

# Vector space of bit vectors

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## Abstract

This article is a supplementary documentation to the Go package `gf2vs` [10]. The package implements data types and functions for the vector space of bit vectors.

## 1 Introduction

Bit vectors are very common in computer science. They are used for integers, combinatorial algorithms, coding theory, and for logical and arithmetic operations [7, 5]. All aspects of the vector space of bit vectors are examined relatively rarely.

Bits are based on the finite set of integers of order 2. This set  $\{0, 1\}$  with the operations addition and multiplication modulo 2 satisfies the axioms of a field. This finite field is named Galois Field [8, 15]<sup>1</sup>  $GF(2) = \mathbb{F}_2$ . Over this field there is the vector space  $\mathbb{F}_2^n$ .

The aim of this article is to document the properties of the vector space of bit vectors  $\mathbb{F}_2^n$ , as implemented in the Go package `gf2vs`. This is a brief reference of the properties collected from several sources.

## 2 Field $GF(2)$

### 2.1 Supporting Set

The supporting set of  $GF(2) = \mathbb{F}_2$  is

$$\mathbb{Z}_2 = \mathbb{Z}/\mathbb{Z}2 = \{0, 1\} \subset \mathbb{Z} \subset \mathbb{R} \quad (1)$$

the subset  $\mathbb{Z}_2$  of  $\mathbb{Z}$ , which is a subset of  $\mathbb{R}$ . This set is equal to  $\mathbb{Z}/\mathbb{Z}2$ , the cyclic group of order 2. This set holds the values of a bit in computer science. In logic we have the boolean values False  $F = 0$  and True  $T = 1$  [6].

### 2.2 Operations

The operations of  $\mathbb{F}_2 = \mathbb{Z}/\mathbb{Z}2$  are defined modulo 2; see [2, ch. 2.2.6] and [5].

The operations addition and multiplication of the field  $\mathbb{F}_2$  satisfies the group axioms [2, 8, 16], both operations are commutative.

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<sup>1</sup>We cite Wikipedia for reused wordings.

Similarly the operations addition and multiplication of the field  $\mathbb{R}$  satisfies the group axioms [2], both operations are commutative.

Please note the different definition of the addition in  $\mathbb{F}_2$  and  $\mathbb{R}$ .

For reference we give here only the operations for  $\mathbb{F}_2$ .

### 2.2.1 Negation

$$- : \mathbb{F}_2 \rightarrow \mathbb{F}_2, \quad -x = x, \quad -0 = 0, \quad -1 = 1. \quad (2)$$

The negation of 1 in  $\mathbb{F}_2$  is computed as  $(-1) \bmod 2 = 1$

### 2.2.2 Complement

$$\neg : \mathbb{F}_2 \rightarrow \mathbb{F}_2, \quad \neg x = 1 - x, \quad \neg 0 = 1, \quad \neg 1 = 0. \quad (3)$$

### 2.2.3 Absolute value

We define the mapping absolute value  $|x|$  of an element  $x$  of  $\mathbb{F}_2$  to  $\mathbb{R}$ :

$$|x| : \mathbb{F}_2 \rightarrow \mathbb{R}, \quad |0| = 0, \quad |1| = 1. \quad (4)$$

We use this mapping, when we need the default classical definition of the addition as in  $\mathbb{R}$ .

### 2.2.4 Addition

The addition is named exclusive disjunction in logic and XOR [6, 14] in computer science. The definition of addition is given in equation 5 obeying  $(1 + 1) \bmod 2 = 0$ .

$$+ : \mathbb{F}_2 \times \mathbb{F}_2 \rightarrow \mathbb{F}_2, \quad 0 + 0 = 0, \quad 0 + 1 = 1, \quad 1 + 0 = 1, \quad 1 + 1 = 0, \quad (5)$$

The group axioms [2, ch. 2.2.8] for the Group  $G = \mathbb{F}_2$  and the operation addition are satisfied:

#### Associativity

$$\forall a, b, c \in G : (a + b) + c = a + (b + c).$$

#### Identity element $e = 0$

$$\exists e \in G, \forall a \in G : e + a = a \text{ and } a + e = a, e = 0, e \text{ is unique.}$$

#### Inverse element $(-a) = a$

$$\forall a \in G \exists (-a) \in G : a + (-a) = e \text{ and } (-a) + a = e, e \text{ identity element, } (-a) = a \text{ is unique for each } a.$$

#### Commutativity

$$a + b = b + a.$$

So  $\mathbb{F}_2$  with the operation addition is an abelian group.

### 2.2.5 Multiplication

The multiplication is named conjunction in logic and AND [6, 18] in computer science. The multiplication is identically defined as in  $\mathbb{Z}$ .

$$\cdot : \mathbb{F}_2 \times \mathbb{F}_2 \rightarrow \mathbb{F}_2, \quad 0 \cdot 0 = 0, \quad 0 \cdot 1 = 0, \quad 1 \cdot 0 = 0, \quad 1 \cdot 1 = 1. \quad (6)$$

We may use the notation  $ab$  instead of  $a \cdot b$ , omitting the multiplication sign if there is no ambiguity.

The group axioms for the Group  $G = \mathbb{F}_2$  and the operation multiplication are satisfied:

#### Associativity

$$\forall a, b, c \in G : (a \cdot b) \cdot c = a \cdot (b \cdot c).$$

#### Identity element $e = 1$

$$\exists e \in G, \forall a \in G : e \cdot a = a \text{ and } a \cdot e = a, e = 1, e \text{ is unique.}$$

#### Inverse element $a^{-1}$

$$\forall a \in G, a \neq 0, \exists a^{-1} \in G : a \cdot a^{-1} = e \text{ and } a^{-1} \cdot a = e, e \text{ is the identity element; the only invertible element is } 1, \text{ hence } a^{-1} = 1 \text{ for } a = 1.$$

#### Commutativity

$$a \cdot b = b \cdot a.$$

So  $\mathbb{F}_2$  with the operation multiplication is an abelian group.

### 2.2.6 Disjunction

In boolean logic we have the operation disjunction, named OR in computer science [6, 19].

$$\vee : \mathbb{F}_2 \times \mathbb{F}_2 \rightarrow \mathbb{F}_2, \quad 0 \vee 0 = 0, \quad 0 \vee 1 = 1, \quad 1 \vee 0 = 1, \quad 1 \vee 1 = 1 \quad (7)$$

The operation  $\vee$  does not satisfy the group axioms; there is no inverse element.

## 2.3 Field axioms

The set  $K := \mathbb{F}_2$  with the operations addition and multiplication satisfies the field axioms [2, 5]. We use  $K$  as symbol for any field satisfying the field axioms.

**K1**  $K$  with the addition  $+$  is an abelian group.

**K2**  $K^* := K \setminus \{0\}$  with the multiplication  $\cdot$  for every element of  $K^*$  is an abelian group.

**K3** distributive property [1, 12] is satisfied  $\forall a, b, c \in K$

$$\begin{aligned} a \cdot (b + c) &= a \cdot b + a \cdot c, \\ (a + b) \cdot c &= a \cdot c + b \cdot c. \end{aligned} \quad (8)$$

### 3 Vector space $\mathbb{F}_2^n$

#### 3.1 Vectors

Bit vectors are the elements of the vector space. We define a bit vector  $x$  of size  $n$  as a tuple  $(x_i)$  of values  $x_i \in \mathbb{F}_2$ :

$$x := (x_i) := (x_1, x_2, \dots, x_n), \quad \forall x_i \in \mathbb{F}_2 \quad (9)$$

We define the set of all bit vectors of size  $n$  see [2, ch 2.4.1]:

$$\mathbb{F}_2^n := \{x = (x_1, \dots, x_n) : x_i \in \mathbb{F}_2\} \quad (10)$$

In addition we define two distinguished constant elements of  $\mathbb{F}_2^n$ :

**Zero**  $\mathbf{0}$  zero vector, all components are 0.

**Ones**  $\mathbf{1}$  ones vector, all components are 1.

#### 3.2 Mapping of Integers to Bit Vectors

We define the mapping  $\varphi$  from  $\mathbb{N}$  to  $\mathbb{F}_2^n$ . For mapping of an integer  $a$  to a bit vector  $x$  we need to compute the length  $n$  of  $x$  with equation (11).

$$n = \lfloor \log_2(a) \rfloor + 1. \quad (11)$$

If we need to map multiple integers  $a_i$  to the same vector space we compute  $n$  from the maximum of the  $a_i$

$$n = \lfloor \log_2(\max(a_i)) \rfloor + 1. \quad (12)$$

Equation (13) defines the mapping for  $a \in \mathbb{N}, a \geq 0$ .

$$\varphi : \mathbb{N} \rightarrow \mathbb{F}_2^n, \quad x = \left( \lfloor \frac{a}{2^0} \rfloor \bmod 2, \lfloor \frac{a}{2^1} \rfloor \bmod 2, \dots, \lfloor \frac{a}{2^{n-1}} \rfloor \bmod 2 \right). \quad (13)$$

The inverse mapping  $\varphi^{-1}$  is given by equation (14)

$$\varphi^{-1} : \mathbb{F}_2^n \rightarrow \mathbb{N}, \quad a = \sum_{i=1}^n x_i 2^{i-1}. \quad (14)$$

In the programming language Go these mapping are used when parsing integer strings and formatting integers. Internally integers are stored as bit vectors.

#### 3.3 Operations on vectors

We define bitwise operations on the bit vectors  $x, y, z$  see [2, ch. 2.4.1] and [7, (1), (2), (3)].

$$\left. \begin{array}{lll} - : \mathbb{F}_2^n \rightarrow \mathbb{F}_2^n, & -x = x & \Leftrightarrow & -x_i = x_i \\ \sim : \mathbb{F}_2^n \rightarrow \mathbb{F}_2^n, & \sim x = y & \Leftrightarrow & \neg x_i = y_i \\ | : \mathbb{F}_2^n \rightarrow \mathbb{R}^n, & |x| = y & \Leftrightarrow & x_i = y_i, \\ \oplus : \mathbb{F}_2^n \times \mathbb{F}_2^n \rightarrow \mathbb{F}_2^n, & x \oplus y = z & \Leftrightarrow & x_i + y_i = z_i, \\ \& : \mathbb{F}_2^n \times \mathbb{F}_2^n \rightarrow \mathbb{F}_2^n, & x \& y = z & \Leftrightarrow & x_i \cdot y_i = z_i, \\ | : \mathbb{F}_2^n \times \mathbb{F}_2^n \rightarrow \mathbb{F}_2^n, & x | y = z & \Leftrightarrow & x_i \vee y_i = z_i, \\ \cdot : \mathbb{F}_2 \times \mathbb{F}_2^n \rightarrow \mathbb{F}_2^n, & \lambda \cdot x = y & \Leftrightarrow & \lambda \cdot x_i = y_i, \end{array} \right\} i = 1, \dots, n. \quad (15)$$

We define the operation  $|$  for formal mapping of a bit vector from  $\mathbb{F}_2^n$  to  $\mathbb{R}^n$ .

For the constants we have:  $\sim 0 = 1$  and  $\sim 1 = 0$ .

We adopt the main identities from [7, (4), ..., (14)] for bit vectors of size  $n$  here:

$$x \oplus y = y \oplus x, \quad x \& y = y \& x, \quad x|y = y|x; \quad (16)$$

$$(x \oplus y) \oplus z = x \oplus (y \oplus z), \quad (x \& y) \& z = x \& (y \& z), \quad (x|y)|z = x|(y|z); \quad (17)$$

$$(x \oplus y) \& z = (x \& z) \oplus (y \& z); \quad (18)$$

$$(x \& y)|z = (x|z) \& (y|z), \quad (x|y) \& z = (x|z) \& (y|z); \quad (19)$$

$$x \oplus y = (x \& y) \oplus (x|y); \quad (20)$$

$$(x \& y)|x = x, \quad (x|y) \& x = x; \quad (21)$$

$$x \oplus 0 = x, \quad x \& 0 = 0, \quad x|0 = x; \quad (22)$$

$$x \oplus x = 0, \quad x \& x = x, \quad x|x = x; \quad (23)$$

$$x \oplus 1 = \sim x, \quad x \& 1 = x, \quad x|1 = 1; \quad (24)$$

$$x \oplus (\sim x) = 1, \quad x \& (\sim x) = 0, \quad x|(\sim x) = 1; \quad (25)$$

$$-(x \oplus y) = (\sim x) \oplus y = x \oplus (\sim y), \quad \sim(x \& y) = (\sim x)|(\sim y), \quad \sim(x|y) = (\sim x) \& (\sim y); \quad (26)$$

### 3.4 Axioms of vector space

The set  $\mathbb{F}_2^n$  with the binary operation of vector addition  $\oplus$  and the binary function of scalar multiplication  $\cdot$ , as given in (15), defines a vector space see [2, 21].

The axioms of a vector space are satisfied for  $\mathbb{F}_2^n$ :

#### Associativity of vector addition

$$u \oplus (v \oplus w) = (u \oplus v) \oplus w, \quad \forall u, v, w \in \mathbb{F}_2^n.$$

#### Commutativity of vector addition

$$u \oplus v = v \oplus u, \quad \forall u, v \in \mathbb{F}_2^n.$$

#### Identity element of vector addition

$$\exists 0 \in \mathbb{F}_2^n : v \oplus 0 = v, \quad \forall v \in \mathbb{F}_2^n.$$

#### Inverse elements of vector addition

$$\forall v \in \mathbb{F}_2^n \exists -v \in \mathbb{F}_2^n : v \oplus (-v) = 0, \text{ and } -v = v, \text{ i.e. each vector is its own additive inverse.}$$

#### Compatibility of scalar multiplication with field multiplication

$$\lambda(\eta v) = (\lambda\eta)v, \quad \lambda, \eta \in \mathbb{F}_2, \quad v \in \mathbb{F}_2^n.$$

#### Identity element of scalar multiplication

$$1v = v, \quad 1 \in \mathbb{F}_2, \quad v \in \mathbb{F}_2^n, \text{ where } 1 \text{ is the multiplicative identity of } \mathbb{F}_2.$$

#### Distributivity of scalar multiplication with respect to vector addition

$$\lambda(u \oplus v) = \lambda u \oplus \lambda v, \quad \lambda \in \mathbb{F}_2, \quad u, v \in \mathbb{F}_2^n.$$

#### Distributivity of scalar multiplication with respect to field addition

$$(\lambda + \eta)v = \lambda v \oplus \eta v, \quad \lambda, \eta \in \mathbb{F}_2, \quad v \in \mathbb{F}_2^n.$$

In the vector space  $\mathbb{F}_2^n$  we are not limited to the operations vector addition and scalar multiplication. We can use the Boolean operations as well.

#### Negation, Complement, Not

$$\sim v = 1 - v = 1 \oplus v, \text{ swap all bits.}$$

### Disjunction, Or

$u|v = (u_i)|(v_i) = (u_i \vee v_i)$ , element-wise Or.

### Exclusive or, Xor

$u \oplus v = u \oplus v = (u_i) \oplus (v_i) = (u_i + v_i)$ , element-wise Xor, equal to vector addition.

### Conjunction, And

$u \wedge v = (u_i) \cdot (v_i) = (u_i \cdot v_i)$ , element-wise And.

As we apply the operations element-wise  $i = 1, \dots, n$ , we satisfy the laws of associativity and commutativity inherited from the field.

## 3.5 Vector space base

We give here the definition of the base, the norm and the scalar product implemented in the Go package. The symbol  $K$  is used in definitions applicable by each of the fields  $\mathbb{F}_2^n, \mathbb{Q}, \mathbb{R}, \mathbb{C}$ .

### 3.5.1 Unit vector

We define the unit vectors  $e_i, i = 1, \dots, n$ , of the vector space as the vectors where the  $i$ th element is  $x_i = 1$  and all other elements are 0.

$$e_i = (x_k), \quad e_i \in K^n, \quad x_k \in K, \quad x_k = \begin{cases} 1, & k = i, \quad \text{identity element of multiplication,} \\ 0, & k \neq i, \quad \text{identity element of addition.} \end{cases} \quad (27)$$

Please note the  $e_i$  are linearly independent.

We observe the identity elements are identical for the fields and the unit vectors are identical for all vector spaces over a field  $K$ .

### 3.5.2 Generating system

We define the subset  $\mathbb{E} := \{e_i\}, \mathbb{E} \subset \mathbb{F}_2^n$ , of unit vectors  $e_i$ . The subset  $\mathbb{E}$  forms a generating system. Each vector  $v$  of  $\mathbb{F}_2^n$  is a linear combination of scalars  $a_i$  and the  $e_i$ :

$$v = \sum_{i=1}^n a_i e_i, \quad a_i \in \mathbb{F}_2, \quad e_i \in \mathbb{E}, \quad \forall v \in \mathbb{F}_2^n. \quad (28)$$

Here the addition is modulo 2.

Equation (28) is used equally for each vector space on any field  $K$  using the operation addition as defined for the field  $K$  and the 1 the identity element of the operation multiplication.

Thus the subset  $\mathbb{E}$  spans  $\mathbb{F}_2^n$ . In this vector space it is one spanning set, and the decomposition of a vector  $v$  into a linear combination of unit vectors  $e_i$  is unique.

### 3.5.3 Base

The subset  $\mathbb{E}$  is one base of the vector space  $\mathbb{F}_2^n$ . As it is a base of every vector space over a field  $K$ .

### 3.5.4 Index

We call  $i = 1, \dots, n$  the index of the unit vector  $e_i$  in the base.

### 3.5.5 Norm via Hamming weight

The addition in the field  $\mathbb{F}_2$  is modulo 2. Hence each sum in  $\mathbb{F}_2$  evaluates to either 0 or 1. In particular, for  $x \in \mathbb{F}_2^n$  we have

$$\left( \sum_{i=1}^n x_i \right) \bmod 2 = \begin{cases} 1, & |\{i \in \{1, \dots, n\} : x_i = 1\}| \text{ is odd,} \\ 0, & |\{i \in \{1, \dots, n\} : x_i = 1\}| \text{ is even.} \end{cases} \quad (29)$$

From this it follows that we cannot directly use the usual norm definitions (as over  $\mathbb{R}$ ) to measure vector length in  $\mathbb{F}_2^n$ .

In coding theory [4, 17] the *Hamming weight*  $w_H$  (number of 1-entries) of  $x \in \{0, 1\}^n$  is defined as:

$$w_H : \mathbb{F}_2^n \rightarrow \mathbb{Z}, \quad w_H(x) = |\{i \in \{1, \dots, n\} : x_i = 1\}|, \quad (30)$$

and the associated *Hamming distance*  $d_H$  is:

$$d_H : \mathbb{F}_2^n \times \mathbb{F}_2^n \rightarrow \mathbb{Z}, \quad d_H(x, y) = w_H(x - y). \quad (31)$$

If we first apply operation  $||$  we map a vector from  $\mathbb{F}_2^n \rightarrow \mathbb{R}^n$ , by embedding  $\mathbb{F}_2^n = \{0, 1\}^n \subset \mathbb{R}^n$ . On vectors in  $\mathbb{R}^n$  norms are defined and we get:

$$\|x\|_1 = \sum_{i=1}^n |x_i| = w_H(x). \quad (32)$$

So the  $l_1$ -norm on  $\mathbb{R}^n$  see [3, 11] equals the Hamming weight on bit vectors. This norm is sometimes called absolute-value norm [20]. The value of the norm is an element of the set  $\{0, 1, \dots, n\} \subset \mathbb{Z} \subset \mathbb{R}$ .

In the programming languages C the function is named `popcount()` and in the language Go it is the function `OnesCount(uint x) uint` in package `math/bits`.

Obviously this norm satisfies the axioms of a norm:

#### Subadditivity / Triangle inequality

$$w_H(x + y) \leq w_H(x) + w_H(y) \quad \forall x, y \in \mathbb{F}_2^n,$$

#### Absolute homogeneity

$$w_H(s \cdot x) = s \cdot w_H(x) \quad \forall s \in \mathbb{F}_2, x \in \mathbb{F}_2^n,$$

#### Positive definiteness

$$w_H(x) = 0 \Rightarrow x = \mathbf{0}.$$

Please note the norm of the unit vectors  $\|e_i\| = 1, \quad \forall i \in \{1, \dots, n\}$ .

### 3.5.6 Scalar product

We define the scalar product, dot product [2, 13] or inner product of two vectors as given in equation (33):

$$\langle \cdot, \cdot \rangle : \mathbb{F}_2^n \times \mathbb{F}_2^n \rightarrow \mathbb{R}, \quad \langle x, y \rangle = \sum_{i=1}^n |x_i \cdot y_i| = \|x \& y\|_1 = w_H(x \& y), \quad (33)$$

The obtained value equals the standard scalar product of  $x, y \in \{0, 1\}^n \subset \mathbb{R}^n$

$$\langle \cdot, \cdot \rangle : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}, \quad \langle x, y \rangle = \sum_{i=1}^n x_i \cdot y_i, \quad (34)$$

We can easily verify by computation the properties of the scalar product:

### Distributivity

$$\langle x \oplus x', y \rangle = \langle x, y \rangle + \langle x', y \rangle,$$

$$\langle x, y \oplus y' \rangle = \langle x, y \rangle + \langle x, y' \rangle,$$

$$\langle \lambda x, y \rangle = \lambda \langle x, y \rangle,$$

$$\langle x, \lambda y \rangle = \lambda \langle x, y \rangle,$$

### Commutativity

$$\langle x, y \rangle = \langle y, x \rangle,$$

### Positive definiteness

$$\langle x, x \rangle \geq 0, \text{ with } \langle x, x \rangle = 0 \text{ only if } x = 0,$$

### Orthogonality

$$\langle x, y \rangle = 0, \text{ we say the two vectors are orthogonal.}$$

In coding theory [9] the orthogonal bit vector is called the dual code.

### 3.5.7 Orthonormal Basis

With the properties of the norm and the scalar product it follows  $\mathbb{E}$  is an orthonormal base, and this is the only orthonormal base of  $\mathbb{F}_2^n$ .

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