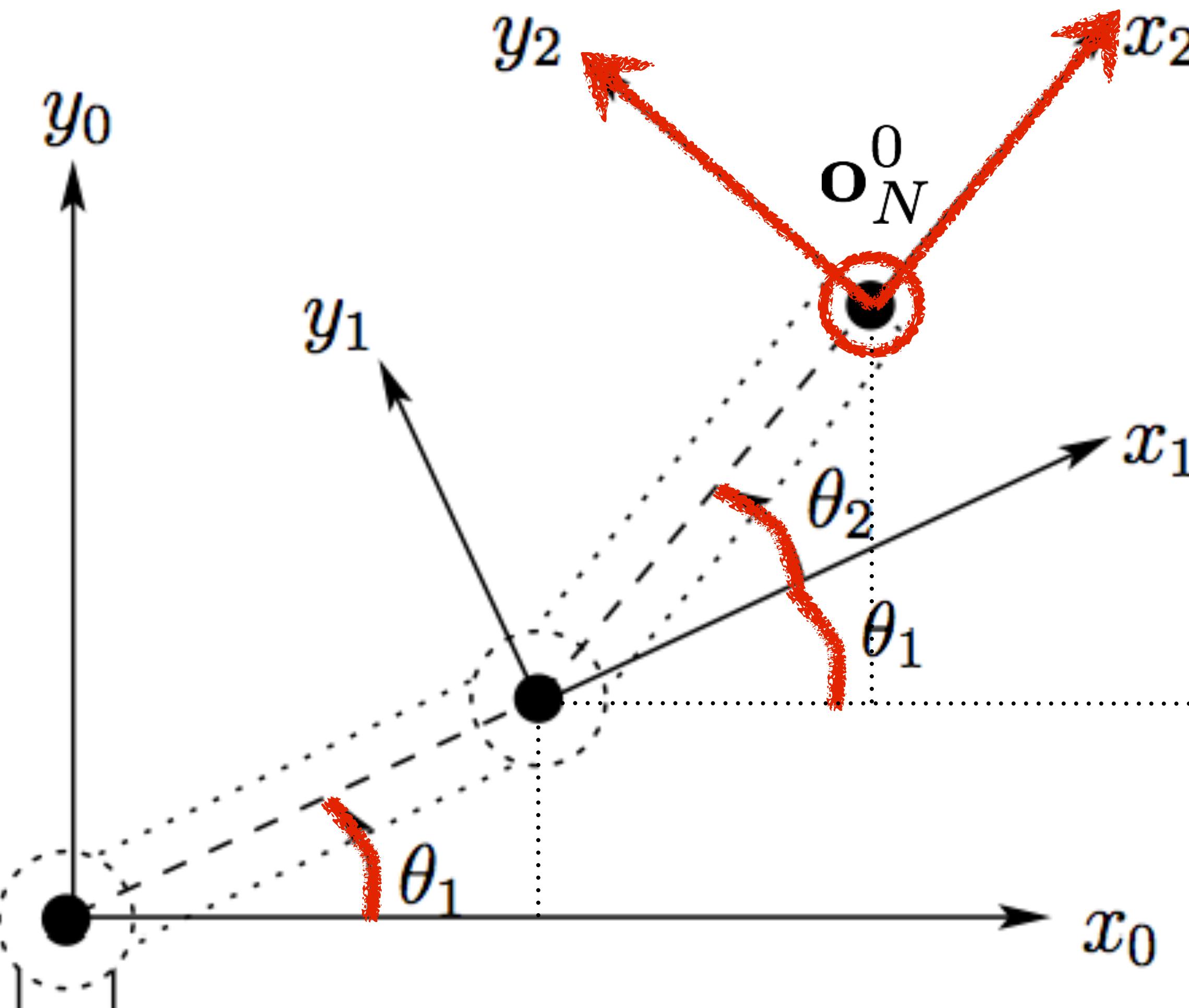
2 angular DOFs
 $q = [\theta_1, \theta_2]$



Robot **endeffector**
is the gripper pose
in world frame

2 Cartesian DOFs
 $o^0_N = p^o = (p_x^o, p_y^o)$

Forward kinematics: $[o^0_N, R^0_N] = f(q)$



What is the position and orientation of the tool wrt. the world?

remember:

$$p^0 = T_1^0 T_2^1 p^2$$

$$R^0_N = \begin{bmatrix} \text{What are the elements of this matrix?} \end{bmatrix}$$

$$o^0_N = \begin{bmatrix} \text{What are the elements of this vector?} \end{bmatrix}$$

Forward kinematics: $[o^0_N, R^0_N] = f(q)$

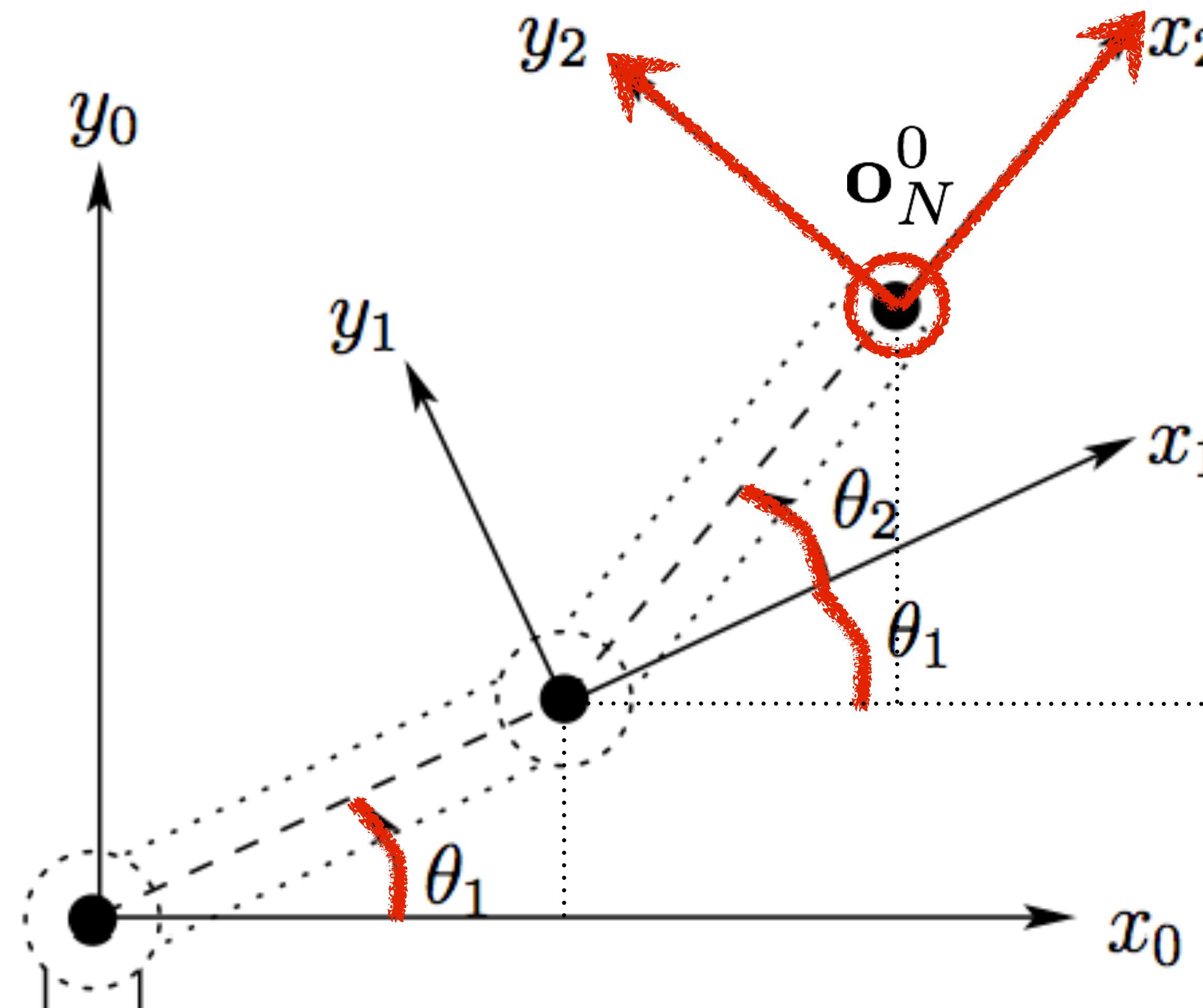
remember:

$$p^0 = T_1^0 T_2^1 p^2$$

$$p^0 = R_2^0 p^2 + d_2^0$$

where

$$R_2^0 = R_1^0 R_2^1$$



$$R_N^0 = \begin{bmatrix} \cos(\theta_1 + \theta_2) & -\sin(\theta_1 + \theta_2) \\ \sin(\theta_1 + \theta_2) & \cos(\theta_1 + \theta_2) \end{bmatrix}$$

$$o_N^0 = \begin{bmatrix} \text{What are the elements} \\ \text{of this vector?} \end{bmatrix}$$

What is the position and orientation of the tool wrt. the world?

Start with:

$$d_2^0 = R_1^0 d_2^1 + d_1^0$$

substitute in variables then perform operations:

$$\begin{bmatrix} \cos\theta_1 & -\sin\theta_1 \\ \sin\theta_1 & \cos\theta_1 \end{bmatrix} \begin{bmatrix} \alpha_2 \cos\theta_2 \\ \alpha_2 \sin\theta_2 \end{bmatrix} + \begin{bmatrix} \alpha_1 \cos\theta_1 \\ \alpha_1 \sin\theta_1 \end{bmatrix}$$

then substitute trig identities

$$\cos(x + y) = \cos(x)\cos(y) - \sin(x)\sin(y)$$

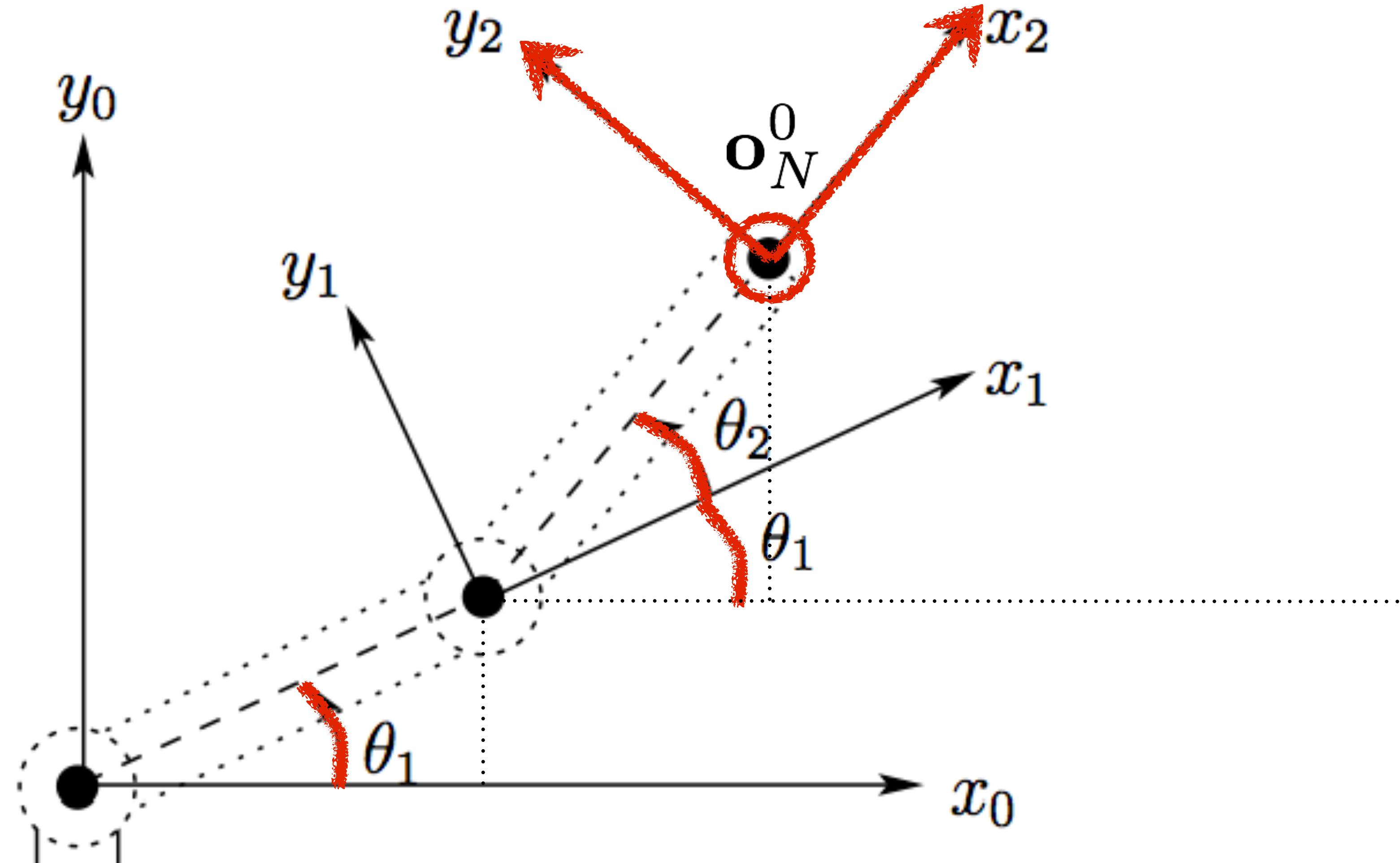
$$\sin(x + y) = \sin(x)\cos(y) + \cos(x)\sin(y)$$

to get:

$$\mathbf{o}_N^0 = \left[\begin{array}{c} \text{What are the elements} \\ \text{of this vector?} \end{array} \right]$$



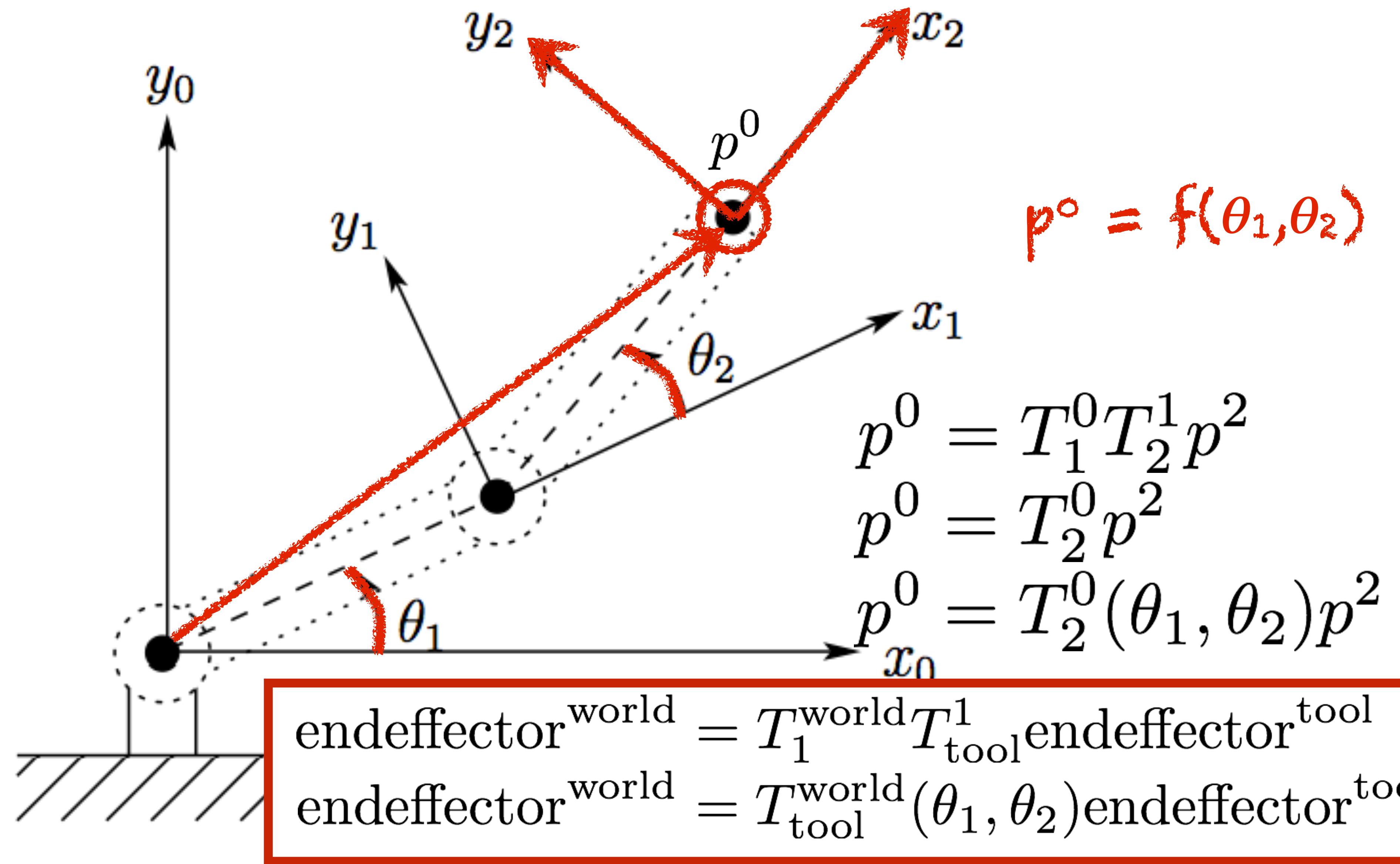
Forward kinematics: $[{\mathbf{o}}^0_N, {\mathbf{R}}^0_N] = f(\mathbf{q})$



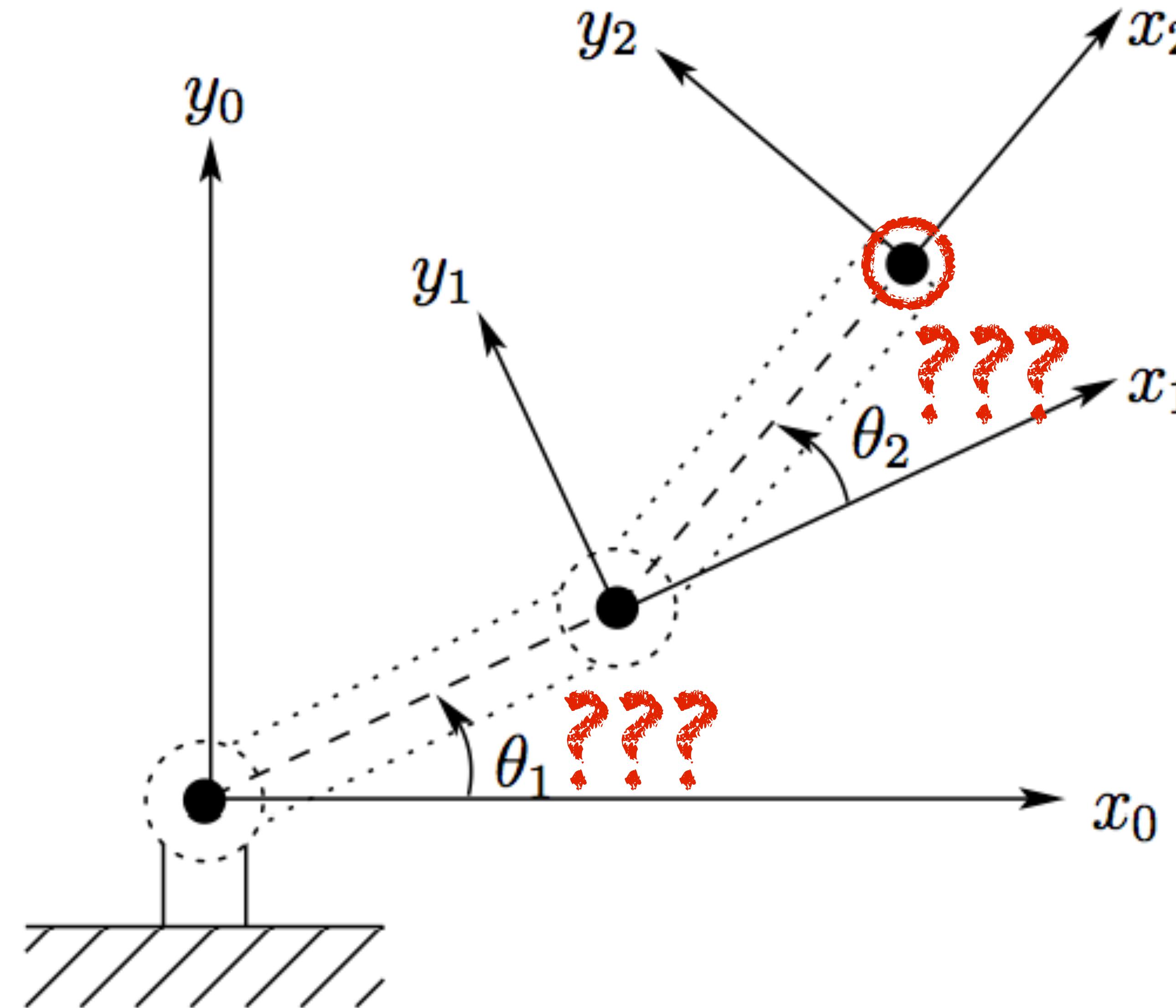
$${\mathbf{R}}^0_N = \begin{bmatrix} \cos(\theta_1 + \theta_2) & -\sin(\theta_1 + \theta_2) \\ \sin(\theta_1 + \theta_2) & \cos(\theta_1 + \theta_2) \end{bmatrix}$$

$${\mathbf{o}}^0_N = \begin{bmatrix} \alpha_1 \cos \theta_1 + \alpha_2 \cos(\theta_1 + \theta_2) \\ \alpha_1 \sin \theta_1 + \alpha_2 \sin(\theta_1 + \theta_2) \end{bmatrix}$$

Forward kinematics: $[o^0_N, R^0_N] = f(q)$

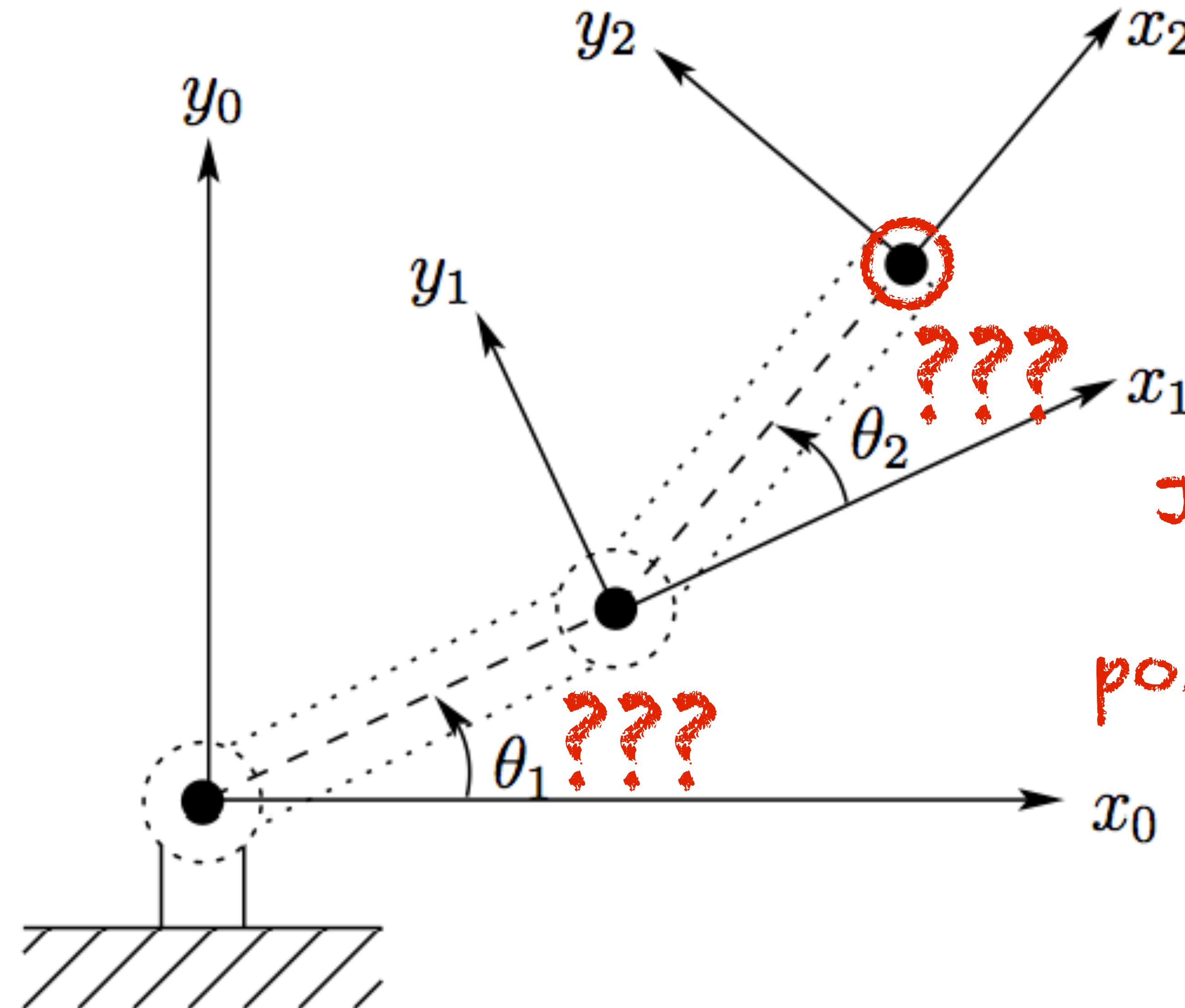


Inverse kinematics: “given endeffector, compute configuration”



Inverse kinematics: $\mathbf{q} = f^{-1}([\mathbf{o}_N^0, \mathbf{R}_N^0])$

$$[\theta_1, \theta_2] = f^{-1}(p^o)$$



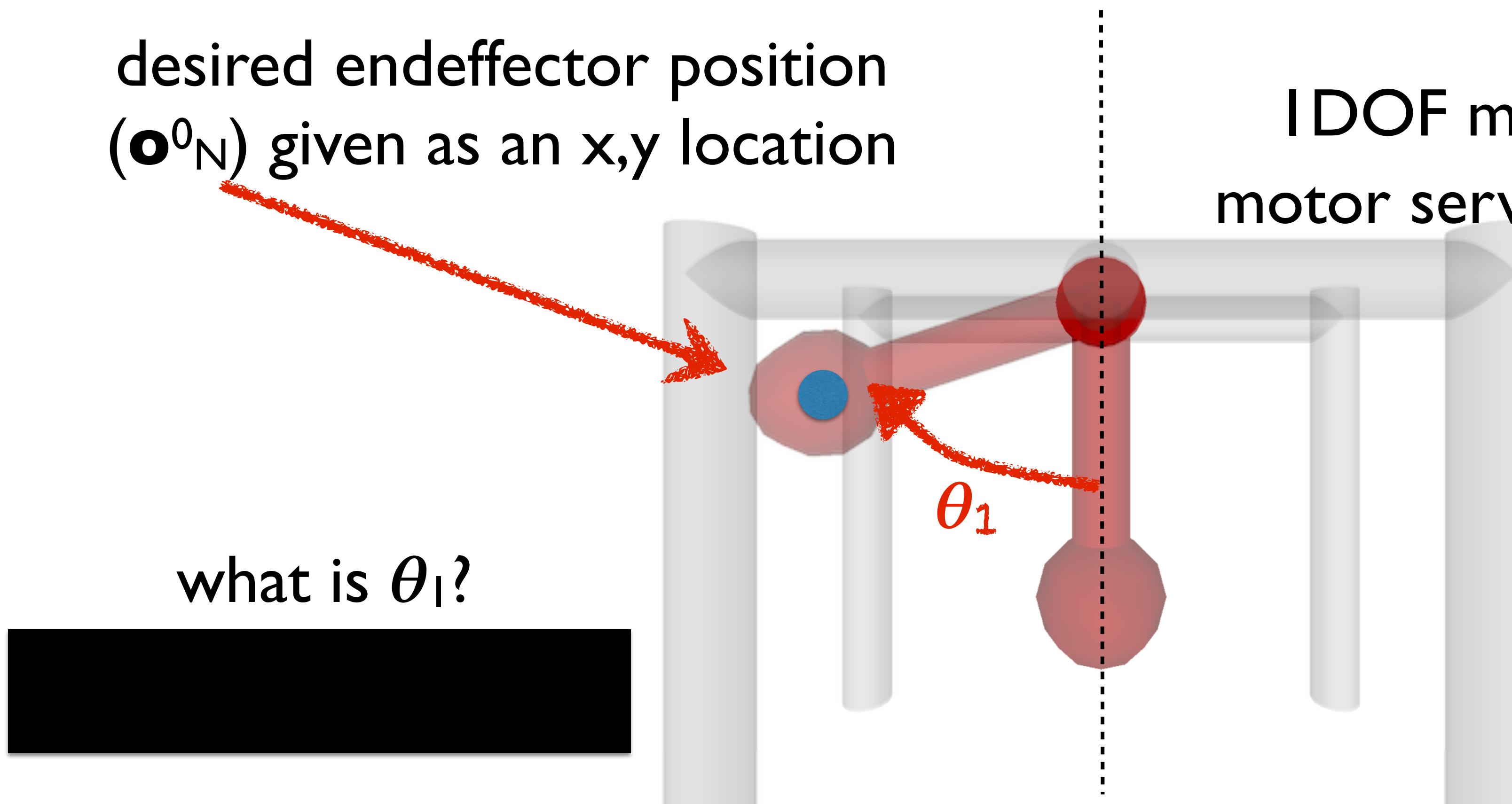
Just consider
endeffector
position for now

1 DOF pendulum example

desired endeffector position
(\bullet^0_N) given as an x,y location

what is θ_1 ?

assume:
1DOF motor at pendulum axis,
motor servo moves arm to angle θ_1



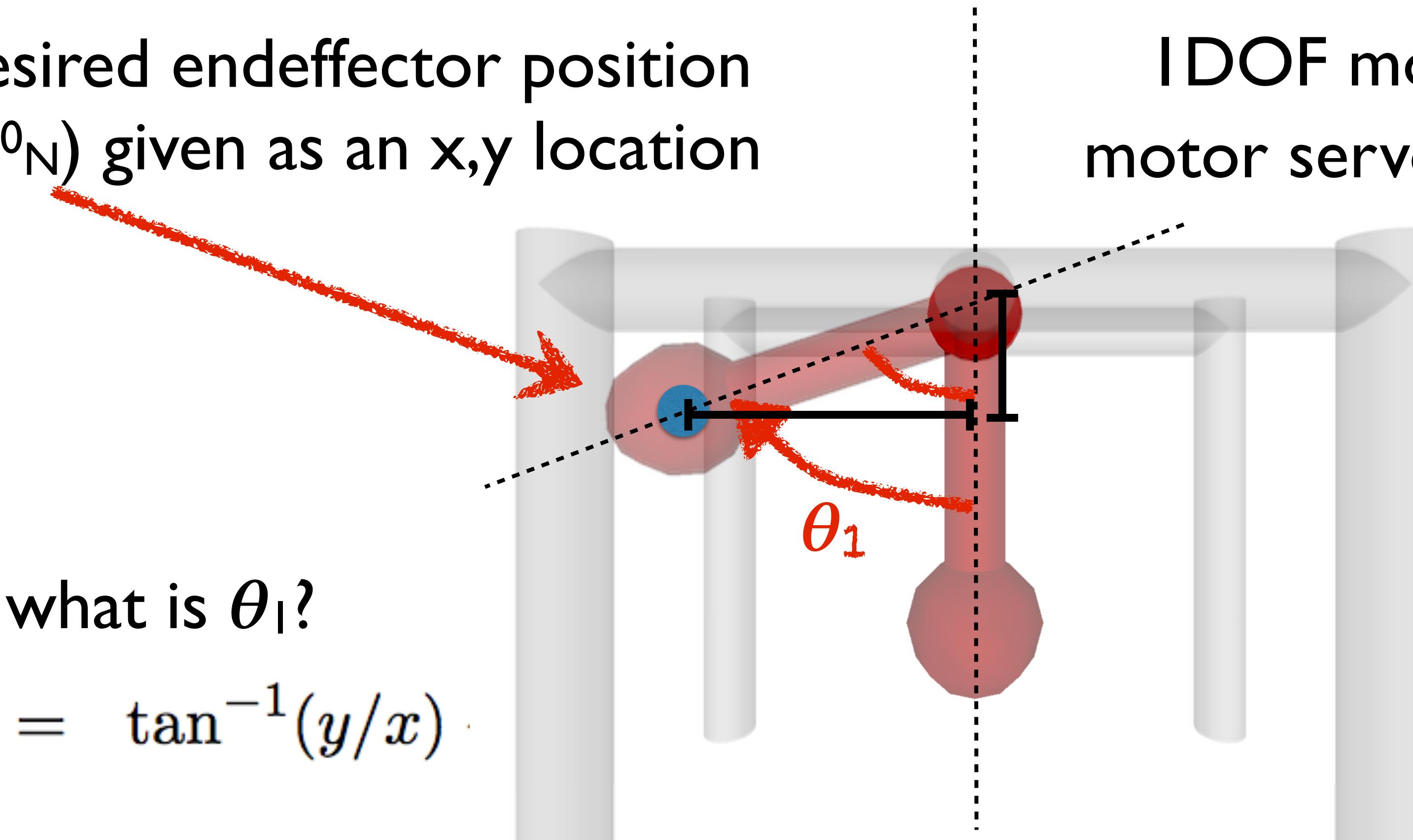
1 DOF pendulum example

desired endeffector position
(\bullet^0_N) given as an x,y location

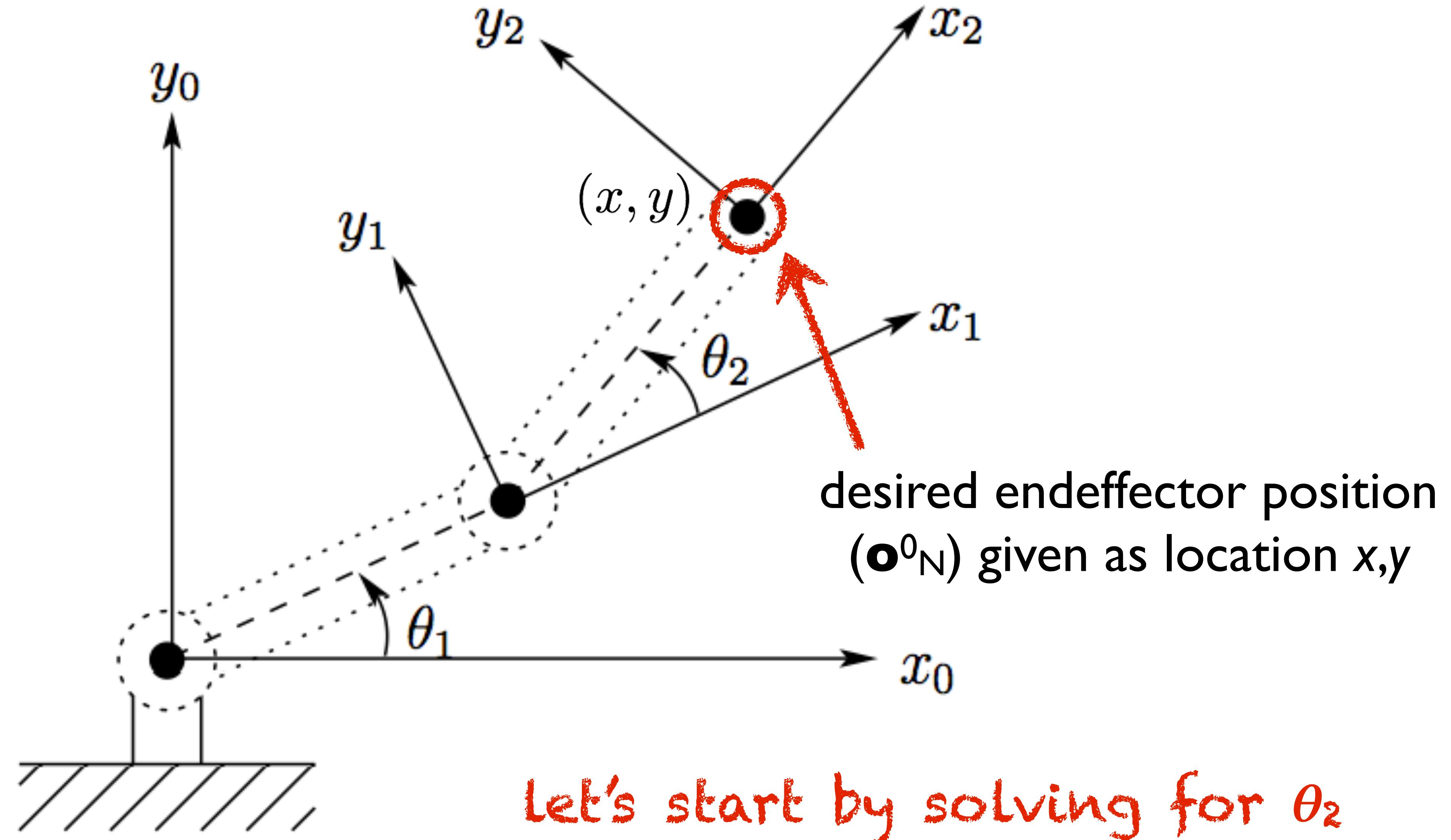
what is θ_1 ?

$$\theta_1 = \tan^{-1}(y/x)$$

assume:
1 DOF motor at pendulum axis,
motor servo moves arm to angle θ_1

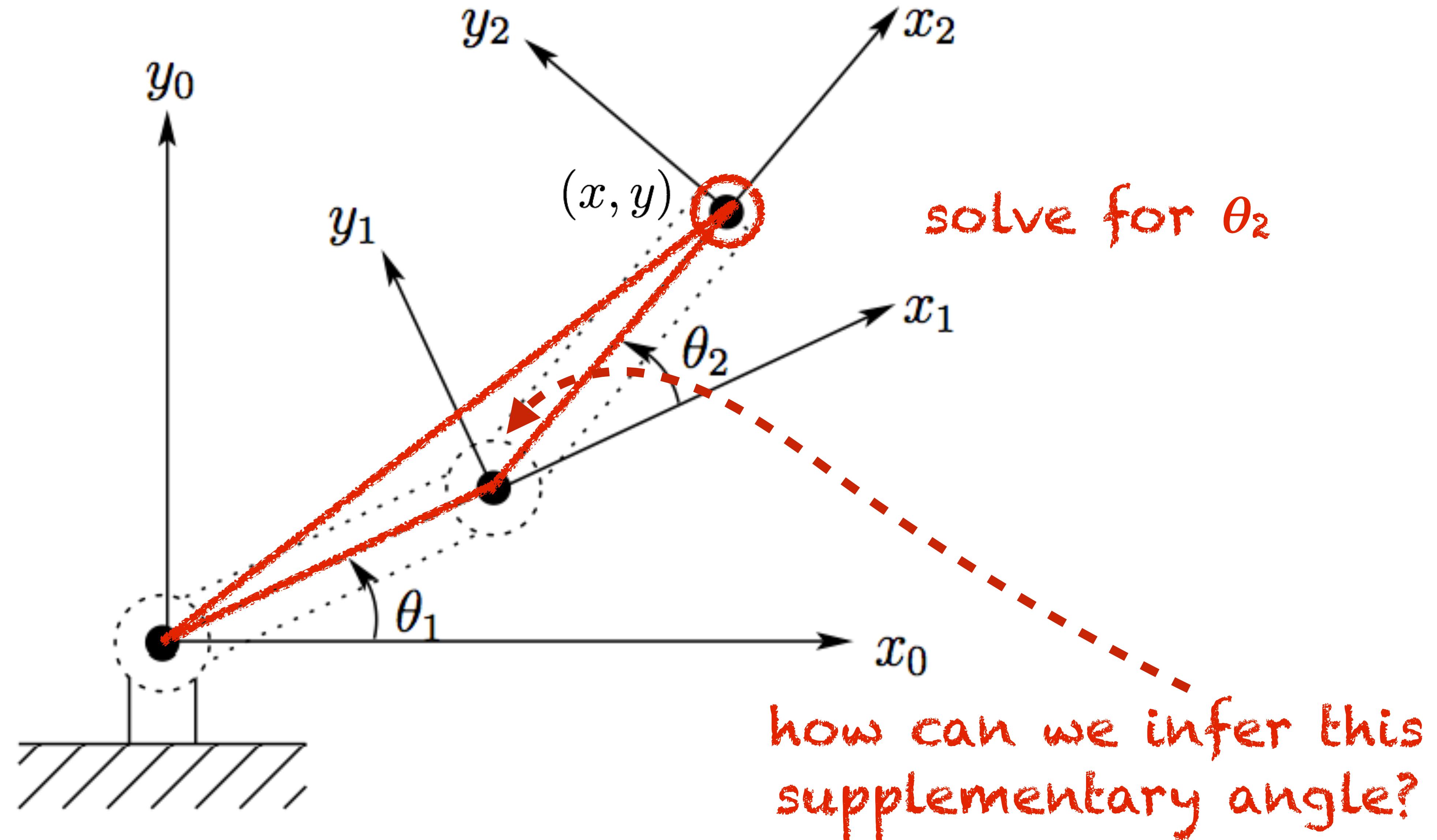


Inverse kinematics: $\mathbf{q} = f^{-1}([\mathbf{o}^0_N, \mathbf{R}^0_N])$ $[\theta_1, \theta_2] = f^{-1}(x, y)$



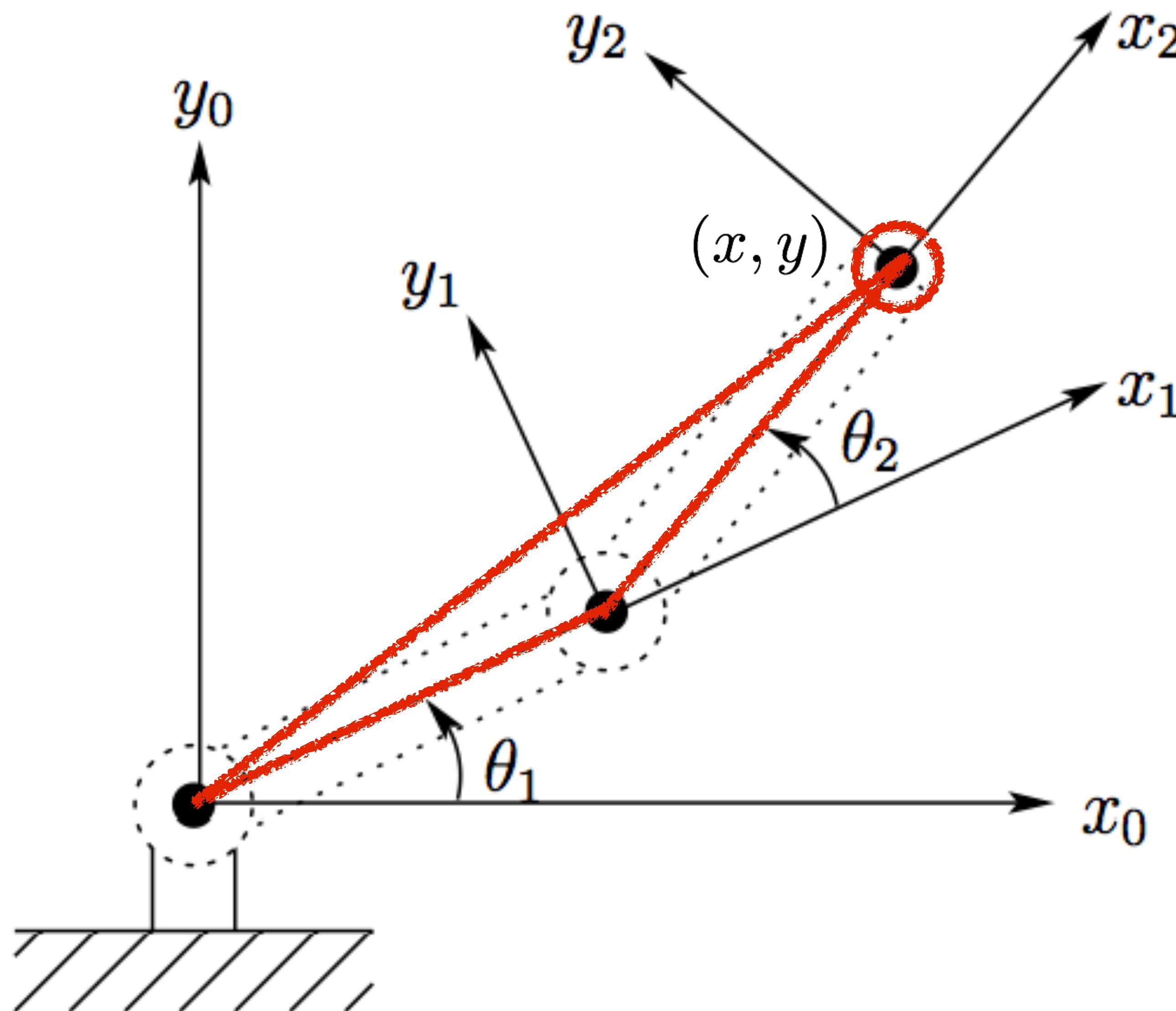
Inverse kinematics: $\mathbf{q} = f^{-1}([\mathbf{o}_N^0, \mathbf{R}_N^0])$

$$[\theta_1, \theta_2] = f^{-1}(x, y)$$



Inverse kinematics: $\mathbf{q} = \mathbf{f}^{-1}([\mathbf{o}_N^0, \mathbf{R}_N^0])$

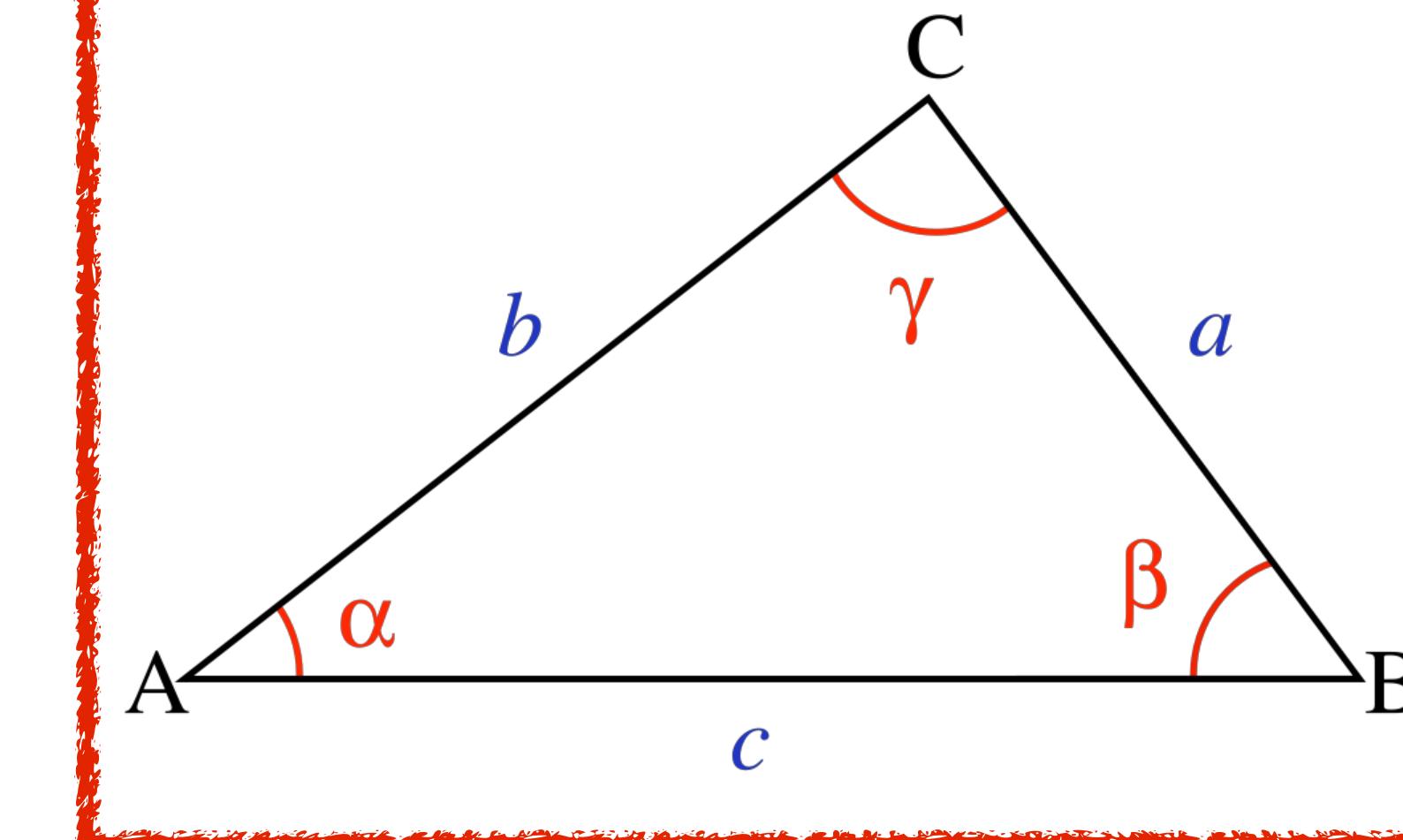
$$[\theta_1, \theta_2] = \mathbf{f}^{-1}(x, y)$$



solve for θ_2

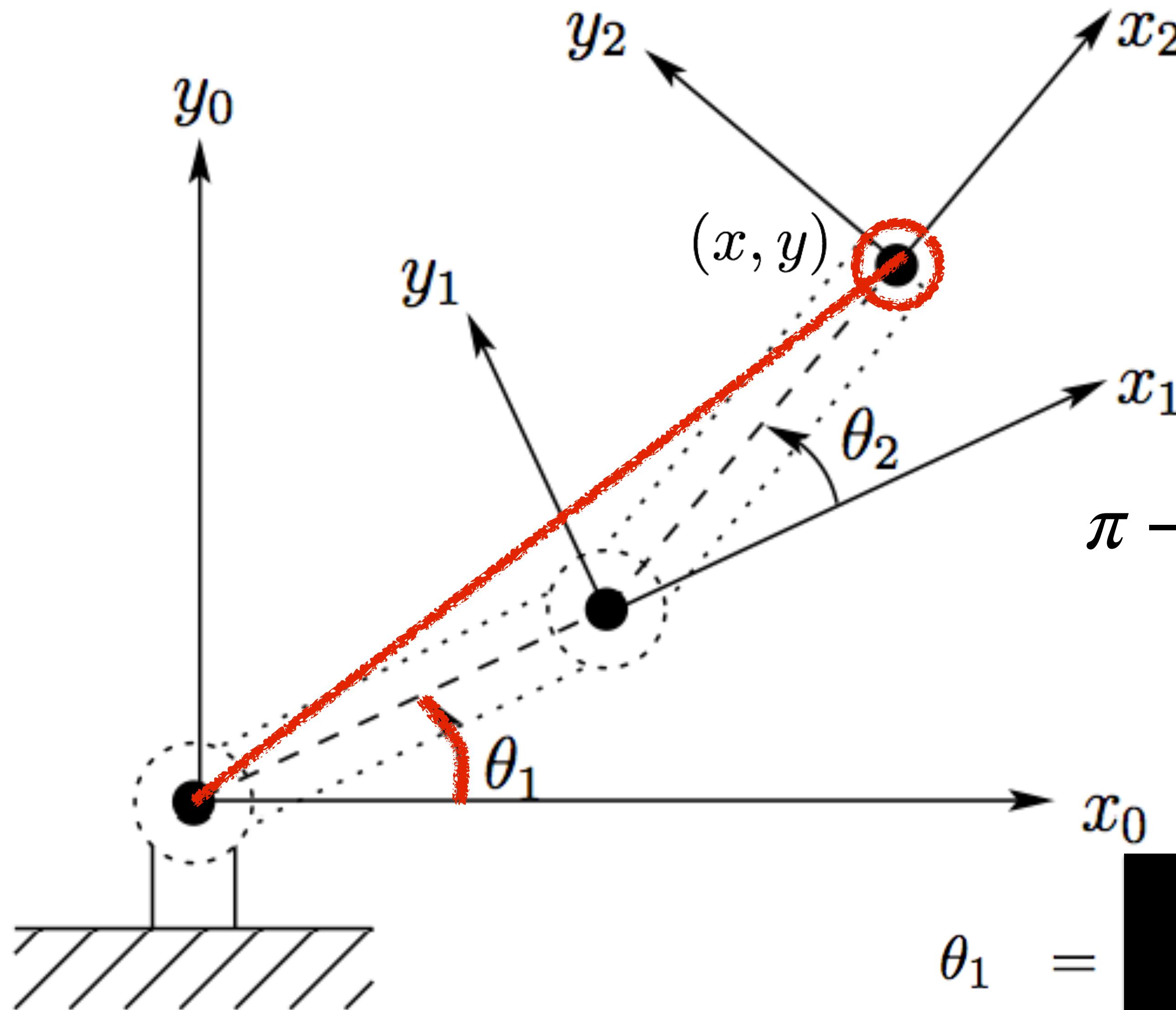
Law of Cosines

$$\gamma = \arccos \left(\frac{a^2 + b^2 - c^2}{2ab} \right)$$



Inverse kinematics: $\mathbf{q} = f^{-1}([\mathbf{o}_N^0, \mathbf{R}_N^0])$

$$[\theta_1, \theta_2] = f^{-1}(x, y)$$



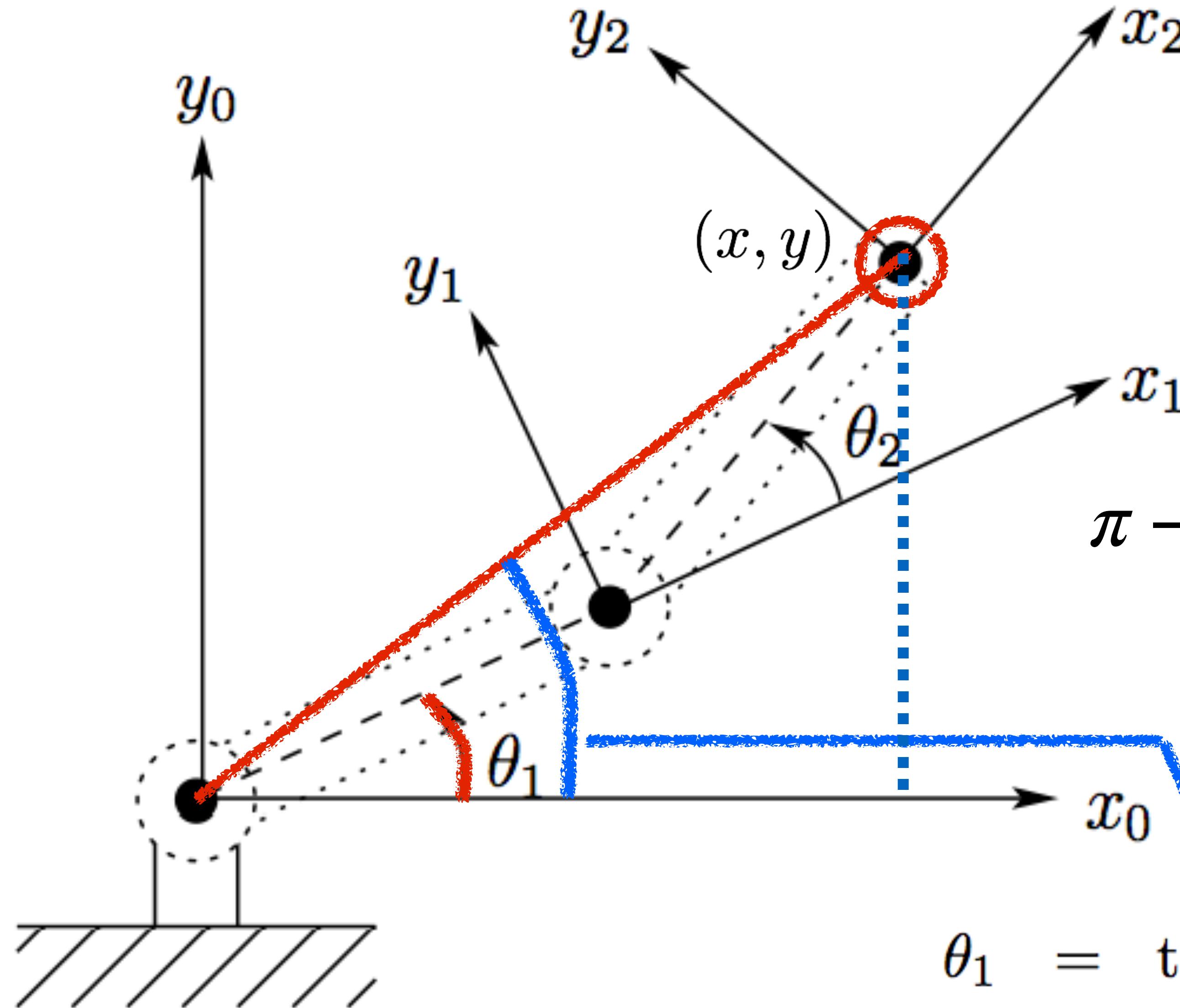
$$\pi - \theta_2 = \cos^{-1}\left(\frac{\alpha_1^2 + \alpha_2^2 - x^2 - y^2}{2\alpha_1\alpha_2}\right)$$

solve for θ_2

$$\theta_1 =$$

Consider two triangles

Inverse kinematics: $\mathbf{q} = f^{-1}([\mathbf{o}_N^0, \mathbf{R}_N^0])$ [θ_1, θ_2] = $f^{-1}(x, y)$



$$\pi - \theta_2 = \cos^{-1}\left(\frac{\alpha_1^2 + \alpha_2^2 - x^2 - y^2}{2\alpha_1\alpha_2}\right)$$

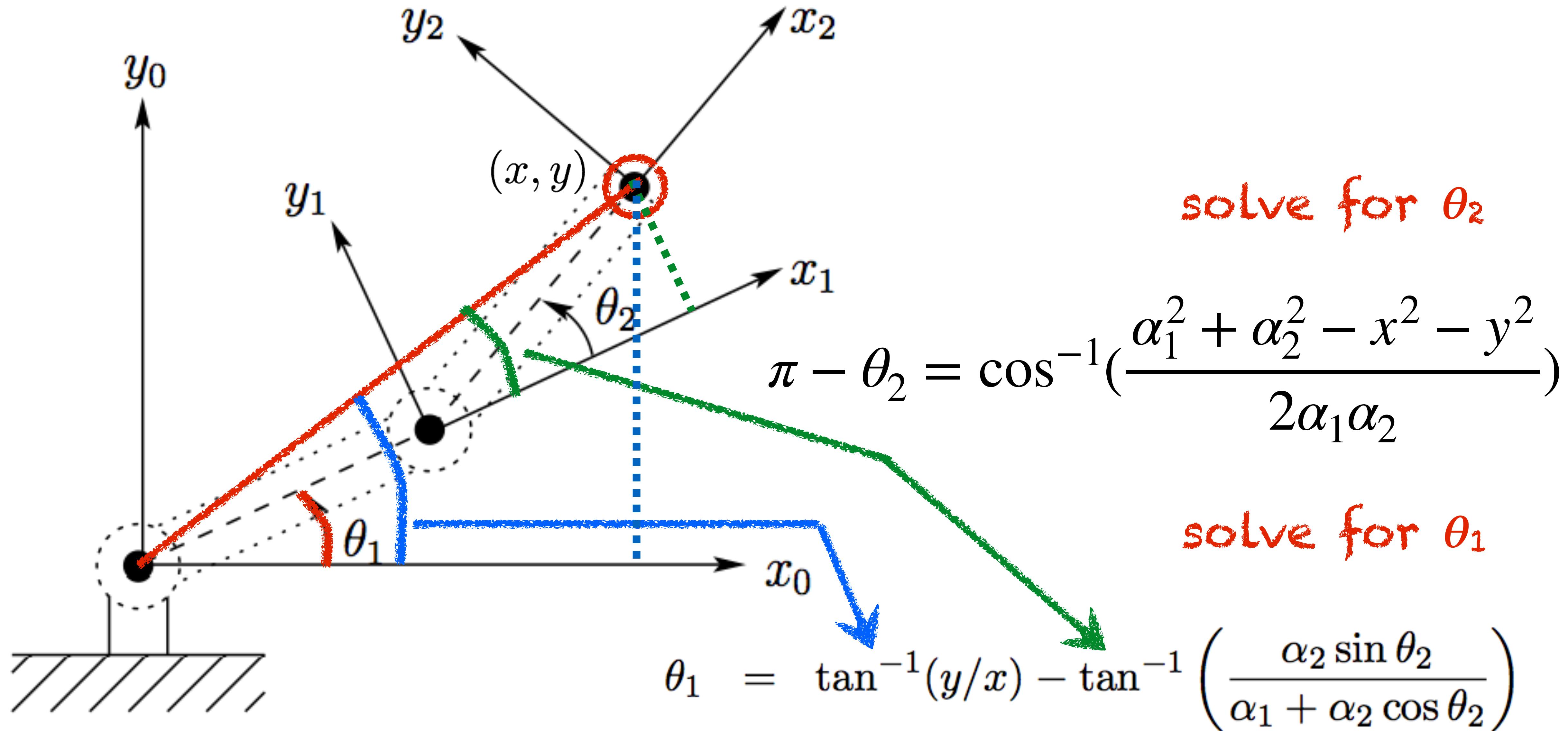
solve for θ_2

$$\theta_1 = \tan^{-1}(y/x) -$$

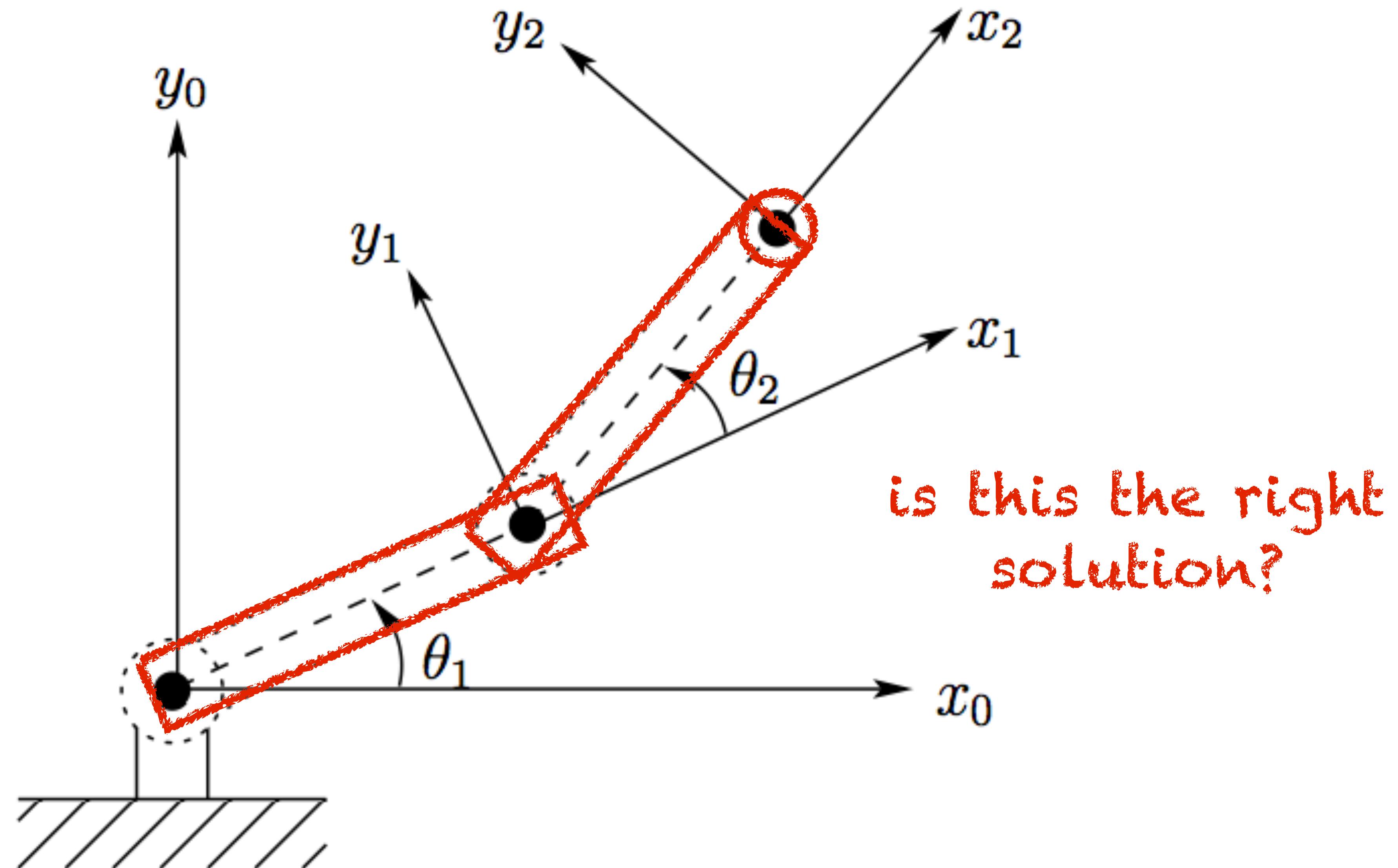
solve for θ_1

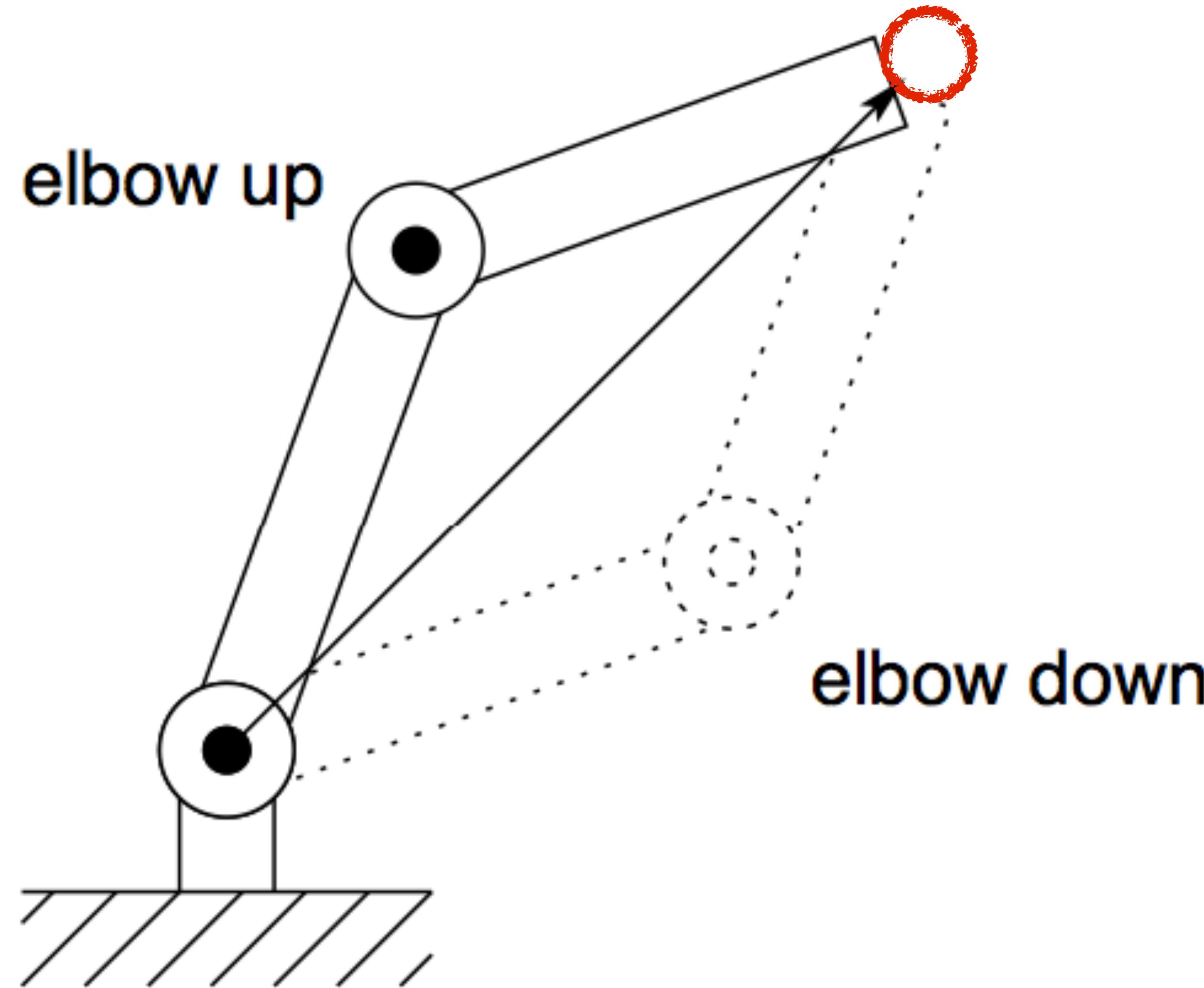
Consider two trian

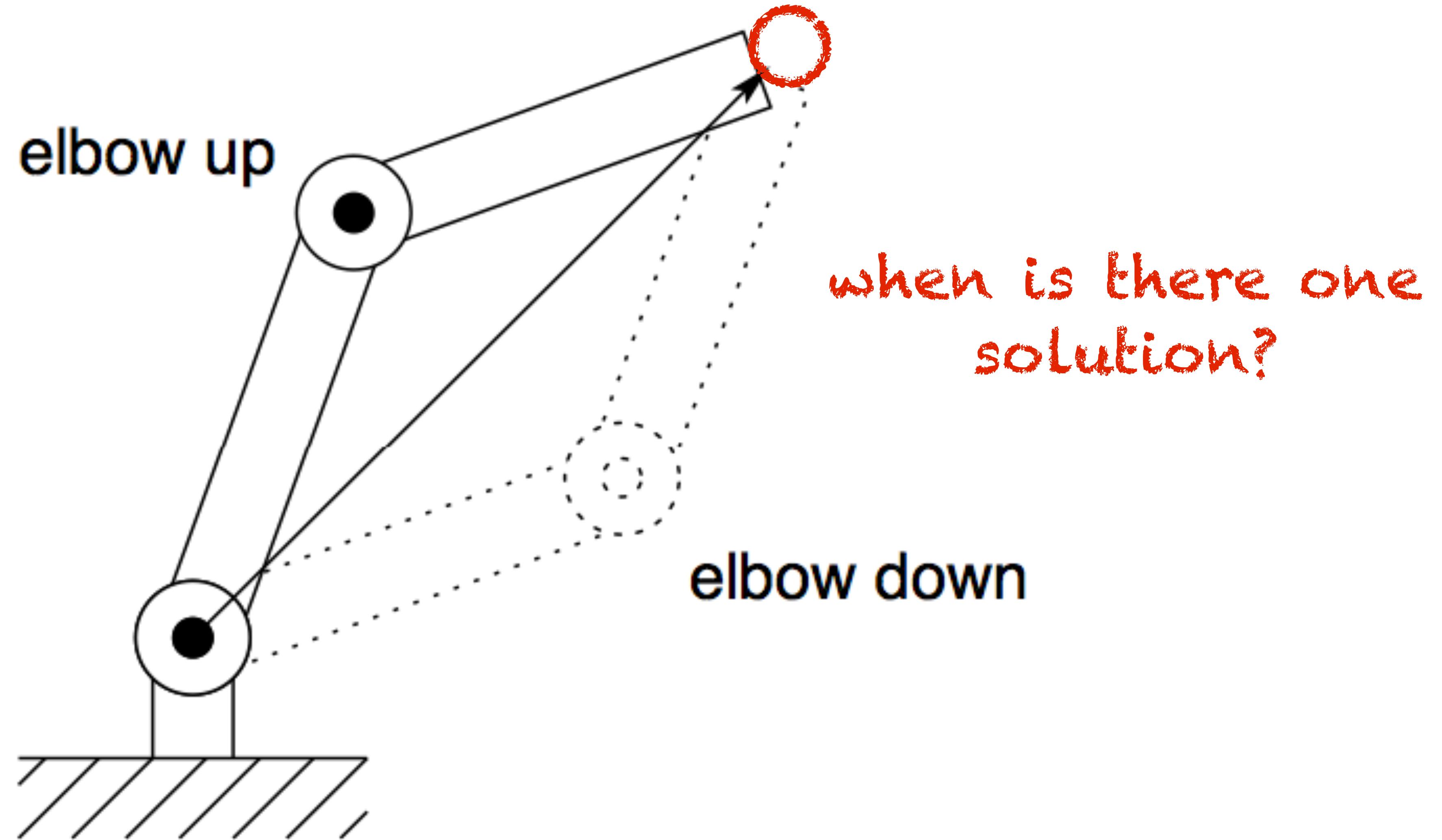
Inverse kinematics: $\mathbf{q} = f^{-1}([\mathbf{o}_N^0, \mathbf{R}_N^0])$ $[\theta_1, \theta_2] = f^{-1}(x, y)$

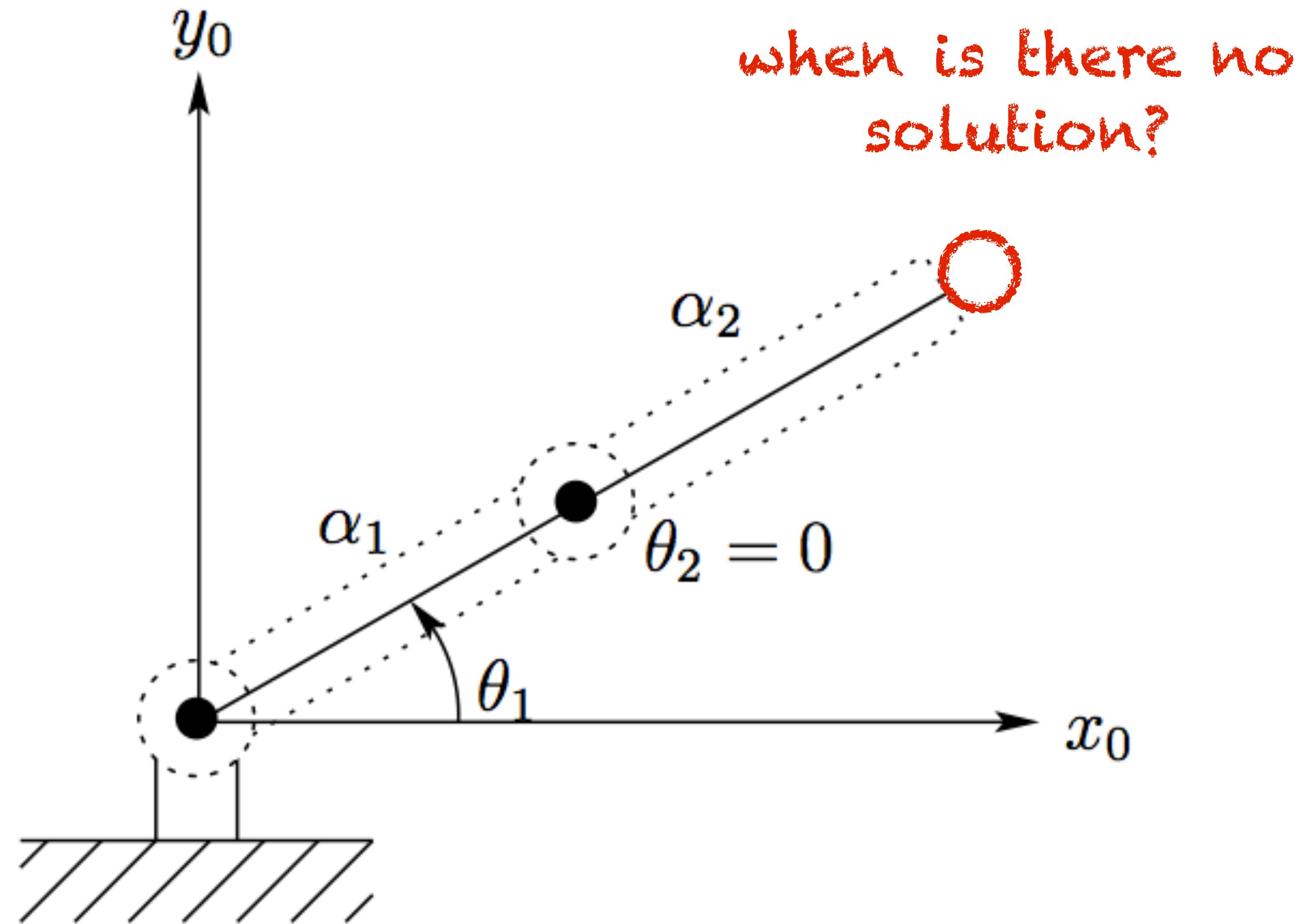


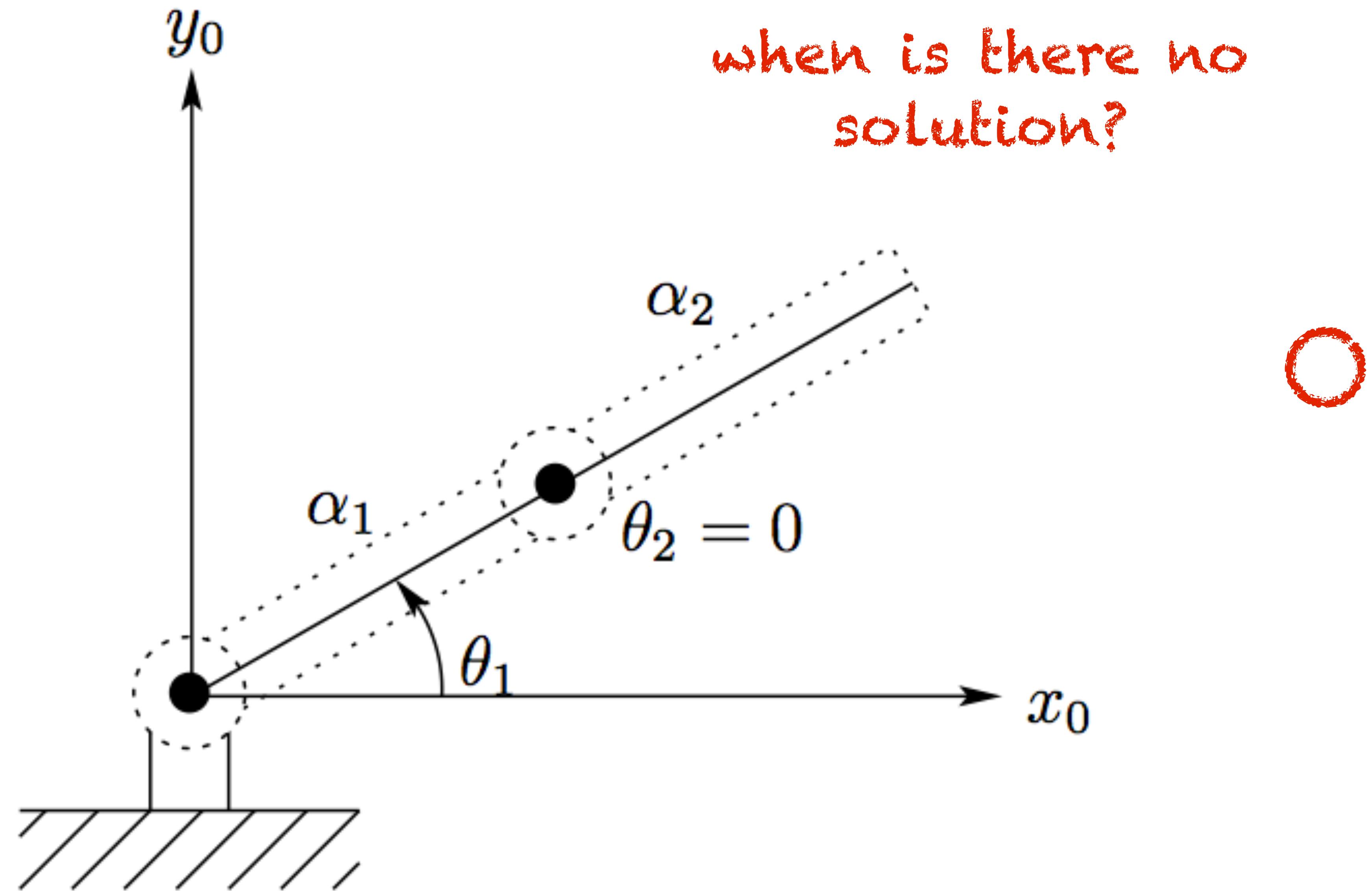
inverse kinematics: $(\theta_1, \theta_2) = f^{-1}(x, y)$







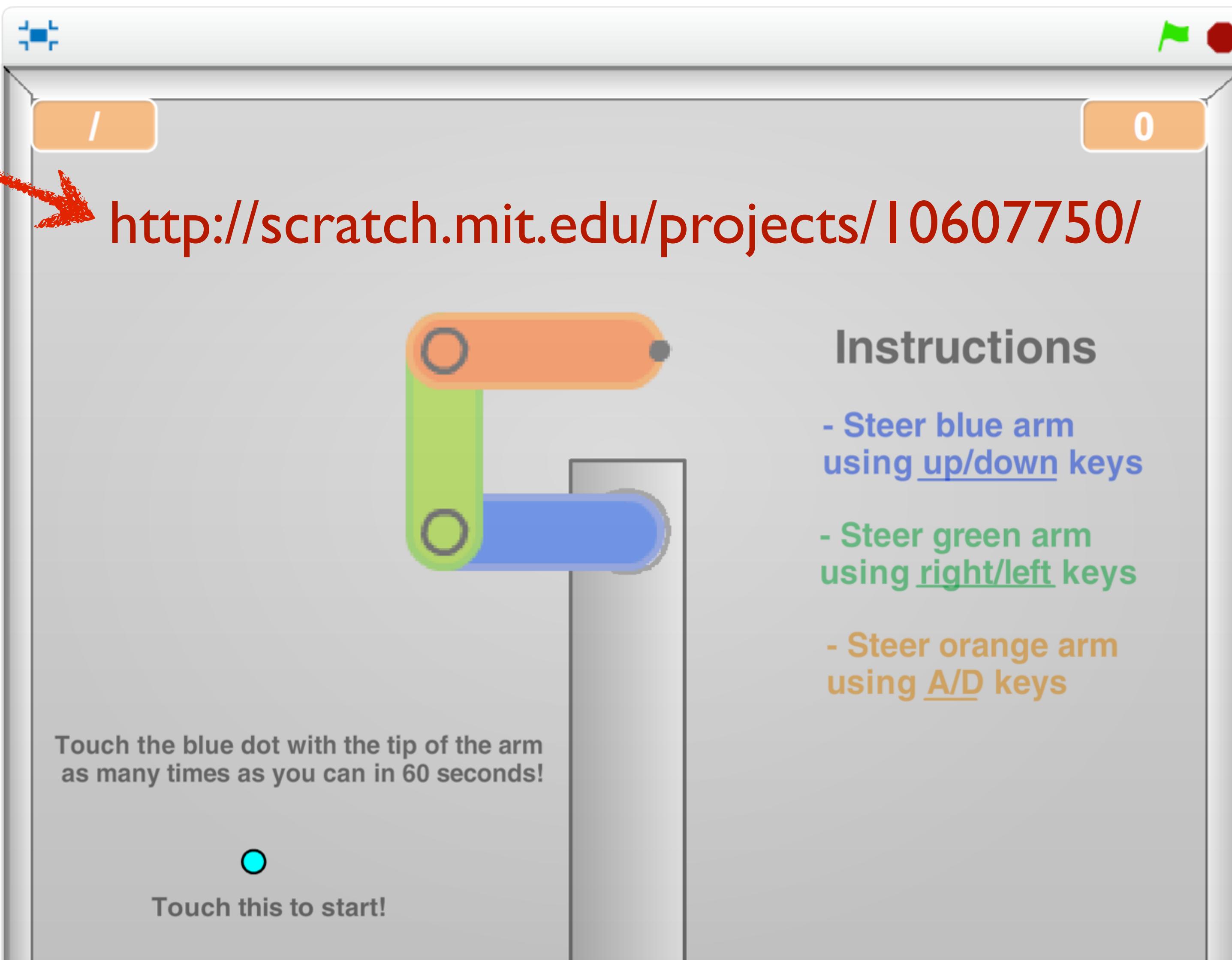




Can we do IK for 3 links?

SHALL HE PLAY A GAME?

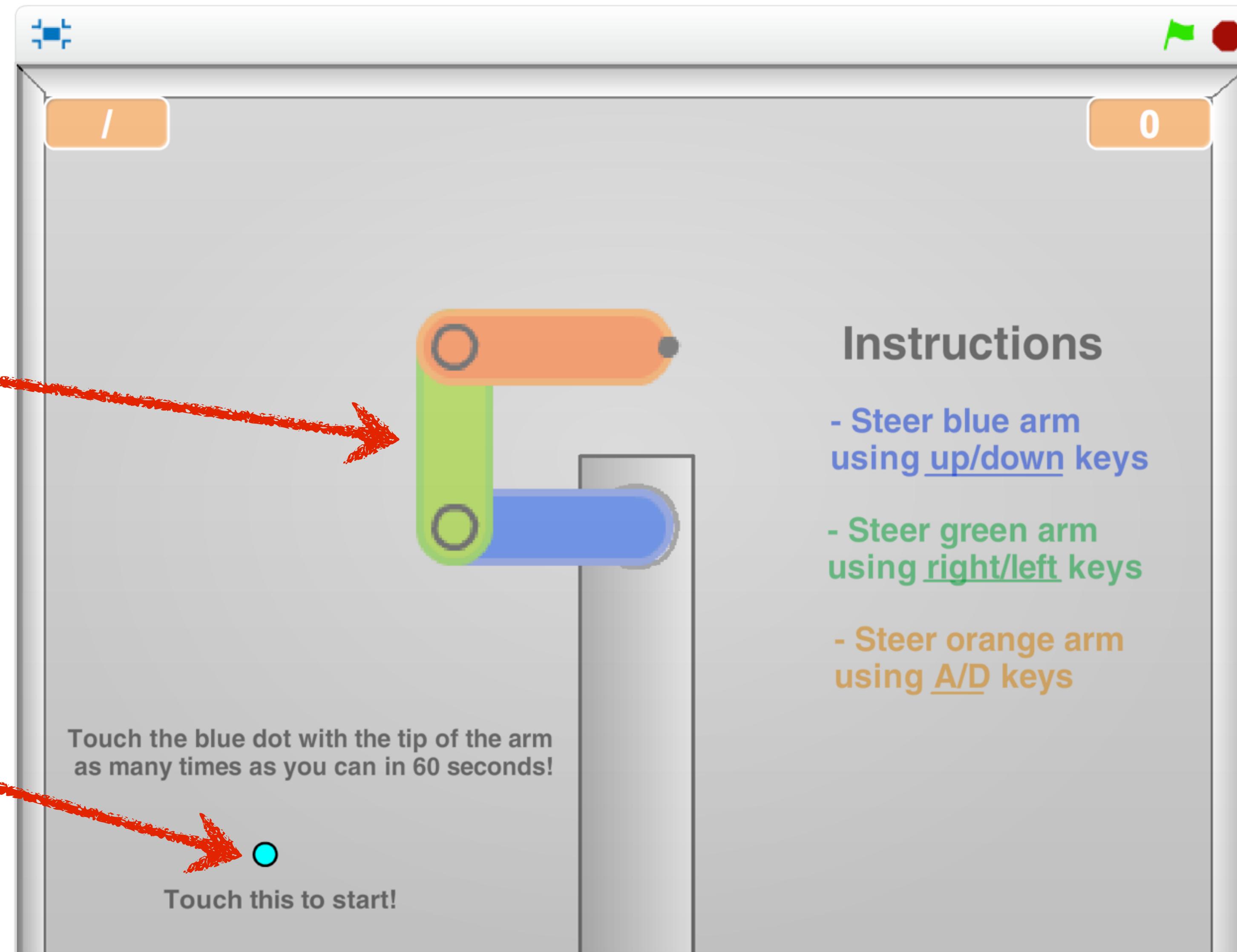
Try this



How many solutions for this arm?

3
unknowns

2
constraints



Instructions

- Steer blue arm using up/down keys
- Steer green arm using right/left keys
- Steer orange arm using A/D keys

Remember:
 $Ax=b$

Inverse Kinematics: 2D

$$T_n^0(q_1, \dots, q_n) = H$$

Inverse Kinematics: 2D

Configuration 

$$T_n^0(q_1, \dots, q_n) = H \quad \text{Transform from endeffector frame to world frame}$$

$$q = \begin{bmatrix} q_1 \\ q_2 \\ \dots \\ q_n \end{bmatrix}$$

$$H = \begin{bmatrix} R & o \\ 0 & 1 \end{bmatrix}$$
$$H = \begin{bmatrix} r_{11} & r_{12} & o_x \\ r_{21} & r_{22} & o_y \\ 0 & 0 & 1 \end{bmatrix}$$

Inverse Kinematics: 2D

Configuration $\rightarrow T_n^0(q_1, \dots, q_n) = H$ Transform from endeffector frame to world frame

$$q = \begin{bmatrix} q_1 \\ q_2 \\ \dots \\ q_n \end{bmatrix}$$

Inverse orientation $R_n^0(q_1, \dots, q_n) = R$

Inverse position $o_n^0(q_1, \dots, q_n) = o$

$$H = \begin{bmatrix} R & o \\ 0 & 1 \end{bmatrix}$$
$$H = \begin{bmatrix} r_{11} & r_{12} & o_x \\ r_{21} & r_{22} & o_y \\ 0 & 0 & 1 \end{bmatrix}$$

Inverse Kinematics: 2D

Configuration

$$T_n^0(q_1, \dots, q_n) = H \quad \text{Transform from endeffector}$$

$$q = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}$$

$$H = \begin{bmatrix} R & o \\ 0 & 1 \end{bmatrix}$$



$$H = \begin{bmatrix} r_{11} & r_{12} & o_x \\ r_{21} & r_{22} & o_y \\ 0 & 0 & 1 \end{bmatrix}$$

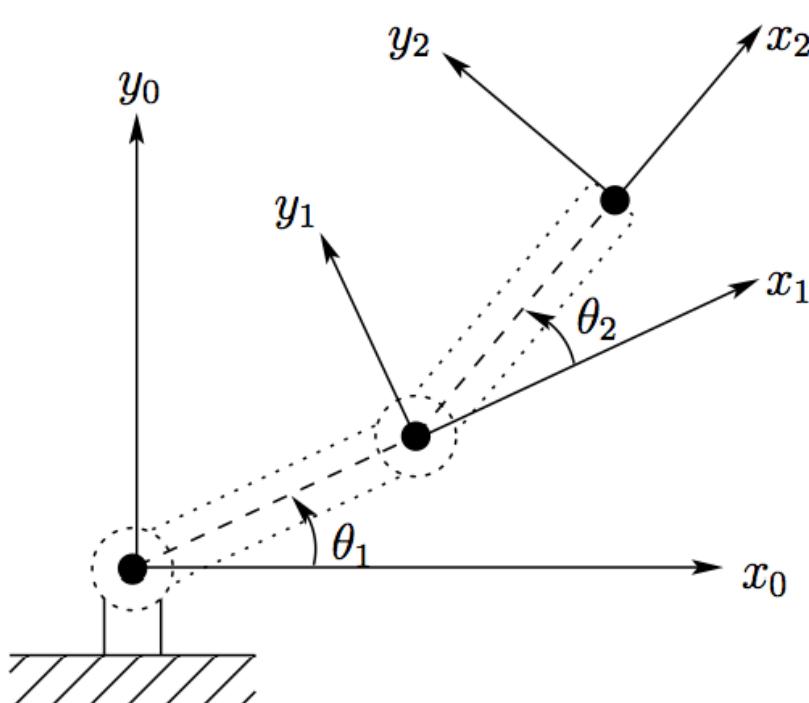
Inverse Kinematics: 2D

Configuration

$$T_n^0(q_1, \dots, q_n) = H \quad \text{Transform from endeffector}$$

$$q = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}$$

$$H = \begin{bmatrix} R & o \\ 0 & 1 \end{bmatrix}$$



Closed form solution

$$\theta_2 = \cos^{-1}\left(\frac{x^2 + y^2 - \alpha_1^2 - \alpha_2^2}{2\alpha_1\alpha_2}\right)$$

$$\theta_1 = \tan^{-1}(y/x) - \tan^{-1}\left(\frac{\alpha_2 \sin \theta_2}{\alpha_1 + \alpha_2 \cos \theta_2}\right)$$

$$H = \begin{bmatrix} r_{11} & r_{12} & o_x \\ r_{21} & r_{22} & o_y \\ 0 & 0 & 1 \end{bmatrix}$$

Inverse Kinematics: 3D

Configuration 

$$T_n^0(q_1, \dots, q_n) = H \quad \text{Transform from endeffector}$$

$$q = \begin{bmatrix} q_1 \\ q_2 \\ \dots \\ q_n \end{bmatrix}$$

6 DOF position and orientation of endeffector

$$H = \begin{bmatrix} R & o \\ 0 & 1 \end{bmatrix}$$

$$H = \begin{bmatrix} r_{11} & r_{12} & r_{13} & o_x \\ r_{21} & r_{22} & r_{23} & o_y \\ r_{31} & r_{32} & r_{33} & o_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Inverse Kinematics: 3D

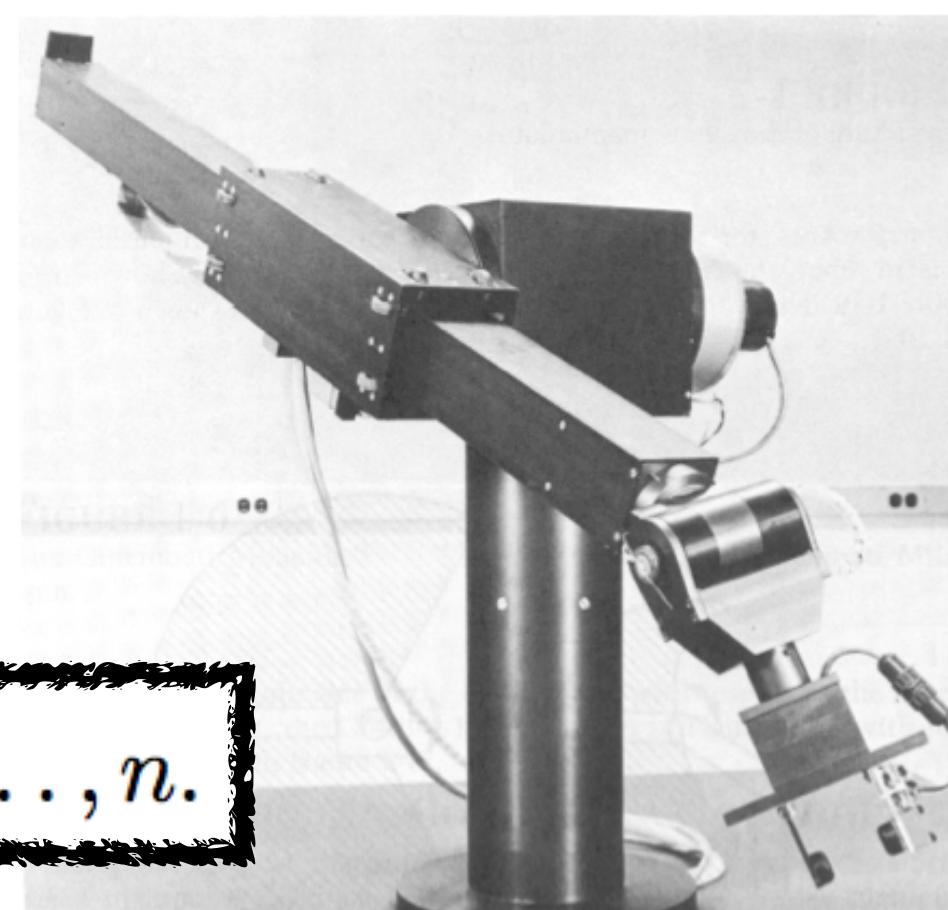
Configuration \rightarrow

$$T_n^0(q_1, \dots, q_n) = H \leftarrow \text{Transform from endeffector}$$

$$q = \begin{bmatrix} q_1 \\ q_2 \\ \dots \\ q_6 \end{bmatrix}$$

Closed form solution?

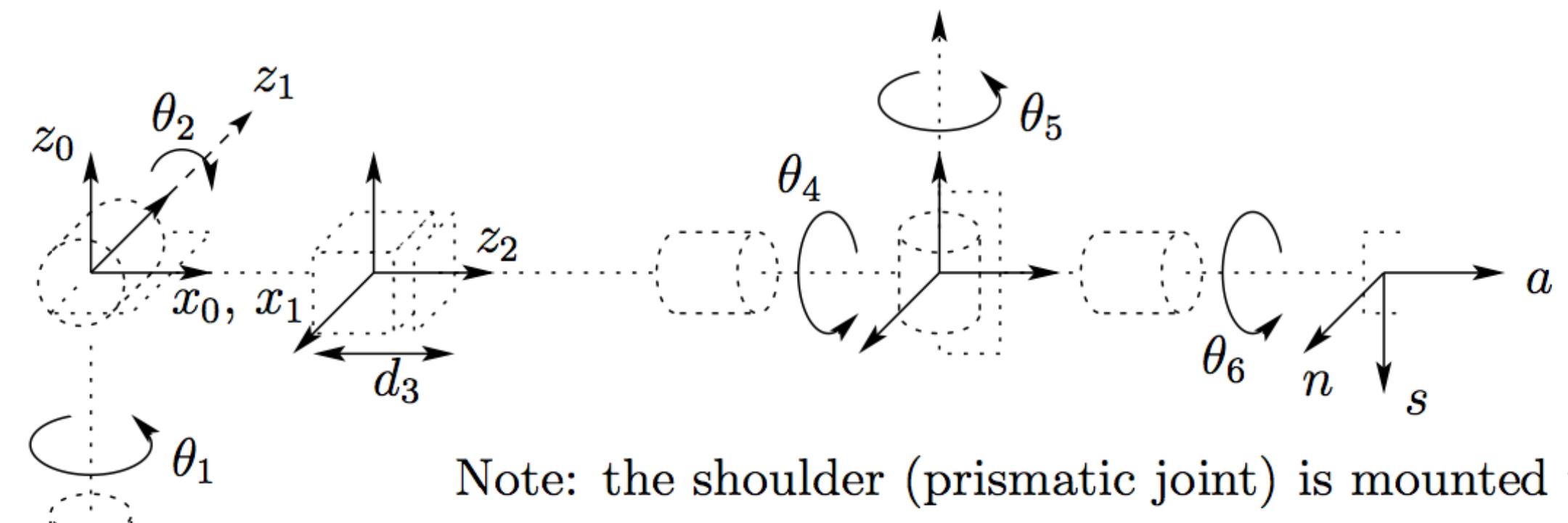
$$q_k = f_k(h_{11}, \dots, h_{34}), \quad k = 1, \dots, n.$$



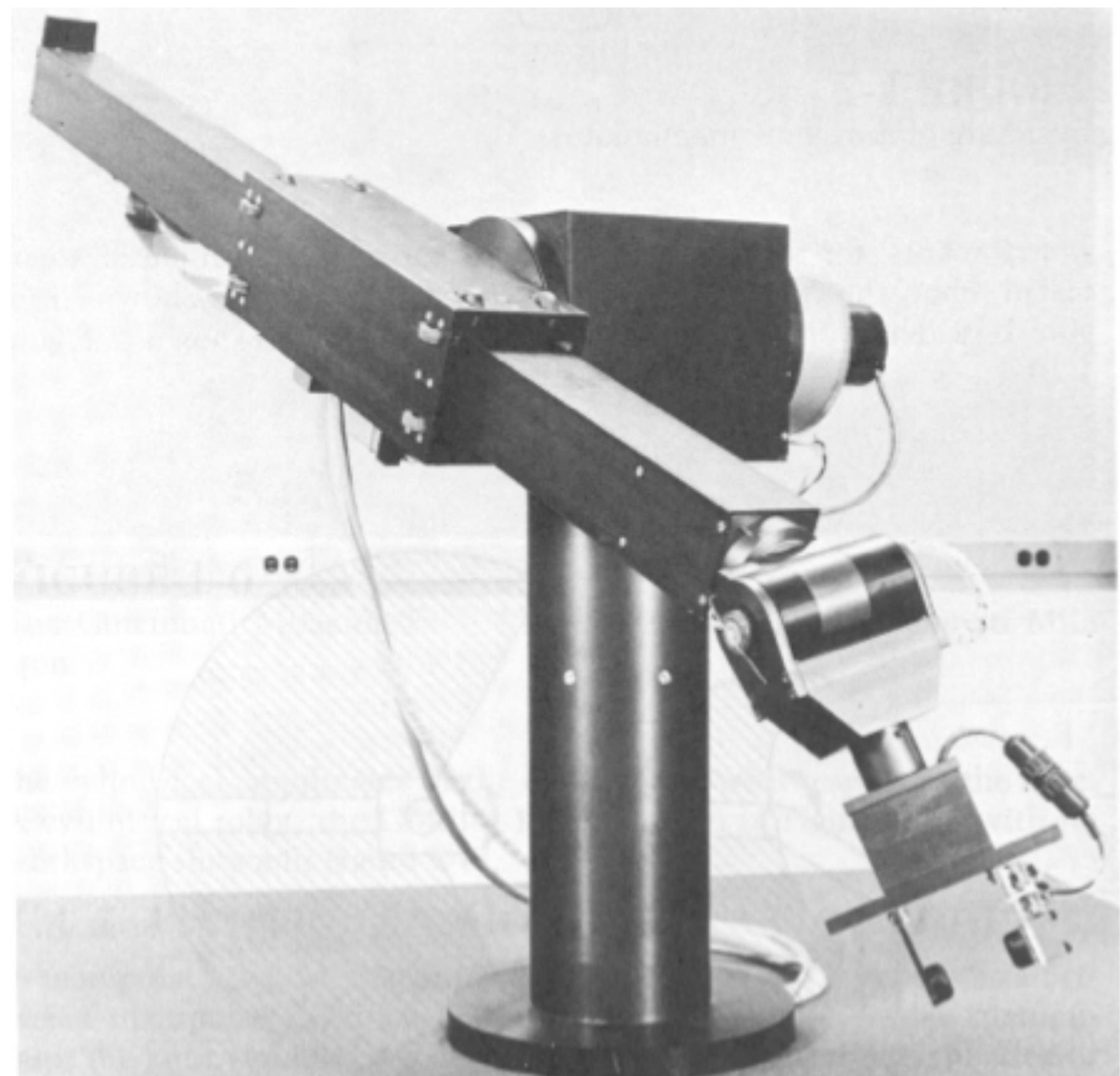
$$H = \begin{bmatrix} R & o \\ 0 & 1 \end{bmatrix}$$

6 DOF position and orientation of endeffector

Stanford Manipulator



Note: the shoulder (prismatic joint) is mounted wrong.



$$\begin{aligned}
 c_1[c_2(c_4c_5c_6 - s_4s_6) - s_2s_5c_6] - s_1(s_4c_5c_6 + c_4s_6) &= r_{11} \\
 s_1[c_2(c_4c_5c_6 - s_4s_6) - s_2s_5c_6] + c_1(s_4c_5c_6 + c_4s_6) &= r_{21} \\
 -s_2(c_4c_5c_6 - s_4s_6) - c_2s_5s_6 &= r_{31} \\
 c_1[-c_2(c_4c_5s_6 + s_4c_6) + s_2s_5s_6] - s_1(-s_4c_5s_6 + c_4c_6) &= r_{12} \\
 s_1[-c_2(c_4c_5s_6 + s_4c_6) + s_2s_5s_6] + c_1(-s_4c_5s_6 + c_4c_6) &= r_{22} \\
 s_2(c_4c_5s_6 + s_4c_6) + c_2s_5s_6 &= r_{32} \\
 c_1(c_2c_4s_5 + s_2c_5) - s_1s_4s_5 &= r_{13} \\
 s_1(c_2c_4s_5 + s_2c_5) + c_1s_4s_5 &= r_{23} \\
 -s_2c_4s_5 + c_2c_5 &= r_{33} \\
 c_1s_2d_3 - s_1d_2 + d_6(c_1c_2c_4s_5 + c_1c_5s_2 - s_1s_4s_5) &= o_x \\
 s_1s_2d_3 + c_1d_2 + d_6(c_1s_4s_5 + c_2c_4s_1s_5 + c_5s_1s_2) &= o_y \\
 c_2d_3 + d_6(c_2c_5 - c_4s_2s_5) &= o_z.
 \end{aligned}$$

assumes D-H frames





Detect, pick, and place each character

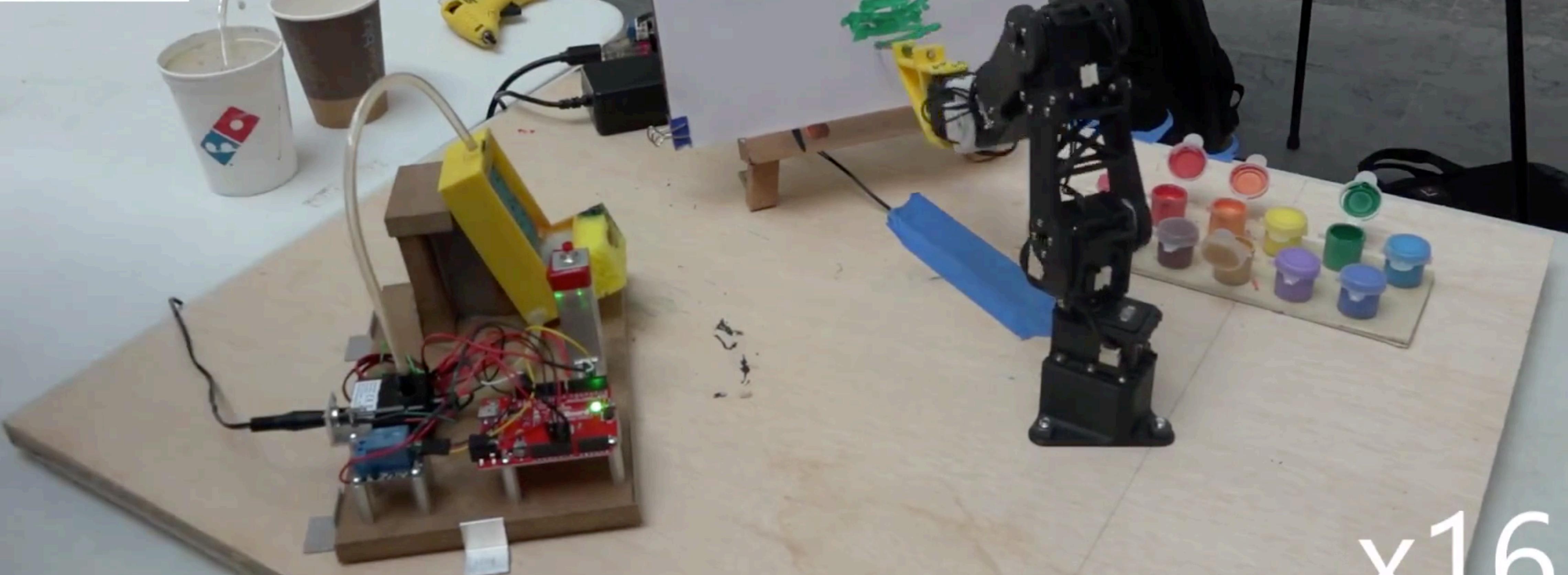
A student project Michigan





A student project Michigan



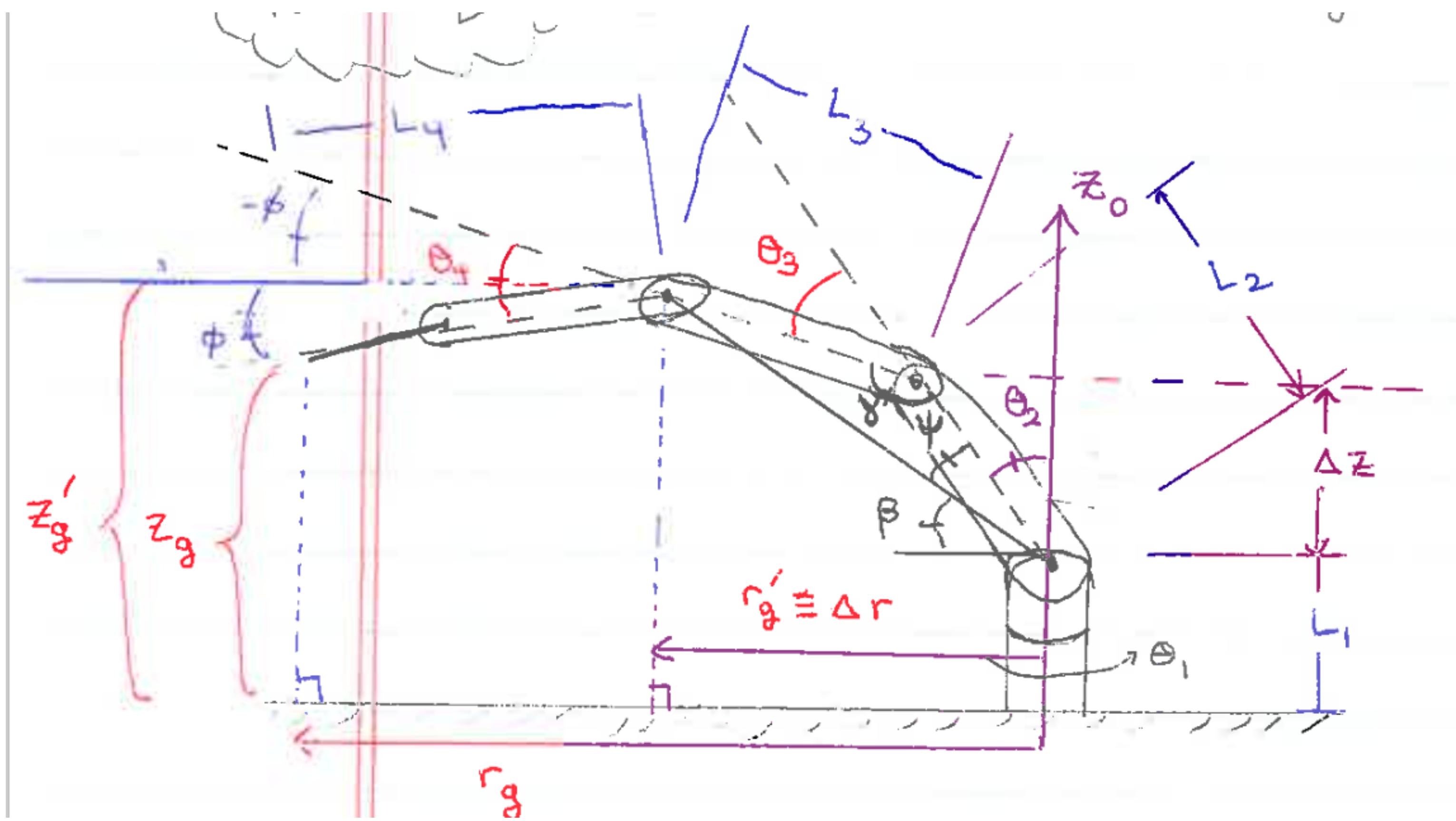


A student project Michigan

x16



RexArm from the above videos



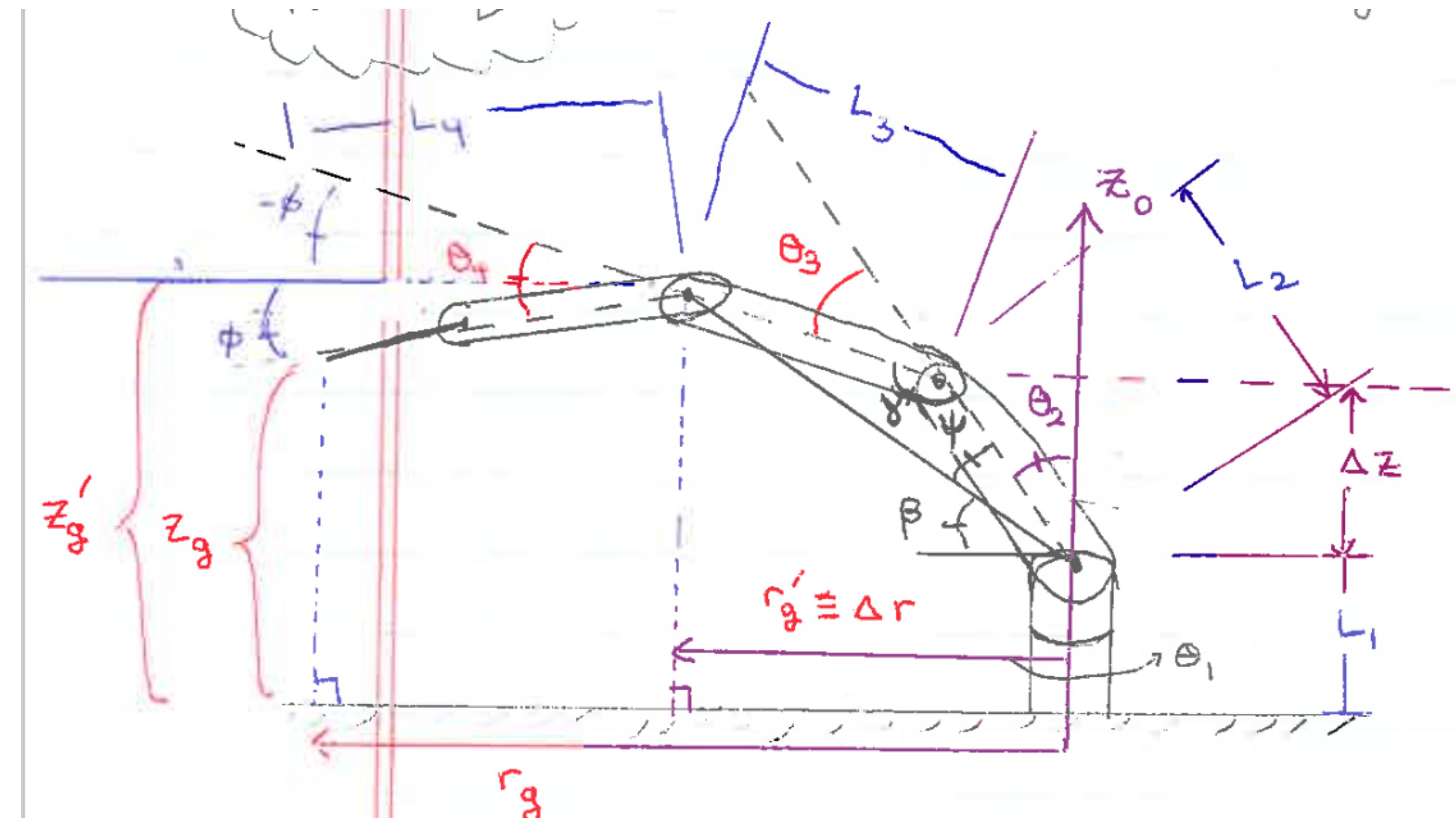
Find: configuration
 $q = [\theta_1 \theta_2 \theta_3 \theta_4]$
as robot joint angles

Given:

Find: configuration

$$\mathbf{q} = [\theta_1 \ \theta_2 \ \theta_3 \ \theta_4]$$

as robot joint angles



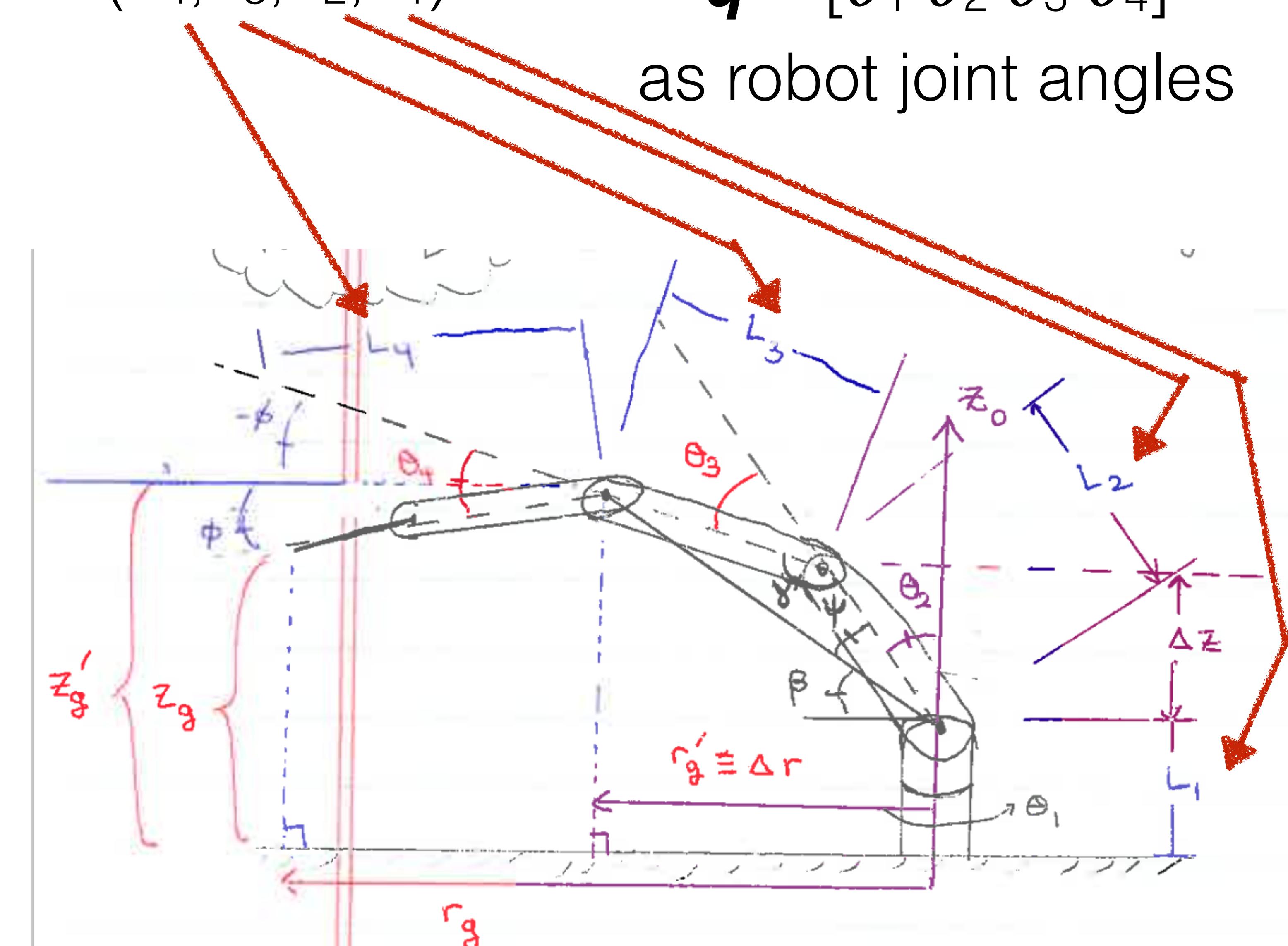
Given:

link lengths (L_4, L_3, L_2, L_1)

Find: configuration

$$\mathbf{q} = [\theta_1 \theta_2 \theta_3 \theta_4]$$

as robot joint angles

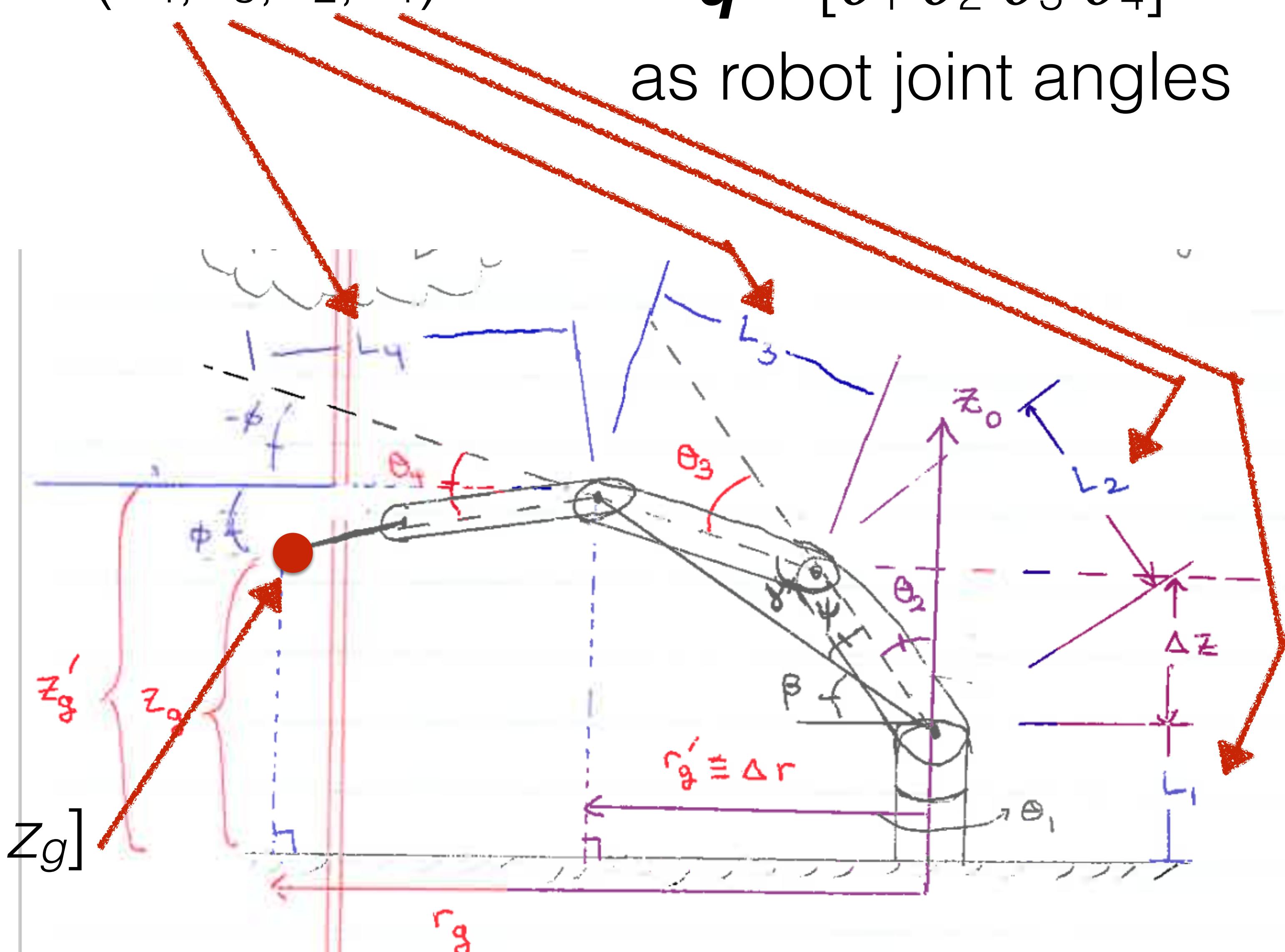


Given:

link lengths (L_4, L_3, L_2, L_1)

endeffector position $[x_g \ y_g \ z_g]$
wrt. base frame

Find: configuration
 $q = [\theta_1 \ \theta_2 \ \theta_3 \ \theta_4]$
as robot joint angles



Given:

endeffector orientation ϕ

as angle wrt. plane
centered at \mathbf{o}_3 and
parallel to ground plane

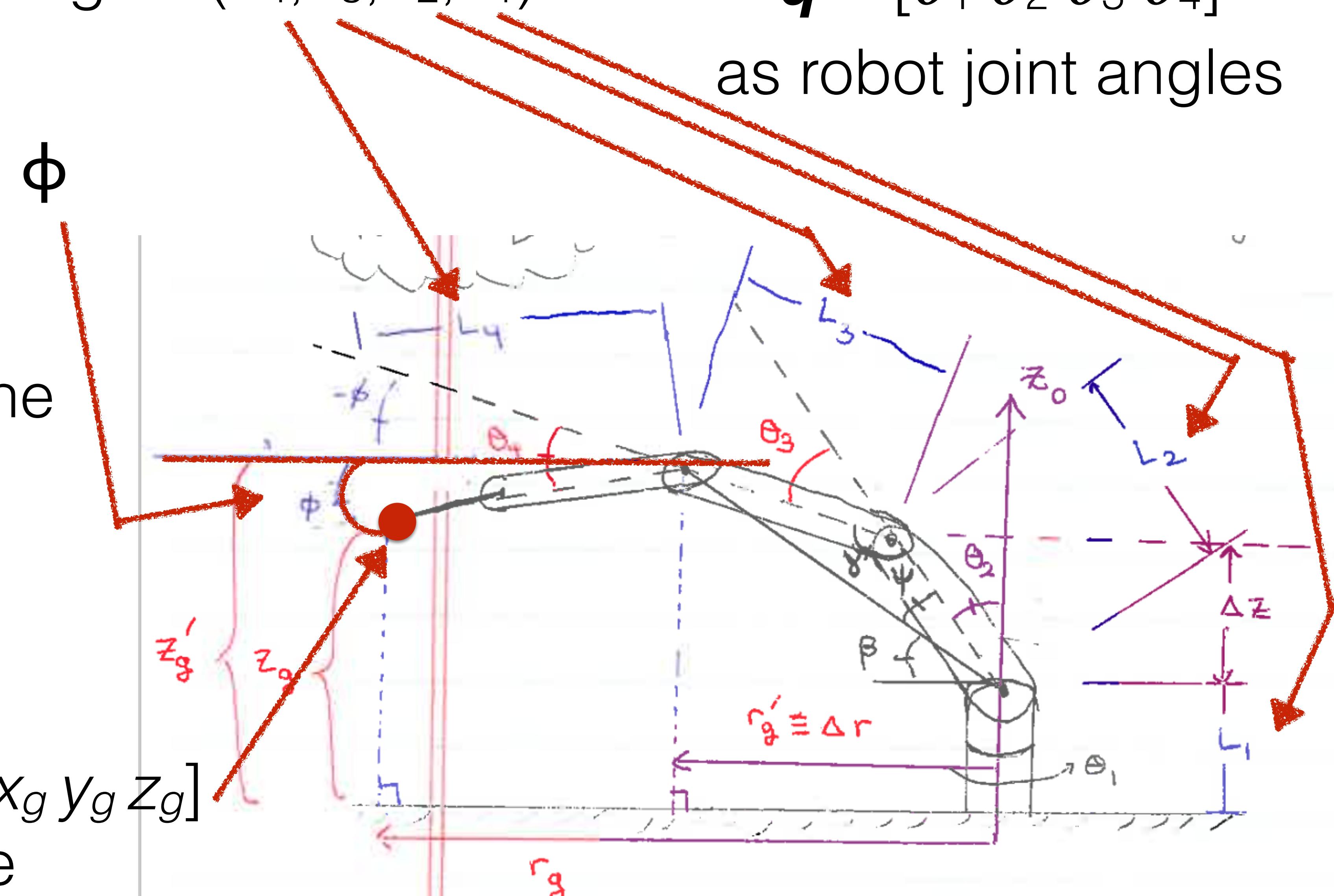
endeffector position $[x_g \ y_g \ z_g]$
wrt. base frame

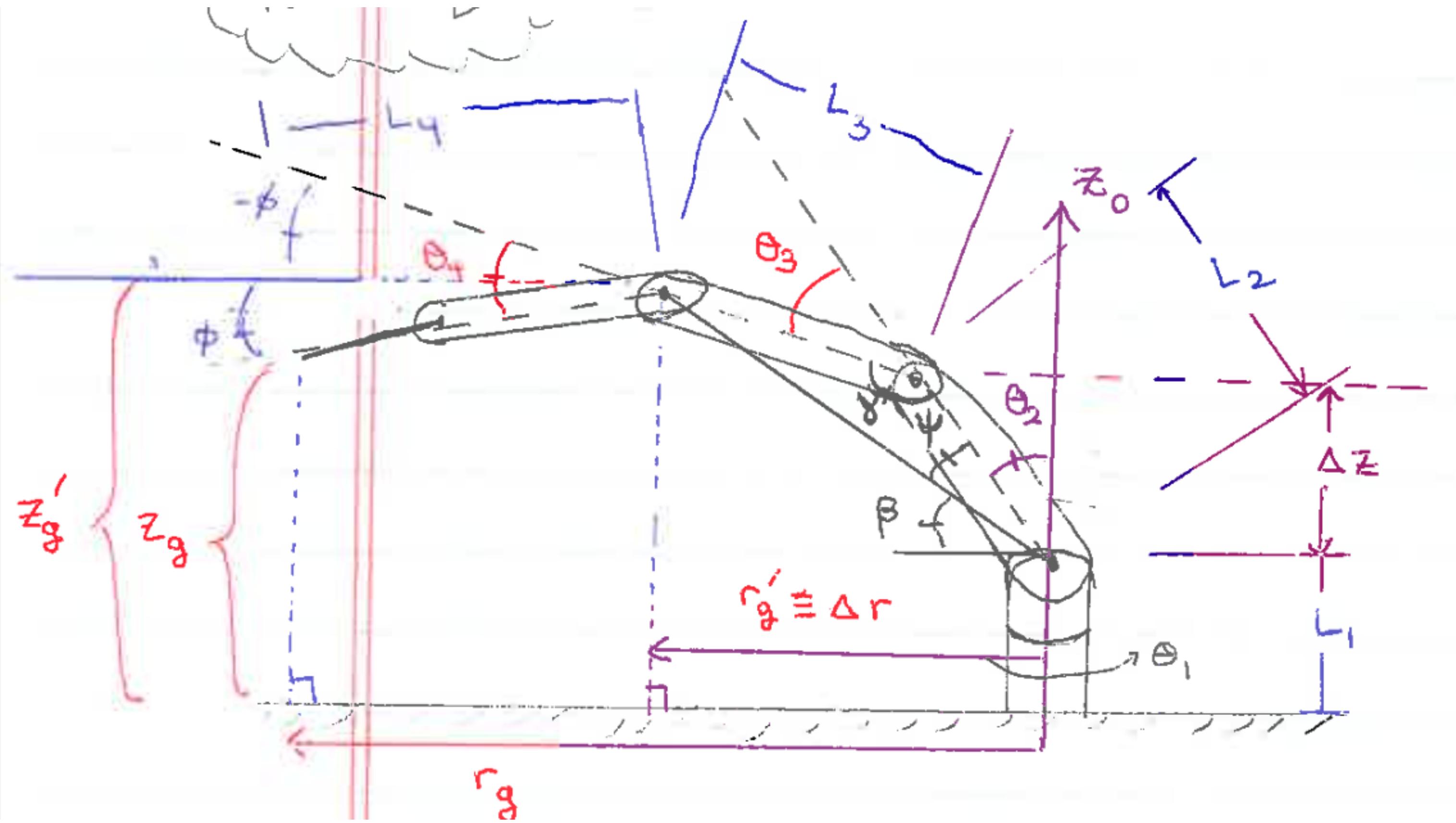
link lengths (L_4, L_3, L_2, L_1)

Find: configuration

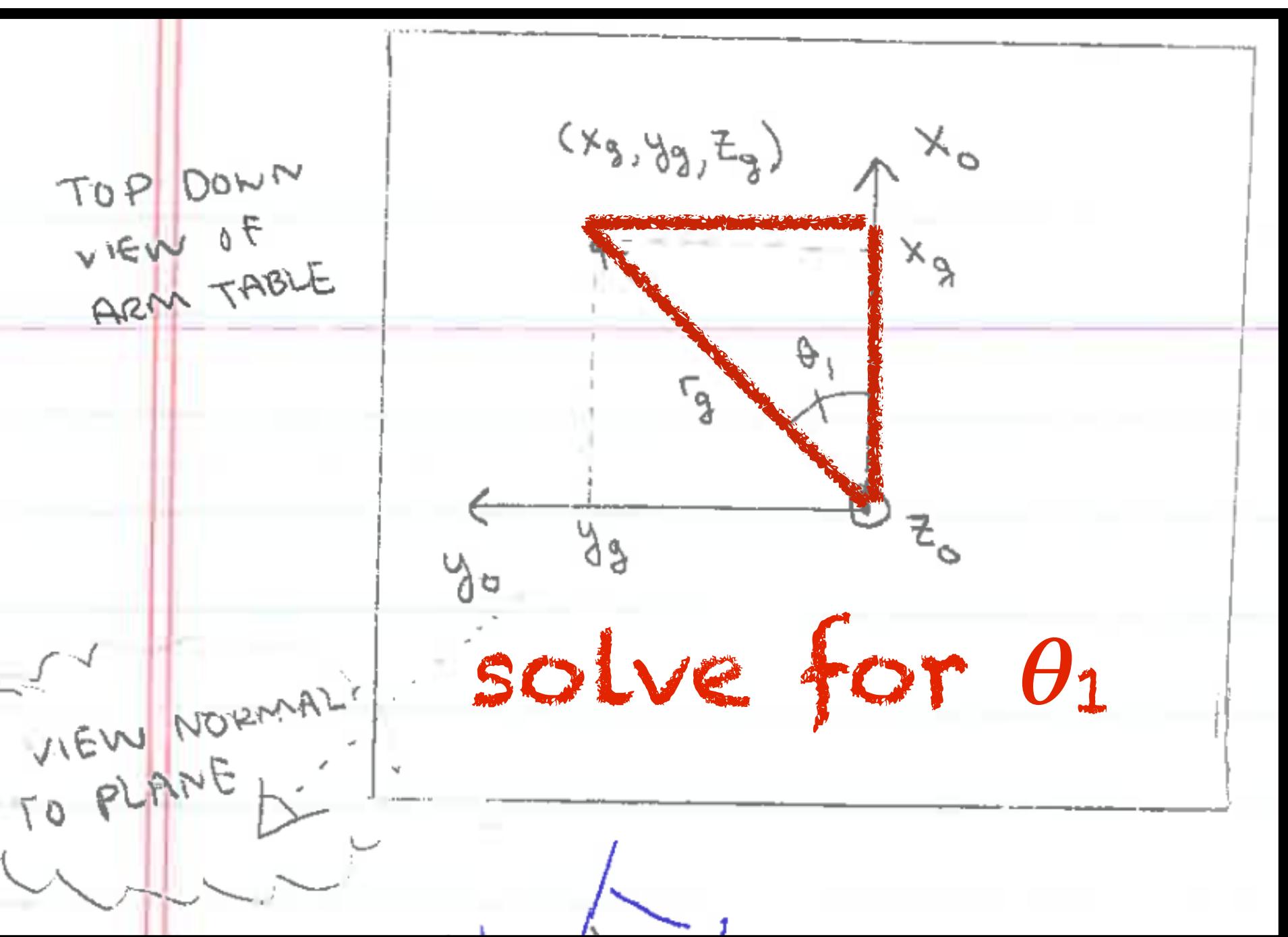
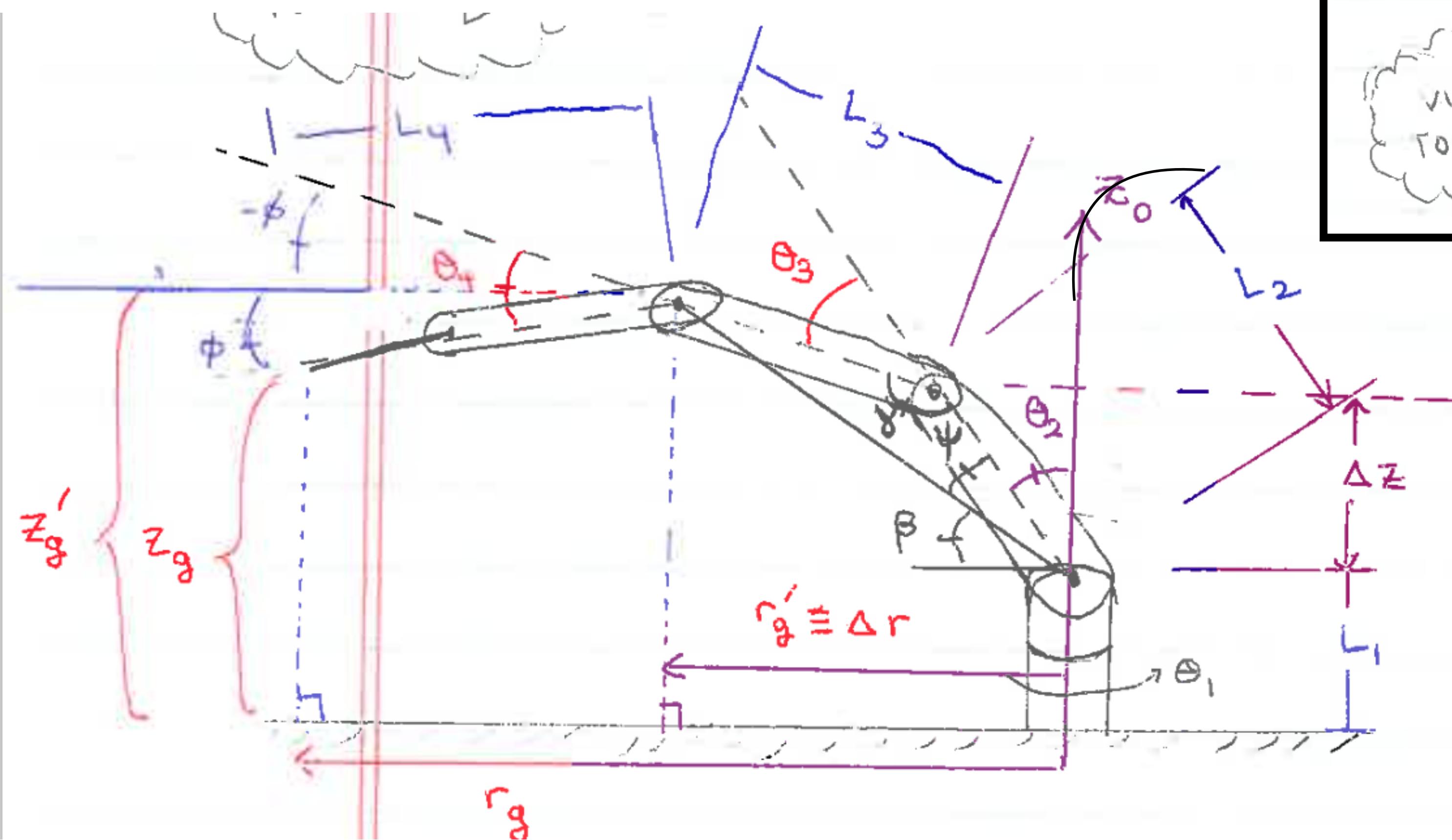
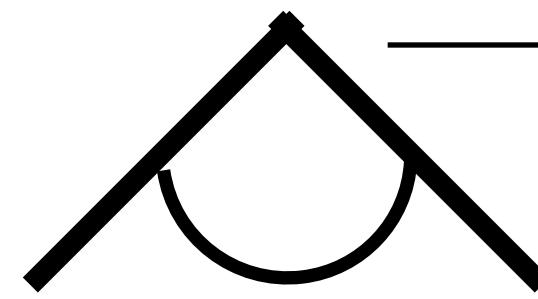
$$\mathbf{q} = [\theta_1 \ \theta_2 \ \theta_3 \ \theta_4]$$

as robot joint angles

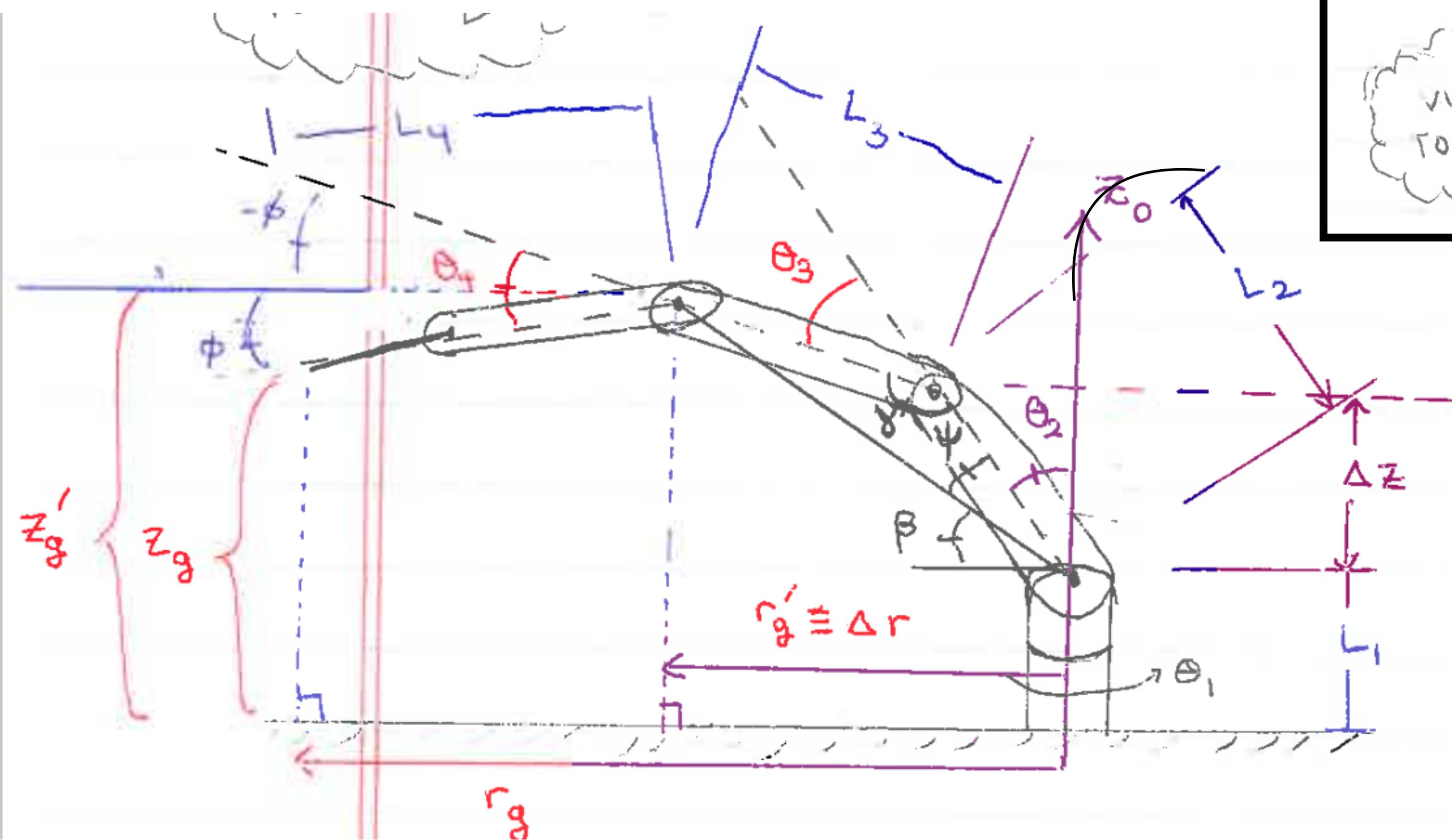
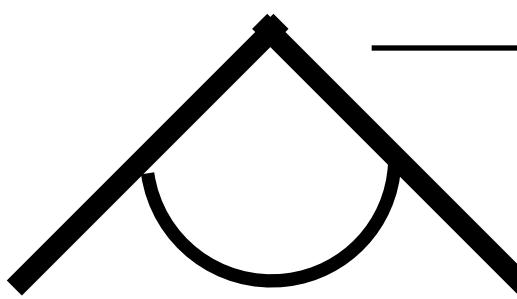




overhead view

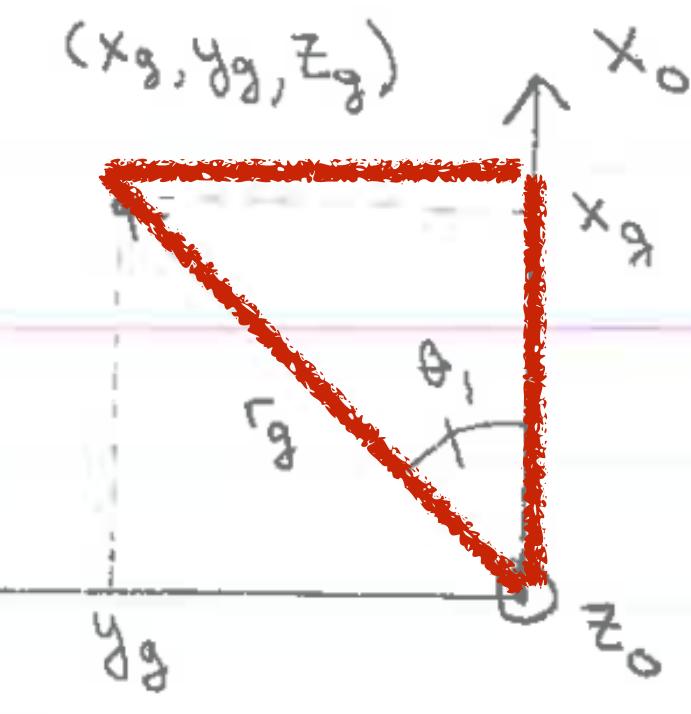


overhead view



TOP DOWN
VIEW OF
ARM TABLE

VIEW NORMAL
TO PLANE

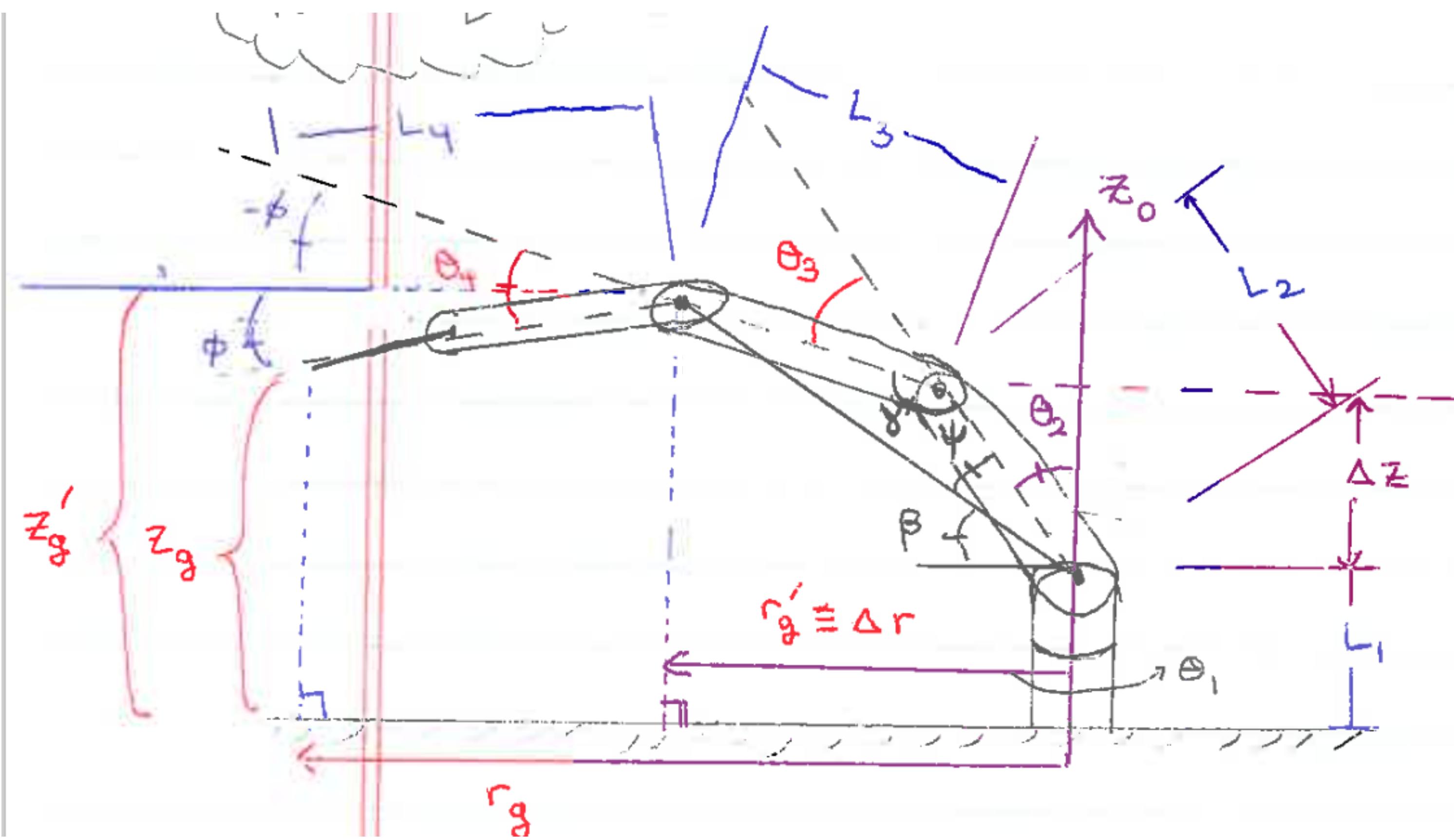


solve for θ_1

$$\theta_1 = \text{atan2}(y_g, x_g)$$

solve for θ_1

$$\theta_1 = \text{atan2}(y_g, x_g)$$



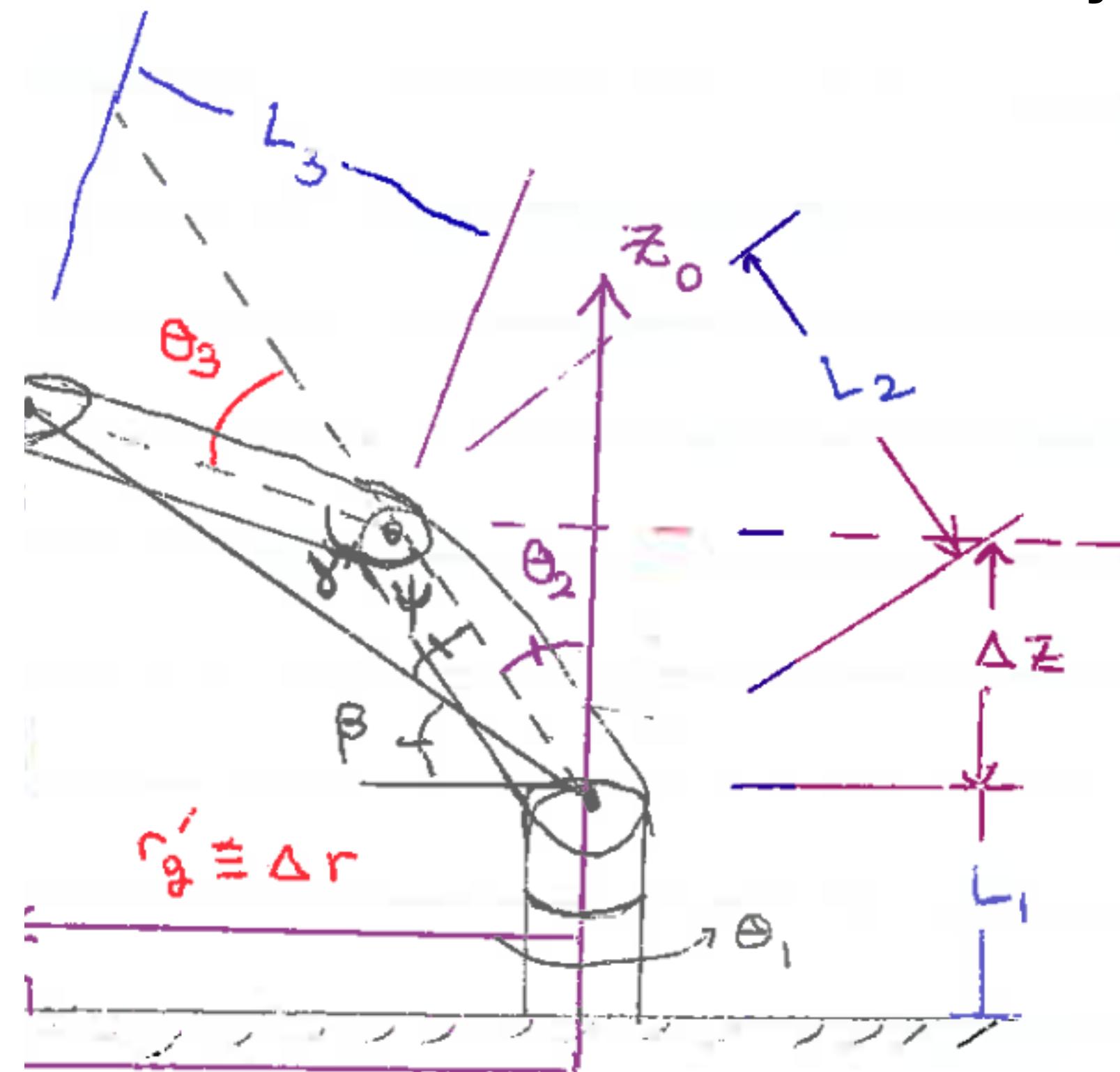
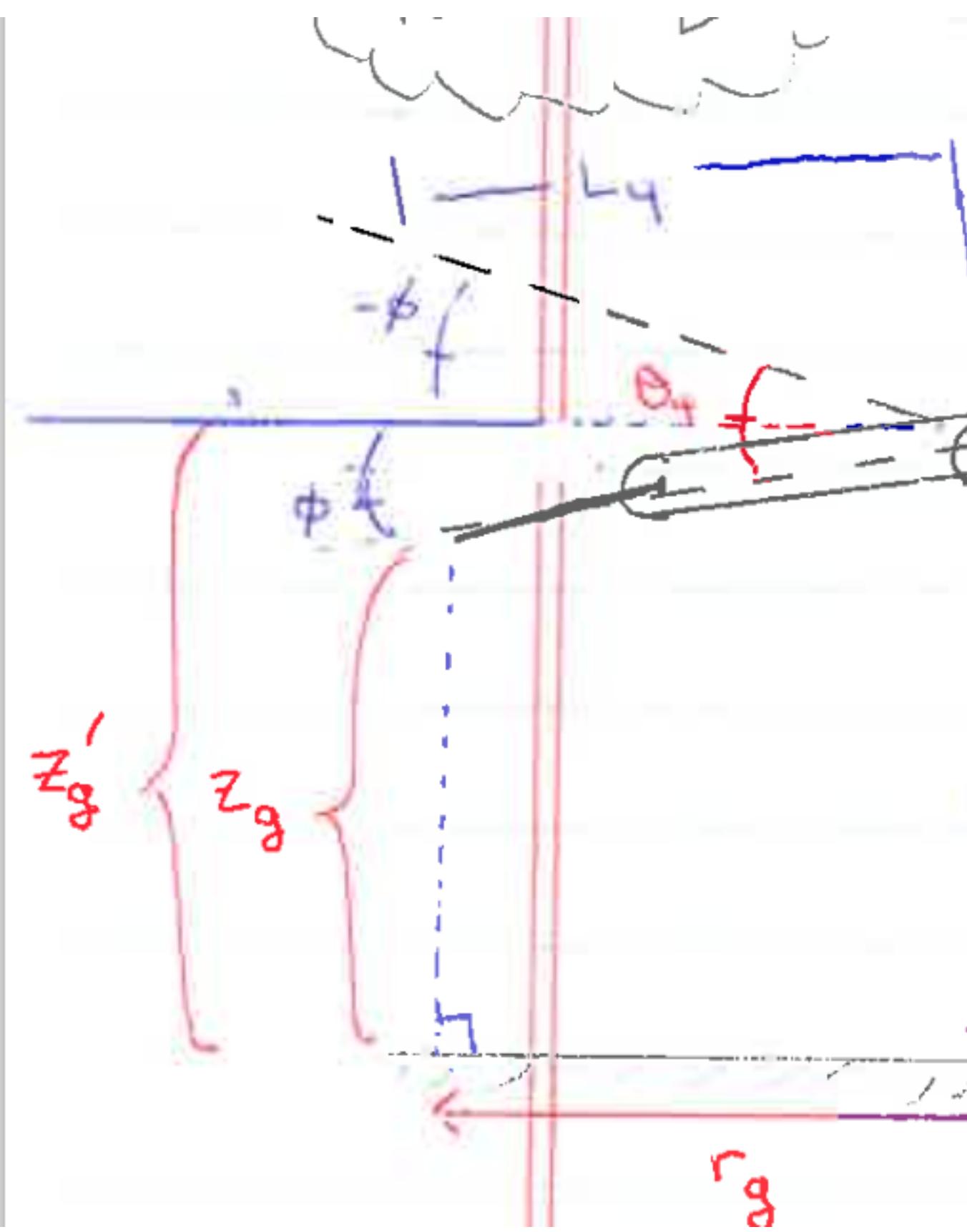
solve for θ_3

solve for θ_1

$$\theta_1 = \text{atan2}(y_g, x_g)$$

Decoupling:
separate endeffector from
rest of the robot at last joint

solve for θ_3

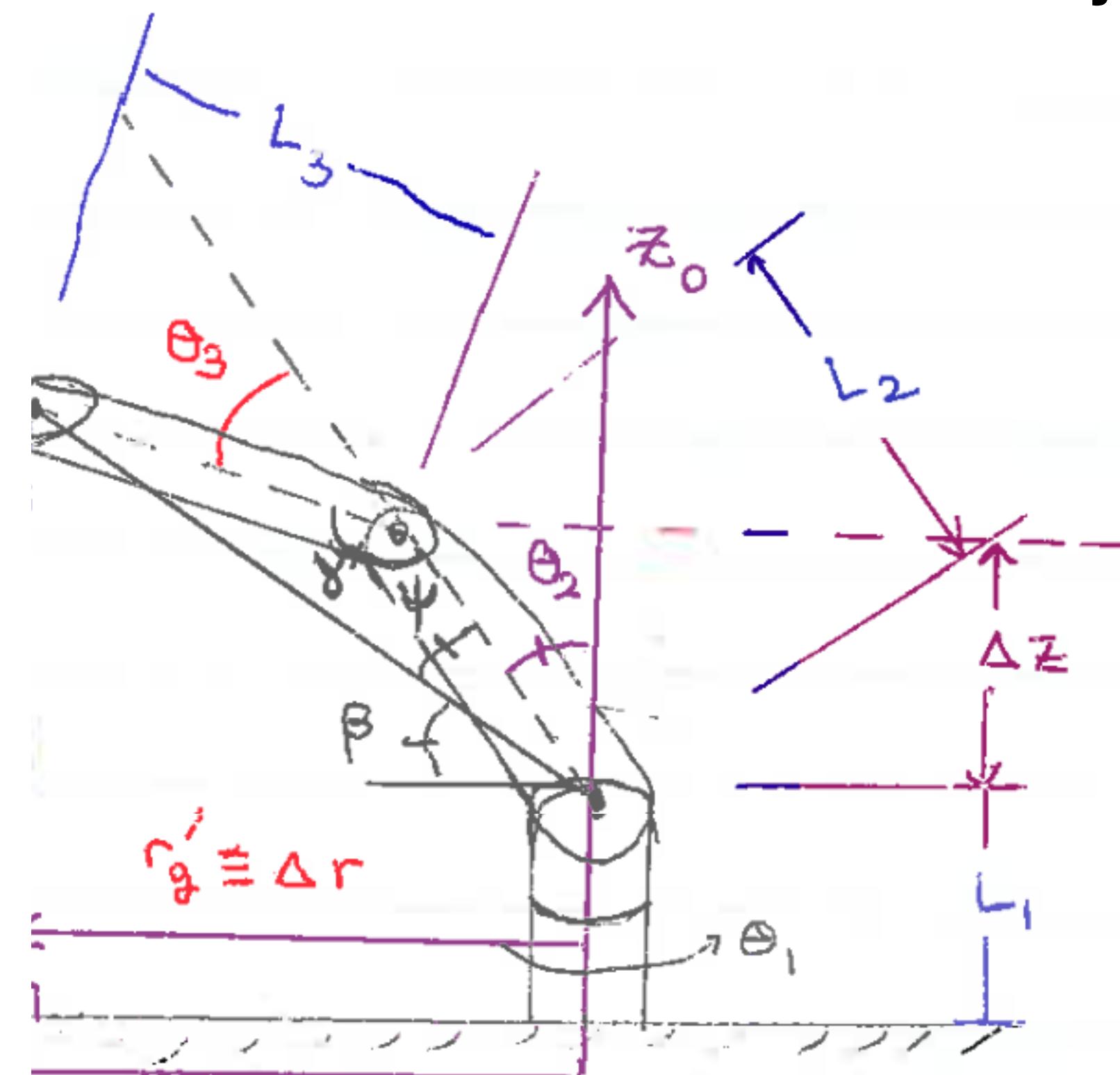
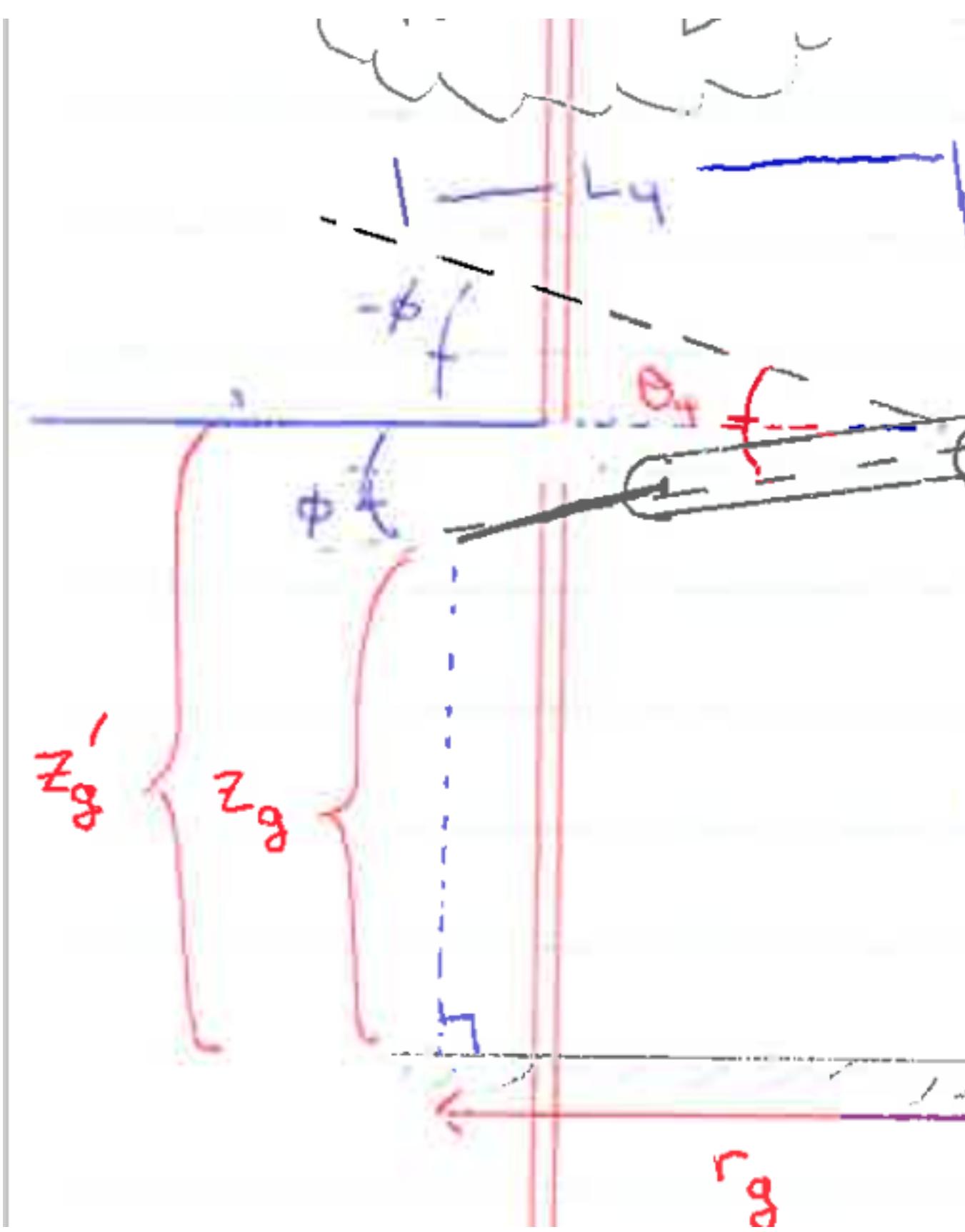


Solve for θ_1

$$\Theta_1 = \text{atan2}(y_g, x_g)$$

Decoupling

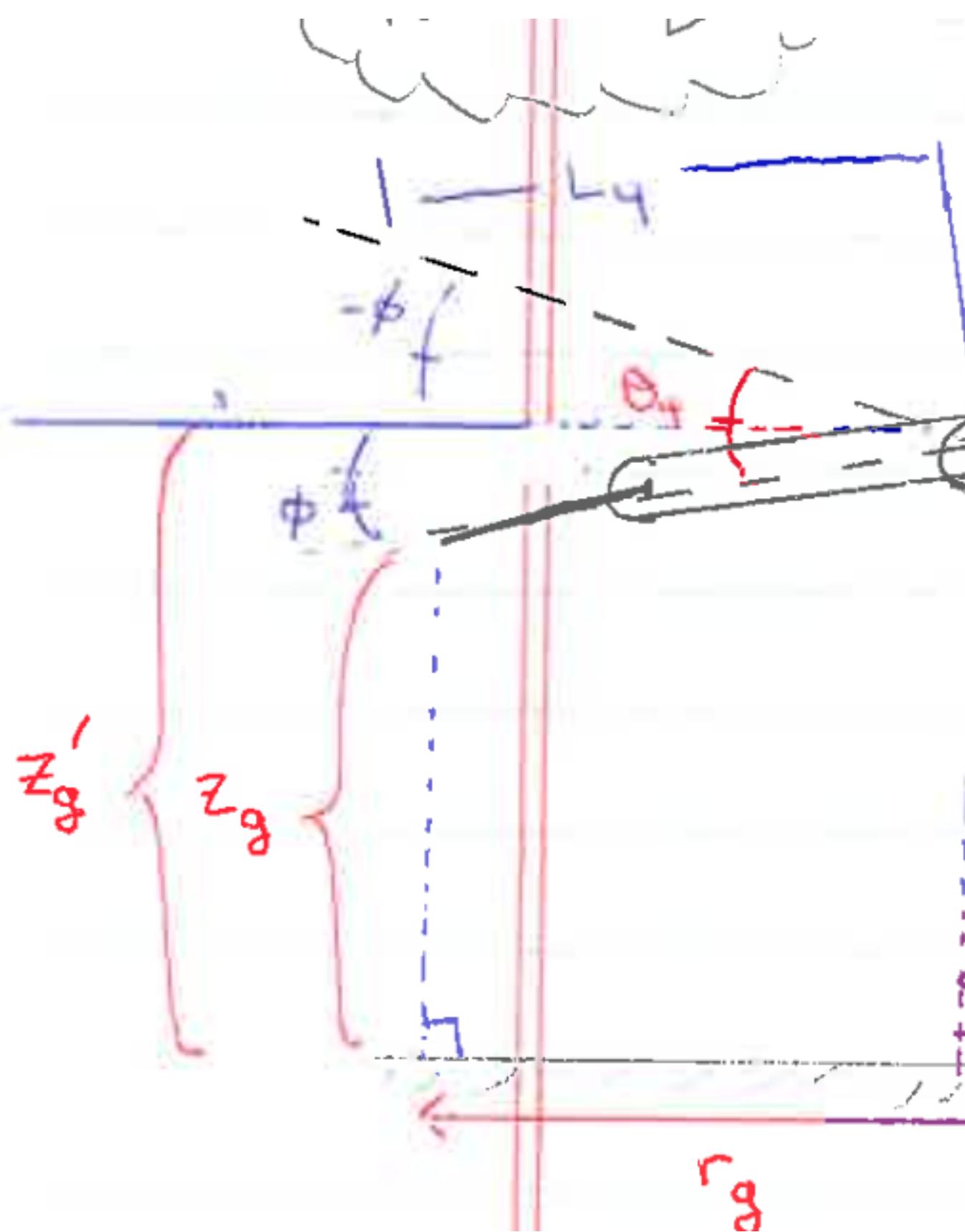
solve for θ_3



and...

solve for θ_1

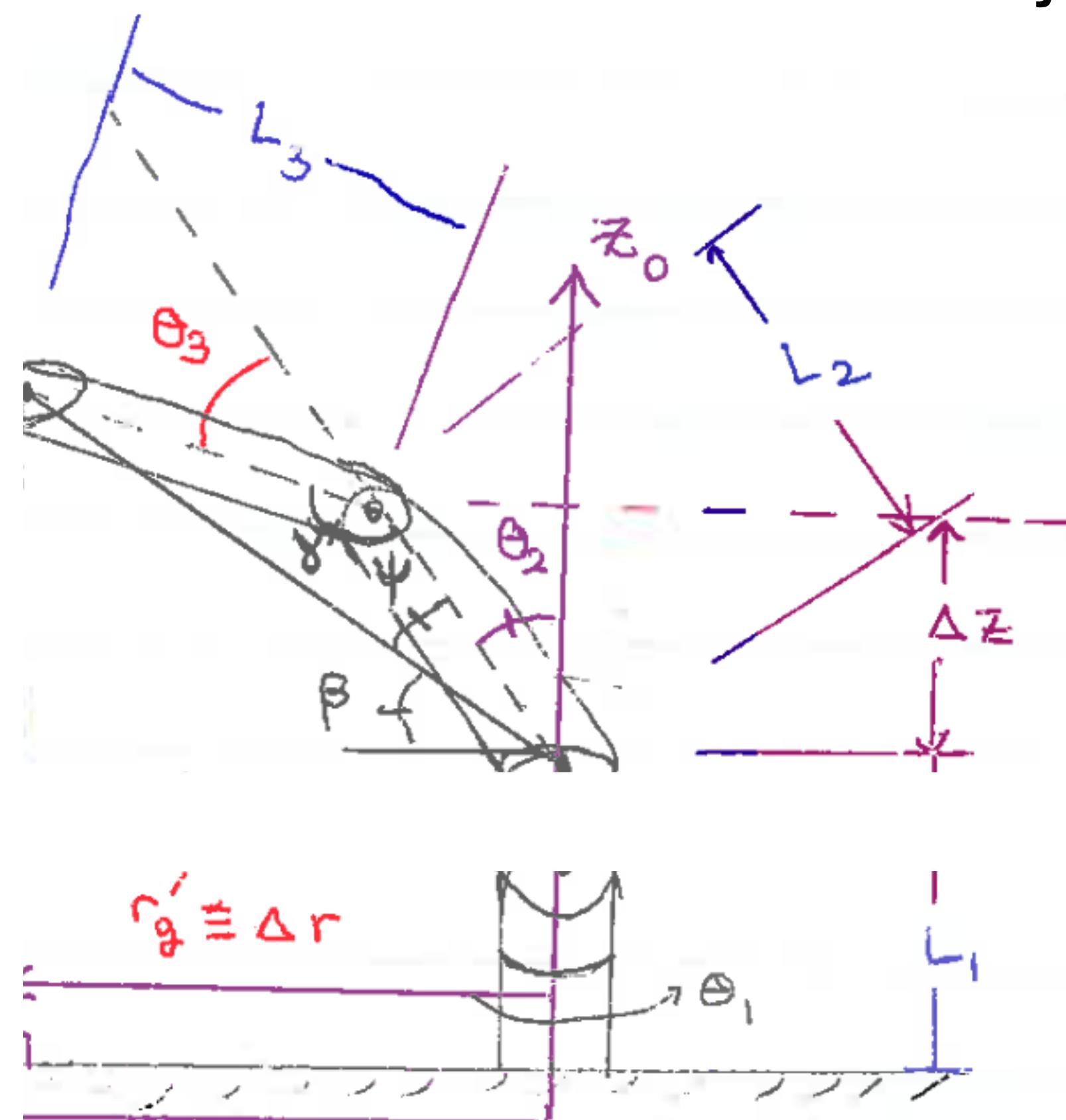
$$\theta_1 = \text{atan2}(y_g, x_g)$$



Decoupling:

separate endeffector from
rest of the robot at last joint

solve for θ_3

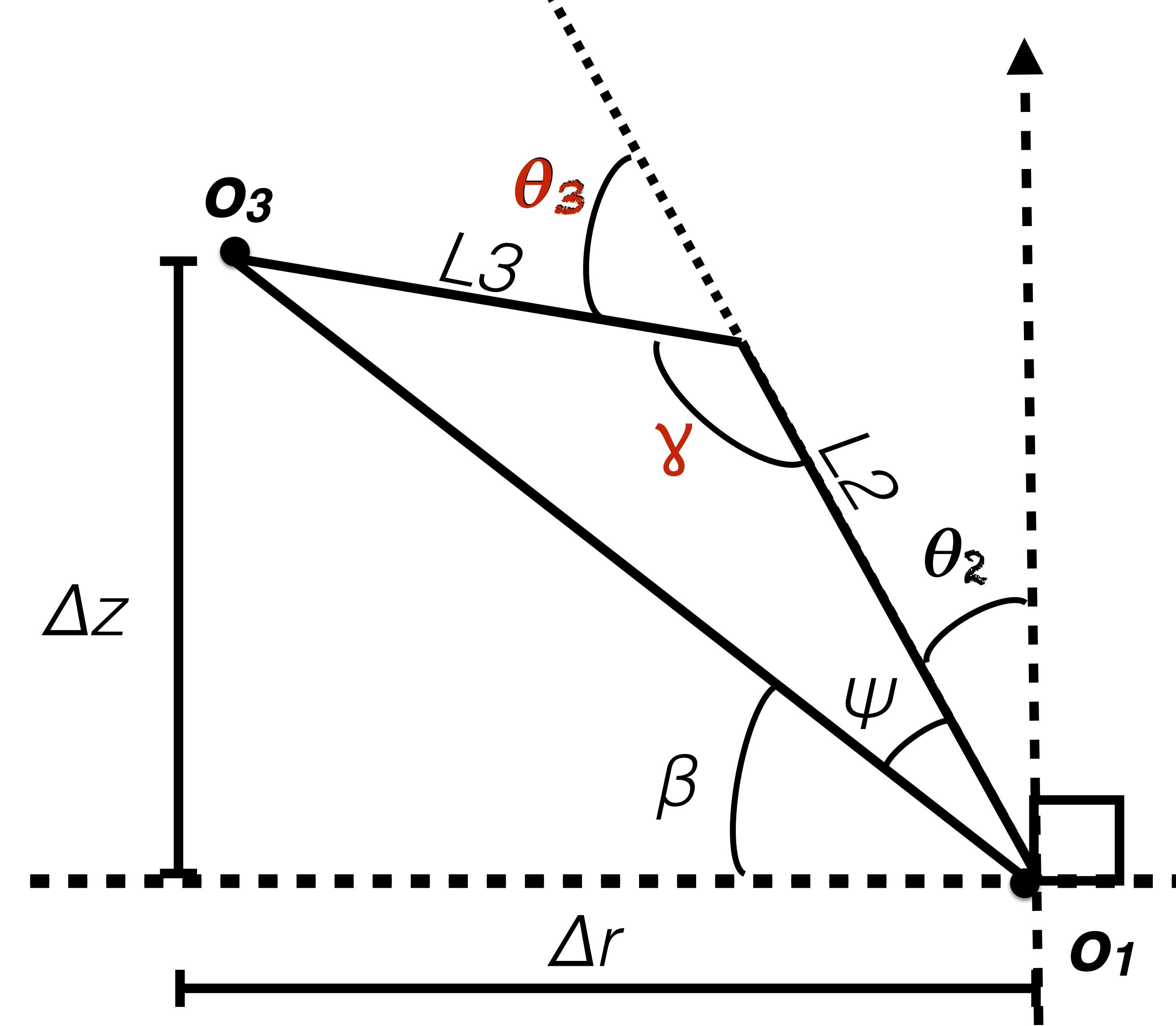
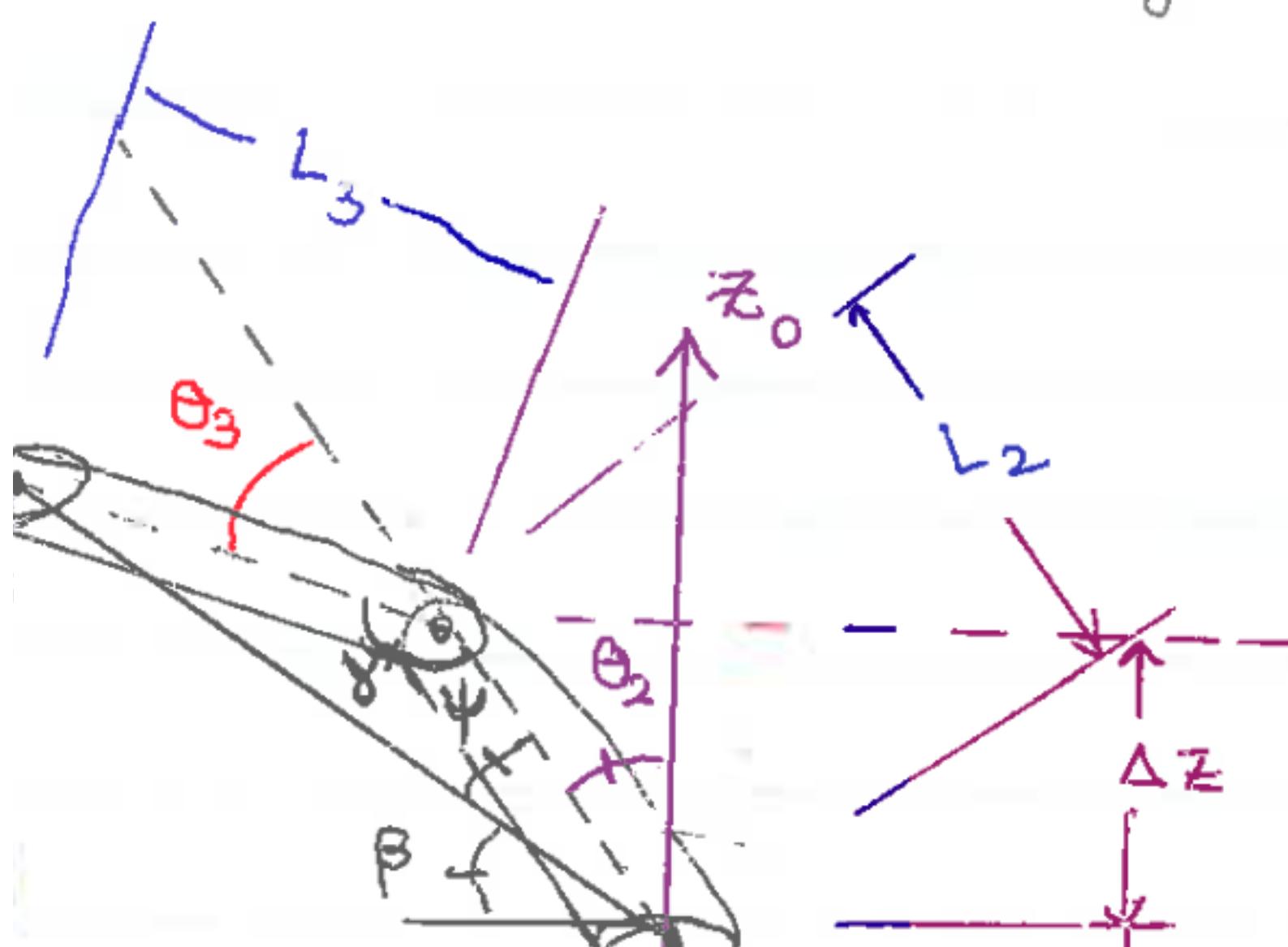


and joint 1 from rest
of robot

solve for θ_1

$$\theta_1 = \text{atan2}(\gamma_g, x_g)$$

solve for θ_3



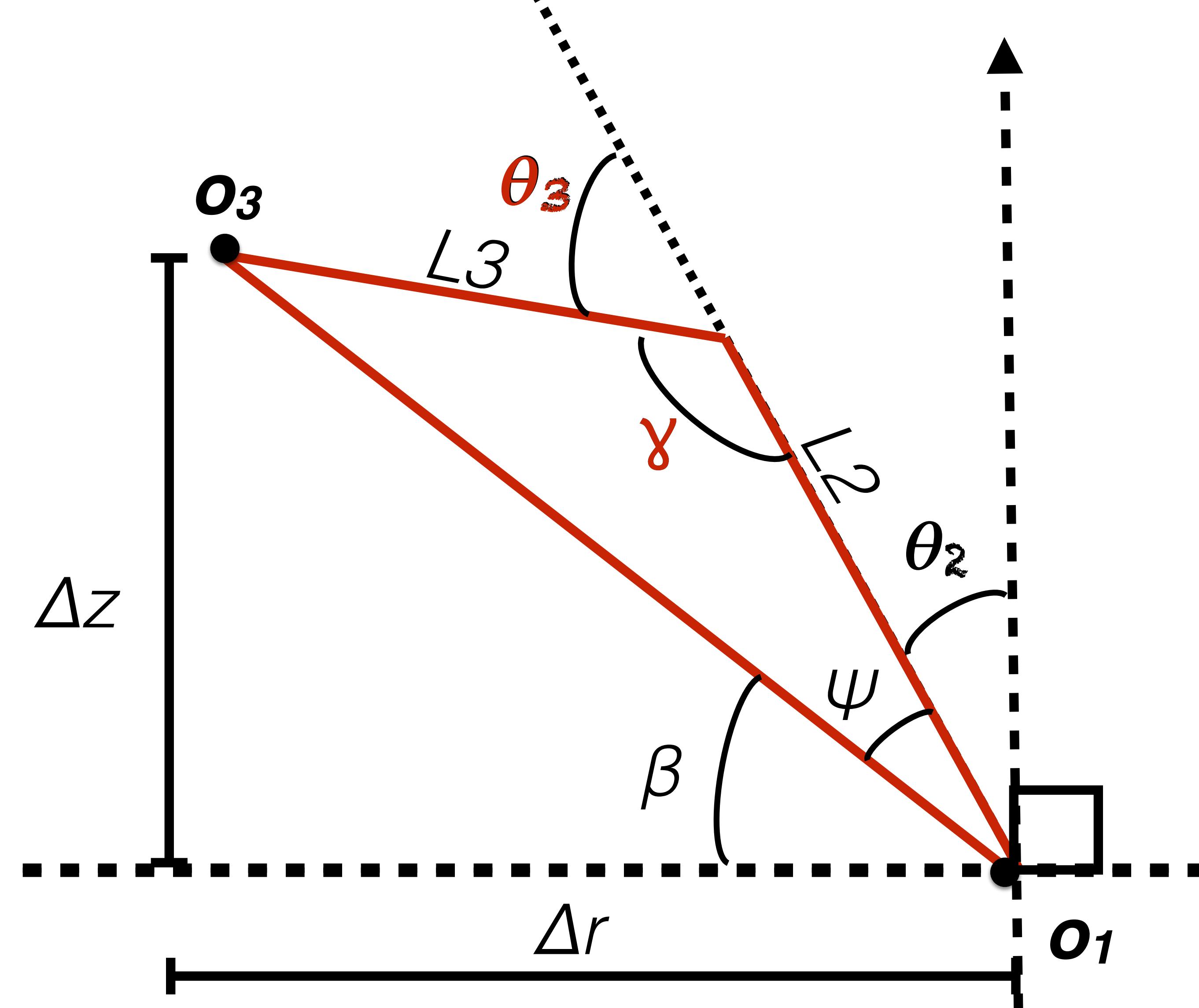
solve for θ_1

$$\theta_1 = \text{atan2}(\gamma_g, x_g)$$

solve for θ_3

(Law of cosines with supplementary angle γ)

$$\cos \theta_3 = \frac{\Delta z^2 + \Delta r^2 - L_2^2 - L_3^2}{2 L_2 L_3}$$



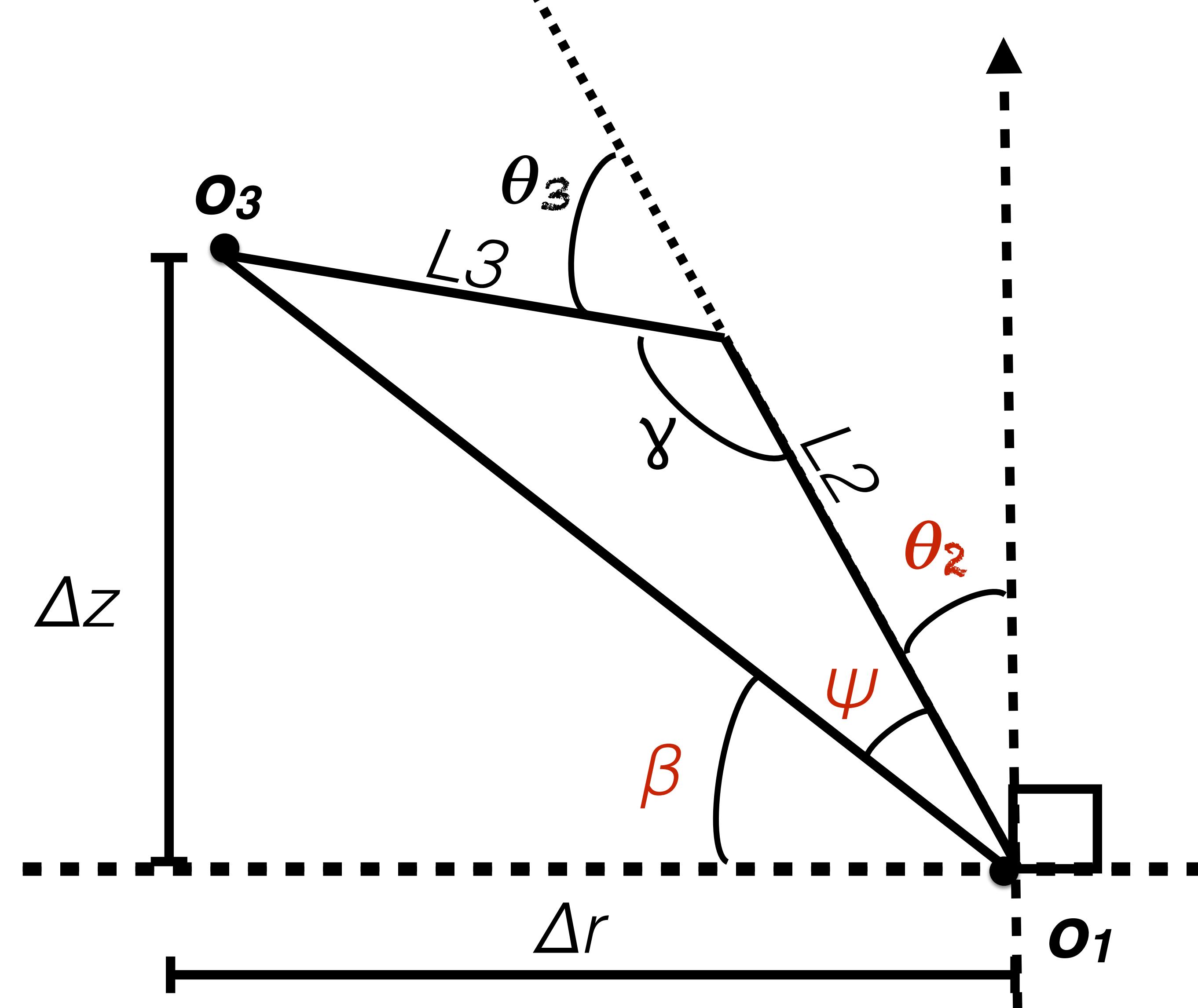
solve for θ_1

$$\theta_1 = \text{atan2}(\gamma_g, x_g)$$

solve for θ_3

$$\cos \theta_3 = \frac{\Delta z^2 + \Delta r^2 - L_2^2 - L_3^2}{2 L_2 L_3}$$

solve for θ_2



solve for θ_1

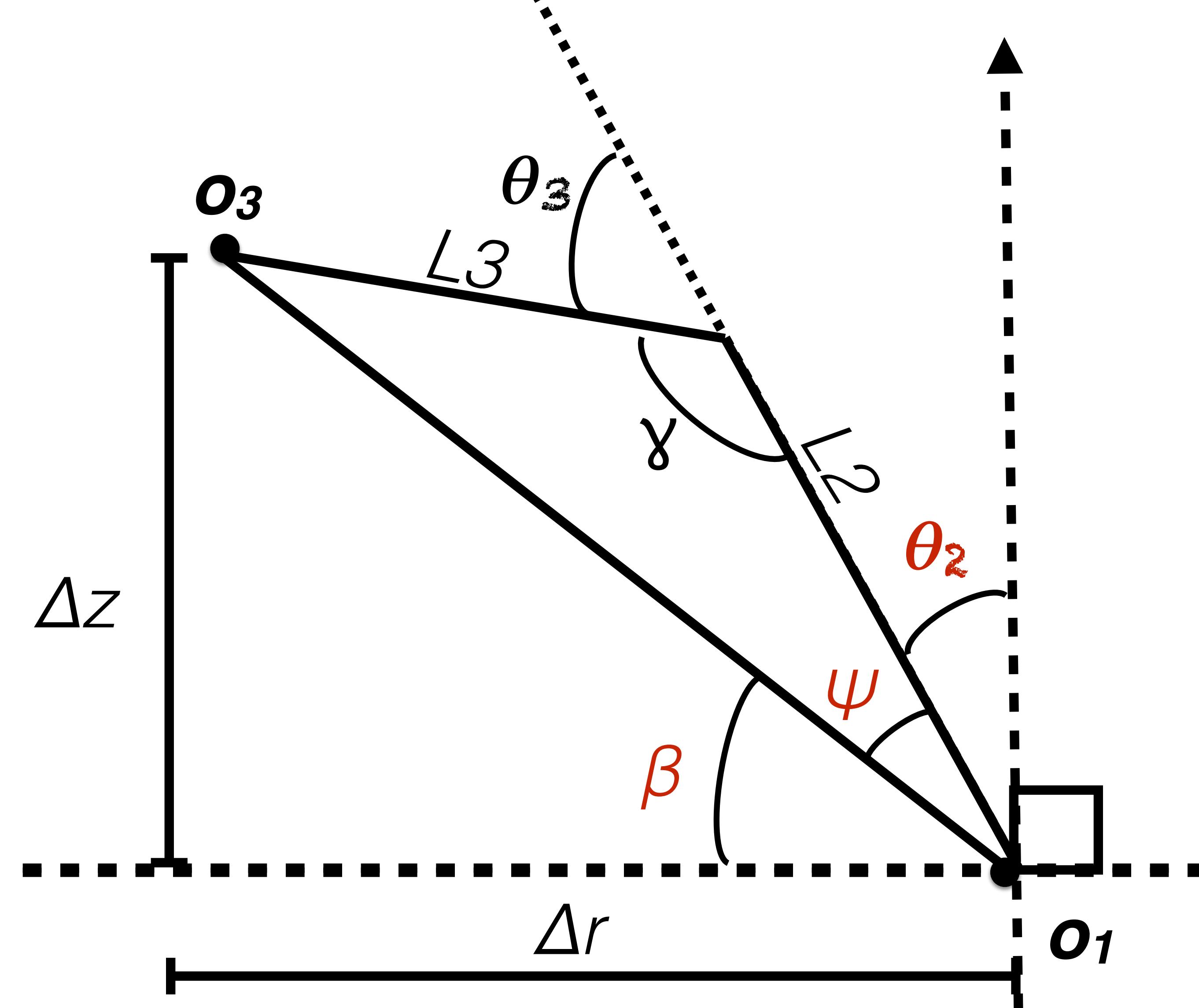
$$\theta_1 = \text{atan2}(\gamma_g, x_g)$$

solve for θ_3

$$\cos \theta_3 = \frac{\Delta z^2 + \Delta r^2 - L_2^2 - L_3^2}{2 L_2 L_3}$$

solve for θ_2

(Law of cosines with angle ψ ,
arctan with angle β)



solve for θ_1

$$\theta_1 = \text{atan2}(\gamma_g, x_g)$$

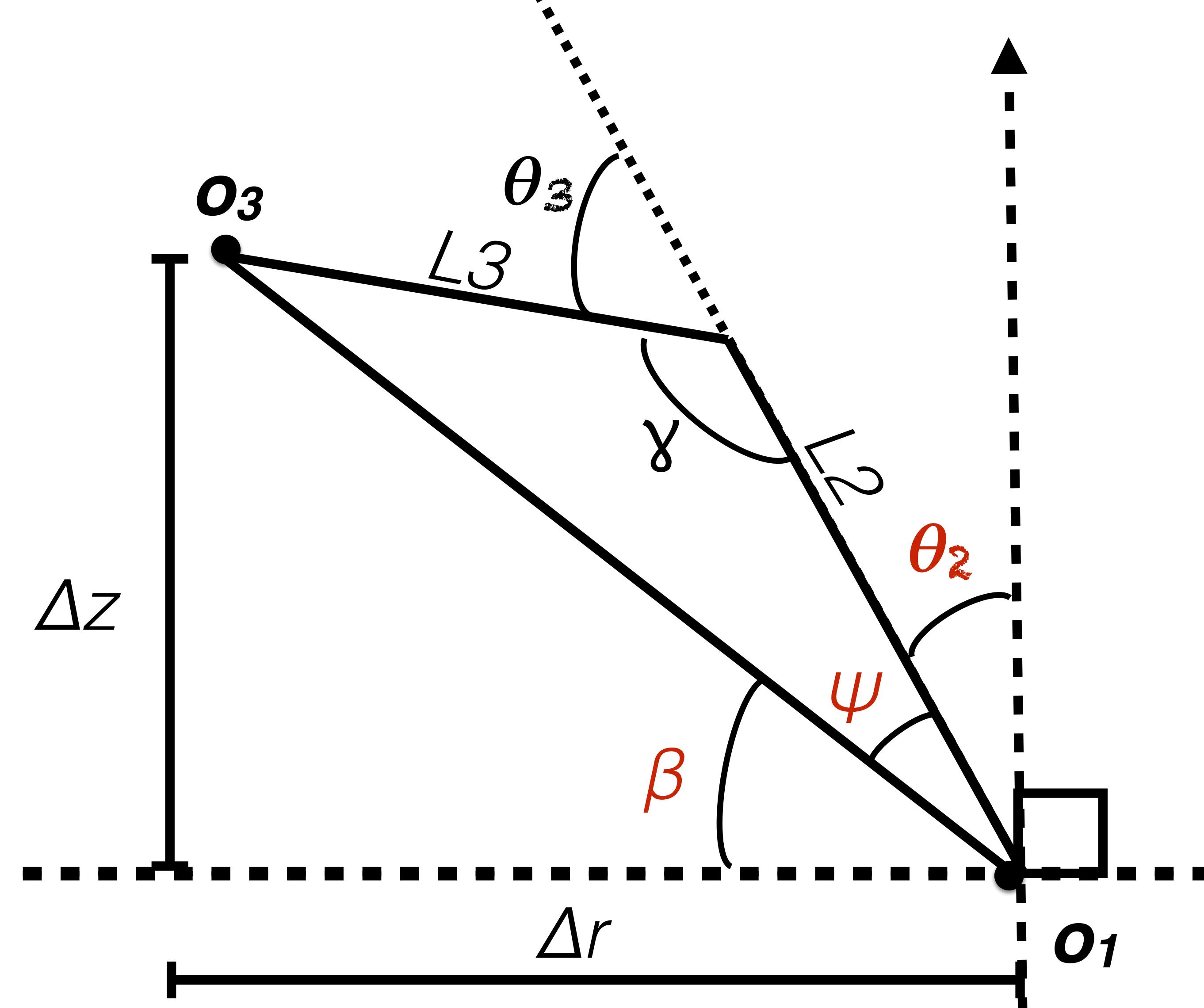
solve for θ_3

$$\cos \theta_3 = \frac{\Delta z^2 + \Delta r^2 - L_2^2 - L_3^2}{2 L_2 L_3}$$

solve for θ_2

$$\theta_2 = \begin{cases} \frac{\pi}{2} - \beta - \psi & \text{if } \theta_3 \geq 0 \quad \text{"Elbow up"} \\ \frac{\pi}{2} - \beta + \psi & \text{if } \theta_3 < 0 \quad \text{"Elbow-down"} \end{cases}$$

two potential solutions
depending on elbow angle



solve for θ_1

$$\theta_1 = \text{atan2}(y_g, x_g)$$

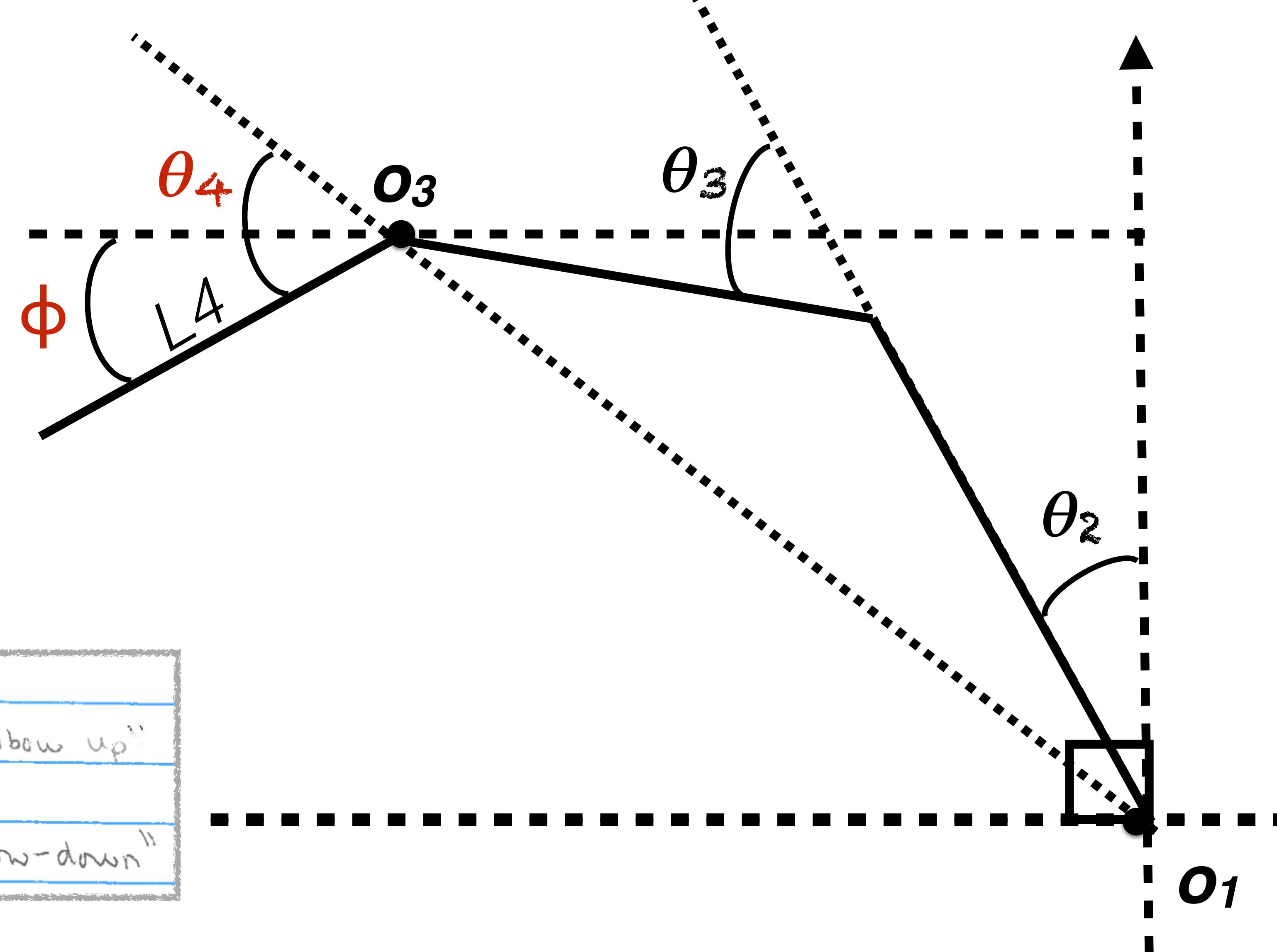
solve for θ_3

$$\cos \theta_3 = \frac{\Delta z^2 + \Delta r^2 - L_2^2 - L_3^2}{2 L_2 L_3}$$

solve for θ_2

$$\theta_2 = \begin{cases} \frac{\pi}{2} - \beta - \psi & \text{if } \theta_3 \geq 0 \quad \text{"Elbow up"} \\ \frac{\pi}{2} - \beta + \psi & \text{if } \theta_3 < 0 \quad \text{"Elbow-down"} \end{cases}$$

solve for θ_4



solve for θ_1

$$\theta_1 = \text{atan2}(y_g, x_g)$$

solve for θ_3

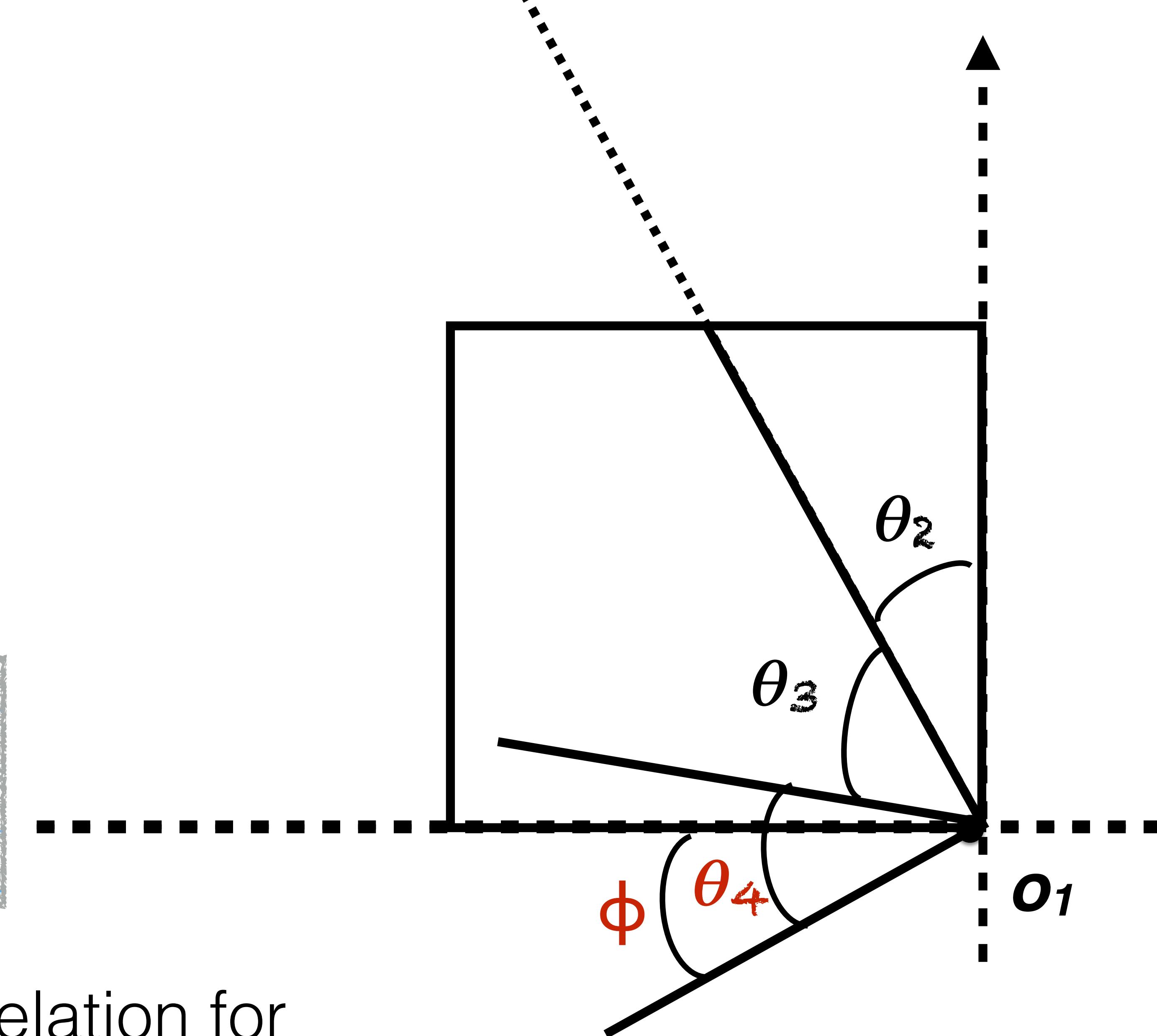
$$\cos \theta_3 = \frac{\Delta z^2 + \Delta r^2 - L_2^2 - L_3^2}{2 L_2 L_3}$$

solve for θ_2

$$\theta_2 = \begin{cases} \frac{\pi}{2} - \beta - \psi & \text{if } \theta_3 \geq 0 \quad \text{"Elbow up"} \\ \frac{\pi}{2} - \beta + \psi & \text{if } \theta_3 < 0 \quad \text{"Elbow-down"} \end{cases}$$

solve for θ_4

(Equivalence relation for
adding angles from \mathbf{z}_0)



solve for θ_1

$$\theta_1 = \text{atan2}(y_g, x_g)$$

solve for θ_3

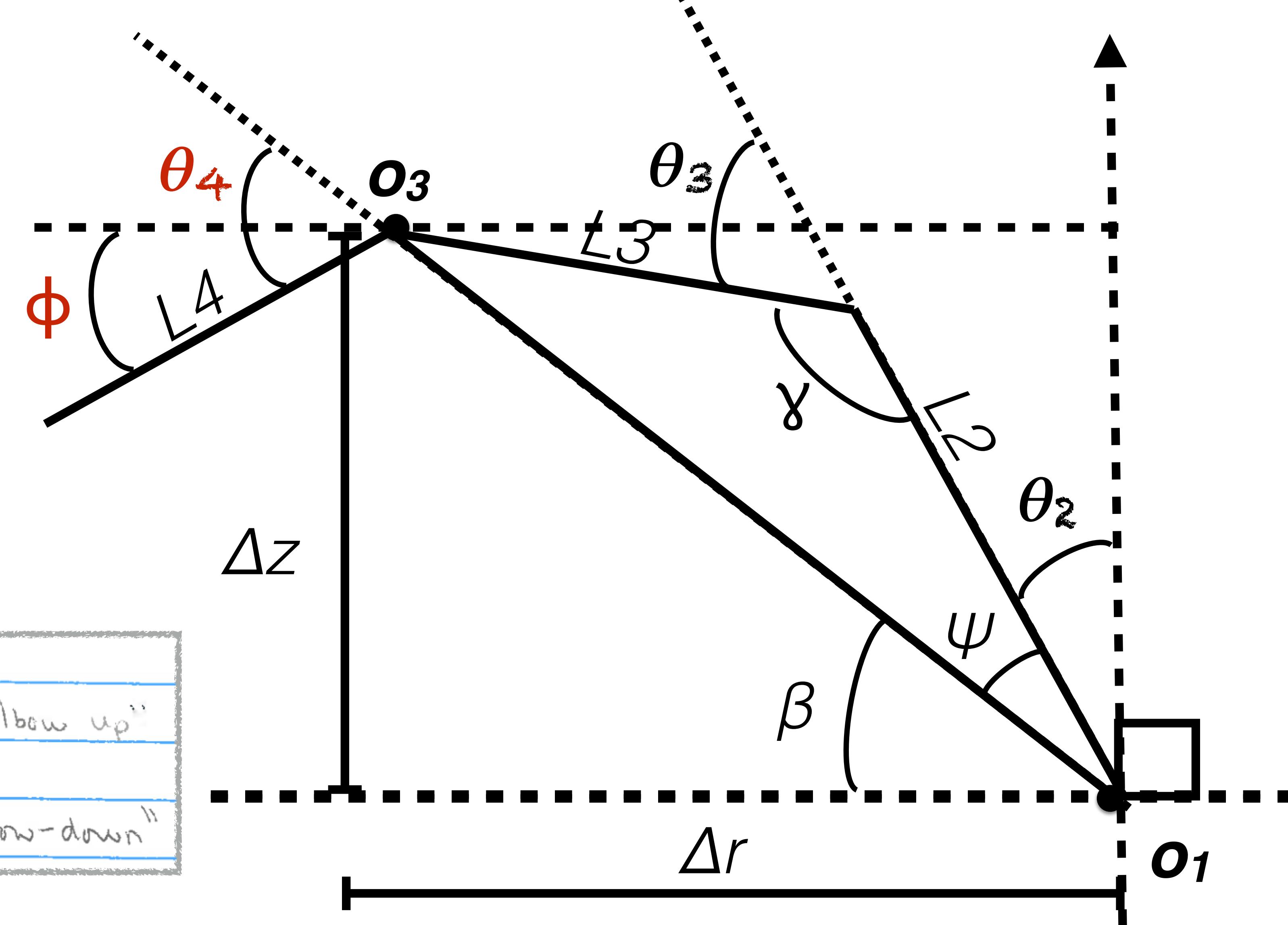
$$\cos \theta_3 = \frac{\Delta z^2 + \Delta r^2 - L_2^2 - L_3^2}{2 L_2 L_3}$$

solve for θ_2

$$\theta_2 = \begin{cases} \frac{\pi}{2} - \beta - \psi & \text{if } \theta_3 \geq 0 \quad \text{"Elbow up"} \\ \frac{\pi}{2} - \beta + \psi & \text{if } \theta_3 < 0 \quad \text{"Elbow-down"} \end{cases}$$

solve for θ_4

$$\theta_4 = \phi - \theta_2 - \theta_3 + \frac{\pi}{2}$$



(Addition of angles in arm plane starting from z_0)

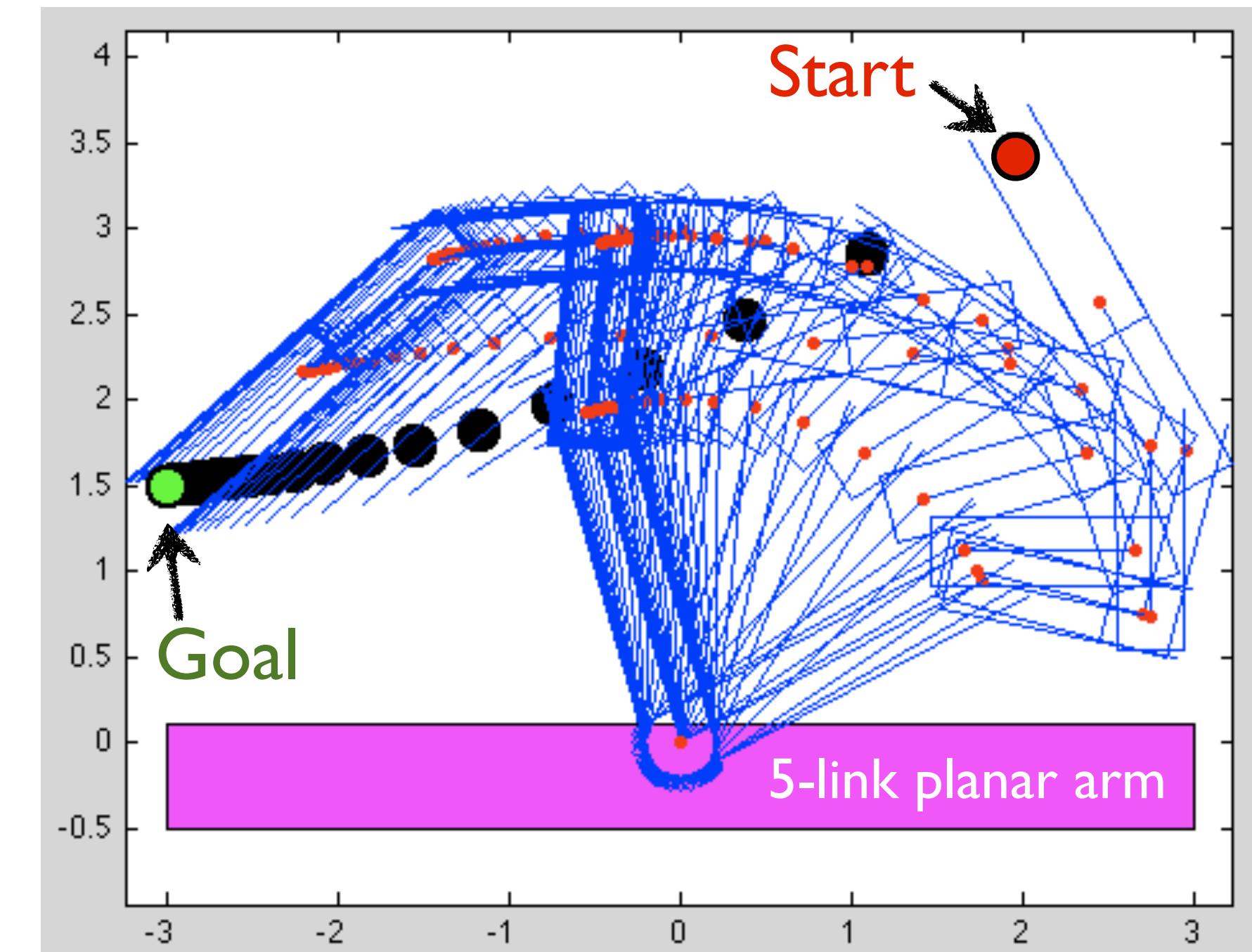
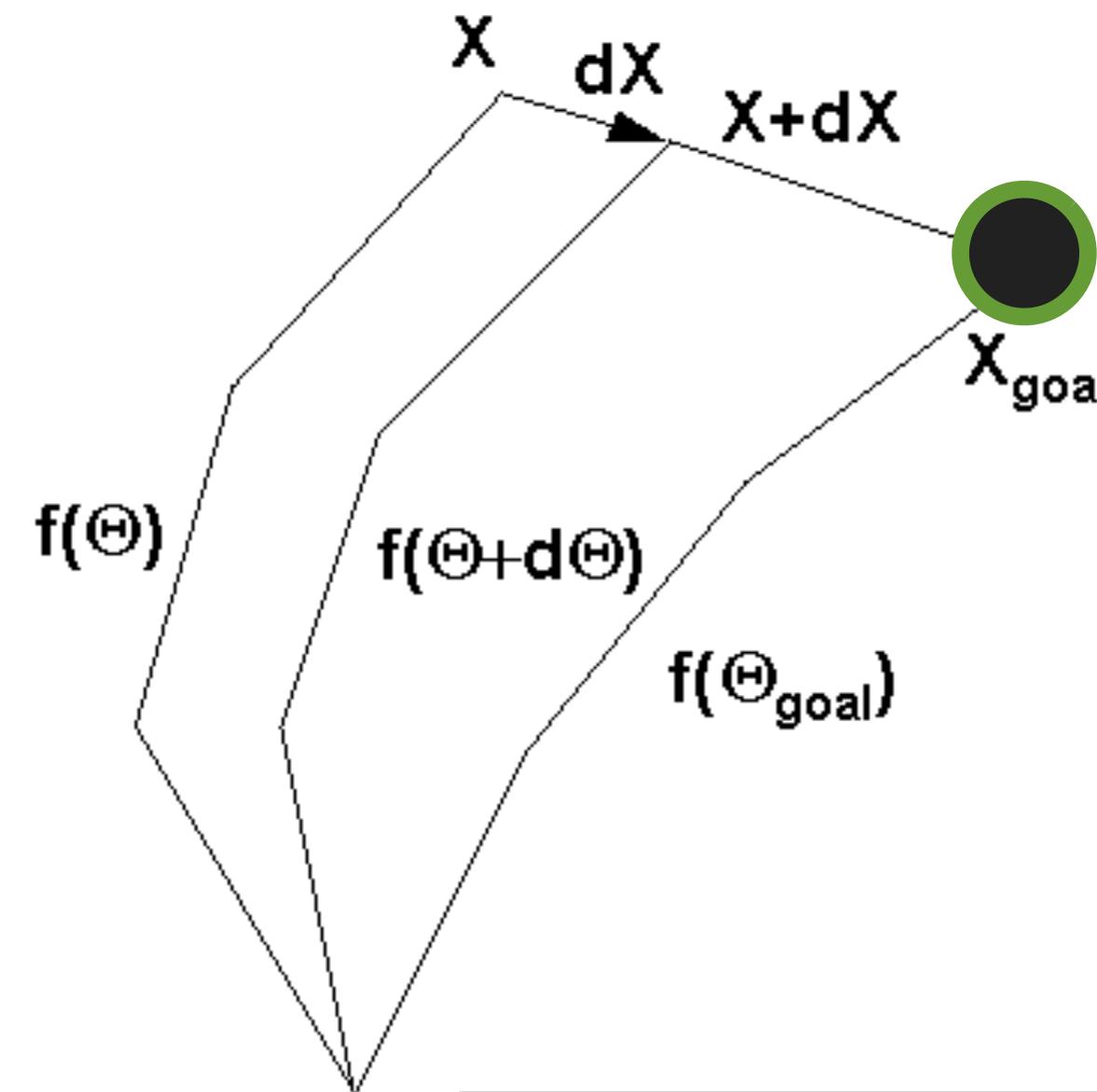
Why Closed Form?

- Advantages
- Speed: IK solution computed in constant time
- Predictability: consistency in selecting satisfying IK solution
- Disadvantage
- Generality: general form for arbitrary kinematics difficult to express



Iterative Solutions to IK

- Minimize error between current endeffector and its desired position
- Transform desired endeffector velocity into configuration space
- Repeatedly step to convergence at desired endeffector position



Inverse Kinematics: 2 possibilities

- **Closed-form solution:** geometrically infer satisfying configuration
 - *Speed:* solution often computed in constant time
 - *Predictability:* solution is selected in a consistent manner
- **Solve by optimization:** minimize error of endeffector to desired pose
 - often some form of Gradient Descent (a la Jacobian Transpose)
 - *Generality:* same solver can be used for many different robots

Next lecture: Inverse Kinematics

