

# Image Denoising and Deblurring

## Applied Math 515 Final Project

Samuel Rudy, Kelsey Maass, Riley Molloy, and Kevin Mueller

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# Image Denoising and Deblurring



# Mathematical Formulation

$$Ax + w = b$$

True image

Observed image

Blur operator

Noise

# Naive Solution

$$x = A^{-1}b$$



True image



Blurred image



Recovered image

# Naive Solution

$$x = A^{-1}(b-w)$$



True image



Blurred and noisy image



Recovered image

Since the naive solution is contaminated by round-off error and noise, a better approach is to solve the problem:

$$\min_x \underbrace{f(Ax - b)}_{\text{Fidelity term}} + \underbrace{\lambda R(x)}_{\text{Regularization}}$$

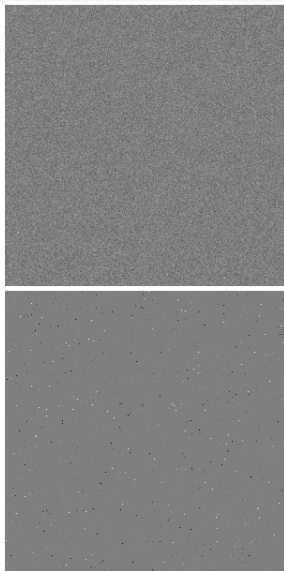
### Fidelity Term

$$f = \begin{cases} \|\cdot\|_F^2 \\ h_\gamma(\cdot) \\ \gamma^{-1} \log(\cosh(\gamma \cdot)) \end{cases}$$

### Regularization Term

$$R = \begin{cases} TV(x) \\ \|Wx\|_1 \end{cases}$$

# Fidelity Term Penalty Functions



What is  $Ax - b$  Why use different functions than frobenius norm? Use pictures as motivation. Show quadratic and huber penalty functions, show different distributions?



# Image Assumptions and Regularizers

Either assume image is sparse in wavelet domain, or assume image is smooth

Talk about choice of  $g$  Haar, FFT What is TV?

Show two different definitions of TV from paper.

# Kelsey's stuff here

- Loss function
- choice of  $\lambda$
- choice of wavelet
- prox gradient method

# Results: Gaussian Noise

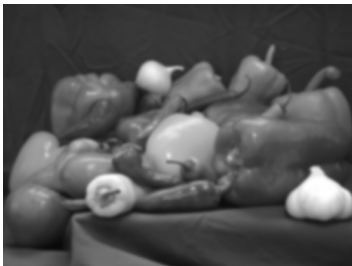
Original Image



Frobenius Loss



Blurred and Noisy



Huber Loss

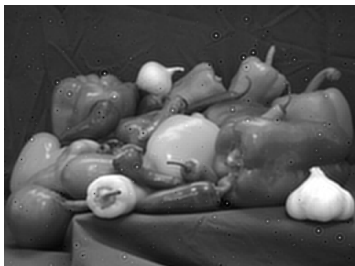


# Results: Student's $t$ Noise

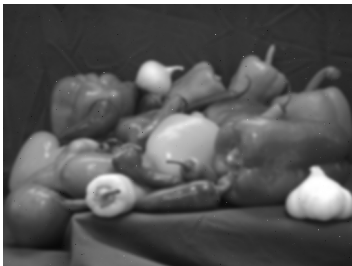
Original Image



Frobenius Loss



Blurred and Noisy



Huber Loss



# Total Variation Regularization

## Loss Function

$$L_b(x) = f(Ax - b) + \lambda \|x\|_{TV} + \delta(x|_{[0,1]})$$

## Proximal Gradient Step

$$\begin{aligned} x^{k+1} &= \text{prox}_{\alpha^{-1}(\lambda \|\cdot\|_{TV} + \delta_{[0,1]})} \left( \underbrace{x^k - \alpha^{-1} A^T \nabla f(Ax^k - b)}_{u^k} \right) \\ &= \arg \min_z \left( \|u^k - z\|_F^2 + \alpha^{-1} \lambda \|z\|_{TV} + \delta(z|_{[0,1]}) \right) \\ &= P_{[0,1]} \left( \arg \min_z \left( \|u^k - z\|_F^2 + \alpha^{-1} \lambda \|z\|_{TV} \right) \right) \end{aligned}$$

# Dual Form of Total Variation

A Few Definitions

weee

Total Variation

blarg

# Dual Form of TV Denoising with $\|\cdot\|_F^2$

# Optimization of Dual Form

## Problem Statement

$$\min_{(p,q) \in \mathcal{P}} \left\{ \|b - \lambda \mathcal{L}(p, q)\|_F^2 - \|(I - P_{[0,1]})(b - \lambda \mathcal{L}(p, q))\|_F^2 \right\}$$

$$\min_{(p,q)} \left\{ \underbrace{\|b - \lambda \mathcal{L}(p, q)\|_F^2 - \|(I - P_{[0,1]})(b - \lambda \mathcal{L}(p, q))\|_F^2}_{=h(p,q)} + \delta((p, q) | \mathcal{P}) \right\}$$

$$\nabla h(p, q) = -2\lambda \mathcal{L}^T P_{[0,1]}(b - \lambda \mathcal{L}(p, q))$$

Lipschitz with constant  $\leq 16\lambda^2$

$\Rightarrow$  Use projected gradient



# Optimization in Dual Form

## Projection onto $\mathcal{P}$

Recall  $\mathcal{P} = (p, q) \in [-1, 1]^{m-1 \times n} \times [-1, 1]^{m \times n-1}$

$$P_{\mathcal{P}}(p, q) = (r, s) \text{ with } \begin{cases} r_{ij} = \text{sgn}(p_{ij}) \min\{1, |p_{ij}|\} \\ s_{ij} = \text{sgn}(q_{ij}) \min\{1, |q_{ij}|\} \end{cases}$$

## Projected Gradient Step

$$(p^{k+1}, q^{k+1}) = P_{\mathcal{P}} \left( (p^k, q^k) + \frac{1}{8\lambda} \mathcal{L}^T P_{[0,1]}(b - \lambda \mathcal{L}(p, q)) \right)$$

# Summary of Fast TV Regularization

**MFISTA**( $b, f, \lambda$ )

$$y^1 = x^0 = b; t^1 = 1$$

$$\alpha \geq \text{Lip}(\nabla f)$$

**for**  $k = 1 : N$  **do**

$$u^k = y^k - \frac{A^T \nabla f(Ay^k - b)}{\alpha}$$

$$z^k = \text{FGP}(u^k, \frac{\lambda}{2\alpha})$$

$$x^k = \underset{x \in \{x^{k-1}, z^k\}}{\text{argmin}} L_b(x)$$

$$t^{k+1} = \frac{1 + \sqrt{1 + 4t^{k2}}}{2}$$

$$y^{k+1} = x^k + \frac{t^k}{t^{k+1}}(z^k - x^k) + \frac{t^{k-1}}{t^{k+1}}(z^k - x^k)$$

**end for**

**return**  $x^N$

**FGP**( $b, \lambda$ )

$$(r_{ij}^1, s_{ij}^1) = (p_{ij}^0, q_{ij}^0) = 0; t^1 = 1$$

**for**  $k = 1 : N$  **do**

$$(p^k, q^k) = P_{\mathcal{P}} \left( (r^k, s^k) - \frac{\mathcal{L}^T P_{[0,1]}(b - \lambda \mathcal{L}(r^k, s^k))}{8\lambda} \right)$$

$$t^{k+1} = \frac{1 + \sqrt{1 + 4t^{k2}}}{2}$$

$$(r^k, s^k) = (p^k, q^k) + \frac{t^k - 1}{t^{k+1}}(p^k - p^{k-1}, q^k - q^{k-1})$$

**end for**

**return**  $P_{[0,1]}(b - \lambda \mathcal{L}(p^N, q^N))$

# Results: Gaussian Noise

Original Image



Frobenius Loss



Blurred and Noisy

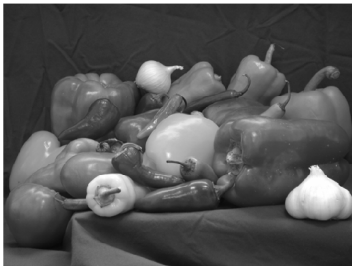


Huber Loss

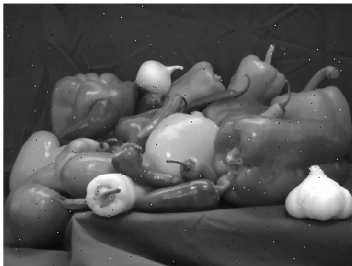


# Results: Student's-t Noise

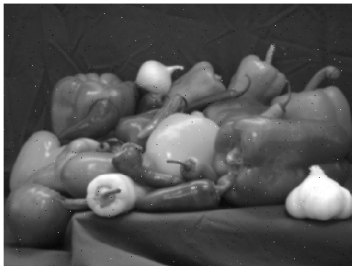
Original Image



Frobenius Loss



Blurred and Noisy



Huber Loss



# Questions?



Codes used to generate figures

<https://github.com/snagcliffs/Amath575project>



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