Image Denoising and Deblurring

Applied Math 515 Final Project

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Contents

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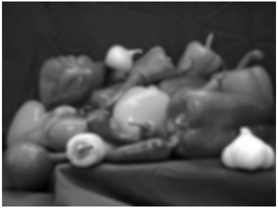
Mathematical Formulation

$$\underbrace{A}_{\text{Blur operator}}\overbrace{x}^{\text{True image}} + \underbrace{w}_{\text{Noise}} = \overbrace{b}^{\text{Observed image}}$$

- Blur: Ax is a discrete convolution of the true image with a Gaussian kernel (reflexive boundary conditions).
- Noise: w is noise drawn from a Gaussian or Student's t distribution.

Naive Solution: $x = A^{-1}b$

True image

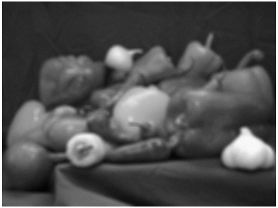


Blurred image

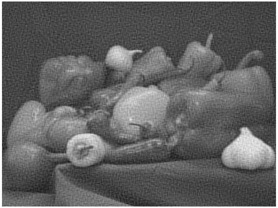
Recovered image

Naive Solution: $x = A^{-1}(b - w)$

True image



Blurred and noisy image



Recovered image

Since the blur operator is ill-conditioned, a better approach is to minimize a regularized loss function.

General Loss Function

$$L_b(x) = \underbrace{f(Ax - b)}_{\text{Fidelity term}} + \underbrace{\lambda R(x)}_{\text{Regularization}}$$

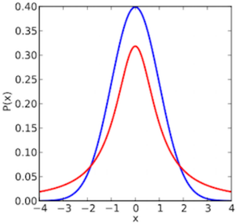
Fidelity Terms

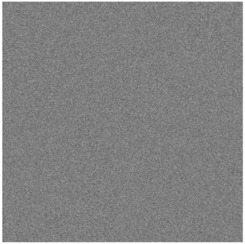
$$f = \begin{cases} \| \cdot \|_F^2 \\ h_\gamma(\cdot) \\ \gamma^{-1} \log(\cosh(\gamma \cdot)) \end{cases}$$

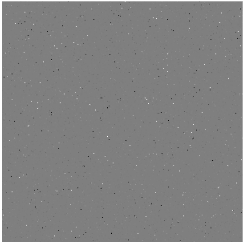
Regularization Terms

$$R = \begin{cases} \|Wx\|_1 \\ TV(x) \end{cases}$$

Fidelity Term Penalty Functions

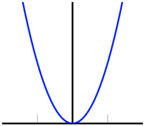






Gaussian Noise

Need: strong mean-centered penalization



Student's t Noise

Need: less penalization of outliers

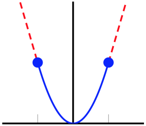
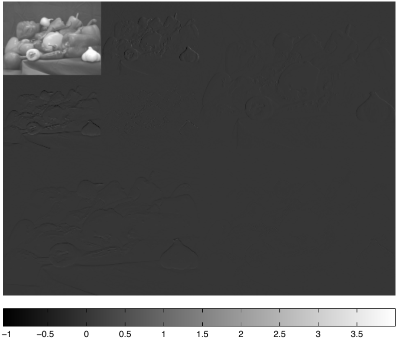
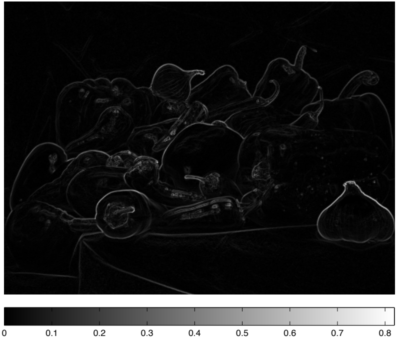


Image Assumptions and Regularizers

2-Level Haar Transform



$|x_{ij} - x_{i+1,j}| + |x_{ij} - x_{i,j+1}|$



Our choices of regularizers assume that images are sparse in wavelet domains and that they are relatively smooth (low total variation). SPLIT THIS INTO 2 SLIDES

L1 Wavelet Regularization

Loss Function

$$L_b(x) = f(Ax - b) + \lambda \| Wx \|_1$$

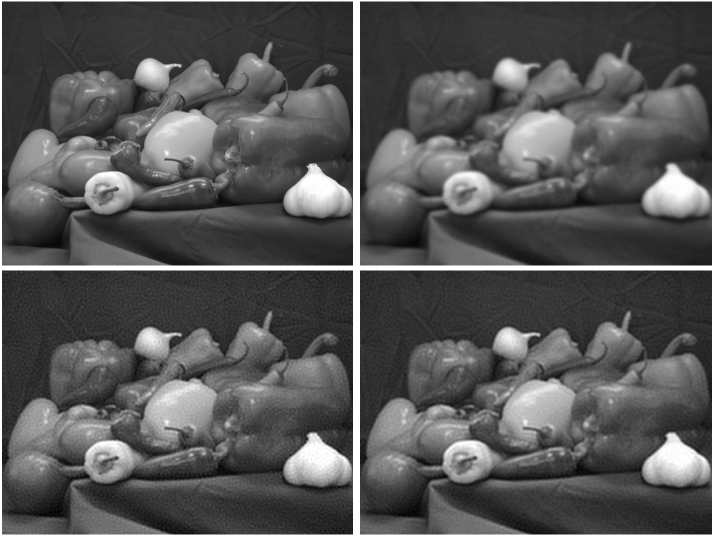
Proximal Gradient Step

$$x^{k+1} = \mathrm{prox}_{\alpha^{-1}\lambda\|W\cdot\|_1}\left(x^k - \alpha^{-1} A^T \nabla f(Ax^k - b)\right)$$

Results: Gaussian Noise

Original Image

Blurred and Noisy



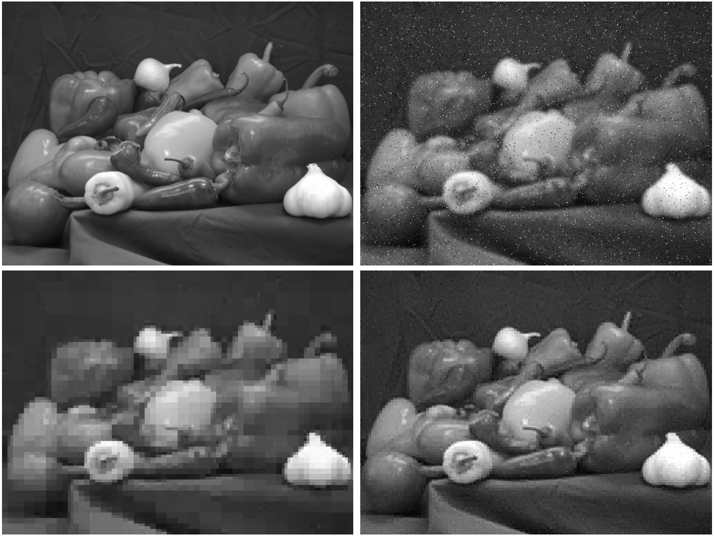
Frobenius Loss

Huber Loss

Results: Student's t Noise

Original Image

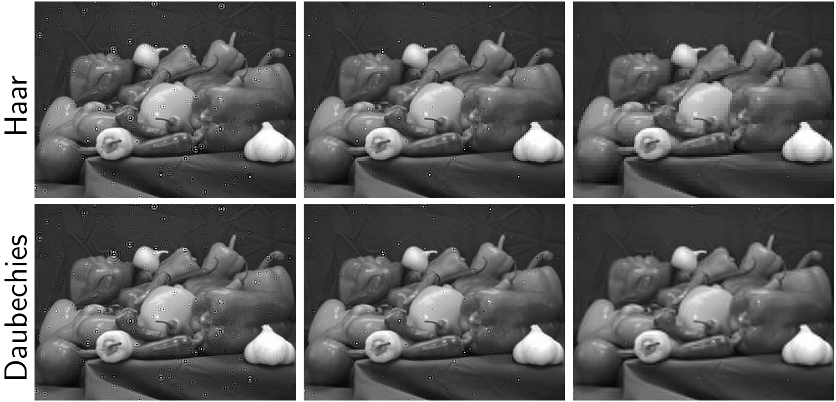
Blurred and Noisy



Frobenius Loss

Huber Loss

Choosing λ



$\lambda = 10^{-4}$ $\lambda = 10^{-3}$ $\lambda = 10^{-2}$

Total Variation Regularization

Loss Function

$$L_b(x) = f(Ax - b) + \lambda \mathrm{TV}(x) + \delta(x|[0,1])$$

Proximal Gradient Step

$$\begin{aligned}
x^{k+1} &= \mathrm{prox}_{\alpha^{-1}(\lambda\|\cdot\|_{TV} + \delta_{[0,1]})}\big(\underbrace{x^k - \alpha^{-1}A^T\nabla f(Ax^k - b)}_{u^k}\big) \\
&= \underset{z}{arg\,min}\left(\|u^k - z\|_F^2 + \alpha^{-1}\lambda\|z\|_{TV} + \delta(z|[0,1])\right) \\
&= P_{[0,1]}\left(\underset{z}{arg\,min}\left(\|u^k - z\|_F^2 + \alpha^{-1}\lambda\|z\|_{TV}\right)\right)
\end{aligned}$$

Dual Form of Total Variation

A Few Definitions

- $\mathcal{P} = \{(p,q) \in \mathbb{R}^{(m-1)\times n} \times \mathbb{R}^{m\times(n-1)} : |p_{i,j}| \leq 1, |p_{i,j}| \leq 1\}$,
- $\mathcal{L} : \mathbb{R}^{(m-1)\times n} \times \mathbb{R}^{m\times(n-1)} \to \mathbb{R}^{m\times n}$ such that

$$\mathcal{L}(p,q)_{i,j} = p_{i,j} + q_{i,j} - p_{i-1,j} - q_{i,j-1}$$

  for $i = 1, \ldots, m$, $j = 1, \ldots, n$, and
  $p_{0,j} = p_{m,j} = q_{i,0} = q_{i,n} = 0$.
- P_C is the usual projection operator onto the set C

Total Variation

$$\mathrm{TV}(x) = max_{p,q\in\mathcal{P}} T(x,p,q) \implies T(x,p,q) = \mathrm{Tr}(\mathcal{L}(p,q)^T x).$$

Dual Form of TV Denoising with $\|\cdot\|_F^2$

The problem:

$$\min_{x \in C} \|x - b\|_F^2 + 2\lambda \mathrm{TV}(x), C = [0, 1]$$

Dual problem:

$$\min_{(p,q) \in \mathcal{P}} \underbrace{-\|H_C(b - \lambda \mathcal{L}(p, q))\|_F^2 + \|b - \lambda \mathcal{L}(p, q)\|_F^2}_{h(p,q)}$$

$$H_C(x) = \underbrace{x - P_C(x)}_{\text{prox}}$$

Optimality conditions:

$$x = P_C(b - \lambda \mathcal{L}(p, q)).$$

Optimization of Dual Form

Problem Statement

$$\min_{(p,q)\in\mathcal{P}} \left\{ \|b - \lambda\mathcal{L}(p,q)\|_F^2 - \|(I - P_{[0,1]})(b - \lambda\mathcal{L}(p,q))\|_F^2 \right\}$$

$$\min_{(p,q)} \left\{ \underbrace{\|b - \lambda\mathcal{L}(p,q)\|_F^2 - \|(I - P_{[0,1]})(b - \lambda\mathcal{L}(p,q))\|_F^2}_{=h(p,q)} + \delta((p,q)|\mathcal{P}) \right\}$$

$$\nabla h(p,q) = -2\lambda\mathcal{L}^T P_{[0,1]}(b - \lambda\mathcal{L}(p,q))$$

Lipschitz with constant $\leq 16\lambda^2$

$\Rightarrow$ Use projected gradient

Optimization in Dual Form

Projection onto $\mathcal{P}$

Recal $\mathcal{P} = (p, q) \in [-1, 1]^{m-1 \times n} \times [-1, 1]^{m \times n-1}$

$$P_{\mathcal{P}}(p, q) = (r, s) \text{ with } \begin{cases} r_{ij} = sgn(p_{ij}) \min\{1, |p_{ij}|\} \\ s_{ij} = sgn(q_{ij}) \min\{1, |q_{ij}|\} \end{cases}$$

Projected Gradient Step

$$(p^{k+1}, q^{k+1}) = P_{\mathcal{P}} \left((p^k, q^k) + \frac{1}{8\lambda} \mathcal{L}^T P_{[0,1]}(b - \lambda \mathcal{L}(p, q)) \right)$$

Summary of Fast TV Regularization

MFISTA(b, f, λ)

$y^1 = x^0 = b;\ t^1 = 1$

$\alpha \geq Lip(\nabla f)$

for $k = 1 : N$ **do**

$$u^k = y^k - \frac{A^T \nabla f(Ay^k - b)}{\alpha}$$

$$z^k = FGP(u^k, \tfrac{\lambda}{2\alpha})$$

$$x^k = \operatorname*{argmin}_{x \in \{x^{k-1}, z^k\}} L_b(x)$$

$$t^{k+1} = \frac{1 + \sqrt{1 + 4{t^k}^2}}{2}$$

$$y^{k+1} = x^k + \frac{t^k}{t^{k+1}}(z^k - x^k) + \frac{t^k - 1}{t^{k+1}}(z^k - x^k)$$

end for

return x^N

FGP(b, λ)

$(r^1_{ij}, s^1_{ij}) = (p^0_{ij}, q^0_{ij}) = 0;\ t^1 = 1$

for $k = 1 : N$ **do**

$$(p^k, q^k) = P_{\mathcal{P}}\left((r^k, s^k) - \frac{\mathcal{L}^T P_{[0,1]}(b - \lambda \mathcal{L}(r^k, s^k))}{8\lambda}\right)$$

$$t^{k+1} = \frac{1 + \sqrt{1 + 4{t^k}^2}}{2}$$

$$(r^k, s^k) = (p^k, q^k) + \frac{t^k - 1}{t^{k+1}}(p^k - p^{k-1}, q^k - q^{k-1})$$

end for

return $P_{[0,1]}(b - \lambda \mathcal{L}(p^N, q^N))$

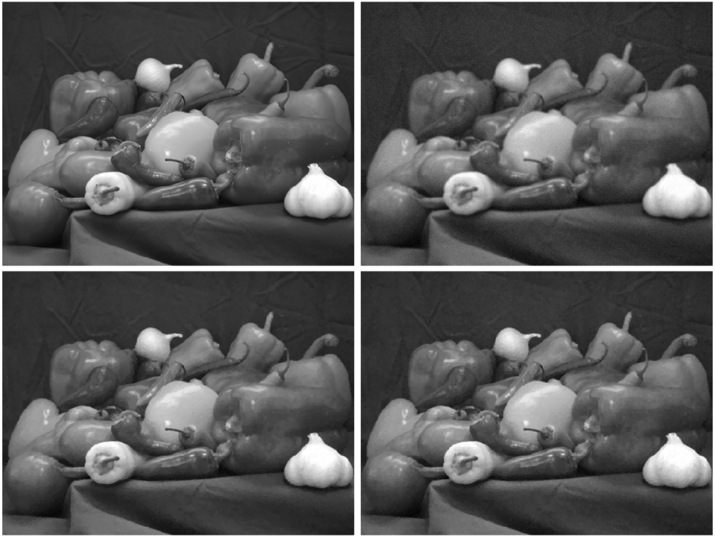
Results: Gaussian Noise

Original Image

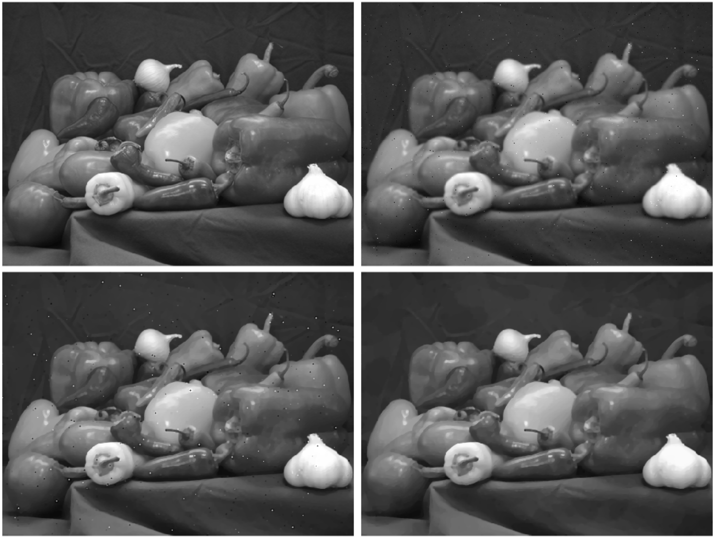
Blurred and Noisy

Frobenius Loss

Huber Loss



Results: Student's-t Noise



Original Image

Blurred and Noisy

Frobenius Loss

Huber Loss

Questions?

Codes used to generate figures
https://github.com/snagcliffs/Amath575project

Guckenheimer, J., Holmes, P. *Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields*. Springer-Verlag, 1983. Print.

Oliveira, D., Leonel, E. (2008) *Braz. J. Phys.* 38(1):62-64

Grassberger, P., Procaccia, I. (1983) *Phys. Rev. Letters.* 50(5):346-349