



# DeepRob

Lecture 5  
Neural Networks  
University of Minnesota





# Project 1 – Reminder

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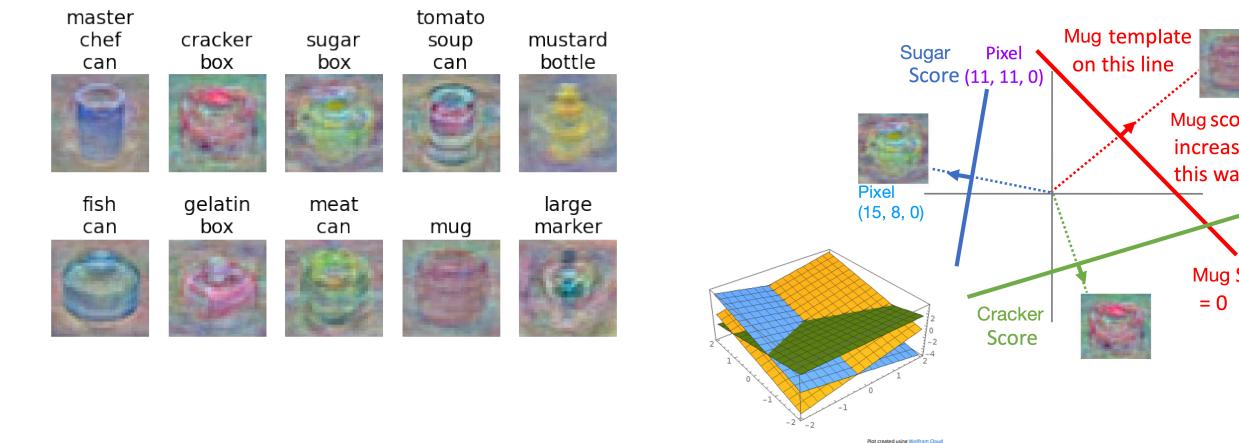
- Instructions and code available on the website
  - Here: <https://rpm-lab.github.io/CSCI5980-F24-DeepRob/projects/project1/>
- Uses Python, PyTorch and Google Colab
- Implement KNN, linear SVM, and linear softmax classifiers
- **Autograder is available!**
- **Due Monday, Sept 29th 11:59 PM CT**



# Recap from Previous Lectures

- Use **Linear Models** for image classification problems.
- Use **Loss Functions** to express preferences over different choices of weights.
- Use **Regularization** to prevent overfitting to training data.
- Use **Stochastic Gradient Descent** to minimize our loss functions and train the model.

$$s = f(x; W) = Wx$$



$$L_i = -\log\left(\frac{\exp^{s_{y_i}}}{\sum_j \exp^{s_j}}\right)$$

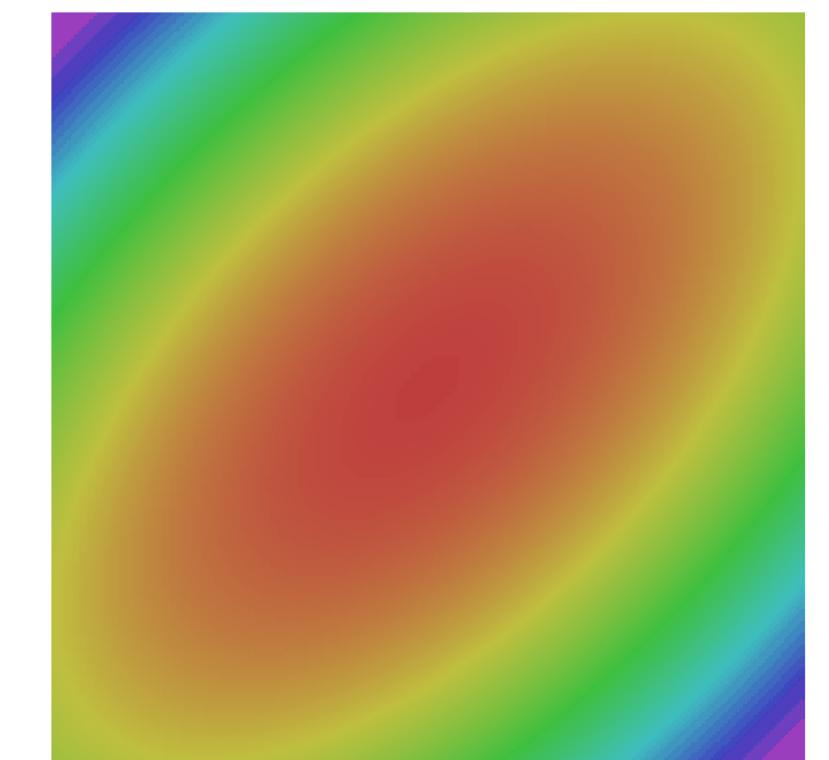
**Softmax**

$$L_i = \sum_{j \neq y_i} \max(0, s_j - s_{y_i} + 1)$$

**SVM**

$$L = \frac{1}{N} \sum_{i=1}^N L_i + R(W)$$

```
v = 0
for t in range(num_steps):
    dw = compute_gradient(w)
    v = rho * v + dw
    w -= learning_rate * v
```





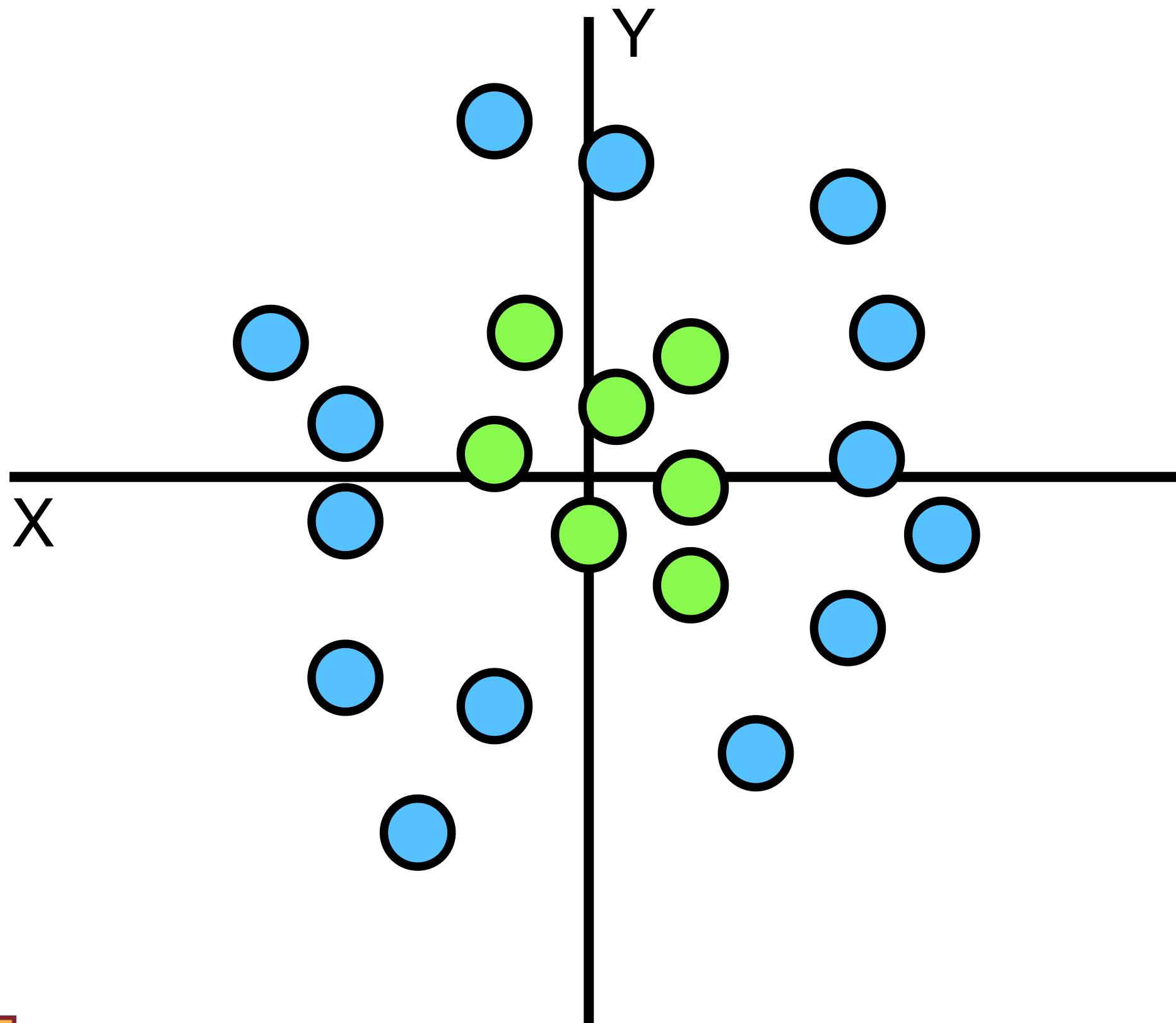
# Neural Networks



# Problem: Linear Classifiers aren't that powerful

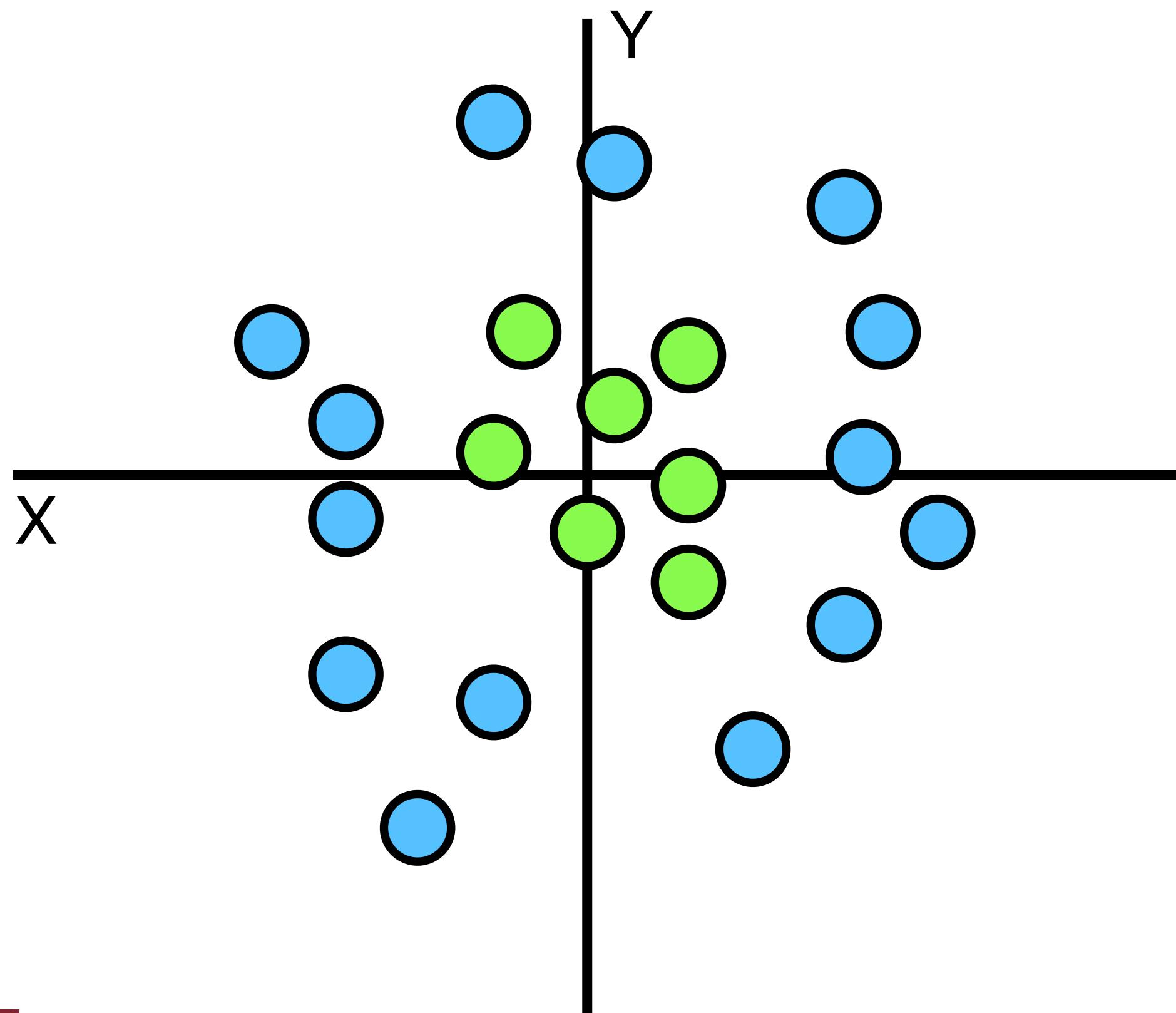
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## Geometric Viewpoint



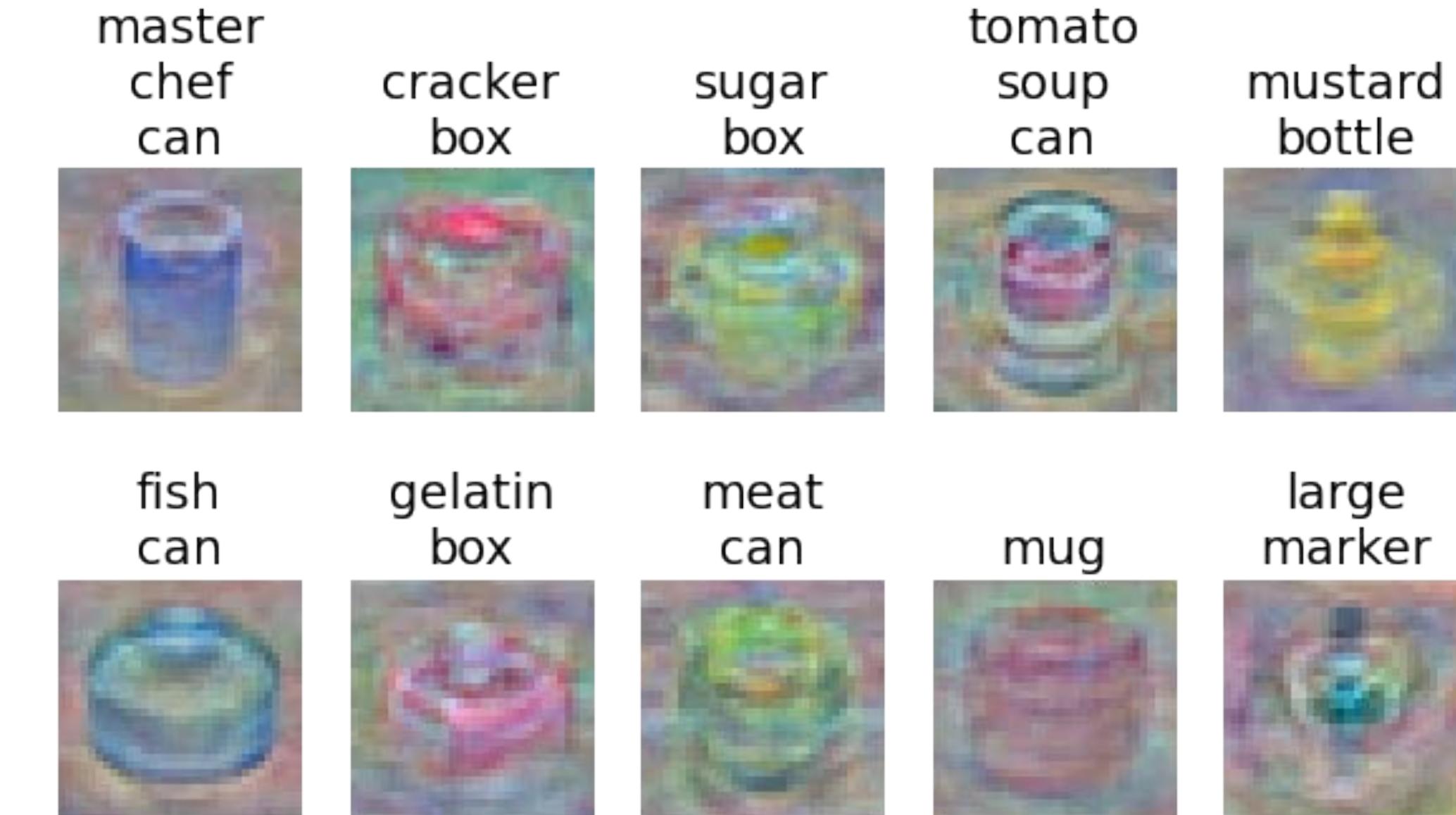
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## Geometric Viewpoint



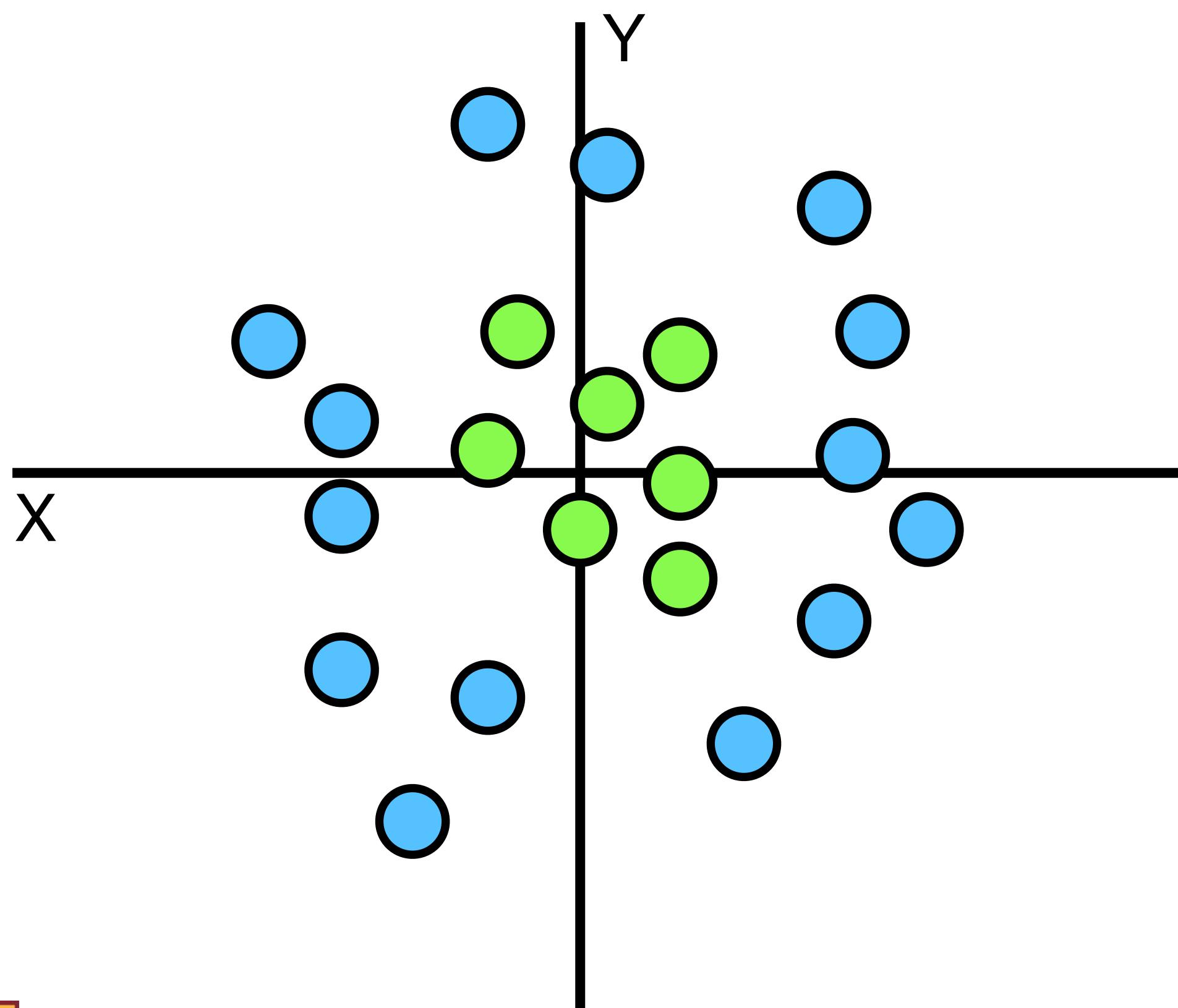
## Visual Viewpoint

One template per class:  
Can't recognize different modes of a  
class



# One solution: Feature Transforms

Original space

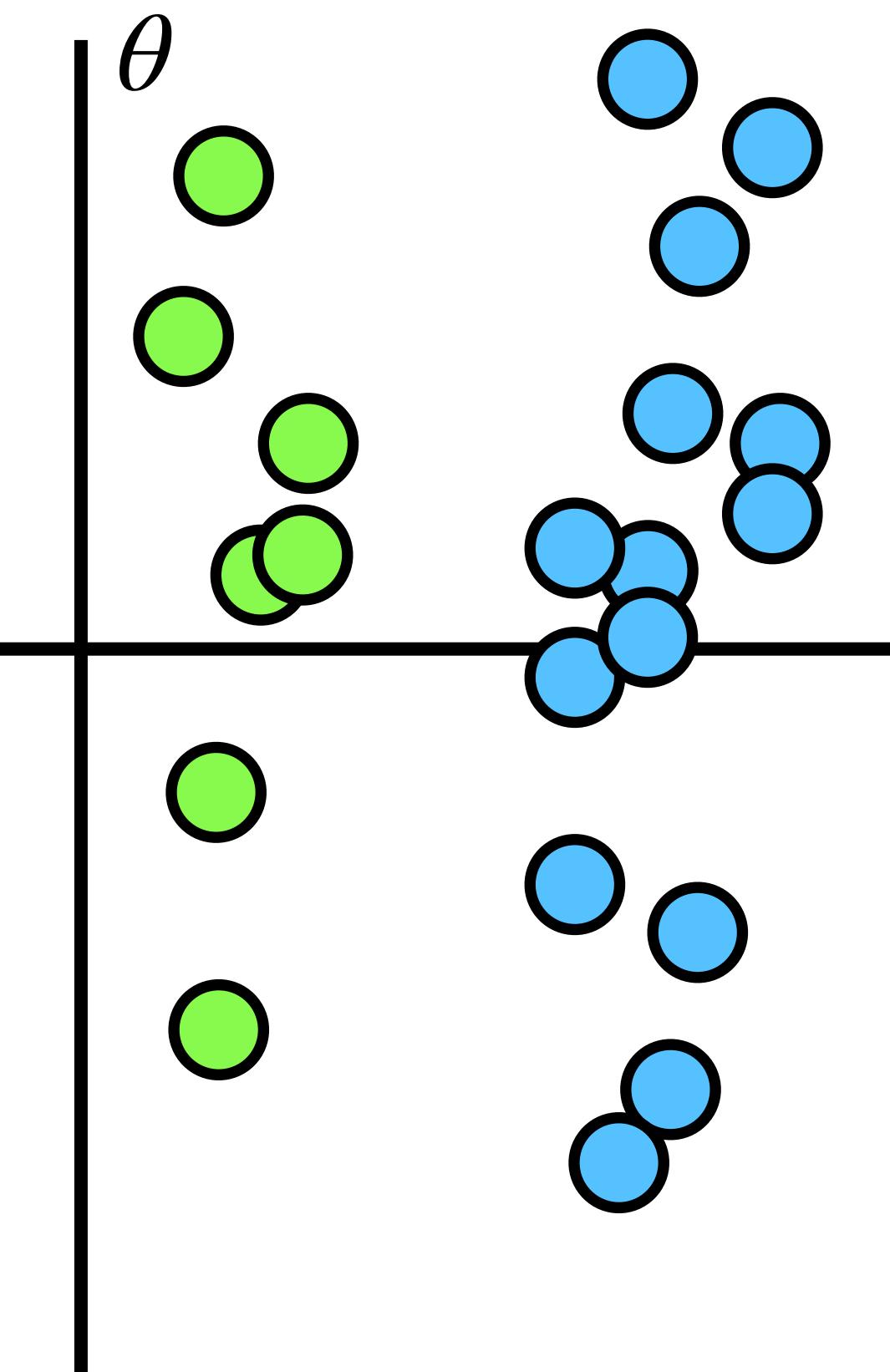


$$r = (x^2 + y^2)^{1/2}$$

$$\theta = \tan^{-1}(y/x)$$

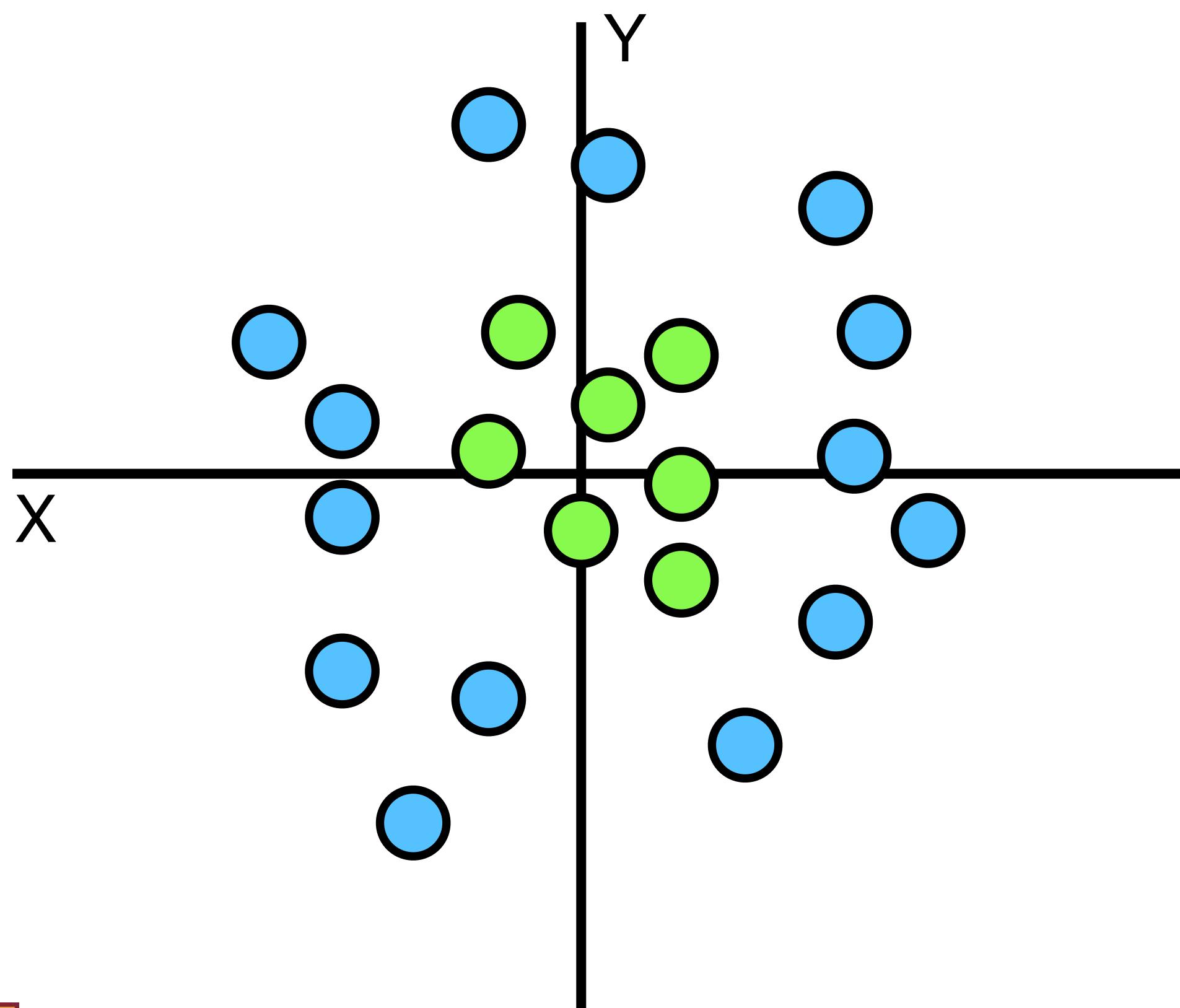
Feature  
Transform

Feature space



# One solution: Feature Transforms

Original space

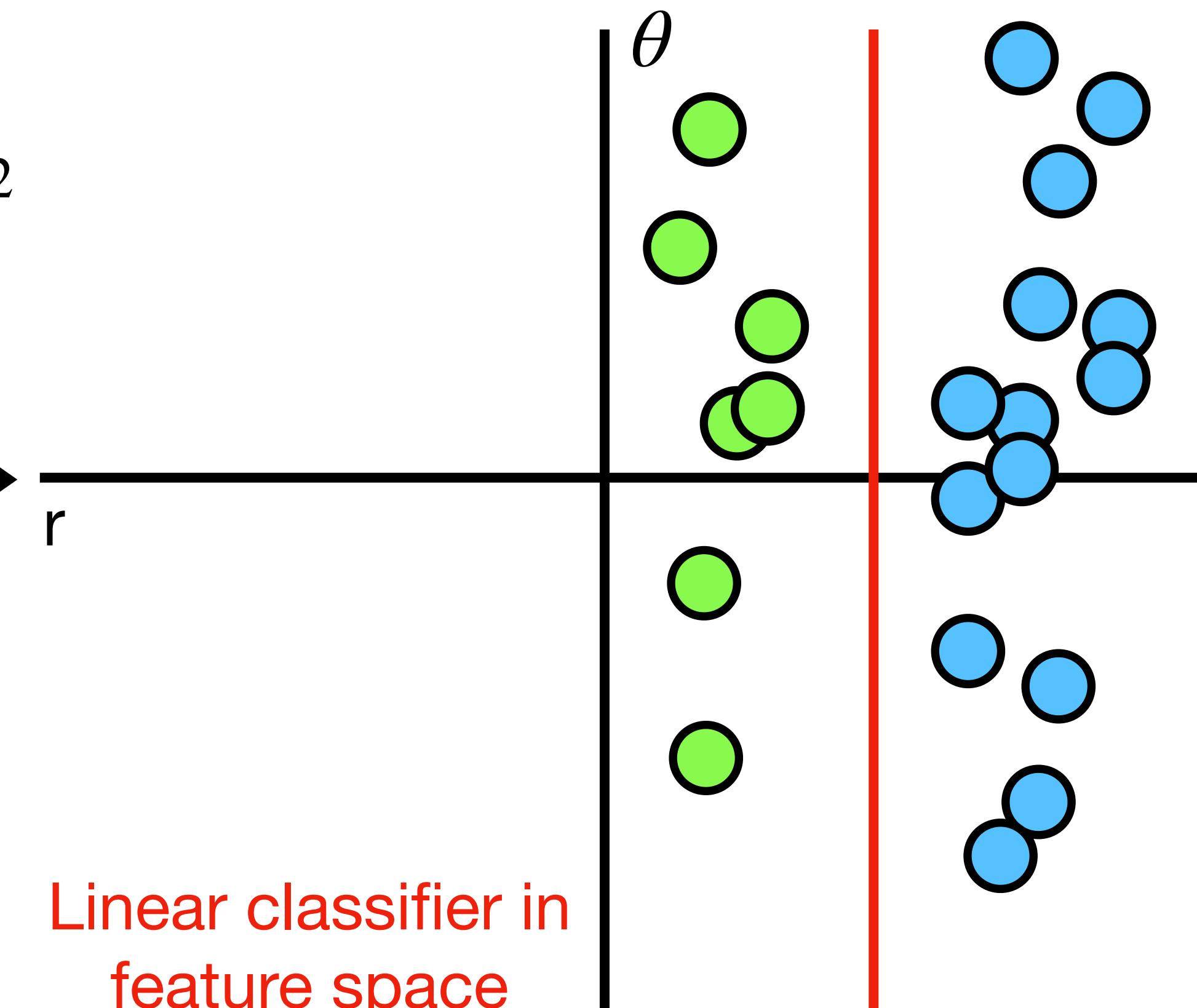


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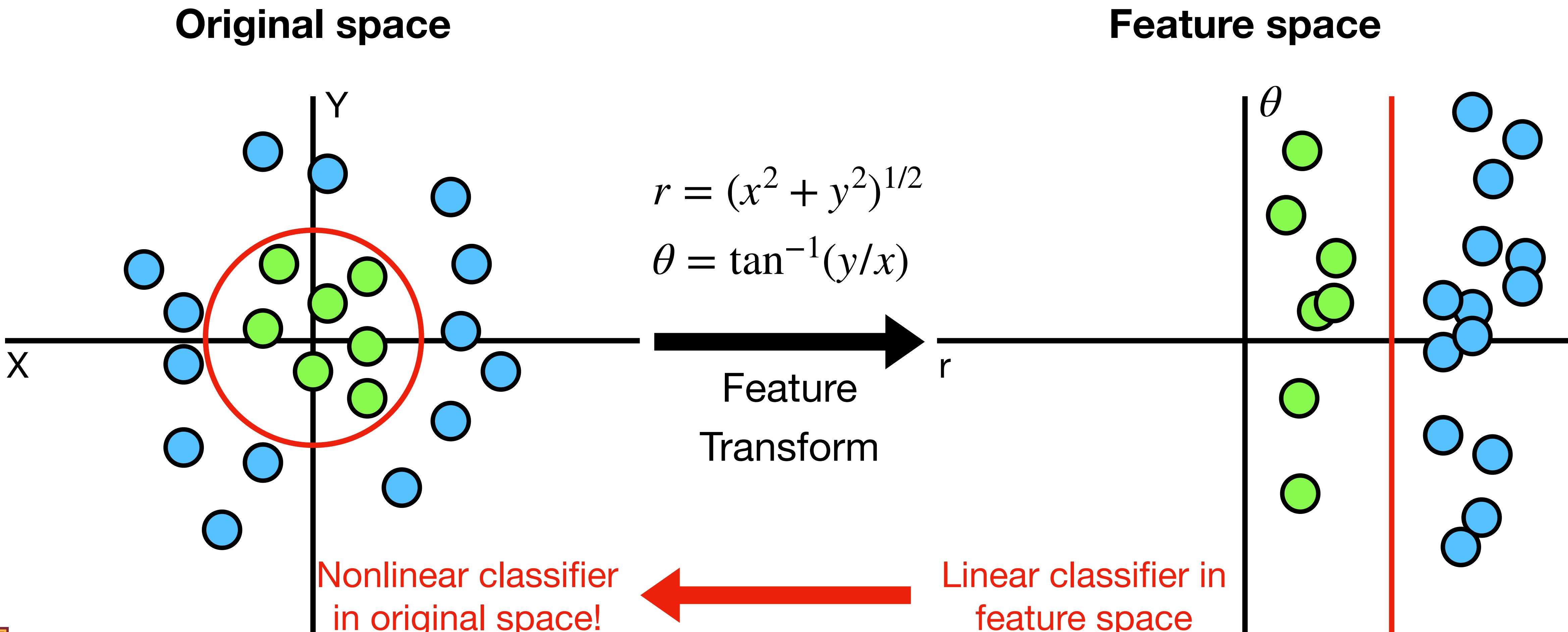
Feature  
Transform

Feature space

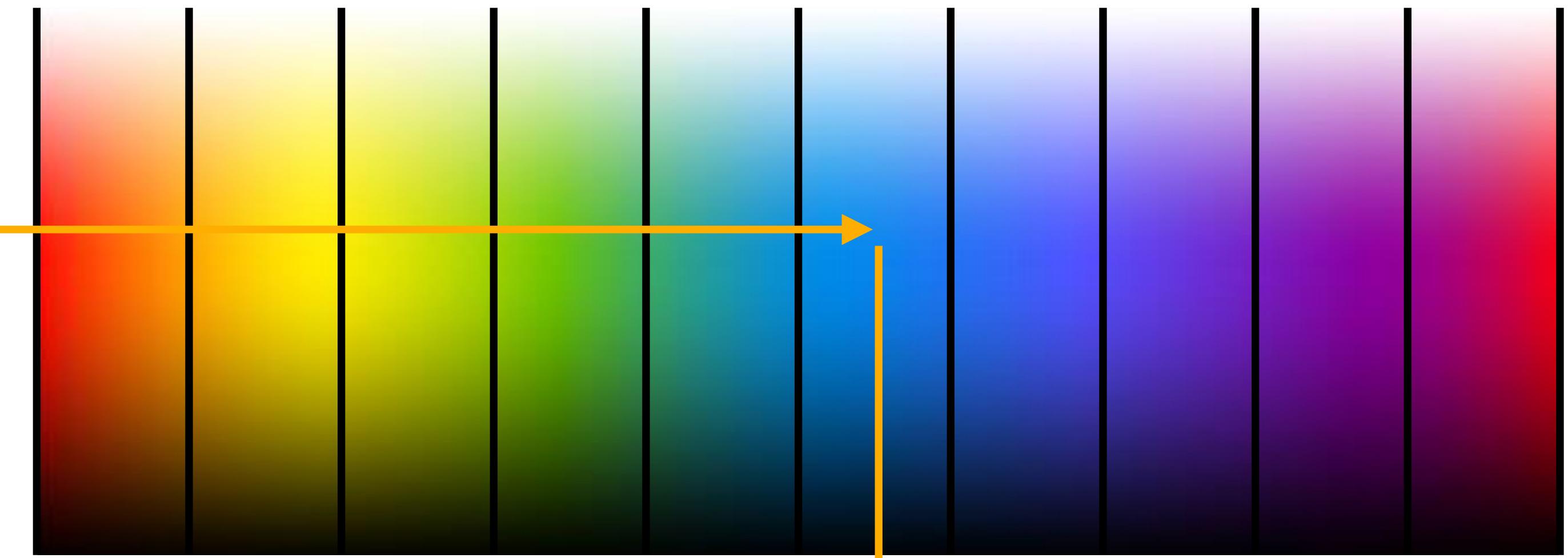


Linear classifier in  
feature space

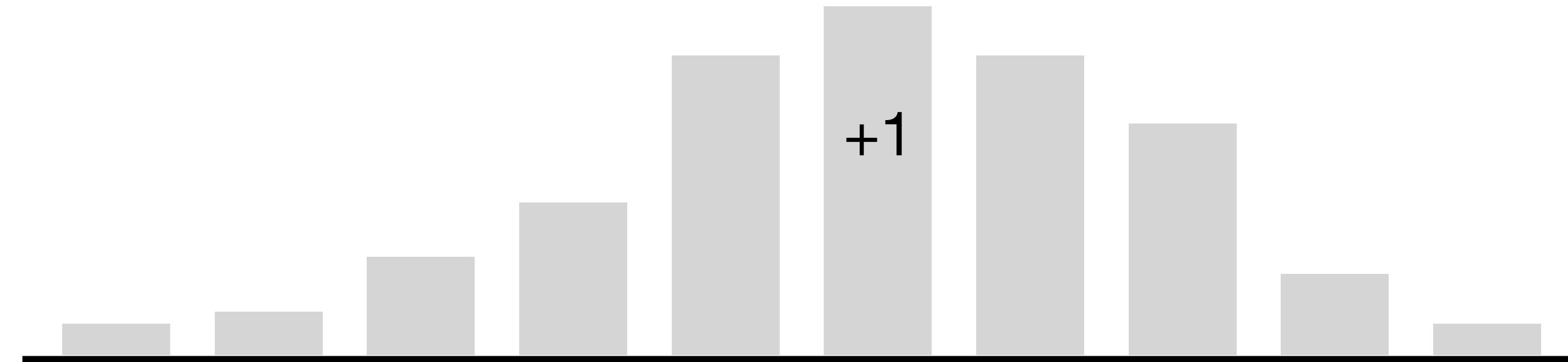
# One solution: Feature Transforms



# Image Features: Color Histogram



Ignores texture,  
spatial positions



[Frog image](#) is in the public domain

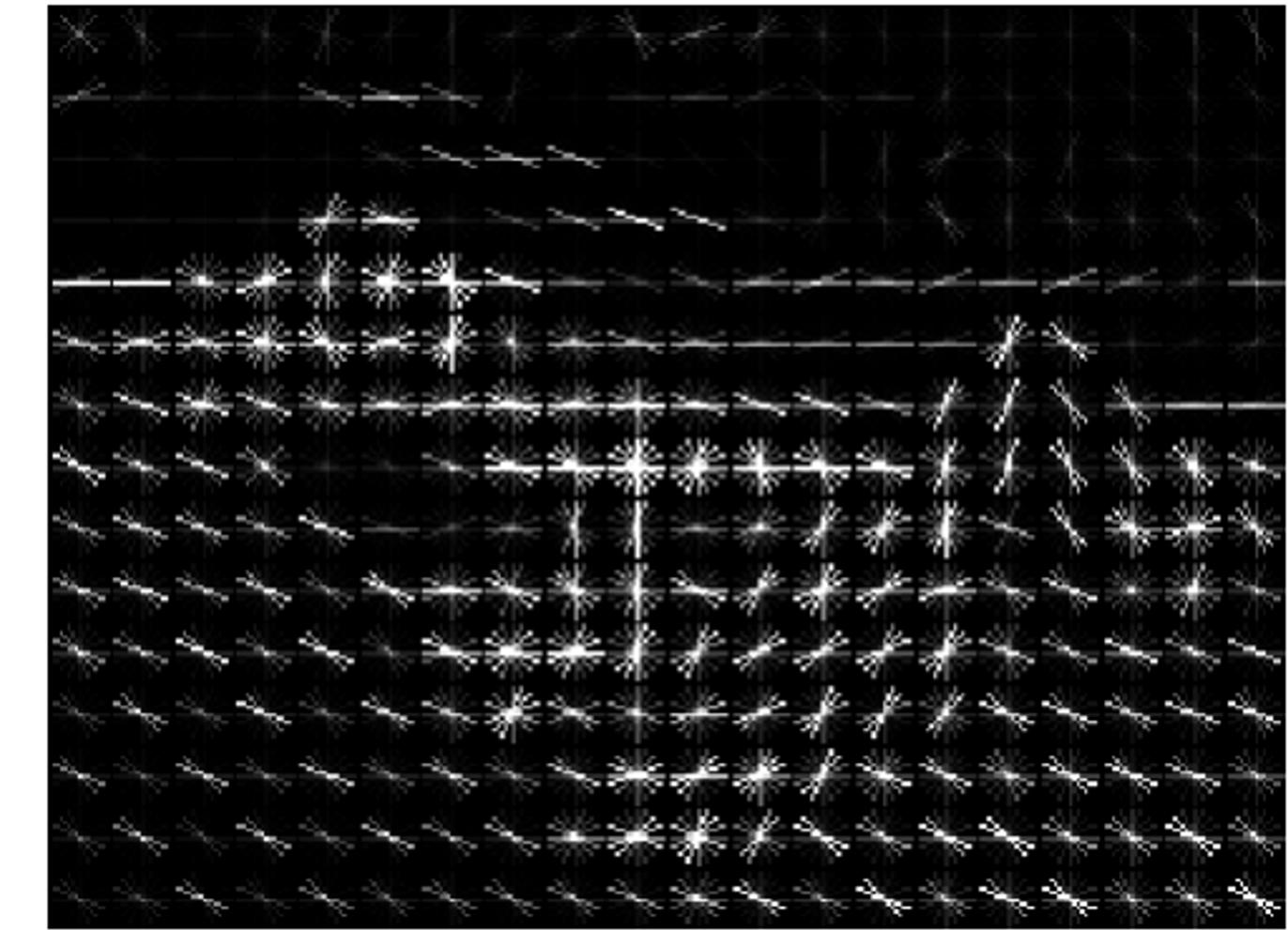
# Image Features: Histogram of Oriented Gradients (HoG)



1. Compute edge direction/  
strength at each pixel
2. Divide image into 8x8 regions
3. Within each region compute a  
histogram of edge direction  
weighted by edge strength



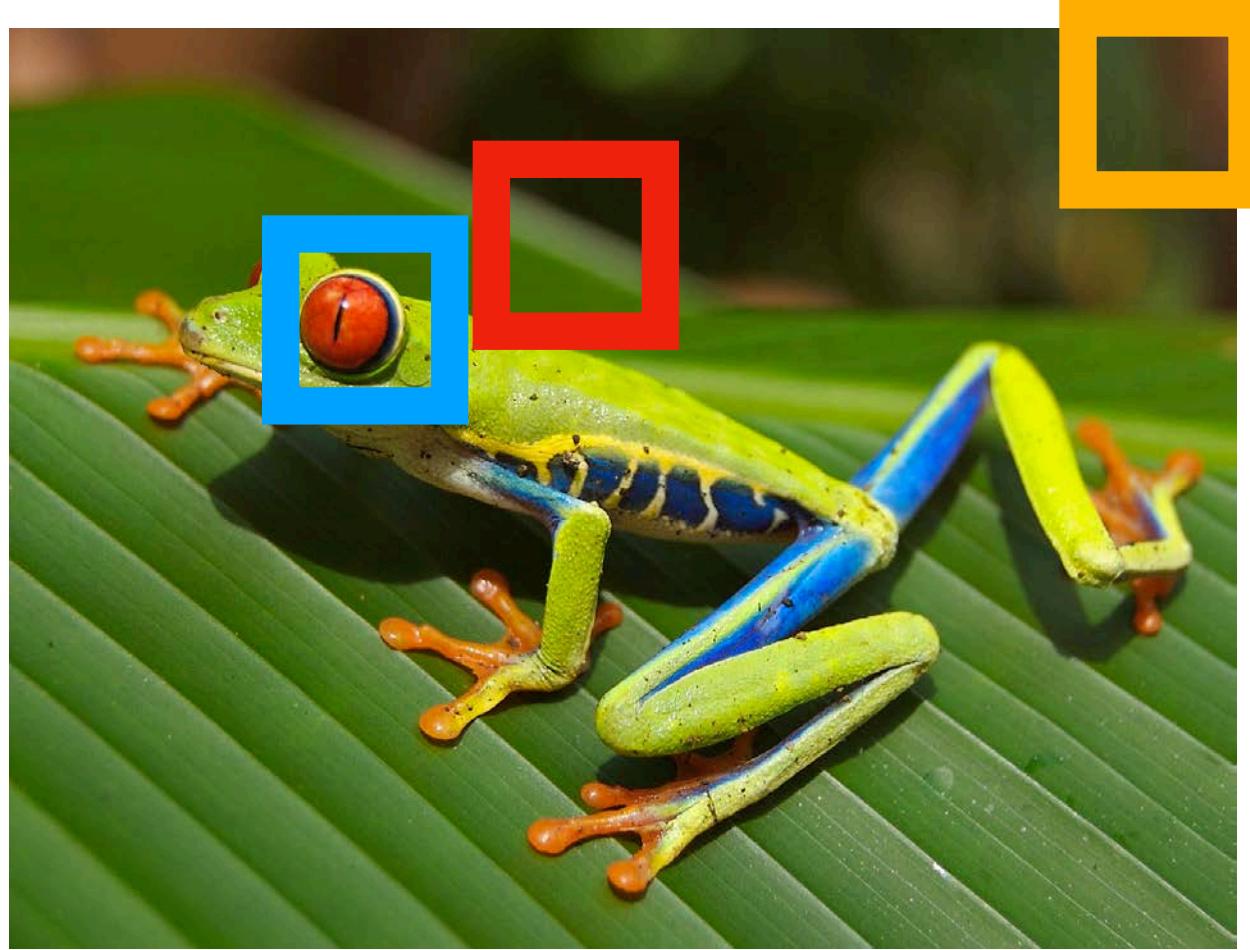
# Image Features: Histogram of Oriented Gradients (HoG)



1. Compute edge direction/strength at each pixel
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Example: 320x240 image gets divided into 40x30 bins; 9 directions per bin; feature vector has  $30 \times 40 \times 9 = 10,800$  numbers

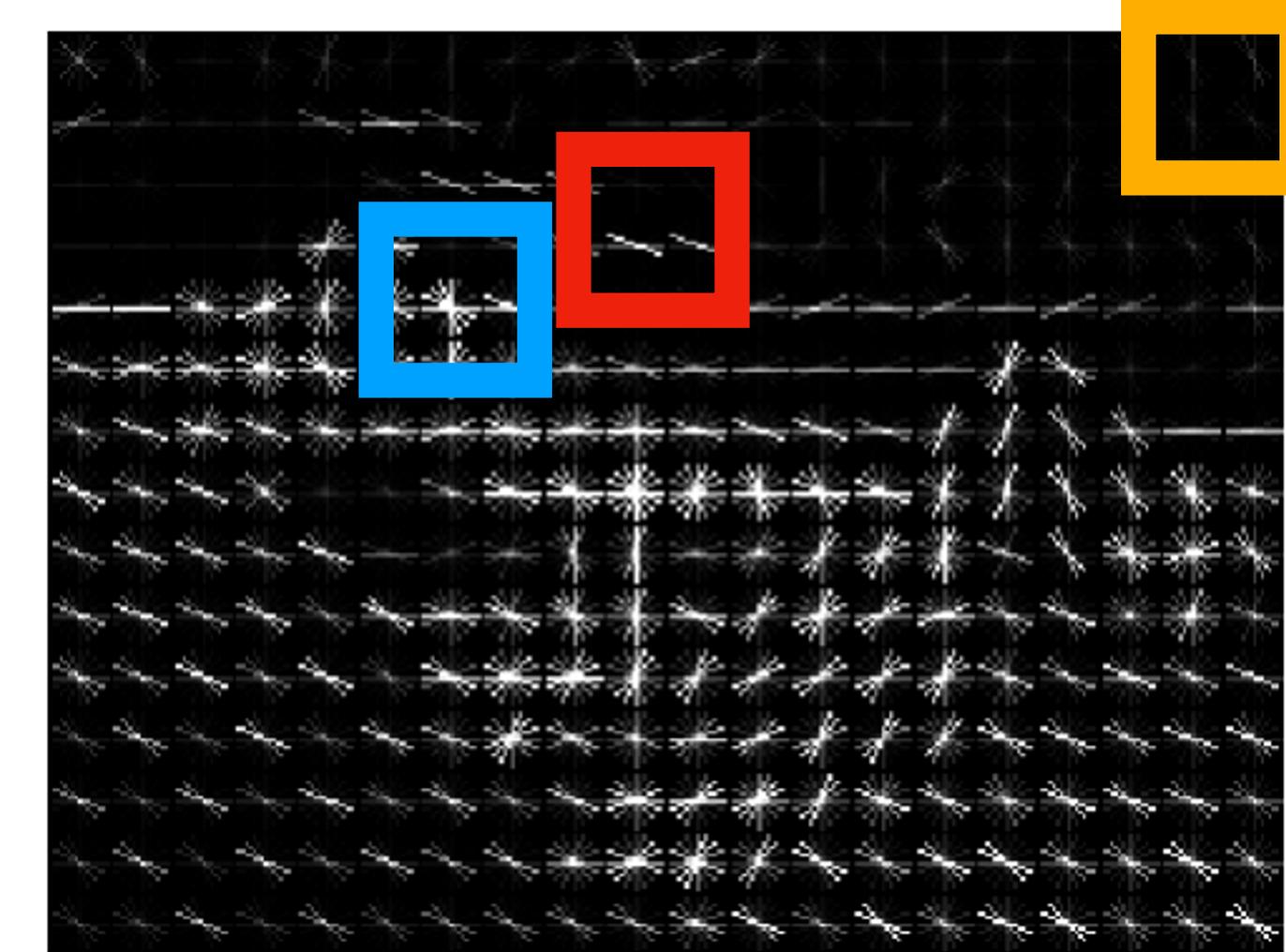
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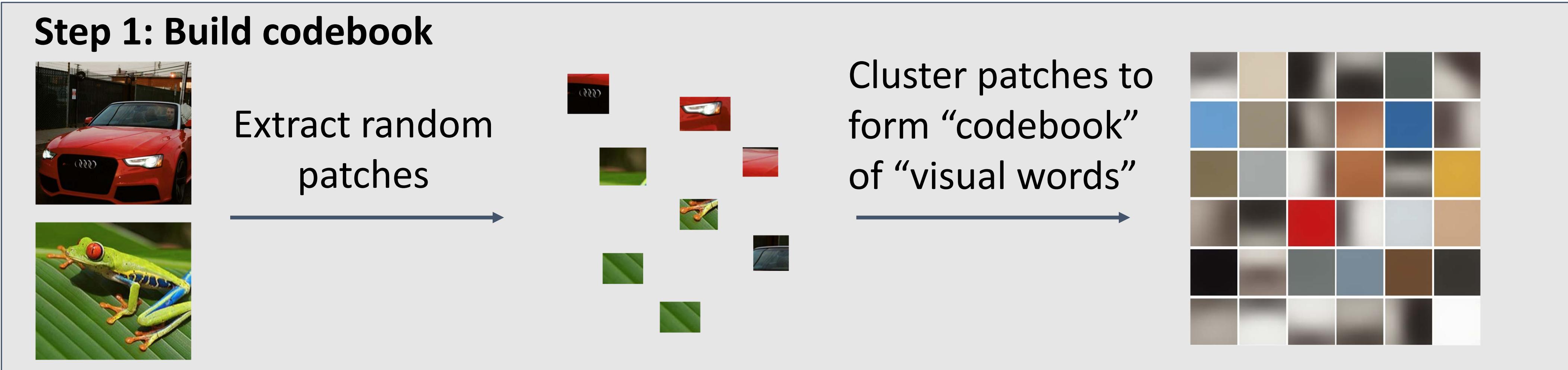
Weak edges  
Strong diagonal edges  
→  
Edges in all directions

Capture texture and position, robust to small image changes

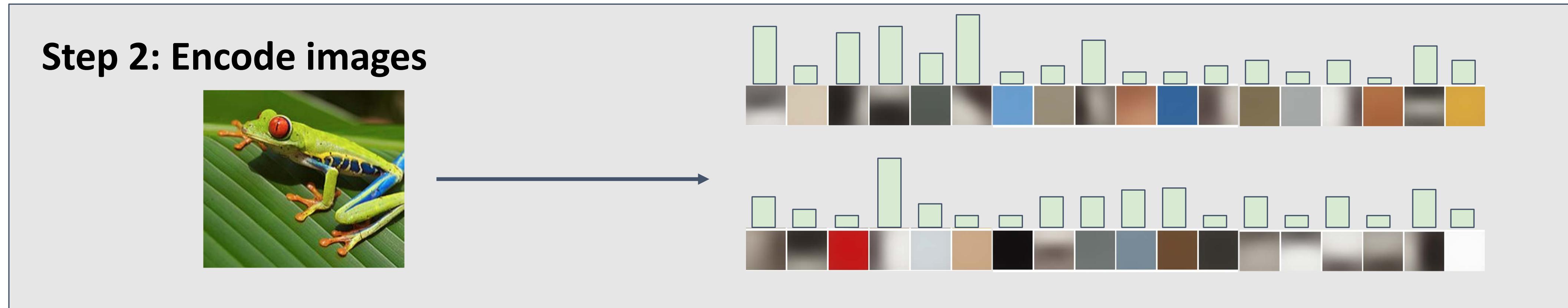
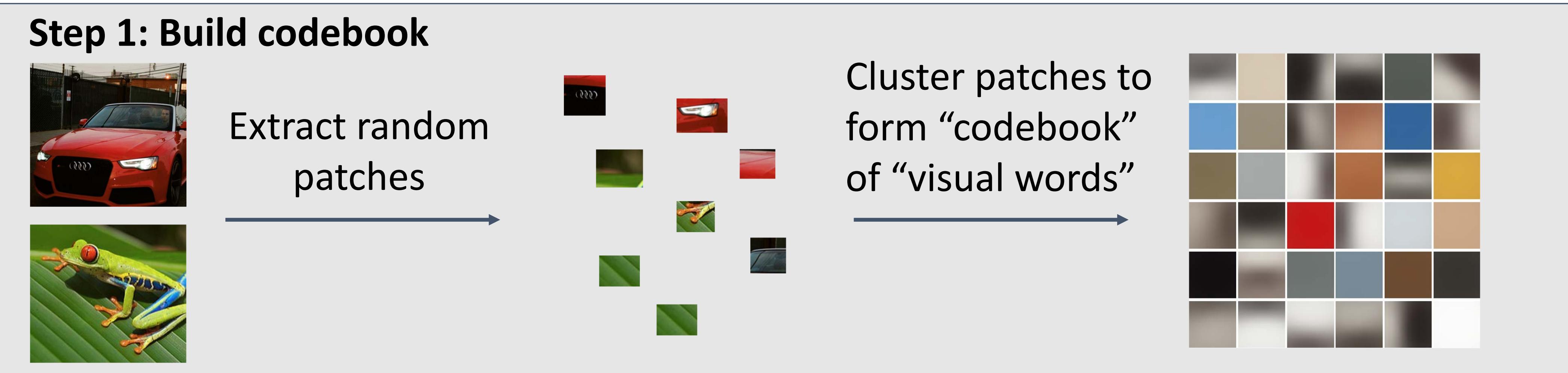


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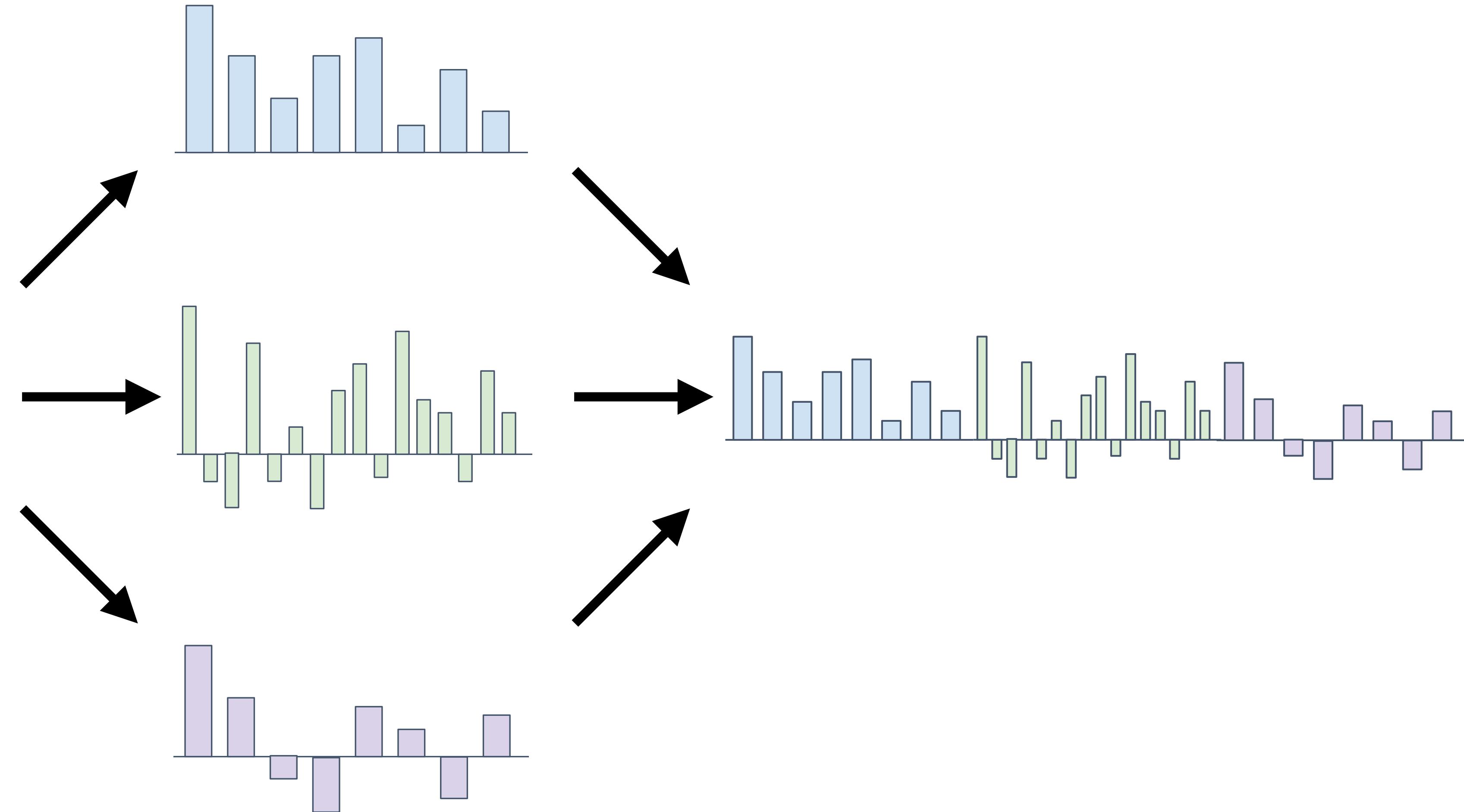
# Image Features: Bag of Words (Data-Driven!)



# Image Features: Bag of Words (Data-Driven!)



# Image Features



# Example: Winner of 2011 ImageNet Challenge

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Low-level feature extraction  $\approx$  10k patches per image

- SIFT: 128-dims
  - Color: 96-dim
- } Reduced to 64-dim with PCA

FV extraction and compression:

- $N=1024$  Gaussians,  $R=4$  regions  $\rightarrow$  520K dim x 2
- Compression:  $G=8$ ,  $b=1$  bit per dimension

One-vs-all SVM learning with SGD

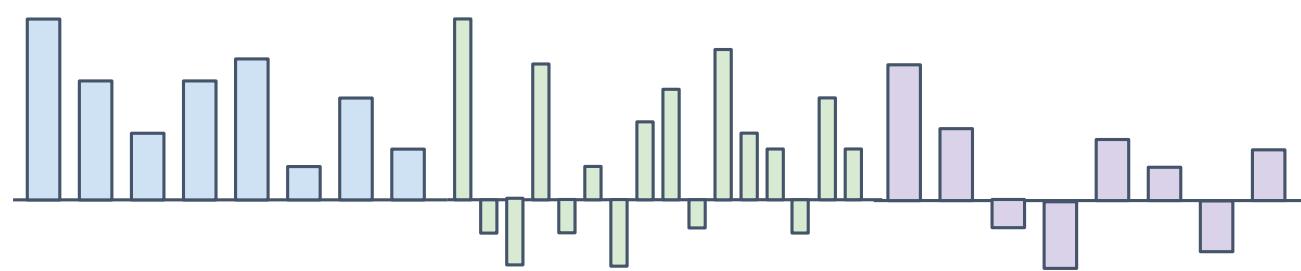
Late fusion of SIFT and color systems



# Image Features



## Feature Extraction



$f$



training

10 numbers giving  
scores for classes

# Image Features vs Neural Networks



$f$

←

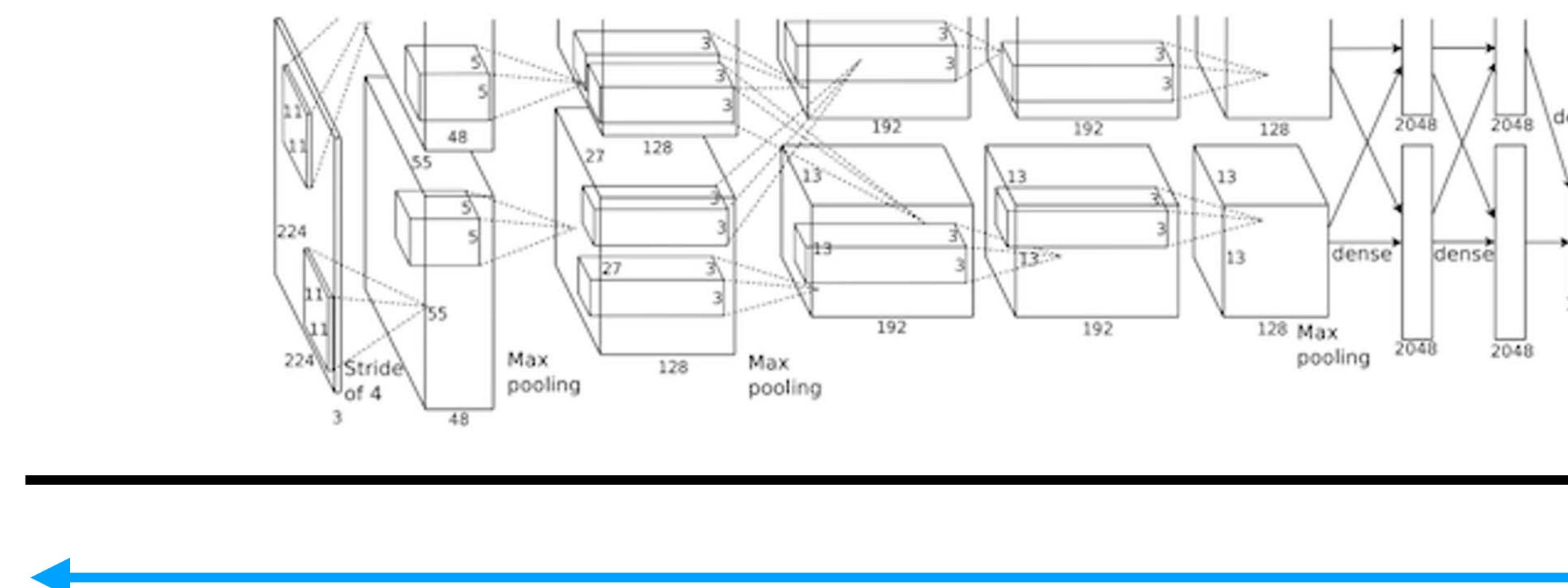
training

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# Image Features vs Neural Networks



10 numbers giving scores for classes



Krizhevsky, Sutskever, and Hinton, "Imagenet classification with deep convolutional neural networks", NIPS 2012.  
Figure copyright Krizhevsky, Sutskever, and Hinton, 2012.  
Reproduced with permission.

10 numbers giving scores for classes



# Neural Networks

---

**Input:**  $x \in \mathbb{R}^D$

**Output:**  $f(x) \in \mathbb{R}^C$

**Before:** Linear Classifier:  $f(x) = Wx + b$

Learnable parameters:  $W \in \mathbb{R}^{D \times C}, b \in \mathbb{R}^C$





# Neural Networks

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Feature Extraction

Linear Classifier

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Or Three-Layer Neural Network:

$$f(x) = W_3 \max(0, W_2 \max(0, W_1 x + b_1) + b_2) + b_3$$



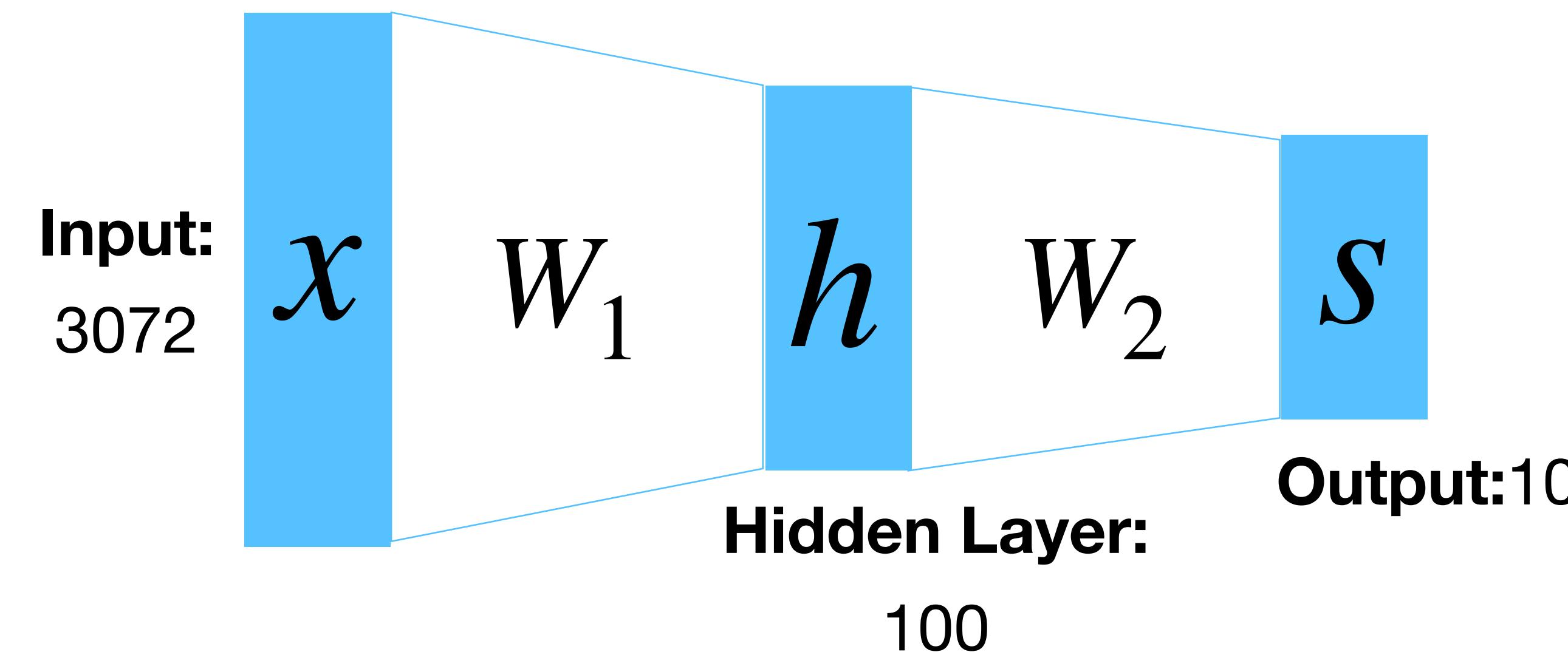
# Neural Networks

**Before:** Linear Classifier:

$$f(x) = Wx + b$$

**Now:** Two-Layer Neural Network:

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$$x \in \mathbb{R}^D, W_1 \in \mathbb{R}^{H \times D}, W_2 \in \mathbb{R}^{C \times H}$$

# Neural Networks

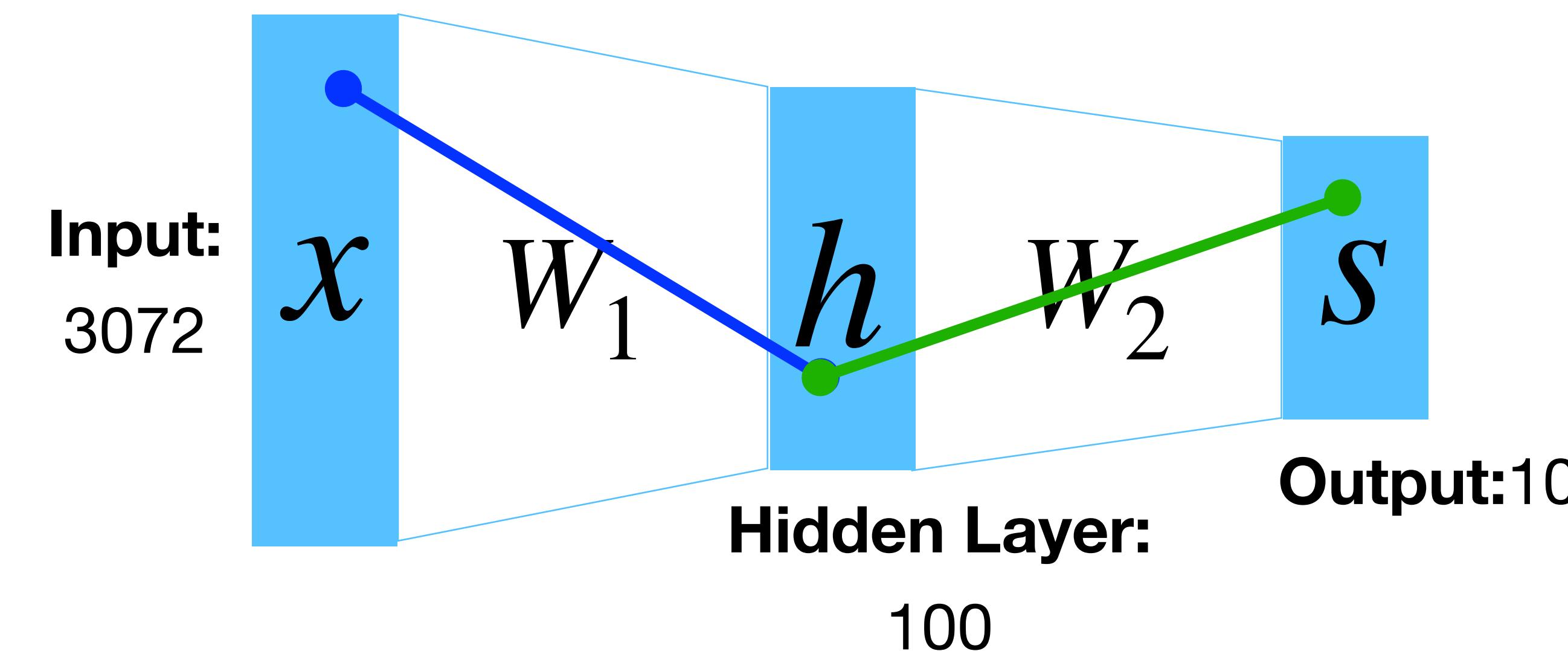
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**Now:** Two-Layer Neural Network:

$$f(x) = W_2 \max(0, W_1 x + b_1) + b_2$$

Element  $(i, j)$  of  $W_1$  gives the effect on  $h_i$  from  $x_j$



Element  $(i, j)$  of  $W_2$  gives the effect on  $s_i$  from  $h_j$

$$x \in \mathbb{R}^D, W_1 \in \mathbb{R}^{H \times D}, W_2 \in \mathbb{R}^{C \times H}$$

# Neural Networks

**Before:** Linear Classifier:

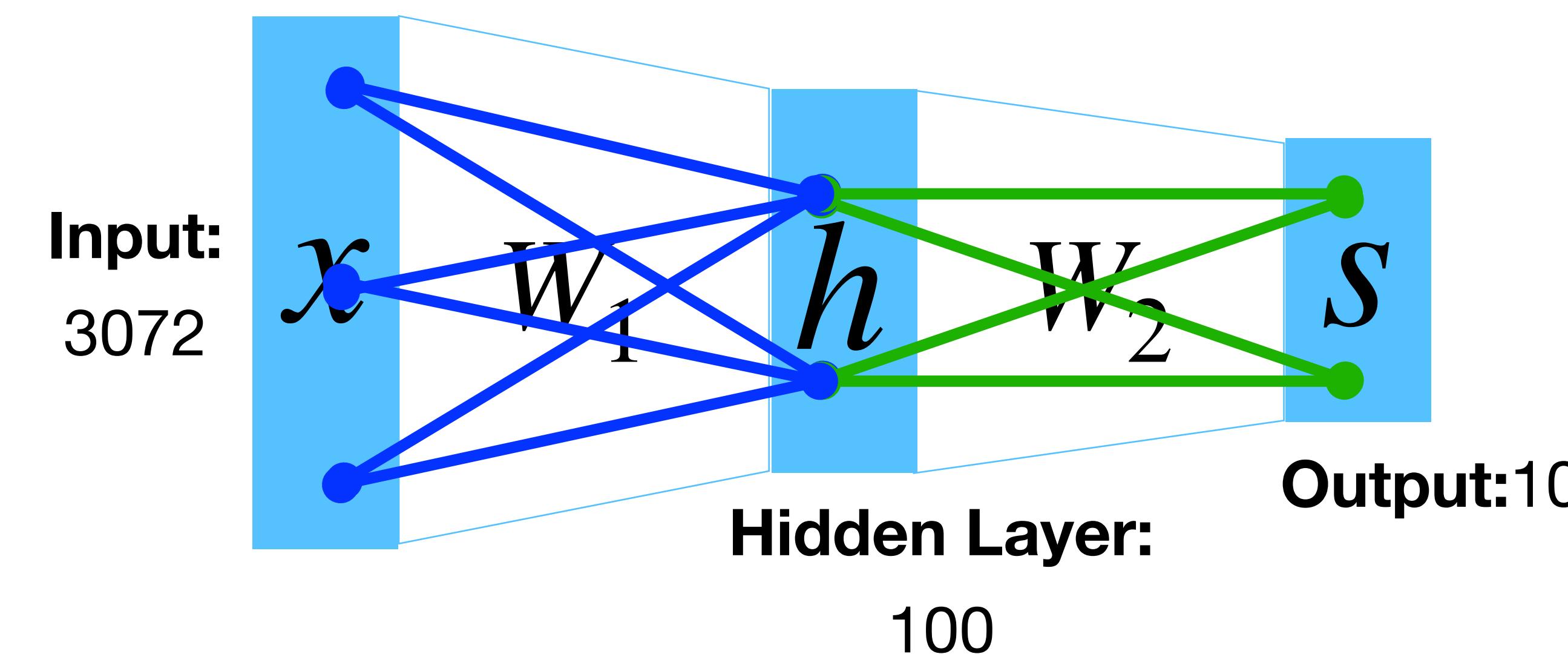
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All elements of  $x$  affect all elements of  $h$



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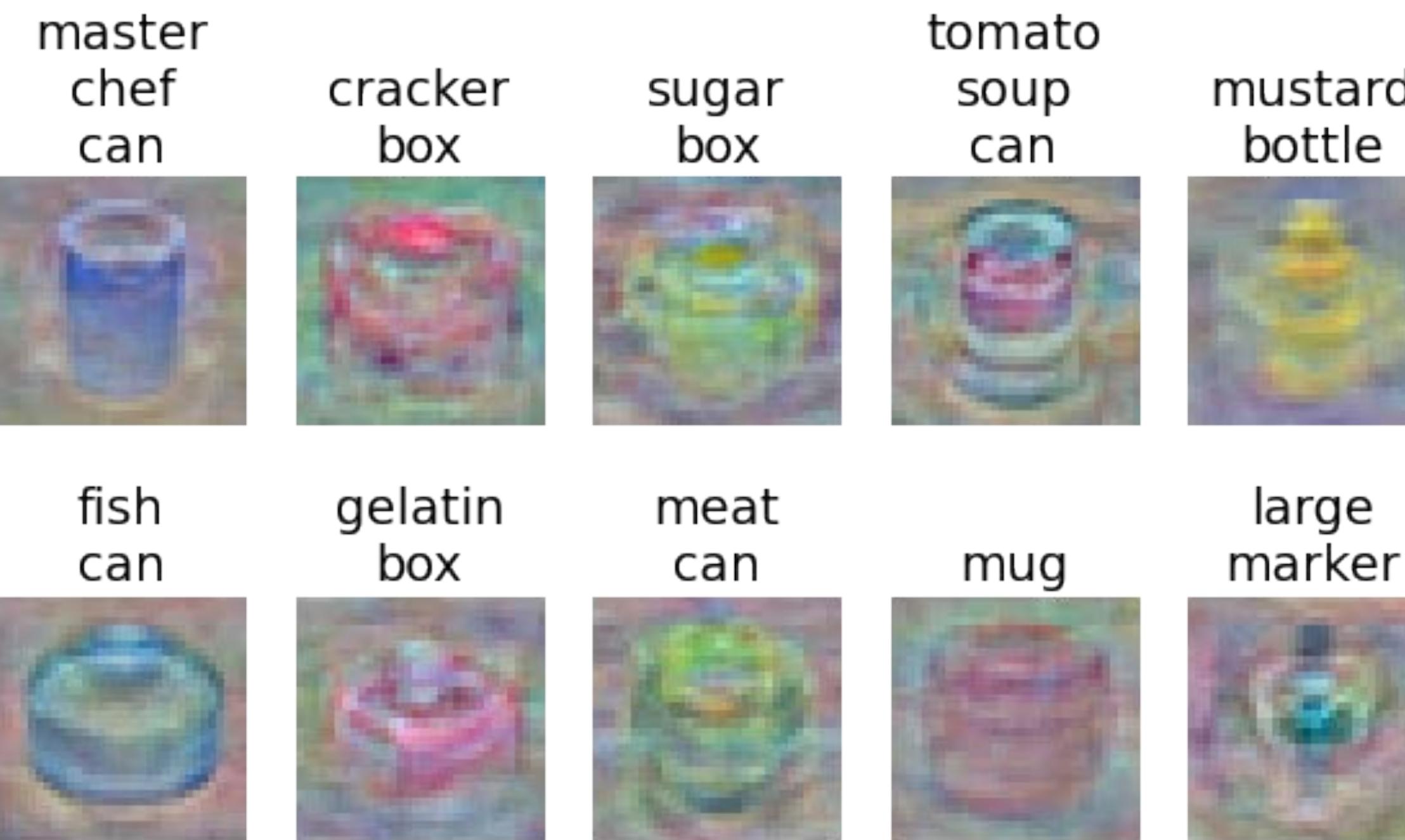
All elements of  $h$  affect all elements of  $s$

Fully-connected neural network also  
“Multi-Layer Perceptron” (MLP)

# Neural Networks

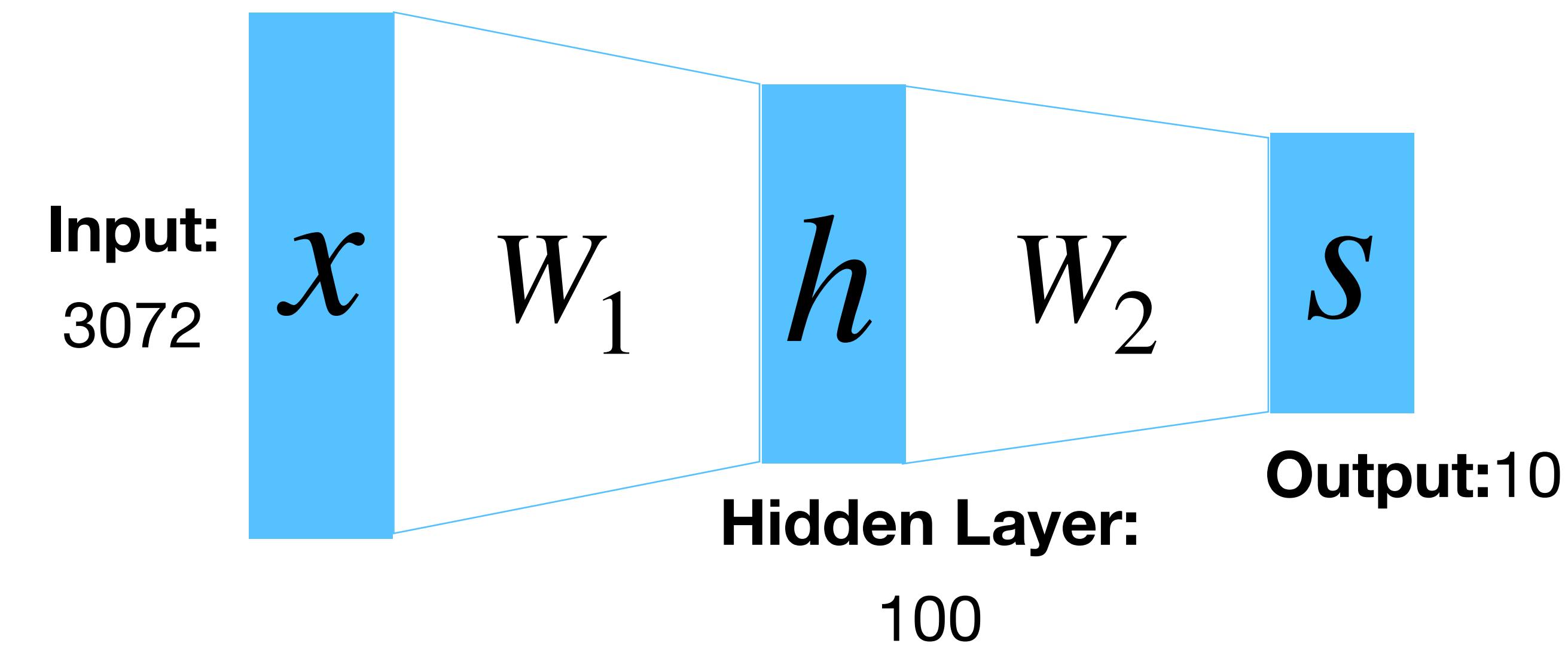
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Linear classifier: One template per class



**Before:** Linear score function

**Now:** Two-Layer Neural Network:



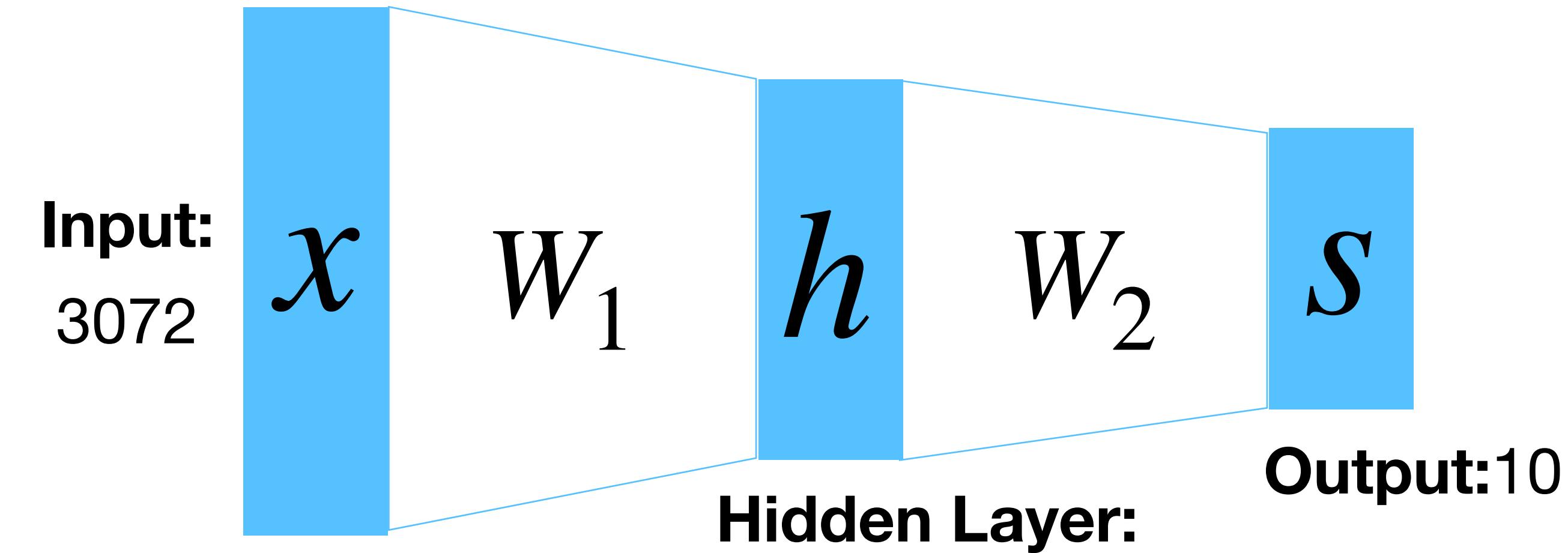
# Neural Networks

Neural net: first layer is bank of templates;  
Second layer recombines templates



**Before:** Linear score function

**Now:** Two-Layer Neural Network:



$$x \in \mathbb{R}^D, W_1 \in \mathbb{R}^{H \times D}, W_2 \in \mathbb{R}^{C \times H}$$

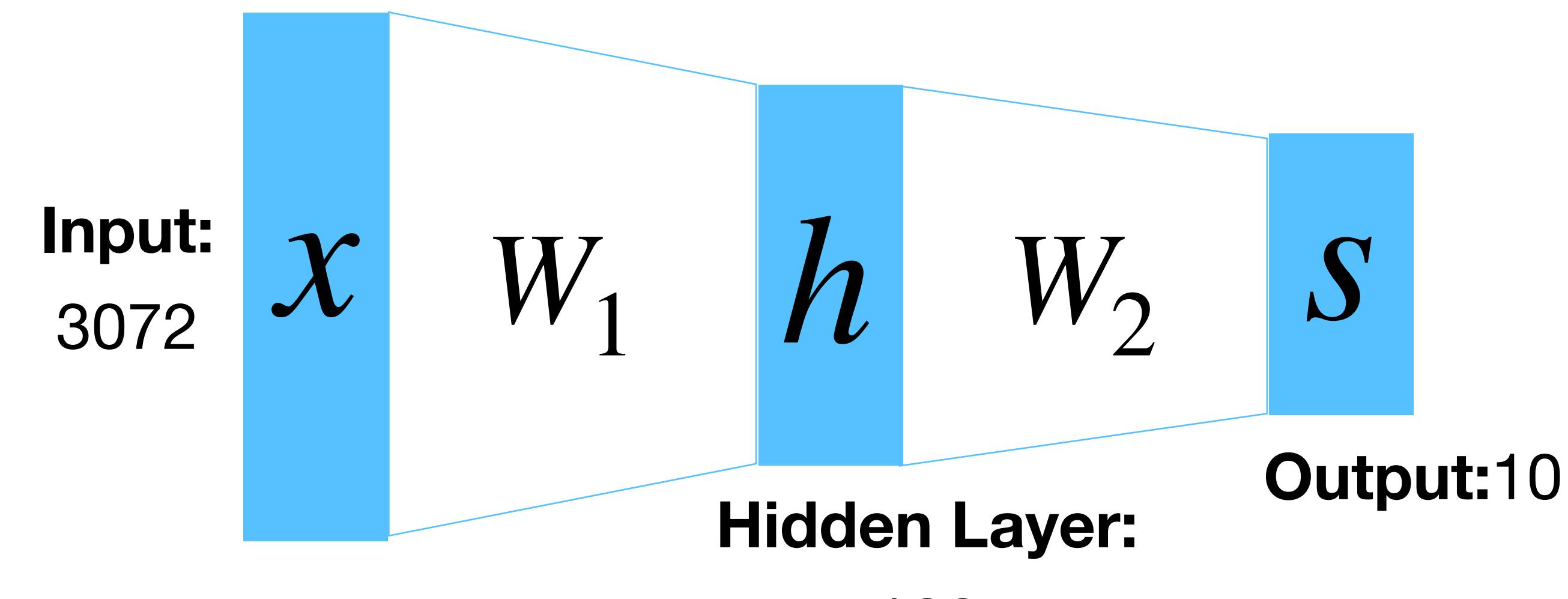
# Neural Networks

Can use different templates to cover multiple modes of a class!



**Before:** Linear score function

**Now:** Two-Layer Neural Network:



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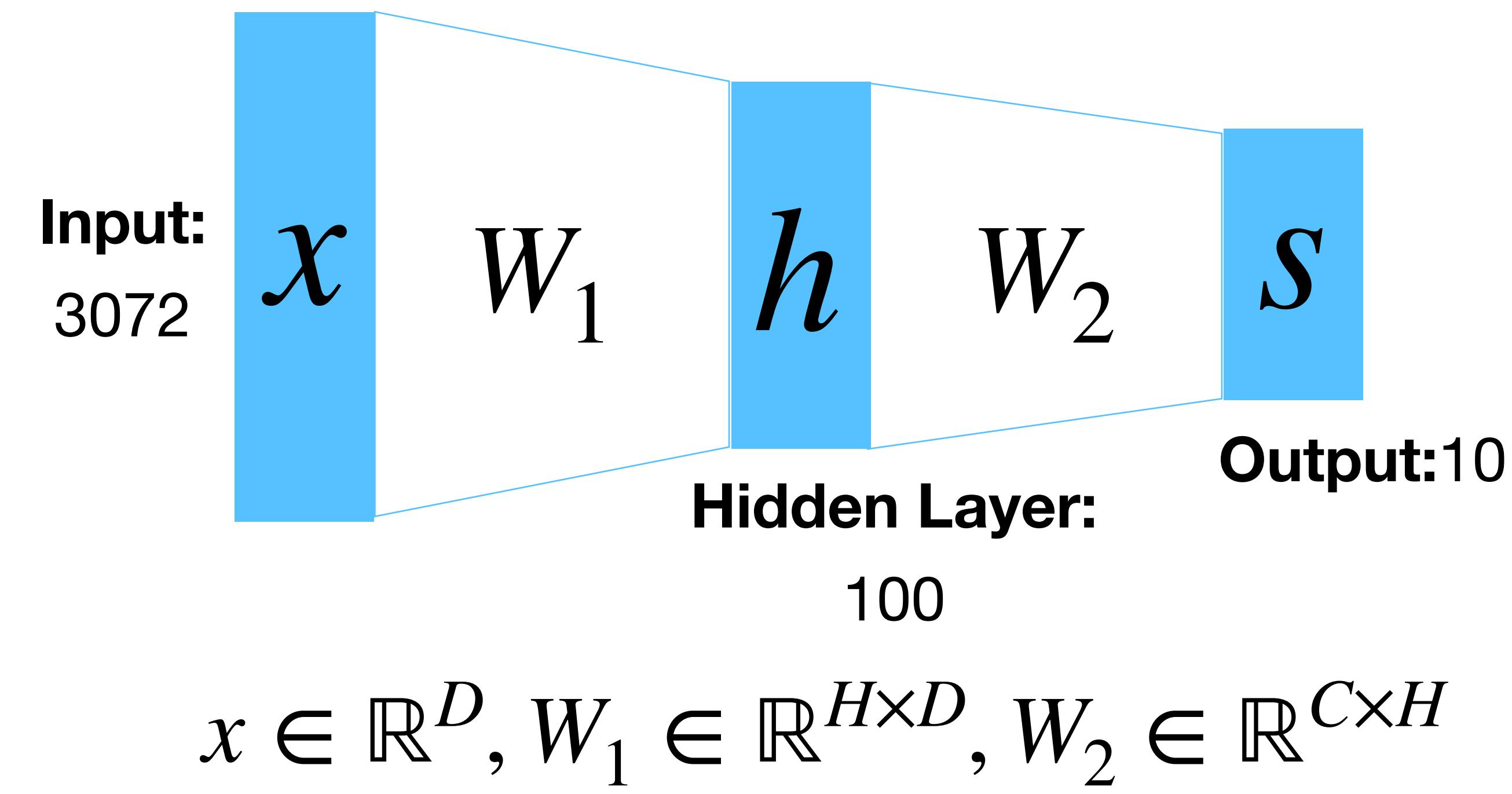
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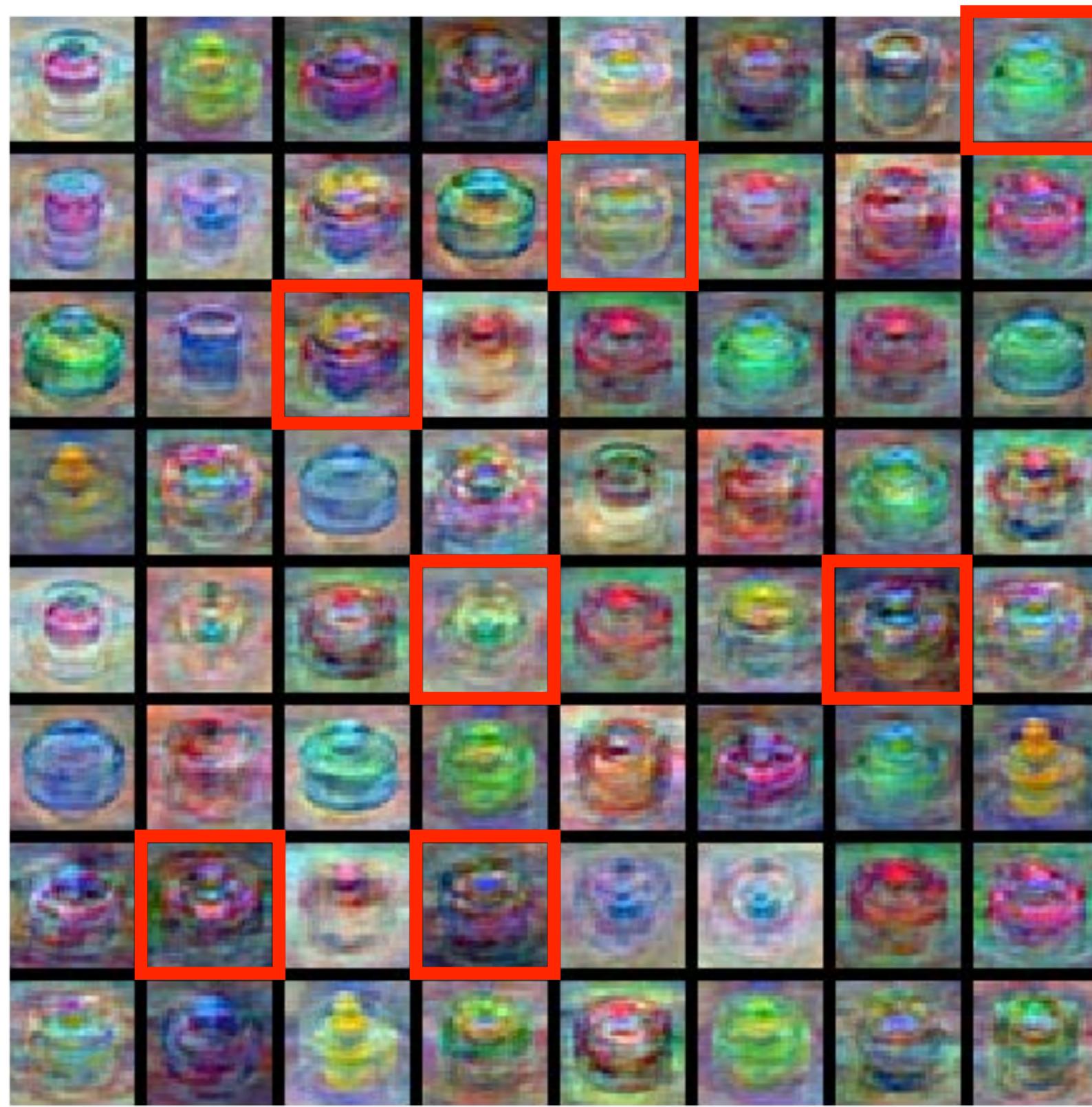
**Before:** Linear score function

**Now:** Two-Layer Neural Network:



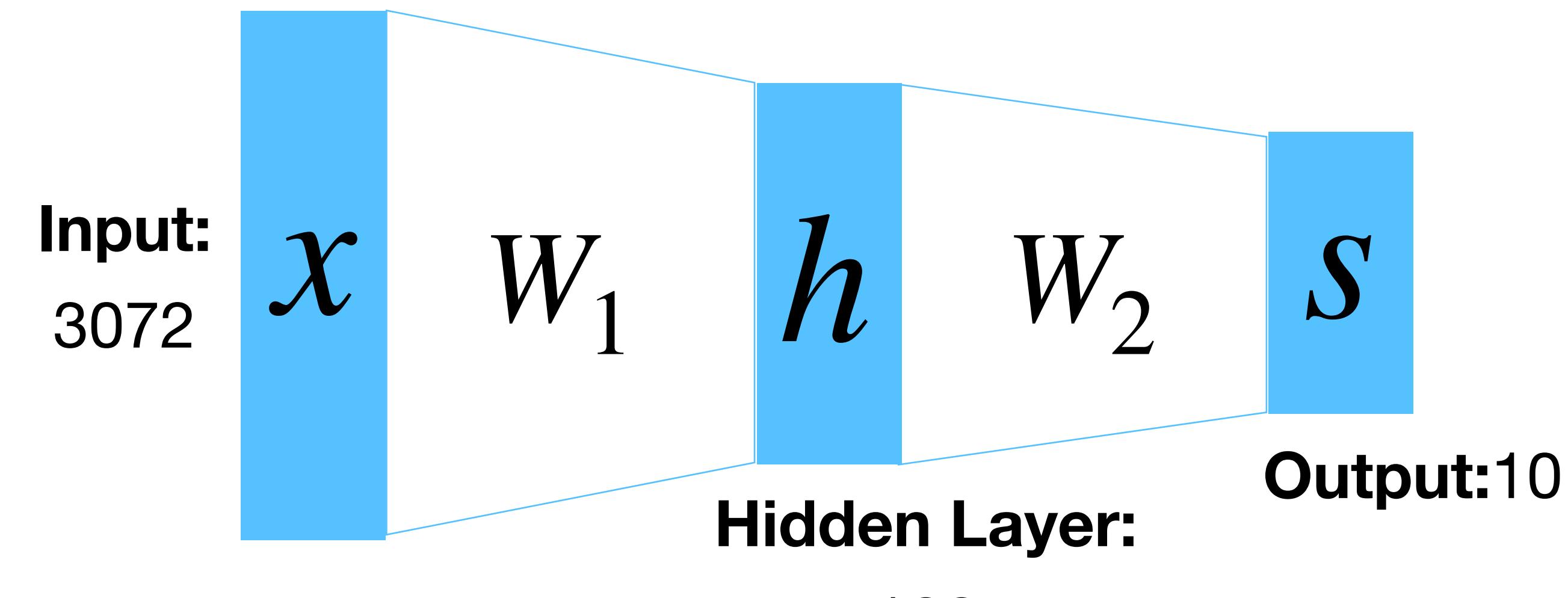
# Neural Networks

“Distributed representation”: Most templates not interpretable!



**Before:** Linear score function

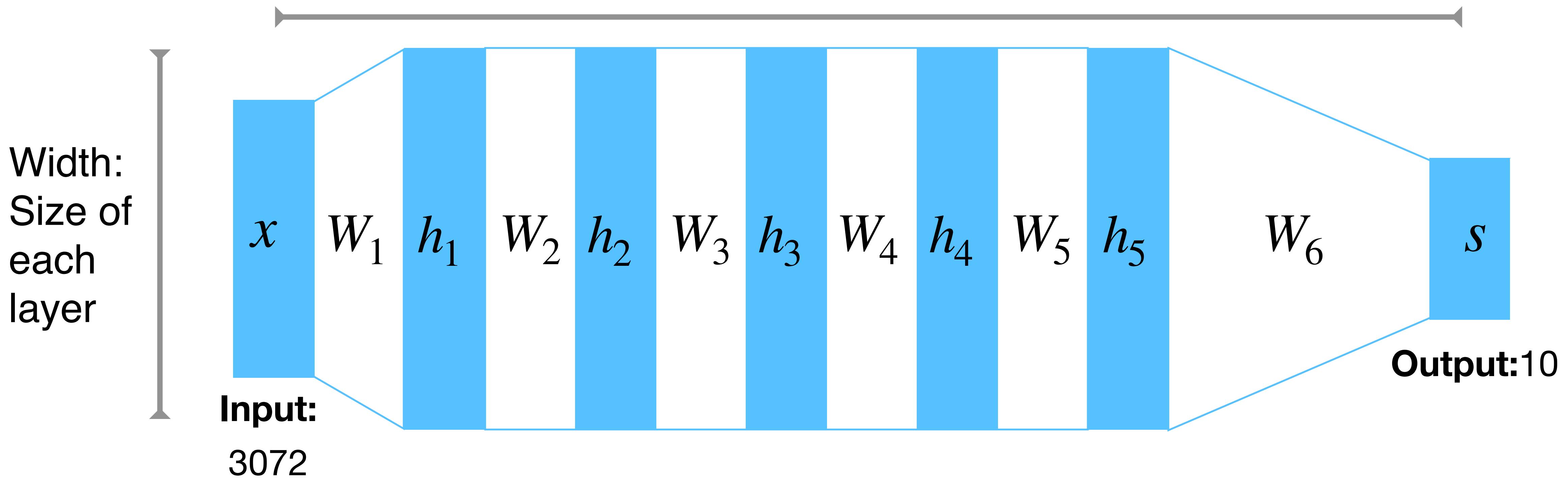
**Now:** Two-Layer Neural Network:



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# Deep Neural Networks

Depth = number of layers



$$s = W_6 \max(0, W_5 \max(0, W_4 \max(0, W_3 \max(0, W_2 \max(0, W_1 x)))))$$

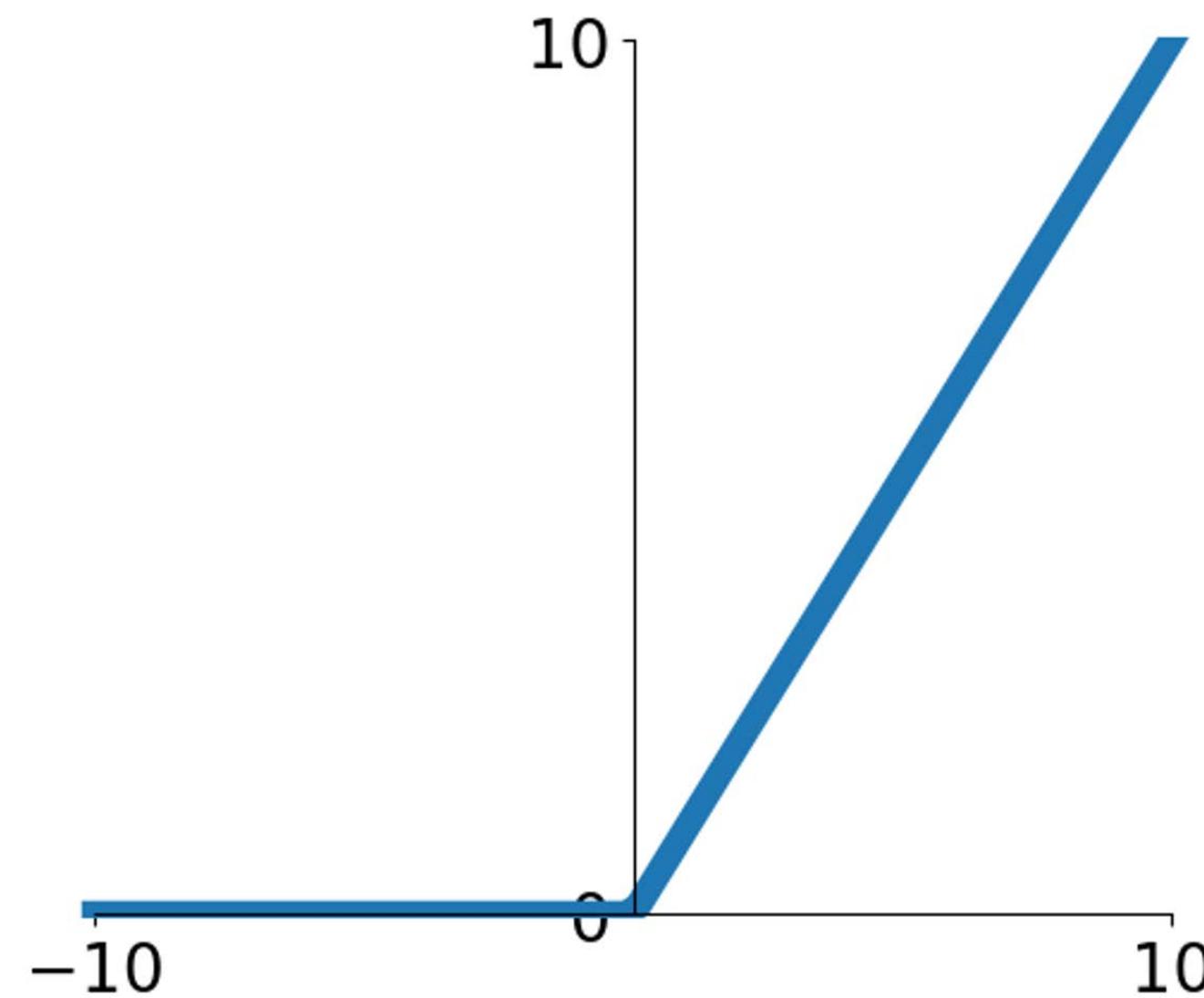
# Activation Functions

2-Layer Neural Network

$$f(x) = W_2 \max(0, W_1 x + b_1) + b_2$$

The auction  $ReLU(z) = \max(0, z)$   
is called “Rectified Linear Unit”

This is called the **activation function**  
of the neural network



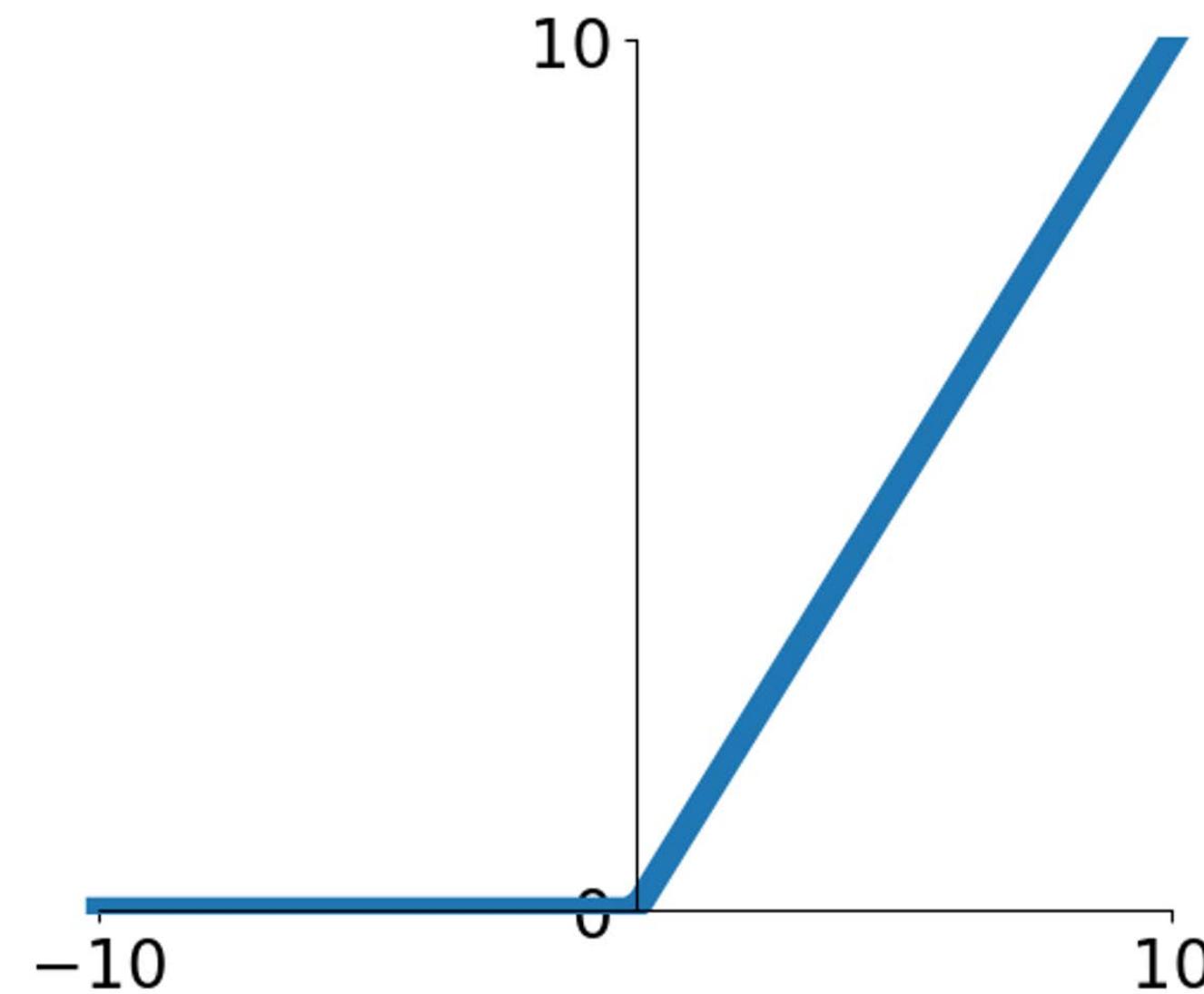
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**Q:** What happens if we build a neural network with no activation function?

$$f(x) = W_2(W_1 x + b_1) + b_2$$

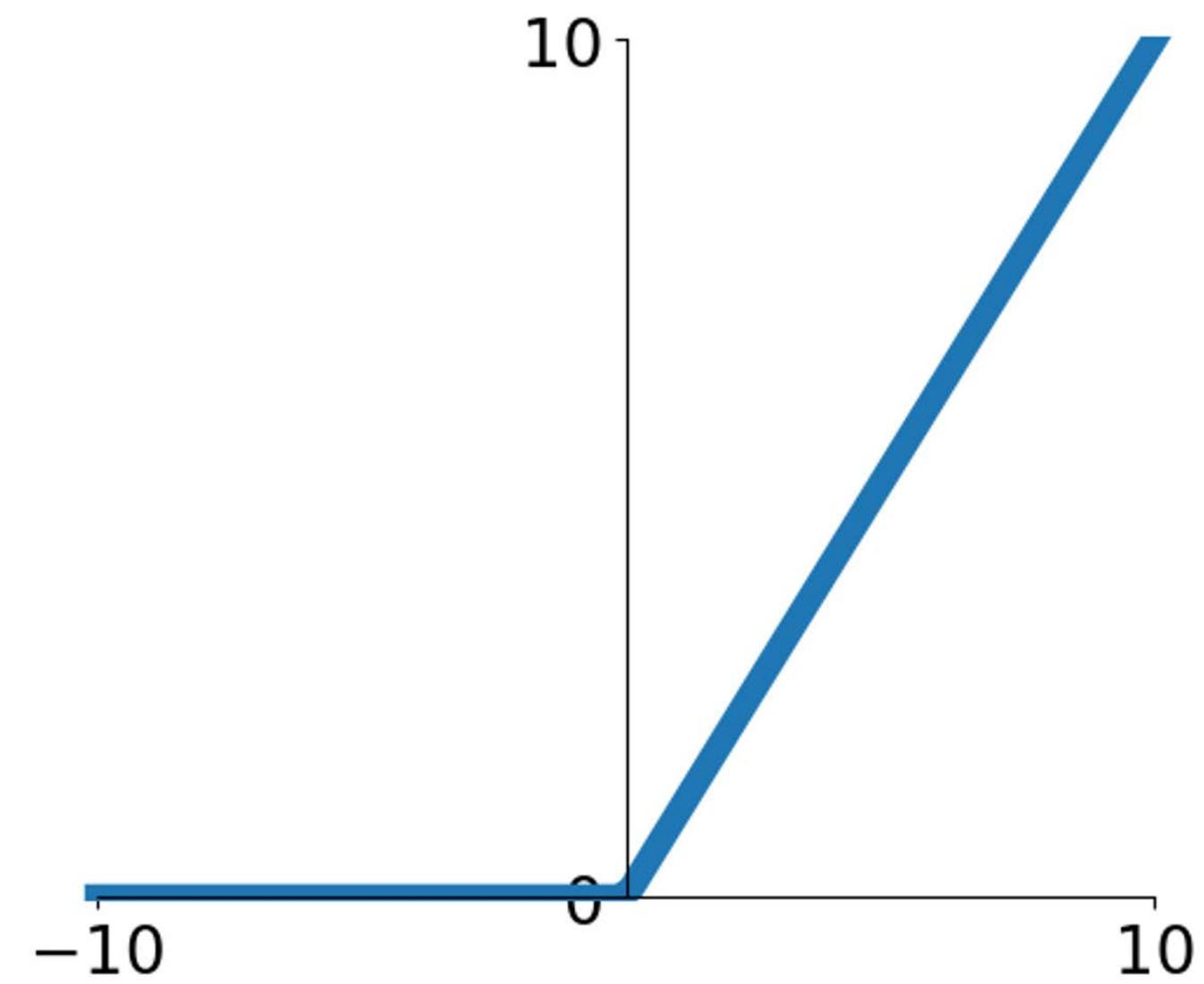
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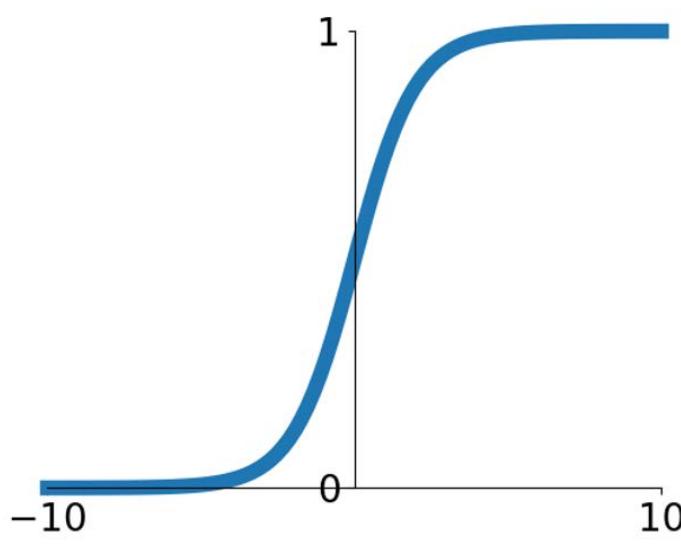
$$\begin{aligned} f(x) &= W_2(W_1 x + b_1) + b_2 \\ &= (W_1 W_2)x + (W_2 b_1 + b_2) \end{aligned}$$

**A:** We end up with a linear classifier

# Activation Functions

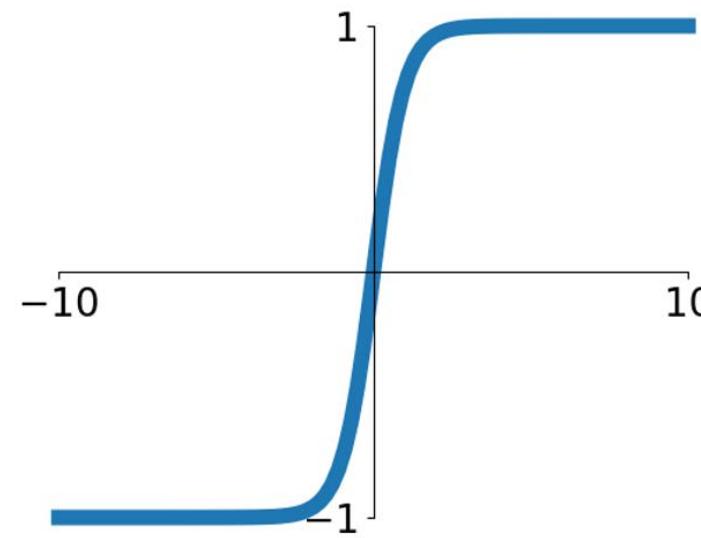
## Sigmoid

$$\sigma(x) = \frac{1}{1 + e^{-x}}$$



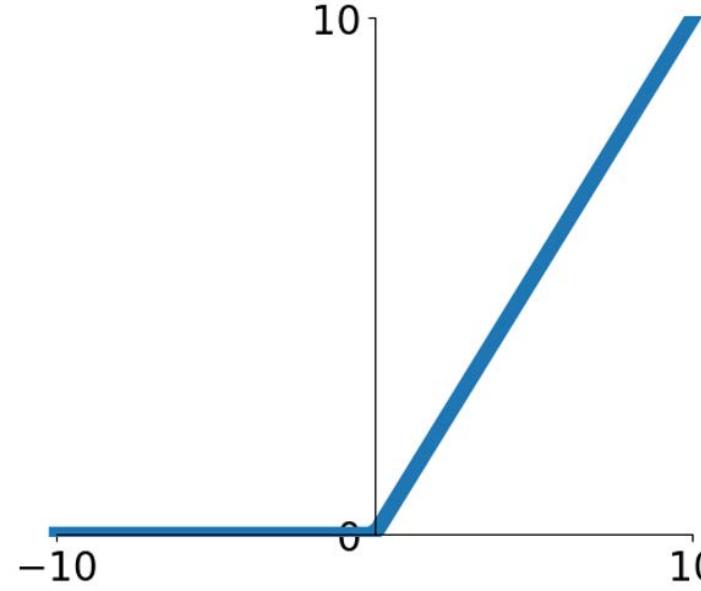
## tanh

$$\tanh(x) = \frac{e^{2x} - 1}{e^{2x} + 1}$$



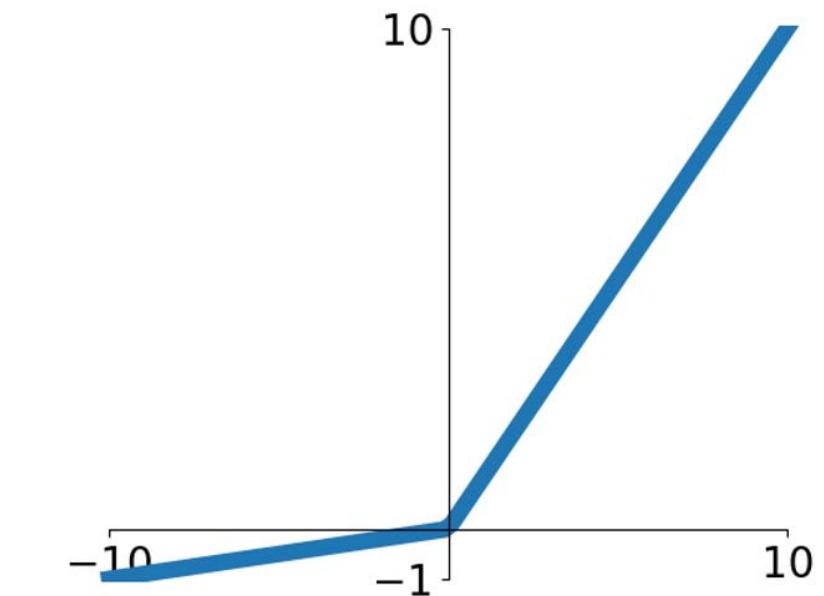
## ReLU

$$\max(0, x)$$



## Leaky ReLU

$$\max(0.2x, x)$$

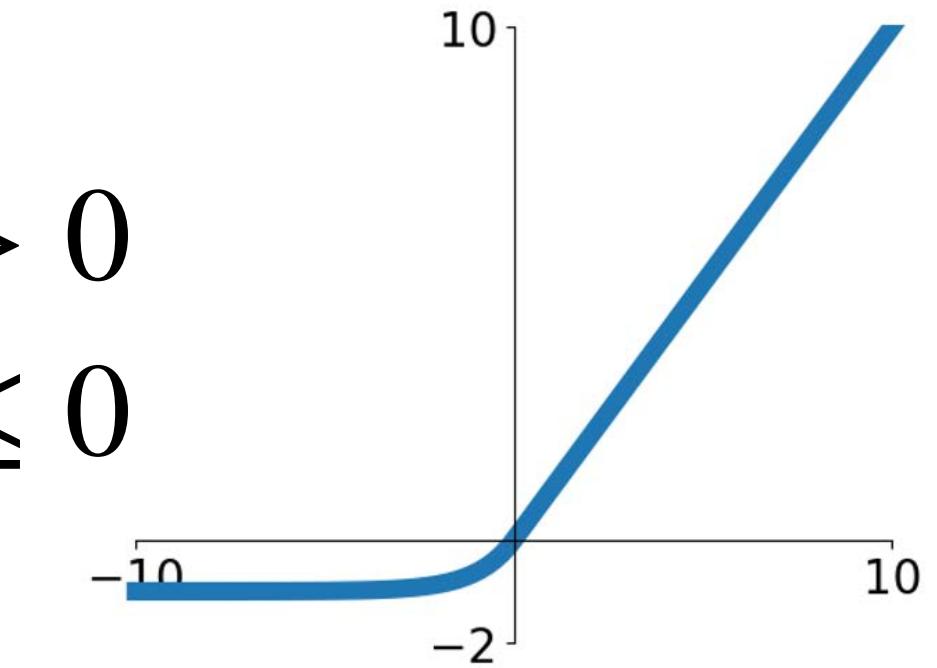


## Softplus

$$\log(1 + \exp(x))$$

## ELU

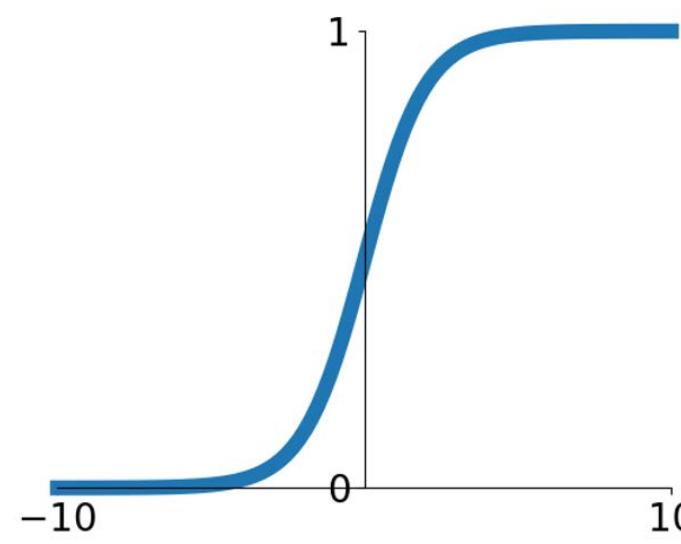
$$f(x) = \begin{cases} x, & x > 0 \\ \alpha(\exp(x) - 1), & x \leq 0 \end{cases}$$



# Activation Functions

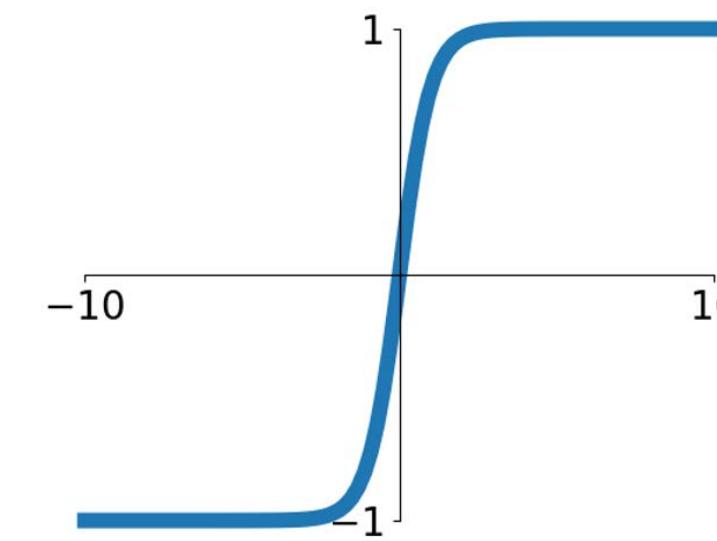
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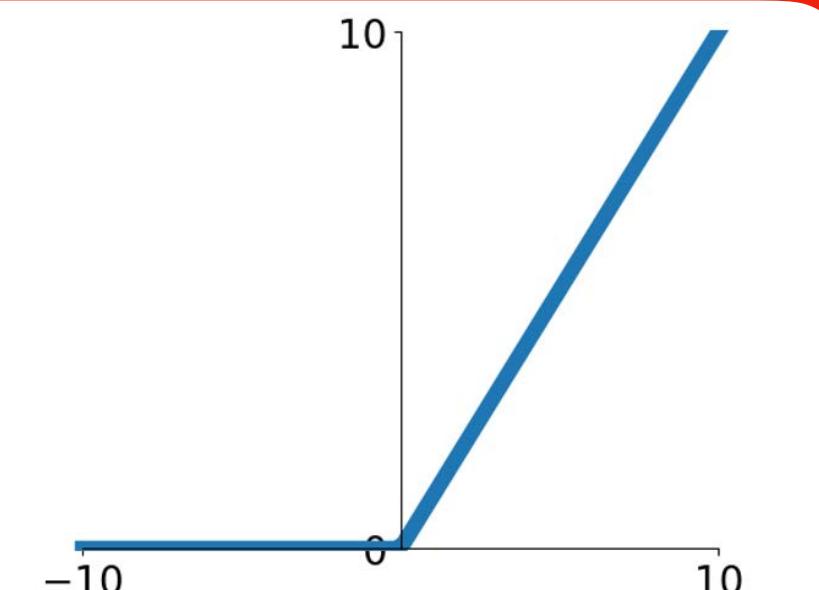
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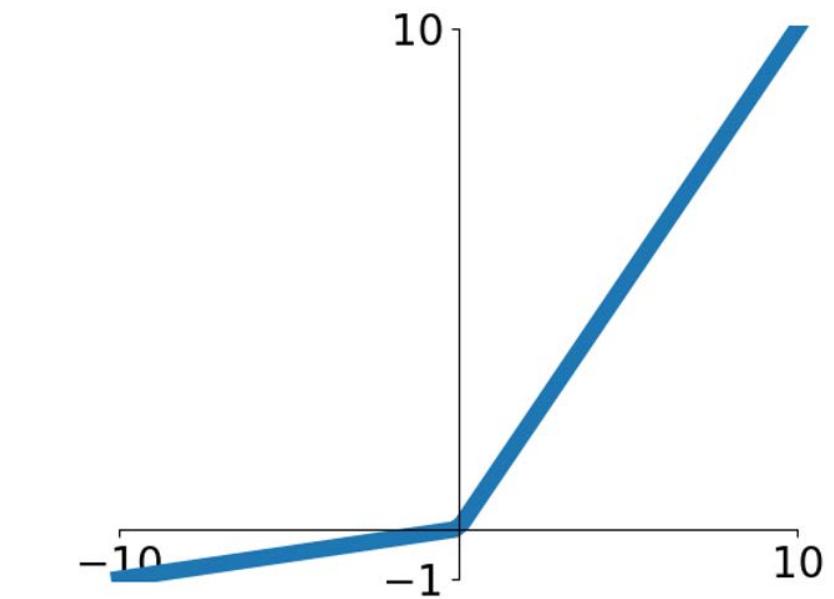
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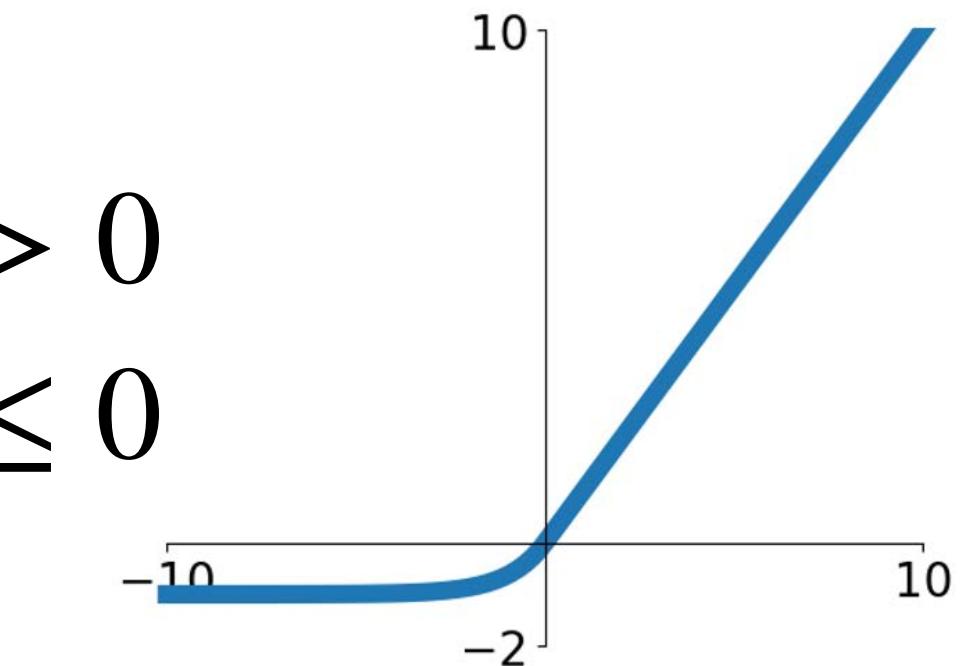


## Softplus

$$\log(1 + \exp(x))$$

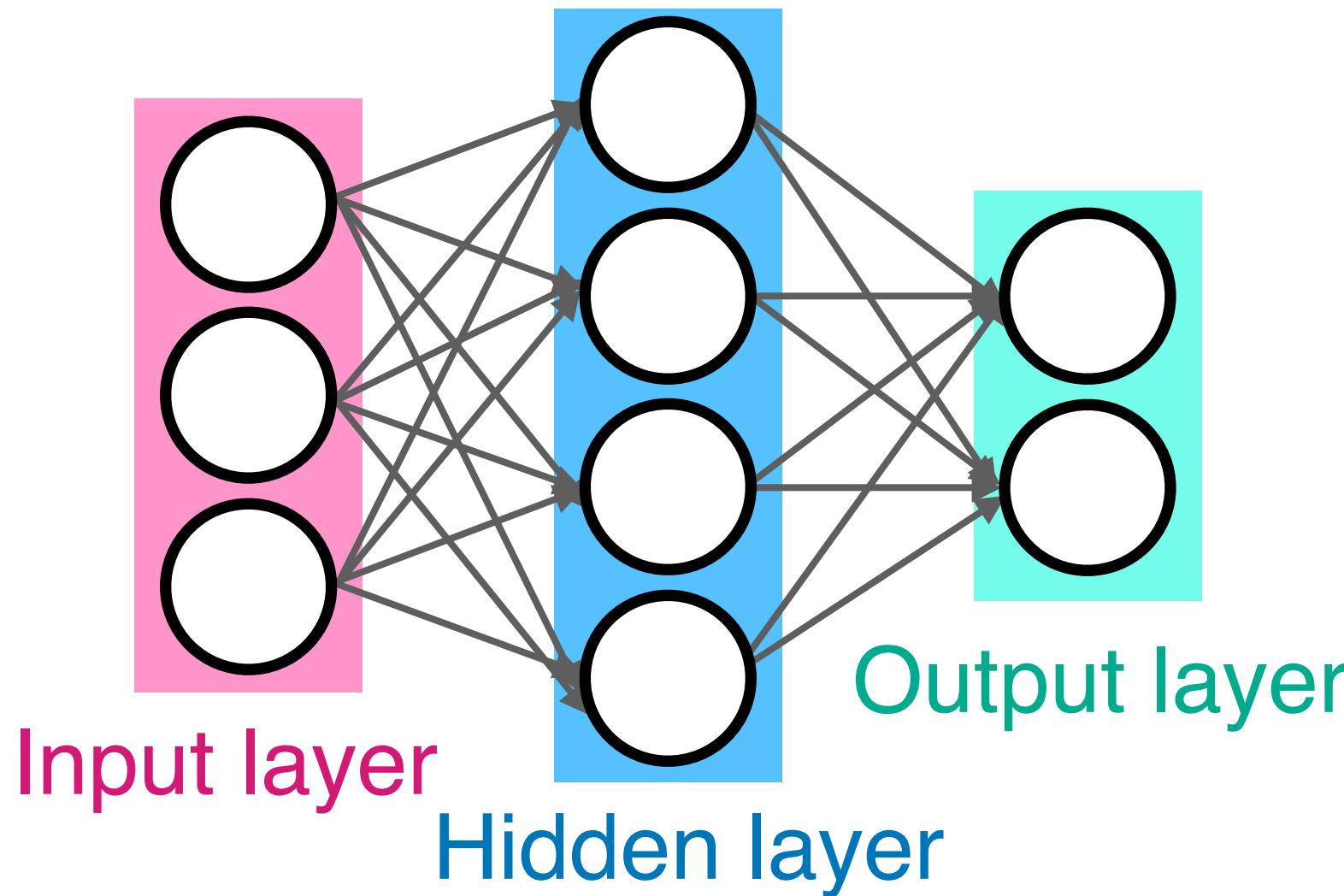
## ELU

$$f(x) = \begin{cases} x, & x > 0 \\ \alpha(\exp(x) - 1), & x \leq 0 \end{cases}$$



ReLU is a good default choice for most problems

# Neural Net in <20 lines!



Initialize weights  
and data

Compute loss (Sigmoid  
activation, L2 loss)

Compute gradients

SGD step

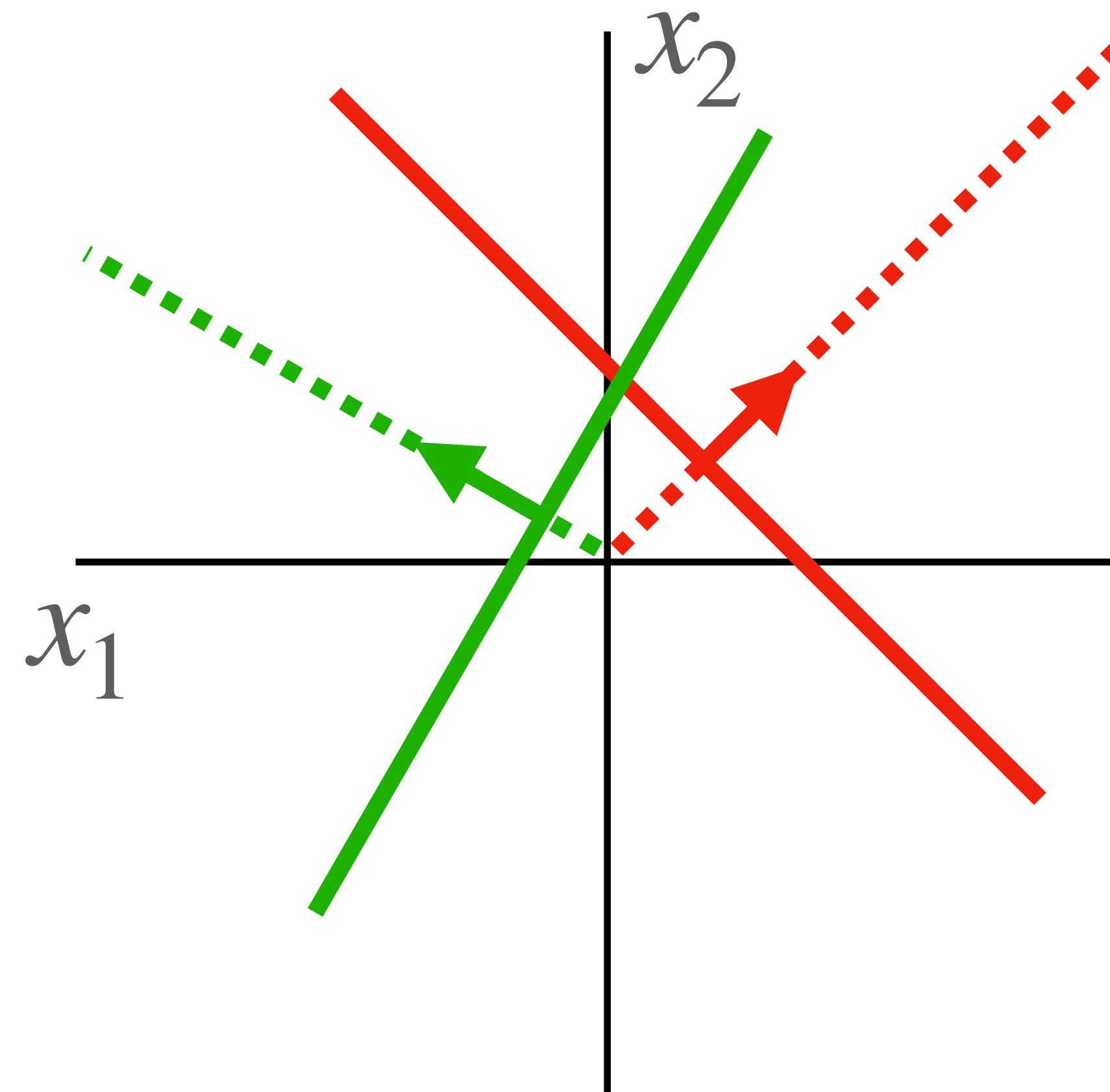
```

1 import numpy as np
2 from numpy.random import randn
3
4 N, Din, H, Dout = 64, 1000, 100, 10
5 x, y = randn(N, Din), randn(N, Dout)
6 w1, w2 = randn(Din, H), randn(H, Dout)
7 for t in range(10000):
8     h = 1.0 / (1.0 + np.exp(-x.dot(w1)))
9     y_pred = h.dot(w2)
10    loss = np.square(y_pred - y).sum()
11    dy_pred = 2.0 * (y_pred - y)
12    dw2 = h.T.dot(dy_pred)
13    dh = dy_pred.dot(w2.T)
14    dw1 = x.T.dot(dh * h * (1 - h))
15    w1 -= 1e-4 * dw1
16    w2 -= 1e-4 * dw2

```

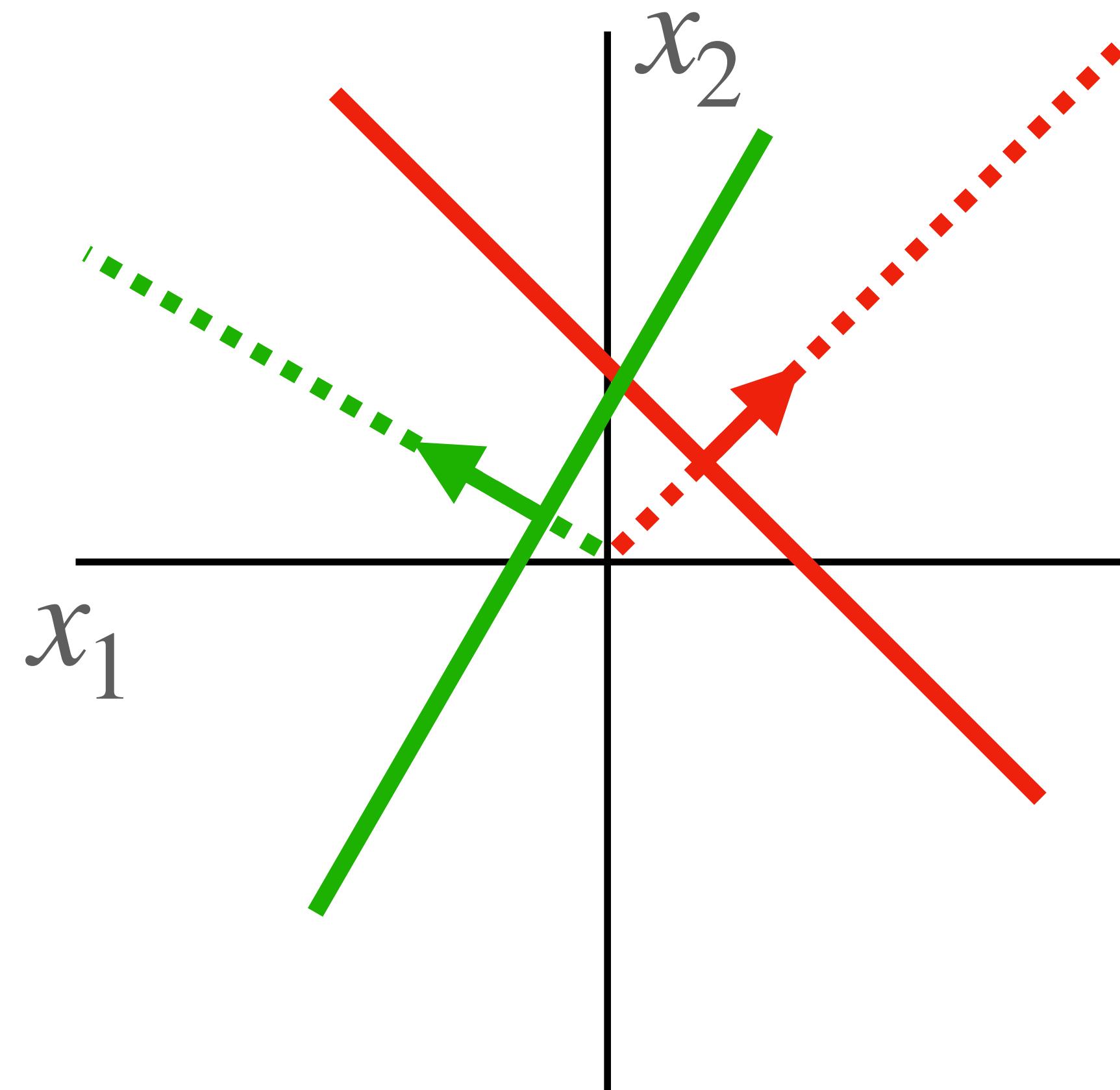
# Space Warping

Consider a linear transform:  $h = Wx + b$   
where  $x, b, h$  are each 2-dimensional



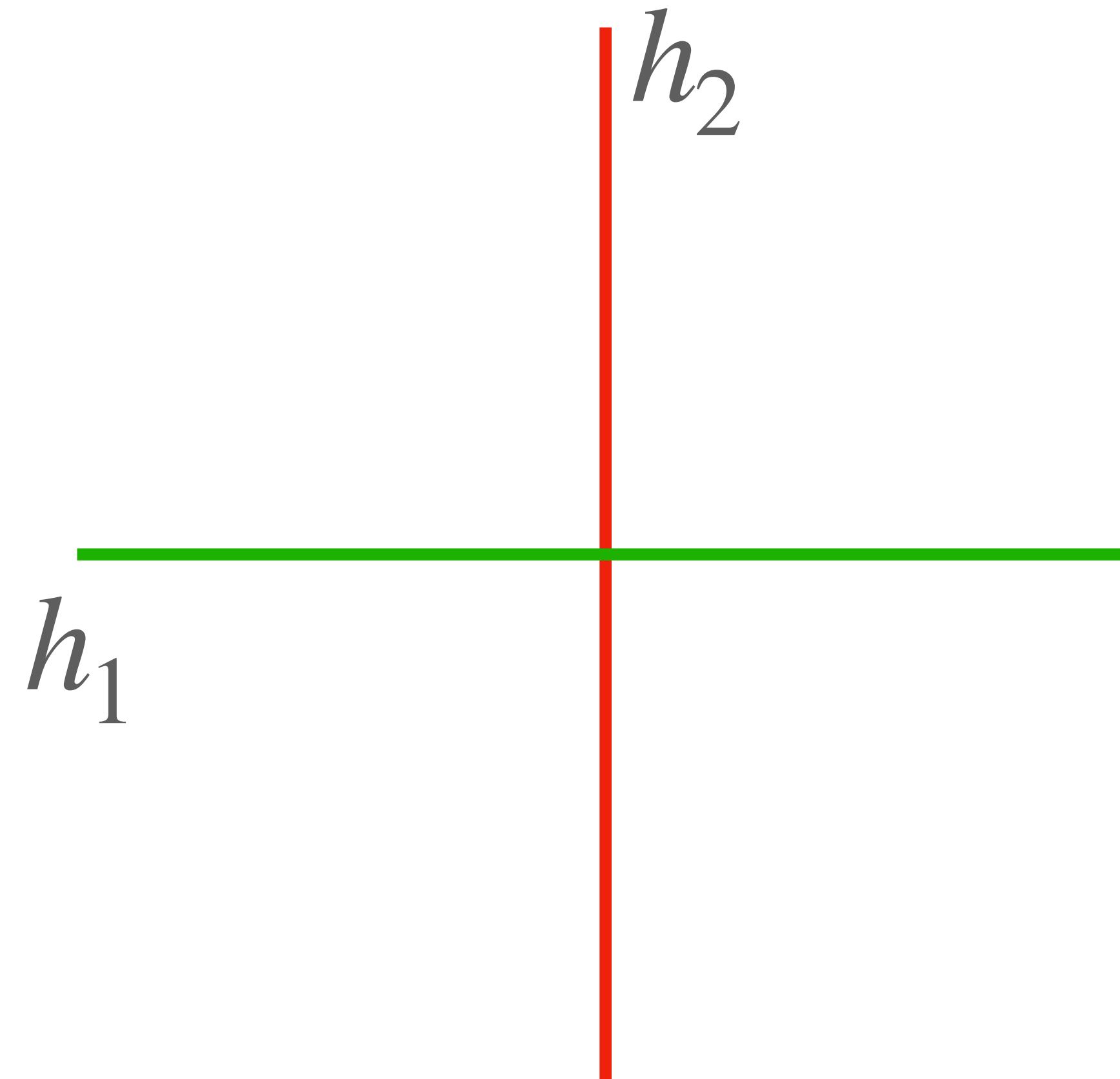
# Space Warping

Consider a linear transform:  $h = Wx + b$   
where  $x, b, h$  are each 2-dimensional



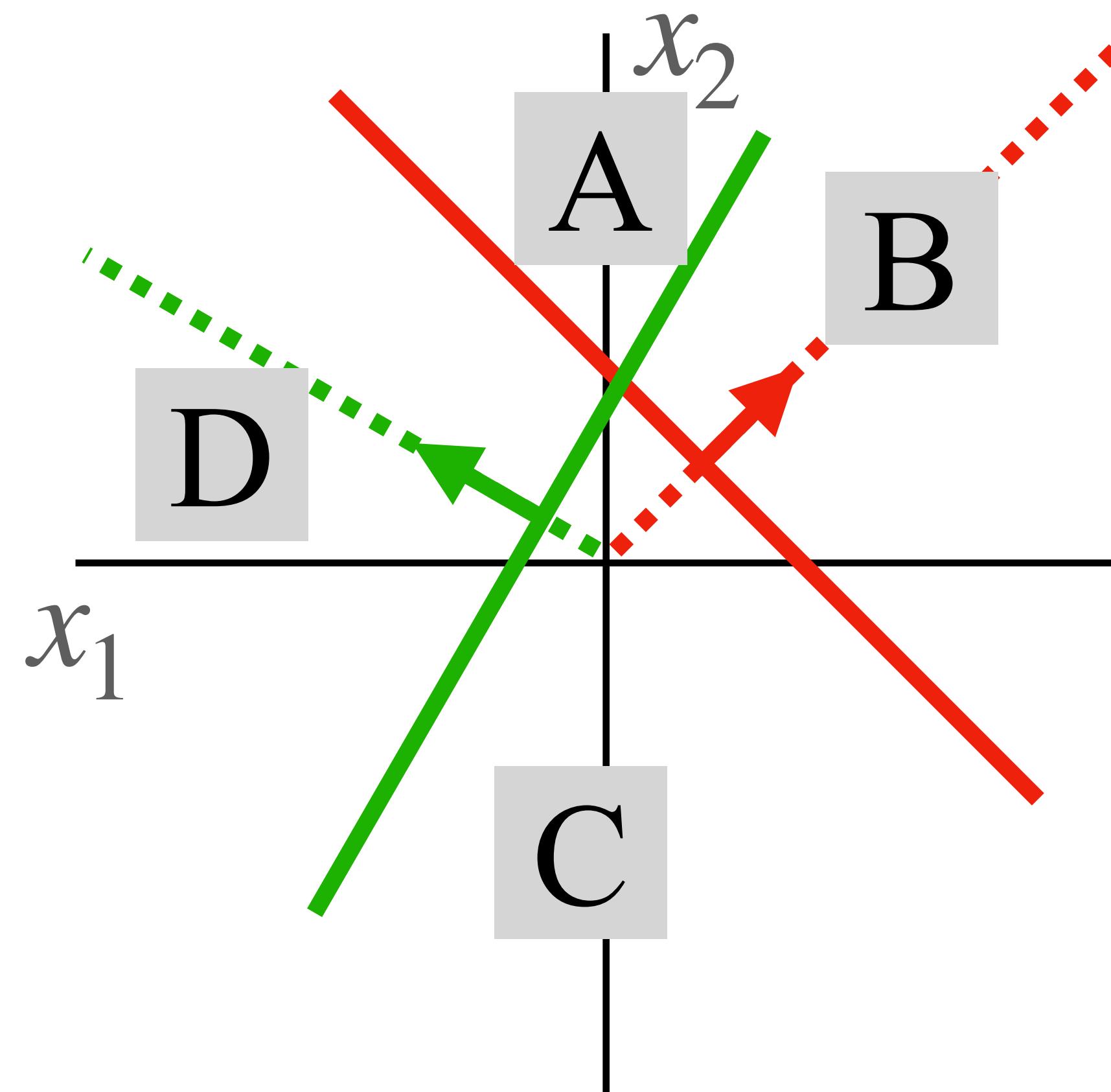
Feature transform:

$$h = Wx + b$$



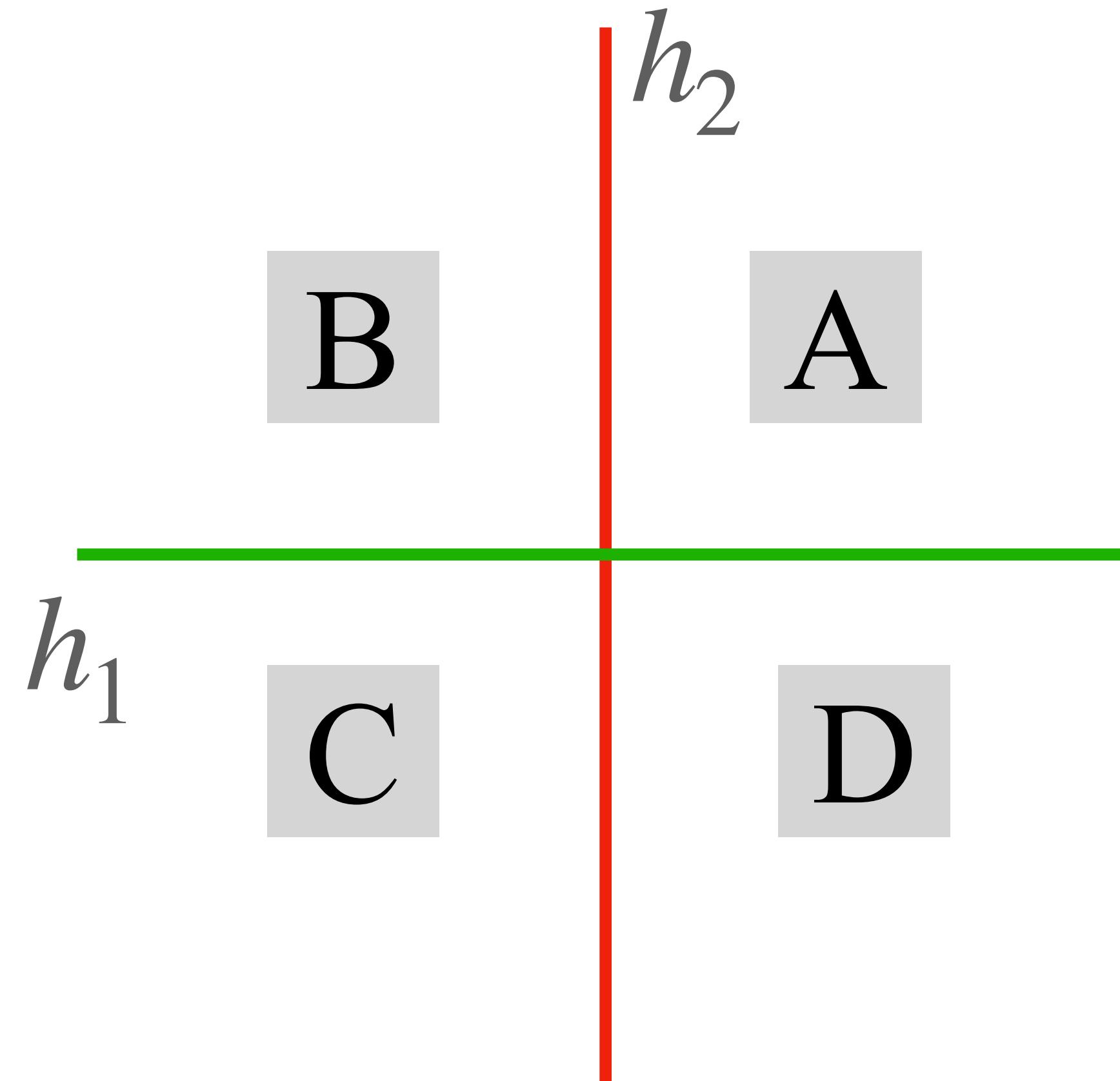
# Space Warping

Consider a linear transform:  $h = Wx + b$   
where  $x, b, h$  are each 2-dimensional



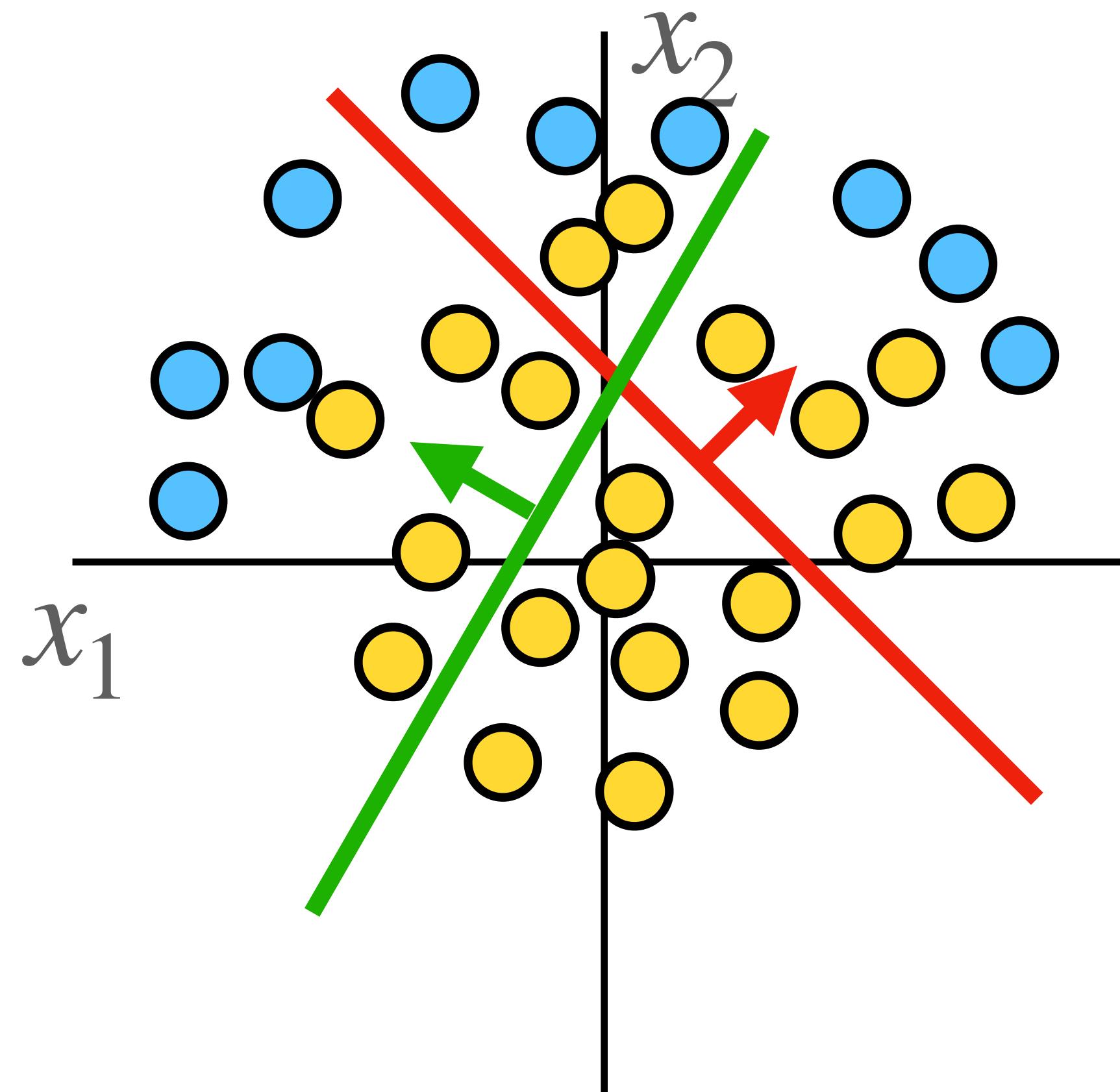
Feature transform:

$$h = Wx + b$$



# Space Warping

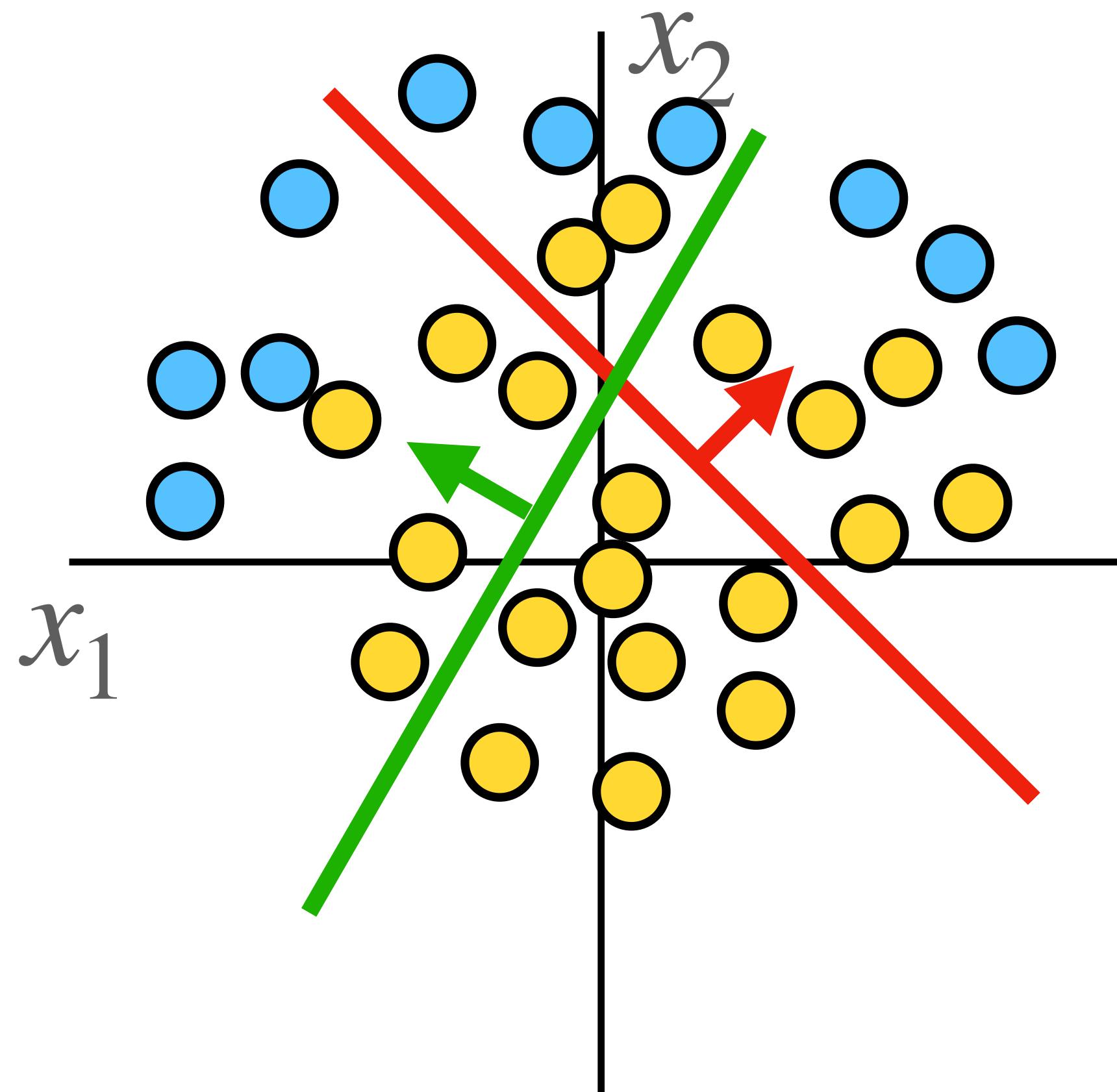
Points not linearly separable  
in original space



Consider a linear transform:  $h = Wx + b$   
where  $x, b, h$  are each 2-dimensional

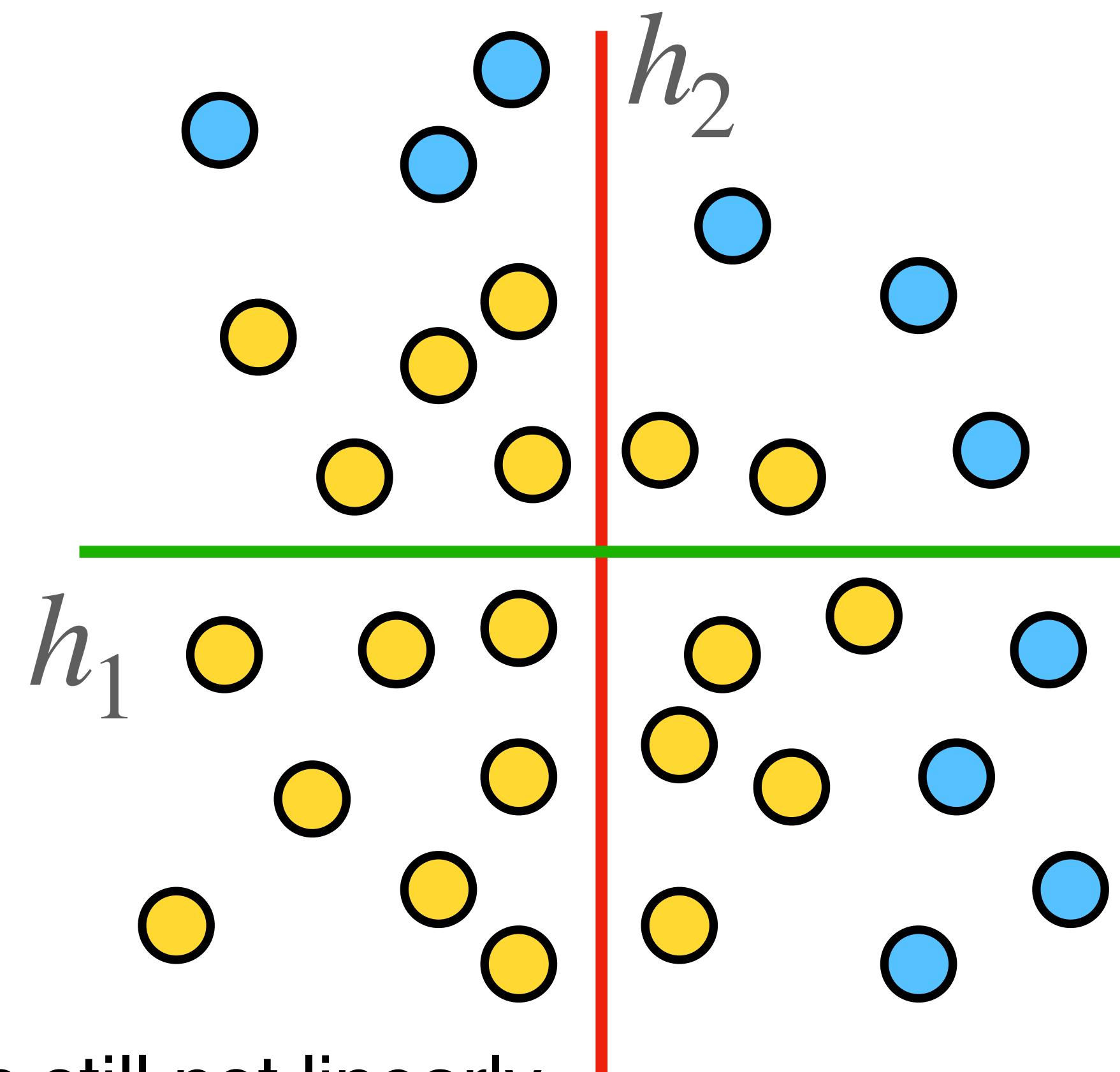
# Space Warping

Points not linearly separable  
in original space



Feature transform:  
 $h = Wx + b$

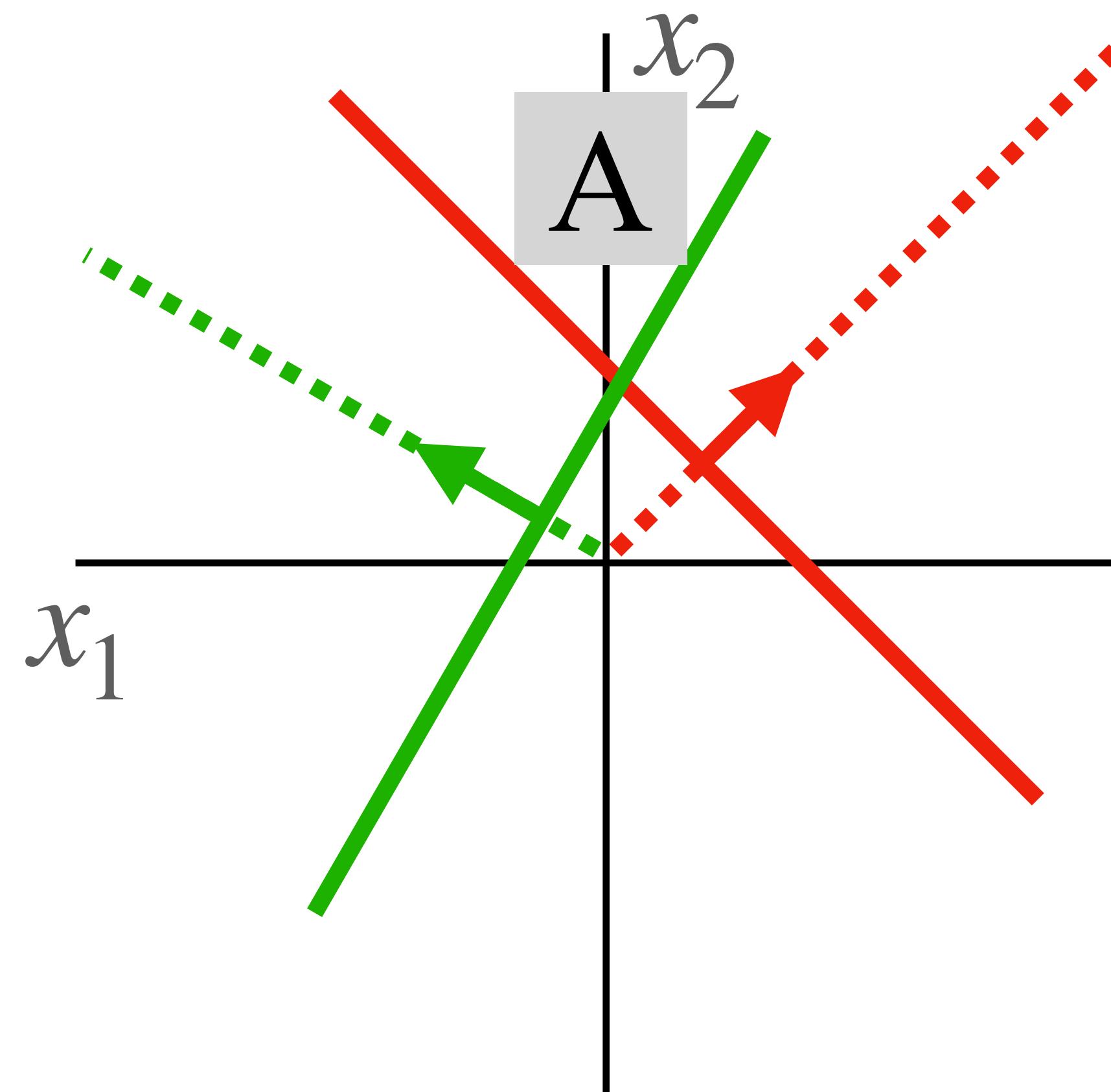
Consider a linear transform:  $h = Wx + b$   
where  $x, b, h$  are each 2-dimensional



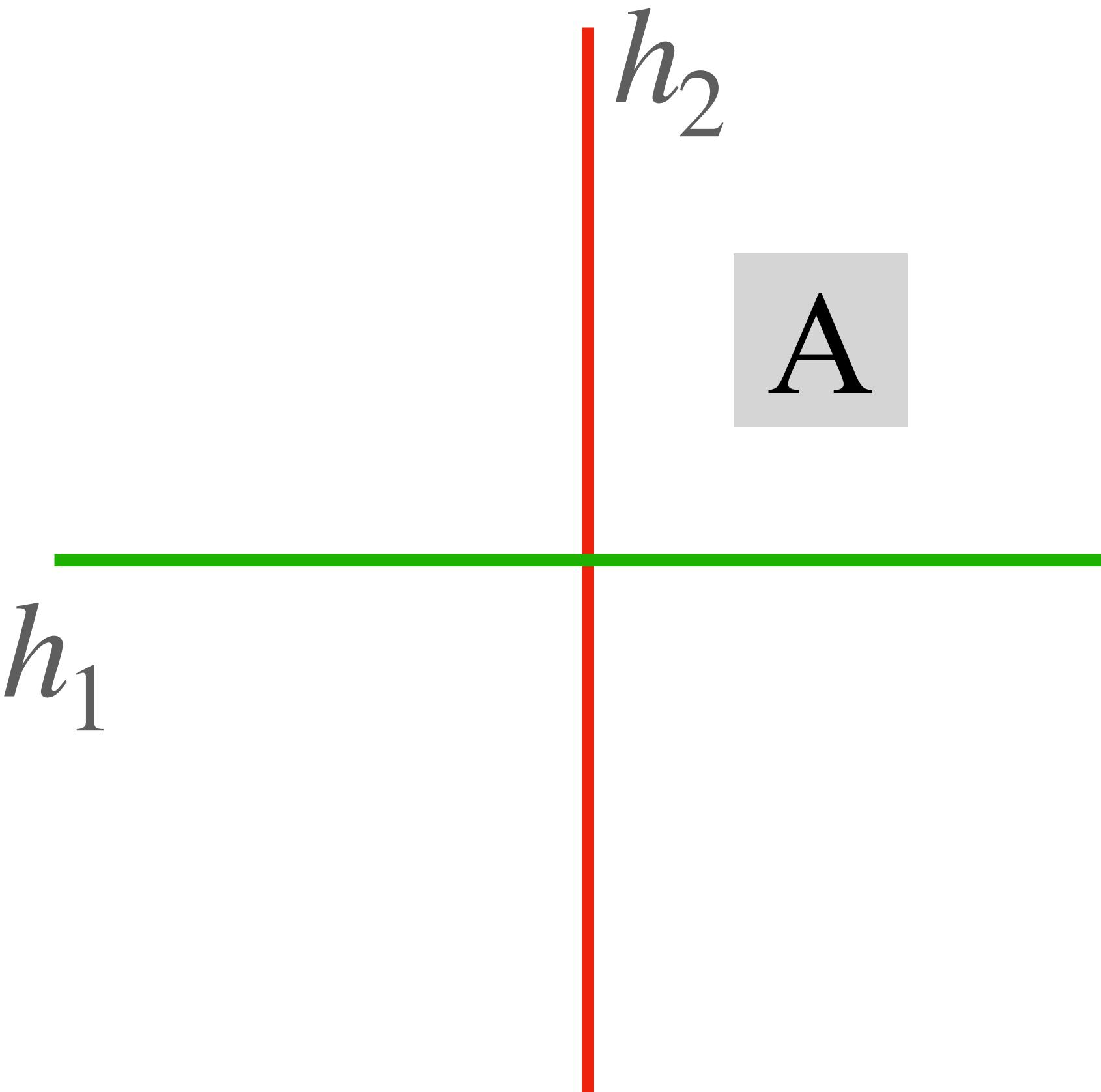
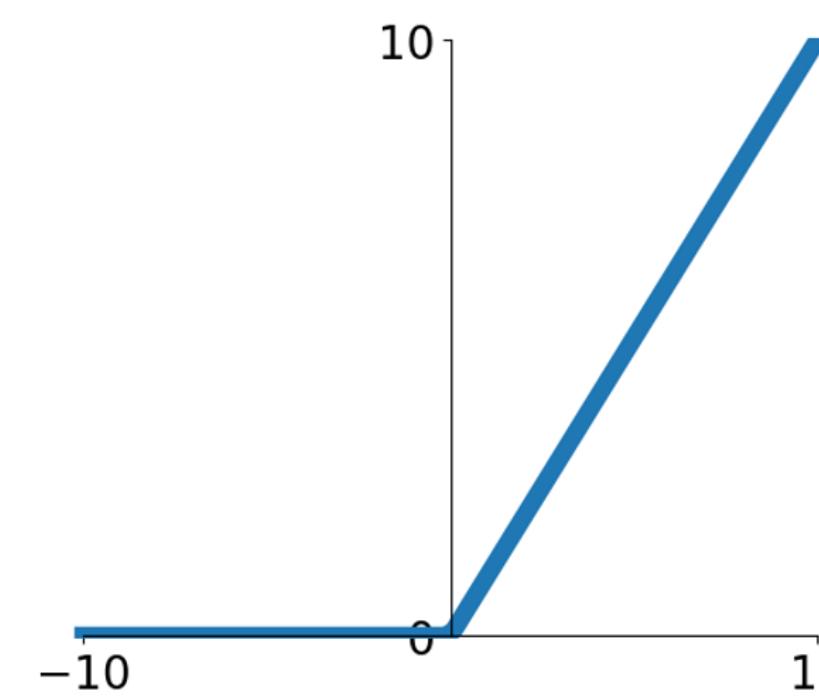
Points still not linearly  
separable in feature space

# Space Warping

Consider a neural net hidden layer:  $h = \text{ReLU}(Wx + b)$   
=  $\max(0, Wx + b)$  where  $x, b, h$  are each 2-dimensional

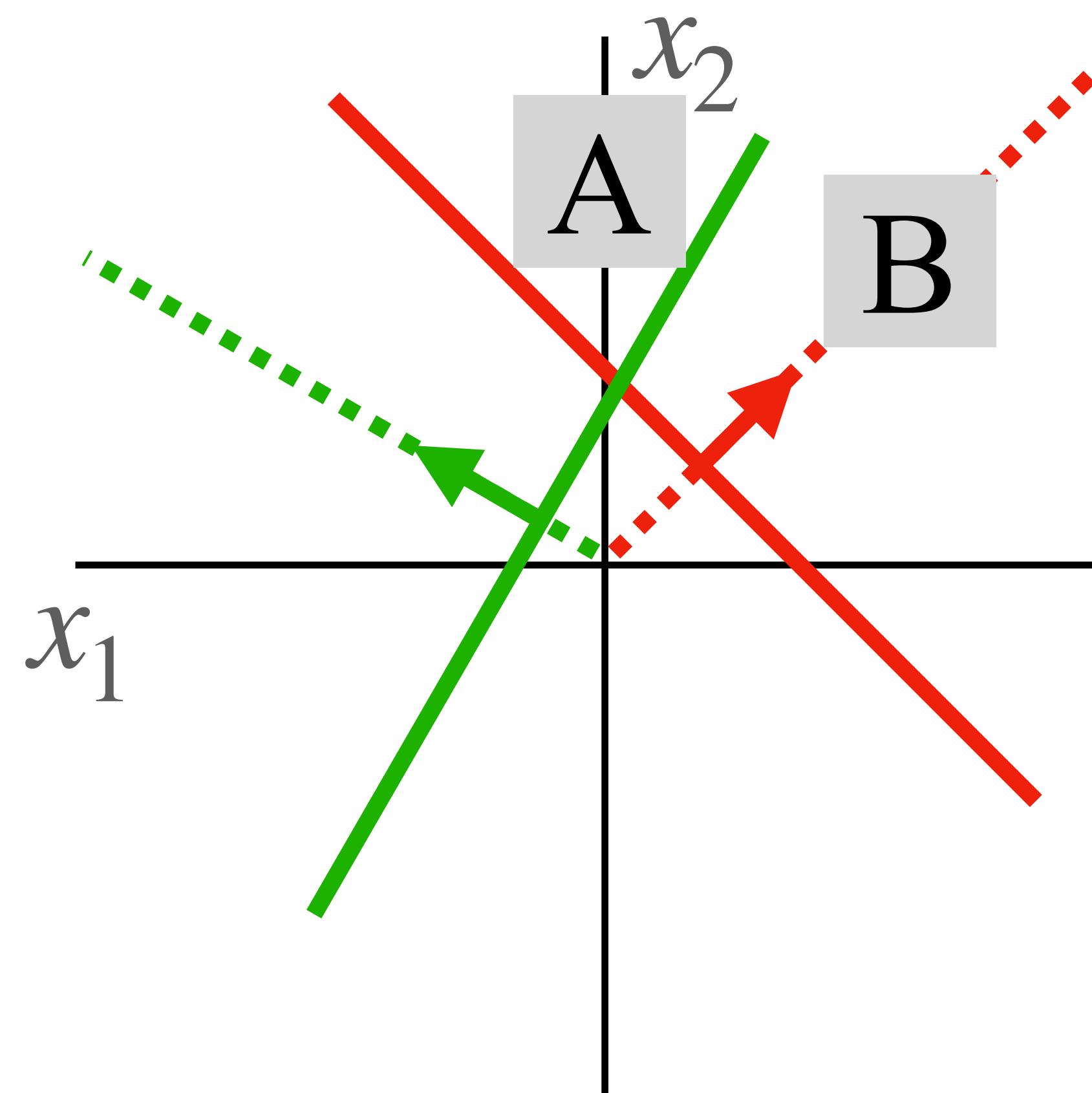


Feature transform:  
 $h = \text{ReLU}(Wx + b)$

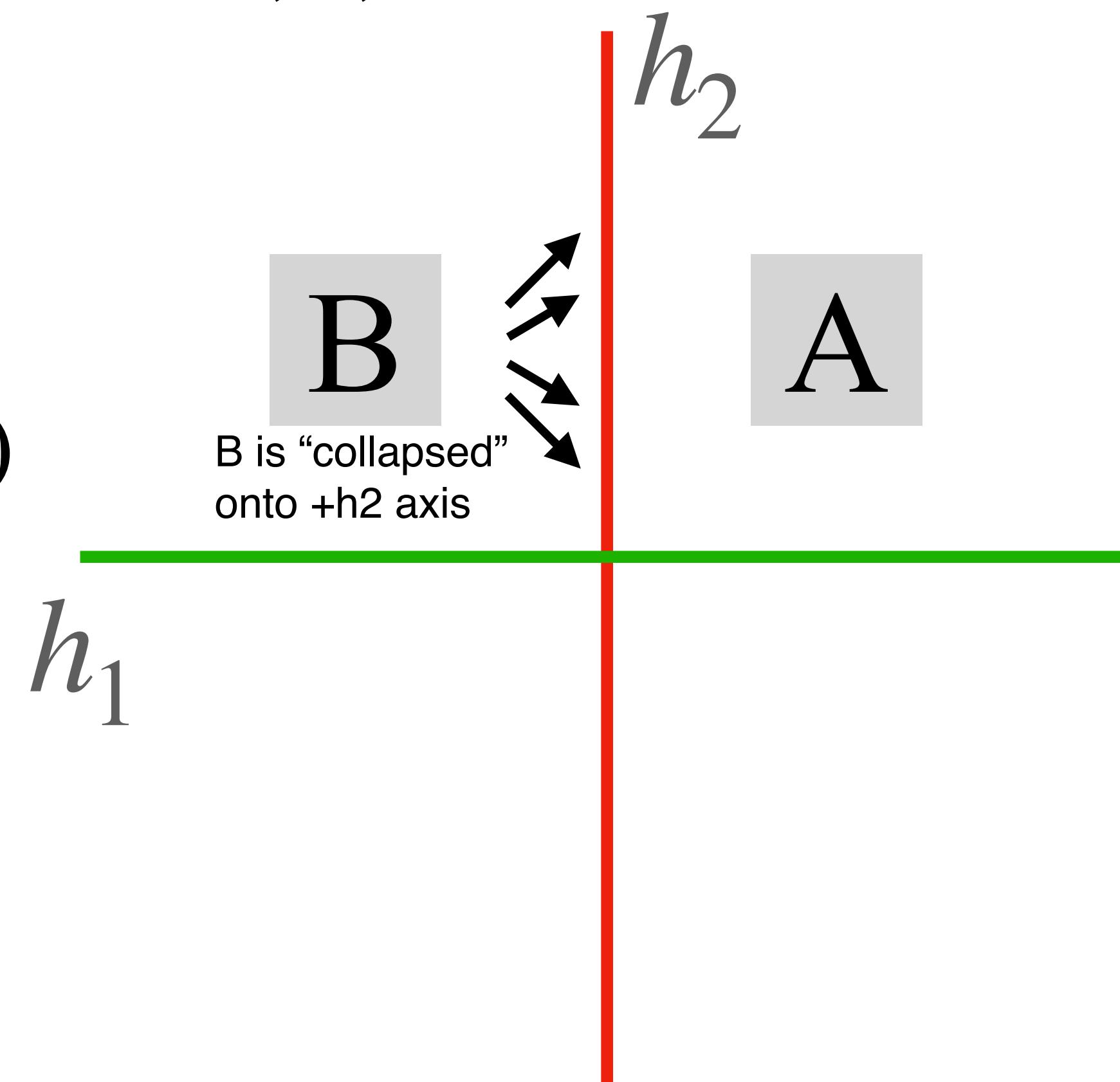
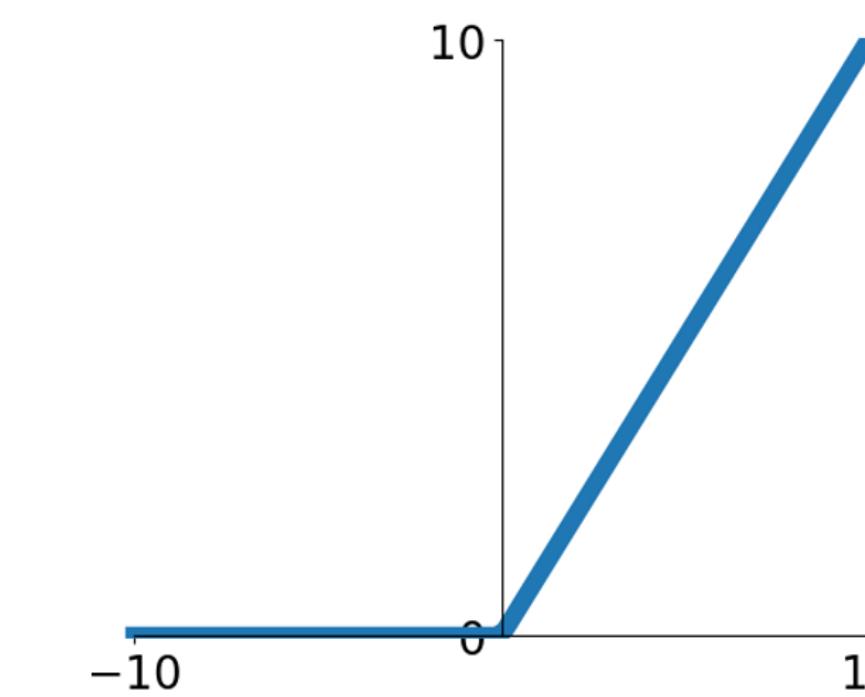


# Space Warping

Consider a neural net hidden layer:  $h = \text{ReLU}(Wx + b)$   
 $= \max(0, Wx + b)$  where  $x, b, h$  are each 2-dimensional

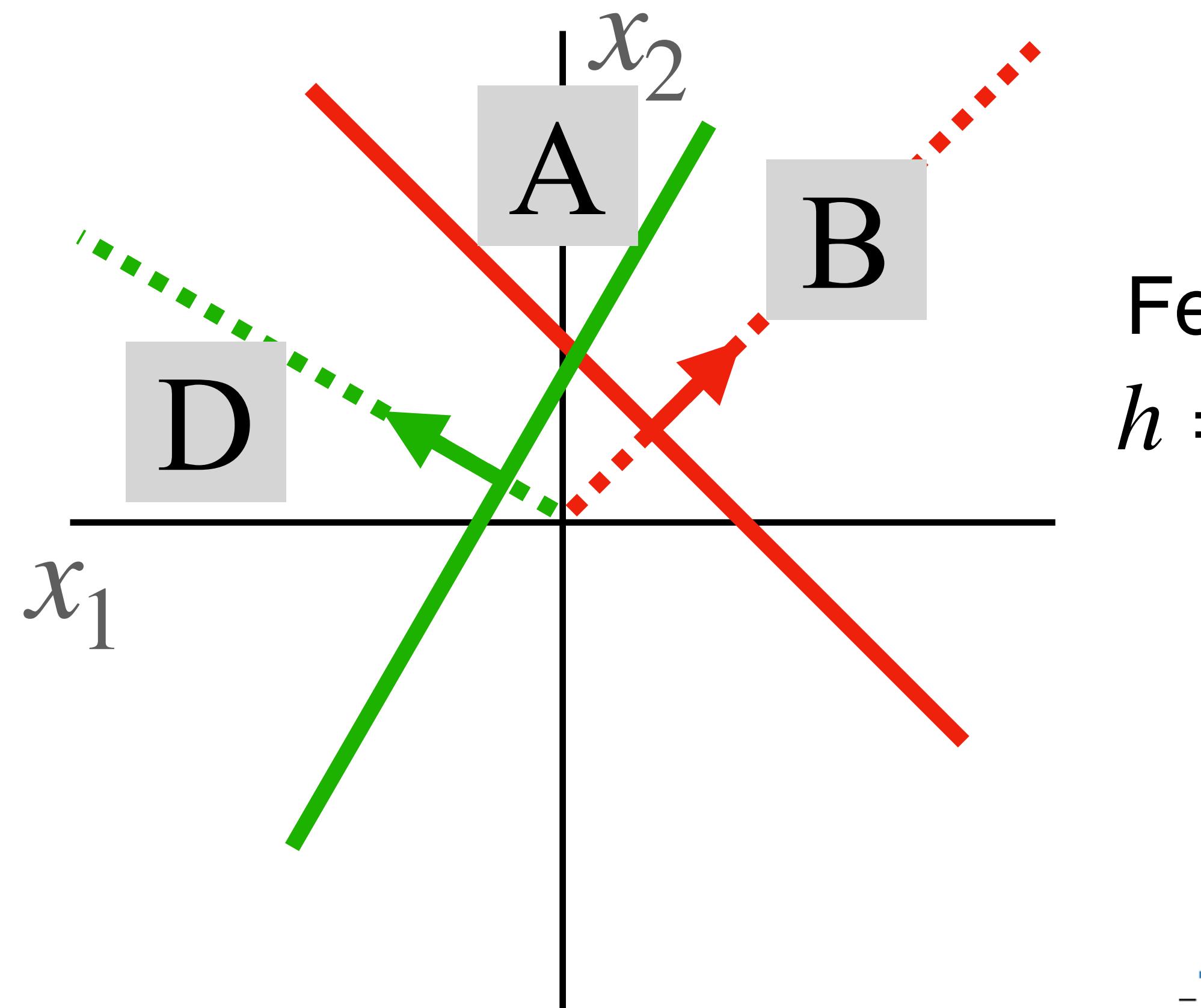


Feature transform:  
 $h = \text{ReLU}(Wx + b)$

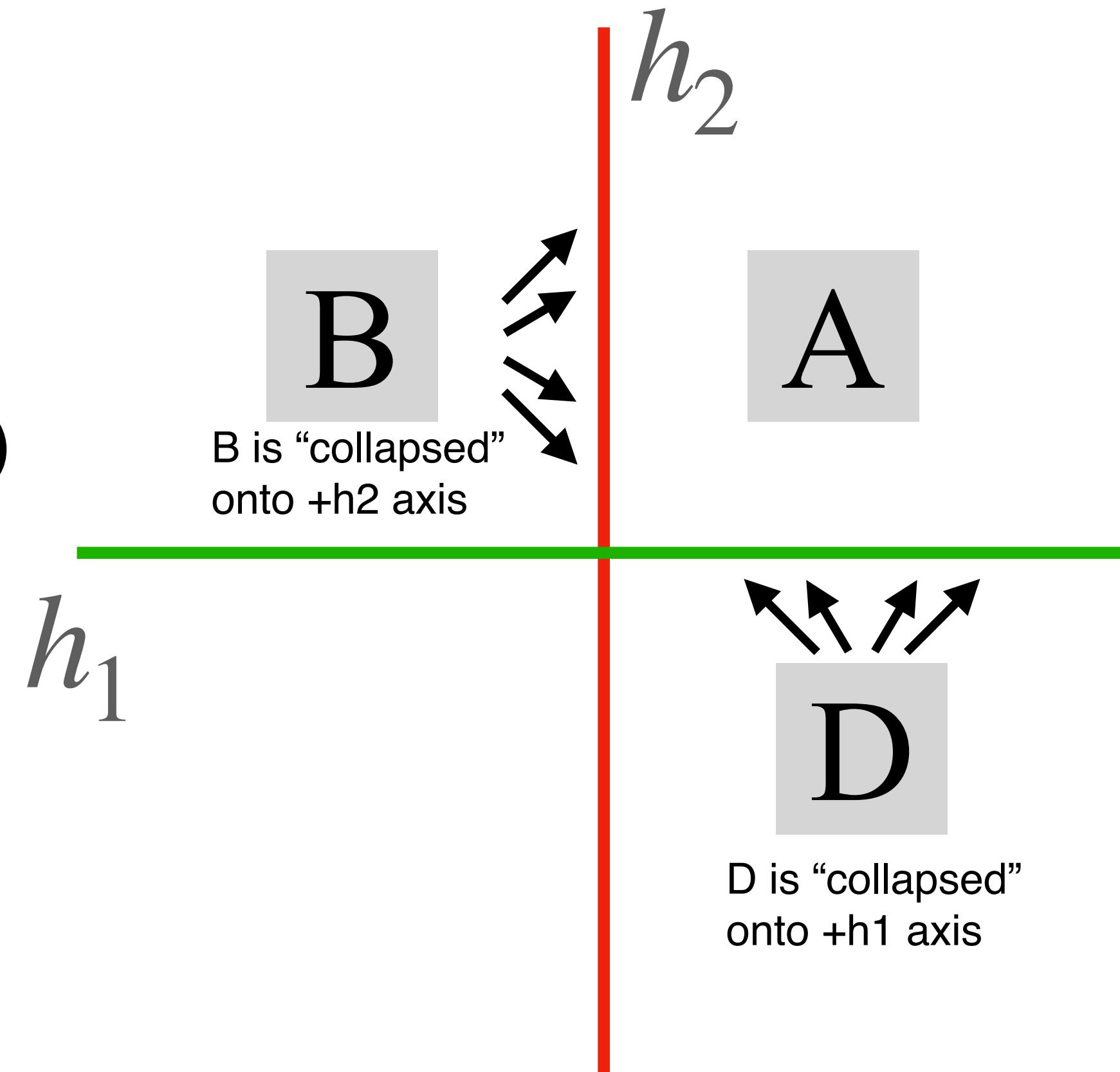
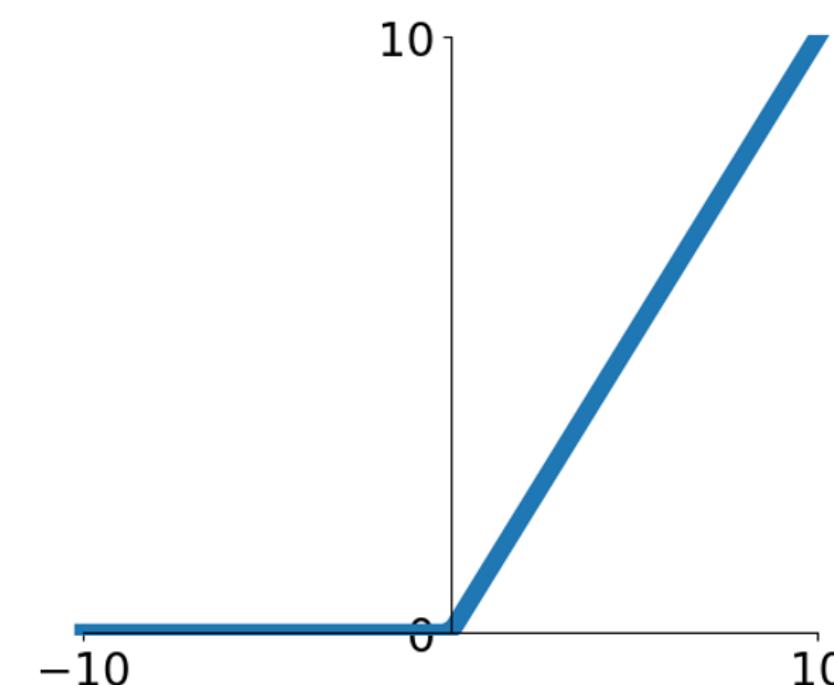


# Space Warping

Consider a neural net hidden layer:  $h = \text{ReLU}(Wx + b)$   
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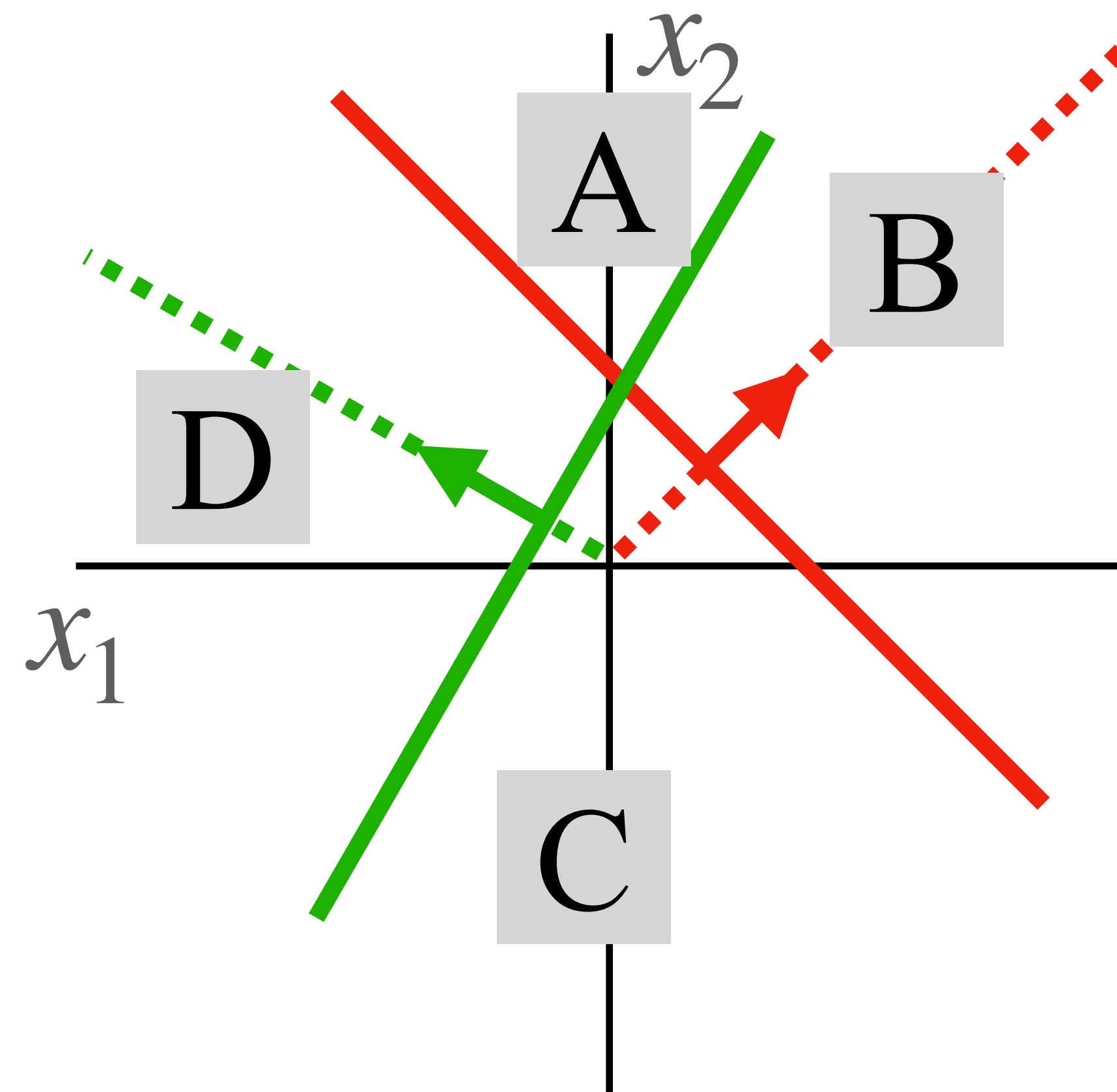


Feature transform:  
 $h = \text{ReLU}(Wx + b)$

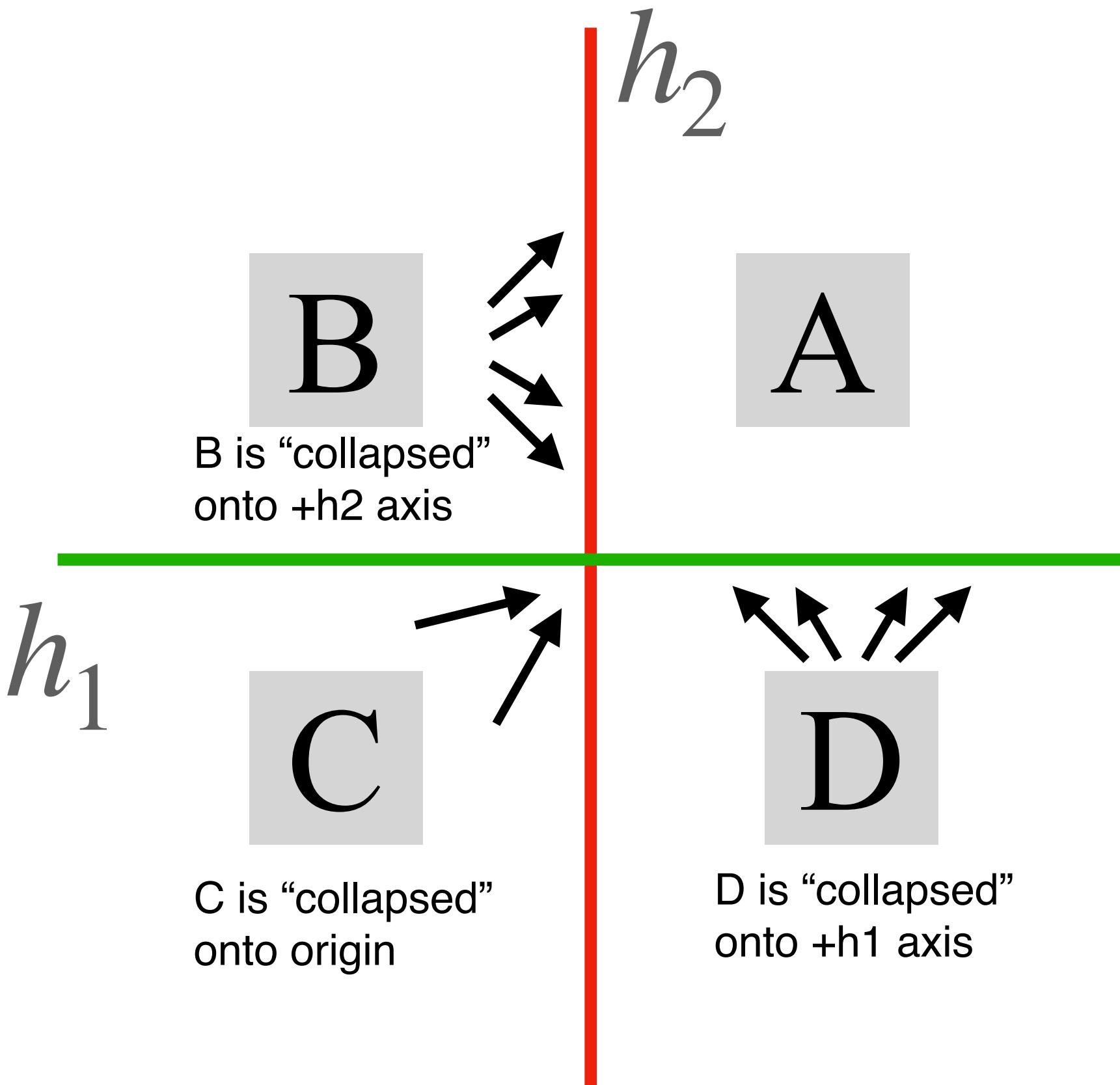
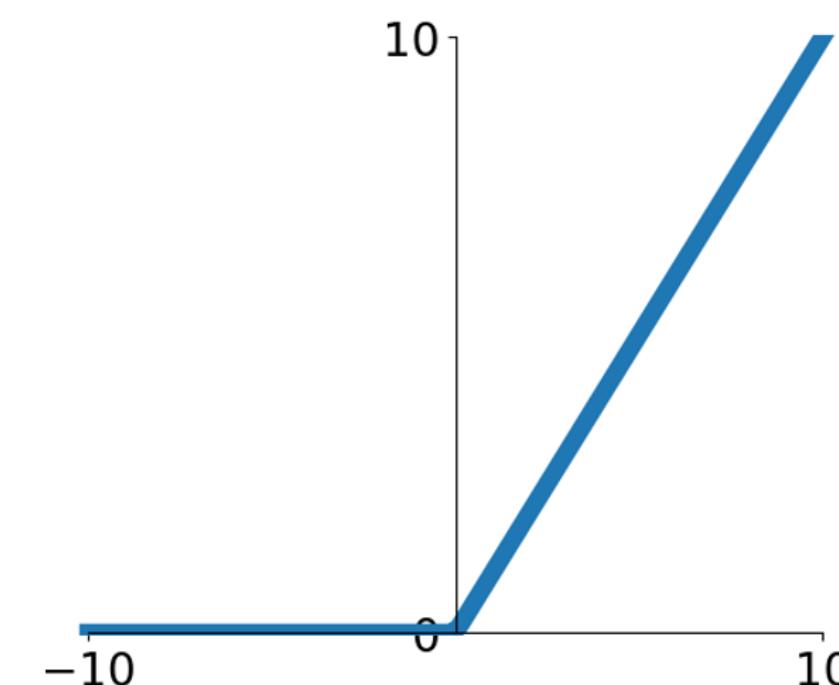


# Space Warping

Consider a neural net hidden layer:  $h = \text{ReLU}(Wx + b)$   
 $= \max(0, Wx + b)$  where  $x, b, h$  are each 2-dimensional

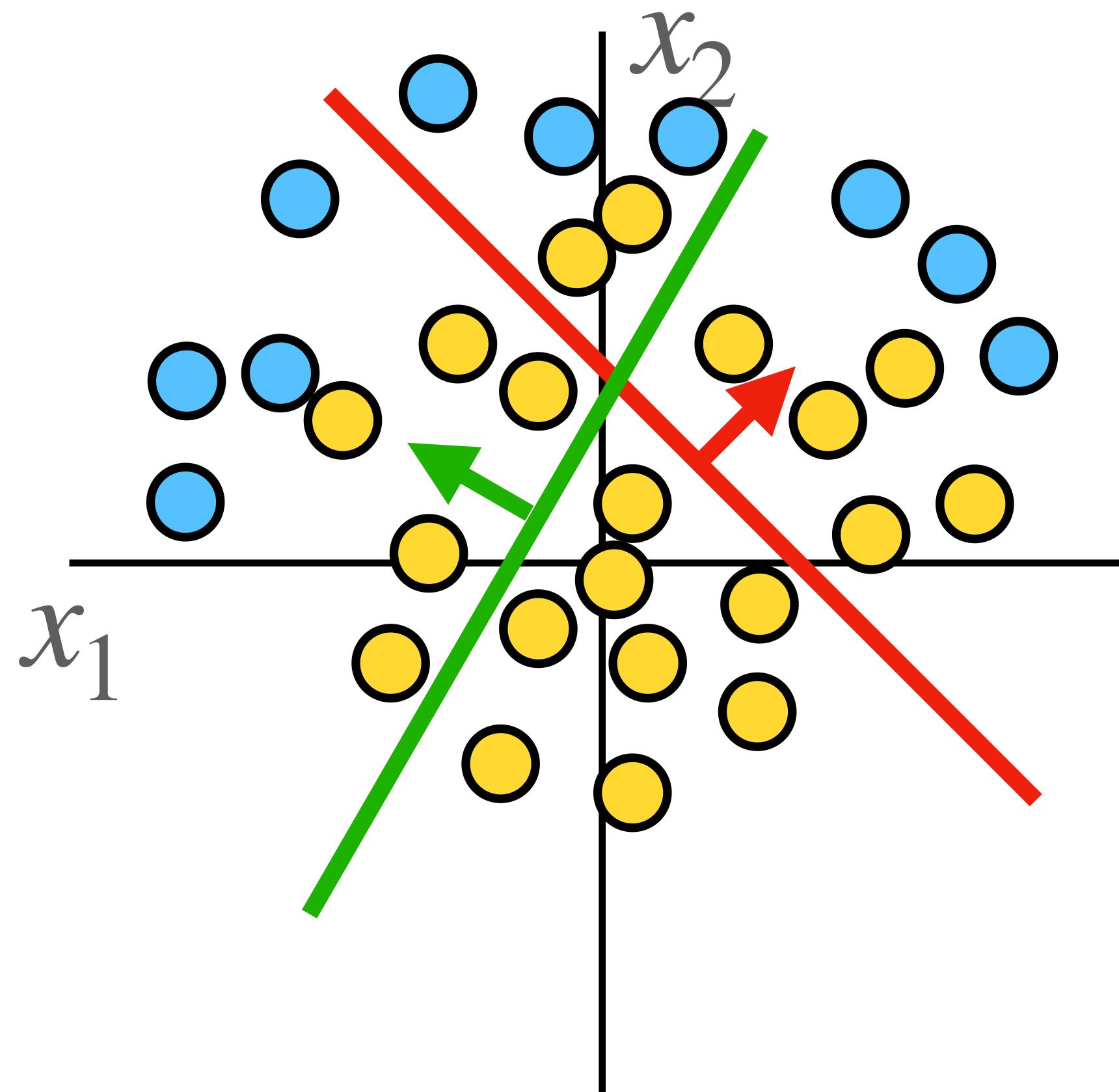


Feature transform:  
 $h = \text{ReLU}(Wx + b)$



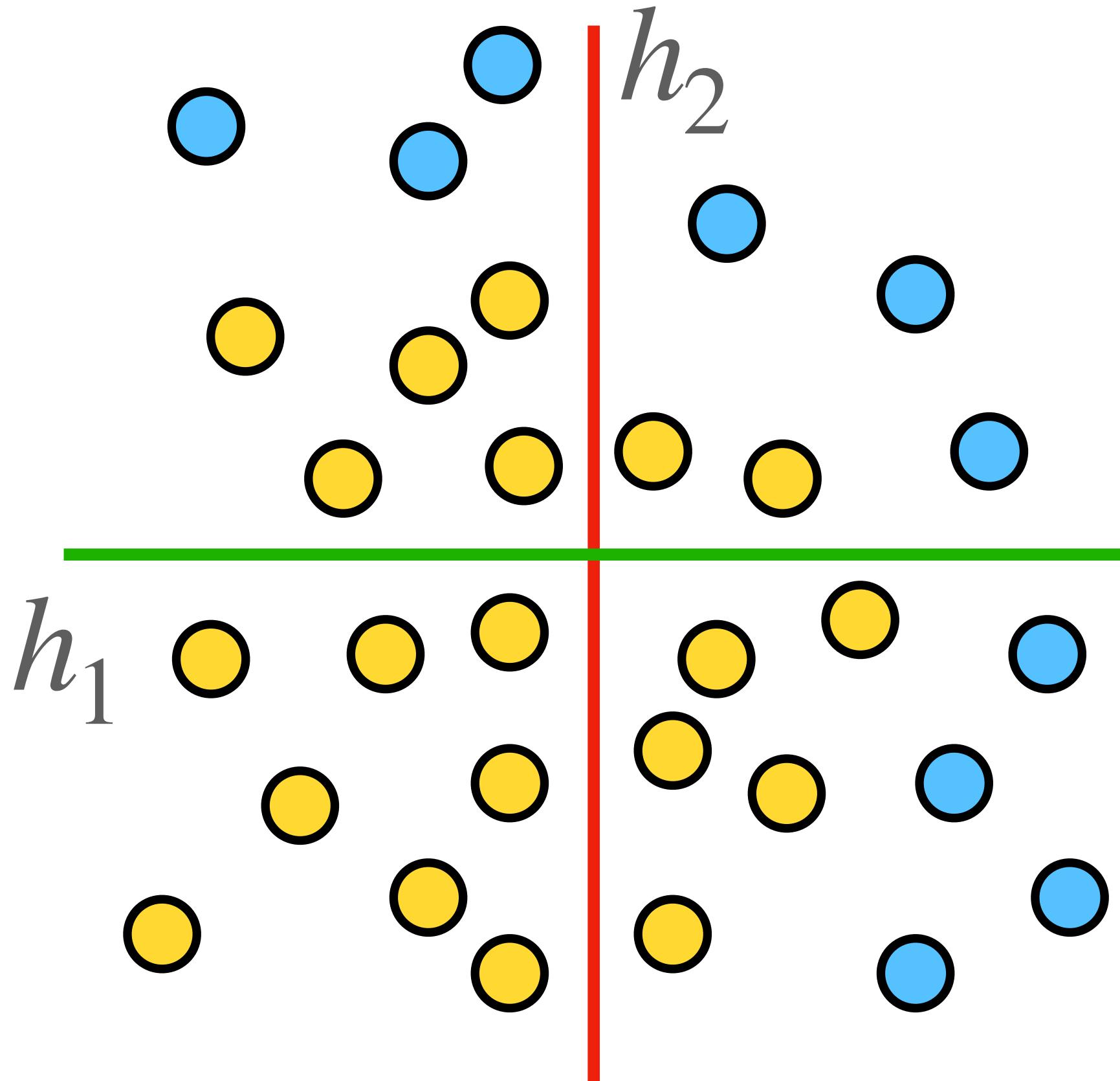
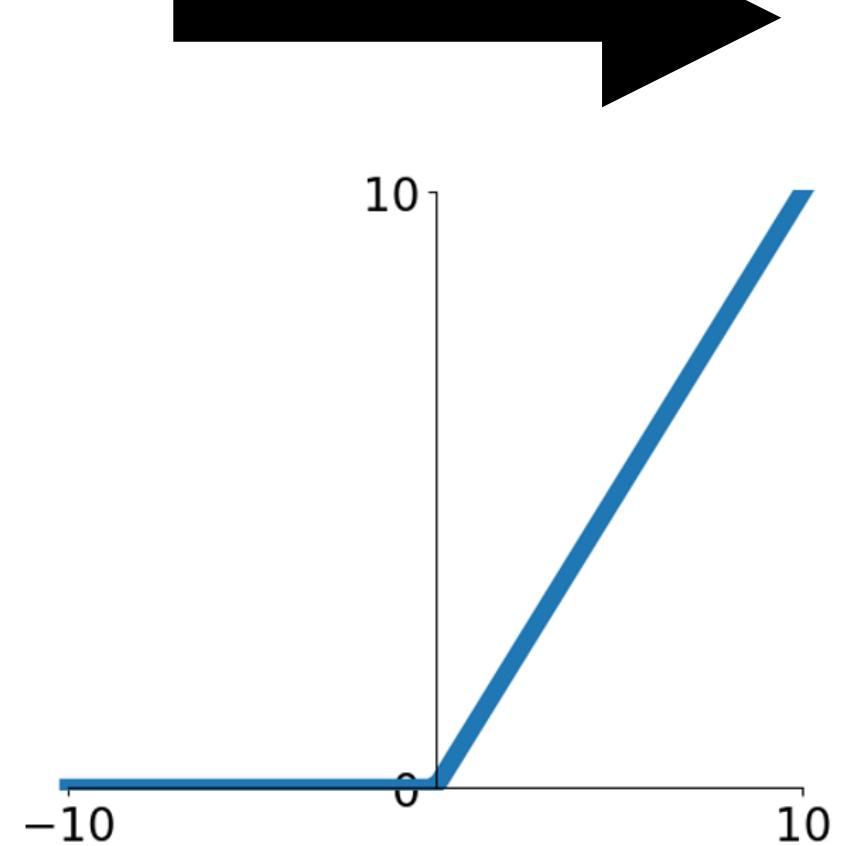
# Space Warping

Points not linearly separable  
in original space



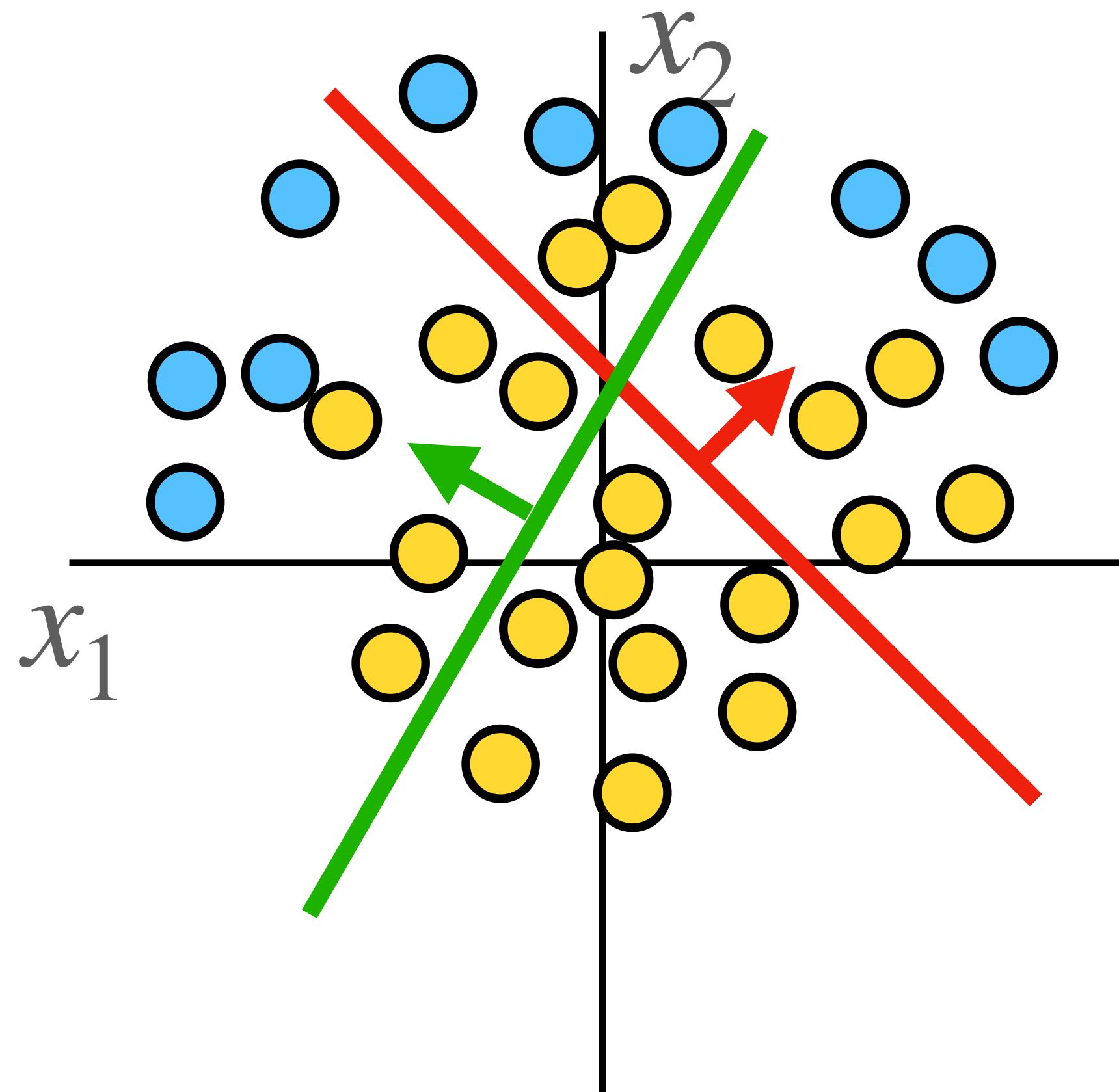
Consider a neural net hidden layer:  $h = \text{ReLU}(Wx + b)$   
 $= \max(0, Wx + b)$  where  $x, b, h$  are each 2-dimensional

Feature transform:  
 $h = \text{ReLU}(Wx + b)$



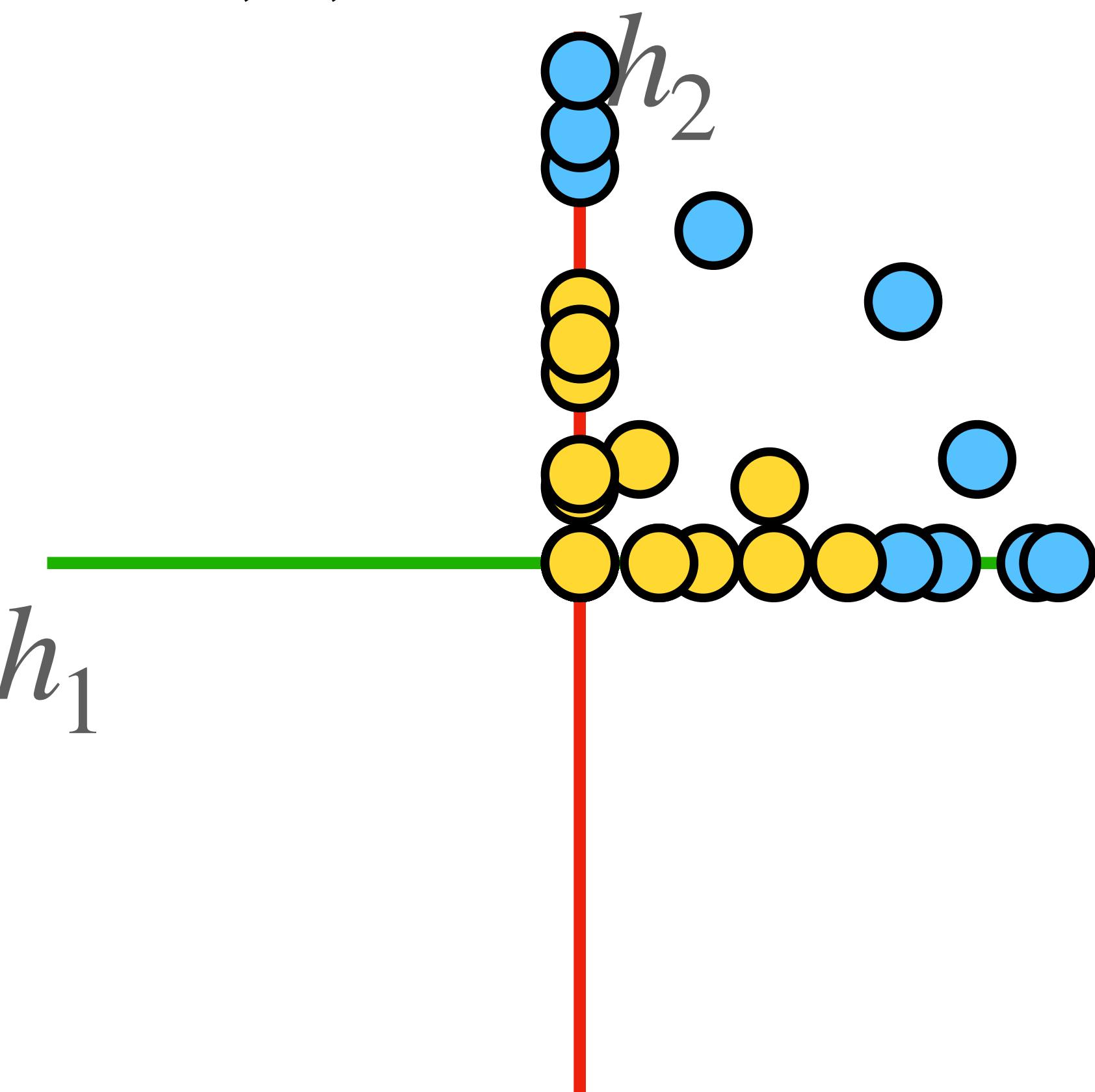
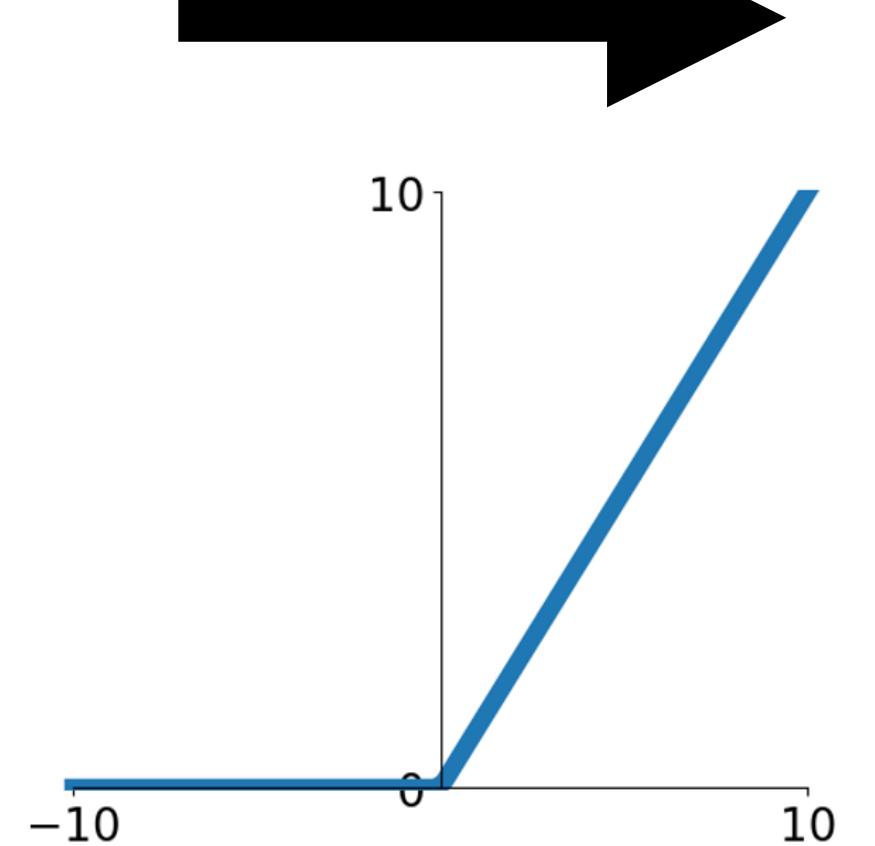
# Space Warping

Points not linearly separable  
in original space



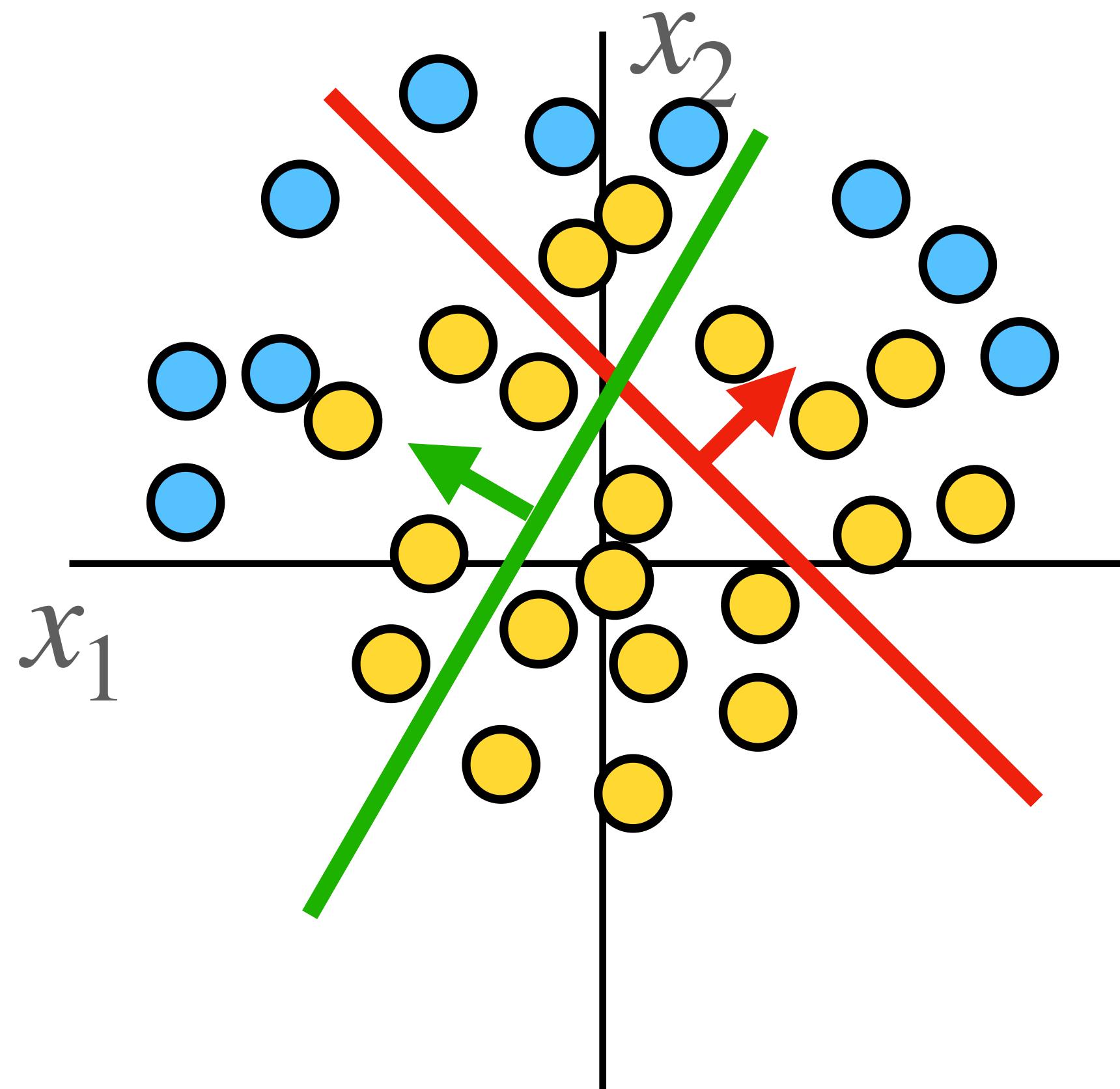
Consider a neural net hidden layer:  $h = \text{ReLU}(Wx + b)$   
 $= \max(0, Wx + b)$  where  $x, b, h$  are each 2-dimensional

Feature transform:  
 $h = \text{ReLU}(Wx + b)$



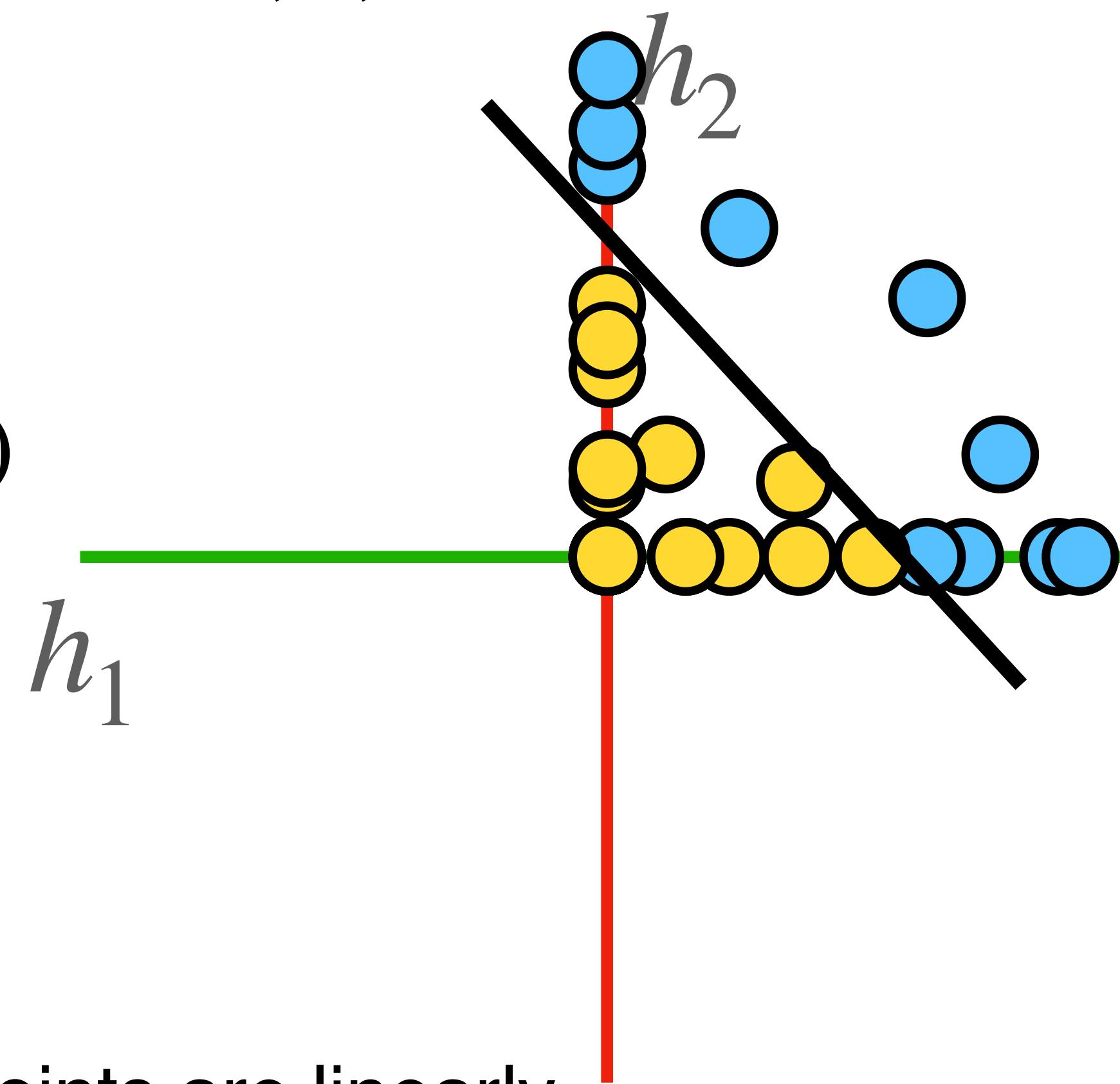
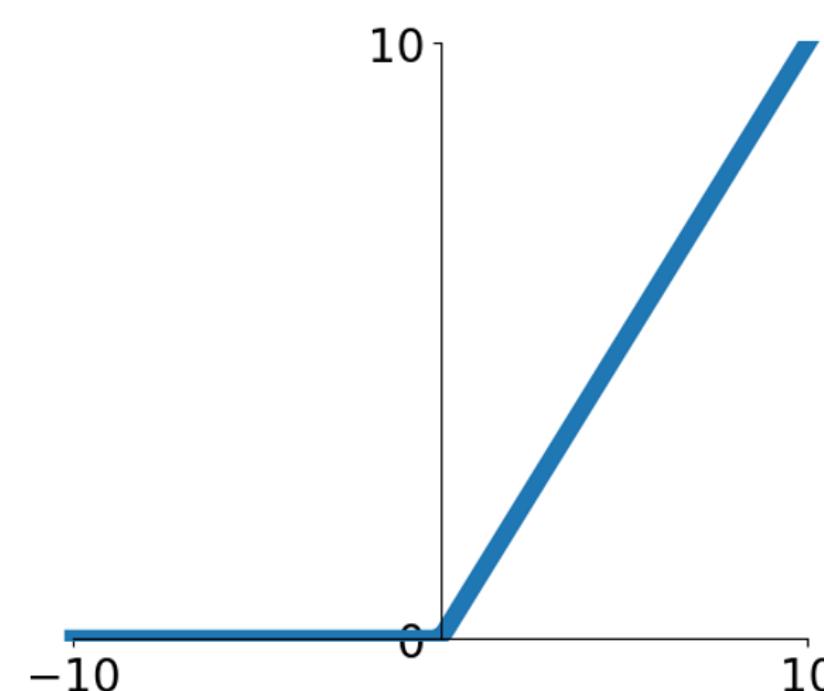
# Space Warping

Points not linearly separable  
in original space



Consider a neural net hidden layer:  $h = \text{ReLU}(Wx + b)$   
 $= \max(0, Wx + b)$  where  $x, b, h$  are each 2-dimensional

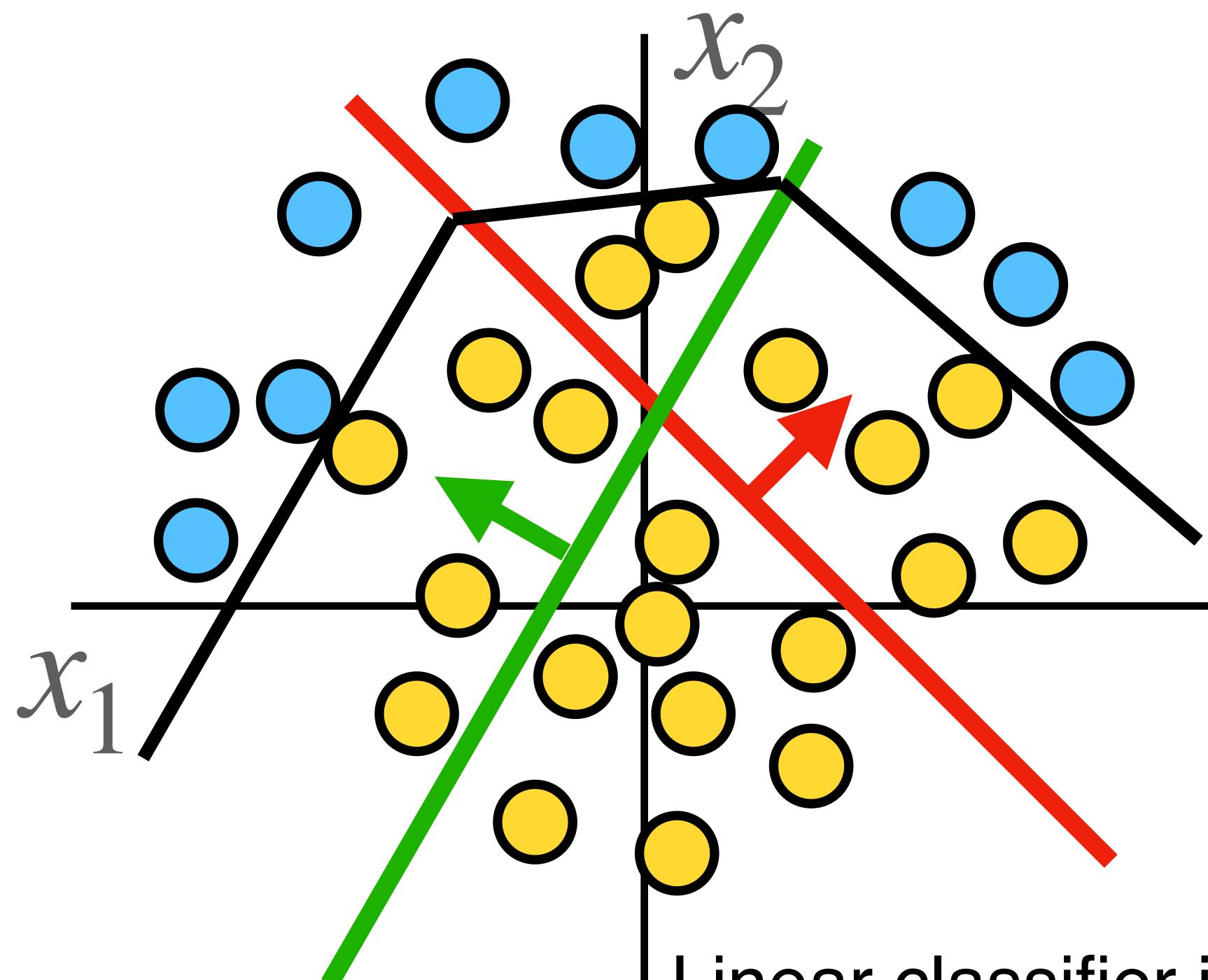
Feature transform:  
 $h = \text{ReLU}(Wx + b)$



Points are linearly  
separable in feature space!

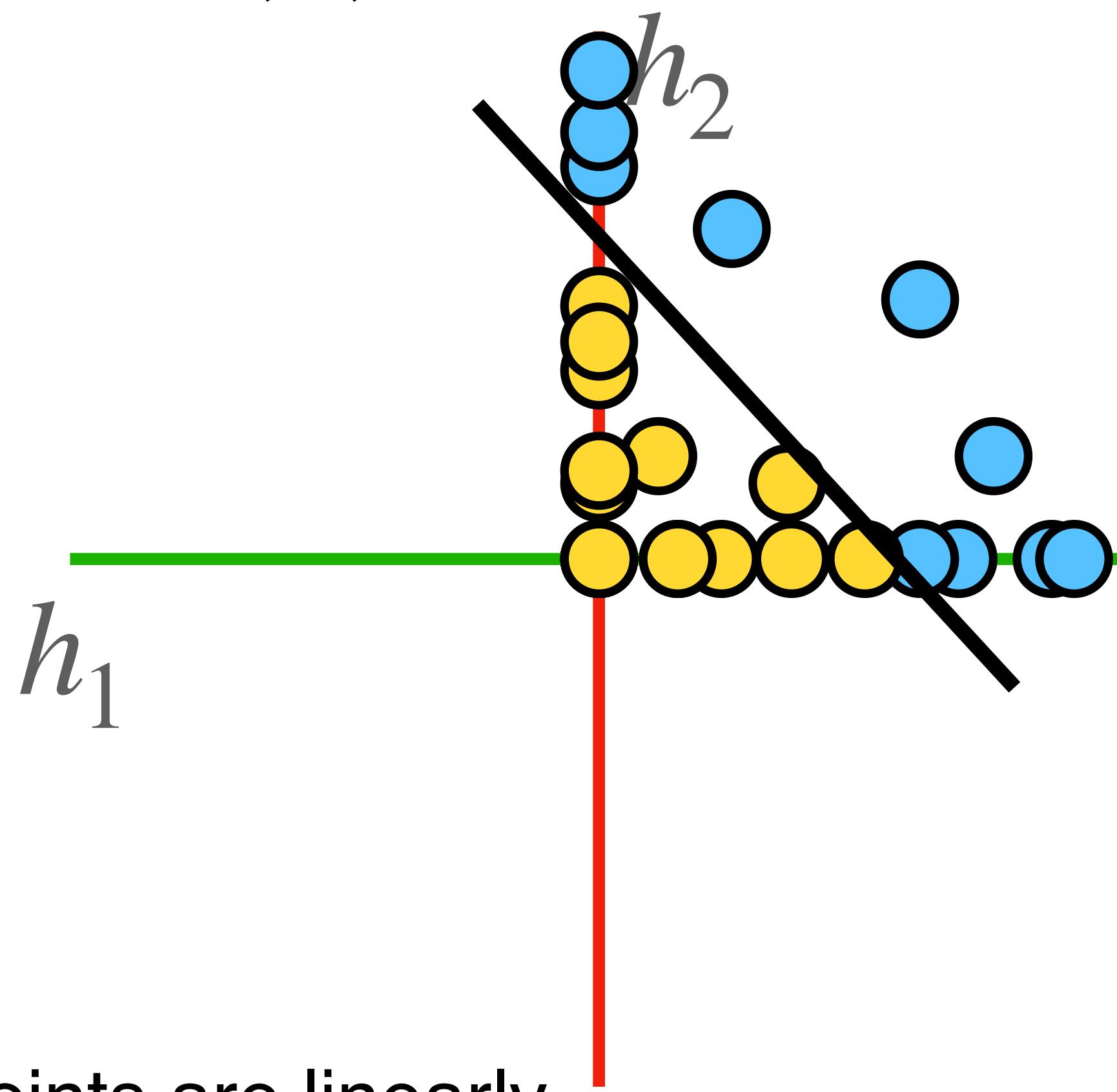
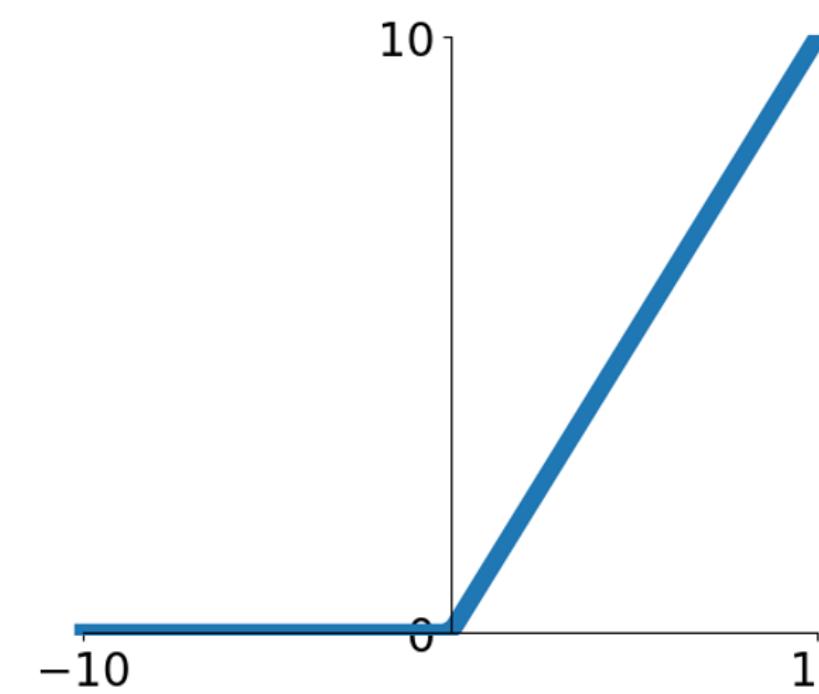
# Space Warping

Points not linearly separable  
in original space

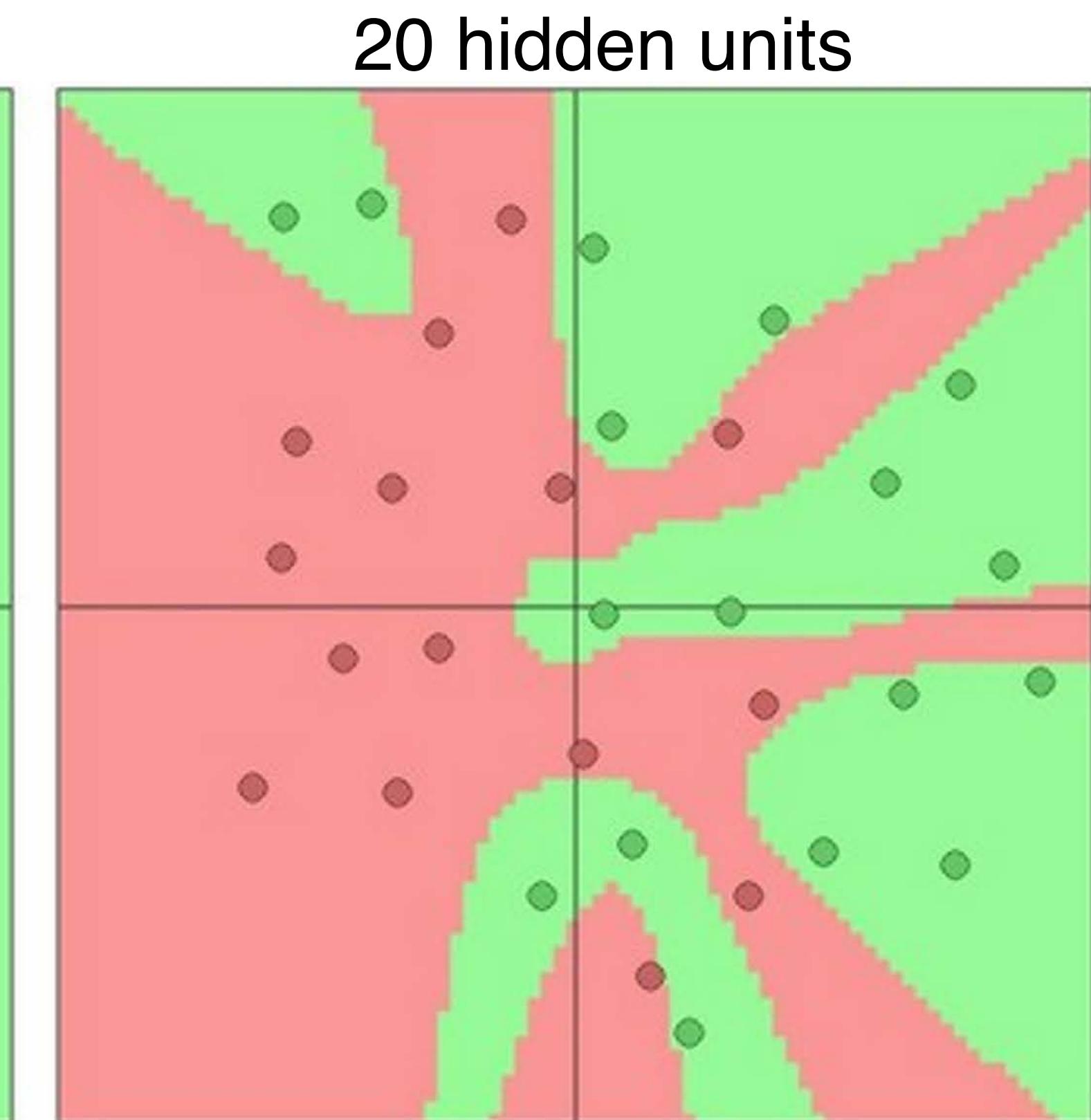
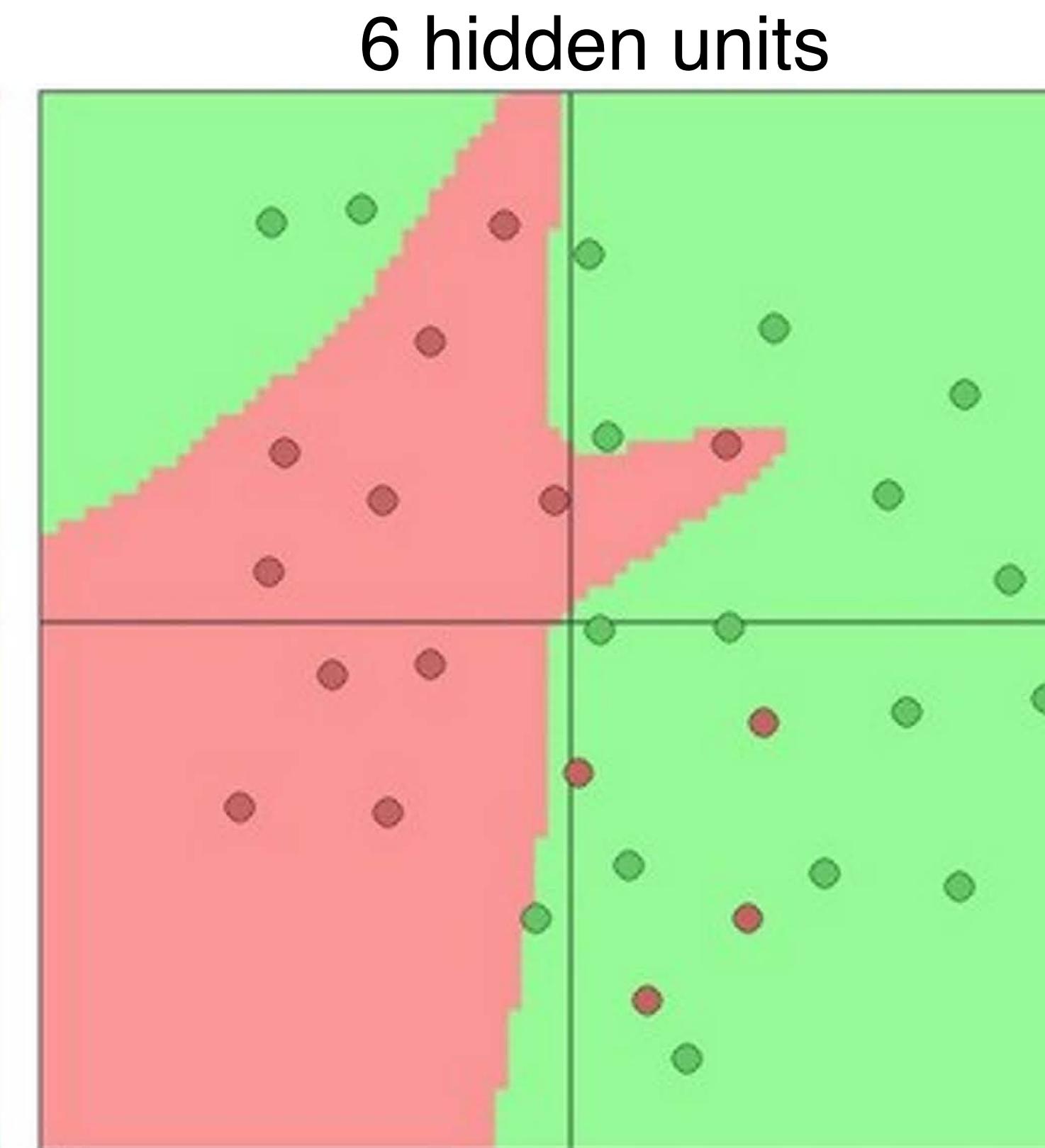
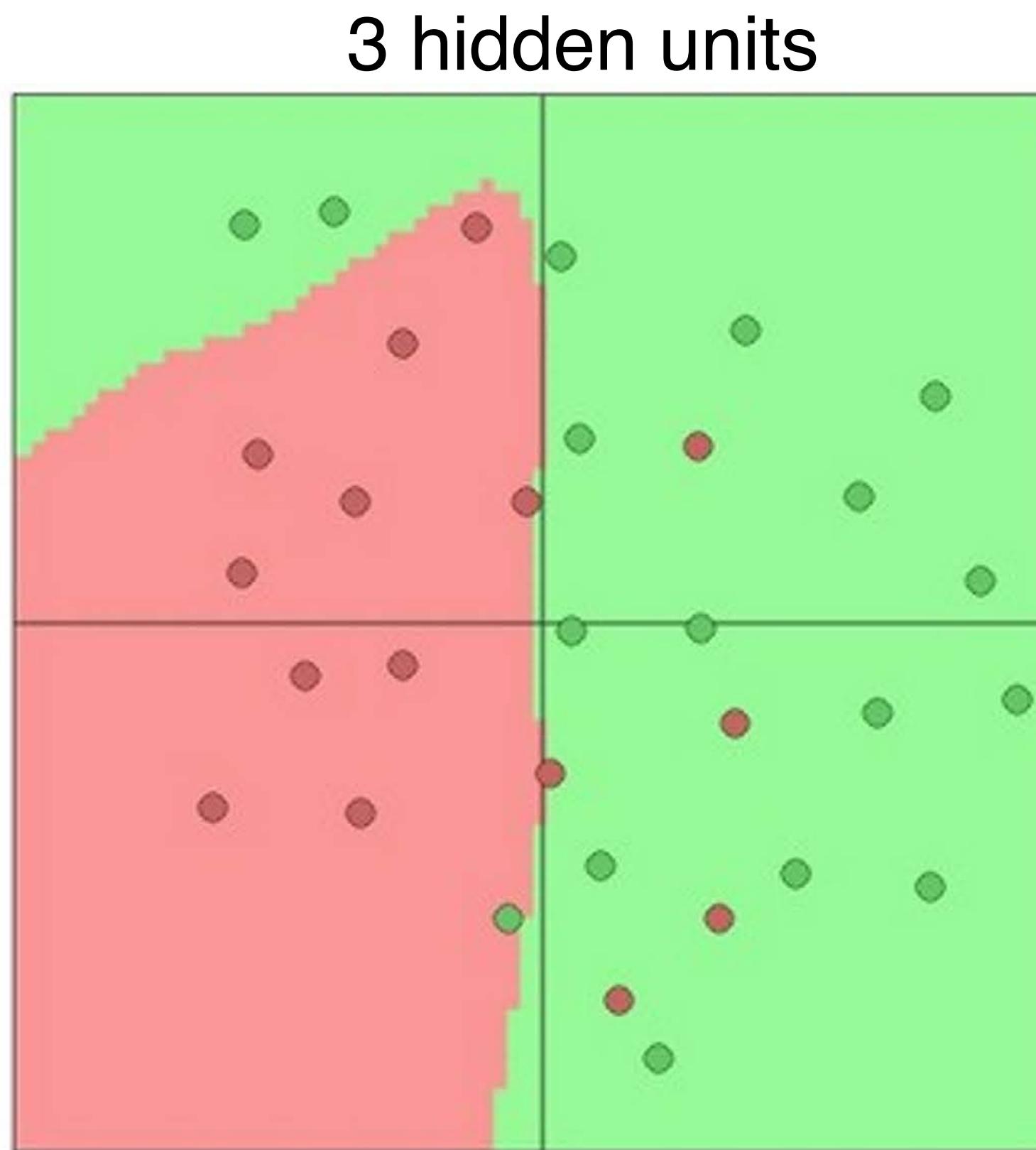


Consider a neural net hidden layer:  $h = \text{ReLU}(Wx + b)$   
=  $\max(0, Wx + b)$  where  $x, b, h$  are each 2-dimensional

Feature transform:  
 $h = \text{ReLU}(Wx + b)$



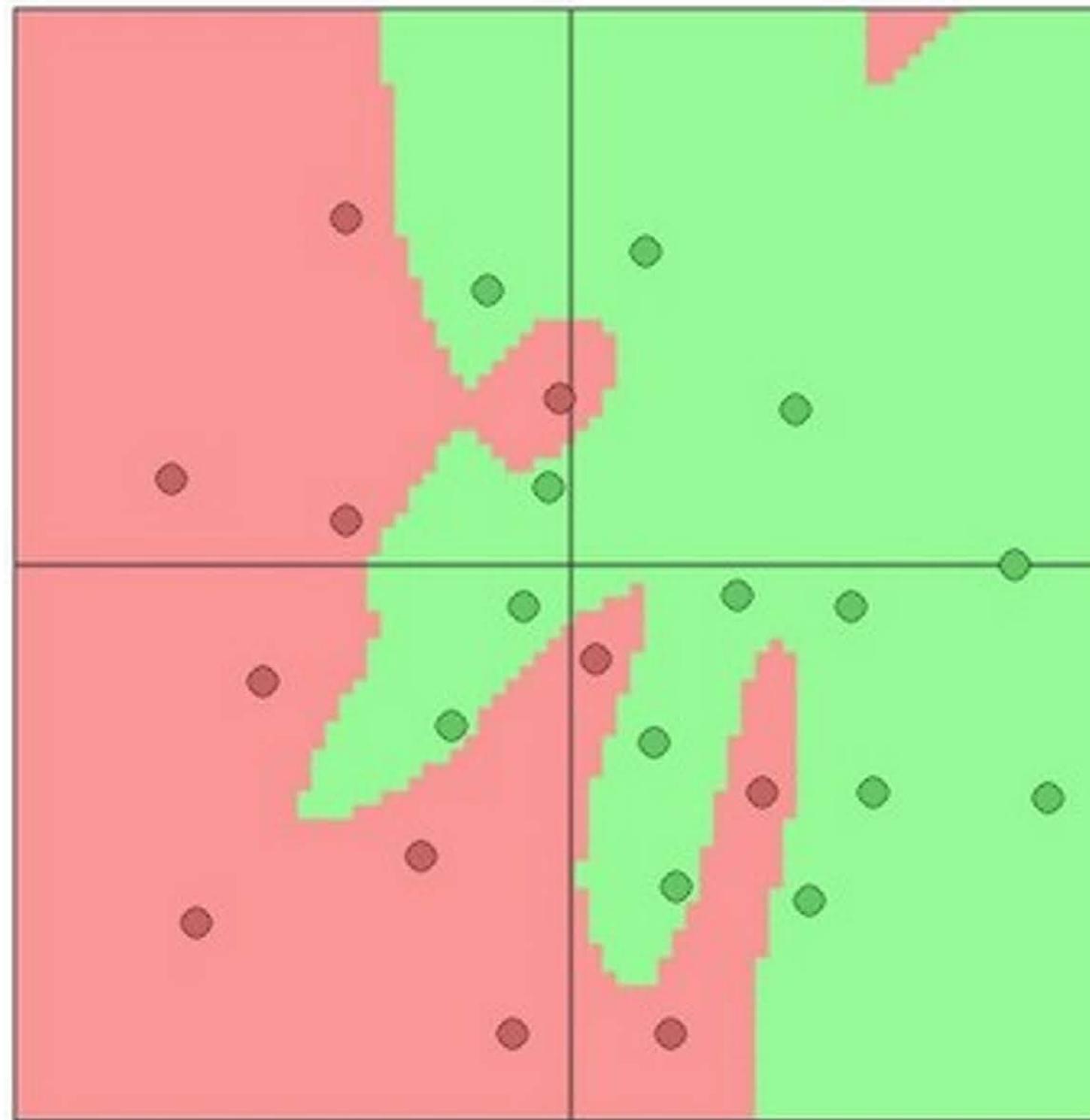
# Setting the number of layers and their sizes



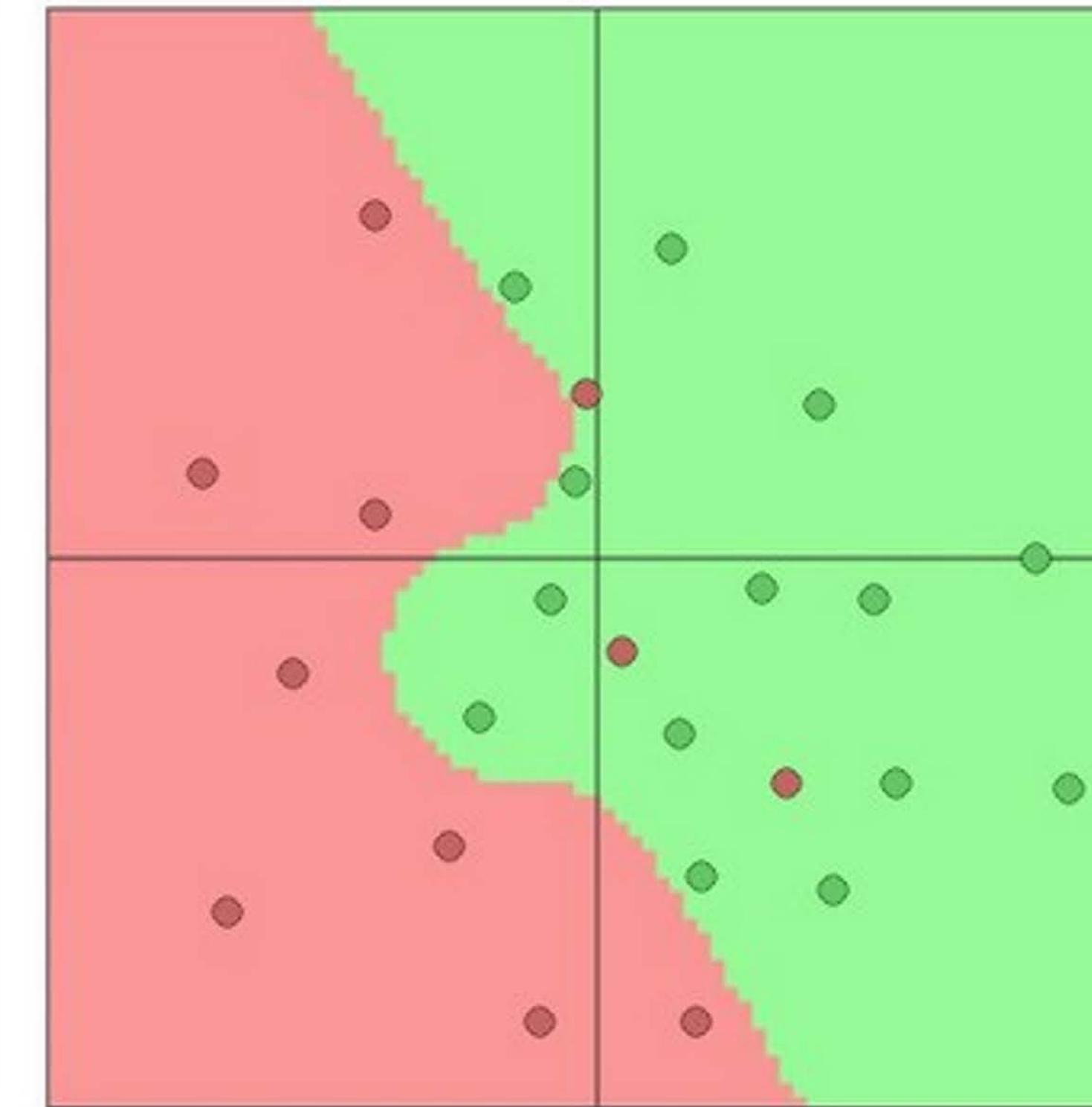
More hidden units = more capacity

# Don't regularize with size; instead use stronger L2

$\lambda = 0.001$



$\lambda = 0.01$



$\lambda = 0.1$



Web demo with ConvNetJS: <https://cs.stanford.edu/people/karpathy/convnetjs/demo/classify2d.html>

# Universal Approximation

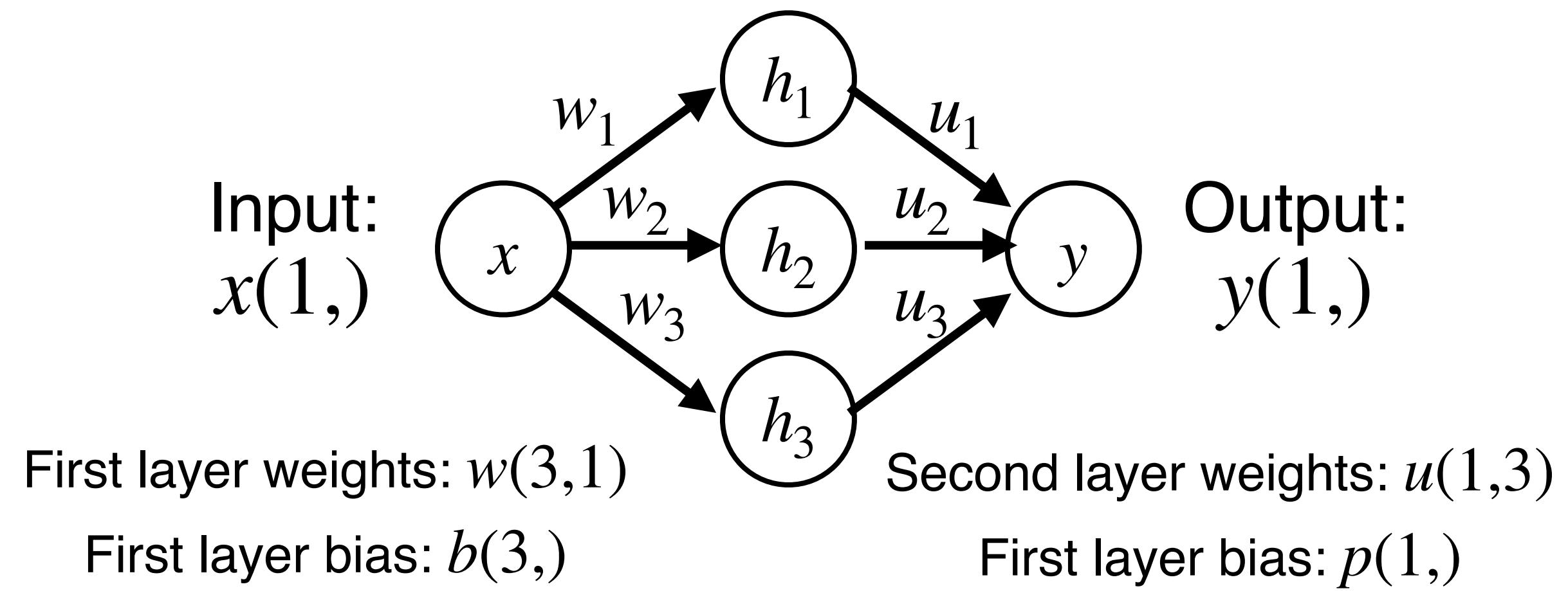
---

A neural network with one hidden layer can approximate any function  $f: \mathbb{R}^N \rightarrow \mathbb{R}^M$  with arbitrary precision\*

\*Many technical conditions: Only holds on compact subsets of  $\mathbb{R}^N$ ; function must be continuous; need to define "arbitrary precision"; etc.

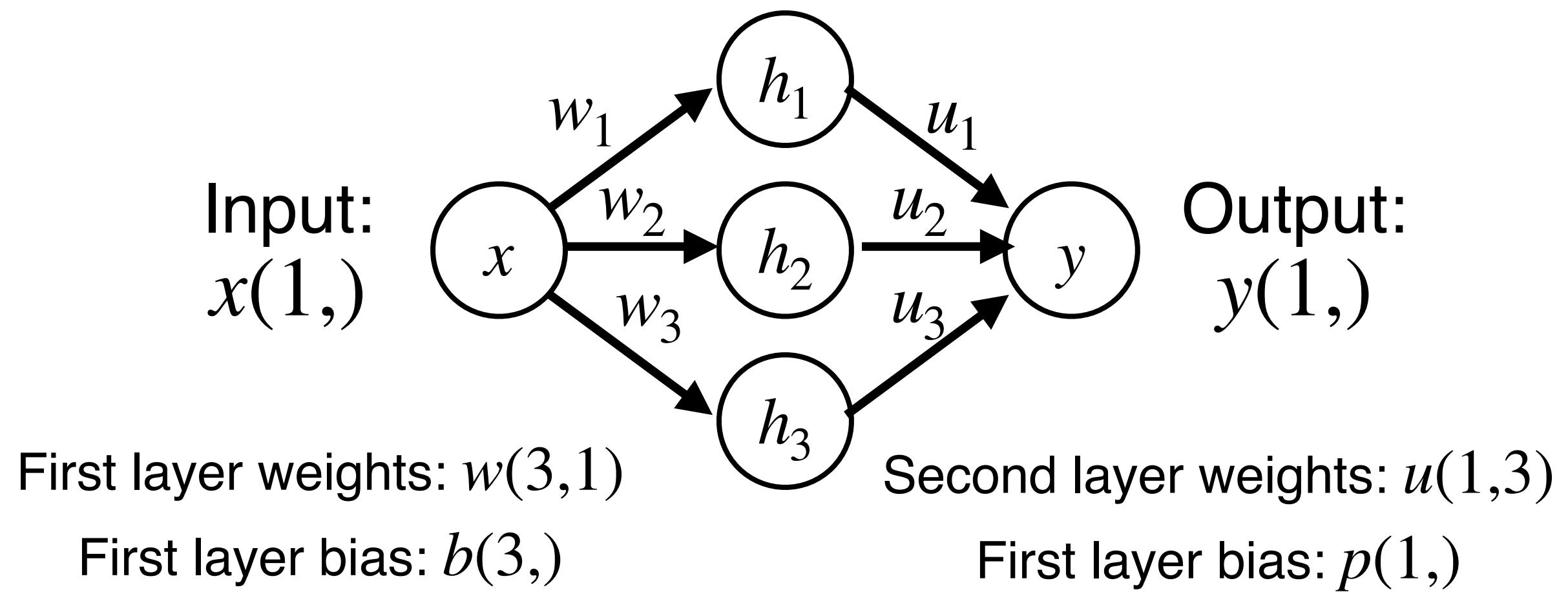
# Universal Approximation

Example: Approximating a function  $f: \mathbb{R} \rightarrow \mathbb{R}$  with a two-layer ReLU network



# Universal Approximation

Example: Approximating a function  $f: \mathbb{R} \rightarrow \mathbb{R}$  with a two-layer ReLU network



$$h_1 = \max(0, w_1 x + b_1)$$

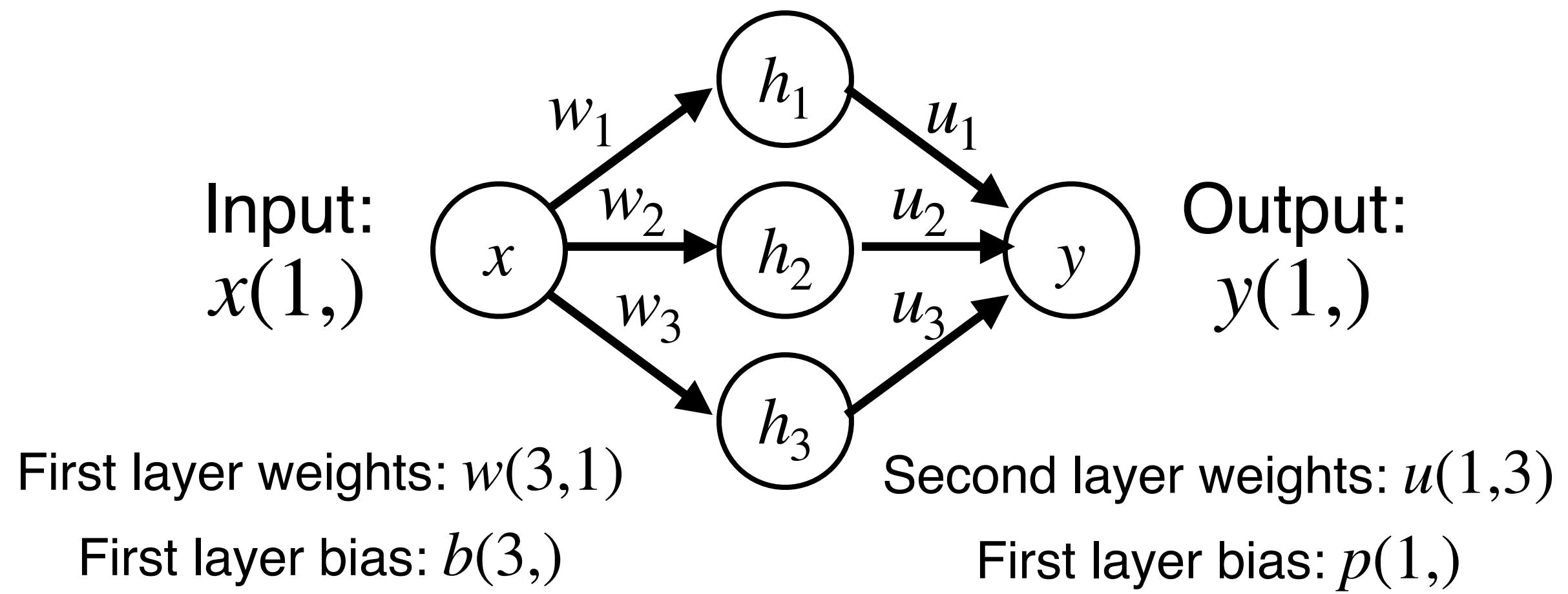
$$h_2 = \max(0, w_2 x + b_2)$$

$$h_3 = \max(0, w_3 x + b_3)$$

$$y = u_1 h_1 + u_2 h_2 + u_3 h_3 + p$$

# Universal Approximation

Example: Approximating a function  $f: \mathbb{R} \rightarrow \mathbb{R}$  with a two-layer ReLU network



$$h_1 = \max(0, w_1 x + b_1)$$

$$h_2 = \max(0, w_2 x + b_2)$$

$$h_3 = \max(0, w_3 x + b_3)$$

$$y = u_1 h_1 + u_2 h_2 + u_3 h_3 + p$$

$$y = u_1 \max(0, w_1 x + b_1)$$

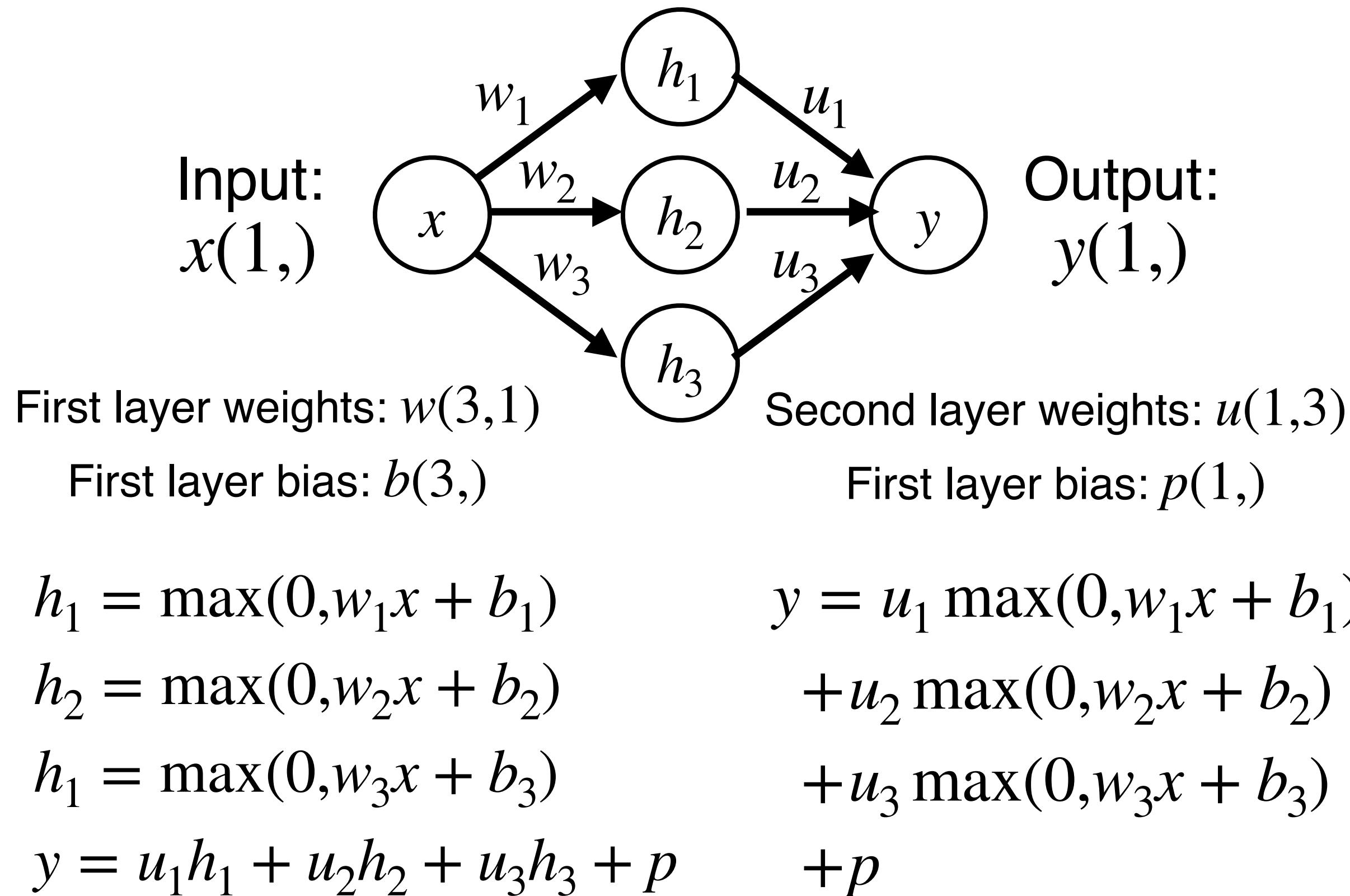
$$+ u_2 \max(0, w_2 x + b_2)$$

$$+ u_3 \max(0, w_3 x + b_3)$$

$$+ p$$

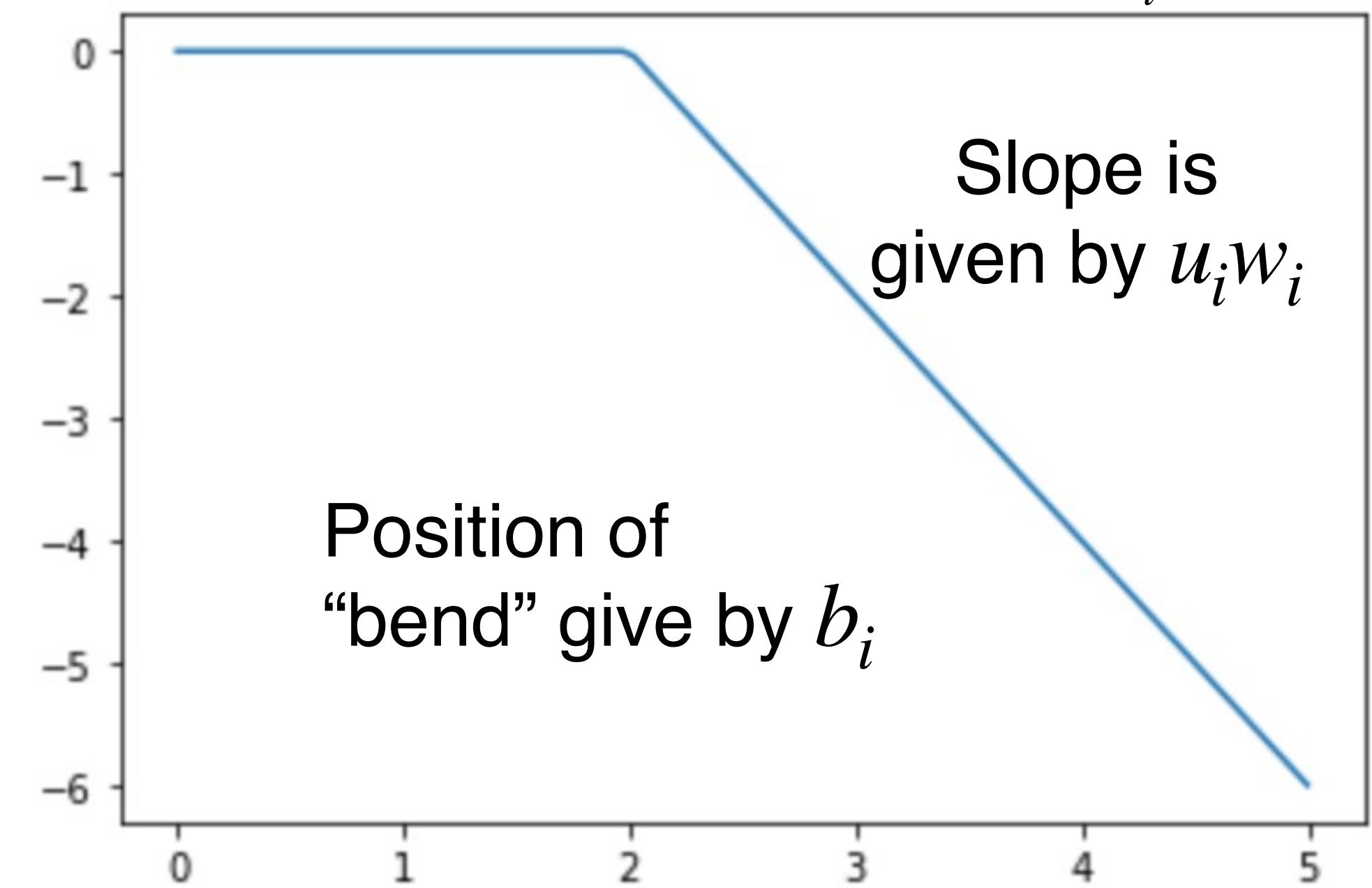
# Universal Approximation

Example: Approximating a function  $f: \mathbb{R} \rightarrow \mathbb{R}$  with a two-layer ReLU network



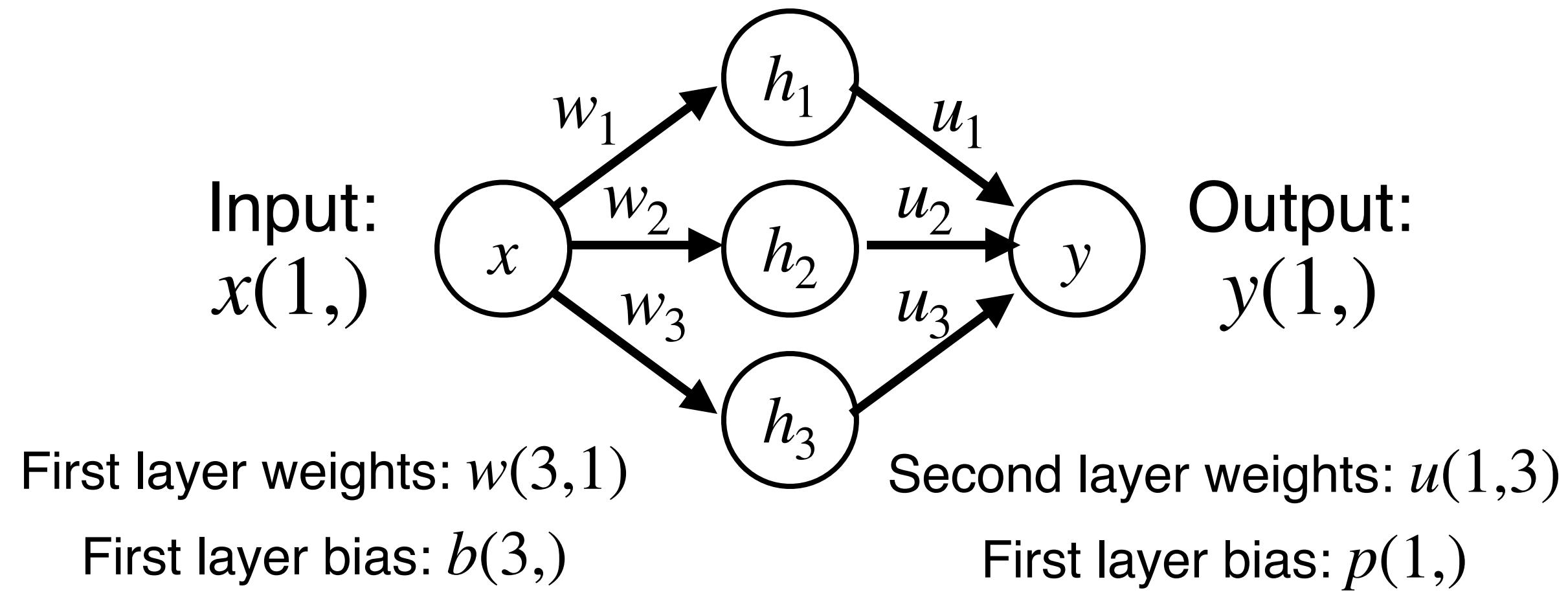
Output is a sum of shifted,  
scaled ReLUs:

Flip left / right based on sign of  $w_i$



# Universal Approximation

Example: Approximating a function  $f: \mathbb{R} \rightarrow \mathbb{R}$  with a two-layer ReLU network



$$h_1 = \max(0, w_1 x + b_1)$$

$$h_2 = \max(0, w_2 x + b_2)$$

$$h_3 = \max(0, w_3 x + b_3)$$

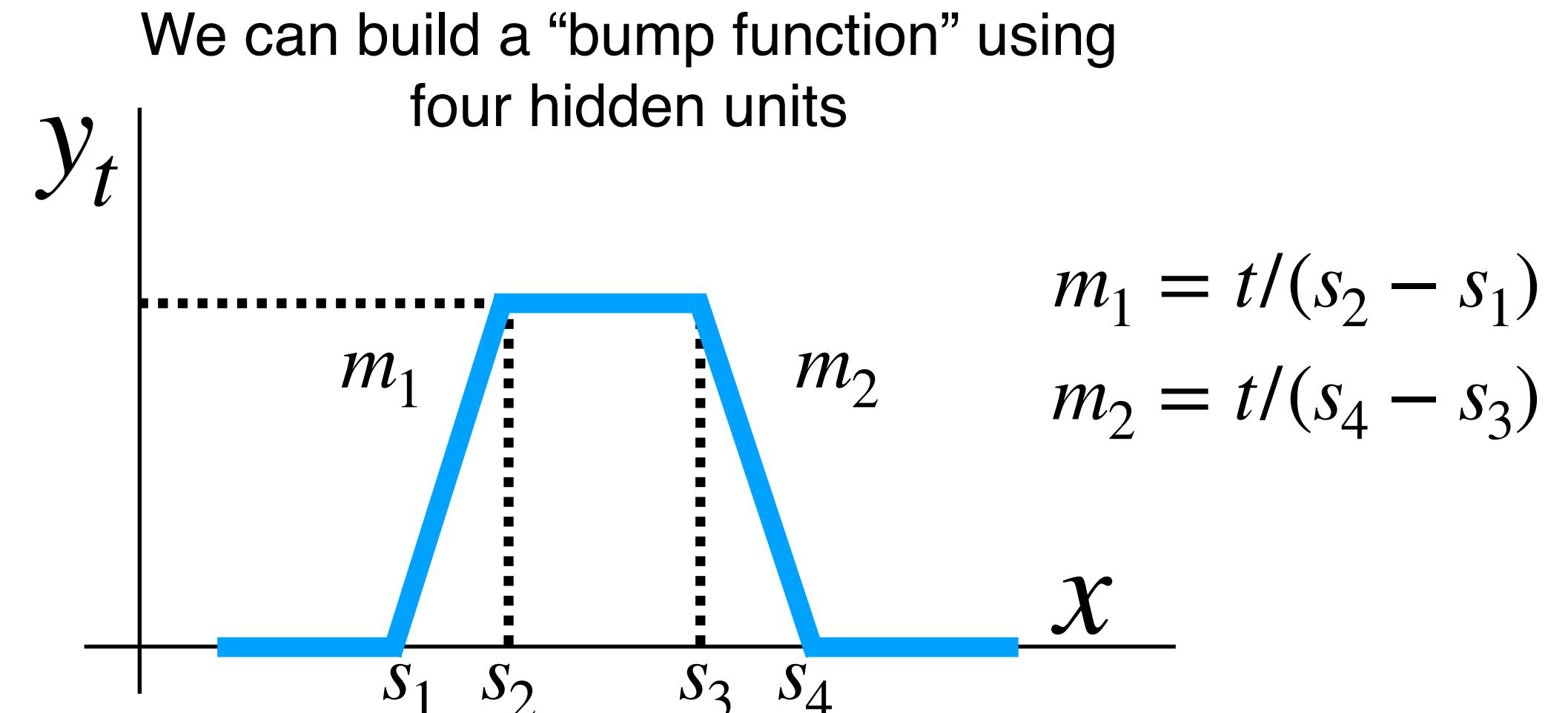
$$y = u_1 h_1 + u_2 h_2 + u_3 h_3 + p$$

$$y = u_1 \max(0, w_1 x + b_1)$$

$$+ u_2 \max(0, w_2 x + b_2)$$

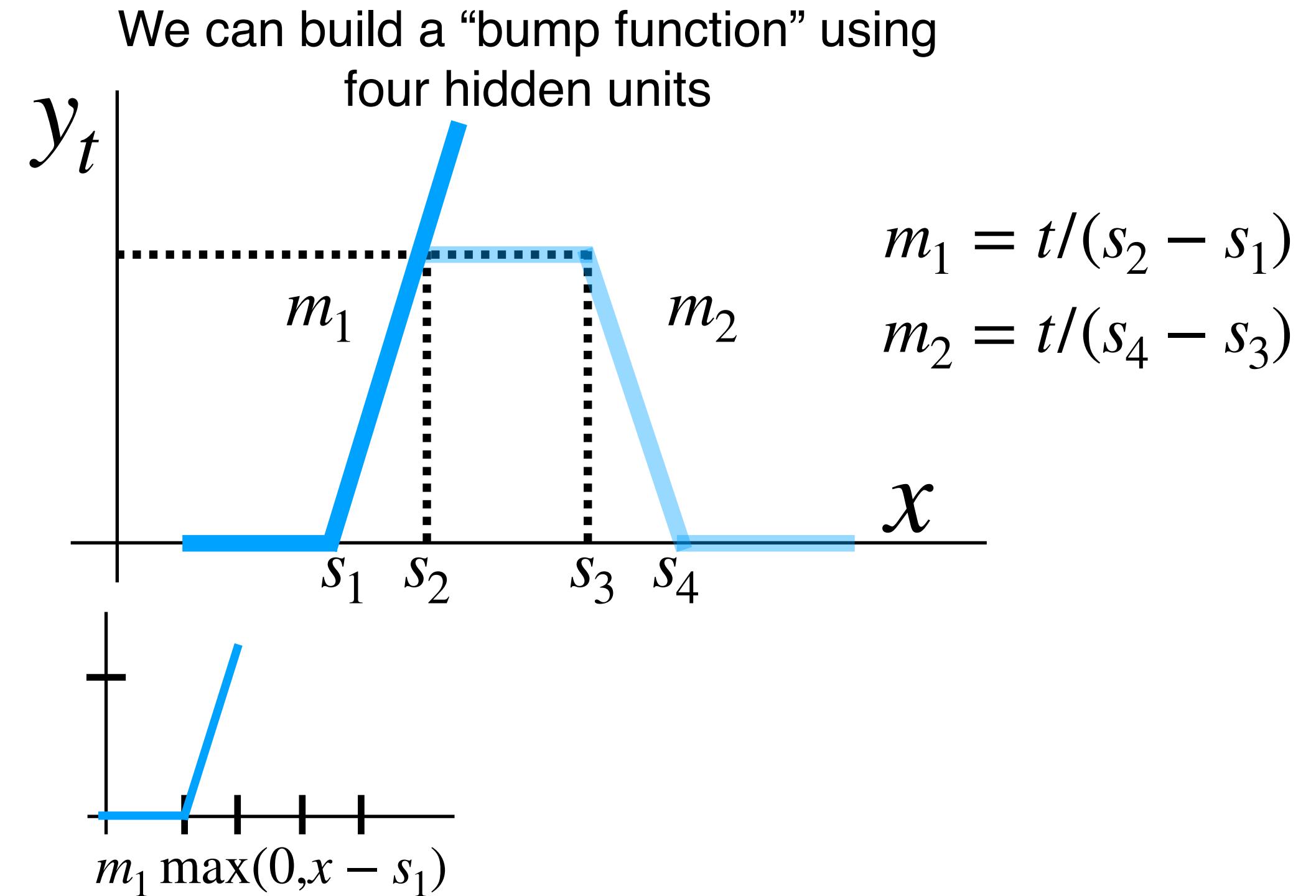
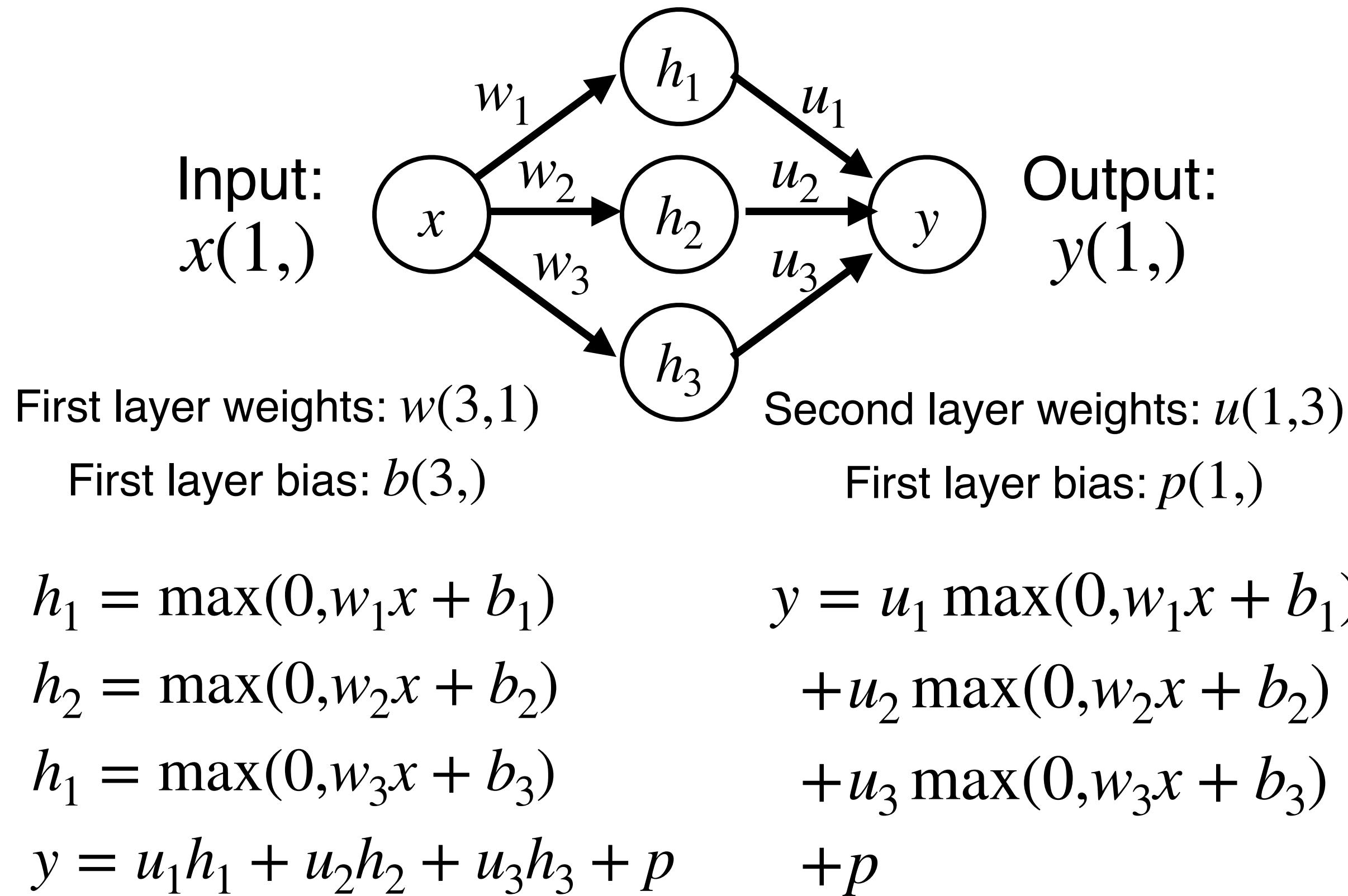
$$+ u_3 \max(0, w_3 x + b_3)$$

$$+ p$$



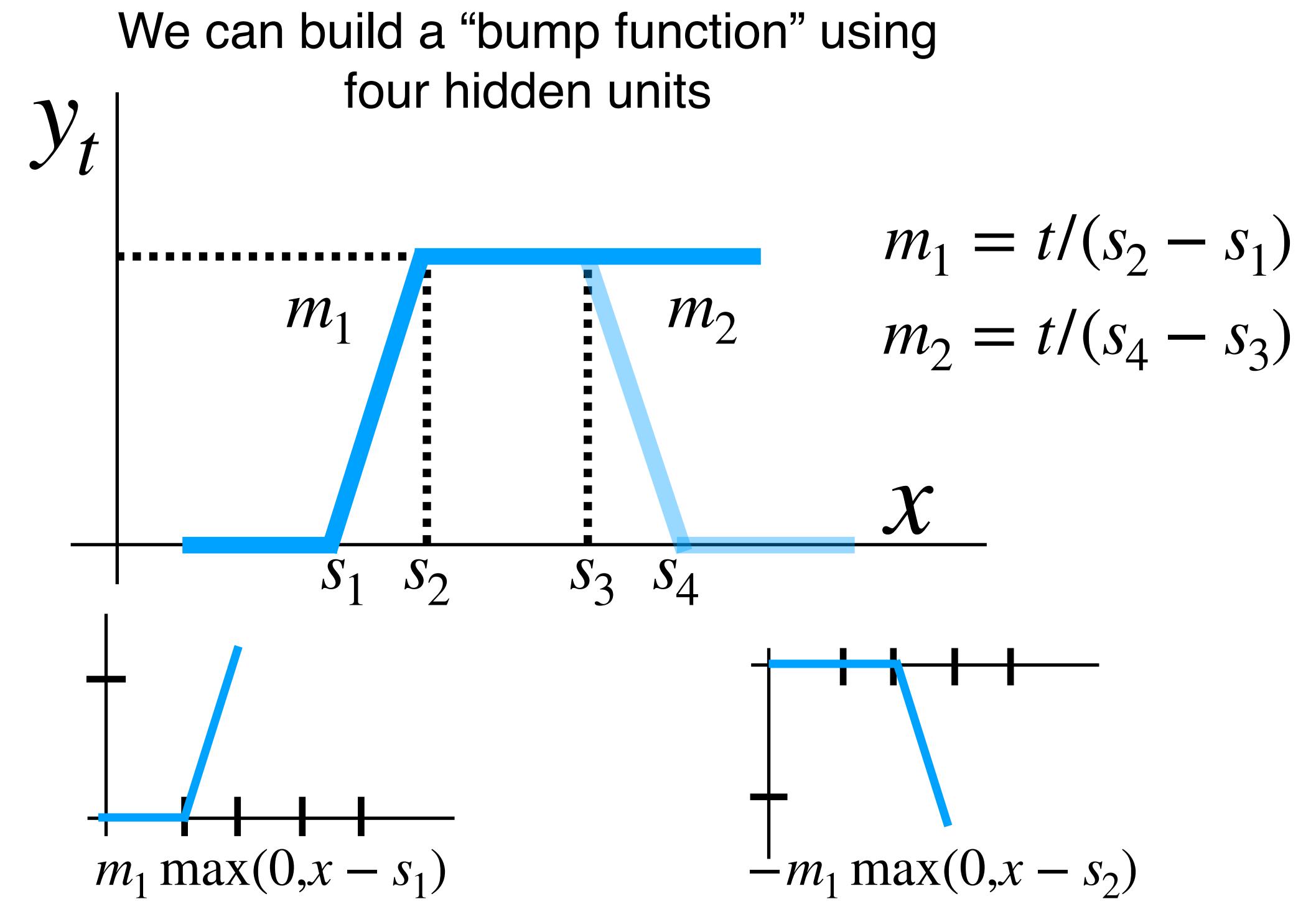
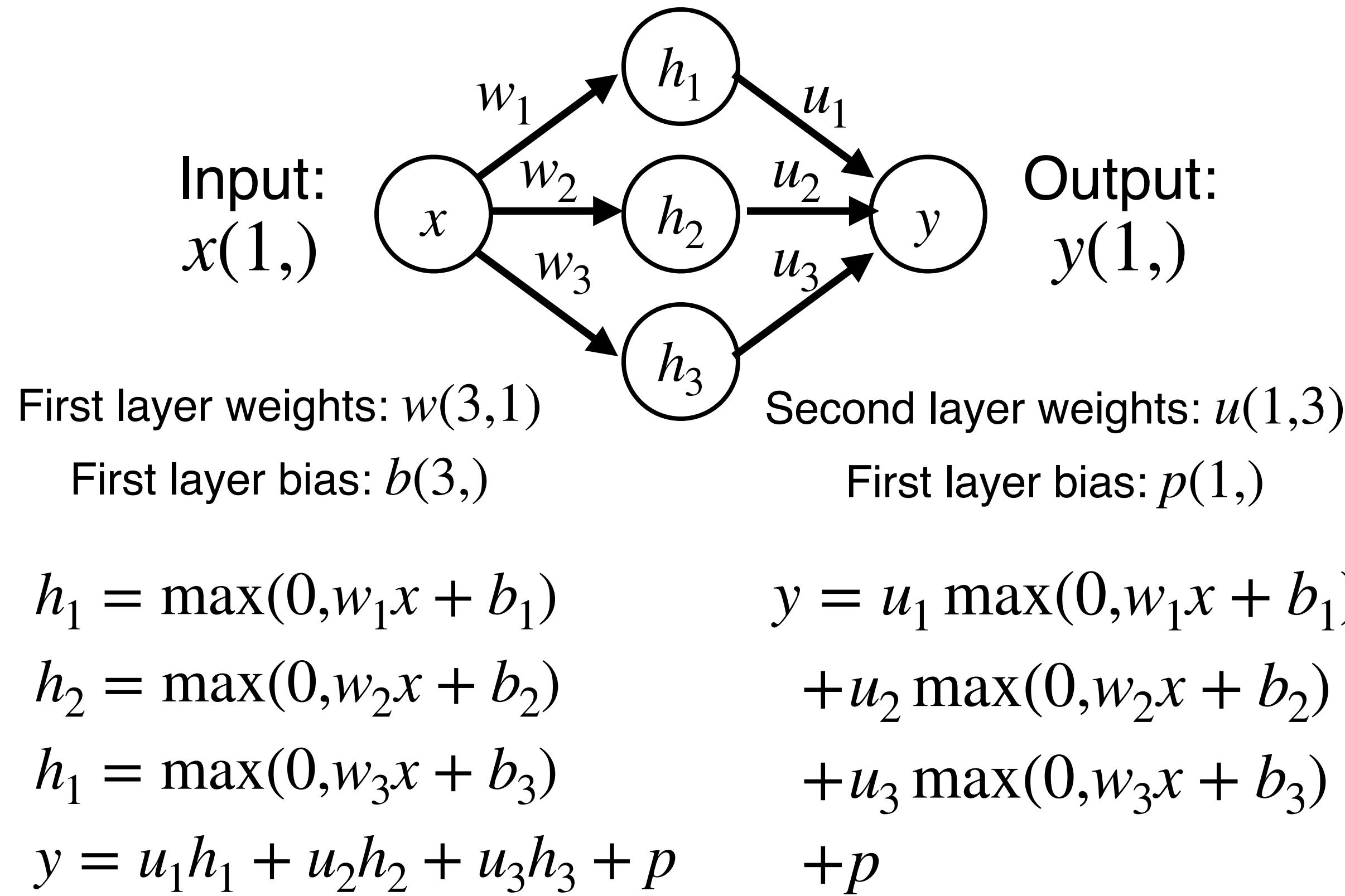
# Universal Approximation

Example: Approximating a function  $f: \mathbb{R} \rightarrow \mathbb{R}$  with a two-layer ReLU network



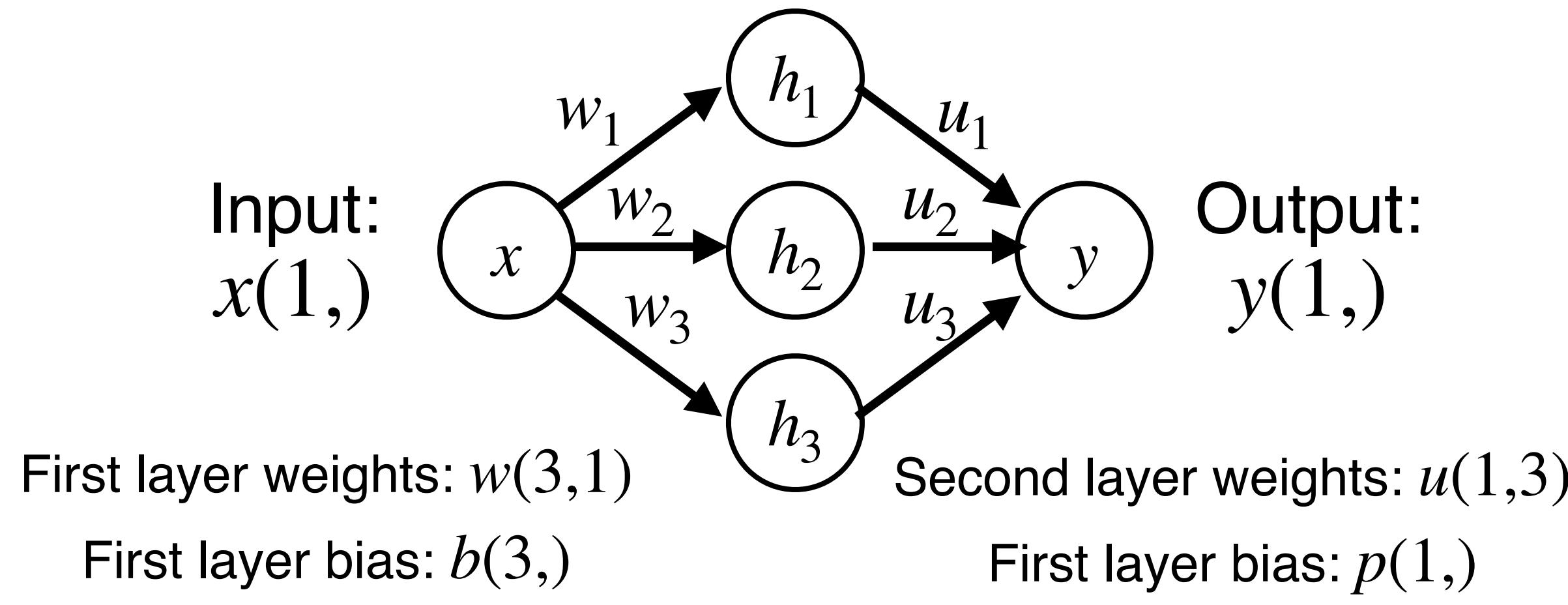
# Universal Approximation

Example: Approximating a function  $f: \mathbb{R} \rightarrow \mathbb{R}$  with a two-layer ReLU network



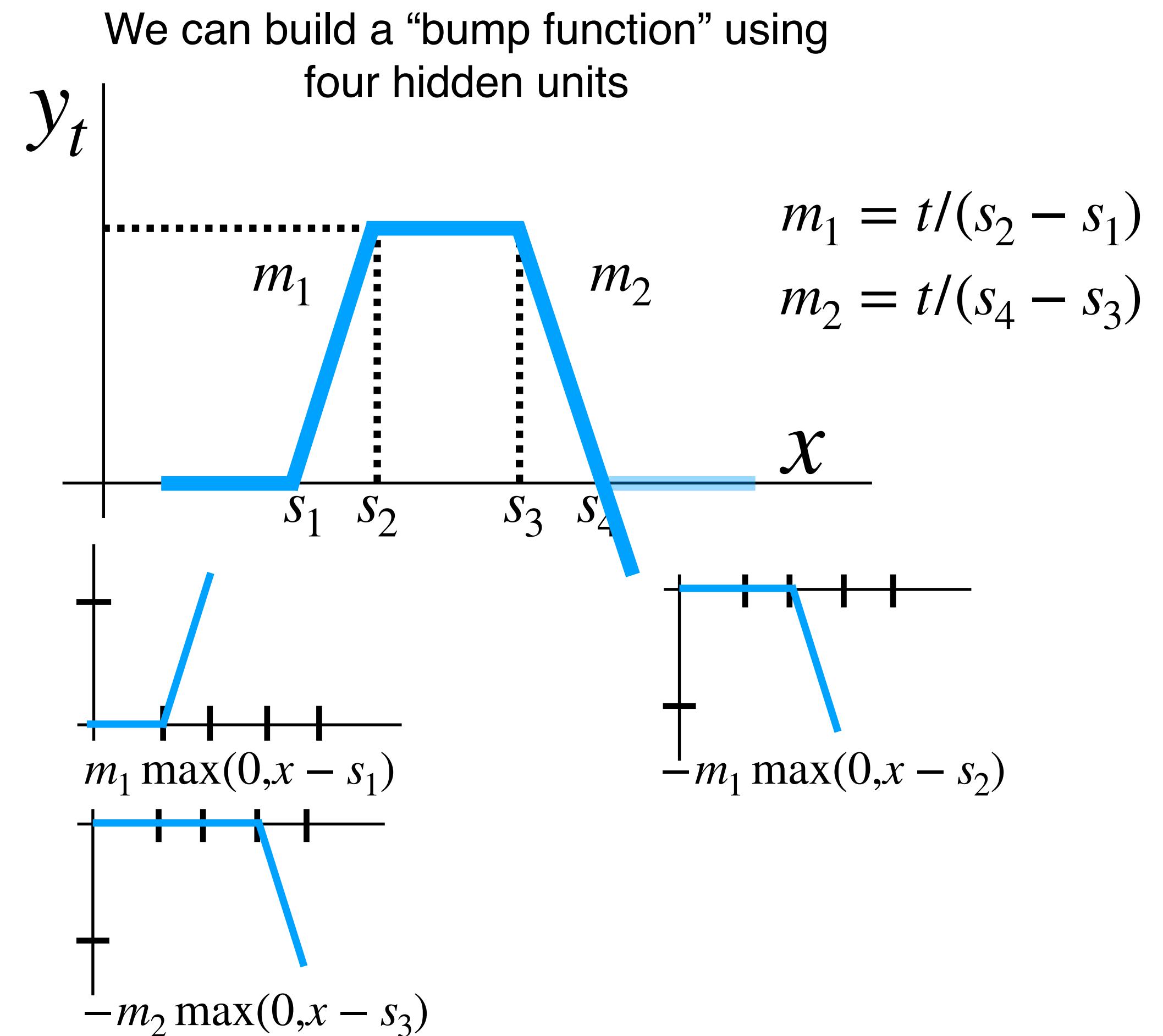
# Universal Approximation

Example: Approximating a function  $f: \mathbb{R} \rightarrow \mathbb{R}$  with a two-layer ReLU network



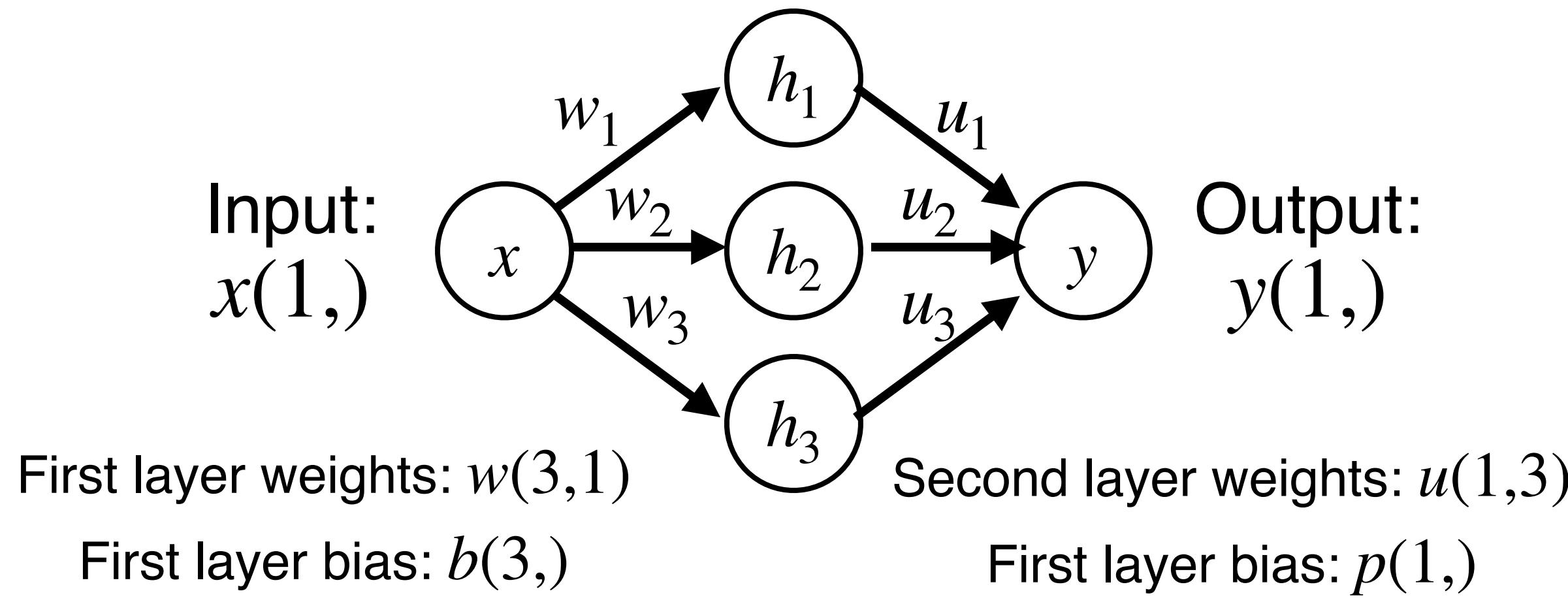
$$\begin{aligned} h_1 &= \max(0, w_1 x + b_1) \\ h_2 &= \max(0, w_2 x + b_2) \\ h_3 &= \max(0, w_3 x + b_3) \\ y &= u_1 h_1 + u_2 h_2 + u_3 h_3 + p \end{aligned}$$

$$\begin{aligned} y &= u_1 \max(0, w_1 x + b_1) \\ &\quad + u_2 \max(0, w_2 x + b_2) \\ &\quad + u_3 \max(0, w_3 x + b_3) \\ &\quad + p \end{aligned}$$



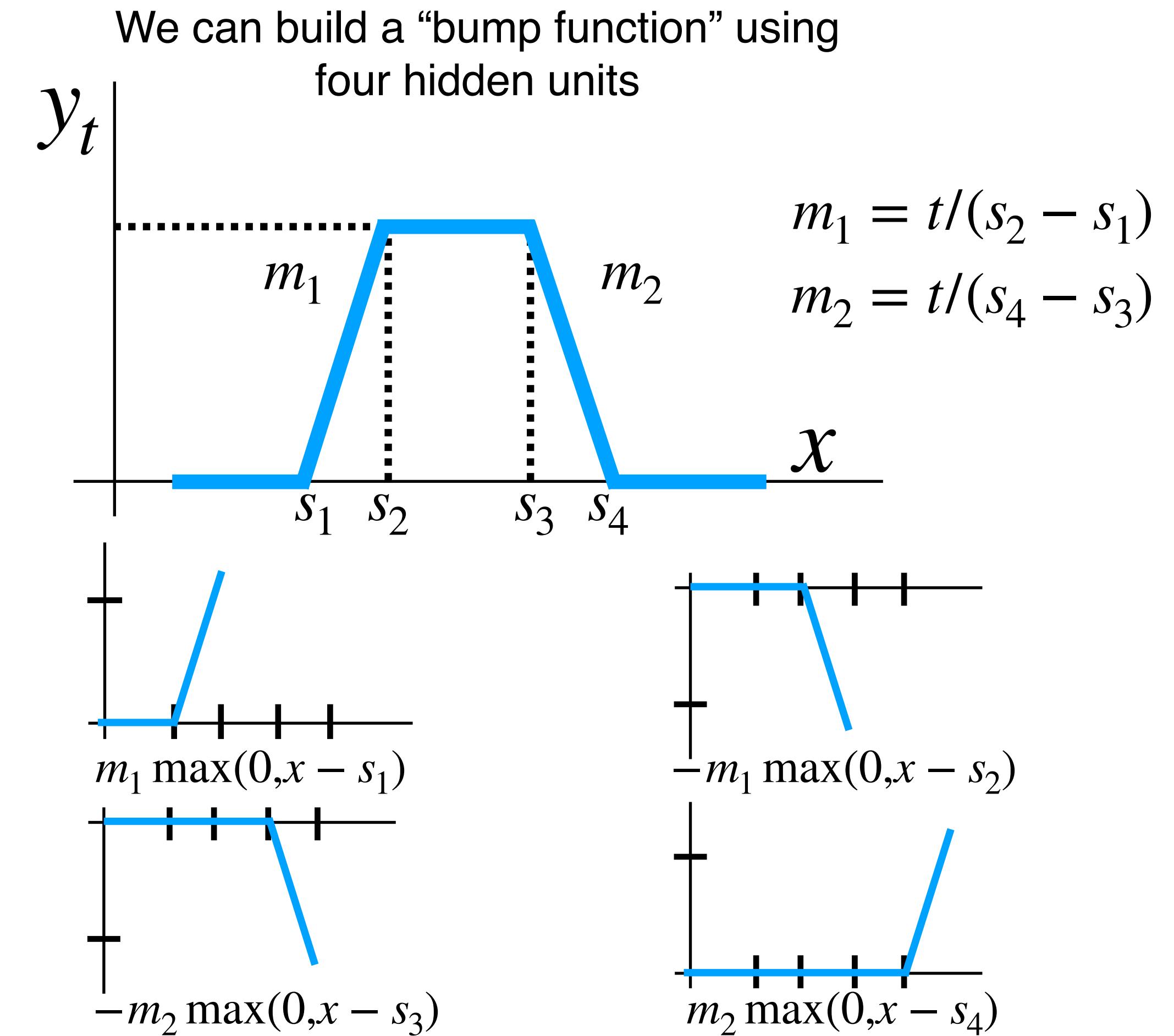
# Universal Approximation

Example: Approximating a function  $f: \mathbb{R} \rightarrow \mathbb{R}$  with a two-layer ReLU network



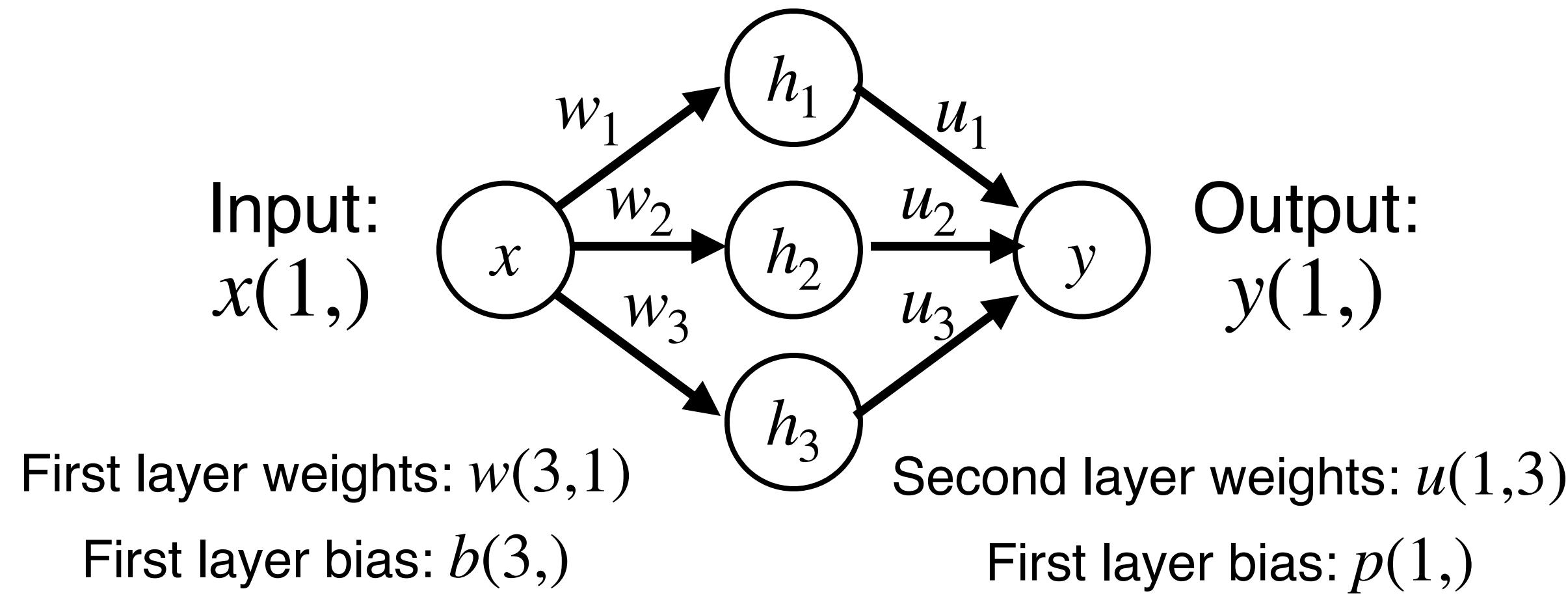
$$\begin{aligned} h_1 &= \max(0, w_1 x + b_1) \\ h_2 &= \max(0, w_2 x + b_2) \\ h_3 &= \max(0, w_3 x + b_3) \\ y &= u_1 h_1 + u_2 h_2 + u_3 h_3 + p \end{aligned}$$

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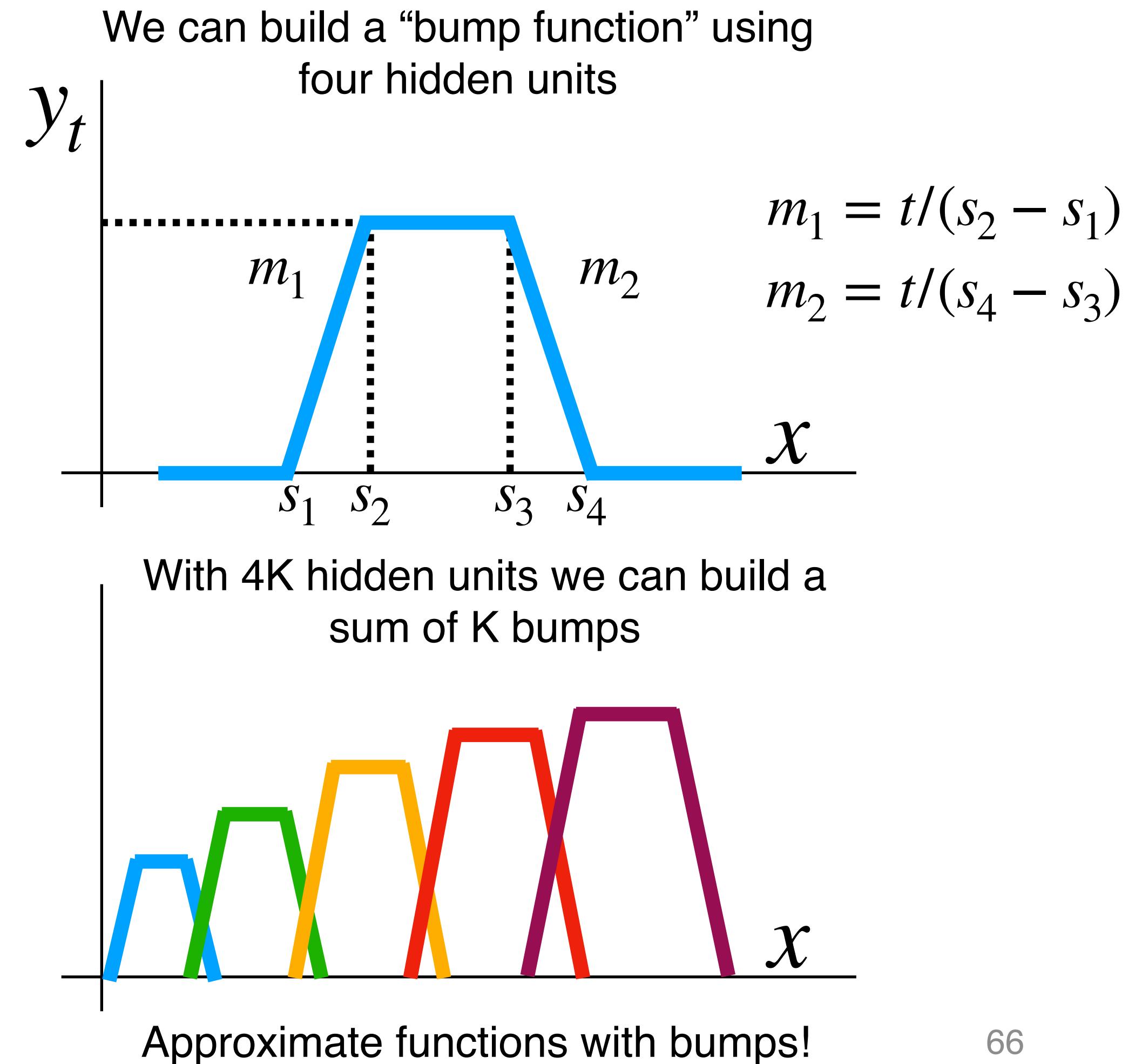
# Universal Approximation

Example: Approximating a function  $f: \mathbb{R} \rightarrow \mathbb{R}$  with a two-layer ReLU network



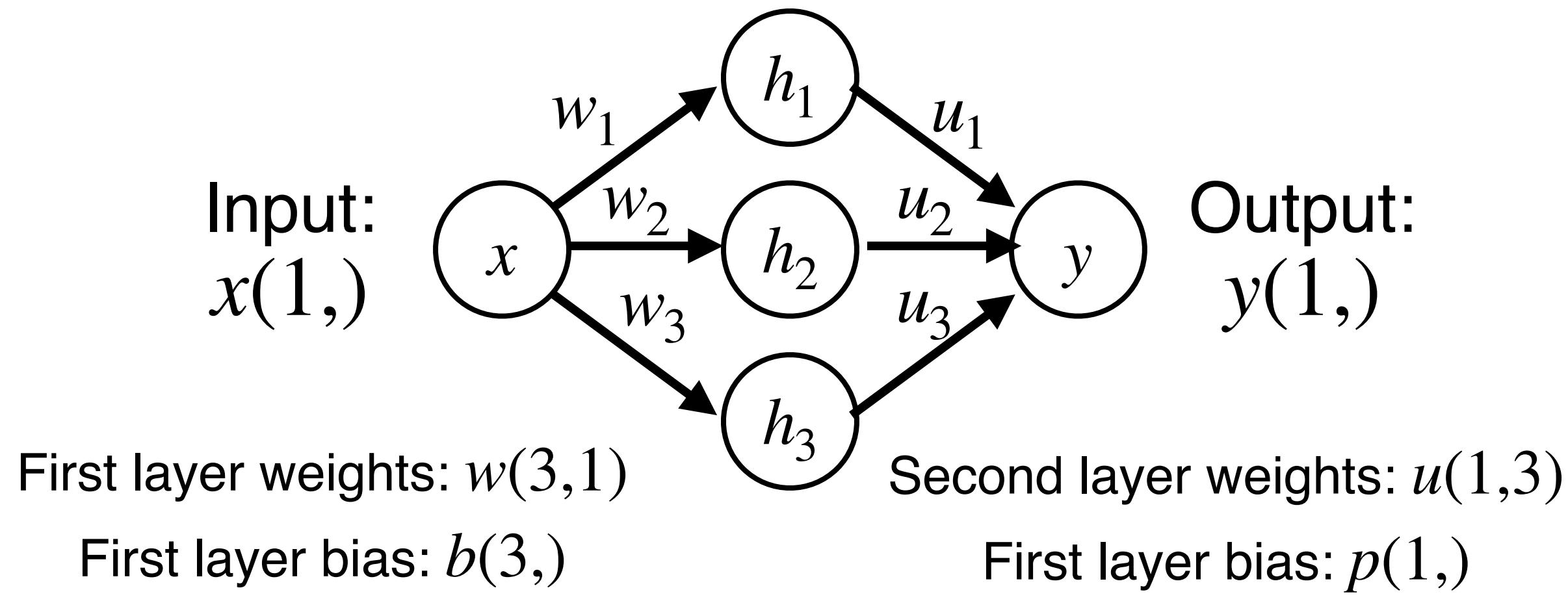
$$\begin{aligned} h_1 &= \max(0, w_1 x + b_1) \\ h_2 &= \max(0, w_2 x + b_2) \\ h_3 &= \max(0, w_3 x + b_3) \\ y &= u_1 h_1 + u_2 h_2 + u_3 h_3 + p \end{aligned}$$

$$\begin{aligned} y &= u_1 \max(0, w_1 x + b_1) \\ &\quad + u_2 \max(0, w_2 x + b_2) \\ &\quad + u_3 \max(0, w_3 x + b_3) \\ &\quad + p \end{aligned}$$



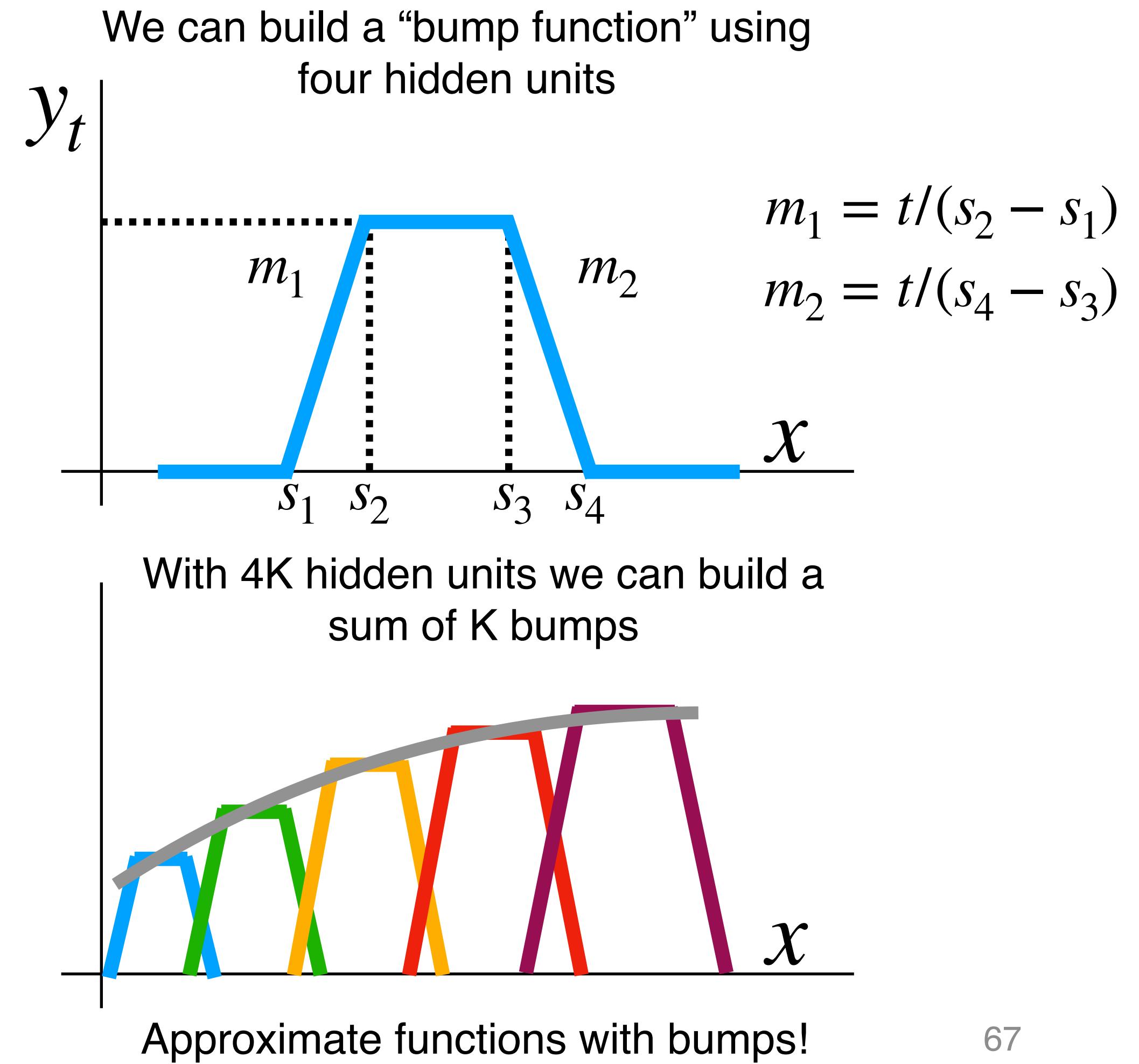
# Universal Approximation

Example: Approximating a function  $f: \mathbb{R} \rightarrow \mathbb{R}$  with a two-layer ReLU network



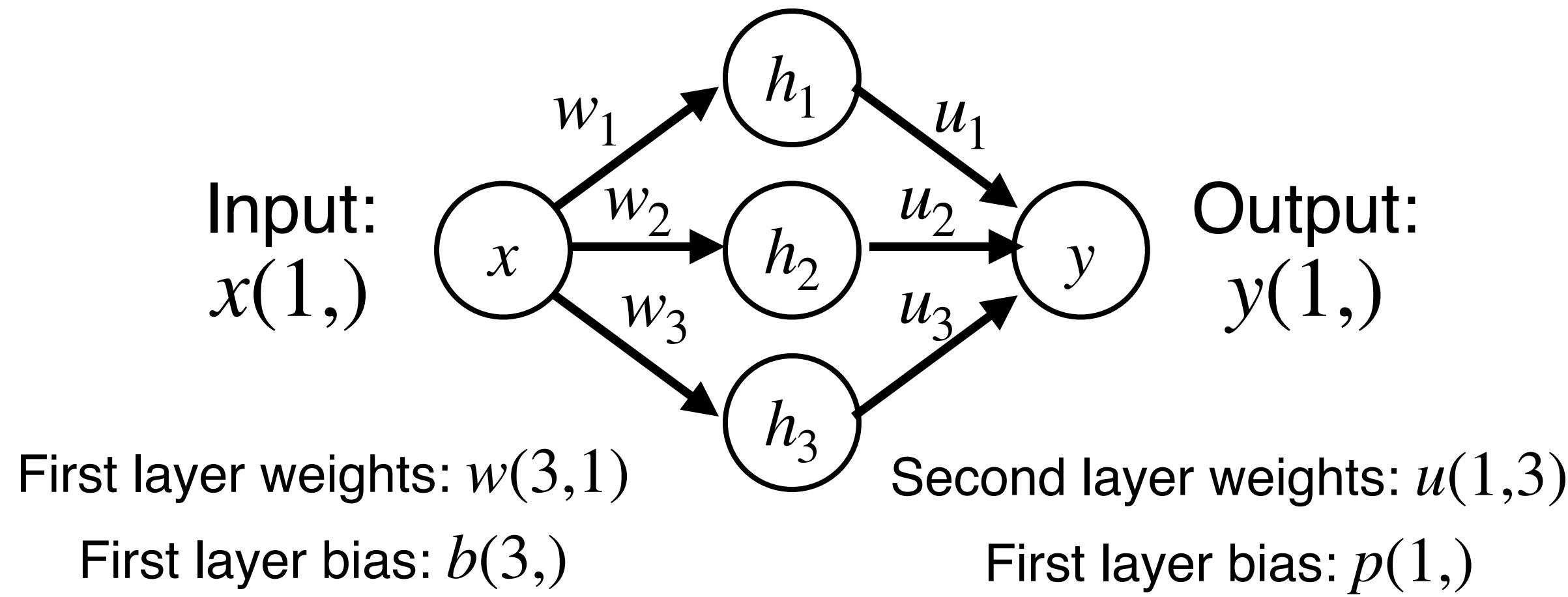
$$\begin{aligned} h_1 &= \max(0, w_1 x + b_1) \\ h_2 &= \max(0, w_2 x + b_2) \\ h_3 &= \max(0, w_3 x + b_3) \\ y &= u_1 h_1 + u_2 h_2 + u_3 h_3 + p \end{aligned}$$

$$\begin{aligned} y &= u_1 \max(0, w_1 x + b_1) \\ &\quad + u_2 \max(0, w_2 x + b_2) \\ &\quad + u_3 \max(0, w_3 x + b_3) \\ &\quad + p \end{aligned}$$



# Universal Approximation

Example: Approximating a function  $f: \mathbb{R} \rightarrow \mathbb{R}$  with a two-layer ReLU network



$$h_1 = \max(0, w_1 x + b_1)$$

$$h_2 = \max(0, w_2 x + b_2)$$

$$h_3 = \max(0, w_3 x + b_3)$$

$$y = u_1 h_1 + u_2 h_2 + u_3 h_3 + p$$

$$y = u_1 \max(0, w_1 x + b_1)$$

$$+ u_2 \max(0, w_2 x + b_2)$$

$$+ u_3 \max(0, w_3 x + b_3)$$

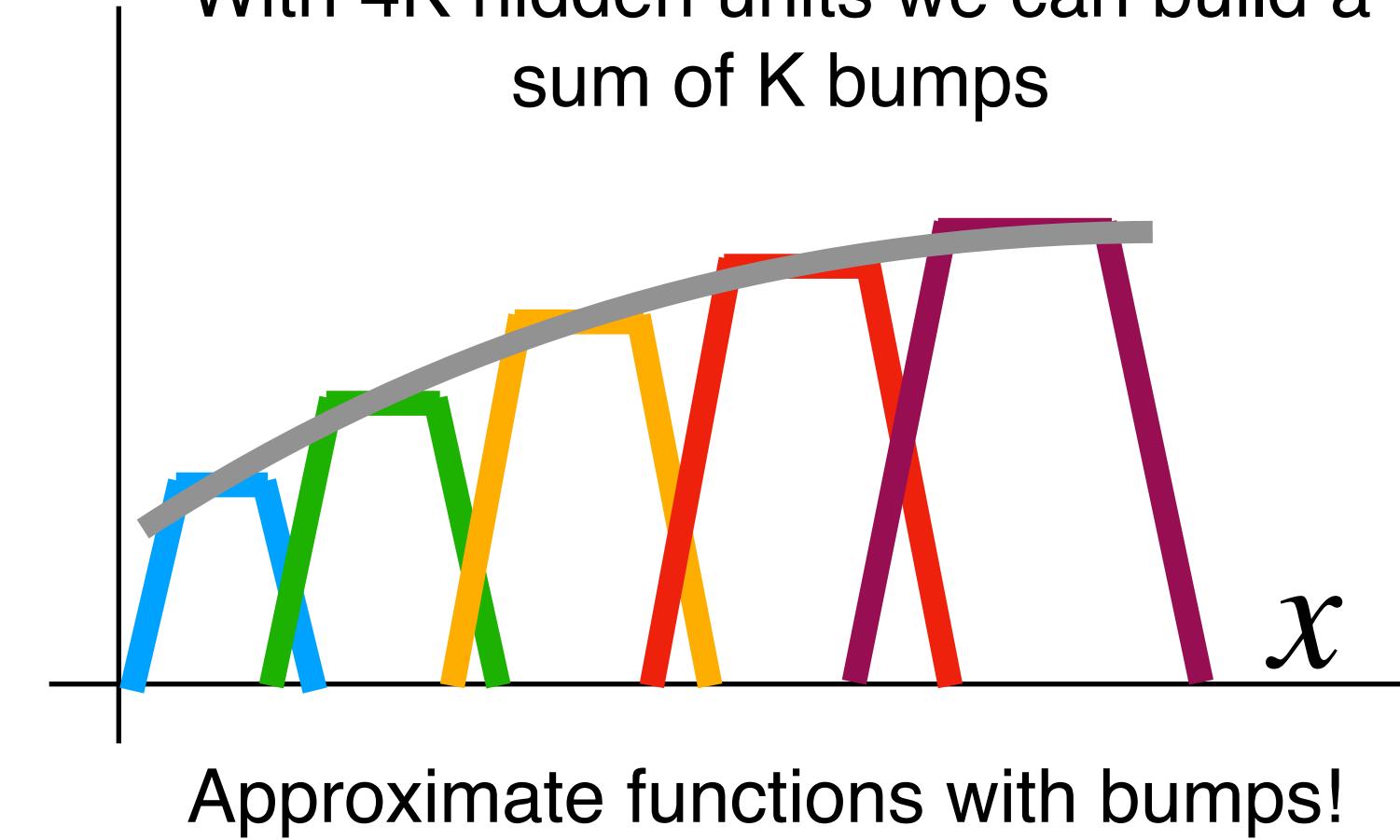
$$+ p$$

What about ...

- Gaps between bumps?
- Other nonlinearities?
- Higher-dimensional functions?

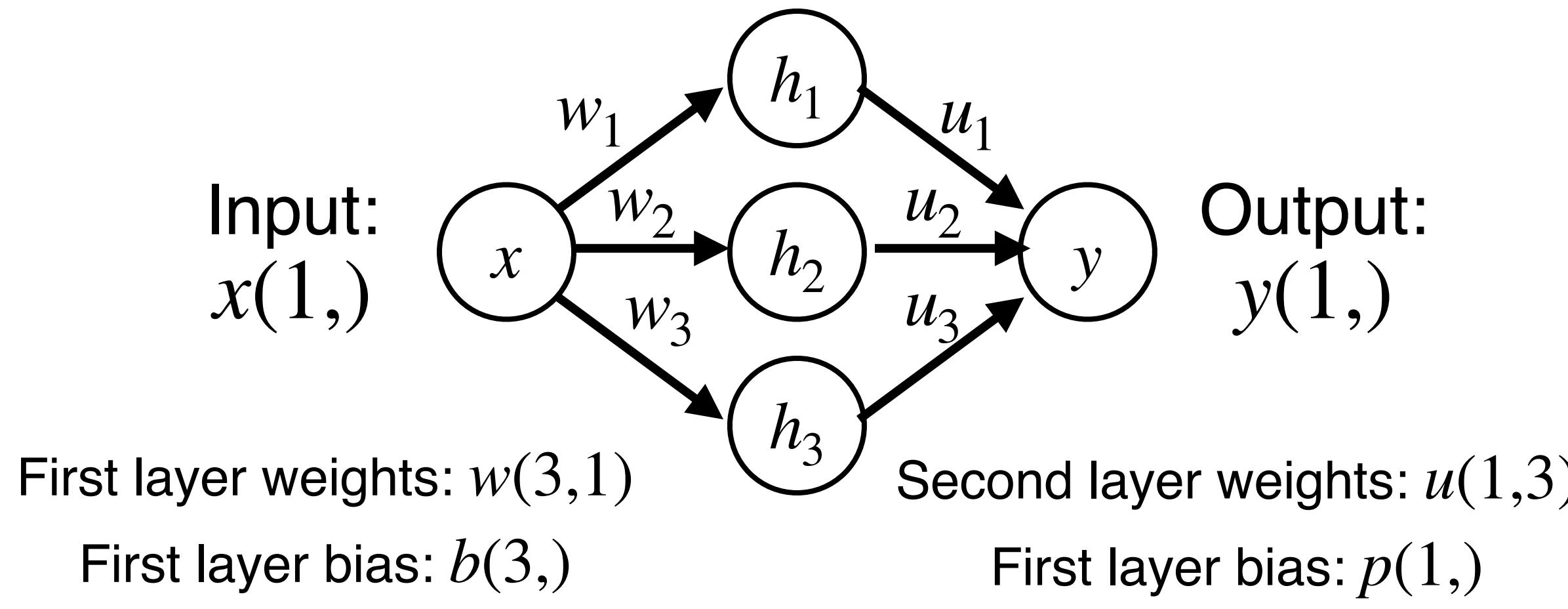
See [Nielsen, Chapter 4](#)

With 4K hidden units we can build a sum of K bumps



# Universal Approximation

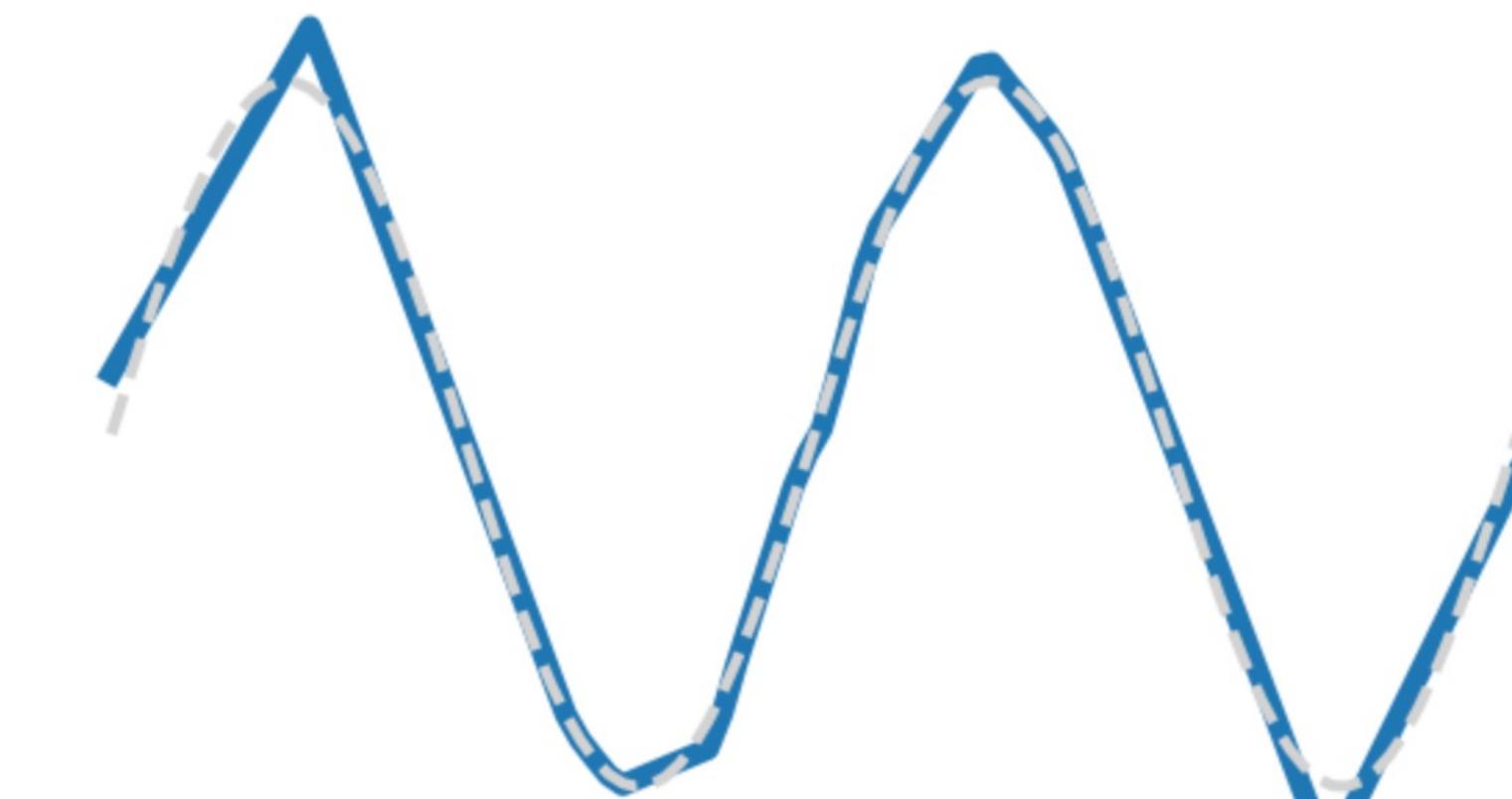
Example: Approximating a function  $f: \mathbb{R} \rightarrow \mathbb{R}$  with a two-layer ReLU network



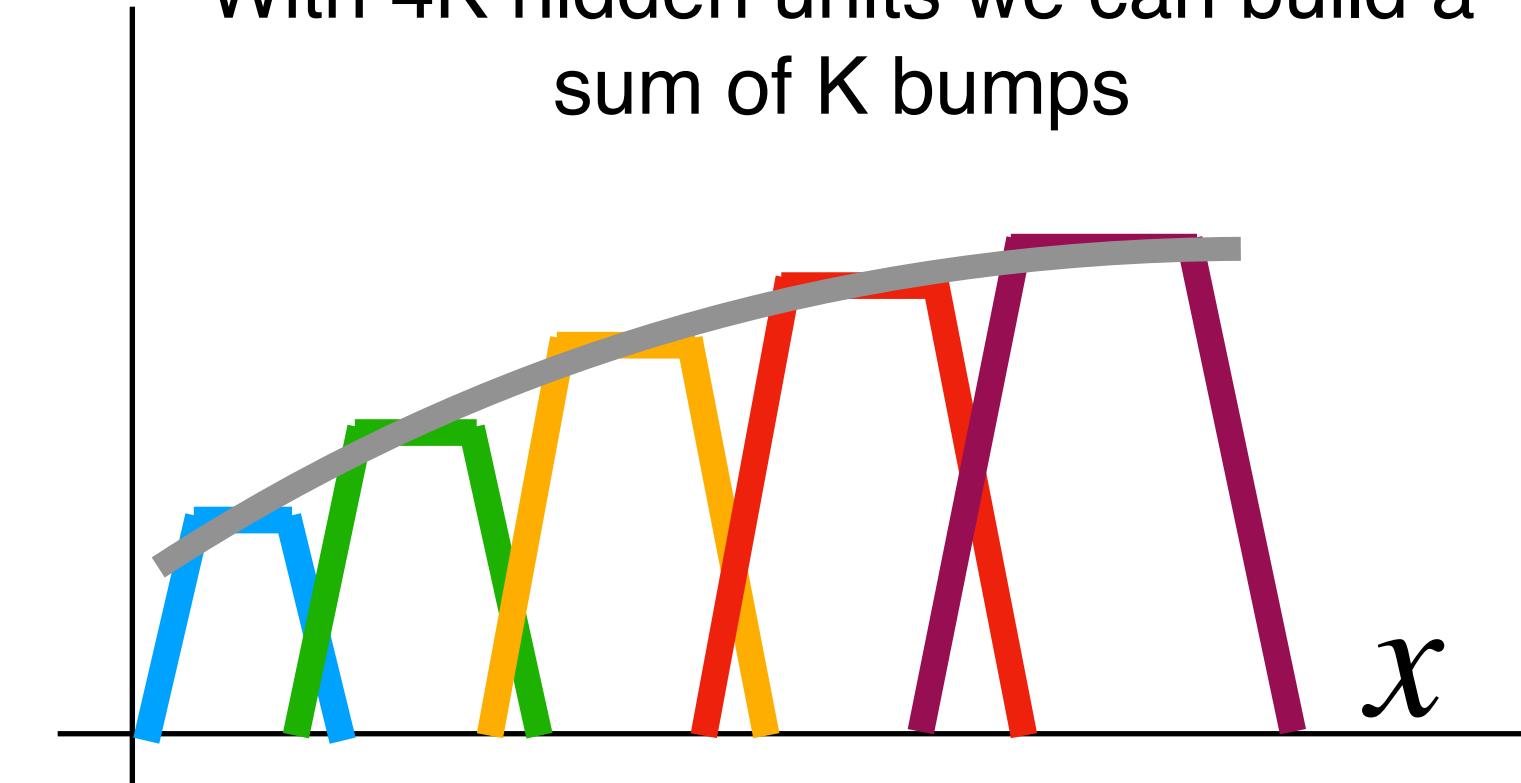
$$\begin{aligned} h_1 &= \max(0, w_1 x + b_1) \\ h_2 &= \max(0, w_2 x + b_2) \\ h_3 &= \max(0, w_3 x + b_3) \\ y &= u_1 h_1 + u_2 h_2 + u_3 h_3 + p \end{aligned}$$

$$\begin{aligned} y &= u_1 \max(0, w_1 x + b_1) \\ &\quad + u_2 \max(0, w_2 x + b_2) \\ &\quad + u_3 \max(0, w_3 x + b_3) \\ &\quad + p \end{aligned}$$

Reality check: Networks don't really learn bumps!



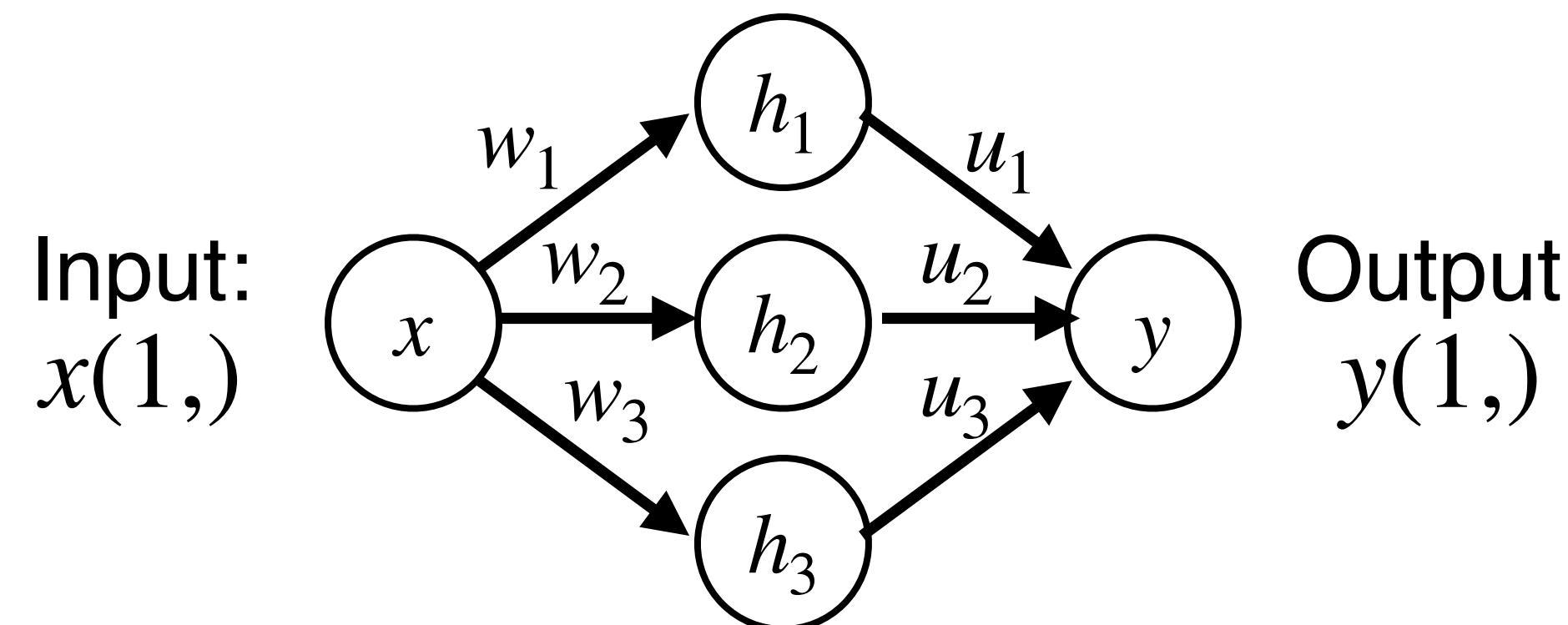
With 4K hidden units we can build a sum of K bumps



Approximate functions with bumps!

# Universal Approximation

Example: Approximating a function  $f: \mathbb{R} \rightarrow \mathbb{R}$  with a two-layer ReLU network



Universal approximation tells us:

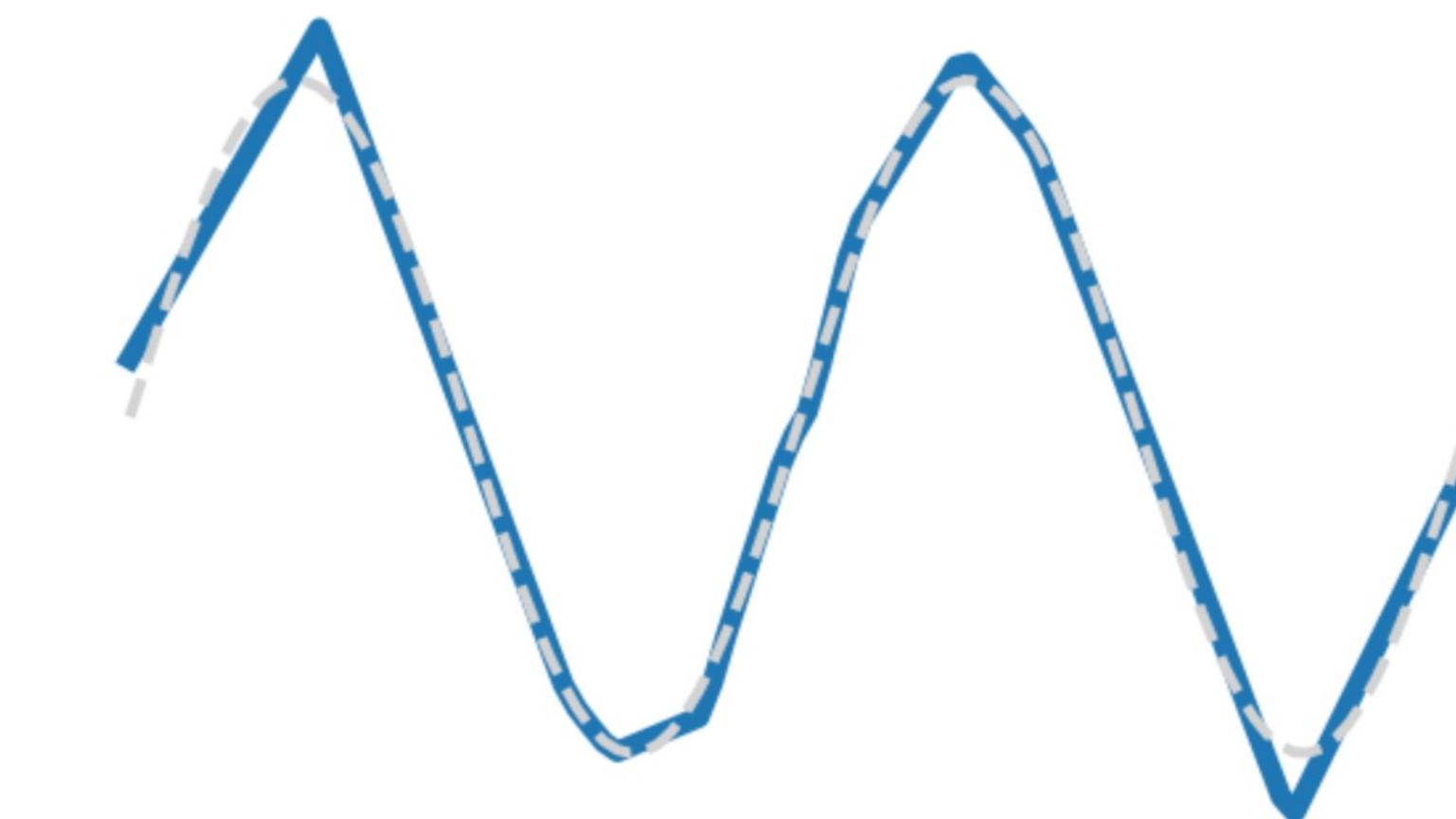
- Neural nets can represent any function

Universal approximation **DOES NOT** tell us:

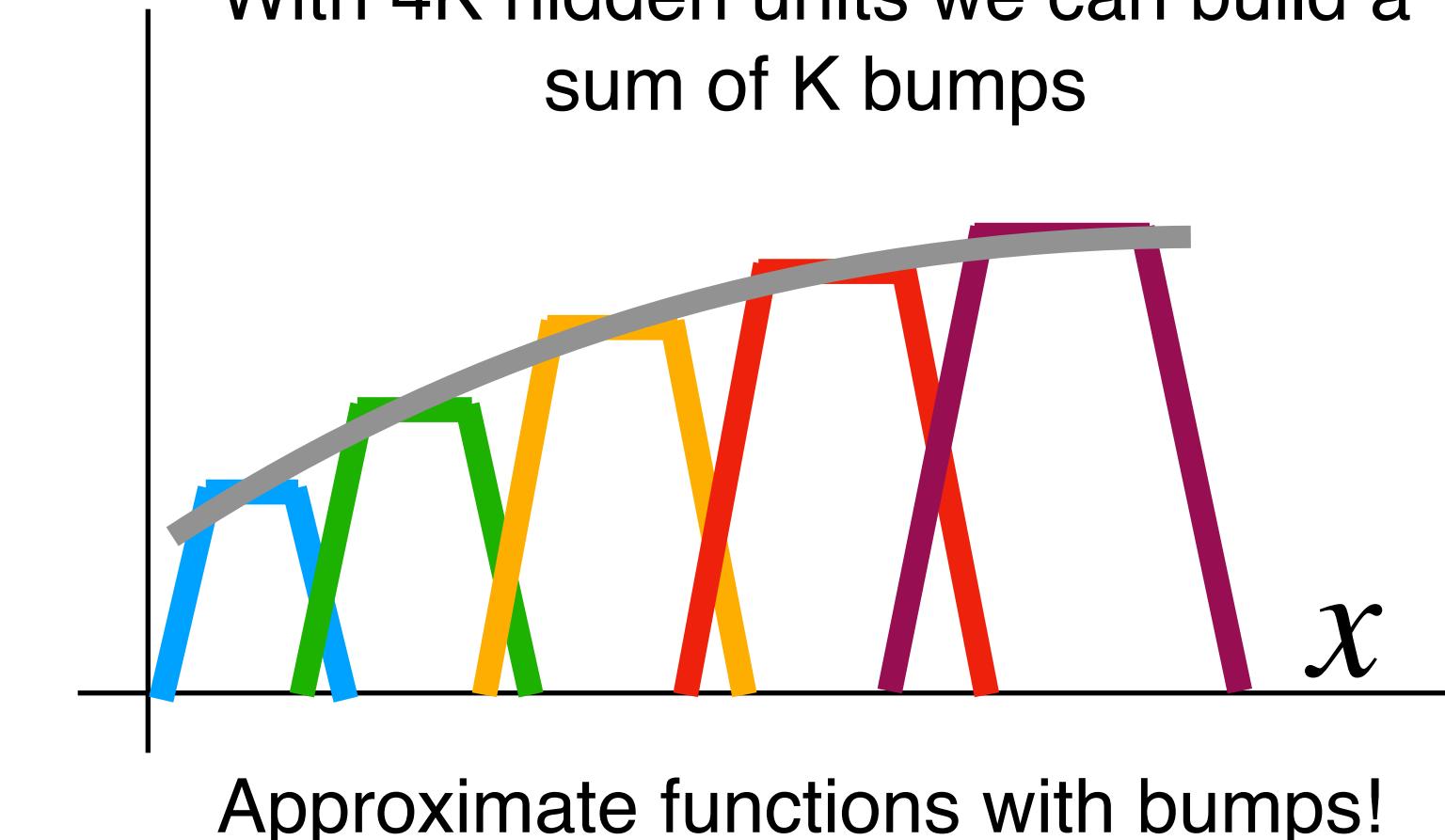
- Whether we can actually learn any function with SGD
- How much data we need to learn a function

Remember: kNN is also a universal approximator!

Reality check: Networks don't really learn bumps!



With 4K hidden units we can build a sum of K bumps

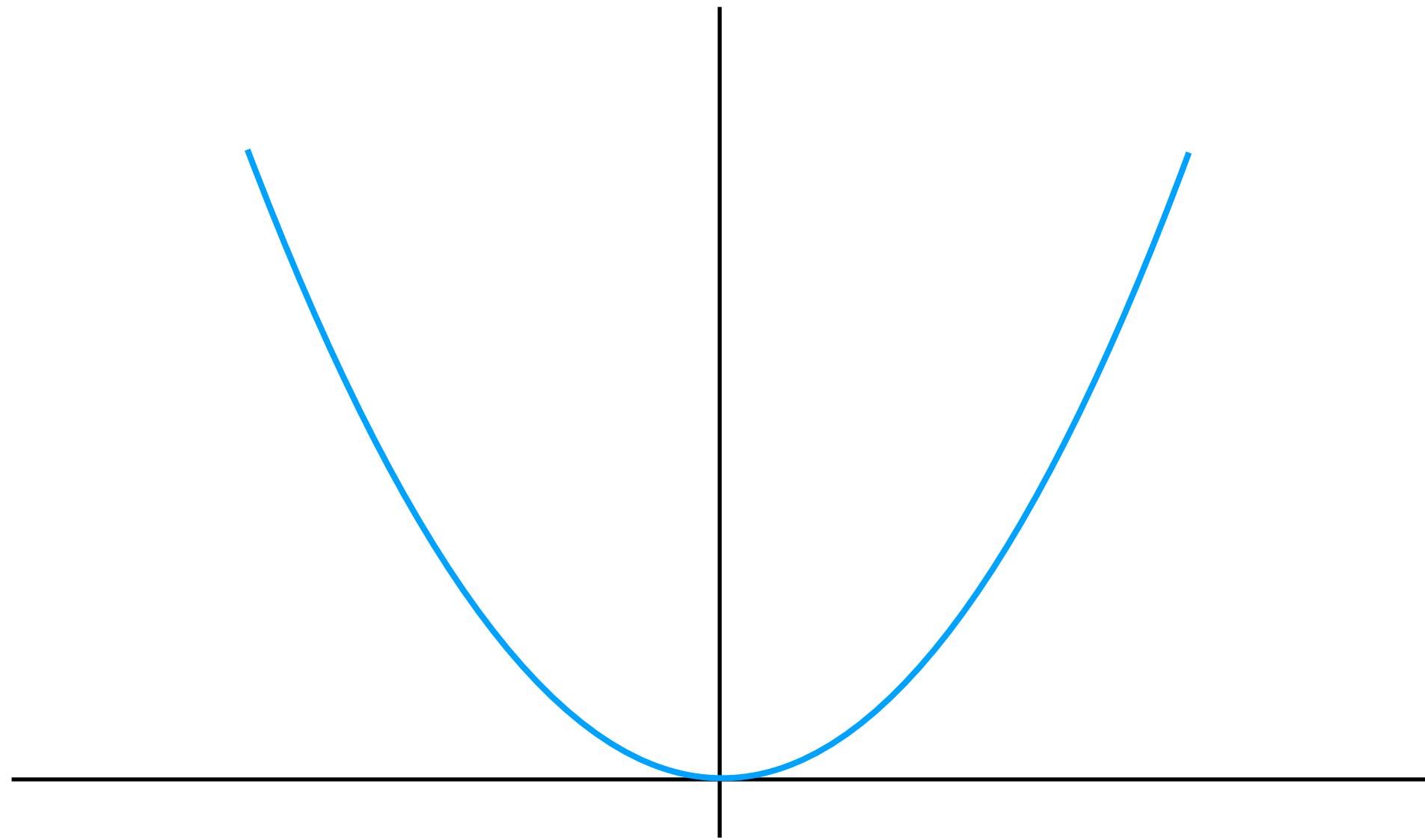


# Convex Functions

A function  $f: X \subseteq \mathbb{R}^N \rightarrow \mathbb{R}$  is **convex** if for all  $x_1, x_2 \in X, t \in [0,1]$ ,

$$f(tx_1 + (1 - t)x_2) \leq tf(x_1) + (1 - t)f(x_2)$$

Example:  $f(x) = x^2$  is convex:

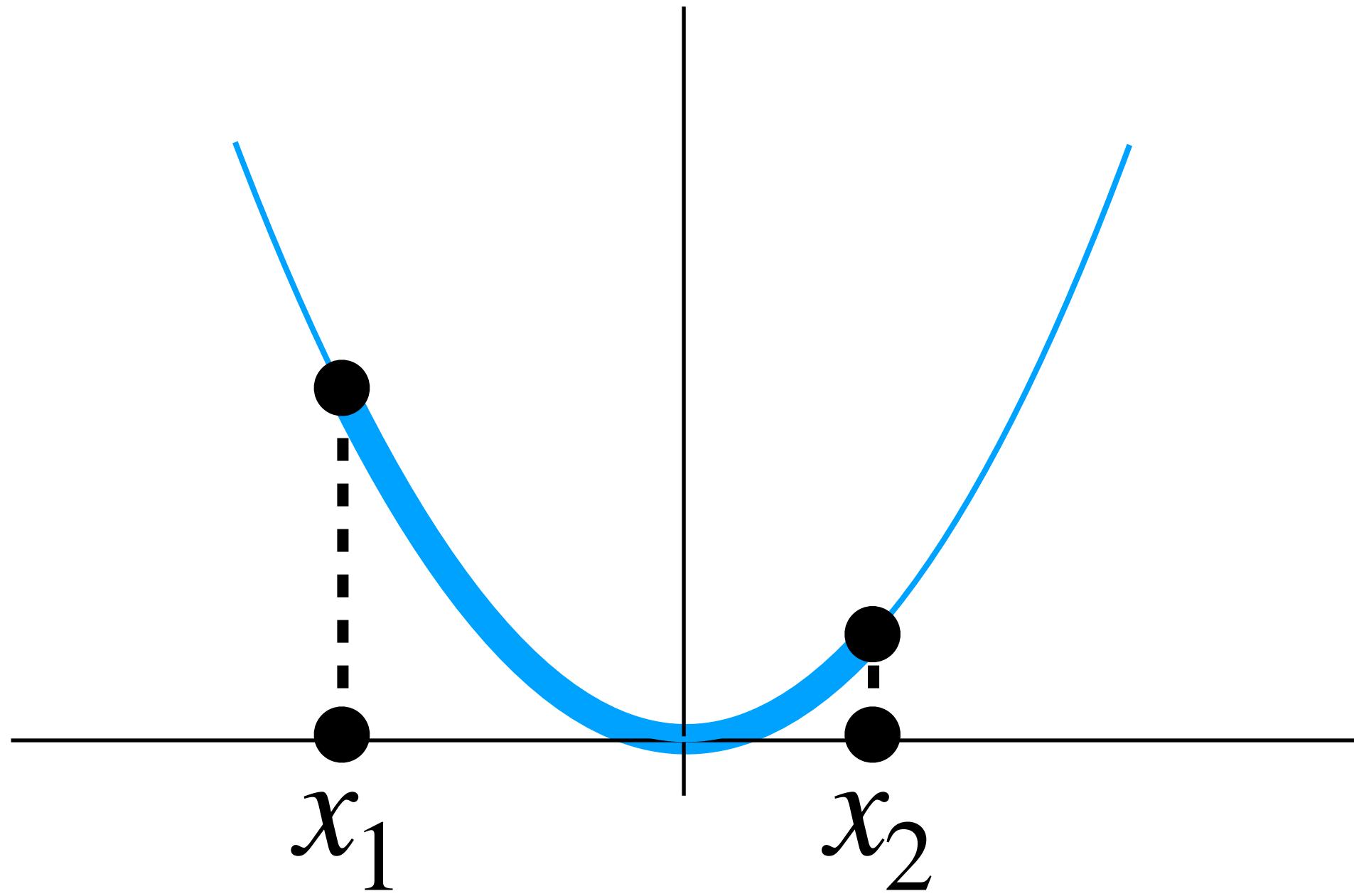


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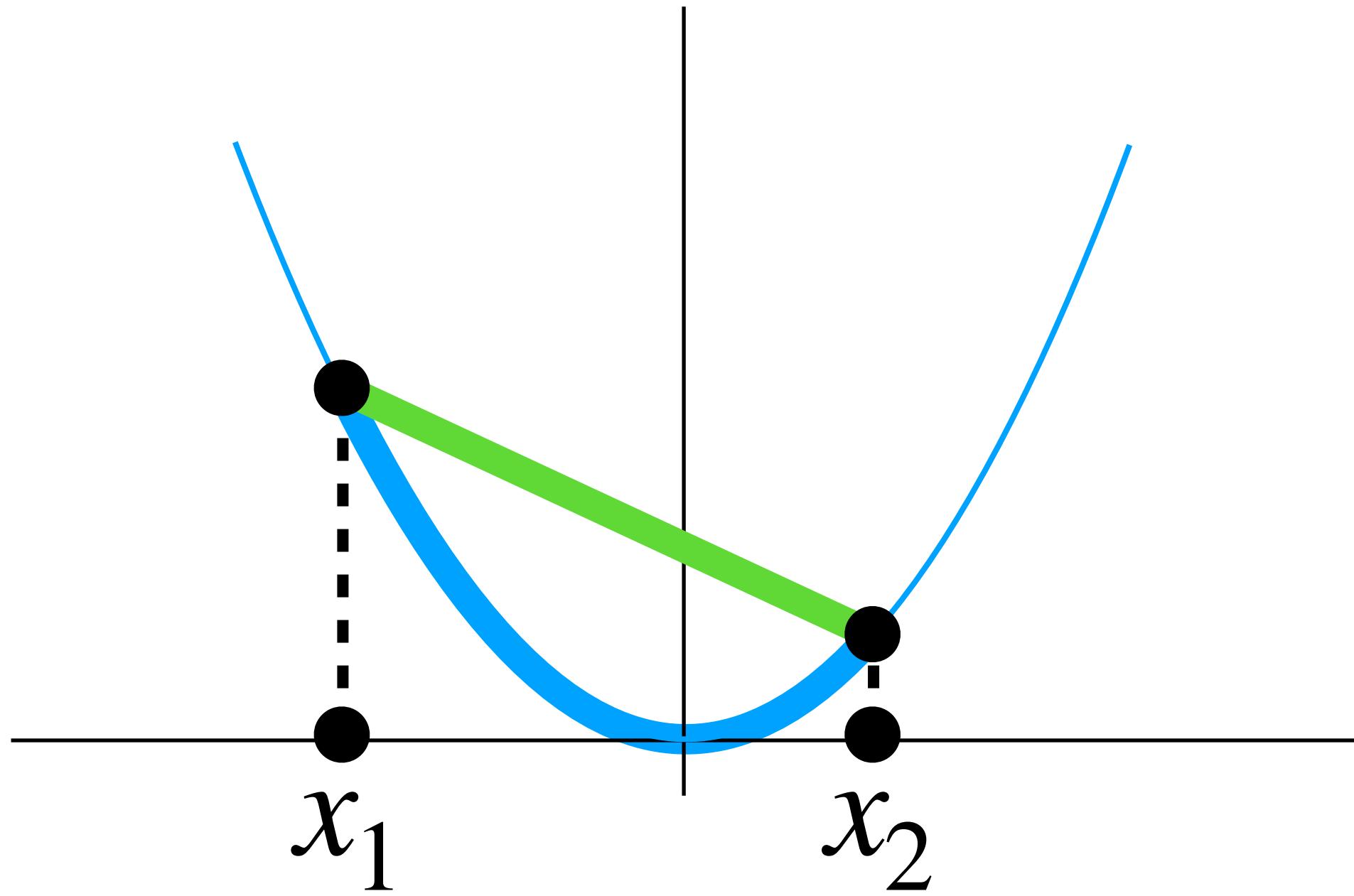


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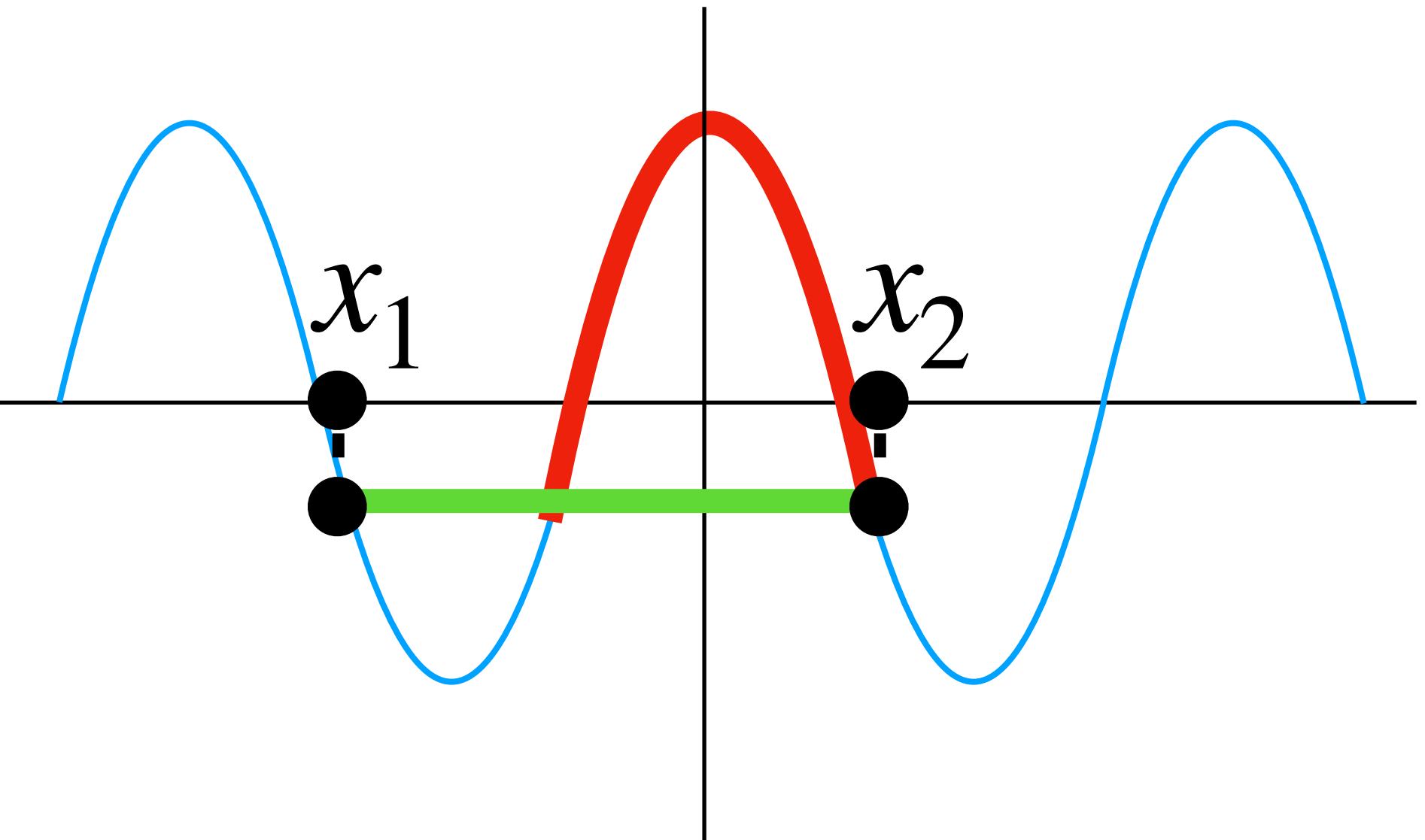


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$$f(tx_1 + (1 - t)x_2) \leq tf(x_1) + (1 - t)f(x_2)$$

Example:  $f(x) = \cos(x)$  is **not** convex:



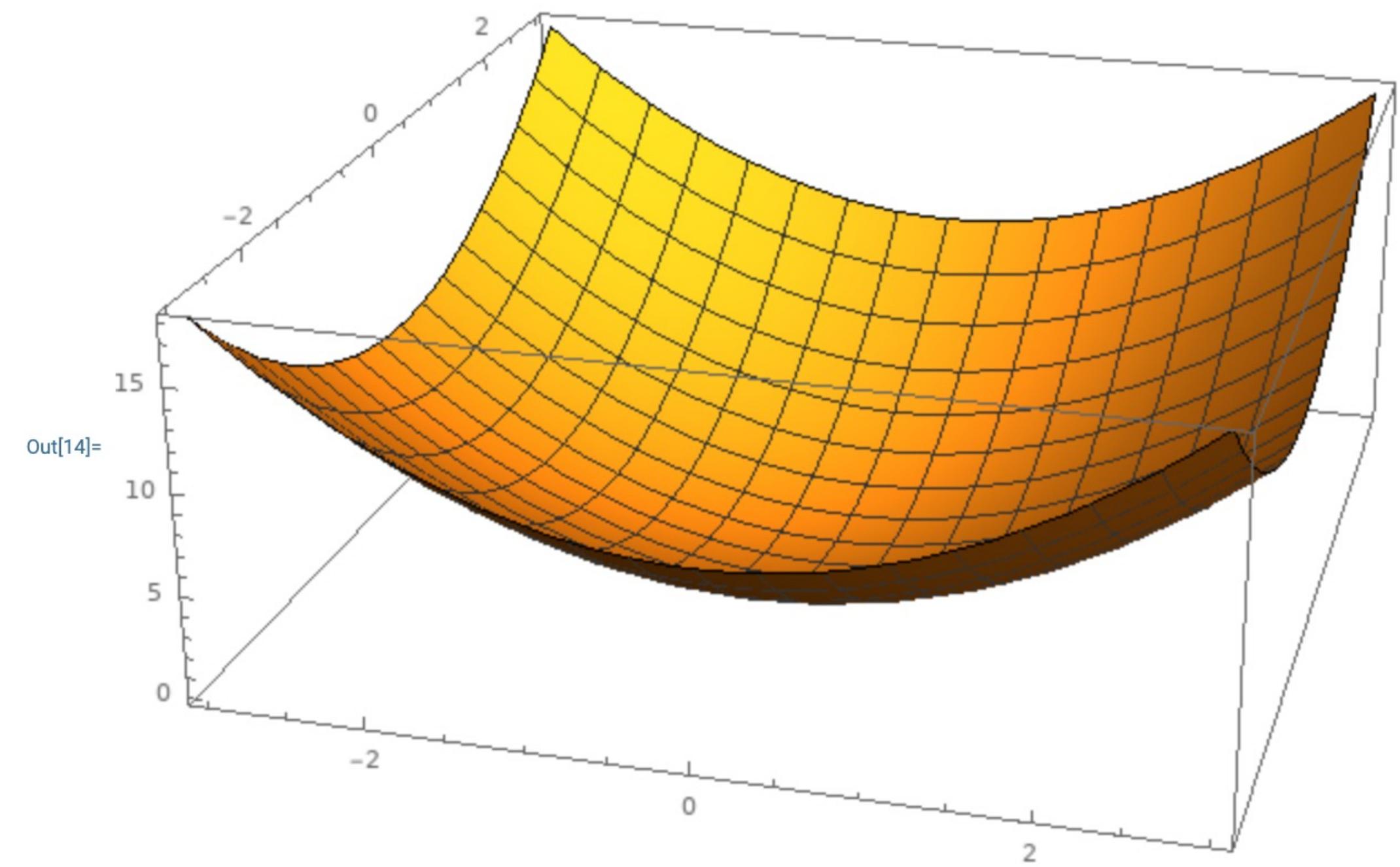
# Convex Functions

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**Intuition:** A convex function is a (multidimensional) bowl

Generally speaking, convex functions are **easy to optimize**: can derive theoretical guarantees about **converging to global minimum\***



# Convex Functions

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**Intuition:** A convex function is a (multidimensional) bowl

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Linear classifiers optimize a **convex function!**

$$s = f(x; W) = Wx$$

$$L_i = -\log\left(\frac{e^{s_{y_i}}}{\sum_j e^{s_j}}\right) \quad \text{Softmax}$$

$$L_i = \sum_{j \neq y_i} \max(0, s_j - s_{y_i} + 1) \quad \text{SVM}$$

$$L = \frac{1}{N} \sum_{i=1}^N L_i + R(W) \quad \text{where } R(W) \text{ is L2 or L1 regularization}$$



# Convex Functions

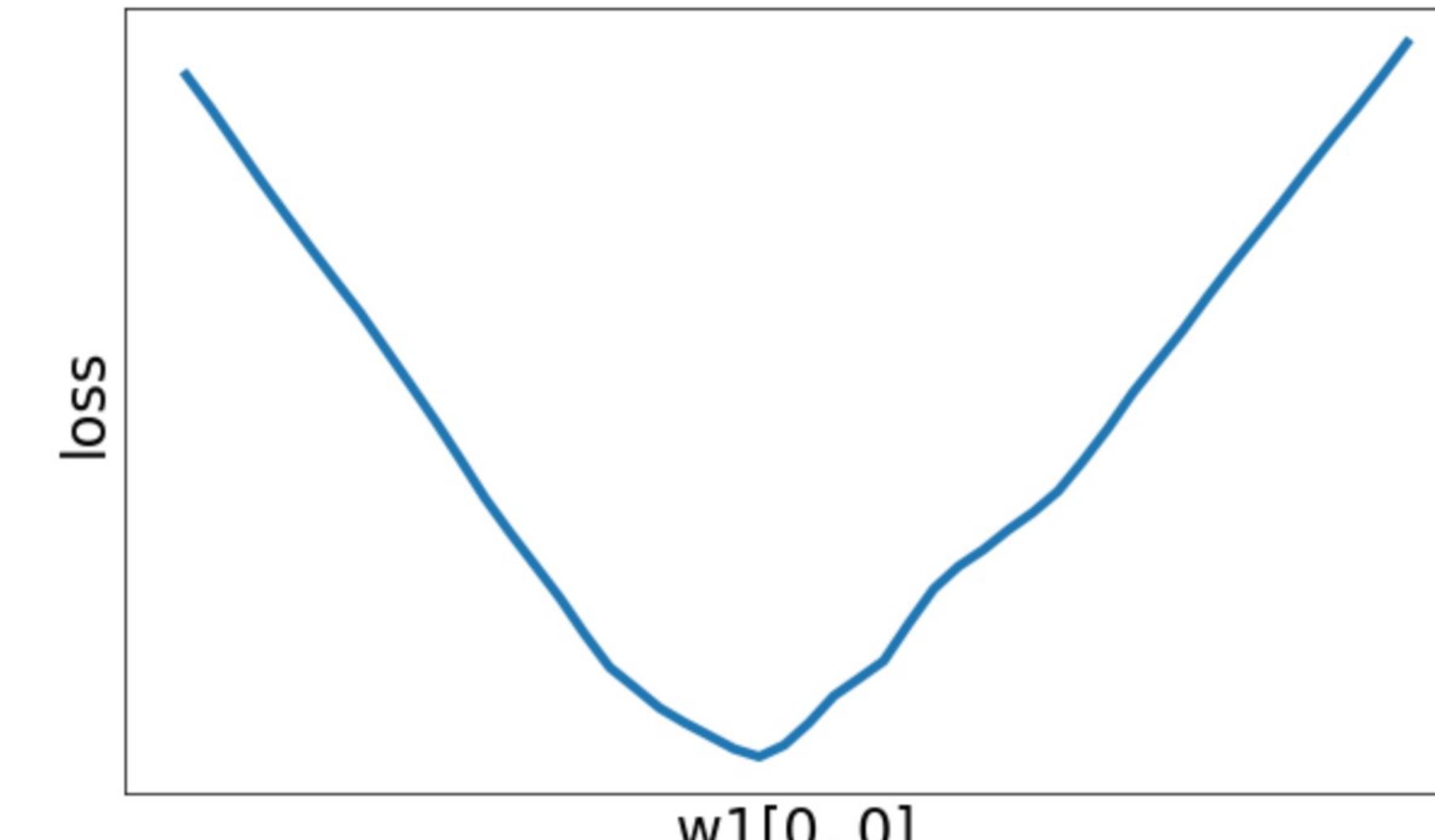
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Generally speaking, convex functions are **easy to optimize**: can derive theoretical guarantees about **converging to global minimum\***

Neural net losses sometimes look convex-ish:



1D slice of loss landscape for a 4-layer ReLU network with 10 input features, 32 units per hidden layer, 10 categories, with softmax loss

# Convex Functions

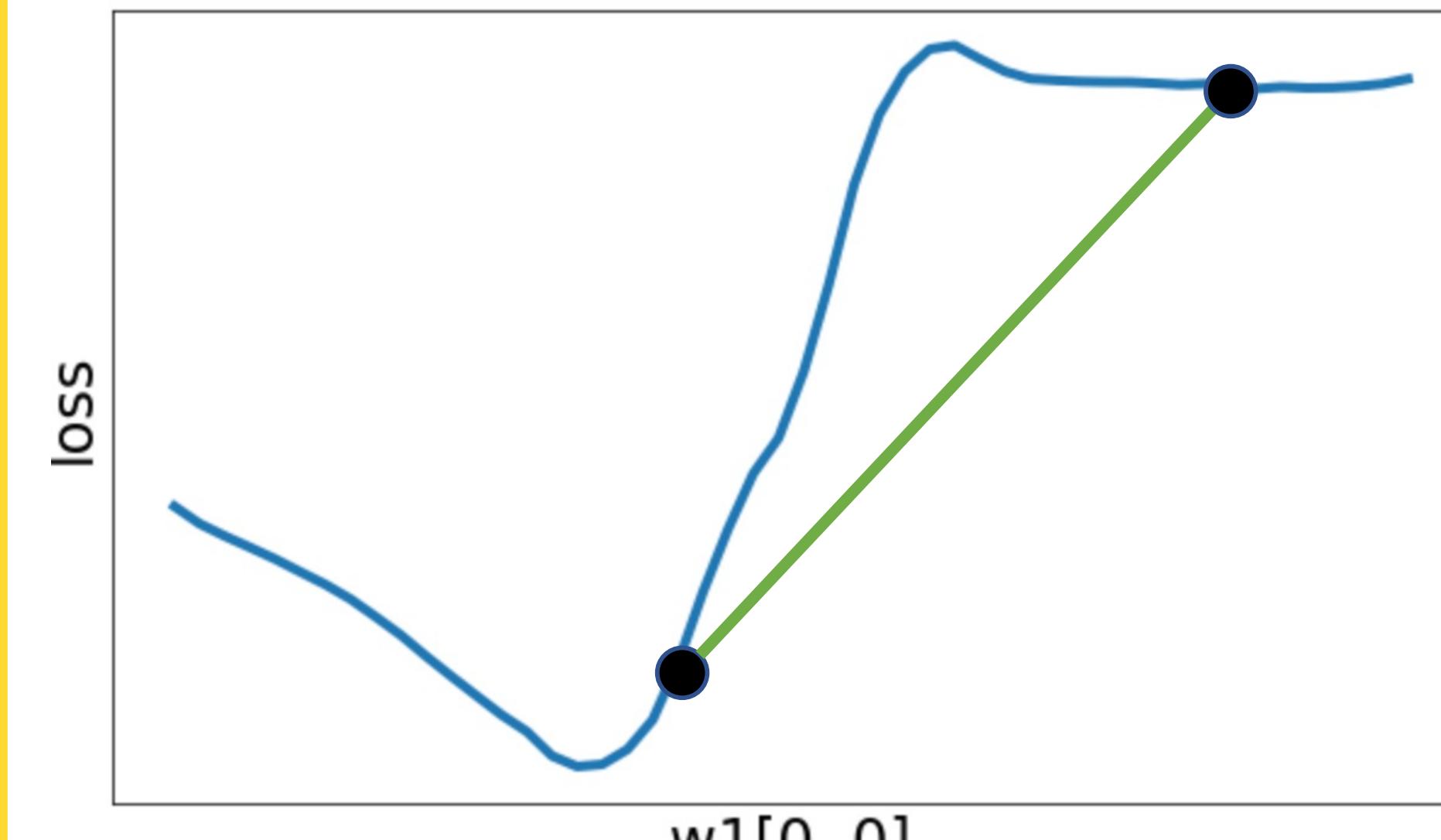
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**Intuition:** A convex function is a (multidimensional) bowl

Generally speaking, convex functions are **easy to optimize**: can derive theoretical guarantees about **converging to global minimum\***

But often clearly nonconvex:



1D slice of loss landscape for a 4-layer ReLU network with 10 input features, 32 units per hidden layer, 10 categories, with softmax loss

# Convex Functions

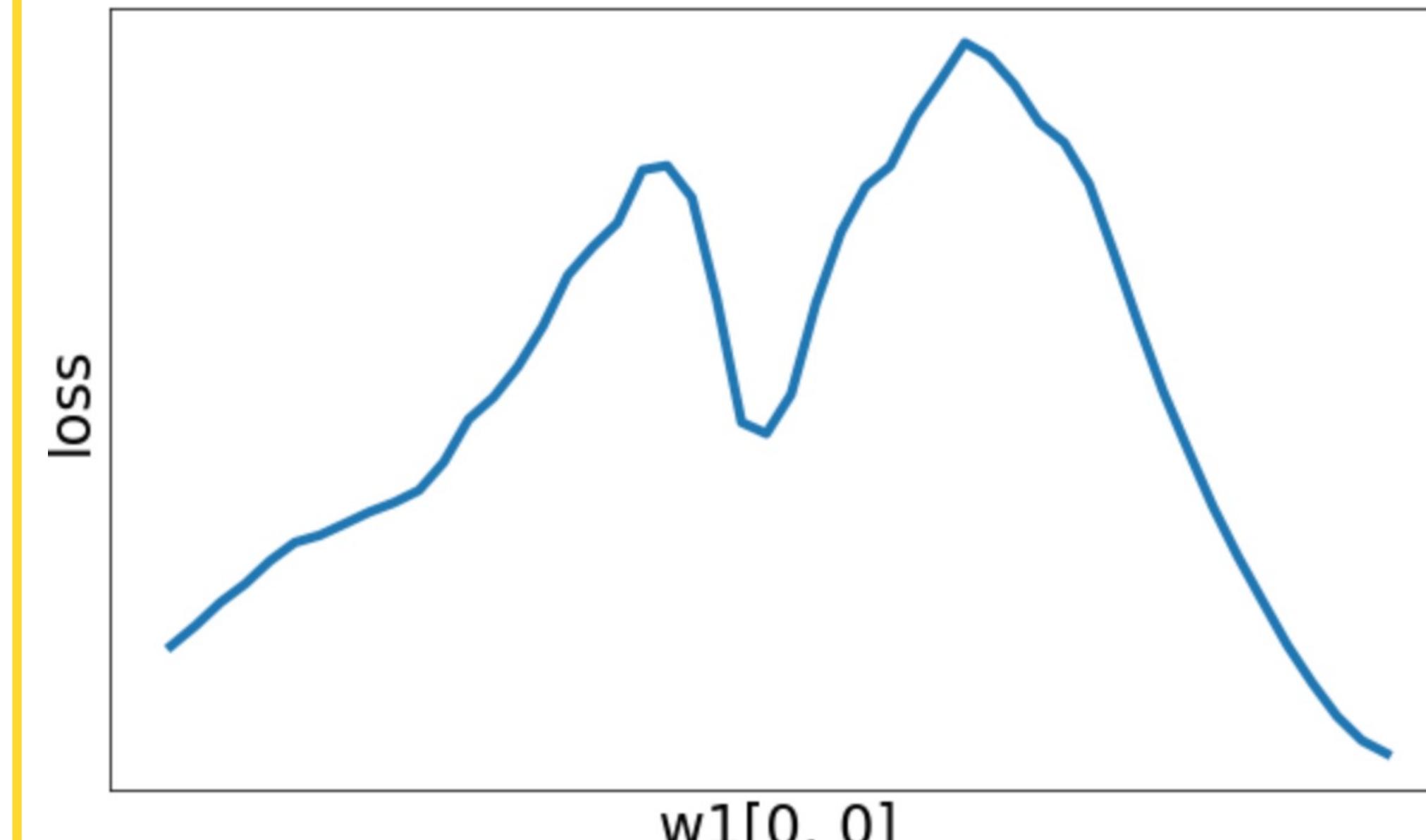
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**Intuition:** A convex function is a (multidimensional) bowl

Generally speaking, convex functions are **easy to optimize**: can derive theoretical guarantees about **converging to global minimum\***

With local minima:



1D slice of loss landscape for a 4-layer ReLU network with 10 input features, 32 units per hidden layer, 10 categories, with softmax loss

# Convex Functions

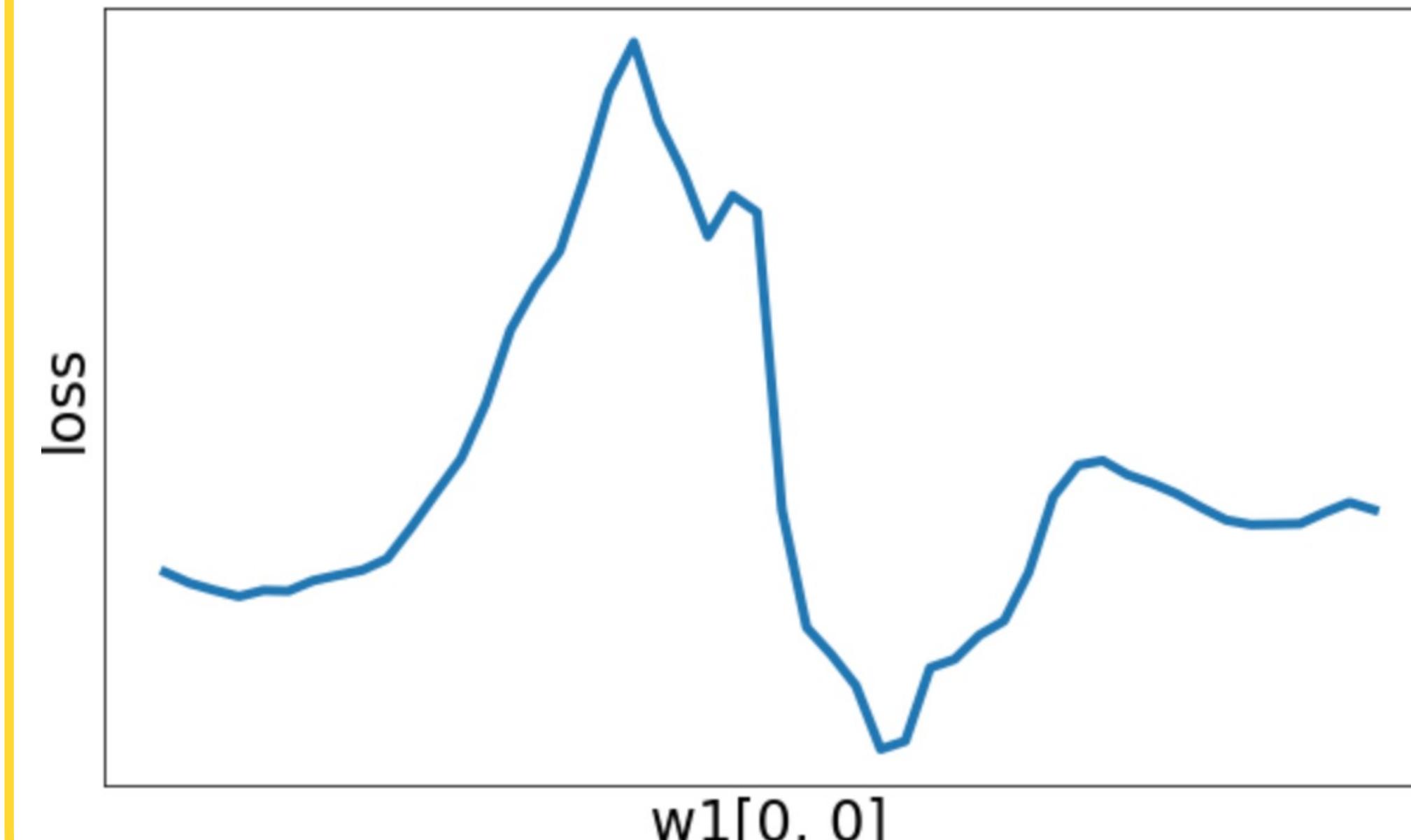
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Generally speaking, convex functions are **easy to optimize**: can derive theoretical guarantees about **converging to global minimum\***

Can get very wild!



1D slice of loss landscape for a 4-layer ReLU network with 10 input features, 32 units per hidden layer, 10 categories, with softmax loss

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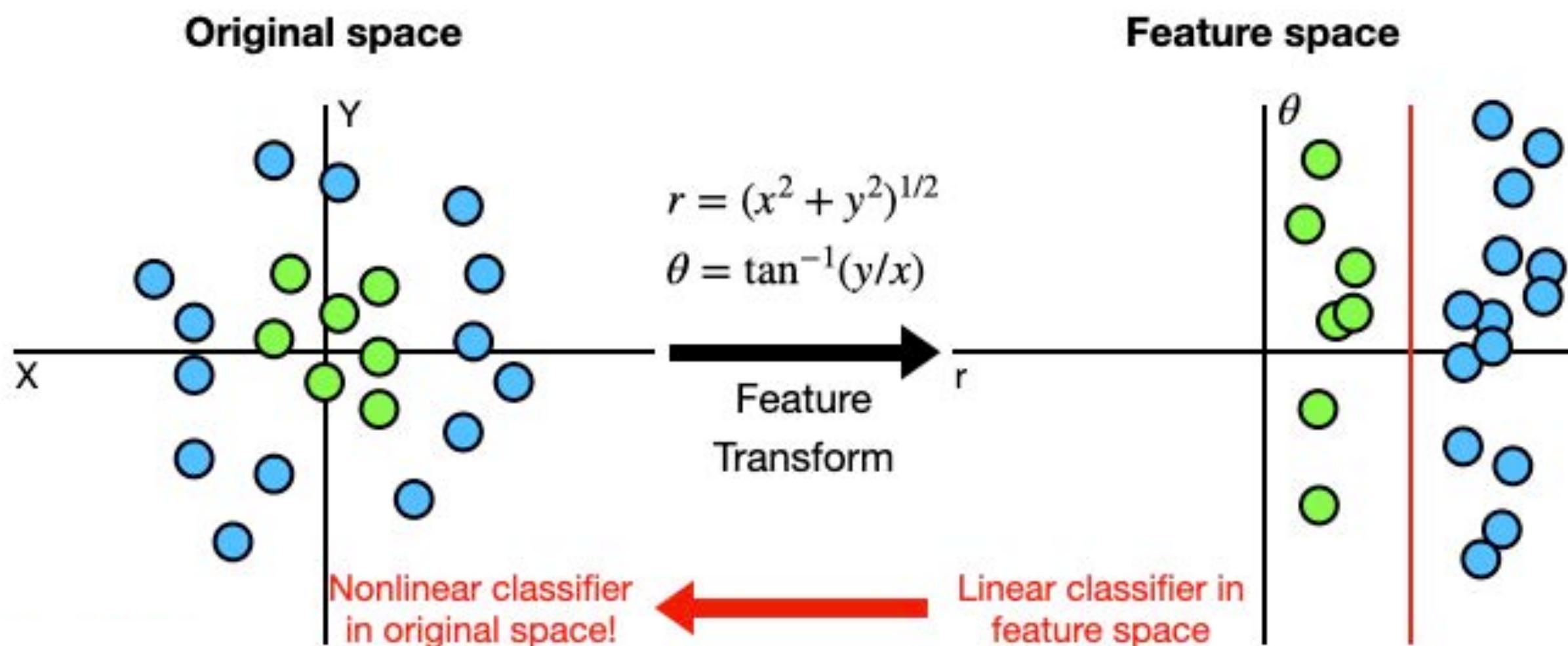
Generally speaking, convex functions are **easy to optimize**: can derive theoretical guarantees about **converging to global minimum\***

Most neural networks need **nonconvex optimization**

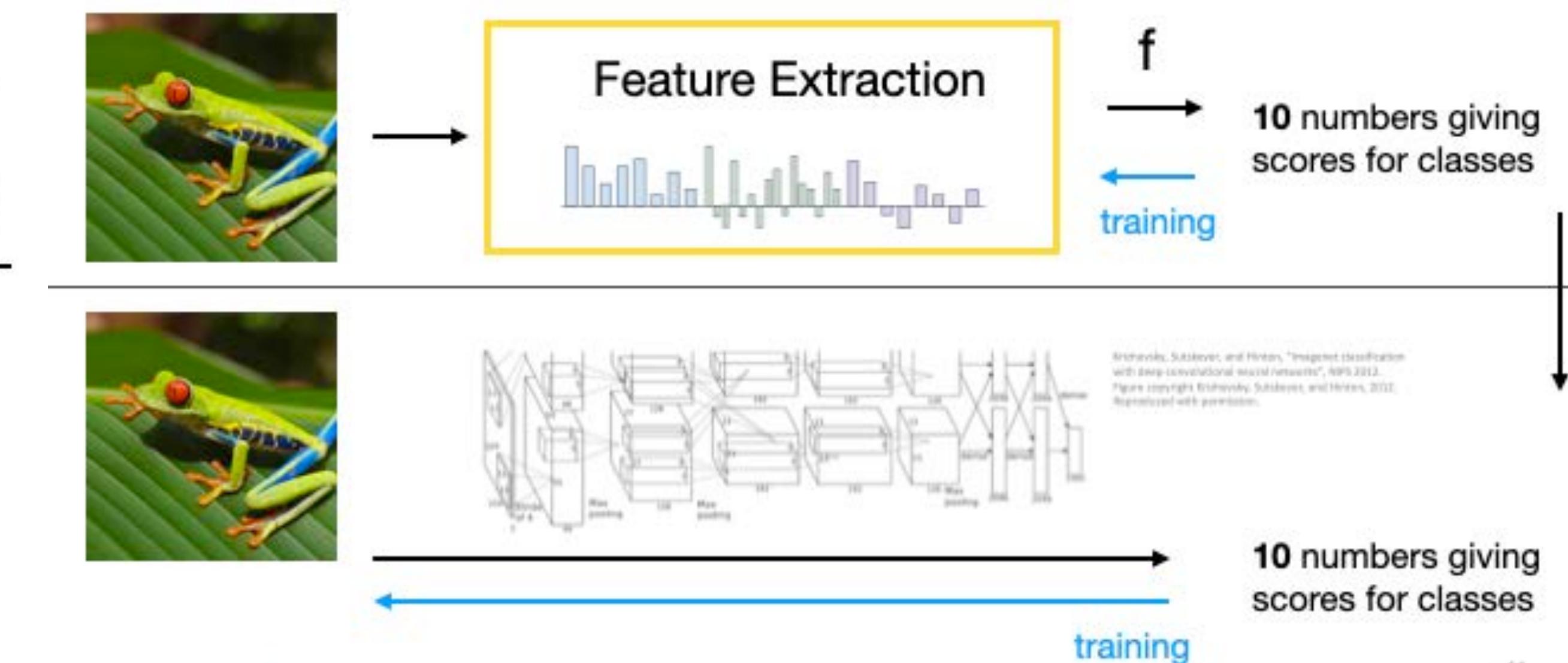
- Few or no guarantees about convergence
- Empirically it seems to work anyway
- Active area of research

# Summary

Feature transform + Linear classifier allows nonlinear decision boundaries



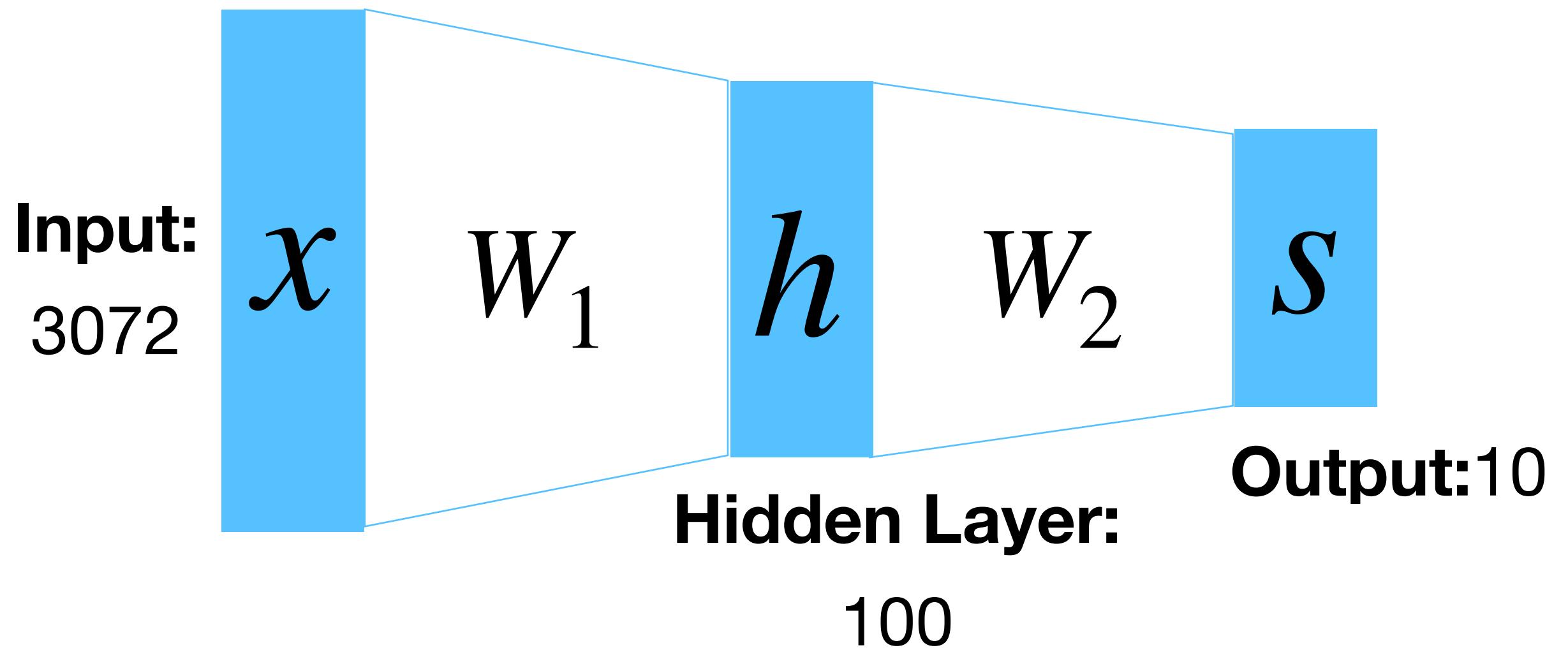
Neural Networks as learnable feature transforms



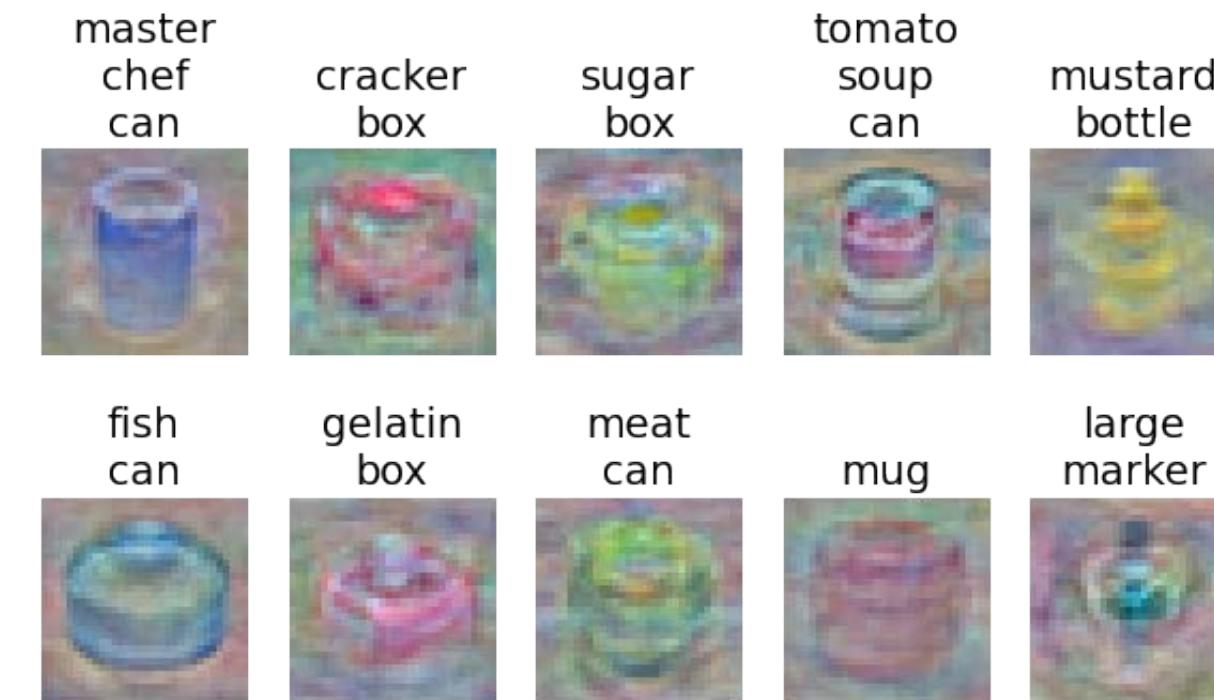
# Summary

From linear classifiers to  
fully-connected networks

$$f(x) = W_2 \max(0, W_1 x + b_1) + b_2$$



Linear classifier: One template per class



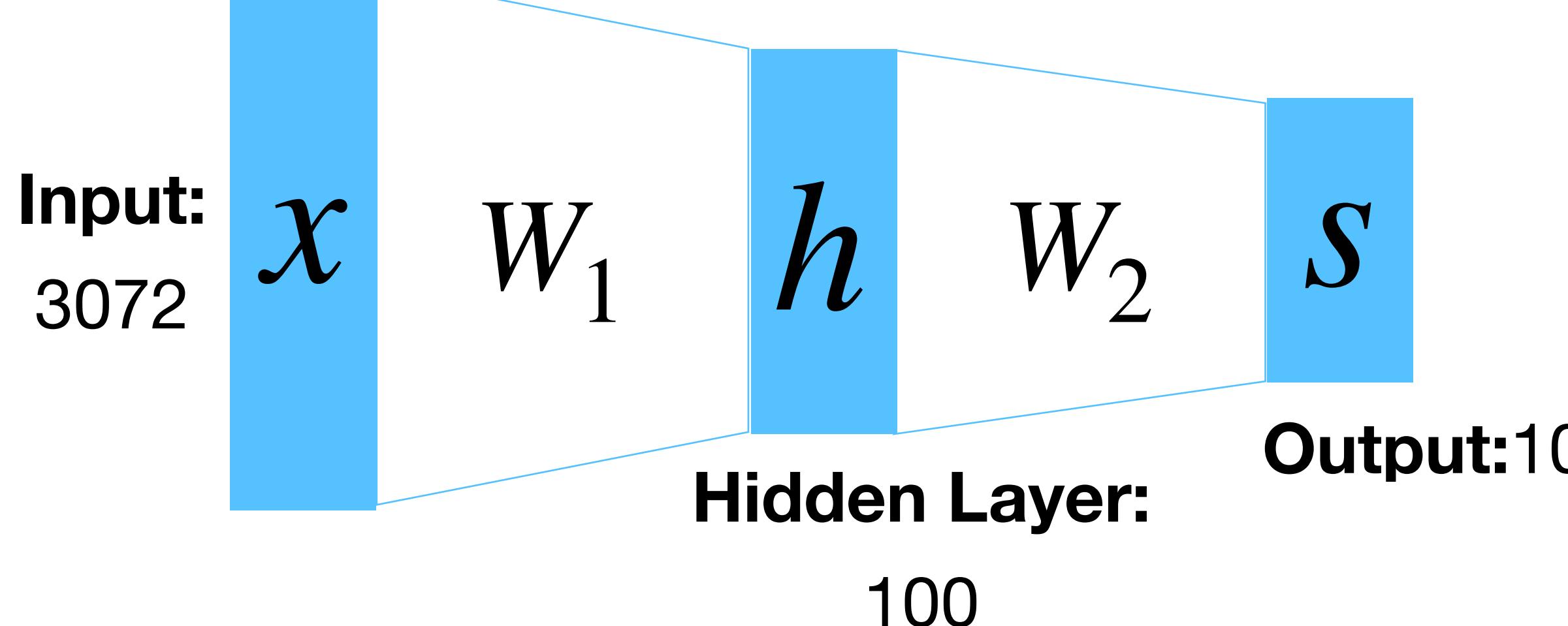
Neural networks: Many reusable templates



# Summary

From linear classifiers to  
fully-connected networks

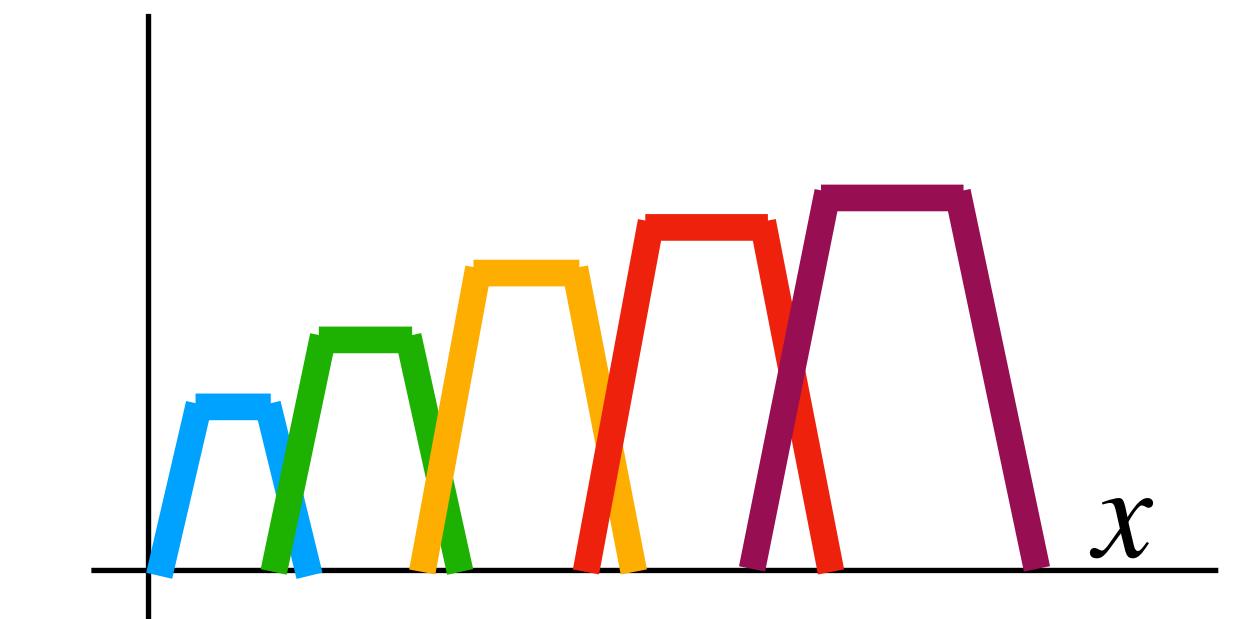
$$f(x) = W_2 \max(0, W_1 x + b_1) + b_2$$



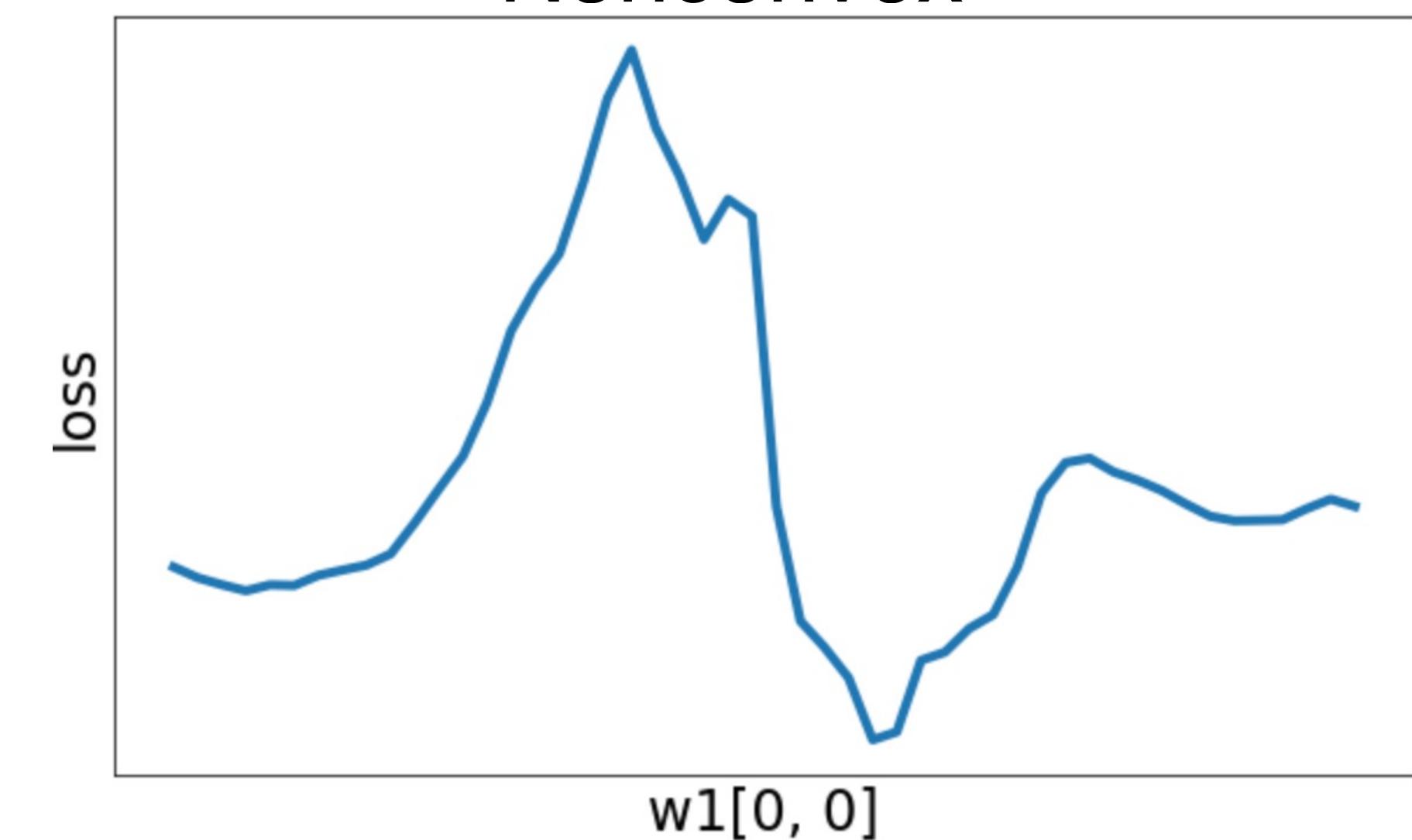
Space Warping



Universal approximation



Nonconvex





# Problem: How to compute gradients?

---

$$s = W_2 \max(0, W_1 x + b_1) + b_2$$

$$L_i = \sum_{j \neq y_i} \max(0, s_j - s_{y_i} + 1)$$

$$R(W) = \sum_k W_k^2$$

$$L(W_1, W_2, b_1, b_2) = \frac{1}{N} \sum_{i=1}^N L_i + \lambda R(W_1) + \lambda R(W_2)$$

If we can compute  $\frac{\delta L}{\delta W_1}, \frac{\delta L}{\delta W_2}, \frac{\delta L}{\delta b_1}, \frac{\delta L}{\delta b_2}$  then we can optimize with SGD

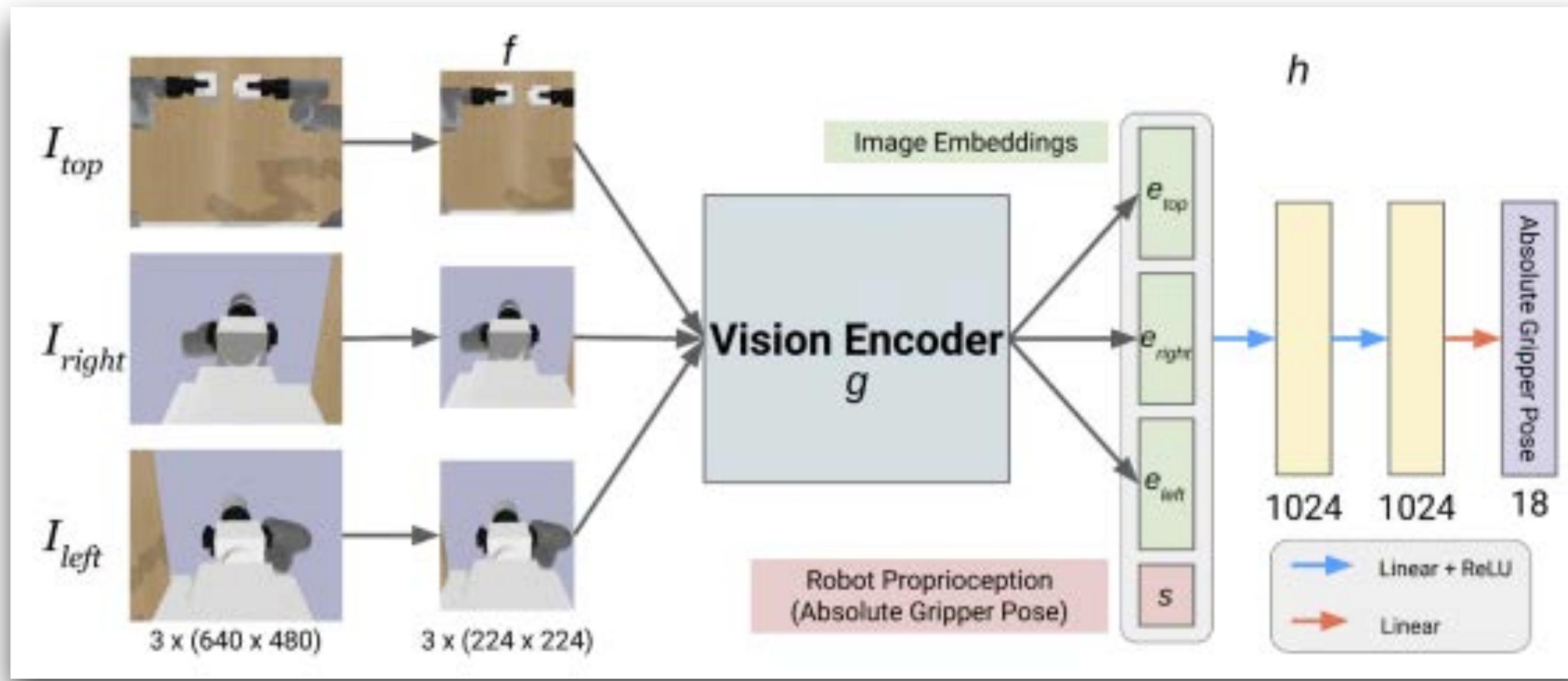




# Next time: Backpropagation



# Problem Statements



C. Ku, C. Winge, R. Diaz, W. Yuan and K. Desingh, "Evaluating Robustness of Visual Representations for Object Assembly Task Requiring Spatio-Geometrical Reasoning," *2024 IEEE International Conference on Robotics and Automation (ICRA)*, Yokohama, Japan, 2024, pp. 831-837, doi: 10.1109/ICRA57147.2024.10610774.

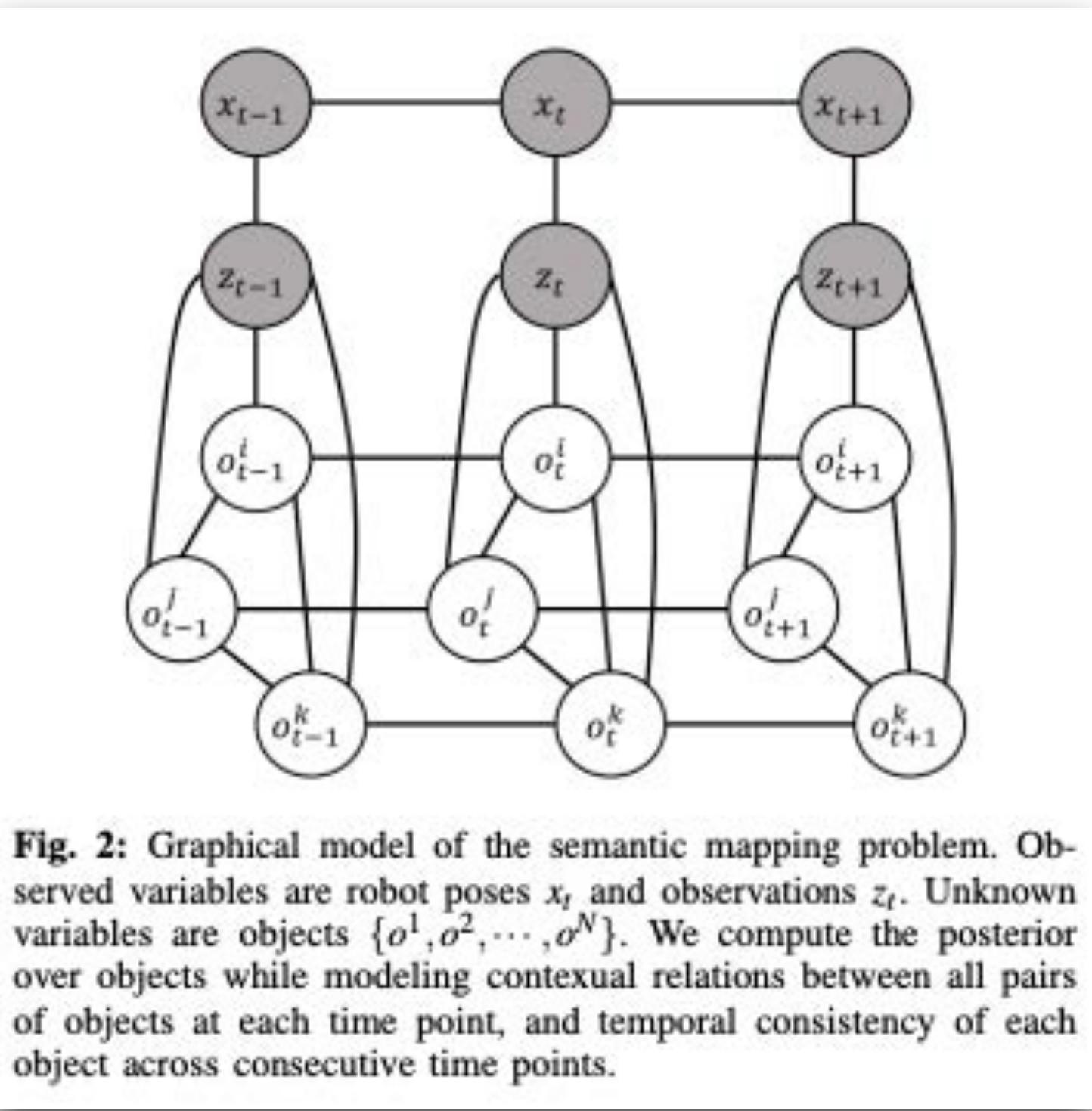
### III. IMITATION LEARNING FRAMEWORK

The goal of imitation learning is to train a policy  $\pi : \mathcal{O} \rightarrow \mathcal{A}$  that maps all observations to an action that will progress the robot towards executing the task. Our dataset  $\mathcal{D} = \{(O_i, a_i)\}_{i=1,\dots,N}$  consists of  $N$  observation-action pairs. Each observation  $O_i = ((I_v)_{v \in V}, s)$  is a tuple of RGB images  $I_v$  from view  $v \in V$  and the current state of the robot  $s \in \mathcal{S}$ . Each action  $a_i \in \mathcal{A}$  is the action the expert performs during demonstration.

Our architecture shown in Fig. 2 is inspired by imitation learning evaluation frameworks from Robomimic [18], R3M [10], and MVP [11]. In our implementation, the policy  $\pi_\theta$  consists of three parts: the image preprocessor  $f$ , the image encoder  $g_\phi$ , and the policy head  $h_\psi$ . The image preprocessor  $f : \mathcal{I}^{640 \times 480} \rightarrow \mathcal{I}^{224 \times 224}$  crops and resizes the original RGB image to a consistent size for fair comparison of all vision encoder models (ViT-B/16 requires this size). The image encoder  $g_\phi : \mathcal{I}^{224 \times 224} \rightarrow \mathbb{R}^D$  is a neural network that deterministically maps an RGB image  $I_v$  to a  $D$ -dimensional image embedding  $e_v$ . The policy head  $h_\psi : \mathbb{R}^D \times \dots \times \mathbb{R}^D \times \mathcal{S} \rightarrow \mathcal{A}$  is a multi-layer perceptron on top of concatenated image embeddings  $(e_v)_{v \in V}$  and robot state  $s$  which produces the final output action  $a$ . In summary, output of the policy  $\pi_\theta(O) = h_\psi(g_\phi(f(I_{v_1})), \dots, g_\phi(f(I_{v_{|V|}))), s) = \hat{a}$  is the predicted action. We train either the parameters  $\psi$  (frozen  $g$ ) or both  $\phi$  and  $\psi$  (unfrozen  $g$ ) by back-propagating the mean squared error loss  $\mathcal{L} = MSE(a, \hat{a})$ .

We choose  $\mathcal{S} \subseteq SE(3) \times SE(3)$  and  $\mathcal{A} \subseteq SE(3) \times SE(3)$  to be absolute gripper poses of both arms. For each arm, there are 3 values representing xyz position and 6 values representing the first two columns of the rotation matrix [25], so we represent both as a 18 dimensional vector. The reason for this choice is explained in Sec. V.

# Problem Statements



**Fig. 2:** Graphical model of the semantic mapping problem. Observed variables are robot poses  $x_t$  and observations  $z_t$ . Unknown variables are objects  $\{o^1, o^2, \dots, o^N\}$ . We compute the posterior over objects while modeling contextual relations between all pairs of objects at each time point, and temporal consistency of each object across consecutive time points.

### III. PROBLEM FORMULATION

We focus on semantic mapping at the object level. Our proposed *CT-Map* method maintains a belief over object classes and poses across an observed scene. We assume that the robot stays localized in the environment through an external localization routine (e.g., ORB-SLAM [23]). The semantic map is composed by a set of  $N$  objects  $O = \{o^1, o^2, \dots, o^N\}$ . Each object  $o^i = \{o^c, o^g, o^\Psi\}$  contains the object class  $o^c \in \mathcal{C}$ , object geometry  $o^g$ , and object pose  $o^\Psi$ , where  $\mathcal{C}$  is the set of object classes  $\mathcal{C} = \{c_1, c_2, \dots, c_n\}$ .

At time  $t$ , the robot is localized at  $x_t$ . The robot observes  $z_t = \{I_t, S_t\}$ , where  $I_t$  is the observed RGB-D image, and  $S_t$  are semantic measurements. The semantic measurements  $s_k = \{s_k^i, s_k^b\} \in S_t$  are returned by an object detector (as explained in section V-A), which contains: 1) a object detection score vector  $s_k^i$ , with each element in  $s_k^i$  denoting the detection confidence of each object class, and 2) a 2D bounding box  $s_k^b$ .

We probabilistically formalize the semantic mapping problem in the form of a CRF, as shown in Figure 2. Robot pose  $x_t$  and observation  $z_t$  are known. The set of objects  $O$  are

unknown variables. We model the contextual dependencies between objects and the temporal consistency of each individual object over time. The posterior probability of the semantic map is expressed as:

$$p(O_{0:T}|x_{0:T}, z_{0:T}) = \frac{1}{Z} \prod_{t=0}^T \prod_{i=1}^N \phi_p(o_t^i, o_{t-1}^i, u_{t-1}^i) \phi_m(o_t^i, x_t, z_t) \prod_{i,j} \phi_c(o_t^i, o_t^j) \quad (1)$$

where  $Z$  is a normalization constant, and action applied to object  $o^i$  at time  $t$  is denoted by  $u_t^i$ .  $\phi_p$  is the *prediction potential* that models the temporal consistency of the object poses.  $\phi_m$  is the *measurement potential* that accounts for the observation model given 3D mesh of objects.  $\phi_c$  is the *context potential* that captures the contextual relations between objects.

Z. Zeng, Y. Zhou, O. C. Jenkins and K. Desingh, "Semantic Mapping with Simultaneous Object Detection and Localization," 2018 IEEE/RSJ International Conference on Intelligent Robots and Systems (IROS), Madrid, Spain, 2018, pp. 911-918, doi: 10.1109/IROS.2018.8594205.

# Problem Statements

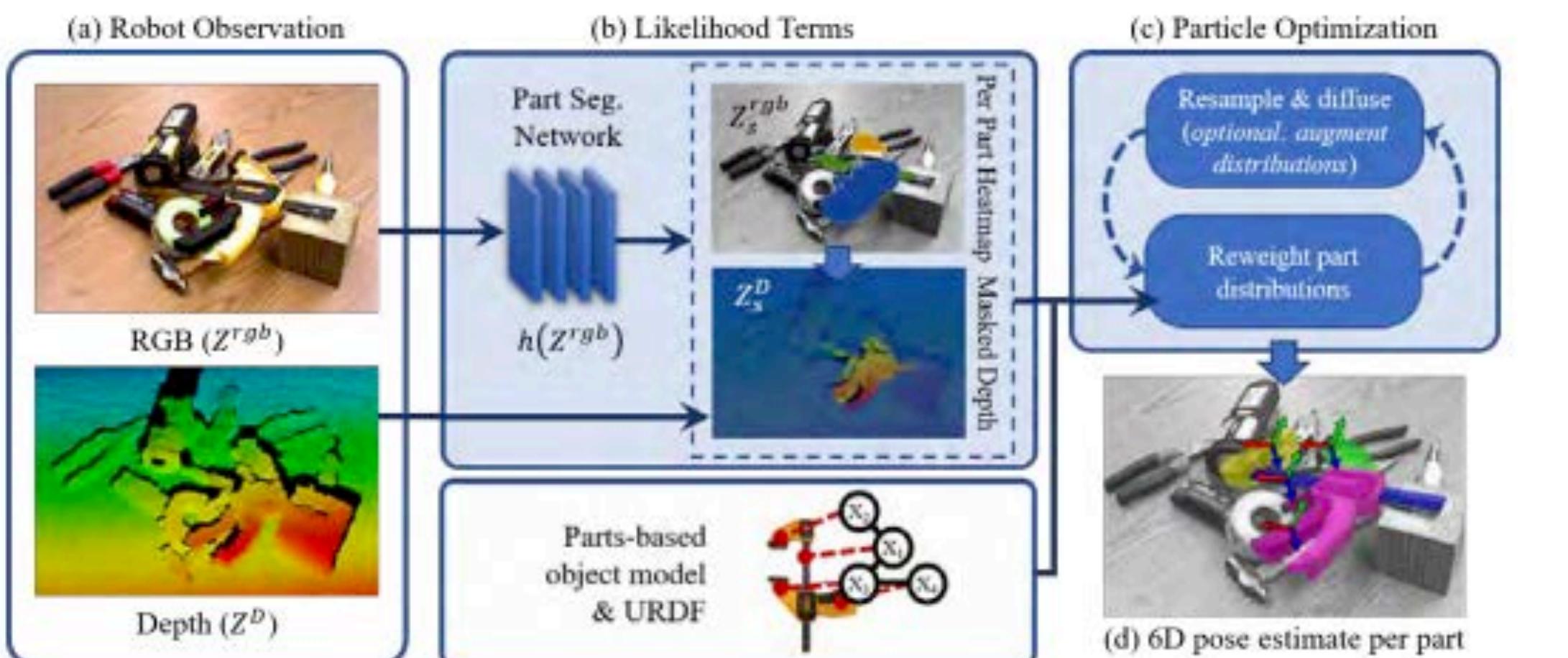


Fig. 3: The inference pipeline. (a) The robot observes a scene as an RGB-D image,  $Z = (Z^{rgb}, Z^D)$ . (b) The RGB image is passed through a trained part segmentation network,  $h(Z^{rgb})$ , that generates a pixel-wise heatmap for the  $P_k$  parts of an object class of interest,  $\{Z_s^{rgb}\}_{s=1}^{P_k}$  (in this example, the clamp, which has one fully occluded part). The heatmaps are used to generate masked depth images,  $\{Z_s^D\}_{s=1}^{P_k}$ . (c) The inference is initialized with part poses using these heatmaps and the depth image. Hypotheses are iteratively reweighted using Equation 4, and resampled with importance sampling. (d) The inference process generates an estimate of the 6D pose of each part. (Best viewed in color).

### III. PROBLEM STATEMENT

Given a scene containing objects  $O$ , such that  $\{O_k\}_{k=1}^K$  is the set of  $K$  relevant objects, we wish to localize each

object  $O_k$ . The state of an object  $O_k$  is represented by the set of part poses  $\mathcal{X} = \{X_i\}_{i=1}^{P_k}$ , where  $X_s$  is the 6D pose of an articulating rigid part  $s$  of  $O_k$ , with  $P_k$  parts. Each object  $O_k$  in the scene is estimated independently.

This estimation problem is formulated as a Markov Random Field (MRF). Let  $G = (V, E)$  denote an undirected graph with nodes  $V$  and edges  $E$ . An example MRF is illustrated in Figure 2. The joint probability of the graph  $G$  is expressed as:

$$p(\mathcal{X}, \mathcal{Z}) \propto \prod_{(s,t) \in E} \psi_{s,t}(X_s, X_t) \prod_{s \in V} \phi_s(X_s, Z_s) \quad (1)$$

where  $\mathcal{X}$  denotes the hidden state variables to be inferred and  $\mathcal{Z}$  denotes the observed sensor information in the form of an RGB-D image. The function  $\psi_{s,t}$  is the *pairwise potential*, describing the correspondence between part poses based on the articulation constraints, and  $\phi_s$  is the *unary potential*, describing the correspondence of a part pose  $X_s$  with its observation  $Z_s$ . The problem of pose estimation of an articulated model  $O_k$  is interpreted as the problem of estimating the marginal distribution of each part pose, called the belief,  $bel(X_s)$ .

In addition to the sensor data, the articulation constraints and 3D geometry of the object, in the form of a Unified Robot Description Format (URDF), and the 3D mesh models of the objects are provided as inputs. We assume that the object articulations are produced by either fixed, prismatic or revolute joints. We consider scenes which contain only one instance of an object. In Section IV, our proposed inference mechanism is detailed, along with a description of our modelling of the potentials in Equation 1.

J. Pavlasek, S. Lewis, K. Desingh and O. C. Jenkins, "Parts-Based Articulated Object Localization in Clutter Using Belief Propagation," 2020 IEEE/RSJ International Conference on Intelligent Robots and Systems (IROS), Las Vegas, NV, USA, 2020, pp. 10595-10602, doi: 10.1109/IROS45743.2020.9340908.

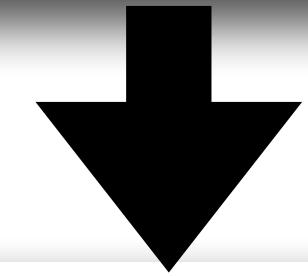


# DR Brainstorming task due 09/23

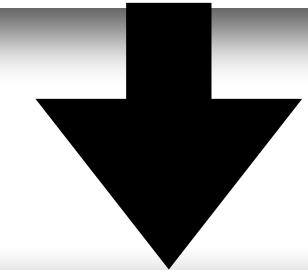
- Pick one of the ideas that you listed.
- Abstract the ideas into a general formulation.
- Write down variables with mathematical notations.
- Write down assumptions and technique that is suitable for this.
- Search for a paper that suits your formulation.

An example:

[1]: **Robot Making Coffee using a Mug and K-Cup Pod Coffee Maker:** A Robot with a single arm and a camera should be able to sense the mug on the table, and pick and place it in the K-Cup Pod Coffee Maker. Then, it should open the K-Cup lid. Pick up the K-cup from a tray, insert it into the coffee maker, closer the lid. The robot should press the button and wait for the coffee to be poured. The robot should pick up the filled coffee mug and place it on the table.



Given an observation of the scene in the form of RGB image  $I \in R^{w \times h \times 3}$ , that contains three objects  $O = \{o_{mug}, o_{kcup}, o_{coffee-m}\}$ , this project aims to develop a learning method that can produce actions  $A = (x, y, z, \pi, \phi, \rho, g)$  where  $x, y, z$  are the gripper position to reach,  $\pi, \phi, \rho$  are the orientation of the gripper, together representing  $SE(3)$  pose of the gripper.  $g$  denotes whether the gripper should be open or close at the end of the action. For this task of coffee making, we aim to develop an end-to-end learning method that can implicitly determine the objects' and their locations, as well as actions to execute from a number of demonstration data that can go from  $I \rightarrow A$ .



VIOLA: Imitation Learning for Vision-Based Manipulation with Object Proposal Priors

Yifeng Zhu<sup>1</sup> Abhishek Joshi<sup>1</sup> Peter Stone<sup>1,2</sup> Yuke Zhu<sup>1</sup>

<sup>1</sup>The University of Texas at Austin <sup>2</sup>Sony AI

[Paper](#) | [Video](#) | [Code](#) | [Bibtex](#)

6th Conference on Robot Learning, Auckland, New Zealand





# DeepRob

Lecture 5  
Neural Networks  
University of Minnesota

