Image Denoising and Deblurring

Applied Math 515 Final Project

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Contents

Image Denoising and Deblurring

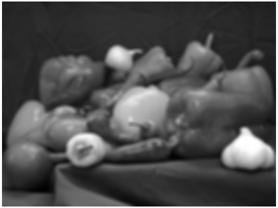
Mathematical Formulation

$$\overset{\text{True image}}{\overbrace{\underset{\text{Blur operator}}{\underbrace{A}}x}} + \underset{\text{Noise}}{\underbrace{w}} = \overset{\text{Observed image}}{\overbrace{b}}$$

- Blur: Ax is a discrete convolution of a Gaussian kernel with symmetric boundary conditions.
- Noise: w is noise drawn from a Gaussian distribution or a Student's t-distribution.

Naive Solution: $x = A^{-1}b$

True image

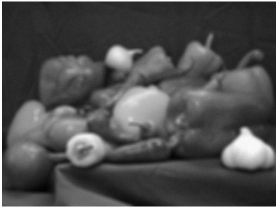


Blurred image

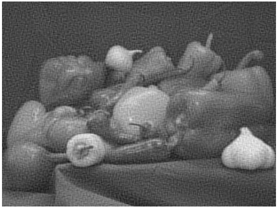
Recovered image

Naive Solution: $x = A^{-1}(b - w)$

True image



Blurred and noisy image



Recovered image

Since the blur operator is ill-conditioned, a better approach is to minimize a regularized loss function.

General Loss Function

$$L_b(x) = \underbrace{f(Ax - b)}_{\text{Fidelity term}} + \underbrace{\lambda R(x)}_{\text{Regularization}}$$

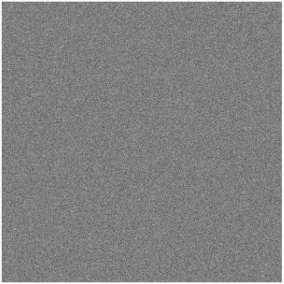
Fidelity Terms

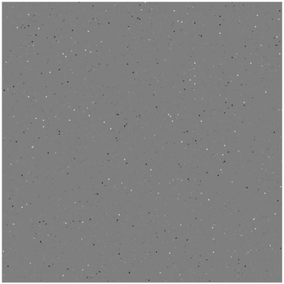
$$f = \begin{cases} \|\cdot\|_F^2 \\ h_\gamma(\cdot) \\ \gamma^{-1}\log(\cosh(\gamma\cdot)) \end{cases}$$

Regularization Terms

$$R = \begin{cases} TV(x) \\ \|Wx\|_1 \end{cases}$$

Fidelity Term Penalty Functions





What is Ax - b Why use different functions than frobenius norm? Use pictures as motivation. Show quadratic and huber penalty functions, show different distributions?

Image Assumptions and Regularizers

Either assume image is sparse in wavelet domain, or assume image is smooth
Talk about choice of g Haar, FFT What is TV?
Show two different definitions of TV from paper.

L1 Wavelet Regularization

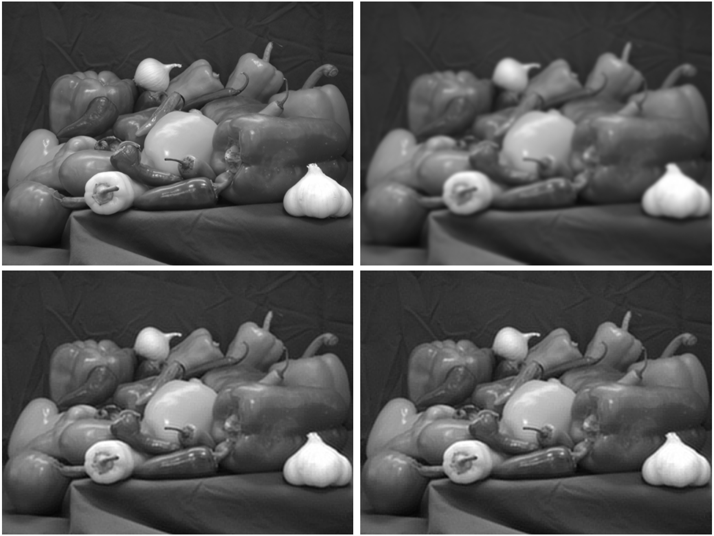
Loss Function

$$L_b(x) = f(Ax - b) + \lambda \|Wx\|_1$$

Proximal Gradient Step

$$x^{k+1} = \text{prox}_{\alpha^{-1}\lambda\|W\cdot\|_1}\left(x^k - \alpha^{-1}A^T\nabla f(Ax^k - b)\right)$$

Results: Gaussian Noise



Original Image

Blurred and Noisy

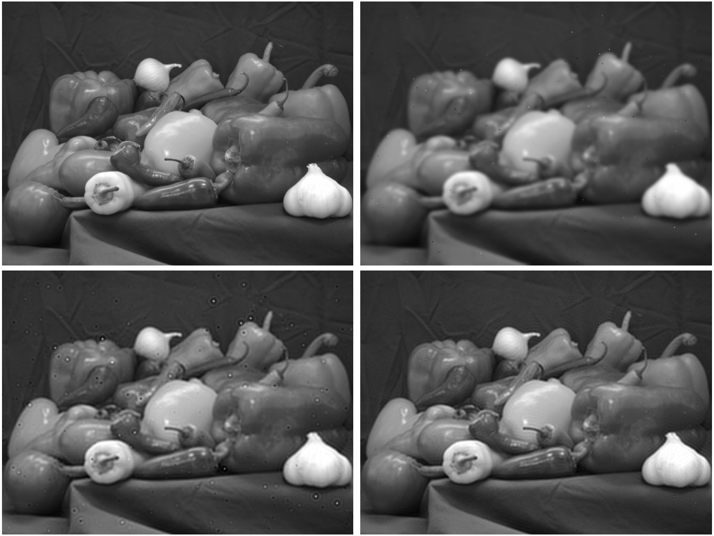
Frobenius Loss

Huber Loss

Results: Student's t Noise

Original Image

Blurred and Noisy



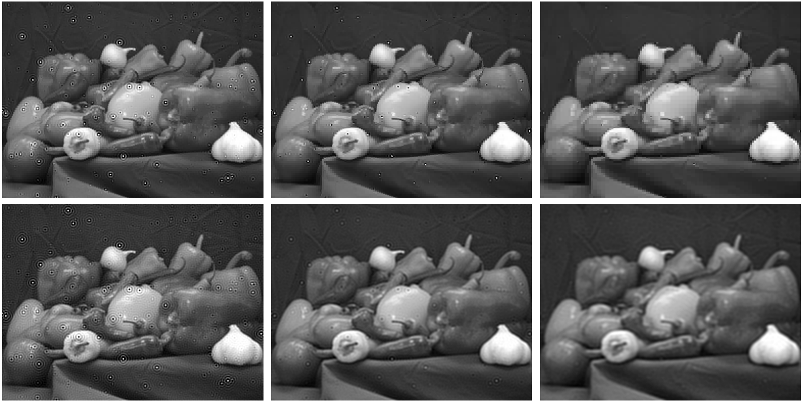
Frobenius Loss

Huber Loss

Choosing λ

Haar

Cosine



$\lambda = 10^{-4}$ $\lambda = 10^{-3}$ $\lambda = 10^{-2}$

Total Variation Regularization

Loss Function

$$L_b(x) = f(Ax - b) + \lambda \|x\|_{TV} + \delta(x|[0,1])$$

Proximal Gradient Step

$$\begin{aligned} x^{k+1} &= \text{prox}_{\alpha^{-1}(\lambda\|\cdot\|_{TV} + \delta_{[0,1]})}(\underbrace{x^k - \alpha^{-1}A^T \nabla f(Ax^k - b)}_{u^k}) \\ &= \underset{z}{arg\,min} \left(\|u^k - z\|_F^2 + \alpha^{-1}\lambda\|z\|_{TV} + \delta(z|[0,1])\right) \\ &= P_{[0,1]}\left(\underset{z}{arg\,min} \left(\|u^k - z\|_F^2 + \alpha^{-1}\lambda\|z\|_{TV}\right)\right) \end{aligned}$$

Dual Form of Total Variation

A Few Definitions

weee

Total Variation

blarg

Dual Form of TV Denoising with $\|\cdot\|_F^2$

Optimization of Dual Form

Problem Statement

$$\min_{(p,q)\in\mathcal{P}} \left\{ \|b-\lambda\mathcal{L}(p,q)\|_F^2 - \|(I-P_{[0,1]})(b-\lambda\mathcal{L}(p,q))\|_F^2 \right\}$$

$$\min_{(p,q)} \left\{ \underbrace{\|b-\lambda\mathcal{L}(p,q)\|_F^2 - \|(I-P_{[0,1]})(b-\lambda\mathcal{L}(p,q))\|_F^2}_{=h(p,q)} + \delta((p,q)|\mathcal{P}) \right\}$$

$$\nabla h(p,q) = -2\lambda\mathcal{L}^T P_{[0,1]}(b-\lambda\mathcal{L}(p,q))$$

Lipschitz with constant $\leq 16\lambda^2$

$\Rightarrow$ Use projected gradient

Optimization in Dual Form

Projection onto $\mathcal{P}$

Recal $\mathcal{P} = (p, q) \in [-1, 1]^{m-1 \times n} \times [-1, 1]^{m \times n-1}$

$$P_{\mathcal{P}}(p, q) = (r, s) \text{ with } \begin{cases} r_{ij} = sgn(p_{ij}) \min\{1, |p_{ij}|\} \\ s_{ij} = sgn(q_{ij}) \min\{1, |q_{ij}|\} \end{cases}$$

Projected Gradient Step

$$(p^{k+1}, q^{k+1}) = P_{\mathcal{P}} \left((p^k, q^k) + \frac{1}{8\lambda} \mathcal{L}^T P_{[0,1]}(b - \lambda \mathcal{L}(p, q)) \right)$$

Summary of Fast TV Regularization

MFISTA(b, f, λ)

$y^1 = x^0 = b;\ t^1 = 1$

$\alpha \geq Lip(\nabla f)$

for $k = 1 : N$ **do**

$u^k = y^k - \frac{A^T \nabla f(Ay^k - b)}{\alpha}$

$z^k = FGP(u^k, \frac{\lambda}{2\alpha})$

$x^k = \underset{x \in \{x^{k-1}, z^k\}}{\text{argmin}} L_b(x)$

$t^{k+1} = \frac{1+\sqrt{1+4{t^k}^2}}{2}$

$y^{k+1} = x^k + \frac{t^k}{t^{k+1}}(z^k - x^k) + \frac{t^k - 1}{t^{k+1}}(z^k - x^k)$

end for

return x^N

FGP(b, λ)

$(r_{ij}^1, s_{ij}^1) = (p_{ij}^0, q_{ij}^0) = 0;\ t^1 = 1$

for $k = 1 : N$ **do**

$(p^k, q^k) = P_{\mathcal{P}}\left((r^k, s^k) - \frac{\mathcal{L}^T P_{[0,1]}(b - \lambda \mathcal{L}(r^k, s^k))}{8\lambda}\right)$

$t^{k+1} = \frac{1+\sqrt{1+4{t^k}^2}}{2}$

$(r^k, s^k) = (p^k, q^k) + \frac{t^k - 1}{t^{k+1}}(p^k - p^{k-1}, q^k - q^{k-1})$

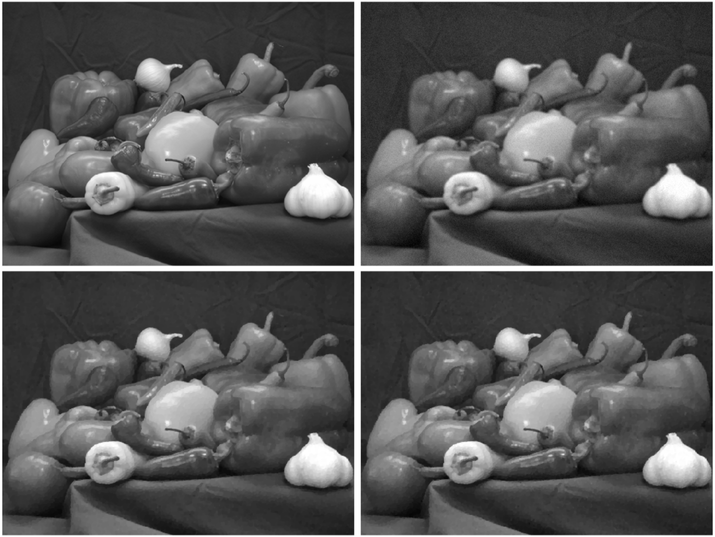
end for

return $P_{[0,1]}(b - \lambda \mathcal{L}(p^N, q^N))$

Results: Gaussian Noise

Original Image

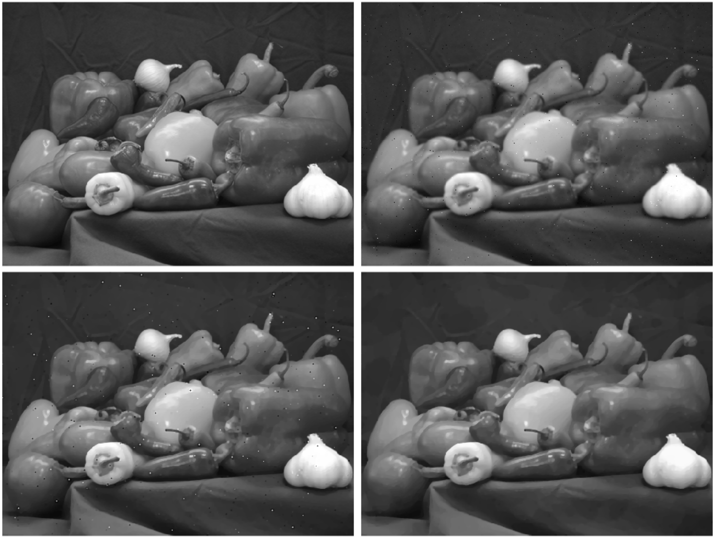
Blurred and Noisy



Frobenius Loss

Huber Loss

Results: Student's-t Noise



Original Image

Blurred and Noisy

Frobenius Loss

Huber Loss

Questions?

- Codes used to generate figures
  `https://github.com/snagcliffs/Amath575project`
- Guckenheimer, J., Holmes, P. *Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields*. Springer-Verlag, 1983. Print.
- Oliveira, D., Leonel, E. (2008) *Braz. J. Phys.* 38(1):62-64
- Grassberger, P., Procaccia, I. (1983) *Phys. Rev. Letters.* 50(5):346-349