Chapter 4

ELECTROSTATIC FIELDS

Take risks: if you win, you will be happy; if you lose you will be wise.

—PETER KREEFT

4.1 INTRODUCTION

Having mastered some essential mathematical tools needed for this course, we are now prepared to study the basic concepts of EM. We shall begin with those fundamental concepts that are applicable to static (or time-invariant) electric fields in free space (or vacuum). An electrostatic field is produced by a static charge distribution. A typical example of such a field is found in a cathode-ray tube.

Before we commence our study of electrostatics, it might be helpful to examine briefly the importance of such a study. Electrostatics is a fascinating subject that has grown up in diverse areas of application. Electric power transmission, X-ray machines, and lightning protection are associated with strong electric fields and will require a knowledge of electrostatics to understand and design suitable equipment. The devices used in solid-state electronics are based on electrostatics. These include resistors, capacitors, and active devices such as bipolar and field effect transistors, which are based on control of electron motion by electrostatic fields. Almost all computer peripheral devices, with the exception of magnetic memory, are based on electrostatic fields. Touch pads, capacitance keyboards, cathode-ray tubes, liquid crystal displays, and electrostatic printers are typical examples. In medical work, diagnosis is often carried out with the aid of electrostatics, as incorporated in electrocardiograms, electroencephalograms, and other recordings of organs with electrical activity including eyes, ears, and stomachs. In industry, electrostatics is applied in a variety of forms such as paint spraying, electrodeposition, electrochemical machining, and separation of fine particles. Electrostatics is used in agriculture to sort seeds, direct sprays to plants, measure the moisture content of crops, spin cotton, and speed baking of bread and smoking of meat.[1,2]

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[1]For various applications of electrostatics, see J. M. Crowley, *Fundamentals of Applied Electrostatics.* New York: John Wiley & Sons, 1986; A. D. Moore, ed., *Electrostatics and Its Applications.* New York: John Wiley & Sons, 1973; and C. E. Jowett, *Electrostatics in the Electronics Environment.* New York: John Wiley & Sons, 1976.

[2]An interesting story on the magic of electrostatics is found in B. Bolton, *Electromagnetism and Its Applications.* London: Van Nostrand, 1980, p. 2.

We begin our study of electrostatics by investigating the two fundamental laws governing electrostatic fields: (1) Coulomb's law, and (2) Gauss's law. Both of these laws are based on experimental studies and they are interdependent. Although Coulomb's law is applicable in finding the electric field due to any charge configuration, it is easier to use Gauss's law when charge distribution is symmetrical. Based on Coulomb's law, the concept of electric field intensity will be introduced and applied to cases involving point, line, surface, and volume charges. Special problems that can be solved with much effort using Coulomb's law will be solved with ease by applying Gauss's law. Throughout our discussion in this chapter, we will assume that the electric field is in a vacuum or free space. Electric field in material space will be covered in the next chapter.

4.2 COULOMB'S LAW AND FIELD INTENSITY

Coulomb's law is an experimental law formulated in 1785 by the French colonel, Charles Augustin de Coulomb. It deals with the force a point charge exerts on another point charge. By a *point charge* we mean a charge that is located on a body whose dimensions are much smaller than other relevant dimensions. For example, a collection of electric charges on a pinhead may be regarded as a point charge. Charges are generally measured in coulombs (C). One coulomb is approximately equivalent to 6×10^{18} electrons; it is a very large unit of charge because one electron charge $e = -1.6019 \times 10^{-19}$ C.

Coulomb's law states that the force F between two point charges Q_1 and Q_2 is:

1. Along the line joining them
2. Directly proportional to the product Q_1Q_2 of the charges
3. Inversely proportional to the square of the distance R between them.[3]

Expressed mathematically,

$$F = \frac{k\, Q_1Q_2}{R^2} \tag{4.1}$$

where k is the proportionality constant. In SI units, charges Q_1 and Q_2 are in coulombs (C), the distance R is in meters (m), and the force F is in newtons (N) so that $k = 1/4\pi\varepsilon_o$. The constant ε_o is known as the *permittivity of free space* (in farads per meter) and has the value

$$\boxed{\begin{aligned} \varepsilon_o &= 8.854 \times 10^{-12} \simeq \frac{10^{-9}}{36\pi} \text{ F/m} \\ \text{or } k &= \frac{1}{4\pi\varepsilon_o} \simeq 9 \times 10^9 \text{ m/F} \end{aligned}} \tag{4.2}$$

[3]Further details of experimental verification of Coulomb's law can be found in W. F. Magie, *A Source Book in Physics.* Cambridge: Harvard Univ. Press, 1963, pp. 408–420.

Thus eq. (4.1) becomes

$$F = \frac{Q_1 Q_2}{4\pi\varepsilon_o R^2} \tag{4.3}$$

If point charges Q_1 and Q_2 are located at points having position vectors $\mathbf{r}_1$ and $\mathbf{r}_2$, then the force $\mathbf{F}_{12}$ on Q_2 due to Q_1, shown in Figure 4.1, is given by

$$\boxed{\mathbf{F}_{12} = \frac{Q_1 Q_2}{4\pi\varepsilon_o R^2}\,\mathbf{a}_{R_{12}}} \tag{4.4}$$

where

$$\mathbf{R}_{12} = \mathbf{r}_2 - \mathbf{r}_1 \tag{4.5a}$$

$$R = |\mathbf{R}_{12}| \tag{4.5b}$$

$$\mathbf{a}_{R_{12}} = \frac{\mathbf{R}_{12}}{R} \tag{4.5c}$$

By substituting eq. (4.5) into eq. (4.4), we may write eq. (4.4) as

$$\mathbf{F}_{12} = \frac{Q_1 Q_2}{4\pi\varepsilon_o R^3}\,\mathbf{R}_{12} \tag{4.6a}$$

or

$$\mathbf{F}_{12} = \frac{Q_1 Q_2\,(\mathbf{r}_2 - \mathbf{r}_1)}{4\pi\varepsilon_o|\mathbf{r}_2 - \mathbf{r}_1|^3} \tag{4.6b}$$

It is worthwhile to note that

1. As shown in Figure 4.1, the force $\mathbf{F}_{21}$ on Q_1 due to Q_2 is given by

$$\mathbf{F}_{21} = |\mathbf{F}_{12}|\mathbf{a}_{R_{21}} = |\mathbf{F}_{12}|(-\mathbf{a}_{R_{12}})$$

or

$$\mathbf{F}_{21} = -\mathbf{F}_{12} \tag{4.7}$$

since

$$\mathbf{a}_{R_{21}} = -\mathbf{a}_{R_{12}}$$

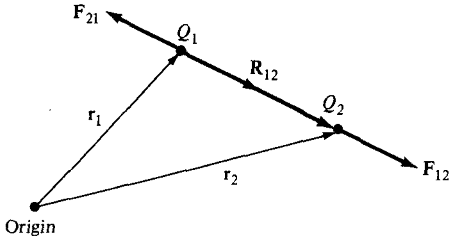

Figure 4.1 Coulomb vector force on point changes Q_1 and Q_2.

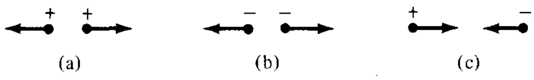

Figure 4.2 **(a)**, **(b)** Like charges repel; **(c)** unlike charges attract.

2. Like charges (charges of the same sign) repel each other while unlike charges attract. This is illustrated in Figure 4.2.
3. The distance R between the charged bodies Q_1 and Q_2 must be large compared with the linear dimensions of the bodies; that is, Q_1 and Q_2 must be point charges.
4. Q_1 and Q_2 must be static (at rest).
5. The signs of Q_1 and Q_2 must be taken into account in eq. (4.4).

If we have more than two point charges, we can use the *principle of superposition* to determine the force on a particular charge. The principle states that if there are N charges $Q_1, Q_2, \ldots, Q_N$ located, respectively, at points with position vectors $\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_N$, the resultant force $\mathbf{F}$ on a charge Q located at point $\mathbf{r}$ is the vector sum of the forces exerted on Q by each of the charges $Q_1, Q_2, \ldots, Q_N$. Hence:

$$\mathbf{F} = \frac{QQ_1(\mathbf{r} - \mathbf{r}_1)}{4\pi\varepsilon_o|\mathbf{r} - \mathbf{r}_1|^3} + \frac{QQ_2(\mathbf{r} - \mathbf{r}_2)}{4\pi\varepsilon_o|\mathbf{r} - \mathbf{r}_2|^3} + \cdots + \frac{QQ_N(\mathbf{r} - \mathbf{r}_n)}{4\pi\varepsilon_o|\mathbf{r} - \mathbf{r}_N|^3}$$

or

$$\boxed{\mathbf{F} = \frac{Q}{4\pi\varepsilon_o}\sum_{k=1}^{N}\frac{Q_k(\mathbf{r} - \mathbf{r}_k)}{|\mathbf{r} - \mathbf{r}_k|^3}} \tag{4.8}$$

We can now introduce the concept of *electric field intensity.*

The electric field intensity (or **electric field strength**) **E** is the force per unit charge when placed in the electric field.

Thus

$$\mathbf{E} = \lim_{Q \to 0} \frac{\mathbf{F}}{Q} \tag{4.9}$$

or simply

$$\boxed{\mathbf{E} = \frac{\mathbf{F}}{Q}} \tag{4.10}$$

The electric field intensity $\mathbf{E}$ is obviously in the direction of the force $\mathbf{F}$ and is measured in newtons/coulomb or volts/meter. The electric field intensity at point $\mathbf{r}$ due to a point charge located at $\mathbf{r}'$ is readily obtained from eqs. (4.6) and (4.10) as

$$\boxed{\mathbf{E} = \frac{Q}{4\pi\varepsilon_o R^2}\mathbf{a}_R = \frac{Q(\mathbf{r} - \mathbf{r}')}{4\pi\varepsilon_o|\mathbf{r} - \mathbf{r}'|^3}} \tag{4.11}$$

For N point charges $Q_1, Q_2, \ldots, Q_N$ located at $\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_N$, the electric field intensity at point $\mathbf{r}$ is obtained from eqs. (4.8) and (4.10) as

$$\mathbf{E} = \frac{Q_1(\mathbf{r} - \mathbf{r}_1)}{4\pi\varepsilon_o|\mathbf{r} - \mathbf{r}_1|^3} + \frac{Q_2(\mathbf{r} - \mathbf{r})_2}{4\pi\varepsilon_o|\mathbf{r} - \mathbf{r}_2|^3} + \cdots + \frac{Q_N(\mathbf{r} - \mathbf{r}_N)}{4\pi\varepsilon_o|\mathbf{r} - \mathbf{r}_N|^3}$$

or

$$\mathbf{E} = \frac{1}{4\pi\varepsilon_o}\sum_{k=1}^{N}\frac{Q_k(\mathbf{r} - \mathbf{r}_k)}{|\mathbf{r} - \mathbf{r}_k|^3} \tag{4.12}$$

EXAMPLE 4.1

Point charges 1 mC and −2 mC are located at (3, 2, −1) and (−1, −1, 4), respectively. Calculate the electric force on a 10-nC charge located at (0, 3, 1) and the electric field intensity at that point.

Solution:

$$\begin{aligned}
\mathbf{F} &= \sum_{k=1,2}\frac{QQ_k}{4\pi\varepsilon_o R^2}\mathbf{a}_R = \sum_{k=1,2}\frac{QQ_k(\mathbf{r} - \mathbf{r}_k)}{4\pi\varepsilon_o|\mathbf{r} - \mathbf{r}_k|^3} \\
&= \frac{Q}{4\pi\varepsilon_o}\left\{\frac{10^{-3}[(0,3,1) - (3,2,-1)]}{|(0,3,1) - (3,2,-1)|^3} - \frac{2.10^{-3}[(0,3,1) - (-1,-1,4)]}{|(0,3,1) - (-1,-1,4)|^3}\right\} \\
&= \frac{10^{-3}\cdot 10\cdot 10^{-9}}{4\pi\cdot\dfrac{10^{-9}}{36\pi}}\left[\frac{(-3,1,2)}{(9+1+4)^{3/2}} - \frac{2(1,4,-3)}{(1+16+9)^{3/2}}\right] \\
&= 9\cdot 10^{-2}\left[\frac{(-3,1,2)}{14\sqrt{14}} + \frac{(-2,-8,6)}{26\sqrt{26}}\right] \\
\mathbf{F} &= -6.507\mathbf{a}_x - 3.817\mathbf{a}_y + 7.506\mathbf{a}_z \text{ mN}
\end{aligned}$$

At that point,

$$\begin{aligned}
\mathbf{E} &= \frac{\mathbf{F}}{Q} \\
&= (-6.507, -3.817, 7.506)\cdot\frac{10^{-3}}{10\cdot 10^{-9}} \\
\mathbf{E} &= -650.7\mathbf{a}_x - 381.7\mathbf{a}_y + 750.6\mathbf{a}_z \text{ kV/m}
\end{aligned}$$

PRACTICE EXERCISE 4.1

Point charges 5 nC and −2 nC are located at (2, 0, 4) and (−3, 0, 5), respectively.

(a) Determine the force on a 1-nC point charge located at (1, −3, 7).

(b) Find the electric field **E** at (1, −3, 7).

Answer: (a) $-1.004\mathbf{a}_x - 1.284\mathbf{a}_y + 1.4\mathbf{a}_z$ nN,
(b) $-1.004\mathbf{a}_x - 1.284\mathbf{a}_y + 1.4\mathbf{a}_z$ V/m.

EXAMPLE 4.2 Two point charges of equal mass m, charge Q are suspended at a common point by two threads of negligible mass and length ℓ. Show that at equilibrium the inclination angle α of each thread to the vertical is given by

$$Q^2 = 16\pi\,\varepsilon_o mg\ell^2 \sin^2\alpha \tan\alpha$$

If α is very small, show that

$$\alpha = \sqrt[3]{\frac{Q^2}{16\pi\varepsilon_o mg\ell^2}}$$

Solution:

Consider the system of charges as shown in Figure 4.3 where F_e is the electric or coulomb force, T is the tension in each thread, and mg is the weight of each charge. At A or B

$$T\sin\alpha = F_e$$

$$T\cos\alpha = mg$$

Hence,

$$\frac{\sin\alpha}{\cos\alpha} = \frac{F_e}{mg} = \frac{1}{mg}\cdot\frac{Q^2}{4\pi\varepsilon_o r^2}$$

But

$$r = 2\ell\sin\alpha$$

Hence,

$$Q^2\cos\alpha = 16\pi\varepsilon_o mg\ell^2\sin^3\alpha$$

or

$$Q^2 = 16\pi\varepsilon_o mg\ell^2\sin^2\alpha\tan\alpha$$

as required. When α is very small

$$\tan\alpha \simeq \alpha \simeq \sin\alpha$$

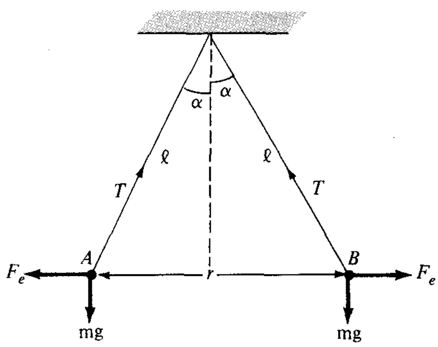

Figure 4.3 Suspended charged particles; for Example 4.2.

and so

$$Q^2 = 16\pi\varepsilon_o mg\ell^2\alpha^3$$

or

$$\alpha = \sqrt[3]{\frac{Q^2}{16\pi\varepsilon_o mg\ell^2}}$$

PRACTICE EXERCISE 4.2

Three identical small spheres of mass m are suspended by threads of negligible masses and equal length ℓ from a common point. A charge Q is divided equally between the spheres and they come to equilibrium at the corners of a horizontal equilateral triangle whose sides are d. Show that

$$Q^2 = 12\pi\varepsilon_o mgd^3\left[\ell^2 - \frac{d^2}{3}\right]^{-1/2}$$

where g = acceleration due to gravity.

Answer: Proof.

EXAMPLE 4.3

A practical application of electrostatics is in electrostatic separation of solids. For example, Florida phosphate ore, consisting of small particles of quartz and phosphate rock, can be separated into its components by applying a uniform electric field as in Figure 4.4. Assuming zero initial velocity and displacement, determine the separation between the particles after falling 80 cm. Take E = 500 kV/m and Q/m = 9 μC/kg for both positively and negatively charged particles.

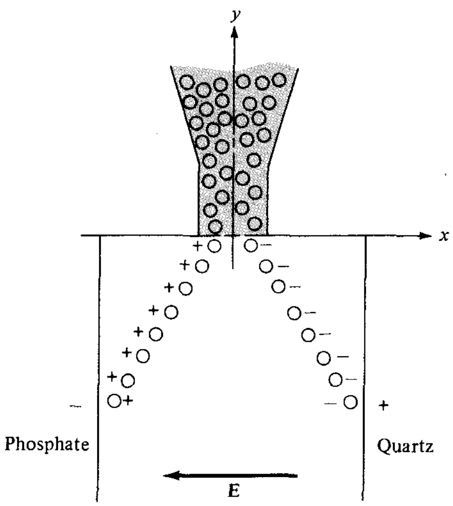

Figure 4.4 Electrostatic separation of solids; for Example 4.3.

Solution:

Ignoring the coulombic force between particles, the electrostatic force is acting horizontally while the gravitational force (weight) is acting vertically on the particles. Thus,

$$Q\mathbf{E} = m\frac{d^2x}{dt^2}\mathbf{a}_x$$

or

$$\frac{d^2x}{dt^2} = \frac{Q}{m}E$$

Integrating twice gives

$$x = \frac{Q}{2m}Et^2 + c_1t + c_2$$

where c_1 and c_2 are integration constants. Similarly,

$$-mg = m\frac{d^2y}{dt^2}$$

or

$$\frac{d^2y}{dt^2} = -g$$

Integrating twice, we get

$$y = -1/2gt^2 + c_3t + c_4$$

Since the initial displacement is zero,

$$x(t=0) = 0 \rightarrow c_2 = 0$$
$$y(t=0) = 0 \rightarrow c_4 = 0$$

Also, due to zero initial velocity,

$$\left.\frac{dx}{dt}\right|_{t=0} = 0 \rightarrow c_1 = 0$$
$$\left.\frac{dy}{dt}\right|_{t=0} = 0 \rightarrow c_3 = 0$$

Thus

$$x = \frac{QE}{2m}t^2 \qquad y = -\frac{1}{2}gt^2$$

When $y = -80 \text{ cm} = -0.8 \text{ m}$

$$t^2 = \frac{0.8 \times 2}{9.8} = 0.1633$$

and

$$x = 1/2 \times 9 \times 10^{-6} \times 5 \times 10^5 \times 0.1633 = 0.3673 \text{ m}$$

The separation between the particles is $2x = 73.47$ cm.

PRACTICE EXERCISE 4.3

An ion rocket emits positive cesium ions from a wedge-shape electrode into the region described by $x > |y|$. The electric field is $\mathbf{E} = -400\mathbf{a}_x + 200\mathbf{a}_y$ kV/m. The ions have single electronic charges $e = -1.6019 \times 10^{-19}$ C and mass $m = 2.22 \times 10^{-25}$ kg and travel in a vacuum with zero initial velocity. If the emission is confined to $-40 \text{ cm} < y < 40$ cm, find the largest value of x which can be reached.

Answer: 0.8 m.

4.3 ELECTRIC FIELDS DUE TO CONTINUOUS CHARGE DISTRIBUTIONS

So far we have only considered forces and electric fields due to point charges, which are essentially charges occupying very small physical space. It is also possible to have continuous charge distribution along a line, on a surface, or in a volume as illustrated in Figure 4.5.

It is customary to denote the line charge density, surface charge density, and volume charge density by ρ_L (in C/m), ρ_S (in C/m^2), and ρ_v (in C/m^3), respectively. These must not be confused with ρ (without subscript) used for radial distance in cylindrical coordinates.

The charge element dQ and the total charge Q due to these charge distributions are obtained from Figure 4.5 as

$$dQ = \rho_L \, dl \rightarrow Q = \int_L \rho_L \, dl \qquad \text{(line charge)} \qquad (4.13a)$$

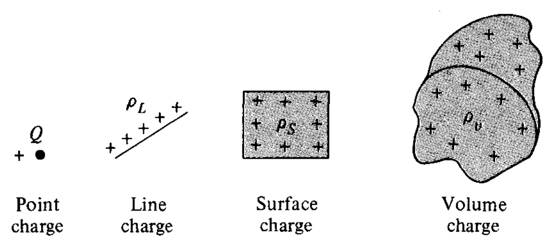

Figure 4.5 Various charge distributions and charge elements.

$$dQ = \rho_S \, dS \rightarrow Q = \int_S \rho_S \, dS \qquad \text{(surface charge)} \tag{4.13b}$$

$$dQ = \rho_v \, dv \rightarrow Q = \int_v \rho_v \, dv \qquad \text{(volume charge)} \tag{4.13c}$$

The electric field intensity due to each of the charge distributions ρ_L, ρ_S, and ρ_v may be regarded as the summation of the field contributed by the numerous point charges making up the charge distribution. Thus by replacing Q in eq. (4.11) with charge element $dQ = \rho L$ dl, $\rho_S \, dS$, or $\rho_v \, dv$ and integrating, we get

$$\mathbf{E} = \int \frac{\rho_L \, dl}{4\pi\varepsilon_o R^2} \mathbf{a}_R \qquad \text{(line charge)} \tag{4.14}$$

$$\mathbf{E} = \int \frac{\rho_S \, dS}{4\pi\varepsilon_o R^2} \mathbf{a}_R \qquad \text{(surface charge)} \tag{4.15}$$

$$\mathbf{E} = \int \frac{\rho_v \, dv}{4\pi\varepsilon_o R^2} \mathbf{a}_R \qquad \text{(volume charge)} \tag{4.16}$$

It should be noted that R^2 and $\mathbf{a}_R$ vary as the integrals in eqs. (4.13) to (4.16) are evaluated. We shall now apply these formulas to some specific charge distributions.

A. A Line Charge

Consider a line charge with uniform charge density ρ_L extending from A to B along the z-axis as shown in Figure 4.6. The charge element dQ associated with element $dl = dz$ of the line is

$$dQ = \rho_L \, dl = \rho_L \, dz$$

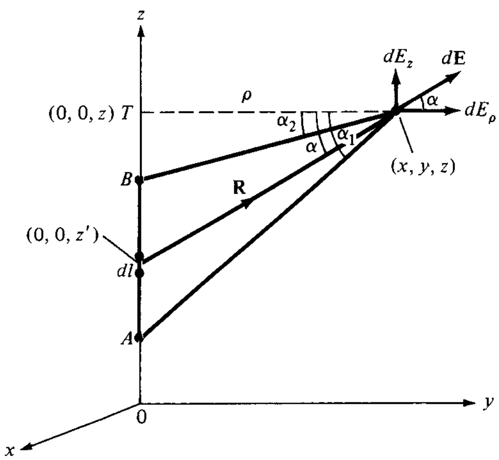

Figure 4.6 Evaluation of the **E** field due to a line charge.

and hence the total charge Q is

$$Q = \int_{z_A}^{z_B} \rho_L \, dz \tag{4.17}$$

The electric field intensity $\mathbf{E}$ at an arbitrary point $P(x, y, z)$ can be found using eq. (4.14). It is important that we learn to derive and substitute each term in eqs. (4.14) to (4.15) for a given charge distribution. It is customary to denote the field point[4] by (x, y, z) and the source point by (x', y', z'). Thus from Figure 4.6,

$$dl = dz'$$

$$\mathbf{R} = (x, y, z) - (0, 0, z') = x\mathbf{a}_x + y\mathbf{a}_y + (z - z')\mathbf{a}_z$$

or

$$\mathbf{R} = \rho\mathbf{a}_\rho + (z - z')\,\mathbf{a}_z$$

$$R^2 = |\mathbf{R}|^2 = x^2 + y^2 + (z - z')^2 = \rho^2 + (z - z')^2$$

$$\frac{\mathbf{a}_R}{R^2} = \frac{\mathbf{R}}{|\mathbf{R}|^3} = \frac{\rho\mathbf{a}_\rho + (z - z')\mathbf{a}_z}{\left[\rho^2 + (z - z')^2\right]^{3/2}}$$

Substituting all this into eq. (4.14), we get

$$\mathbf{E} = \frac{\rho_L}{4\pi\varepsilon_o} \int \frac{\rho\mathbf{a}_\rho + (z - z')\,\mathbf{a}_z}{\left[\rho^2 + (z - z')^2\right]^{3/2}}\, dz' \tag{4.18}$$

To evaluate this, it is convenient that we define α, α_1, and α_2 as in Figure 4.6.

$$R = \left[\rho^2 + (z - z')^2\right]^{1/2} = \rho \sec \alpha$$

$$z' = OT - \rho \tan \alpha, \qquad dz' = -\rho \sec^2 \alpha \, d\alpha$$

Hence, eq. (4.18) becomes

$$\begin{aligned} \mathbf{E} &= \frac{-\rho_L}{4\pi\varepsilon_o} \int_{\alpha_1}^{\alpha_2} \frac{\rho \sec^2 \alpha \,[\cos \alpha \, \mathbf{a}_\rho + \sin \alpha \, \mathbf{a}_z]\, d\alpha}{\rho^2 \sec^2 \alpha} \\ &= -\frac{\rho_L}{4\pi\varepsilon_o\rho} \int_{\alpha_1}^{\alpha_2} [\cos \alpha \, \mathbf{a}_\rho + \sin \alpha \, \mathbf{a}_z]\, d\alpha \end{aligned} \tag{4.19}$$

Thus for a *finite line charge,*

$$\mathbf{E} = \frac{\rho_L}{4\pi\varepsilon_o\rho}\left[-(\sin \alpha_2 - \sin \alpha_1)\mathbf{a}_\rho + (\cos \alpha_2 - \cos \alpha_1)\mathbf{a}_z\right] \tag{4.20}$$

[4] The field point is the point at which the field is to be evaluated.

As a special case, for an *infinite line charge,* point B is at $(0, 0, \infty)$ and A at $(0, 0, -\infty)$ so that $\alpha_1 = \pi/2$, $\alpha_2 = -\pi/2$; the z-component vanishes and eq. (4.20) becomes

$$\boxed{\mathbf{E} = \frac{\rho_L}{2\pi\varepsilon_o\rho}\,\mathbf{a}_\rho} \tag{4.21}$$

Bear in mind that eq. (4.21) is obtained for an infite line charge along the z-axis so that ρ and $\mathbf{a}_\rho$ have their usual meaning. If the line is not along the z-axis, ρ is the perpendicular distance from the line to the point of interest and $\mathbf{a}_\rho$ is a unit vector along that distance directed from the line charge to the field point.

B. A Surface Charge

Consider an infinite sheet of charge in the xy-plane with uniform charge density ρ_S. The charge associated with an elemental area dS is

$$dQ = \rho_S\, dS$$

and hence the total charge is

$$Q = \int \rho_S\, dS \tag{4.22}$$

From eq. (4.15), the contribution to the $\mathbf{E}$ field at point $P(0, 0, h)$ by the elemental surface 1 shown in Figure 4.7 is

$$d\mathbf{E} = \frac{dQ}{4\pi\varepsilon_o R^2}\,\mathbf{a}_R \tag{4.23}$$

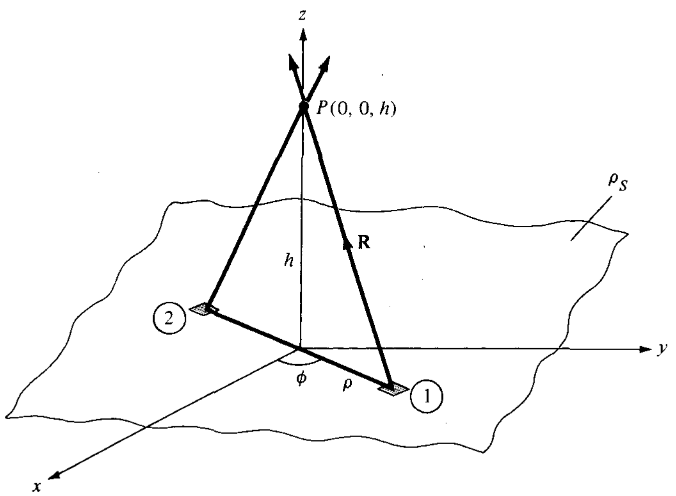

Figure 4.7 Evaluation of the **E** field due to an infinite sheet of charge.

From Figure 4.7,

$$\mathbf{R} = \rho(-\mathbf{a}_\rho) + h\mathbf{a}_z, \qquad R = |\mathbf{R}| = \left[\rho^2 + h^2\right]^{1/2}$$

$$\mathbf{a}_R = \frac{\mathbf{R}}{R}, \qquad dQ = \rho_S\, dS = \rho_S\, \rho\, d\phi\, d\rho$$

Substitution of these terms into eq. (4.23) gives

$$d\mathbf{E} = \frac{\rho_S\, \rho\, d\phi\, d\rho\left[-\rho\mathbf{a}_\rho + h\mathbf{a}_z\right]}{4\pi\varepsilon_o\left[\rho^2 + h^2\right]^{3/2}} \tag{4.24}$$

Due to the symmetry of the charge distribution, for every element 1, there is a corresponding element 2 whose contribution along $\mathbf{a}_\rho$ cancels that of element 1, as illustrated in Figure 4.7. Thus the contributions to E_ρ add up to zero so that $\mathbf{E}$ has only z-component. This can also be shown mathematically by replacing $\mathbf{a}_\rho$ with $\cos\phi\, \mathbf{a}_x + \sin\phi\, \mathbf{a}_y$. Integration of $\cos\phi$ or $\sin\phi$ over $0 < \phi < 2\pi$ gives zero. Therefore,

$$\begin{aligned}
\mathbf{E} &= \int d\mathbf{E}_z = \frac{\rho_S}{4\pi\varepsilon_o}\int_{\phi=0}^{2\pi}\int_{\rho=0}^{\infty}\frac{h\rho\, d\rho\, d\phi}{\left[\rho^2 + h^2\right]^{3/2}}\,\mathbf{a}_z \\
&= \frac{\rho_S h}{4\pi\varepsilon_o}\,2\pi\int_0^{\infty}\left[\rho^2 + h^2\right]^{-3/2}\frac{1}{2}\,d(\rho^2)\,\mathbf{a}_z \\
&= \frac{\rho_S h}{2\varepsilon_o}\left\{-\left[\rho^2 + h^2\right]^{-1/2}\right\}_0^{\infty}\mathbf{a}_z
\end{aligned}$$

$$\mathbf{E} = \frac{\rho_S}{2\varepsilon_o}\,\mathbf{a}_z \tag{4.25}$$

that is, $\mathbf{E}$ has only z-component if the charge is in the xy-plane. In general, for an *infinite sheet* of charge

$$\boxed{\mathbf{E} = \frac{\rho_S}{2\varepsilon_o}\,\mathbf{a}_n} \tag{4.26}$$

where $\mathbf{a}_n$ is a unit vector normal to the sheet. From eq. (4.25) or (4.26), we notice that the electric field is normal to the sheet and it is surprisingly independent of the distance between the sheet and the point of observation P. In a parallel plate capacitor, the electric field existing between the two plates having equal and opposite charges is given by

$$\mathbf{E} = \frac{\rho_S}{2\varepsilon_o}\,\mathbf{a}_n + \frac{-\rho_S}{2\varepsilon_o}(-\mathbf{a}_n) = \frac{\rho_S}{\varepsilon_o}\,\mathbf{a}_n \tag{4.27}$$

C. A Volume Charge

Let the volume charge distribution with uniform charge density ρ_v be as shown in Figure 4.8. The charge dQ associated with the elemental volume dv is

$$dQ = \rho_v\, dv$$

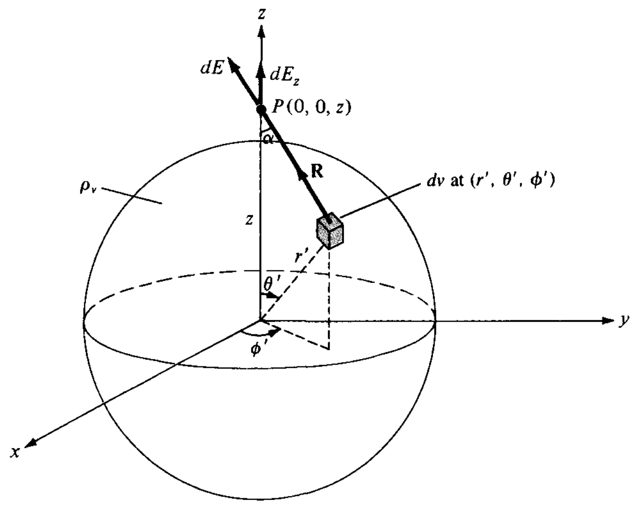

Figure 4.8 Evaluation of the **E** field due to a volume charge distribution.

and hence the total charge in a sphere of radius a is

$$Q = \int \rho_v \, dv = \rho_v \int dv \qquad (4.28)$$
$$= \rho_v \frac{4\pi a^3}{3}$$

The electric field $d\mathbf{E}$ at $P(0, 0, z)$ due to the elementary volume charge is

$$d\mathbf{E} = \frac{\rho_v \, dv}{4\pi\varepsilon_o R^2} \mathbf{a}_R$$

where $\mathbf{a}_R = \cos\alpha \, \mathbf{a}_z + \sin\alpha \, \mathbf{a}_\rho$. Due to the symmetry of the charge distribution, the contributions to E_x or E_y add up to zero. We are left with only E_z, given by

$$E_z = \mathbf{E} \cdot \mathbf{a}_z = \int dE \cos\alpha = \frac{\rho_v}{4\pi\varepsilon_o} \int \frac{dv \cos\alpha}{R^2} \qquad (4.29)$$

Again, we need to derive expressions for dv, R^2, and $\cos\alpha$.

$$dv = r'^2 \sin\theta' \, dr' \, d\theta' \, d\phi' \qquad (4.30)$$

Applying the cosine rule to Figure 4.8, we have

$$R^2 = z^2 + r'^2 - 2zr' \cos\theta'$$
$$r'^2 = z^2 + R^2 - 2zR \cos\alpha$$

It is convenient to evaluate the integral in eq. (4.29) in terms of R and r'. Hence we express $\cos\theta'$, $\cos\alpha$, and $\sin\theta'\,d\theta'$ in terms of R and r', that is,

$$\cos\alpha = \frac{z^2 + R^2 - r'^2}{2zR} \tag{4.31a}$$

$$\cos\theta' = \frac{z^2 + r'^2 - R^2}{2zr'} \tag{4.31b}$$

Differentiating eq. (4.31b) with respect to θ' keeping z and r' fixed, we obtain

$$\sin\theta'\,d\theta' = \frac{R\,dR}{z\,r'} \tag{4.32}$$

Substituting eqs. (4.30) to (4.32) into eq. (4.29) yields

$$\begin{aligned}
E_z &= \frac{\rho_v}{4\pi\varepsilon_o}\int_{\phi'=0}^{2\pi} d\phi' \int_{r'=0}^{a}\int_{R=z-r'}^{z+r'} r'^2\,\frac{R\,dR}{zr'}\,dr'\,\frac{z^2 + R^2 - r'^2}{2zR}\,\frac{1}{R^2} \\
&= \frac{\rho_v 2\pi}{8\pi\varepsilon_o z^2}\int_{r'=0}^{a}\int_{R=z-r'}^{z+r'} r'\left[1 + \frac{z^2 - r'^2}{R^2}\right] dR\,dr' \\
&= \frac{\rho_v\pi}{4\pi\varepsilon_o z^2}\int_0^a r'\left[R - \frac{(z^2 - r'^2)}{R}\right]_{z-r'}^{z+r'} dr' \\
&= \frac{\rho_v\pi}{4\pi\varepsilon_o z^2}\int_0^a 4r'^2\,dr' = \frac{1}{4\pi\varepsilon_o}\frac{1}{z^2}\left(\frac{4}{3}\pi a^3\rho_v\right)
\end{aligned}$$

or

$$\mathbf{E} = \frac{Q}{4\pi\varepsilon_o z^2}\,\mathbf{a}_z \tag{4.33}$$

This result is obtained for $\mathbf{E}$ at $P(0, 0, z)$. Due to the symmetry of the charge distribution, the electric field at $P(r, \theta, \phi)$ is readily obtained from eq. (4.33) as

$$\mathbf{E} = \frac{Q}{4\pi\varepsilon_o r^2}\,\mathbf{a}_r \tag{4.34}$$

which is identical to the electric field at the same point due to a point charge Q located at the origin or the center of the spherical charge distribution. The reason for this will become obvious as we cover Gauss's law in Section 4.5.

EXAMPLE 4.4

A circular ring of radius a carries a uniform charge ρ_L C/m and is placed on the xy-plane with axis the same as the z-axis.

(a) Show that

$$\mathbf{E}(0, 0, h) = \frac{\rho_L a h}{2\varepsilon_o[h^2 + a^2]^{3/2}}\,\mathbf{a}_z$$

(b) What values of h gives the maximum value of $\mathbf{E}$?

(c) If the total charge on the ring is Q, find $\mathbf{E}$ as $a \to 0$.

Solution:

(a) Consider the system as shown in Figure 4.9. Again the trick in finding $\mathbf{E}$ using eq. (4.14) is deriving each term in the equation. In this case,

$$dl = a\,d\phi, \qquad \mathbf{R} = a(-\mathbf{a}_\rho) + h\mathbf{a}_z$$

$$R = |\mathbf{R}| = \left[a^2 + h^2\right]^{1/2}, \qquad \mathbf{a}_R = \frac{\mathbf{R}}{R}$$

or

$$\frac{\mathbf{a}_R}{R^2} = \frac{\mathbf{R}}{|\mathbf{R}|^3} = \frac{-a\mathbf{a}_\rho + h\mathbf{a}_z}{\left[a^2 + h^2\right]^{3/2}}$$

Hence

$$\mathbf{E} = \frac{\rho_L}{4\pi\varepsilon_o} \int_{\phi=0}^{2\pi} \frac{(-a\mathbf{a}_\rho + h\mathbf{a}_z)}{\left[a^2 + h^2\right]^{3/2}}\, a\,d\phi$$

By symmetry, the contributions along $\mathbf{a}_\rho$ add up to zero. This is evident from the fact that for every element dl there is a corresponding element diametrically opposite it that gives an equal but opposite dE_ρ so that the two contributions cancel each other. Thus we are left with the z-component. That is,

$$\mathbf{E} = \frac{\rho_L a h \mathbf{a}_z}{4\pi\varepsilon_o\left[h^2 + a^2\right]^{3/2}} \int_0^{2\pi} d\phi = \frac{\rho_L a h \mathbf{a}_z}{2\varepsilon_o\left[h^2 + a^2\right]^{3/2}}$$

as required.

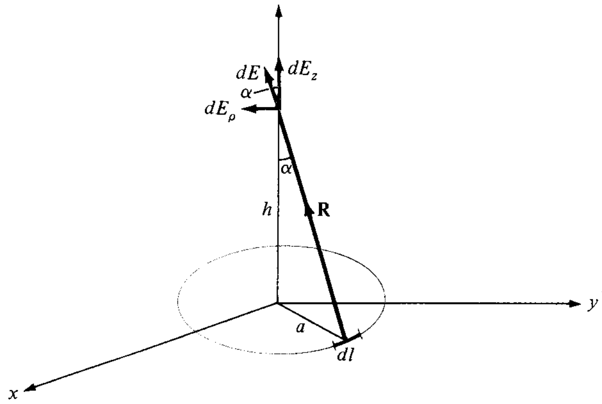

Figure 4.9 Charged ring; for Example 4.4.

(b)

$$\frac{d|\mathbf{E}|}{dh} = \frac{\rho_L a}{2\varepsilon_o}\left\{\frac{[h^2 + a^2]^{3/2}(1) - \frac{3}{2}(h)2h[h^2 + a^2]^{1/2}}{[h^2 + a^2]^3}\right\}$$

For maximum $\mathbf{E}$, $\dfrac{d|\mathbf{E}|}{dh} = 0$, which implies that

$$[h^2 + a^2]^{1/2}[h^2 + a^2 - 3h^2] = 0$$

$$a^2 - 2h^2 = 0 \qquad \text{or} \qquad h = \pm\frac{a}{\sqrt{2}}$$

(c) Since the charge is uniformly distributed, the line charge density is

$$\rho_L = \frac{Q}{2\pi a}$$

so that

$$\mathbf{E} = \frac{Qh}{4\pi\varepsilon_o[h^2 + a^2]^{3/2}}\,\mathbf{a}_z$$

As $a \to 0$

$$\mathbf{E} = \frac{Q}{4\pi\varepsilon_o h^2}\,\mathbf{a}_z$$

or in general

$$\mathbf{E} = \frac{Q}{4\pi\varepsilon_o r^2}\,\mathbf{a}_R$$

which is the same as that of a point charge as one would expect.

PRACTICE EXERCISE 4.4

A circular disk of radius a is uniformly charged with ρ_S C/m^2. If the disk lies on the $z = 0$ plane with its axis along the z-axis,

(a) Show that at point $(0, 0, h)$

$$\mathbf{E} = \frac{\rho_S}{2\varepsilon_o}\left\{1 - \frac{h}{[h^2 + a^2]^{1/2}}\right\}\mathbf{a}_z$$

(b) From this, derive the $\mathbf{E}$ field due to an infinite sheet of charge on the $z = 0$ plane.

(c) If $a \ll h$, show that $\mathbf{E}$ is similar to the field due to a point charge.

Answer: (a) Proof, (b) $\dfrac{\rho_S}{2\varepsilon_o}\,\mathbf{a}_z$, (c) Proof

EXAMPLE 4.5

The finite sheet $0 \leq x \leq 1,\ 0 \leq y \leq 1$ on the $z = 0$ plane has a charge density $\rho_S = xy(x^2 + y^2 + 25)^{3/2}$ nC/m². Find

(a) The total charge on the sheet

(b) The electric field at (0, 0, 5)

(c) The force experienced by a -1 mC charge located at (0, 0, 5)

Solution:

(a) $$Q = \int \rho_S \, dS = \int_0^1 \int_0^1 xy(x^2 + y^2 + 25)^{3/2} \, dx \, dy \text{ nC}$$

Since $x\,dx = 1/2\, d(x^2)$, we now integrate with respect to x^2 (or change variables: $x^2 = u$ so that $x\,dx = du/2$).

$$\begin{aligned}
Q &= \frac{1}{2} \int_0^1 y \int_0^1 (x^2 + y^2 + 25)^{3/2} \, d(x^2) \, dy \text{ nC} \\
&= \frac{1}{2} \int_0^1 y \frac{2}{5} (x^2 + y^2 + 25)^{5/2} \Big|_0^1 dy \\
&= \frac{1}{5} \int_0^1 \frac{1}{2} \left[(y^2 + 26)^{5/2} - (y^2 + 25)^{5/2} \right] d(y^2) \\
&= \frac{1}{10} \cdot \frac{2}{7} \left[(y^2 + 26)^{7/2} - (y^2 + 25)^{7/2} \right] \Big|_0^1 \\
&= \frac{1}{35} \left[(27)^{7/2} + (25)^{7/2} - 2(26)^{7/2} \right] \\
Q &= 33.15 \text{ nC}
\end{aligned}$$

(b) $$\mathbf{E} = \int \frac{\rho_S \, dS \, \mathbf{a}_R}{4\pi\varepsilon_o r^2} = \int \frac{\rho_S \, dS \, (\mathbf{r} - \mathbf{r}')}{4\pi\varepsilon_o |\mathbf{r} - \mathbf{r}'|^3}$$

where $\mathbf{r} - \mathbf{r}' = (0, 0, 5) - (x, y, 0) = (-x, -y, 5)$. Hence,

$$\begin{aligned}
\mathbf{E} &= \int_0^1 \int_0^1 \frac{10^{-9} xy(x^2 + y^2 + 25)^{3/2} (-x\mathbf{a}_x - y\mathbf{a}_y + 5\mathbf{a}_z) dx \, dy}{4\pi \cdot \dfrac{10^{-9}}{36\pi} (x^2 + y^2 + 25)^{3/2}} \\
&= 9 \left[-\int_0^1 x^2 \, dx \int_0^1 y \, dy \, \mathbf{a}_x - \int_0^1 x \, dx \int_0^1 y^2 dy \, \mathbf{a}_y + 5 \int_0^1 x \, dx \int_0^1 y \, dy \, \mathbf{a}_z \right] \\
&= 9 \left(\frac{-1}{6}, \frac{-1}{6}, \frac{5}{4} \right) \\
&= (-1.5, -1.5, 11.25) \text{ V/m}
\end{aligned}$$

(c) $\mathbf{F} = q\mathbf{E} = (1.5, 1.5, -11.25)$ mN

PRACTICE EXERCISE 4.5

A square plate described by $-2 \leq x \leq 2$, $-2 \leq y \leq 2$, $z = 0$ carries a charge $12\,|y|$ mC/m^2. Find the total charge on the plate and the electric field intensity at (0, 0, 10).

Answer: 192 mC, 16.46 $\mathbf{a}_z$ MV/m.

EXAMPLE 4.6

Planes $x = 2$ and $y = -3$, respectively, carry charges 10 nC/m^2 and 15 nC/m^2. If the line $x = 0$, $z = 2$ carries charge 10π nC/m, calculate $\mathbf{E}$ at $(1, 1, -1)$ due to the three charge distributions.

Solution:

Let

$$\mathbf{E} = \mathbf{E}_1 + \mathbf{E}_2 + \mathbf{E}_3$$

where $\mathbf{E}_1$, $\mathbf{E}_2$, and $\mathbf{E}_3$ are, respectively, the contributions to $\mathbf{E}$ at point $(1, 1, -1)$ due to the infinite sheet 1, infinite sheet 2, and infinite line 3 as shown in Figure 4.10(a). Applying eqs. (4.26) and (4.21) gives

$$\mathbf{E}_1 = \frac{\rho_{S_1}}{2\varepsilon_o}(-\mathbf{a}_x) = -\frac{10 \cdot 10^{-9}}{2 \cdot \dfrac{10^{-9}}{36\pi}}\mathbf{a}_x = -180\pi\mathbf{a}_x$$

$$\mathbf{E}_2 = \frac{\rho_{S_2}}{2\varepsilon_o}\mathbf{a}_y = \frac{15 \cdot 10^{-9}}{2 \cdot \dfrac{10^{-9}}{36\pi}}\mathbf{a}_y = 270\pi\,\mathbf{a}_y$$

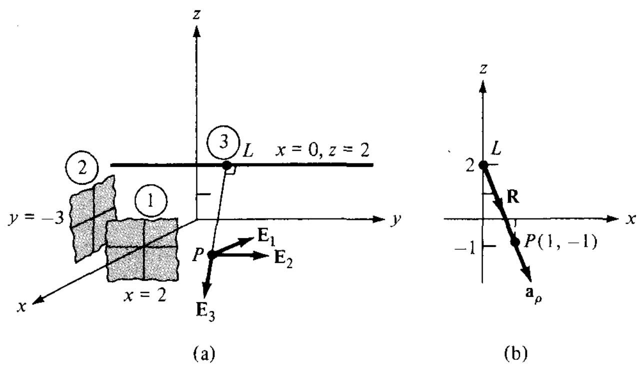

Figure 4.10 For Example 4.6: **(a)** three charge distributions; **(b)** finding ρ and $\mathbf{a}_\rho$ on plane $y = 1$.

and

$$\mathbf{E}_3 = \frac{\rho_L}{2\pi\varepsilon_o\rho}\,\mathbf{a}_\rho$$

where $\mathbf{a}_\rho$ (not regular $\mathbf{a}_\rho$ but with a similar meaning) is a unit vector along LP perpendicular to the line charge and ρ is the length LP to be determined from Figure 4.10(b). Figure 4.10(b) results from Figure 4.10(a) if we consider plane $y = 1$ on which $\mathbf{E}_3$ lies. From Figure 4.10(b), the distance vector from L to P is

$$\mathbf{R} = -3\mathbf{a}_z + \mathbf{a}_x$$

$$\rho = |\mathbf{R}| = \sqrt{10}, \qquad \mathbf{a}_\rho = \frac{\mathbf{R}}{|\mathbf{R}|} = \frac{1}{\sqrt{10}}\mathbf{a}_x - \frac{3}{\sqrt{10}}\mathbf{a}_z$$

Hence,

$$\begin{aligned}\mathbf{E}_3 &= \frac{10\pi \cdot 10^{-9}}{2\pi \cdot \dfrac{10^{-9}}{36\pi}} \cdot \frac{1}{10}(\mathbf{a}_x - 3\mathbf{a}_z) \\ &= 18\pi(\mathbf{a}_x - 3\mathbf{a}_z)\end{aligned}$$

Thus by adding $\mathbf{E}_1$, $\mathbf{E}_2$, and $\mathbf{E}_3$, we obtain the total field as

$$\mathbf{E} = -162\pi\mathbf{a}_x + 270\pi\mathbf{a}_y - 54\pi\mathbf{a}_z \text{ V/m}$$

Note that to obtain $\mathbf{a}_r$, $\mathbf{a}_\rho$, or $\mathbf{a}_n$, which we always need for finding $\mathbf{F}$ or $\mathbf{E}$, we must go from the charge (at position vector $\mathbf{r}'$) to the field point (at position vector $\mathbf{r}$); hence $\mathbf{a}_r$, $\mathbf{a}_\rho$, or $\mathbf{a}_n$ is a unit vector along $\mathbf{r} - \mathbf{r}'$. Observe this carefully in Figures 4.6 to 4.10.

PRACTICE EXERCISE 4.6

In Example 4.6 if the line $x = 0$, $z = 2$ is rotated through 90° about the point $(0, 2, 2)$ so that it becomes $x = 0$, $y = 2$, find $\mathbf{E}$ at $(1, 1, -1)$.

Answer: $-282.7\mathbf{a}_x + 564.5\mathbf{a}_y$ V/m.

4.4 ELECTRIC FLUX DENSITY

The flux due to the electric field $\mathbf{E}$ can be calculated using the general definition of flux in eq. (3.13). For practical reasons, however, this quantity is not usually considered as the most useful flux in electrostatics. Also, eqs. (4.11) to (4.16) show that the electric field intensity is dependent on the medium in which the charge is placed (free space in this chapter). Suppose a new vector field $\mathbf{D}$ independent of the medium is defined by

$$\boxed{\mathbf{D} = \varepsilon_o\mathbf{E}} \tag{4.35}$$

We define *electric flux* Ψ in terms of $\mathbf{D}$ using eq. (3.13), namely,

$$\Psi = \int \mathbf{D} \cdot d\mathbf{S} \tag{4.36}$$

In SI units, one line of electric flux emanates from $+1$ C and terminates on -1 C. Therefore, the electric flux is measured in coulombs. Hence, the vector field $\mathbf{D}$ is called the *electric flux density* and is measured in coulombs per square meter. For historical reasons, the electric flux density is also called *electric displacement.*

From eq. (4.35), it is apparent that all the formulas derived for $\mathbf{E}$ from Coulomb's law in Sections 4.2 and 4.3 can be used in calculating $\mathbf{D}$, except that we have to multiply those formulas by ε_o. For example, for an infinite sheet of charge, eqs. (4.26) and (4.35) give

$$\mathbf{D} = \frac{\rho_S}{2} \mathbf{a}_n \tag{4.37}$$

and for a volume charge distribution, eqs. (4.16) and (4.35) give

$$\mathbf{D} = \int \frac{\rho_v \, dv}{4\pi R^2} \mathbf{a}_R \tag{4.38}$$

Note from eqs. (4.37) and (4.38) that $\mathbf{D}$ is a function of charge and position only; it is independent of the medium.

EXAMPLE 4.7

Determine $\mathbf{D}$ at (4, 0, 3) if there is a point charge -5π mC at (4, 0, 0) and a line charge 3π mC/m along the y-axis.

Solution:

Let $\mathbf{D} = \mathbf{D}_Q + \mathbf{D}_L$ where $\mathbf{D}_Q$ and $\mathbf{D}_L$ are flux densities due to the point charge and line charge, respectively, as shown in Figure 4.11:

$$\mathbf{D}_Q = \varepsilon_o \mathbf{E} = \frac{Q}{4\pi R^2} \mathbf{a}_R = \frac{Q\,(\mathbf{r} - \mathbf{r}')}{4\pi |\mathbf{r} - \mathbf{r}'|^3}$$

where $\mathbf{r} - \mathbf{r}' = (4, 0, 3) - (4, 0, 0) = (0, 0, 3)$. Hence,

$$\mathbf{D}_Q = \frac{-5\pi \cdot 10^{-3}(0, 0, 3)}{4\pi |(0, 0, 3)|^3} = -0.138\,\mathbf{a}_z \text{ mC/m}^2$$

Also

$$\mathbf{D}_L = \frac{\rho_L}{2\pi\rho} \mathbf{a}_\rho$$

In this case

$$\mathbf{a}_\rho = \frac{(4, 0, 3) - (0, 0, 0)}{|(4, 0, 3) - (0, 0, 0)|} = \frac{(4, 0, 3)}{5}$$

$$\rho = |(4, 0, 3) - (0, 0, 0)| = 5$$

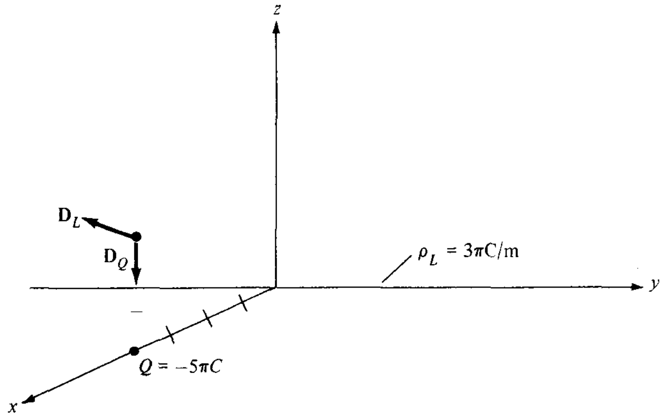

Figure 4.11 Flux density **D** due to a point charge and an infinite line charge.

Hence,

$$\mathbf{D}_L = \frac{3\pi}{2\pi(25)}(4\mathbf{a}_x + 3\mathbf{a}_z) = 0.24\mathbf{a}_x + 0.18\mathbf{a}_z \text{ mC/m}^2$$

Thus

$$\begin{aligned}\mathbf{D} &= \mathbf{D}_Q + \mathbf{D}_L \\ &= 240\mathbf{a}_x + 42\mathbf{a}_z \ \mu\text{C/m}^2\end{aligned}$$

PRACTICE EXERCISE 4.7

A point charge of 30 nC is located at the origin while plane $y = 3$ carries charge 10 nC/m^2. Find **D** at (0, 4, 3).

Answer: $5.076\mathbf{a}_y + 0.0573\mathbf{a}_z$ nC/m^2.

4.5 GAUSS'S LAW—MAXWELL'S EQUATION

Gauss's[5] law constitutes one of the fundamental laws of electromagnetism.

Gauss's law states that the total electric flux Ψ through any *closed* surface is equal to the total charge enclosed by that surface.

[5]Karl Friedrich Gauss (1777–1855), a German mathematician, developed the divergence theorem of Section 3.6, popularly known by his name. He was the first physicist to measure electric and magnetic quantities in absolute units. For details on Gauss's measurements, see W. F. Magie, *A Source Book in Physics.* Cambridge: Harvard Univ. Press, 1963, pp. 519–524.

Thus

$$\Psi = Q_{\text{enc}} \tag{4.39}$$

that is,

$$\begin{aligned}\Psi = \oint d\Psi &= \oint_S \mathbf{D} \cdot d\mathbf{S} \\ &= \text{Total charge enclosed } Q = \int \rho_v \, dv\end{aligned} \tag{4.40}$$

or

$$\boxed{Q = \oint_S \mathbf{D} \cdot d\mathbf{S} = \int_v \rho_v \, dv} \tag{4.41}$$

By applying divergence theorem to the middle term in eqs. (4.41)

$$\oint_S \mathbf{D} \cdot d\mathbf{S} = \int_v \nabla \cdot \mathbf{D} \, dv \tag{4.42}$$

Comparing the two volume integrals in eqs. (4.41) and (4.42) results in

$$\boxed{\rho_v = \nabla \cdot \mathbf{D}} \tag{4.43}$$

which is the first of the four *Maxwell's equations* to be derived. Equation (4.43) states that the volume charge density is the same as the divergence of the electric flux density. This should not be surprising to us from the way we defined the divergence of a vector in eq. (3.32) and from the fact that ρ_v at a point is simply the charge per unit volume at that point.

Note that:

1. Equations (4.41) and (4.43) are basically stating Gauss's law in different ways; eq. (4.41) is the integral form, whereas eq. (4.43) is the differential or point form of Gauss's law.
2. Gauss's law is an alternative statement of Coulomb's law; proper application of the divergence theorem to Coulomb's law results in Gauss's law.
3. Gauss's law provides an easy means of finding **E** or **D** for symmetrical charge distributions such as a point charge, an infinite line charge, an infinite cylindrical surface charge, and a spherical distribution of charge. A continuous charge distribution has rectangular symmetry if it depends only on x (or y or z), cylindrical symmetry if it depends only on ρ, or spherical symmetry if it depends only on r (independent of θ and ϕ). It must be stressed that whether the charge distribution is symmetric or not, Gauss's law always holds. For example, consider the charge distribution in Figure 4.12 where v_1 and v_2 are closed surfaces (or volumes). The total flux leaving v_1 is $10 - 5 = 5$ nC because only 10 nC and -5 nC charges are enclosed by v_1. Although charges 20 nC and 15 nC outside v_1 do contribute to the flux crossing v_1, the net flux crossing v_1, according to Gauss's law, is irrespective of those charges outside v_1. Similarly, the total flux leaving v_2 is zero

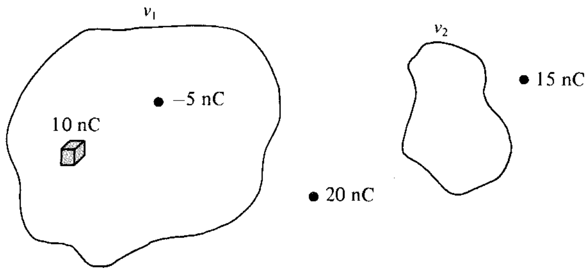

Figure 4.12 Illustration of Gauss's law; flux leaving v_1 is 5 nC and that leaving v_2 is 0 C.

because no charge is enclosed by v_2. Thus we see that Gauss's law, $\Psi = Q_{\text{enclosed}}$, is still obeyed even though the charge distribution is not symmetric. However, we cannot use the law to determine **E** or **D** when the charge distribution is not symmetric; we must resort to Coulomb's law to determine **E** or **D** in that case.

4.6 APPLICATIONS OF GAUSS'S LAW

The procedure for applying Gauss's law to calculate the electric field involves first knowing whether symmetry exists. Once symmetric charge distribution exists, we construct a mathematical closed surface (known as a *Gaussian surface*). The surface is chosen such that **D** is normal or tangential to the Gaussian surface. When **D** is normal to the surface, $\mathbf{D} \cdot d\mathbf{S} = D\, dS$ because **D** is constant on the surface. When **D** is tangential to the surface, $\mathbf{D} \cdot d\mathbf{S} = 0$. Thus we must choose a surface that has some of the symmetry exhibited by the charge distribution. We shall now apply these basic ideas to the following cases.

A. Point Charge

Suppose a point charge Q is located at the origin. To determine **D** at a point P, it is easy to see that choosing a spherical surface containing P will satisfy symmetry conditions. Thus, a spherical surface centered at the origin is the Gaussian surface in this case and is shown in Figure 4.13.

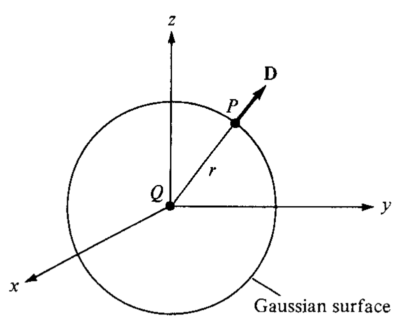

Figure 4.13 Gaussian surface about a point charge.

Since $\mathbf{D}$ is everywhere normal to the Gaussian surface, that is, $\mathbf{D} = D_r\mathbf{a}_r$, applying Gauss's law ($\Psi = Q_{\text{enclosed}}$) gives

$$Q = \oint \mathbf{D} \cdot d\mathbf{S} = D_r \oint dS = D_r\, 4\pi r^2 \tag{4.44}$$

where $\oint dS = \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} r^2 \sin\theta \, d\theta \, d\phi = 4\pi r^2$ is the surface area of the Gaussian surface. Thus

$$\mathbf{D} = \frac{Q}{4\pi r^2}\,\mathbf{a}_r \tag{4.45}$$

as expected from eqs. (4.11) and (4.35).

B. Infinite Line Charge

Suppose the infinite line of uniform charge ρ_L C/m lies along the z-axis. To determine $\mathbf{D}$ at a point P, we choose a cylindrical surface containing P to satisfy symmetry condition as shown in Figure 4.14. $\mathbf{D}$ is constant on and normal to the cylindrical Gaussian surface; that is, $\mathbf{D} = D_\rho \mathbf{a}_\rho$. If we apply Gauss's law to an arbitrary length ℓ of the line

$$\rho_L \ell = Q = \oint \mathbf{D} \cdot d\mathbf{S} = D_\rho \oint dS = D_\rho\, 2\pi\rho\ell \tag{4.46}$$

where $\oint dS = 2\pi\rho\ell$ is the surface area of the Gaussian surface. Note that $\int \mathbf{D} \cdot d\mathbf{S}$ evaluated on the top and bottom surfaces of the cylinder is zero since $\mathbf{D}$ has no z-component; that means that $\mathbf{D}$ is tangential to those surfaces. Thus

$$\mathbf{D} = \frac{\rho_L}{2\pi\rho}\,\mathbf{a}_\rho \tag{4.47}$$

as expected from eqs. (4.21) and (4.35).

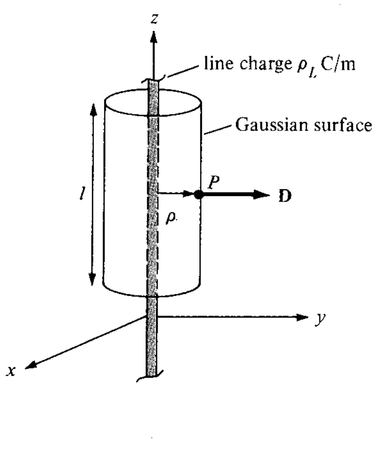

Figure 4.14 Gaussian surface about an infinite line charge.

C. Infinite Sheet of Charge

Consider the infinite sheet of uniform charge ρ_S C/m^2 lying on the $z = 0$ plane. To determine **D** at point P, we choose a rectangular box that is cut symmetrically by the sheet of charge and has two of its faces parallel to the sheet as shown in Figure 4.15. As **D** is normal to the sheet, $\mathbf{D} = D_z\mathbf{a}_z$, and applying Gauss's law gives

$$\rho_S \int dS = Q = \oint \mathbf{D} \cdot d\mathbf{S} = D_z\left[\int_{\text{top}} dS + \int_{\text{bottom}} dS\right] \tag{4.48}$$

Note that $\mathbf{D} \cdot d\mathbf{S}$ evaluated on the sides of the box is zero because **D** has no components along $\mathbf{a}_x$ and $\mathbf{a}_y$. If the top and bottom area of the box each has area A, eq. (4.48) becomes

$$\rho_S A = D_z\,(A + A) \tag{4.49}$$

and thus

$$\mathbf{D} = \frac{\rho_S}{2}\,\mathbf{a}_z$$

or

$$\mathbf{E} = \frac{\mathbf{D}}{\varepsilon_o} = \frac{\rho_S}{2\varepsilon_o}\,\mathbf{a}_z \tag{4.50}$$

as expected from eq. (4.25).

D. Uniformly Charged Sphere

Consider a sphere of radius a with a uniform charge ρ_v C/m^3. To determine **D** everywhere, we construct Gaussian surfaces for cases $r \leq a$ and $r \geq a$ separately. Since the charge has spherical symmetry, it is obvious that a spherical surface is an appropriate Gaussian surface.

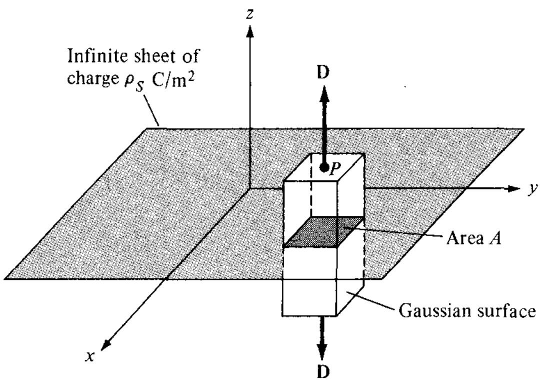

Figure 4.15 Gaussian surface about an infinite line sheet of charge.

For $r \leq a$, the total charge enclosed by the spherical surface of radius r, as shown in Figure 4.16 (a), is

$$\begin{aligned} Q_{\text{enc}} &= \int \rho_v\, dv = \rho_v \int dv = \rho_v \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} \int_{r=0}^{r} r^2 \sin\theta\, dr\, d\theta\, d\phi \\ &= \rho_v \frac{4}{3} \pi r^3 \end{aligned} \tag{4.51}$$

and

$$\begin{aligned} \Psi &= \oint \mathbf{D} \cdot d\mathbf{S} = D_r \oint dS = D_r \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} r^2 \sin\theta\, d\theta\, d\phi \\ &= D_r\, 4\pi r^2 \end{aligned} \tag{4.52}$$

Hence, $\Psi = Q_{\text{enc}}$ gives

$$D_r\, 4\pi r^2 = \frac{4\pi r^3}{3} \rho_v$$

or

$$\mathbf{D} = \frac{r}{3} \rho_v\, \mathbf{a}_r \qquad 0 < r \leq a \tag{4.53}$$

For $r \geq a$, the Gaussian surface is shown in Figure 4.16(b). The charge enclosed by the surface is the entire charge in this case, that is,

$$\begin{aligned} Q_{\text{enc}} &= \int \rho_v\, dv = \rho_v \int dv = \rho_v \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} \int_{r=0}^{a} r^2 \sin\theta\, dr\, d\theta\, d\phi \\ &= \rho_v \frac{4}{3} \pi a^3 \end{aligned} \tag{4.54}$$

while

$$\Psi = \oint \mathbf{D} \cdot d\mathbf{S} = D_r\, 4\pi r^2 \tag{4.55}$$

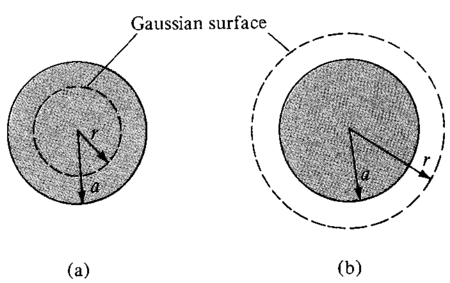

Figure 4.16 Gaussian surface for a uniformly charged sphere when: **(a)** $r \geq a$ and **(b)** $r \leq a$.

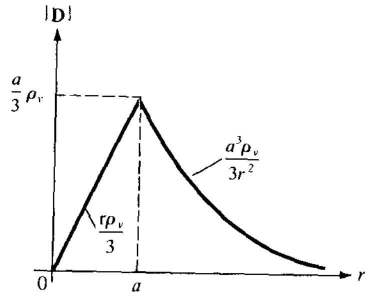

Figure 4.17 Sketch of $|\mathbf{D}|$ against r for a uniformly charged sphere.

just as in eq. (4.52). Hence:

$$D_r\, 4\pi r^2 = \frac{4}{3}\pi a^3 \rho_v$$

or

$$\mathbf{D} = \frac{a^3}{3r^2}\rho_v \mathbf{a}_r \qquad r \geqslant a \tag{4.56}$$

Thus from eqs. (4.53) and (4.56), **D** everywhere is given by

$$\mathbf{D} = \begin{cases} \dfrac{r}{3}\rho_v\, \mathbf{a}_r & 0 < r \leqslant a \\ \dfrac{a^3}{3r^2}\rho_v\, \mathbf{a}_r & r \geqslant a \end{cases} \tag{4.57}$$

and $|\mathbf{D}|$ is as sketched in Figure 4.17.

Notice from eqs. (4.44), (4.46), (4.48), and (4.52) that the ability to take **D** out of the integral sign is the key to finding **D** using Gauss's law. In other words, **D** must be constant on the Gaussian surface.

EXAMPLE 4.8

Given that $\mathbf{D} = z\rho \cos^2\phi\, \mathbf{a}_z$ C/m^2, calculate the charge density at $(1, \pi/4, 3)$ and the total charge enclosed by the cylinder of radius 1 m with $-2 \leq z \leq 2$ m.

Solution:

$$\rho_v = \nabla \cdot \mathbf{D} = \frac{\partial D_z}{\partial z} = \rho \cos^2 \phi$$

At $(1, \pi/4, 3)$, $\rho_v = 1 \cdot \cos^2(\pi/4) = 0.5$ C/m^3. The total charge enclosed by the cylinder can be found in two different ways.

Method 1: This method is based directly on the definition of the total volume charge.

$$
\begin{aligned}
Q &= \int_v \rho_v \, dv = \int_v \rho \cos^2 \phi \, \rho \, d\phi \, d\rho \, dz \\
&= \int_{z=-2}^{2} dz \int_{\phi=0}^{2\pi} \cos^2 \phi \, d\phi \int_{\rho=0}^{1} \rho^2 \, d\rho = 4(\pi)(1/3) \\
&= \frac{4\pi}{3} \text{ C}
\end{aligned}
$$

Method 2: Alternatively, we can use Gauss's law.

$$
\begin{aligned}
Q = \Psi &= \oint \mathbf{D} \cdot d\mathbf{S} = \left[\int_s + \int_t + \int_b \right] \mathbf{D} \cdot d\mathbf{S} \\
&= \Psi_s + \Psi_t + \Psi_b
\end{aligned}
$$

where Ψ_s, Ψ_t, and Ψ_b are the flux through the sides, the top surface, and the bottom surface of the cylinder, respectively (see Figure 3.17). Since $\mathbf{D}$ does not have component along $\mathbf{a}_\rho$, $\Psi_s = 0$, for Ψ_t, $d\mathbf{S} = \rho \, d\phi \, d\rho \, \mathbf{a}_z$ so

$$
\begin{aligned}
\Psi_t &= \int_{\rho=0}^{1} \int_{\phi=0}^{2\pi} z\rho \cos^2 \phi \, \rho \, d\phi \, d\rho \bigg|_{z=2} = 2 \int_0^1 \rho^2 d\rho \int_0^{2\pi} \cos^2 \phi \, d\phi \\
&= 2\left(\frac{1}{3}\right)\pi = \frac{2\pi}{3}
\end{aligned}
$$

and for Ψ_b, $d\mathbf{S} = -\rho \, d\phi \, d\rho \, \mathbf{a}_z$, so

$$
\begin{aligned}
\Psi_b &= -\int_{\rho=0}^{1} \int_{\phi=0}^{2\pi} z\rho \cos^2 \phi \, \rho \, d\phi \, d\rho \bigg|_{z=-2} = 2 \int_0^1 \rho^2 \, d\rho \int_0^{2\pi} \cos^2 \phi \, d\phi \\
&= \frac{2\pi}{3}
\end{aligned}
$$

Thus

$$
Q = \Psi = 0 + \frac{2\pi}{3} + \frac{2\pi}{3} = \frac{4\pi}{3} \text{ C}
$$

as obtained previously.

PRACTICE EXERCISE 4.8

If $\mathbf{D} = (2y^2 + z)\mathbf{a}_x + 4xy\mathbf{a}_y + x\mathbf{a}_z$ C/m^2, find

(a) The volume charge density at $(-1, 0, 3)$

(b) The flux through the cube defined by $0 \leq x \leq 1, 0 \leq y \leq 1, 0 \leq z \leq 1$

(c) The total charge enclosed by the cube

Answer: (a) -4 C/m^3, (b) 2 C, (c) 2 C.

EXAMPLE 4.9

A charge distribution with spherical symmetry has density

$$\rho_v = \begin{cases} \dfrac{\rho_o r}{R}, & 0 \leq r \leq R \\ 0, & r > R \end{cases}$$

Determine **E** everywhere.

Solution:

The charge distribution is similar to that in Figure 4.16. Since symmetry exists, we can apply Gauss's law to find **E**.

$$\varepsilon_o \oint \mathbf{E} \cdot d\mathbf{S} = Q_{\text{enc}} = \int \rho_v \, dv$$

(a) For $r < R$

$$\varepsilon_o E_r \, 4\pi r^2 = Q_{\text{enc}} = \int_0^r \int_0^\pi \int_0^{2\pi} \rho_v \, r^2 \sin\theta \, d\phi \, d\theta \, dr$$

$$= \int_0^r 4\pi r^2 \frac{\rho_o r}{R} \, dr = \frac{\rho_o \pi r^4}{R}$$

or

$$\mathbf{E} = \frac{\rho_o r^2}{4\varepsilon_o R} \mathbf{a}_r$$

(b) For $r > R$,

$$\varepsilon_o E_r 4\pi r^2 = Q_{\text{enc}} = \int_0^r \int_0^\pi \int_0^{2\pi} \rho_v r^2 \sin\theta \, d\phi \, d\theta \, dr$$

$$= \int_0^R \frac{\rho_o r}{R} 4\pi r^2 \, dr + \int_R^r 0 \cdot 4\pi r^2 \, dr$$

$$= \pi \rho_o R^3$$

or

$$\mathbf{E} = \frac{\rho_o R^3}{4\varepsilon_o r^2} \mathbf{a}_r$$

PRACTICE EXERCISE 4.9

A charge distribution in free space has $\rho_v = 2r$ nC/m^3 for $0 \leq r \leq 10$ m and zero otherwise. Determine **E** at $r = 2$ m and $r = 12$ m.

Answer: $226\mathbf{a}_r$ V/m, $3.927\mathbf{a}_r$ kV/m.