Dynamics of a single Vortex in the Thomas-Fermi Profile

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Abstract

Here we made a study and implementation of suitable phases for a better description of physical systems using the variational method of variable parameters. Generally the condensate phases must be polynomials of degree equal to or greater than two. Thus taking a new appropriate phase was possible to calculate the collective modes of a condensate containing a single vortex along of the axial axis (cylindrical symmetry), as well as its free expansion. As result, a degeneracy was opened on each oscillatory mode already known, monopole (breathing) and quadrupole, being associated to vortex core oscillation in or out phase with respect to the axial radius. There is always at least one degenerate mode in the system, when excited presents a collective mode which is the sum of two modes with the same energy.

Contents

I. INTRODUCING THE PROBLEM

We are interested in observing the dynamics of a trapped condensate containing a line vortex at its center, i.e., obtaining the collective oscillation modes of the system. The interest in this problem come from the fact that these oscillation can be measured when the atomic cloud is moved out its equilibrium point using the Feshbach resonance to modulate the scattering length. From the theoretical point of view we want to know how the vortex core oscillates with respect to the collective modes in cylindrical coordinates (breathing mode and quadrupole mode). The quadrupole mode is when the components of condensate oscillate out phase, being this one the mode with smaller frequency of oscillation. However, the mode which requires more energy to be excited is the breathing mode, because the high density of atomic cloud imposes a greater resistance to go out of balance.

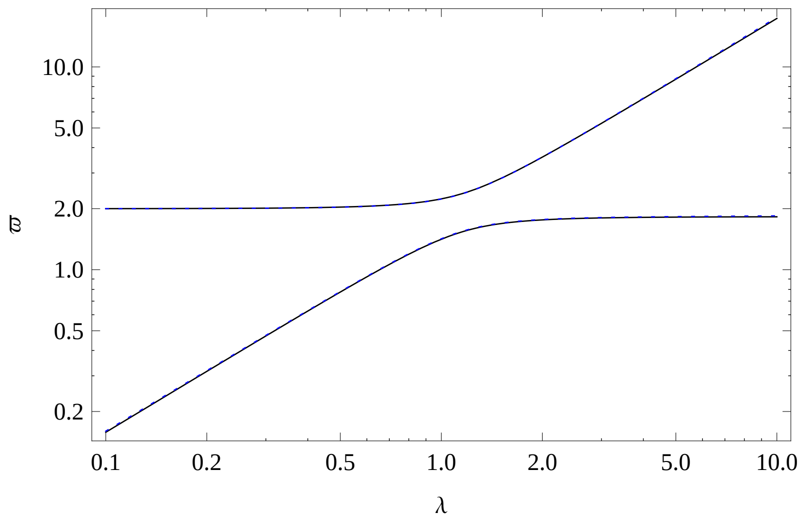

Figure 1: The upper lines relate to the frequencies of the breathing mode as a function of the anisotropy of the harmonic potential (trap), whereas the lower lines represent the frequencies of the quadrupole mode. The solid (black) lines were used to the case of vortex-free TF-profile, and the dotted (blue) lines describe the Gaussian approximation for a profile with a single vortex. This approximation becomes poor above $\lambda \approx 1.5$ because the Gaussian line crosses the TF line for breathing mode, although it is unnoticeable on graph. Necessarily the presence of the vortex must decrease the frequency of breathing mode (does not agree with the region of oblate condensates), and increase the frequency of quadrupole mode (in agreement), being these the shift of the frequencies. Note that ϖ is normalized by the frequency of the radial direction ω_ρ.

Preliminary calculations using a Gaussian Ansatz, which does not take into account the size of the vortex core, shows a small shift in frequencies (Figure 1). Thus we can expect that the frequency of the monopole (breathing) falls while the quadrupole frequency increases in the presence of the vortex.

To calculate this in a way more consistent with the physical reality, we use then a Thomas-Fermi (TF) Ansatz,

$$\psi\left(\rho,\varphi,z,t\right)=A\left(t\right)\left[\frac{\rho^2}{\rho^2+\xi\left(t\right)^2}\right]^{\frac{1}{2}}e^{i\varphi}\sqrt{1-\frac{\rho^2}{R_\rho\left(t\right)^2}-\frac{z^2}{R_z\left(t\right)^2}}\exp\left[iB_\rho\left(t\right)\frac{\rho^2}{2}+iB_z\left(t\right)\frac{z^2}{2}\right], \tag{1}$$

and calculate the equation of motions for the parameters of the condensate. Continuing the calculations, these motion equations were linearized thereby obtaining a dispersion relation that resulted in negative frequencies which made no sense. Therefore, we had the linearized equations given by

$$M\ddot{\delta}+V\delta=0, \tag{2}$$

and its dispersion relation

$$\varpi^2 = \det\left(M^{-1}V\right) = \det M^{-1}\det V = \frac{\det V}{\det M}. \tag{3}$$

In order to have positive frequencies and physical meaning, both determinants above must necessarily be positive. In this case was not happening. Here we have that $\det M^{-1} < 0$, which indicates there is something wrong with our Ansatz. To reverse this problem we need to change the phase of this Ansatz. Then we will choose a suitable phase subsequently a study about the phase, which is done in the next section.

II. ENGINEERING PHASE

To evaluate possible phases and subsequently build our wave function, we use the continuity equation

$$\frac{\partial n}{\partial t} + \nabla\cdot(n\mathbf{v}) = 0 \tag{4}$$

$$\frac{\partial n}{\partial t} + \frac{\hbar}{m}\left[(\nabla n)(\nabla S) + n\nabla^2 S\right] = 0. \tag{5}$$

In this way, by introducing the density of condensate inside of the above equation we can calculate its phase. Firstly we have calculated the phase for a condensate on lab referential, i.e. the center of mass is relevant. So this density has following form

$$n(r,t) = A(t)\left\{1 - \left[\frac{r-r_0(t)}{R(t)}\right]^2\right\}, \tag{6}$$

where r_0 is the center of mass displacement from the center of the trapping potential. We obtained the phase equation given by

$$\begin{aligned}&\left(\frac{r-r_0}{R}\right)^2\left[\frac{m}{\hbar}\frac{\dot{R}}{R} + (r-r_0)^{-1}\left(\frac{m}{\hbar}\dot{r_0} - \frac{\partial S}{\partial r}\right)\right]\\ &+\frac{1}{2}\left[1-\left(\frac{r-r_0}{R}\right)^2\right]\left(\frac{m}{\hbar}\frac{\dot{A}}{A} + \frac{\partial^2 S}{\partial r^2} + \frac{1}{r}\frac{\partial S}{\partial r}\right) = 0.\end{aligned} \tag{7}$$

Since we are interested in a phase which describes the behavior at the edge of the condensate, as the gradient at its center has no significant variations. Then we can simplify the above expression when we are looking at radius of the condensate, i.e., $1-(r-r_0)^2/R^2 = 0$.

$$\frac{\partial S}{\partial r} = \frac{m}{\hbar}\frac{\dot{R}}{R}(r-r_0) + \frac{m}{\hbar}\dot{r_0} \tag{8}$$

$$S(r) = \frac{m}{\hbar}\left(\dot{r_0} - r_0\frac{\dot{R}}{R}\right)r + \frac{m}{\hbar}\frac{\dot{R}}{R}\frac{r^2}{2} \tag{9}$$

$$= \left(\frac{m}{\hbar}\dot{r_0} - r_0 B_r\right)r + B_r\frac{r^2}{2} \tag{10}$$

Hence we recovered the phases used in [1] where we notice that the radius of the condensate has its velocity field proportional to the square of respective coordinate. Being the displacement r_0 proportional to its coordinate, remembering that the movement of the center of mass is uncoupled with the dynamics of the condensate radius. So that the phase of condensate must have $b(t)r^2/2$ to describe its radial velocity (curvature of the wave function), and a term $a(t)r$ which depicting the movement of the center of mass (slope of the wave function).

Now we do the same calculation for a TF-density containing a single central vortex along the z axis,

$$n\left(\rho,\varphi,z,t\right)=A(t)e^{i\ell\varphi}\left\{1-\left[\frac{\xi(t)}{\rho}\right]^2-\left[\frac{\rho}{R_\rho(t)}\right]^2-\left[\frac{z}{R_z(t)}\right]^2\right\}, \tag{11}$$

whose its continuity equation is being

$$\begin{aligned}&\frac{m}{\hbar}\left(-\frac{\xi\dot{\xi}}{\rho^2}+\frac{\dot{R}_\rho}{R_\rho^3}\rho^2+\frac{\dot{R}_z}{R_z^3}z^2\right)+\left(\frac{\xi^2}{\rho^3}-\frac{\rho}{R_\rho^2}\right)\frac{\partial S}{\partial\rho}-\frac{z}{R_z^2}\frac{\partial S}{\partial z}\\&+\frac{1}{2}\left(1-\frac{\xi^2}{\rho^2}-\frac{\rho^2}{R_\rho^2}-\frac{z^2}{R_z^2}\right)\left(\frac{m}{\hbar}\frac{\dot{A}}{A}+\frac{\partial^2 S}{\partial\rho^2}+\frac{\partial^2 S}{\partial z^2}+\frac{1}{\rho}\frac{\partial S}{\partial\rho}\right)=0.\end{aligned} \tag{12}$$

Using the same previous argument to this case, $1-\frac{\xi^2}{\rho^2}-\frac{\rho^2}{R_\rho^2}-\frac{z^2}{R_z^2}=0$ (if we were dealing with the limit of ideal gas, where the density is a Gaussian function, the bound conditions would be $\rho\to 0$ e $\rho\to\infty$).

$$\left(-\frac{\xi^2}{\rho^3}+\frac{\rho}{R_\rho^2}\right)\frac{\partial S}{\partial\rho}+\frac{z}{R_z^2}\frac{\partial S}{\partial z}-\frac{m}{\hbar}\left(-\frac{\xi\dot{\xi}}{\rho^2}+\frac{\dot{R}_\rho}{R_\rho^3}\rho^2+\frac{\dot{R}_z}{R_z^3}z^2\right)=0 \tag{13}$$

Solving the equation (13) we obtain

$$S\left(\rho,z\right)=\frac{m}{4\hbar}\left(\xi\dot{R}_\rho-R_\rho\dot{\xi}\right)\left[\ln\left(\frac{R_\rho\xi-\rho^2}{R_\rho\xi+\rho^2}\right)+\ln\left(-1\right)\right]+\frac{m}{\hbar}\frac{\dot{R}_\rho}{R_\rho}\frac{\rho^2}{2}+\frac{m}{\hbar}\frac{\dot{R}_z}{R_z}\frac{z^2}{2} \tag{14}$$

$$=\frac{m}{4\hbar}\left(\xi\dot{R}_\rho-R_\rho\dot{\xi}\right)\left[\ln\left(\frac{R_\rho\xi-\rho^2}{R_\rho\xi+\rho^2}\right)+i\pi\right]+B_\rho\frac{\rho^2}{2}+B_z\frac{z^2}{2}. \tag{15}$$

$$S\left(\rho,z\right)\simeq\frac{m}{2\hbar}\left(\xi\dot{R}_\rho-R_\rho\dot{\xi}\right)\left[i\pi-\frac{R\xi}{\rho^2}+O\left(\xi^3\right)\right]+B_\rho\frac{\rho^2}{2}+B_z\frac{z^2}{2} \tag{16}$$

$$S\left(\rho,z\right)\simeq\frac{m}{2\hbar}\left(\xi\dot{R}_\rho-R_\rho\dot{\xi}\right)\left[\frac{i\pi}{2}-\frac{\rho^2}{R\xi}+O\left(R^{-3}\right)\right]+B_\rho\frac{\rho^2}{2}+B_z\frac{z^2}{2} \tag{17}$$

When it is used a phase proportional to ρ^{-2} (Figure 2) the result is just available for two-dimensional framework. As we are looking for a three-dimensional description for the issue in question, thus this phase presents convergence problems to evaluate the Lagrangian. With a $C(t)\rho^{-1}$ type phase imaginary terms appears in Lagrangian and consequently in motion equations, which has no physical meaning. The phase $C(t)\ln\rho$ is not trivial to work for with a TF-Ansatz where its complication is on evaluate the temporal part of the Lagrangian

$$L_{time}=\frac{i\hbar}{2}\int d^3r\left[\psi^*\left(\mathbf{r},t\right)\frac{\partial\psi\left(\mathbf{r},t\right)}{\partial t}+c.c.\right]. \tag{18}$$

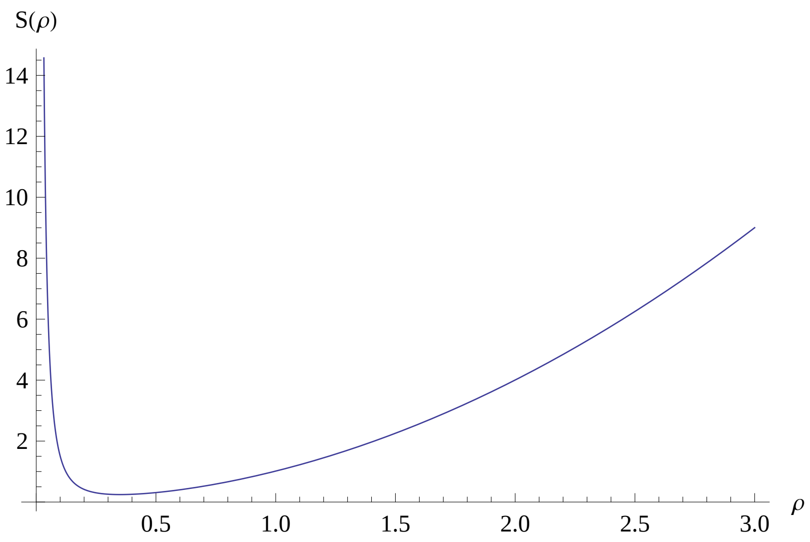

Figure 2: Outline of (16) in the radial direction.

Therefore the phase must be a polynomial of fourth degree,

$$S(\rho, z, t) = B_\rho(t)\frac{\rho^2}{2} + C(t)\frac{\rho^4}{4} + B_z(t)\frac{z^2}{2}, \tag{19}$$

describing approximately the radial part of (15).

III. CALCULATING THE LAGRANGIAN

Completed the study of phase we have implemented the polynomial phase (19) in our TF-Ansatz which is as follows:

$$\psi(\mathbf{r}, t) = \sqrt{\frac{N}{\pi R_\rho(t)^2 R_z(t) A_0(t)}} \left[\frac{\rho^2}{\rho^2 + \xi(t)^2}\right]^{\frac{1}{2}} e^{i\ell\varphi} \sqrt{1 - \frac{\rho^2}{R_\rho(t)^2} - \frac{z^2}{R_z(t)^2}} \\ \times \exp\left[iB_\rho(t)\frac{\rho^2}{2} + iC(t)\frac{\rho^4}{4} + iB_z(t)\frac{z^2}{2}\right], \tag{20}$$

where A_0 (All functions $A_i(\alpha)$ are calculated in Appendix B), which is determined by the normalization condition

$$N = \int \psi(\mathbf{r}, t)\, d\mathbf{r}, \tag{21}$$

is a function of $\alpha(t) \equiv \xi(t)/R_\rho(t)$, being $\xi(t)$ the vortex core, $R_\rho(t)$ the radius of the condensate in $\hat{\rho}$ direction and $R_z(t)$ is the condensate radius in axial direction ($\hat{z}$). This wave function has the integration limits determined by $1 - \frac{\rho^2}{R_\rho^2} - \frac{z^2}{R_z^2} \geq 0$, i.e., the wave function is approximately an inverted parabola (TF-shape), unless the central vortex, whose the extension in the region of negative numbers is null. By dealing with a Bose-Einstein condensate which has a density given by a TF-shape, the trapping potential in cylindrical coordinates is described by $V(\rho, z) =$

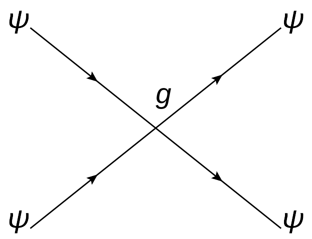

Figure 3: Feynman's diagram of the atom-atom interaction in the Bose-Einstein condensates subject.

$\frac{1}{2}m\omega_\rho^2\left(\rho^2+\lambda^2z^2\right)$ where the anisotropy is $\lambda=\omega_z/\omega_\rho$, that is, for each value of λ the magneto-optical trap (MOT) features different shapes. For λ is equal to unity, the harmonic potential is isotropic. As for $\lambda<1$ it shows itself as a prolate shape where it is elongated along of $\hat{z}$, and being flatted at the opposite direction when $\lambda>1$. The shape of trapping potential obviously sets the condensate format. Note that when we sets a value for λ, obligatorily greater than zero, indeed it means that ω_ρ is fixed and we are setting the value of ω_z which eventually determines the shape of the sample.

The atom-atom interaction is $V_{int}=g\left|\psi\left(\mathbf{r},t\right)\right|^2$ (Figure 3), whose physical meaning is the scattering in s-wave, i.e., low energy scattering. Because of this, the interaction parameter is proportional to the scattering length a_s, which is $g=\frac{4\pi\hbar^2a_s}{m}$.

The Lagrangian density,

$$\mathcal{L}=\mathcal{L}_{time}+\mathcal{L}_{kin}+\mathcal{L}_{pot}+\mathcal{L}_{int}, \tag{22}$$

has four components: temporal, kinetic, potential and interaction potential. Similarly we can split our Lagrangian,

$$L=\int\mathcal{L}d\mathbf{r}. \tag{23}$$

Following this way we have computed all parts of the Lagrangian.

Temporal term:

$$\begin{aligned}\mathcal{L}_{time}=&\frac{i\hbar}{2}\left[\psi^*\left(\mathbf{r},t\right)\frac{\partial\psi\left(\mathbf{r},t\right)}{\partial t}-\psi\left(\mathbf{r},t\right)\frac{\partial\psi^*\left(\mathbf{r},t\right)}{\partial t}\right]\\ L_{time}=&-\frac{N\hbar}{2}\left(D_1\dot{B}_\rho R_\rho^2+D_2\dot{B}_zR_z^2+\frac{1}{2}D_3\dot{C}R_\rho^4\right).\end{aligned} \tag{24}$$

Kinetic term:

$$\begin{aligned}\mathcal{L}_{kin}=&-\frac{\hbar^2}{2m}\left[\nabla\psi^*\left(\mathbf{r},t\right)\right]\left[\nabla\psi\left(\mathbf{r},t\right)\right]\\ L_{kin}=&-\frac{N\hbar^2}{2m}\left[D_1B_\rho^2R_\rho^2+D_2B_z^2R_z^2+2D_3B_\rho CR_\rho^4+R_\rho^{-2}\left(\ell^2D_4+D_5\right)+D_6C^2R_\rho^6\right].\end{aligned} \tag{25}$$

Potential term (harmonic trap):

$$\begin{aligned}\mathcal{L}_{pot} =& -\frac{1}{2}m\omega_\rho^2\left(\rho^2+\lambda^2z^2\right)\psi^*\left(\mathbf{r},t\right)\psi\left(\mathbf{r},t\right)\\ L_{pot} =& -\frac{N}{2}m\omega_\rho^2\left(D_1R_\rho^2+\lambda^2D_2R_z^2\right).\end{aligned} \tag{26}$$

Atom-atom interaction potential term:

$$\begin{aligned}\mathcal{L}_{int} =& -\frac{g}{2}\left[\psi^*\left(\mathbf{r},t\right)\psi\left(\mathbf{r},t\right)\right]^2\\ L_{int} =& -\frac{N^2gD_7}{2\pi R_\rho^2R_z}.\end{aligned} \tag{27}$$

The functions of α, D_i, are: $D_i = A_i/A_0$, with exception of $D_7 = A_7/A_0^2$. So the Lagrangian is the sum of each contribution, $L = L_{time} + L_{kin} + L_{pot} + L_{int}$. For simplicity we can scale the parameters of Lagrangian to make them dimensionless, which are the following changes:

$$\begin{aligned}R_\rho(t) \to& a_{osc}r_\rho(t),\\ R_z(t) \to& a_{osc}r_z(t),\\ \xi(t) \to& a_{osc}r_\xi(t),\\ B_\rho(t) \to& a_{osc}^{-2}\beta_\rho(t),\\ B_z(t) \to& a_{osc}^{-2}\beta_z(t),\\ C(t) \to& a_{osc}^{-4}\zeta(t)\end{aligned}$$

and

$$t \to \omega_\rho^{-1}\tau;$$

where the harmonic oscillator length is $a_{osc} = \sqrt{\hbar/m\omega_\rho}$, and the dimensionless parameter of interaction is $\gamma = Na_s/a_{osc}$. Thus the Lagrangian becomes

$$\begin{aligned}L =& -N\hbar\omega_\rho\Big[D_1r_\rho^2\left(\dot{\beta}_\rho+\beta_\rho^2+1\right)+D_2r_z^2\left(\dot{\beta}_z+\beta_z^2+\lambda^2\right)\\ &+D_3r_\rho^4\left(\frac{1}{2}\dot{\zeta}+2\beta_\rho\zeta\right)+r_\rho^{-2}\left(\ell^2D_4+D_5\right)+D_6\zeta^2r_\rho^6+\frac{4D_7\gamma}{r_\rho^2r_z}\Big].\end{aligned} \tag{28}$$

IV. MOTION EQUATIONS

Taking the Euler-Lagrange equations,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{q}_i}\right)-\frac{\partial L}{\partial q_i} = 0, \tag{29}$$

for each one of the six variational parameters from Lagrangian (28) hence six differential equations:

$$\beta_\rho - \frac{\dot{r_\rho}}{r_\rho} - \frac{D_1'\dot{\alpha}}{2D_1} + \frac{D_3 r_\rho^2 \zeta}{D_1} = 0 \tag{30}$$

$$\beta_z - \frac{\dot{r_z}}{r_z} - \frac{D_2'\dot{\alpha}}{2D_2} = 0 \tag{31}$$

$$\zeta - \frac{D_3\dot{r_\rho}}{D_6 r_\rho} - \frac{D_3\dot{\alpha}}{4D_6 r_\rho^2} + \frac{D_3\beta_\rho}{D_6 r_\rho^2} = 0 \tag{32}$$

$$\begin{aligned} D_1 r_\rho \left(\dot{\beta}_\rho + \beta_\rho^2 + 1\right) + D_3 r_\rho^3 \left(\dot{\zeta} + 4\beta_\rho\zeta\right) - \\ r_\rho^{-3}\left(\ell^2 D_4 + D_5\right) + 3D_6\zeta^2 r_\rho^5 - \frac{4D_7\gamma}{r_\rho^3 r_z} = 0 \end{aligned} \tag{33}$$

$$D_2 r_z \left(\dot{\beta}_z + \beta_z^2 + \lambda^2\right) - \frac{2D_{67}\gamma}{r_\rho^2 r_z^2} = 0 \tag{34}$$

$$\begin{aligned} D_1' r_\rho^2 \left(\dot{\beta}_\rho + \beta_\rho^2 + 1\right) + D_2' r_z^2 \left(\dot{\beta}_z + \beta_z^2 + \lambda^2\right) + D_3' r_\rho^4 \left(\frac{1}{2}\dot{\zeta} + 2\beta_\rho\zeta\right) + \\ r_\rho^{-2}\left(\ell^2 D_4' + D_5'\right) + D_6'\zeta^2 r_\rho^6 - \frac{4D_7'\gamma}{r_\rho^2 r_z} = 0. \end{aligned} \tag{35}$$

Solving the equations for the parameters arising from the phase, which are related with the variation of the curvature of the wave function, we have:

$$\beta_\rho = \frac{\dot{r_\rho}}{r_\rho} + F_1\dot{\alpha}, \tag{36}$$

$$\beta_z = \frac{\dot{r_z}}{r_z} + F_2\dot{\alpha} \tag{37}$$

and

$$\zeta = F_3 \frac{\dot{\alpha}}{r_\rho^2}; \tag{38}$$

where

$$F_1 = \frac{D_3' D_3 - 2D_1' D_6}{4\left(D_3^2 - D_1 D_6\right)}, \tag{39}$$

$$F_2 = \frac{D_2'}{2D_2} \tag{40}$$

and

$$F_3 = \frac{2D_1' D_3 - D_1 D_3'}{4\left(D_3^2 - D_1 D_6\right)}. \tag{41}$$

Replacing (36), (37) and (38) in equations (33), (34) and (35), we reduce our six coupled equations in only three equations, also coupled, which are given by:

$$D_1\left(\ddot{r_\rho}+r_\rho\right)+G_1 r_\rho\ddot{\alpha}+G_2 r_\rho\dot{\alpha}^2+G_3\dot{r_\rho}\dot{\alpha}-\frac{G_4}{r_\rho^3}-\frac{4D_7\gamma}{r_\rho^3 r_z}=0 \tag{42}$$

$$D_2\left(\ddot{r_z}+\lambda^2 r_z\right)+G_5 r_z\ddot{\alpha}+G_6 r_z\dot{\alpha}^2+G_7\dot{r_z}\dot{\alpha}-\frac{2D_6\gamma}{r_\rho^2 r_z^2}=0 \tag{43}$$

$$\begin{aligned} D_1' r_\rho\left(\ddot{r_\rho}+r_\rho\right)+D_2' r_z\left(\ddot{r_z}+\lambda^2 r_z\right)+\left(G_8 r_\rho^2+G_9 r_z^2\right)\ddot{\alpha}+\left(G_{10} r_\rho^2+G_{11} r_z^2\right)\dot{\alpha}^2 \\ +\left(G_{12} r_\rho\dot{r_\rho}+G_{13} r_z\dot{r_z}\right)\dot{\alpha}+\frac{G_{14}}{r_\rho^2}+\frac{4D_6'\gamma}{r_\rho^2 r_z}=0, \end{aligned} \tag{44}$$

with

$$G_1 = D_1 F_1 + D_3 F_3 \tag{45}$$

$$G_2 = D_1\left(F_1^2+F_1'\right)+D_3\left(4F_1F_3+F_3'\right)+3D_6F_3^2 \tag{46}$$

$$G_3 = 2\left(D_1F_1+D_3F_3\right)=2G_1 \tag{47}$$

$$G_4 = \ell^2 D_4 + D_5 \tag{48}$$

$$G_5 = D_2F_2 \tag{49}$$

$$G_6 = D_2\left(F_2^2+F_2'\right) \tag{50}$$

$$G_7 = 2D_2F_2 = 2G_5 \tag{51}$$

$$G_8 = D_1'F_1+\frac{1}{2}D_3'F_3 \tag{52}$$

$$G_9 = D_2'F_2 \tag{53}$$

$$G_{10} = D_1'\left(F_1^2+F_1'\right)+D_3'\left(\frac{1}{2}F_3'+2F_1F_3\right)+D_6'F_3^2 \tag{54}$$

$$G_{11} = D_2'\left(F_2^2+F_2'\right) \tag{55}$$

$$G_{12} = 2D_1'F_1+D_3'F_3 \tag{56}$$

$$G_{13} = 2D_2'F_2 = 2G_9 \tag{57}$$

$$G_{14} = \ell^2 D_4' + D_5'. \tag{58}$$

The terms $D_1 r_\rho$, $D_2\lambda^2 r_z$, $D_1' r_\rho^2$ and $D_2' r_z^2$ are responsible for condensate trapping, if they are thrown away we have the free expansion equations. The parameter γ indicates the elements that give the contribution of the atomic interaction potential, while the fractions proportional to r_ρ^{-2} and r_ρ^{-3} are result of the kinetic contribution due to the presence of the vortex, having the effect of adding a quantum pressure. The remaining factors represent the coupling effect between the radii of the condensate and the vortex core.

Making the velocities ($\dot{r_\rho}$, $\dot{r_z}$ e $\dot{\alpha}$) and accelerations ($\ddot{r_\rho}$, $\ddot{r_z}$ e $\ddot{\alpha}$) equal to zero then we have the equations for the stationary solution.

$$D_1\rho_0 = \frac{G_4}{\rho_0^3} + \frac{4D_7\gamma}{\rho_0^3 z_0} \tag{59}$$

$$D_2\lambda^2 z_0 = \frac{2D_7\gamma}{\rho_0^2 z_0^2} \tag{60}$$

$$D_1'\rho_0^2 + D_2'\lambda^2 z_0^2 = -\frac{G_{14}}{\rho_0^3} - \frac{4D_7'\gamma}{\rho_0^2 z_0} \tag{61}$$

Remembering that now the α functions become functions of α_0, which is time-independent. We improve the Newton numerical method to solve the coupled stationary equations above.

V. COLLECTIVE MODES

When the vortex is introduced on system it breaks the symmetry of the phase which leads to open a degenerescence onto collective modes of the condensate. Thus we expect to find two modes equivalent to monopole (breathing), and two others equivalent to quadrupole. To obtain from the motion equations a response of the condensate subjected to an external perturbation, we have linearized these equations which is equivalent to calculate the Bogoliubov spectrum. Therefore, we say that the condensate is slightly displaced from its equilibrium position, which in mathematical language is $r_\rho(t) \to \rho_0 + \delta_\rho(t)$, $r_z(t) \to z_0 + \delta_z(t)$ e $\alpha(t) \to \alpha_0 + \delta_\alpha(t)$. Following these changes and neglecting second order terms we have:

$$\begin{aligned} D_1\ddot{\delta}_\rho + G_1\rho_0\ddot{\delta}_\alpha + \left(D_1 + \frac{3G_4}{\rho_0^4} + \frac{12D_7\gamma}{\rho_0^4 z_0}\right)\delta_\rho + \\ \left(\frac{4D_7\gamma}{\rho_0^3 z_0^2}\right)\delta_z + \left(D_1'\rho_0 - \frac{G_4'}{\rho_0^3} - \frac{4D_7'\gamma}{\rho_0^3 z_0}\right)\delta_\alpha = 0 \end{aligned} \tag{62}$$

$$\begin{aligned} D_2\ddot{\delta}_z + G_5 z_0\ddot{\delta}_\alpha + \left(\frac{4D_7\gamma}{\rho_0^3 z_0^2}\right)\delta_\rho + \left(D_2\lambda^2 + \frac{4D_7\gamma}{\rho_0^2 z_0^3}\right)\delta_z + \\ \left(D_2'\lambda^2 z_0 - \frac{2D_7'\gamma}{\rho_0^2 z_0^2}\right)\delta_\alpha = 0 \end{aligned} \tag{63}$$

$$\begin{aligned} D_1'\rho_0\ddot{\delta}_\rho + D_2' z_0\ddot{\delta}_z + \left(G_8\rho_0^2 + G_9 z_0^2\right)\ddot{\delta}_\alpha + \left(2D_1'\rho_0 - \frac{2G_{14}}{\rho_0^3} - \frac{8D_7'\gamma}{\rho_0^3 z_0}\right)\delta_\rho + \\ \left(2D_4'\lambda^2 z_0 - \frac{4D_7'\gamma}{\rho_0^2 z_0^2}\right)\delta_z + \left(D_1''\rho_0^2 + D_2''\lambda^2 z_0^2 + \frac{G_{14}'}{\rho_0^2} + \frac{4D_7''\gamma}{\rho_0^2 z_0}\right)\delta_\alpha = 0. \end{aligned} \tag{64}$$

The zeroth order terms form the stationary solution, and they vanish. As mentioned in the previous section, the functions of α become functions of α_0 which is time-independent. For

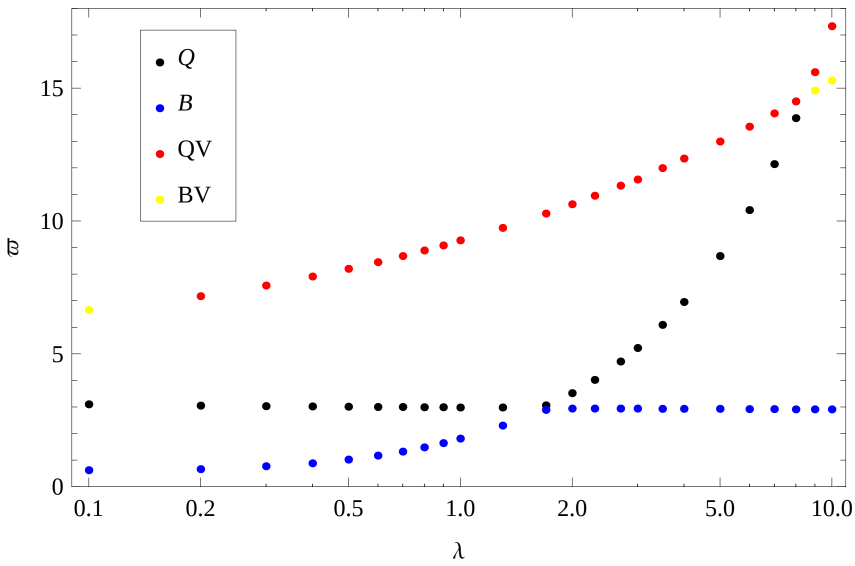

Figure 4: The frequencies of oscillation modes of a condensate containing a single vortex at its center.

convenience we're changing to the matrix notation,

$$M\ddot{\delta} + V\delta = 0 \tag{65}$$

$$\begin{pmatrix} m_{11} & 0 & m_{13} \\ 0 & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{pmatrix} \begin{pmatrix} \ddot{\delta}_\rho \\ \ddot{\delta}_z \\ \ddot{\delta}_\alpha \end{pmatrix} + \begin{pmatrix} v_{11} & v_{12} & v_{13} \\ v_{21} & v_{22} & v_{23} \\ v_{31} & v_{32} & v_{33} \end{pmatrix} \begin{pmatrix} \delta_\rho \\ \delta_z \\ \delta_\alpha \end{pmatrix} = 0. \tag{66}$$

Calculating the dispersion relation,

$$\det\left(M^{-1}V - \varpi^2 I\right) = 0, \tag{67}$$

this results in the frequency of the collective modes of oscillation. The above determinant is a cubic function of ϖ^2, thus

$$-\varpi^6 + b\varpi^4 - c\varpi^2 + d = 0, \tag{68}$$

whose solution is in the Appendix C. Now the determinants $\det M$ and $\det V$ are, both, positive for $\ell = 1$. Meaning that we are in the lower energy state for the case of a central vortex in a Bose-Einstein condensate therefore the frequencies ϖ^2 shall also be positive. According to this result we have three frequencies which can not be linked to only a single mode oscillation, namely we have three frequencies and four modes of oscillation in total, of which only three modes can be observed according to the anisotropy of harmonic potential as it is shown in Figure 4. From the four modes, two of them represent the monopolo oscillation, and the other two represent the quadrupole oscillation for the atomic cloud. The first two are when the core of the vortex is in phase with the direction of the radius ρ (B and Q in Figure 4), and the last

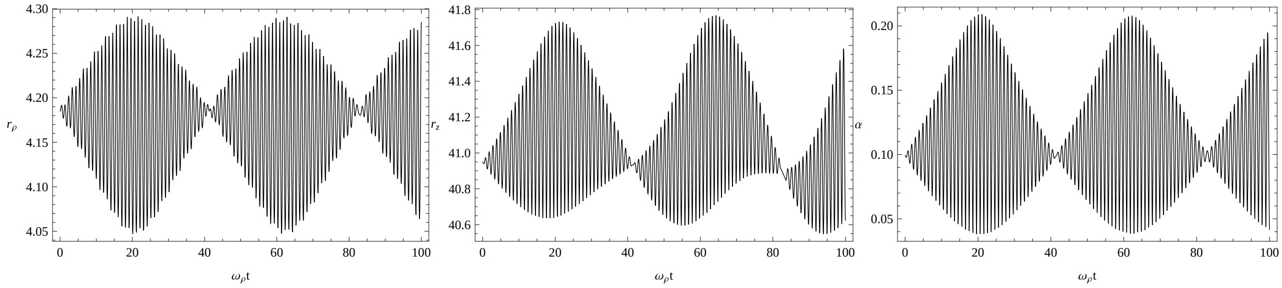

Figure 5: Collective excitation for $\gamma_0 = 800$, $\delta\gamma = 24$, $\lambda = 0.1$ e $\Omega = 6.1$.

two have the core of the vortex in phase with the axial radius (B and Q in Figure 4). Note that for $\lambda \approx 1.9$ the monopole and quadrupole are degenerate, as well as three another modes (Q, QV and BV) are degenerate in $\lambda \approx 8.5$. Paying greater attention in $\lambda < 8.5$ where the modes BV and QV are always degenerate, above this value Q and QV that become degenerate.

VI. MODULATING THE SCATTERING LENGTH

The mechanisms of excitation of collective modes, which were described in the previous section, is via a modulation of scattering length, i.e.

$$a_s(t) = a_0 + \delta a \cos(\Omega t), \tag{69}$$

this is equivalent to do $\gamma \to \gamma(t)$ with the same form of $a_s(t)$,

$$\gamma(t) = \gamma_0 + \delta\gamma \cos(\Omega t). \tag{70}$$

Where γ_0 is the average of the interaction parameter, $\delta\gamma$ is the modulation amplitude and the frequency Ω with which excites the modes.

In Figure 5 is excited through of non-linear equations the degenerate state to a condensate with anisotropy $\lambda = 0.1$. It can be seen that its oscillation is similar to a beat wave, because we have three oscillation modes simultaneously, that is, this beat wave is composed of the beat frequency added the predominant frequency (the frequency close to the resonance) and the other two frequencies as well as their harmonics.

VII. FREE EXPANSION

The interest in free expansion comes from the fact that we can only make measurements of condensate after switching off the trapping potential. For this purpose, as previously mentioned,

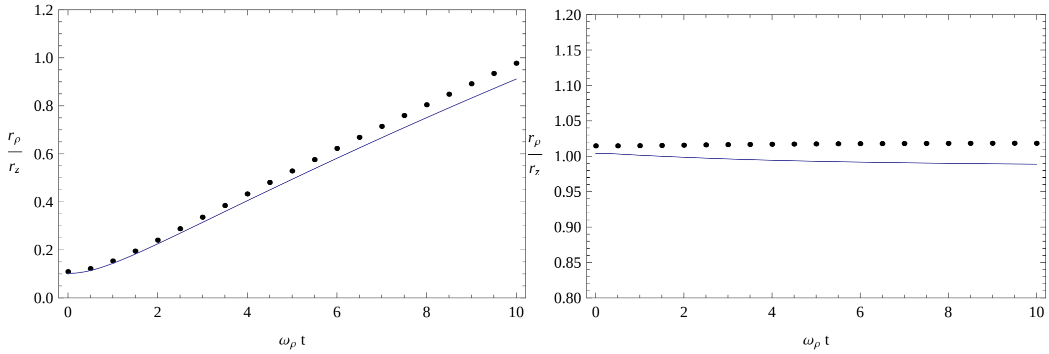

Figure 6: Both graphics show the expansion of aspect ratio to a single vortex condensate ($\ell = 1$), initially trapped in a potential with anisotropy equal to $\lambda = 0.1$ (left) and $\lambda = 1$ (right). We have calculated for an interaction of $\gamma = 800$, which corresponds to the approximate value in the case of a condensate produced with the characteristics of the BEC-I experiment. The dotted line represents the data from numerical simulation, and the solid line is the result by using the variational method.

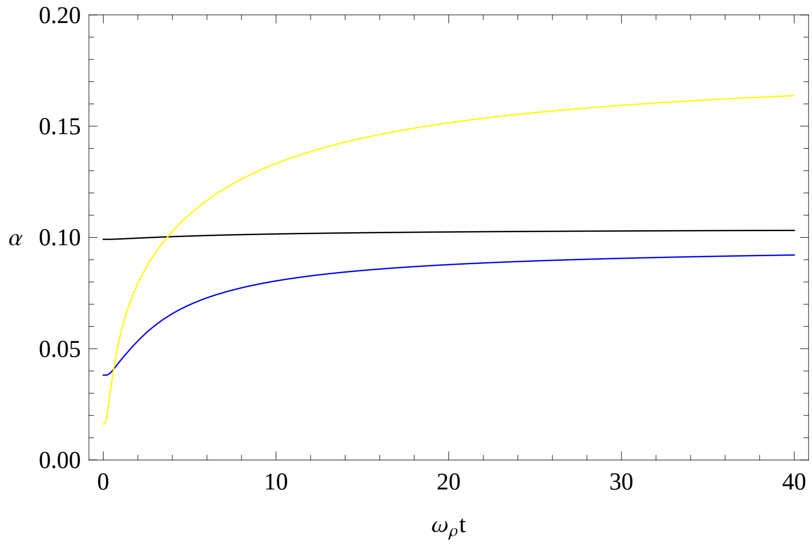

Figure 7: This graphic shows the free expansion of the $\alpha(t) = \xi(t)/R_\rho(t)$, being the black line to a prolate condensate ($\lambda = 0.1$), the blue line to the isotropic case ($\lambda = 1$) and the yellow line to a oblate condensate ($\lambda = 8$).

simply use the equations of motion ((42), (43) e (44)) without the fraction arising from the harmonic potential, it means

$$D_1\ddot{r_\rho}+G_1r_\rho\ddot{\alpha}+G_2r_\rho\dot{\alpha}^2+G_3\dot{r_\rho}\dot{\alpha}-\frac{G_4}{r_\rho^3}-\frac{4D_7\gamma}{r_\rho^3r_z}=0 \tag{71}$$

$$D_2\ddot{r_z}+G_5r_z\ddot{\alpha}+G_6r_z\dot{\alpha}^2+G_7\dot{r_z}\dot{\alpha}-\frac{2D_7\gamma}{r_\rho^2r_z^2}=0 \tag{72}$$

$$D_1'r_\rho\ddot{r_\rho}+D_2'r_z\ddot{r_z}+\left(G_8r_\rho^2+G_9r_z^2\right)\ddot{\alpha}+\left(G_{10}r_\rho^2+G_{11}r_z^2\right)\dot{\alpha}^2 \\ +\left(G_{12}r_\rho\dot{r_\rho}+G_{13}r_z\dot{r_z}\right)\dot{\alpha}+\frac{G_{14}}{r_\rho^3}+\frac{4D_7'\gamma}{r_\rho^2r_z}=0, \tag{73}$$

whose initial conditions are given by the stationary solution. This result agrees with the numerical simulation as can be seen in Figure 6, and the free expansion of the vortex core is given on Figure 7.

Appendix A: Two Thomas-Fermi approaches for the profile containing a single vortex

Two approaches to the density profile can be obtained from the GPE when there is a singularity at the center of the condensate. First we write the time-independent GPE in cylindrical coordinates,

$$\mu\psi\left(\rho,z\right)=-\frac{\hbar^2}{2m}\nabla^2_{\rho,z}\psi\left(\rho,z\right)+\frac{\hbar^2\ell^2}{2m\rho^2}\psi\left(\rho,z\right)+V\left(\rho,z\right)\psi\left(\rho,z\right)+g\left|\psi\left(\rho,z\right)\right|^2\psi\left(\rho,z\right), \tag{A1}$$

where $\nabla^2_{\rho,z}$ is the laplacian just for ρ and z coordinates. Consider then the effective potential much greater than the kinetic energy for a harmonic and isotropic trap,

$$\mu=\frac{\hbar^2\ell^2}{2m\rho^2}+\frac{1}{2}m\omega_\rho^2\left(\rho^2+\lambda^2z^2\right)+gn\left(\rho,z\right), \tag{A2}$$

being $n\left(\rho,z\right)=\left|\psi\left(\rho,z\right)\right|^2$. There is two ways to follow. The fist one is the most usual, and it consists in isolate $n\left(\rho,z\right)$ that gives

$$n\left(\rho,z\right)=\frac{\mu}{g}\left(1-\frac{\xi^2}{\rho^2}-\frac{\rho^2}{R_\rho^2}-\frac{z^2}{R_z^2}\right)\Theta\left(1-\frac{\xi^2}{\rho^2}-\frac{\rho^2}{R_\rho^2}-\frac{z^2}{R_z^2}\right), \tag{A3}$$

with Θ being the step function, and the radii of TF are:

$$R_\rho=\sqrt{\frac{2\mu}{m\omega_\rho^2}},\ R_z=\sqrt{\frac{2\mu}{m\lambda^2\omega_\rho^2}}\text{ and }\xi=\sqrt{\frac{\hbar^2\ell^2}{2m\mu}}. \tag{A4}$$

Now the wave function must have the phase $\ell\varphi$,

$$\psi\left(\rho,\varphi,z\right)=e^{i\ell\varphi}\left(\frac{\mu}{g}\right)^{\frac{1}{2}}\sqrt{1-\frac{\xi^2}{\rho^2}-\frac{\rho^2}{R_\rho^2}-\frac{z^2}{R_z^2}}, \tag{A5}$$

and its limits is described for this region of space: $1-\frac{\xi^2}{\rho^2}-\frac{\rho^2}{R_\rho^2}-\frac{z^2}{R_z^2}\geq 0$.

The second approach is similar, nevertheless it is fairly simple to work when we are considering of a condensate in three dimensions. In (A2) we use relation of the healing length,

$$\frac{\hbar^2\ell^2}{2m\xi^2} = gn\left(\rho, z\right), \tag{A6}$$

rendering it follows:

$$\mu = \frac{1}{2}m\omega_\rho^2\left(\rho^2 + \lambda^2 z^2\right) + \left(\frac{\xi^2}{\rho^2} + 1\right) gn\left(\rho, z\right). \tag{A7}$$

Now we have isolated the density $n\left(\rho, z\right)$, thus

$$n\left(\rho, z\right) = \frac{\mu}{g}\left(\frac{\rho^2}{\rho^2 + \xi^2}\right)\left(1 - \frac{\rho^2}{R_\rho^2} - \frac{z^2}{R_z^2}\right)\Theta\left(1 - \frac{\rho^2}{R_\rho^2} - \frac{z^2}{R_z^2}\right), \tag{A8}$$

where the radii are given as the previous ones, except by the vortex core size which does not depend on μ. In this description the vortex core size is associated with the interaction parameter g, and it may vary with the ratio μ/g. The same phase has to be added when we use the wave function. The second advantage of this second way of wringing a TF-density profile is the fact that we can separate the density of vortex from the back ground density[2]. We shall call

$$n_{TF}\left(\rho, z\right) = \frac{\mu}{g}\left(1 - \frac{\rho^2}{R_\rho^2} - \frac{z^2}{R_z^2}\right)\Theta\left(1 - \frac{\rho^2}{R_\rho^2} - \frac{z^2}{R_z^2}\right) \tag{A9}$$

and now follows

$$\begin{aligned} n\left(\rho, z\right) &= \left(\frac{\rho^2}{\rho^2 + \xi^2}\right) n_{TF}\left(\rho, z\right) \\ &= \left(1 - \frac{\xi^2}{\rho^2 + \xi^2}\right) n_{TF}\left(\rho, z\right) \\ &= n_{bg}\left(\rho, z\right) + n_v\left(\rho, z\right), \end{aligned} \tag{A10}$$

where $n_{bg}\left(\rho, z\right) = n_{TF}\left(\rho, z\right)$ e

$$n_v\left(\rho, z\right) = \left(-\frac{\xi^2}{\rho^2 + \xi^2}\right) n_{TF}\left(\rho, z\right). \tag{A11}$$

The difference between the energies calculated by both approaches is less than 1%.

Appendix B: Functions $A_i(\alpha)$

The functions A_i are results after the TF-functions are integrated, being each one related to one part of the Lagrangian except A_0 which arising from the normalization condition (21). Starting with this one which is more detailed since the procedure is the same for the others mentioned. Doing $\rho \to \rho R_\rho$ and $z \to zR_z$, the normalization integral is

$$A_0(\alpha) = 2\pi \int\int \left(\frac{\rho^2}{\rho^2 + \alpha^2}\right)\left(1 - \rho^2 - z^2\right)\rho d\rho dz. \tag{B1}$$

By changing the variables from cylindrical coordinates to spherical coordinates, $\rho = r\sin\theta$ and $z = r\cos\theta$, the limit of integration is simplified.

$$\begin{aligned} A_0(\alpha) =& 2\pi \int_0^\pi \int_0^1 \left(\frac{r^2\sin^2\theta}{r^2\sin^2\theta+\alpha^2}\right)\left(1-r^2\right) r^2 \sin\theta dr d\theta \\ =& \frac{8}{45}\left[3+20\alpha^2+15\alpha^4-15\alpha^2\left(1+\alpha^2\right)^{\frac{3}{2}}\operatorname{acoth}\left(\sqrt{1+\alpha^2}\right)\right]. \end{aligned} \tag{B2}$$

Note the following relation:

$$\operatorname{acoth}\left(\sqrt{1+\alpha^2}\right) = -\frac{1}{4}\log\left[\frac{\left(1-\sqrt{1+\alpha^2}\right)^2}{\left(1+\sqrt{1+\alpha^2}\right)^2}\right], \tag{B3}$$

which the utility is set $\alpha \in$ Real, preventing that A_i becomes imaginary. Proceeding, the next A_i are:

$$\begin{aligned} A_1(\alpha) =& 2\pi \int_0^\pi \int_0^1 \left(\frac{r^4\sin^4\theta}{r^2\sin^2\theta+\alpha^2}\right)\left(1-r^2\right) r^2 \sin\theta dr d\theta \\ =& \frac{8}{315}\left[6-7\alpha^2\left(3+20\alpha^2+15\alpha^4\right)+105\alpha^4\left(1+\alpha^2\right)^{\frac{3}{2}}\operatorname{acoth}\left(\sqrt{1+\alpha^2}\right)\right] \end{aligned} \tag{B4}$$

$$\begin{aligned} A_2(\alpha) =& 2\pi \int_0^\pi \int_0^1 \left(\frac{r^4\sin^2\theta\cos^2\theta}{r^2\sin^2\theta+\alpha^2}\right)\left(1-r^2\right) r^2 \sin\theta dr d\theta \\ =& \frac{8}{1575}\left[15+7\alpha^2\left(23+35\alpha^2+15\alpha^4\right)-105\alpha^2\left(1+\alpha^2\right)^{\frac{3}{2}}\operatorname{acoth}\left(\sqrt{1+\alpha^2}\right)\right] \end{aligned} \tag{B5}$$

$$\begin{aligned} A_3(\alpha) =& 2\pi \int_0^\pi \int_0^1 \left(\frac{r^6\sin^6\theta}{r^2\sin^2\theta+\alpha^2}\right)\left(1-r^2\right) r^2 \sin\theta dr d\theta \\ =& \frac{8}{945}\left[8-18\alpha^2+63\alpha^4+420\alpha^6+315\alpha^8-315\alpha^6\left(1+\alpha^2\right)^{\frac{3}{2}}\operatorname{acoth}\left(\sqrt{1+\alpha^2}\right)\right] \end{aligned} \tag{B6}$$

$$\begin{aligned} A_4(\alpha) =& 2\pi \int_0^\pi \int_0^1 \left(\frac{1}{r^2\sin^2\theta+\alpha^2}\right)\left(1-r^2\right) r^2 \sin\theta dr d\theta \\ =& -\frac{8}{9}\left[4+3\alpha^2-3\left(1+\alpha^2\right)^{\frac{3}{2}}\operatorname{acoth}\left(\sqrt{1+\alpha^2}\right)\right] \end{aligned} \tag{B7}$$

$$A_5(\alpha) = 2\pi \int_0^\pi \int_0^1 \left(\frac{1}{r^2\sin^2\theta+\alpha^2}\right)\left(1-r^2\right) r^2 \sin\theta dr d\theta \tag{B8}$$

$$= \frac{3}{2}-\alpha^2+\alpha^4\left(1+\alpha^2\right)^{-\frac{1}{2}}\operatorname{acoth}\left(\sqrt{1+\alpha^2}\right) \tag{B9}$$

$$\begin{aligned} A_6(\alpha) =& 2\pi \int_0^\pi \int_0^1 \left(\frac{r^8\sin^8\theta}{r^2\sin^2\theta+\alpha^2}\right)\left(1-r^2\right) r^2 \sin\theta dr d\theta \\ =& \frac{8}{10395}\left[48-11\alpha^2\left(8-18\alpha^2+63\alpha^4+420\alpha^6+315\alpha^8\right)+3465\alpha^8\left(1+\alpha^2\right)^{\frac{3}{2}}\operatorname{acoth}\left(\sqrt{1+\alpha^2}\right)\right] \end{aligned} \tag{B10}$$

$$\begin{aligned} A_7(\alpha) =& 2\pi \int_0^\pi \int_0^1 \left(\frac{r^2\sin^2\theta}{r^2\sin^2\theta+\alpha^2}\right)^2\left(1-r^2\right)^2 r^2 \sin\theta dr d\theta \\ =& \frac{16}{1575}\left[30+749\alpha^2+1680\alpha^4+945\alpha^6-105\alpha^2\left(4+9\alpha^2\right)\left(1+\alpha^2\right)^{\frac{3}{2}}\operatorname{acoth}\left(\sqrt{1+\alpha^2}\right)\right]. \end{aligned} \tag{B11}$$

Appendix C: Cubic equation solution

A forma canônica da equação cúbica é

$$ax^3 + bx^2 + cx + d = 0 \tag{C1}$$

com $a \neq 0$. Como solução usual faremos uma transformação no x, $x \to t + h$.

$$a\,(t+h)^3 + b\,(t+h)^2 + c\,(t+h) + d = 0 \tag{C2}$$

$$a\left(t^3 + 3ht^2 + 3h^2t + h^3\right) + b\left(t^2 + 2ht + h^2\right) + c\,(t+h) + d = 0 \tag{C3}$$

$$at^3 + (3ah+b)\,t^2 + \left(3ah^2 + 2bh + c\right)t + d + ah^3 + bh^2 + ch = 0 \tag{C4}$$

$$t^3 + \left(3h + \frac{b}{a}\right)t^2 + \left(3h^2 + 2h\frac{b}{a} + \frac{c}{a}\right)t + \frac{d}{a} + h^3 + \frac{b}{a}h^2 + \frac{c}{a}h = 0 \tag{C5}$$

Escolhendo $h = -\frac{b}{3a}$, reduzimos o grau da equação acima.

$$t^3 + pt + q = 0, \tag{C6}$$

onde

$$q = \frac{2b^3}{27a^3} - \frac{bc}{3a^2} + \frac{d}{a} \tag{C7}$$

e

$$p = \frac{c}{a} - \frac{b^2}{3a^2}. \tag{C8}$$

Se exprimimos $t = u + v$, a equação (C6) transformar-se-á em

$$u^3 + v^3 + q + (3uv + p)\,(u+v) = 0, \tag{C9}$$

cuja solução é dada em termos de um sistema de u e v

$$\begin{cases} u^3 + v^3 & = -q \\ u^3v^3 & = -\frac{p^3}{27} \end{cases} \tag{C10}$$

então conduzimos o sistema para uma solução da qual sabemos a soma de dois números $S = u^3 + v^3 = -q$ e o produto dos mesmos $P = u^3v^3 = -\frac{p^3}{27}$, ou seja, uma equação quadrática

$$Y^2 - SY + P = 0. \tag{C11}$$

Podemos escrever sem perda de generalidade,

$$u^3 = Y_+ = \frac{q}{2} + \frac{1}{2}\sqrt{q^2 + \frac{4p^3}{27}} \tag{C12}$$

$$v^3 = Y_- = \frac{q}{2} - \frac{1}{2}\sqrt{q^2 + \frac{4p^3}{27}}. \tag{C13}$$

Assim já temos a primeira solução

$$t_1 = \left(\frac{q}{2} + \frac{1}{2}\sqrt{q^2 + \frac{4p^3}{27}} \right)^{\frac{1}{3}} + \left(\frac{q}{2} - \frac{1}{2}\sqrt{q^2 + \frac{4p^3}{27}} \right)^{\frac{1}{3}}. \tag{C14}$$

Conhecida a solução t_1 podemos obter as outras duas decompondo (C6) como o produto de todas soluções, isto é,

$$(t - t_1)(t - t_2)(t - t_3) = t^3 + pt + q. \tag{C15}$$

$$t^3 - (t_1 + t_2 + t_3) t^2 + (t_1 t_2 + t_2 t_3 + t_3 t_1) t - t_1 t_2 t_3 = t^3 + pt + q \tag{C16}$$

Os dois polinômios são equivalentes de tiverem coeficientes homólogos iguais:

$$\begin{cases} t_2 + t_3 & = -t_1 \\ t_2 t_3 & = -\frac{q}{t_1} \end{cases} \tag{C17}$$

da mesma maneira que foi feito anteriormente, temos uma soma e um produto de dois números o que é equivalente a uma equação do segundo grau,

$$Z^2 + t_1 Z - \frac{q}{t_1} = 0. \tag{C18}$$

Assim a solução que faltava é

$$t_1 = Z_+ = -\frac{t_1}{2} + \sqrt{\frac{t_1}{4} + \frac{q}{t_1}} \tag{C19}$$

$$t_2 = Z_- = -\frac{t_1}{2} - \sqrt{\frac{t_1}{4} + \frac{q}{t_1}}. \tag{C20}$$

Portanto, as três soluções da equação cúbica (C1), no caso de discriminante $p > 0$, são

$$x_i = t_i - \frac{b}{3a} \tag{C21}$$

para $i = 1, 2, 3$.

No caso de discriminante negativo $p < 0$ convertemos os complexos conjugados Y_+ e Y_- à sua forma trigonométrica

$$Y_+ = |Y_+| \cos\theta + i \sin\theta \tag{C22}$$

$$Y_- = |Y_-| \cos\theta - i \sin\theta. \tag{C23}$$

O que no fim das contas resultará na seguinte solução:

$$x_1 = 2\sqrt{-\frac{p}{3}} \cos\left[\frac{1}{3} \arccos\left(-\frac{q}{2}\sqrt{-\frac{27}{p^3}} \right) \right] - \frac{b}{3a} \tag{C24}$$

$$x_2 = 2\sqrt{-\frac{p}{3}} \cos\left[\frac{1}{3} \arccos\left(-\frac{q}{2}\sqrt{-\frac{27}{p^3}} \right) + \frac{2\pi}{3} \right] - \frac{b}{3a} \tag{C25}$$

$$x_3 = 2\sqrt{-\frac{p}{3}} \cos\left[\frac{1}{3} \arccos\left(-\frac{q}{2}\sqrt{-\frac{27}{p^3}} \right) + \frac{4\pi}{3} \right] - \frac{b}{3a}. \tag{C26}$$

Appendix D: Cross-Check

Para verificarmos, em partes, se estamos indo pelo caminho certo, precisamos fazer um cross-check. No caso, isso seria validar as equações quando $\ell = 0$ e $\alpha \to 0$, assim as equações de movimento (42), (43) e (44) se tornarão

$$\ddot{r_\rho} + r_\rho = \lim_{\alpha\to 0} \frac{4D_6(\alpha)\gamma}{D_1(\alpha)r_\rho^3 r_z} \tag{D1}$$

$$\ddot{r_z} + \lambda^2 r_z = \lim_{\alpha\to 0} \frac{2D_6(\alpha)\gamma}{D_2(\alpha)r_\rho^2 r_z^2}, \tag{D2}$$

ou seja,

$$\ddot{r_\rho} + r_\rho = \frac{15\gamma}{r_\rho^3 r_z} \tag{D3}$$

$$\ddot{r_z} + \lambda^2 r_z = \frac{15\gamma}{r_\rho^2 r_z^2}. \tag{D4}$$

É fácil de perceber que (59), (60) e (61) são reduzidas à

$$\rho_0 = \frac{15\gamma}{\rho_0^3 z_0} \tag{D5}$$

$$\lambda^2 z_0 = \frac{15\gamma}{\rho_0^2 z_0^2}, \tag{D6}$$

já que $\lim_{\alpha\to 0} G_4 = 0$.

[1] PÉREZ-GARCÍA, V. M. et al. Dynamics of bose-einstein condensates: variational solutions of the gross-pitaevskii equations. *Physical Review A*, v. 56, n. 2, p. 1424–1432, August 1997.

[2] O'DELL, D. H. J.; EBERLEIN, C. Vortex in a trapped bose-einstein condensate with dipole-dipole interactions. *Physical Review A*, v. 75, n. 1, p. 013604, 2007.