Image Denoising and Deblurring
Applied Math 515 Final Project

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Image Denoising and Deblurring

Mathematical Formulation

True image — Observed image

$$Ax + w = b$$

Blur operator — Noise

Naive Solution

$$x = A^{-1}b$$

True image

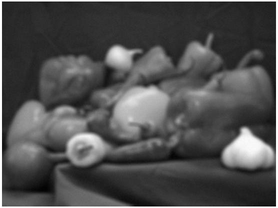
Blurred image

Recovered image

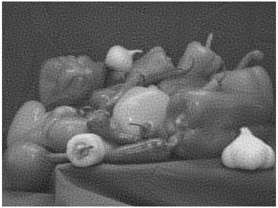
Naive Solution

$$x = A^{-1}(b-w)$$

True image



Blurred and noisy image



Recovered image

Since the naive solution is contaminated by round-off error and noise, a better approach is to solve the problem:

$$\min_x \quad \underbrace{f(Ax-b)}_{\text{Fidelity term}} + \underbrace{\lambda R(x)}_{\text{Regularization}}$$

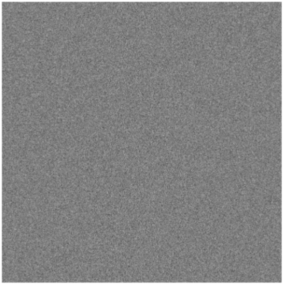
Fidelity Term

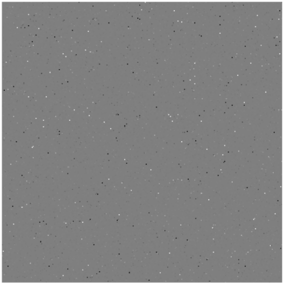
$$f = \begin{cases} \|\cdot\|_F^2 \\ h_\gamma(\cdot) \\ \gamma^{-1}\log(\cosh(\gamma\cdot)) \end{cases}$$

Regularization Term

$$R = \begin{cases} TV(x) \\ \|Wx\|_1 \end{cases}$$

Fidelity Term Penalty Functions





What is Ax - b Why use different functions than frobenius norm? Use pictures as motivation. Show quadratic and huber penalty functions, show different distributions?

Image Assumptions and Regularizers

Either assume image is sparse in wavelet domain, or assume image is smooth
Talk about choice of g Haar, FFT What is TV?
Show two different definitions of TV from paper.

Kelsey's stuff here

- Loss function
- choice of lambda
- choice of wavelet
- prox gradient method

Total Variation Regularization

Loss Function

$$L_b(x) = f(Ax - b) + \lambda\|x\|_{TV} + \delta(x|[0,1])$$

Proximal Gradient Step

$$\begin{aligned}
x^{k+1} &= \text{prox}_{\mathcal{L}^{-1}(\lambda\|\cdot\|_{TV} + \delta_{[0,1]})}(\underbrace{x^k - \mathcal{L}^{-1}A^T\nabla f(Ax^k - b)}_{u^k}) \\
&= \underset{z}{arg\ min}\left(\|u^k - z\|_F^2 + \lambda\|z\|_{TV} + \delta(z|[0,1])\right) \\
&= P_{[0,1]}\left(\underset{z}{arg\ min}\left(\|u^k - z\|_F^2 + \lambda\|z\|_{TV}\right)\right)
\end{aligned}$$

Dual Form of Total Variation

A Few Definitions

weee

Total Variation

blarg

Dual Form of TV Denoising with $\|\cdot\|_F^2$

Optimization of Dual Form

Problem Statement

$$\min_{(p,q)\in\mathcal{P}} \left\{ \|b - \lambda\mathcal{L}(p,q)\|_F^2 - \|(I - P_{[0,1]})(b - \lambda\mathcal{L}(p,q))\|_F^2 \right\}$$

$$\min_{(p,q)} \left\{ \underbrace{\|b - \lambda\mathcal{L}(p,q)\|_F^2 - \|(I - P_{[0,1]})(b - \lambda\mathcal{L}(p,q))\|_F^2}_{=h(p,q)} + \delta((p,q)|\mathcal{P}) \right\}$$

$$\nabla h(p,q) = -2\lambda\mathcal{L}^T P_{[0,1]}(b - \lambda\mathcal{L}(p,q))$$

Lipschitz with constant $\leq 16\lambda^2$

$\Rightarrow$ Use projected gradient

Optimization in Dual Form

Projection onto $\mathcal{P}$

Recal $\mathcal{P} = (p, q) \in [-1, 1]^{m-1 \times n} \times [-1, 1]^{m \times n-1}$

$$P_{\mathcal{P}}(p, q) = (r, s) \text{ with } \begin{cases} r_{ij} = sgn(p_{ij}) \min\{1, |p_{ij}|\} \\ s_{ij} = sgn(q_{ij}) \min\{1, |q_{ij}|\} \end{cases}$$

Projected Gradient Step

$$(p^{k+1}, q^{k+1}) = P_{\mathcal{P}} \left((p^k, q^k) + \frac{1}{8\lambda} \mathcal{L}^T P_{[0,1]}(b - \lambda \mathcal{L}(p, q)) \right)$$

Summary of Fast TV Regularization

MFISTA(b, f, λ)

$y^1 = x^0 = b;\ t^1 = 1$

$\alpha \geq Lip(\nabla f)$

for $k = 1 : N$ **do**

$$u^k = y^k - \frac{A^T \nabla f(Ay^k - b)}{\alpha}$$

$$z^k = FGP(u^k, \tfrac{\lambda}{2\alpha})$$

$$x^k = \underset{x \in \{x^{k-1}, z^k\}}{\operatorname{argmin}} L_b(x)$$

$$t^{k+1} = \frac{1+\sqrt{1+4{t^k}^2}}{2}$$

$$y^{k+1} = x^k + \frac{t^k}{t^{k+1}}(z^k - x^k) + \frac{t^k - 1}{t^{k+1}}(z^k - x^k)$$

end for

FGP(b, λ)

$(r^1_{ij}, s^1_{ij}) = (p^0_{ij}, q^0_{ij}) = 0;\ t^1 = 1$

for $k = 1 : N$ **do**

$$(p^k, q^k) = P_{\mathcal{P}}\left((r^k, s^k) - \frac{\mathcal{L}^T P_{[0,1]}(b - \lambda \mathcal{L}(r^k, s^k))}{8\lambda}\right)$$

$$t^{k+1} = \frac{1+\sqrt{1+4{t^k}^2}}{2}$$

$$(r^k, s^k) = (p^k, q^k) + \frac{t^k - 1}{t^{k+1}}(p^k - p^{k-1}, q^k - q^{k-1})$$

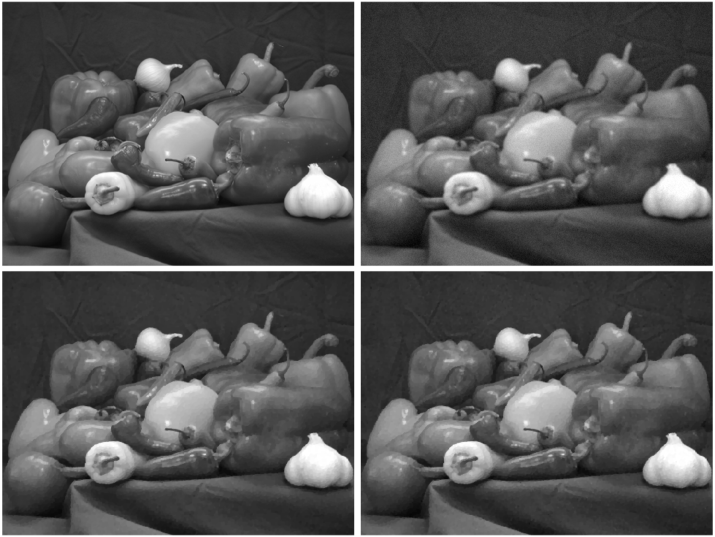
end for

return $P_{[0,1]}(b - \lambda \mathcal{L}(p^N, q^N))$

Results: Gaussian Noise

Original Image

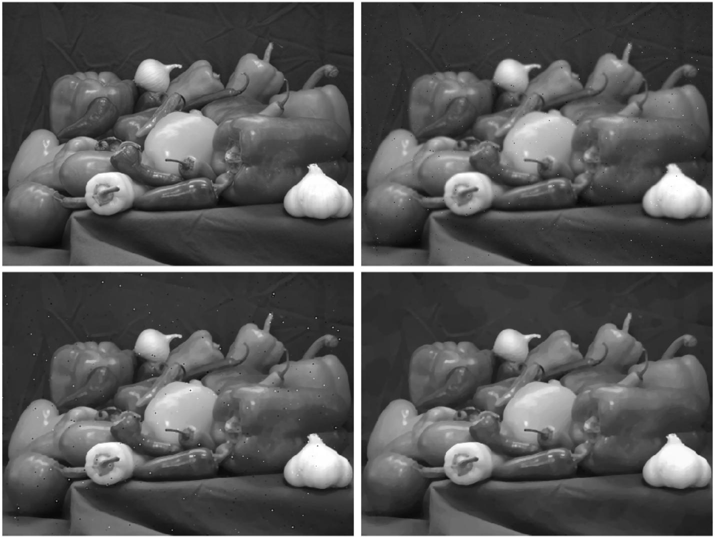
Blurred and Noisy



Frobenius Loss

Huber Loss

Results: Student's-t Noise



Questions?

- Codes used to generate figures
  https://github.com/snagcliffs/Amath575project
- Guckenheimer, J., Holmes, P. *Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields*. Springer-Verlag, 1983. Print.
- Oliveira, D., Leonel, E. (2008) *Braz. J. Phys.* 38(1):62-64
- Grassberger, P., Procaccia, I. (1983) *Phys. Rev. Letters.* 50(5):346-349