

Image Denoising and Deblurring

Applied Math 515 Final Project

Samuel Rudy, Kelsey Maass, Riley Molloy, and Kevin Mueller

March 12 2016

Contents

- 1 Motivation
- 2 De(noise/blur)ing Objective Functions
- 3 Optimization with L1 Wavelet Regularization
- 4 Optimization with Total Variation Regularization
- 5 Discussion
- 6 Discussion

Image Denoising and Deblurring



Mathematical Formulation

$$Ax + w = b$$

Diagram illustrating the mathematical formulation of image denoising:

- Ax is labeled "Blur operator" (indicated by a downward arrow).
- w is labeled "Noise" (indicated by a downward arrow).
- b is labeled "Observed image" (indicated by an upward arrow).
- The term x is labeled "True image" (indicated by an upward arrow).

- Blur: Ax is a discrete convolution of the true image with a Gaussian kernel (reflexive boundary conditions).
- Noise: w is noise drawn from a Gaussian or Student's t distribution.

Naive Solution: $x = A^{-1}b$



True image



Blurred image

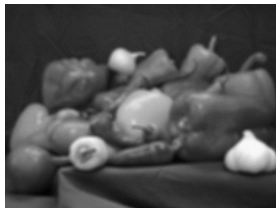


Recovered image

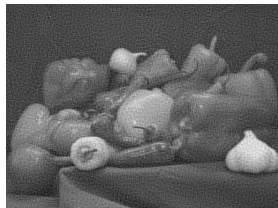
Naive Solution: $x = A^{-1}(b - w)$



True image



Blurred and noisy image



Recovered image

Since the blur operator is ill-conditioned, a better approach is to minimize a regularized loss function.

General Loss Function

$$L_b(x) = \underbrace{f(Ax - b)}_{\text{Fidelity term}} + \underbrace{\lambda R(x)}_{\text{Regularization}}$$

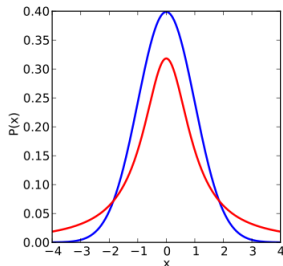
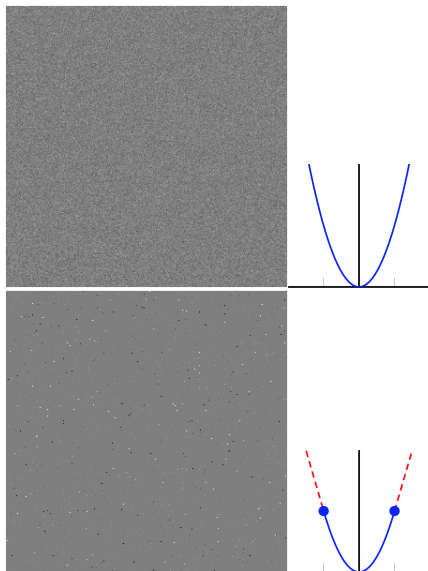
Fidelity Terms

$$f = \begin{cases} \|\cdot\|_F^2 \\ h_\gamma(\cdot) \\ \gamma^{-1} \log(\cosh(\gamma \cdot)) \end{cases}$$

Regularization Terms

$$R = \begin{cases} \|Wx\|_1 \\ TV(x) \end{cases}$$

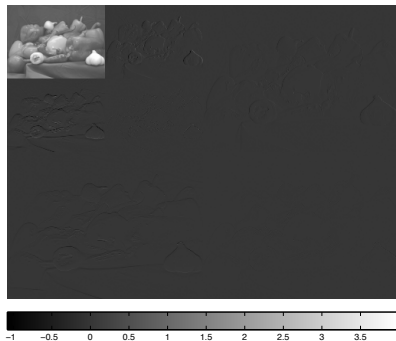
Fidelity Term Penalty Functions



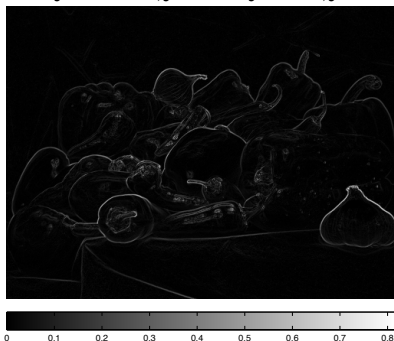
- Gaussian \implies strong mean-centered penalization
- Student's t \implies less penalization of outliers

Image Assumptions and Regularizers

2-Level Haar Transform



$$|x_{ij} - x_{i+1,j}| + |x_{ij} - x_{i,j+1}|$$



Our choices of regularizers assume that images are sparse in wavelet domains and that they are relatively smooth (low total variation).

SPLIT THIS INTO 2 SLIDES

L1 Wavelet Regularization

Loss Function

$$L_b(x) = f(Ax - b) + \lambda \|Wx\|_1$$

Proximal Gradient Step

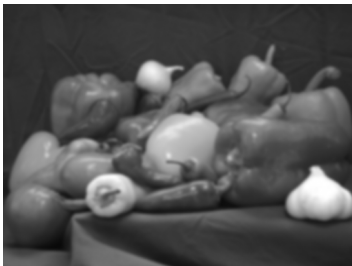
$$x^{k+1} = \text{prox}_{\alpha^{-1}\lambda\|W\cdot\|_1} \left(x^k - \alpha^{-1} A^T \nabla f(Ax^k - b) \right)$$

Results: Gaussian Noise

Original Image



Blurred and Noisy



Frobenius Loss



Huber Loss

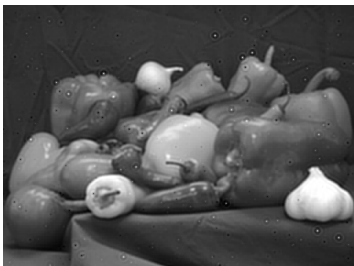


Results: Student's t Noise

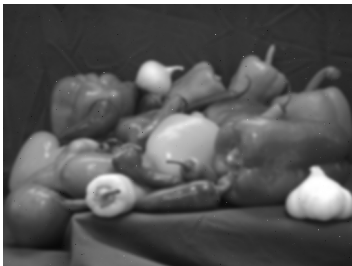
Original Image



Frobenius Loss



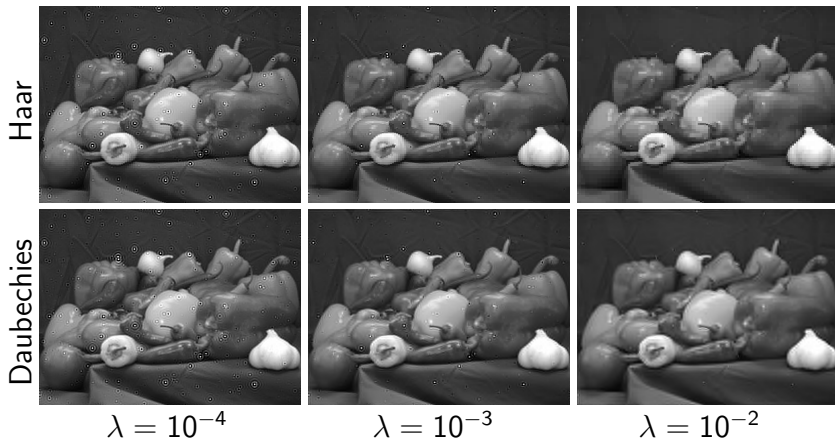
Blurred and Noisy



Huber Loss



Choosing λ



Total Variation Regularization

Loss Function

$$L_b(x) = f(Ax - b) + \lambda \text{TV}(x) + \delta(x|[0, 1])$$

Proximal Gradient Step

$$\begin{aligned} x^{k+1} &= \text{prox}_{\alpha^{-1}(\lambda \|\cdot\|_{\text{TV}} + \delta_{[0,1]})} \left(\underbrace{x^k - \alpha^{-1} A^T \nabla f(Ax^k - b)}_{u^k} \right) \\ &= \arg \min_z \left(\|u^k - z\|_F^2 + \alpha^{-1} \lambda \|z\|_{\text{TV}} + \delta(z|[0, 1]) \right) \\ &= P_{[0,1]} \left(\arg \min_z \left(\|u^k - z\|_F^2 + \alpha^{-1} \lambda \|z\|_{\text{TV}} \right) \right) \end{aligned}$$

Dual Form of Total Variation

A Few Definitions

- $\mathcal{P} = \{(p, q) \in \mathbb{R}^{(m-1) \times n} \times \mathbb{R}^{m \times (n-1)} : |p_{i,j}| \leq 1, |q_{i,j}| \leq 1\}$,
- $\mathcal{L} : \mathbb{R}^{(m-1) \times n} \times \mathbb{R}^{m \times (n-1)} \rightarrow \mathbb{R}^{m \times n}$ such that

$$\mathcal{L}(p, q)_{i,j} = p_{i,j} + q_{i,j} - p_{i-1,j} - q_{i,j-1}$$

for $i = 1, \dots, m, j = 1, \dots, n$, and

$$p_{0,j} = p_{m,j} = q_{i,0} = q_{i,n} = 0.$$

- P_C is the usual projection operator onto the set C

Total Variation

$$\text{TV}(x) = \max_{p,q \in \mathcal{P}} T(x, p, q) \implies T(x, p, q) = \text{Tr}(\mathcal{L}(p, q)^T x).$$

Dual Form of TV Denoising with $\|\cdot\|_F^2$

The problem:

$$\min_{x \in C} \|x - b\|_F^2 + 2\lambda \text{TV}(x), C = [0, 1]$$

Dual problem:

$$\min_{(p,q) \in \mathcal{P}} \underbrace{-\|H_C(b - \lambda \mathcal{L}(p, q))\|_F^2 + \|b - \lambda \mathcal{L}(p, q)\|_F^2}_{h(p,q)}$$

$$H_C(x) = \underbrace{x - P_C(x)}_{\text{prox}}$$

Optimality conditions:

$$x = P_C(b - \lambda \mathcal{L}(p, q)).$$

Optimization of Dual Form

Problem Statement

$$\min_{(p,q) \in \mathcal{P}} \left\{ \|b - \lambda \mathcal{L}(p, q)\|_F^2 - \|(I - P_{[0,1]})(b - \lambda \mathcal{L}(p, q))\|_F^2 \right\}$$

$$\min_{(p,q)} \left\{ \underbrace{\|b - \lambda \mathcal{L}(p, q)\|_F^2 - \|(I - P_{[0,1]})(b - \lambda \mathcal{L}(p, q))\|_F^2}_{=h(p,q)} + \delta((p, q) | \mathcal{P}) \right\}$$

$$\nabla h(p, q) = -2\lambda \mathcal{L}^T P_{[0,1]}(b - \lambda \mathcal{L}(p, q))$$

Lipschitz with constant $\leq 16\lambda^2$

\Rightarrow Use projected gradient

Optimization in Dual Form

Projection onto \mathcal{P}

Recall $\mathcal{P} = (p, q) \in [-1, 1]^{m-1 \times n} \times [-1, 1]^{m \times n-1}$

$$P_{\mathcal{P}}(p, q) = (r, s) \text{ with } \begin{cases} r_{ij} = \text{sgn}(p_{ij}) \min\{1, |p_{ij}|\} \\ s_{ij} = \text{sgn}(q_{ij}) \min\{1, |q_{ij}|\} \end{cases}$$

Projected Gradient Step

$$(p^{k+1}, q^{k+1}) = P_{\mathcal{P}} \left((p^k, q^k) + \frac{1}{8\lambda} \mathcal{L}^T P_{[0,1]}(b - \lambda \mathcal{L}(p, q)) \right)$$

Summary of Fast TV Regularization

MFISTA(b, f, λ)

$$y^1 = x^0 = b; t^1 = 1$$

$$\alpha \geq \text{Lip}(\nabla f)$$

for $k = 1 : N$ **do**

$$u^k = y^k - \frac{A^T \nabla f(Ay^k - b)}{\alpha}$$

$$z^k = \text{FGP}(u^k, \frac{\lambda}{2\alpha})$$

$$x^k = \underset{x \in \{x^{k-1}, z^k\}}{\text{argmin}} L_b(x)$$

$$t^{k+1} = \frac{1 + \sqrt{1 + 4t^{k2}}}{2}$$

$$y^{k+1} = x^k + \frac{t^k}{t^{k+1}}(z^k - x^k) + \frac{t^{k-1}}{t^{k+1}}(z^k - x^k)$$

end for

return x^N

FGP(b, λ)

$$(r_{ij}^1, s_{ij}^1) = (p_{ij}^0, q_{ij}^0) = 0; t^1 = 1$$

for $k = 1 : N$ **do**

$$(p^k, q^k) = P_{\mathcal{P}} \left((r^k, s^k) - \frac{\mathcal{L}^T P_{[0,1]}(b - \lambda \mathcal{L}(r^k, s^k))}{8\lambda} \right)$$

$$t^{k+1} = \frac{1 + \sqrt{1 + 4t^{k2}}}{2}$$

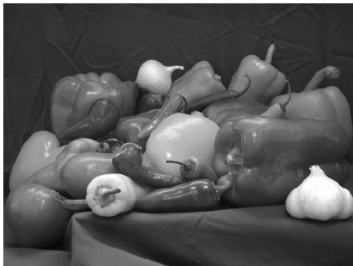
$$(r^k, s^k) = (p^k, q^k) + \frac{t^k - 1}{t^{k+1}}(p^k - p^{k-1}, q^k - q^{k-1})$$

end for

return $P_{[0,1]}(b - \lambda \mathcal{L}(p^N, q^N))$

Results: Gaussian Noise

Original Image



Frobenius Loss



Blurred and Noisy

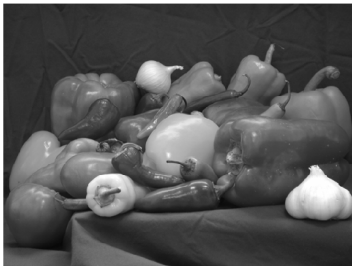


Huber Loss

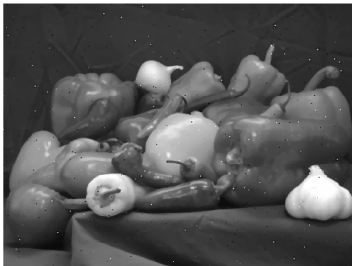


Results: Student's-t Noise

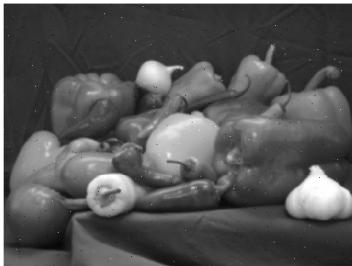
Original Image



Frobenius Loss



Blurred and Noisy



Huber Loss



Questions?



Codes used to generate figures

<https://github.com/snagcliffs/Amath575project>



Guckenheimer, J., Holmes, P. *Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields*. Springer-Verlag, 1983. Print.



Oliveira, D., Leonel, E. (2008) *Braz. J. Phys.* 38(1):62-64



Grassberger, P., Procaccia, I. (1983) *Phys. Rev. Letters*. 50(5):346-349