

Vector space \mathbb{F}_2^n

Ralf Pöppel

<mailto:ralf@poepfel-familie.de>

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Abstract

This article is a supplemental documentation to the go package `gf2vs` [1]. The package implements data types and functions modeling the vector space \mathbb{F}_2^n . The vector space \mathbb{F}_2^n of size n is based on the finite field of order 2 or Galois field $GF(2)$ [3]. We use $GF(2)$ to model binary values, or bits. We consider the properties of the vector space of bit vectors.

Field \mathbb{F}_2

The finite field of order 2 has 2 elements $\mathbb{F}_2 = \{0, 1\}$ and the operations addition $+$ and multiplication \cdot . For the definition see equation (1).

$$\begin{aligned} + : & \quad 0 + 0 = 0, \quad 0 + 1 = 1, \quad 1 + 0 = 1, \quad 1 + 1 = 0, \\ \cdot : & \quad 0 \cdot 0 = 0, \quad 0 \cdot 1 = 0, \quad 1 \cdot 0 = 0, \quad 1 \cdot 1 = 1. \end{aligned} \tag{1}$$

We may use the notation ab instead of $a \cdot b$ omitting the multiplication sign if there is no ambiguity.

Each of the 2 operations of the field \mathbb{F}_2 satisfy the group axioms [4] for the groups $G_+ : (\mathbb{F}_2, +)$ and $G_\cdot : (\mathbb{F}_2, \cdot)$. In addition both operations are commutative. For reference the group axioms are repeated here. We use the symbol \circ to denote the binary operations $+$, \cdot .

Associativity

$$\forall a, b, c \in G : (a \circ b) \circ c = a \circ (b \circ c).$$

Identity element e

$$\exists e \in G, \forall a \in G : e \circ a = a \text{ and } a \circ e = a, e \text{ is unique.}$$

Inverse element a^{-1}

$$\forall a \in G \quad \exists b \in G : a \circ b = e \text{ and } b \circ a = e, e \text{ identity element, } b \text{ is unique } \forall a, \text{ notation } b = a^{-1}.$$

Commutativity

$$a \circ b = b \circ a.$$

We can look at the field from an algebraic point of view or from a logic view. In logic the field can be seen as the boolean variables $F = 0$ and $T = 1$. The boolean operations are disjunction \vee [7], contravalence \oplus [2] and conjunction \wedge [6]. The definition is repeated in equation (2).

$$\begin{aligned} \vee : & \quad 0 \vee 0 = 0, \quad 0 \vee 1 = 1, \quad 1 \vee 0 = 1, \quad 1 \vee 1 = 1, \\ \oplus : & \quad 0 \oplus 0 = 0, \quad 0 \oplus 1 = 1, \quad 1 \oplus 0 = 1, \quad 1 \oplus 1 = 0, \\ \wedge : & \quad 0 \wedge 0 = 0, \quad 0 \wedge 1 = 0, \quad 1 \wedge 0 = 0, \quad 1 \wedge 1 = 1. \end{aligned} \tag{2}$$

Please note the operations \oplus and \wedge are identically defined as $+$ and \cdot and hence satisfy the group axioms. But the operation \vee does not satisfy the group axioms, there is no inverse element. In the remaining chapters we will use the notation $+$, \cdot , \vee for the operations only.

Vector Space \mathbb{F}_2^n

We define the vector space \mathbb{F}_2^n over the field \mathbb{F}_2 as set V of vectors v of n elements of the field together with the binary operation addition and the binary function scalar multiplication ??.

$$u + v = w, \quad u, v, w \in V, \quad a \cdot v = w, \quad a \in \mathbb{F}_2, v, w \in V. \quad (3)$$

We apply the addition element-wise and we multiply the scalar with each element of the vector. This definition is similar to the definition in [8].

We use the notation $(v_i) := v$ for the vector v with the components v_i .

In addition we define 2 constants of \mathbb{F}_2^n :

Zeros

$\mathbf{0}$ Zero vector where all components are 0.

Ones

$\mathbf{1}$ Vector where all components are 1.

The axioms of a vector space are satisfied [8]:

Associativity of vector addition

$$u + (v + w) = (u + v) + w, \quad \forall u, v, w \in \mathbb{F}_2^n.$$

Commutativity of vector addition

$$u + v = v + u, \quad \forall u, v \in \mathbb{F}_2^n.$$

Identity element of vector addition

$$\exists \mathbf{0} \in \mathbb{F}_2^n : v + \mathbf{0} = v, \quad \forall v \in \mathbb{F}_2^n.$$

Inverse elements of vector addition

$$\forall v \in \mathbb{F}_2^n \quad \exists -v \in \mathbb{F}_2^n : v + (-v) = \mathbf{0}, \quad -v = v, \quad \text{each vector is its own additive inverse.}$$

Compatibility of scalar multiplication with field multiplication

$$a(bv) = (ab)v, \quad a, b \in \mathbb{F}_2, v \in \mathbb{F}_2^n.$$

Identity element of scalar multiplication

$$1v = v, \quad 1 \in \mathbb{F}_2, v \in \mathbb{F}_2^n, \quad 1 \text{ is the multiplicative identity of } \mathbb{F}_2.$$

Distributivity of scalar multiplication with respect to vector addition

$$a(u + v) = au + av, \quad a \in \mathbb{F}_2, u, v \in \mathbb{F}_2^n.$$

Distributivity of scalar multiplication with respect to field addition

$$(a + b)v = av + bv, \quad a, b \in \mathbb{F}_2, v \in \mathbb{F}_2^n.$$

In this vector space we are not limited to the operations vector addition and scalar multiplication. We can use the boolean operations too.

Complement, Not

$$\bar{v} = \mathbf{1} - v = \mathbf{1} + v, \quad \text{swap all bits.}$$

Disjunction, Or

$u \vee v = (u_i) \vee (v_i) = (u_i \vee v_i)$, element wise Or.

Contravalence, Xor

$u \oplus v = u + v = (u_i) + (v_i) = (u_i + v_i)$, element wise xor, duplicate of vector addition.

Conjunction, And

$u \wedge v = (u_i) \cdot (v_i) = (u_i \cdot v_i)$, element wise And.

As we apply the operations element wise, we satisfy the laws of associativity and commutativity.

We use some more definitions to cover the further properties of a vector space:

Unit vector

We define the unit vectors $e_i, i = 1, \dots, n$ of the vector space as the vectors where the i th element is $x_i = 1$ and all other elements are 0.

$$e_i = (x_k),$$

$$x_k = \begin{cases} 1, & k = i, \\ 0, & k \neq i, \end{cases} \quad x_k \in \mathbb{F}_2, \quad e_i \in \mathbb{F}_2^n.$$

Generating system

We define the subspace $\mathbb{E} = \{e_i\}$, $\mathbb{E} \subset \mathbb{F}_2^n$ of vectors e_i . The subspace \mathbb{E} forms a generating system. Obviously each vector v of \mathbb{F}_2^n is a linear combination of the scalars a_i , and the e_i .

$$v = \sum_{i=1}^n a_i e_i, \quad a_i \in \mathbb{F}_2, \quad e_i \in \mathbb{E}, \quad \forall v \in \mathbb{F}_2^n.$$

So the subset \mathbb{E} is a span of \mathbb{F}_2^n . In this vector space it is the only span. And the decomposition of a vector v in a linear combination of unit vectors e_i is unique.

Basis

The subspace \mathbb{E} is the one and only basis of the vector space \mathbb{F}_2^n .

Index

We name $i = 1, \dots, n$ of e_i the index of a unit vector in the basis.

Norm

We define the Norm $|v|$ of a vector $v \in \mathbb{F}_2^n$ to be its Hamming weight [5]. In this case the count of ones of the vector. The value of the norm is an element of the set $\{0, 1, \dots, n\} \neq \mathbb{F}_2, n > 2$. In contrast to usual vector spaces for example on \mathbb{Z} , where the norm of a vector is an element of \mathbb{Z} .

Inner product We define the inner product of 2 vectors, to be the norm of the product of 2 vectors:

$$\langle u, v \rangle = |u \cdot v|.$$

Orthogonality

$$\langle u, v \rangle = 0,$$

we say 2 vectors are orthogonal if the inner product is 0. Please note the inner product of any vector with $\mathbf{0}$ is 0.

References

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