Vortex lattices in rapidly rotating Bose-Einstein condensates: Modes and correlation functions

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After delineating the physical regimes that vortex lattices encounter in rotating Bose-Einstein condensates as the rotation rate Ω increases, we derive the normal modes of the vortex lattice in two dimensions at zero temperature. Taking into account effects of the finite compressibility, we find an inertial mode of frequency $\gtrsim 2\Omega$, and a primarily transverse Tkachenko mode, whose frequency goes from being linear in the wave vector in the slowly rotating regime, where Ω is small compared with the lowest compressional mode frequency, to quadratic in the wave vector in the opposite limit. We calculate the correlation functions of vortex displacements and phase, density and superfluid velocities, and find that the zero-point excitations of the soft quadratic Tkachenko modes lead in a large system to a loss of long-range phase correlations, growing logarithmically with distance, and hence lead to a fragmented state at zero temperature. The vortex positional ordering is preserved at zero temperature, but the thermally excited Tkachenko modes cause the relative positional fluctuations to grow logarithmically with separation at finite temperature. The superfluid density, defined in terms of the transverse velocity autocorrelation function, vanishes at all temperatures. Finally we construct the long-wavelength single-particle Green's function in the rotating system and calculate the condensate depletion as a function of temperature.

I. INTRODUCTION

Under rapid rotation, a superfluid forms a triangular lattice of quantized vortices carrying the angular momentum of the system. Such structure was seen experimentally in superfluid helium in Refs. [1]. Rapidly rotating Bose-Einstein condensates of cold atoms [2–4] open up the study of the physics of vortex lattices in regimes well beyond those achievable in superfluid helium [5–7]. With increasing rotational frequency Ω, condensates go through a number of different regimes. At Ω small compared with the lowest compressional frequencies, $\Omega \ll sk_0$, where s is the sound velocity and $k_0 \sim \pi/R_\perp$ is the lowest wave number in the finite geometry, with $R_\perp$ the size of the system perpendicular to the rotation axis, the system is in the "stiff" Thomas-Fermi regime, and responds to rotation effectively as an incompressible fluid. As Ω becomes larger than the lowest sound mode frequencies, $\Omega \gtrsim sk_0$, or essentially that the outer edge of the cloud moves supersonically, the system enters the "soft" Thomas-Fermi regime, where the compressibility becomes important in the response to rotation.

When, in harmonically trapped condensates, Ω approaches the transverse trapping frequency $\omega_\perp$, the centrifugal force begins to balance the trapping force, and the system flattens towards a lower density and therefore more compressible two-dimensional cloud. For $\Omega \gg ms$, where m is the particle mass, the condensate wave function becomes formed primarily of particle orbits in the lowest Landau level of the Coriolis force—the mean-field quantum Hall state [8]—with approximate order parameter, for rotation about the z axis,

$$\Psi(r) = \mathcal{C}\Pi_j(u - u_j)e^{-|u|^2/2d_\perp^2}, \quad u = x + iy, \quad (1)$$

where the $u_j = x_j + iy_j$ are the vortex locations in the usual complex notation, $d_\perp = (m\omega_\perp)^{-1/2}$ is the transverse oscillator length, and $\mathcal{C}$ is a normalization constant. The structure of the trapped condensate in the axial direction approaches a Gaussian shape as $\Omega \to \omega_\perp$. Although Eq. (1) predicts that the transverse structure is also Gaussian, the transverse structure, if Thomas-Fermi in the nonrotating cloud, remains Thomas-Fermi in this limit [9], accompanied by a small relaxation of the lattice from perfect triangularity [38]. Experiments in this regimes are reported in Ref. [6]. With further increase of Ω, the vortex lattice is expected to melt, at the point where the number of vortices becomes of order 10% of the number of particles [10,11], and the system eventually enters new highly correlated bosonic quantum Hall many-particle states, no longer describable in mean field [12–16].

Prior to discussing the modes of the lattice, it is useful to lay out the demarcations between the various regimes in a harmonically trapped gas. We assume that the interaction between the particles, of total number N, is described by a repulsive s-wave interaction parameter $g = 4\pi a_s/m$, where a_s is the scattering length; we use units in which $\hbar = 1$. In Thomas-Fermi, the transverse radius $R_\perp$ is given by [17],

$$R_\perp^2 = \frac{d_\perp^2 \tau}{[1 - (\Omega/\omega_\perp)^2]^{3/5}}, \quad (2)$$

where $\tau = [(15Nba_s/d_\perp)(\omega_z/\omega_\perp)]^{2/5}$, and ω_z is the axial trapping frequency. The factor $b \gtrsim 1$ describes the renormalization of the interaction energy of long-wavelength density fluctuations in the system [18]. Furthermore,

$$\frac{\Omega}{ms(0)^2} = \frac{2}{\tau}\frac{\Omega}{\omega_\perp}[1 - (\Omega/\omega_\perp)^2]^{2/5}, \quad (3)$$

where $s(0)$ is the sound velocity in the center of the cloud. If we write $k_0 = \alpha/R_\perp$, where $\alpha \simeq 5.45$ [19], the criterion for being in the soft regime, that Ω/sk_0 be large, becomes

$$\frac{\Omega/\omega_\perp}{[1-(\Omega/\omega_\perp)^2]^{1/2}} \gg \frac{1}{\sqrt{2}\alpha} \simeq 0.13. \quad (4)$$

The experiments of Ref. [5] on Tkachenko modes reach $\Omega/sk_0 \sim 1.15$.

The criterion to be in the mean-field quantum Hall regime, $\Omega/ms^2 \gg 1$, can be conveniently written in terms of the filling factor $\nu = N/N_v$, where N_v is the total number of vortices in the rotating cloud. Since

$$N_v = \pi n_v R_\perp^2 = m\Omega R_\perp^2 = \frac{\Omega}{\omega_\perp}\frac{\tau}{[1-(\Omega/\omega_\perp)^2]^{3/5}}, \quad (5)$$

where $n_v = m\Omega/\pi$ is the density of vortices, we find

$$\frac{\Omega}{ms(0)^2} = \frac{2}{\nu^{2/3}}\left(\frac{\Omega}{\omega_\perp}\right)^{1/3}\left(\frac{d_\perp \omega_\perp}{15ba_s\omega_z}\right)^{2/3}. \quad (6)$$

This result is independent of the number of particles, as long as the system is in the Thomas-Fermi regime. For the parameters $\omega_\perp/\omega_z = 8.3/5.2$ of the Tkachenko mode experiment of Ref. [5] in ^{87}Rb, we find $\Omega/ms(0)^2 \simeq 200/\nu^{2/3}$ in the limit $\Omega \to \omega_\perp$.

The vortex lattice supports a number of modes, first discussed by Tkachenko [20] for a two-dimensional incompressible fluid, and later in Ref. [21] at finite temperature with full effects of the normal fluid, dissipation and Kelvin oscillations of the vortex lines in three dimensions. The low-frequency in-plane Tkachenko mode is an elliptically polarized oscillation of the vortex lines, with the semimajor axis of the ellipse orthogonal to the direction of propagation. The Tkachenko mode is linear at small wave vector k in the transverse plane,

$$\omega_T = \left(\frac{2C_2}{mn}\right)^{1/2}k \to \left(\frac{\Omega}{4m}\right)^{1/2}k, \quad (7)$$

where n is the particle density and C_2 is the elastic shear modulus of the vortex lattice; at slow rotation, $C_2 = n\Omega/8$. In the soft regime, the dispersion relation instead becomes quadratic [11,22],

$$\omega_T = \left(\frac{s^2C_2}{2\Omega^2 nm}\right)^{1/2}k^2. \quad (8)$$

Were it possible to rotate helium sufficiently rapidly one would also see this very long-wavelength quadratic behavior.

Such Tkachenko soft modes can play havoc with the stability of a large system, causing loss of long-range phase coherence even at zero temperature; they are eventually responsible for the melting of the lattice [11]. In a recent paper [22] we derived the modes of the vortex lattice for all rotation rates, through constructing the conservation laws and superfluid acceleration equation describing the long wavelength behavior of the system. In this paper we focus on deriving the correlation functions of density, superfluid phase and velocity, and vortex displacements from equilibrium, which enable us to understand the effects of the soft infrared structure on the stability of the superfluid and lattice. This work is a generalization of Ref. [23], which discussed the effects of the oscillations of the vortex lines at finite temperature in liquid helium on the long-ranged phase correlations of the superfluid. As we shall see, the long wavelength Tkachenko modes lead to fragmentation of the condensate, even at zero temperature. Whether the system loses phase coherence over its volume or the lattice melts first depends on the number of particles in the system and its rotation rate.

In Sec. II, we review the basic equations describing the dynamics, restricting the analysis to linearized motion in two dimensions, and neglecting the normal component of the superfluid as well as dissipative terms. The analysis of the full three-dimensional problem will be published separately [24]. In Sec. III we construct the correlation functions of the physical quantities of interest, and in Sec. IV study the condensate depletion by constructing the single-particle Green's function for the rotating superfluid.

II. CONDENSATE PHASE AND CONSERVATION LAWS

Let us first recapitulate the conservation laws and equation for the superfluid phase which govern the long-wavelength behavior of the system. While much of this material has been given in Ref. [22], we include it here in order to facilitate the derivation of the correlation functions. The basic formalism given here applies to general bosonic superfluids.

A. Condensate phase

We work in the frame corotating with the lattice and describe the deviations of the vortices from their home positions by the continuum displacement field $\vec{\epsilon}(r,t)$. In linear order in the vortex displacements, the long-wavelength superfluid velocity $\vec{v}(r,t)$ can be written, following Ref. [23], in terms of the long-wavelength vortex-lattice displacement field and the phase $\Phi(r,t)$ of the order parameter, as

$$\vec{v} + 2\vec{\Omega} \times \vec{\epsilon} = \vec{\nabla}\Phi/m. \quad (9)$$

The curl of this equation is

$$\vec{\nabla} \times \vec{v} = -2\Omega\vec{\nabla} \cdot \vec{\epsilon}. \quad (10)$$

The origin of Eq. (9) is the law of conservation of vorticity, $\vec{\varpi} \equiv \vec{\nabla} \times \vec{v}$:

$$\frac{\partial \vec{\varpi}}{\partial t} + \vec{\nabla} \times (\vec{\varpi} \times \dot{\vec{\epsilon}}) = 0, \quad (11)$$

where here $\dot{\vec{\epsilon}}$ tells the rate at which the vorticity moves about. Since under uniform rotation, $\vec{\varpi} = 2\vec{\Omega}$, the time derivative of the curl of Eq. (9) is just this equation linearized. The longitudinal part of the left side of Eq. (9) is trivially the gradient of a scalar. Equation (9) constrains the number of degrees of freedom in two dimensions to four from the original five − n, $\vec{v}$, and $\vec{\epsilon}$.

The time derivative of Eq. (9) is the superfluid acceleration equation,

$$m\left(\frac{\partial \vec{v}}{\partial t} + 2\vec{\Omega} \times \dot{\vec{\epsilon}}\right) = -\vec{\nabla}(\mu_s + V_{\text{eff}}), \qquad (12)$$

where μ_s is the chemical potential of the matter [39]. For an axially symmetric harmonic confining trap of frequency $\omega_\perp$ in the transverse direction and ω_z in the axial direction,

$$V_{\text{eff}} = \frac{m}{2}[(\omega_\perp^2 - \Omega^2)r^2 + \omega_z^2 z^2], \qquad (13)$$

where $\vec{r}$ denotes (x, y). In the frame corotating with the vortex lattice, the chemical potential μ_s is related to the phase by

$$\mu_s(r,t) + V_{\text{eff}} = -\frac{\hbar}{m}\frac{\partial \Phi(r,t)}{\partial t}. \qquad (14)$$

B. Conservation laws

The dynamics are specified by the conservation laws of particles and momentum, together with the superfluid acceleration equation, Eq. (12). The continuity equation takes its usual form,

$$\frac{\partial n(r,t)}{\partial t} + \vec{\nabla} \cdot \vec{j}(r,t) = 0, \qquad (15)$$

where n is the density and $\vec{j} = n\vec{v}$ the particle current. Conservation of momentum reads

$$m\frac{\partial \vec{j}}{\partial t} + 2m\vec{\Omega} \times \vec{j} + \vec{\nabla}P + n\vec{\nabla}V_{\text{eff}} = -\vec{\sigma} - \vec{\zeta}. \qquad (16)$$

Here P is the pressure and $\vec{\sigma}$ is the elastic stress tensor, discussed below. At zero temperature, $\vec{\nabla}P = n\vec{\nabla}\mu_s$, while in equilibrium, $\vec{\nabla}P + n\vec{\nabla}V_{\text{eff}} = 0$. To calculate the displacement autocorrelation functions, we include here, as in Ref. [23], an external driving force $-\vec{\zeta}(\vec{r},t)$, acting on the lattice, derived from an external perturbation $H' = \vec{\zeta} \cdot \vec{\epsilon}$.

The elastic stress tensor is derived from the elastic energy density of the lattice, which in two dimensions has the form (in the notation of Ref. [21]),

$$\mathcal{E}(r) = 2C_1(\vec{\nabla} \cdot \vec{\epsilon})^2 + C_2\left[\left(\frac{\partial \epsilon_x}{\partial x} - \frac{\partial \epsilon_y}{\partial y}\right)^2 + \left(\frac{\partial \epsilon_x}{\partial y} + \frac{\partial \epsilon_y}{\partial x}\right)^2\right], \qquad (17)$$

where C_1 is the compressional modulus and C_2 the shear modulus of the vortex lattice. The elastic constants are density-dependent properties of the fluid. Then the elastic stress tensor σ_i is given in terms of the total elastic energy $E_{\text{el}} = \int d^2 r \mathcal{E}(r)$ by

$$\sigma_i(r,t) = \frac{\delta E_{\text{el}}}{\delta \epsilon_i} = -4\nabla_i(C_1\vec{\nabla} \cdot \vec{\epsilon}) - 2\vec{\nabla} \cdot (C_2\vec{\nabla}\epsilon_i), \qquad (18)$$

where we allow for effects of density gradients entering through the elastic constants. In an incompressible fluid, $C_2 = n\Omega/8 = -C_1$. On the other hand, in the quantum Hall regime, the shear modulus is determined by the deviations of

the interaction energy caused by distorting the vortex lattice, and [25]

$$C_2 \simeq \frac{81}{80\pi^4}ms^2 n, \qquad (19)$$

in agreement with the shear modulus given numerically in Ref. [11]. Calculation of the elastic constants C_1 and C_2 over the full range of Ω from the stiff Thomas-Fermi to the quantum Hall limits will be given in Ref. [25].

Subtraction of Eq. (16) divided by n, from the superfluid acceleration equation (12) yields

$$2\vec{\Omega} \times (\dot{\vec{\epsilon}} - \vec{v}) = \frac{\vec{\sigma} + \vec{\zeta}}{mn}. \qquad (20)$$

Equations (12), (15), (18), and (20) fully specify the problem for a trapped system with a nonuniform density.

C. Modes

In the following, we neglect effects of nonuniformity of the equilibrium density for simplicity and proceed to derive the modes as in Ref. [22]. The curl of Eq. (20) is

$$\vec{\nabla} \cdot (\dot{\vec{\epsilon}} - \vec{v}) = \frac{1}{2\Omega nm}\vec{\nabla} \times (\vec{\sigma} + \vec{\zeta}), \qquad (21)$$

while its divergence, together with Eq. (10), yields,

$$\vec{\nabla} \times \dot{\vec{\epsilon}} + 2\vec{\Omega}\vec{\nabla} \cdot \vec{\epsilon} = -\frac{1}{2\Omega nm}\vec{\nabla} \cdot (\vec{\sigma} + \vec{\zeta}), \qquad (22)$$

where $\vec{\nabla} \times \vec{\sigma} = -2C_2\nabla^2(\vec{\nabla} \times \vec{\epsilon})$ and $\vec{\nabla} \cdot \vec{\sigma} = -2(C_2 + 2C_1)\nabla^2(\vec{\nabla} \cdot \vec{\epsilon})$. To eliminate $\vec{v}$ from Eq. (21), we note that the density oscillations are governed by

$$\left(-\frac{\partial^2}{\partial t^2} + s^2\nabla^2\right)n = 2n\Omega\vec{\nabla} \times \dot{\vec{\epsilon}}, \qquad (23)$$

where the sound speed s is given by $ms^2 = \partial P/\partial n$ [26]. In terms of the frequency ω and wave vector k we have then

$$\vec{k} \cdot \vec{v} = \frac{2\omega^2}{\omega^2 - s^2 k^2} \vec{\Omega} \cdot \vec{k} \times \epsilon, \tag{24}$$

so that in terms of longitudinal and transverse components [27],

$$-i\omega\epsilon_T + \left(2\Omega + \frac{C_2 + 2C_1}{\Omega nm}k^2\right)\epsilon_L = -\frac{\zeta_L}{2\Omega nm},$$

$$-i\omega\epsilon_L - \left(\frac{2\omega^2\Omega}{\omega^2 - s^2 k^2} + \frac{C_2}{\Omega mn}k^2\right)\epsilon_T = \frac{\zeta_T}{2\Omega nm}. \tag{25}$$

Solving for ϵ_L and ϵ_T we have

$$\epsilon_L = \frac{1}{nmD}\left\{\left(\omega^2 + (\omega^2 - s^2 k^2)\frac{C_2}{2\Omega^2 nm}k^2\right)\zeta_L \right.$$
$$\left. + i\omega\frac{\omega^2 - s^2 k^2}{2\Omega}\zeta_T\right\},$$

$$\epsilon_T = \frac{1}{nmD}\left\{(\omega^2 - s^2 k^2)\left(1 + \frac{C_2 + 2C_1}{2\Omega^2 nm}k^2\right)\zeta_T \right.$$
$$\left. - i\omega\frac{\omega^2 - s^2 k^2}{2\Omega}\zeta_L\right\}, \tag{26}$$

where the secular determinant, whose zeroes determine the mode frequencies, is

$$D(k,\omega) \equiv \omega^4 - \omega^2\left[4\Omega^2 + \left(s^2 + \frac{4}{nm}(C_1 + C_2)\right)k^2\right] + \frac{2s^2 C_2}{nm}k^4$$
$$= (\omega^2 - \omega_I^2)(\omega^2 - \omega_T^2) = 0; \tag{27}$$

we have dropped terms of second order in the elastic constants.

For $2s^2 C_2 k^4/nm \ll (4\Omega^2 + [s^2 + 4(C_1 + C_2)/nm]k^2)^2$, as is always the case at long wavelengths in both the incompressible and quantum Hall limits, the mode frequencies are given by

$$\omega_I^2 = 4\Omega^2 + \left(s^2 + \frac{4(C_1 + C_2)}{nm}\right)k^2, \tag{28}$$

and

$$\omega_T^2 = \frac{2C_2}{nm}\frac{s^2 k^4}{\{4\Omega^2 + [s^2 + 4(C_1 + C_2)/nm]k^2\}}. \tag{29}$$

The first mode is the standard inertial mode of a rotating fluid; for $\Omega \ll s^2 k^2$ the mode is a sound wave, while for $\Omega \gg s^2 k^2$, the mode frequencies begin essentially at twice the axial trapping frequency. This mode has been calculated in realistic trapping geometries in Refs. [28] and [29]. The second mode is the elliptically polarized Tkachenko mode. Equation (22) implies that the inertial mode is circularly polarized: $\epsilon_L/\epsilon_T \simeq i$, and in the Tkachenko mode, $\epsilon_L/\epsilon_T \simeq i\omega_T/2\Omega$; the small longitudinally polarized component is $\pi/2$ out of phase with the transversely polarized component. In the limit of an incompressible fluid ($s^2 \to \infty$),

$$\omega_I^2 = 4\Omega^2 + \left(s^2 + \frac{4C_2}{nm}\right)k^2, \tag{30}$$

and the Tkachenko frequency ω_T is linear in k, Eq. (7). In the soft limit, by contrast

$$\omega_T^2 = \frac{s^2 C_2}{2\Omega^2 nm}k^4, \tag{31}$$

unlike in the stiff Thomas-Fermi regime, the mode frequency is quadratic in k at long wavelengths; using Eq. (19) for C_2 we have

$$\omega_T \simeq \frac{9}{4\pi^2\sqrt{10}}\frac{s^2 k^2}{\Omega}. \tag{32}$$

The present results for the modes are valid for a uniform system over the entire range of rotation frequencies, from the slowly rotating stiff regime up to the melting of the vortex lattice. The Tkachenko mode has been calculated numerically for realistic trapping geometries in Ref. [19] in the stiff limit and more generally in Refs. [30] and [31].

III. CORRELATION FUNCTIONS

We turn to determining the effects of the lattice modes on the lattice ordering and the phase coherence and condensate fraction of the rotating superfluid. To do so we construct the correlation functions of the density, superfluid velocity, vortex displacements, and phase from the dynamical equations in the preceding section. All the correlation functions of interest, including the single-particle Green's function, can be written in terms of the density-density and displacement-displacement correlation functions. In the following we let $\langle AB\rangle(k,z)$ denote the Fourier transform in space and the analytic continuation of the Fourier transformation in imaginary time to complex frequency z of the correlation function $\langle A(rt)B(r't')\rangle - \langle A(rt)\rangle\langle B(r't')\rangle$.

The density-density correlation function $\langle nn\rangle(k,z)$ is readily found from the response of $\langle n(r,t)\rangle$ to an external potential $U(r,t)$ coupled to the density. Using Eq. (24) to eliminate ϵ_T from Eq. (23), we have

$$\langle nn\rangle(k,z) = \int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\frac{B(b,\omega)}{z - \omega} = \frac{nk^2}{mD(k,z)}\left(z^2 - \frac{2C_2 k^2}{nm}\right), \tag{33}$$

with D given by Eq. (27). As $z \to \infty$, $\langle nn\rangle(k,z) \to nk^2/mz^2$, which is the expected f-sum rule on the spectral weight $B(k,\omega)$,

$$\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\omega B(k,\omega) = \frac{nk^2}{m}. \tag{34}$$

Similarly, $\langle nn\rangle(k \to 0,0) \to -n/ms^2$, yielding the correct compressibility sum rule,

$$\lim_{k \to 0} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{B(b,\omega)}{\omega} = \frac{n}{ms^2}. \tag{35}$$

As expected, the f-sum rule is dominated by the high-frequency inertial mode; the low-frequency Tkachenko mode dominates the compressibility sum rule [32].

The correlations of the longitudinal velocity are given in terms of $\langle nn \rangle(k,\omega)$, as usual, by

$$\langle v_L n \rangle(k,z) = \frac{z}{nk} \langle nn \rangle(k,z), \tag{36}$$

and

$$\langle v_L v_L \rangle(k,z) = \frac{z^2}{n^2 k^2} \langle nn \rangle(k,z) - \frac{1}{nm} \tag{37}$$

$$= \frac{1}{nmD} \left\{ z^2 \left(4\Omega^2 + s^2 k^2 + \frac{2}{nm}(2C_1 + C_2)k^2 \right) \right.$$
$$\left. - \frac{2s^2 C_2}{nm} k^4 \right\}. \tag{38}$$

The correlation functions of the elastic displacements are given by

$$\langle \epsilon_i \epsilon_j \rangle = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{B_{ij}(k,\omega)}{z-\omega} = \frac{\delta \langle \epsilon_i \rangle}{\delta \zeta_j}, \tag{39}$$

where $\langle \epsilon_i \rangle$ is the displacement in the ith direction induced by the force $\vec{\zeta}$. Equations (39) and (25) then imply

$$\langle \epsilon_L \epsilon_L \rangle(k,z) = \frac{1}{nmD(k,z),} \left(z^2 + (z^2 - s^2 k^2)\frac{C_2}{2\Omega^2 nm}k^2 \right), \tag{40}$$

$$\langle \epsilon_T \epsilon_T \rangle(k,z) = \frac{z^2 - s^2 k^2}{nmD(k,z)} \left(1 + \frac{C_2 + 2C_1}{2\Omega^2 nm}k^2 \right), \tag{41}$$

$$\langle \epsilon_L \epsilon_T \rangle(k,z) = iz\frac{z^2 - s^2 k^2}{2\Omega nmD} = \langle \epsilon_T \epsilon_L \rangle^*. \tag{42}$$

We note, for later calculation of the phase correlations, that the displacement-density correlation function, found from Eq. (23) together with the continuity equation, is

$$\langle n\epsilon_T \rangle(k,z) = \langle \epsilon_T n \rangle(k,z) = \frac{2n\Omega zk}{z^2 - s^2 k^2} \langle \epsilon_T \epsilon_T \rangle, \tag{43}$$

and thus the displacement-longitudinal velocity correlation is

$$\langle \epsilon_T v_L \rangle(k,z) = \langle v_L \epsilon_T \rangle(k,z) = \frac{2\Omega z^2}{z^2 - s^2 k^2} \langle \epsilon_T \epsilon_T \rangle. \tag{44}$$

A. Lattice displacements

We first address the effects of the vortex modes on the lattice displacements from equilibrium. At finite temperature T the equal-time displacement correlations are given by

$$\langle [\epsilon_i(r) - \epsilon_i(r')]^2 \rangle = 2 \int \frac{d^2 k}{(2\pi)^2 Z}(1 - \cos \vec{k} \cdot \vec{R}) \int_0^{\infty} \frac{d\omega}{2\pi} B_{ii}(k,\omega)$$
$$\times [1 + 2f(\omega)], \tag{45}$$

where Z is the thickness of the system in the z direction, $\vec{R} = \vec{r} - \vec{r}'$, and $f(\omega) = 1/(e^{\beta\omega} - 1)$, with $\beta = 1/k_B T$. The spectral weights B_{ij} are found from Eqs. (40)–(42) by letting $z \to \omega$ and

$$\frac{1}{D(k,z)} \to \frac{2\pi}{\omega_I^2 - \omega_T^2} \left\{ \frac{1}{2\omega_I}[\delta(\omega - \omega_I) - \delta(\omega - \omega_I)] \right.$$
$$\left. - \frac{1}{2\omega_T}[\delta(\omega - \omega_T) - \delta(\omega - \omega_T)] \right\}. \tag{46}$$

The leading terms in the mean displacement of a single vortex from equilibrium due to excitations of the modes are

$$\langle \vec{\epsilon}^2 \rangle = \int \frac{d^2 k}{(2\pi)^2 Z} \frac{1}{\omega_I^2 nm} \left[\omega_I[1 + 2f(\omega_I)] + \frac{s^2 k^2}{2\omega_T}[1 + 2f(\omega_T)] \right]. \tag{47}$$

The mean displacement is convergent at zero temperature. However, at finite temperature it diverges logarithmically with system size if the Tkachenko mode spectrum reaches down into the soft quadratic regime, then

$$\frac{\langle \vec{\epsilon}^2 \rangle}{\ell^2} \sim \frac{Tm\Omega}{8\pi Z C_2} \ln N_v. \tag{48}$$

The relative separation, with only the leading terms kept,

$$\langle [\vec{\epsilon}(r) - \vec{\epsilon}(r')]^2 \rangle = 2 \int \frac{d^2 k}{(2\pi)^2 Z} \frac{1 - \cos \vec{k} \cdot \vec{R}}{nm\omega_I^2} \left[\omega_I[1 + 2f(\omega_I)] \right.$$
$$\left. + \frac{s^2 k^2}{2\omega_T}[1 + 2f(\omega_T)] \right], \tag{49}$$

converges at zero temperature, and the lattice preserves long-range positional order at large separation; at finite temperature, however, $\langle (\vec{\epsilon}(r) - \vec{\epsilon}(r'))^2 \rangle$ grows logarithmically with separation, $\sim (Tm\Omega/4\pi Z C_2)\ln N_v(R)$, where $N_v(R)$ is the number of vortices within radius R. In the stiff Thomas-Fermi limit, this expression becomes $(4\pi/nZ\lambda^2)\ln N_v(R)$, where λ is the thermal wavelength; in the quantum Hall limit we have rather $(20\pi^4/81\nu)(T/ms^2)\ln N_v(R)$.

Equation (47) may be used with a Lindemann criterion to estimate the point where the lattice melts at $T=0$ in an extended system [10,11]. The zero-temperature displacements are sensitive to the entire spectrum of modes up to the lattice Debye vector, $k_d = (4\pi n_v)^{1/2}$, whereas the mode frequencies derived here are valid only for $k \ll k_d$. For a first estimate, we replace the integrand by its infrared limit, letting $\int d^2 k/(2\pi)^2 \to n_v$. Then using the Tkachenko frequency at rapid rotation, Eq. (32), we have

$$\frac{\langle \vec{\epsilon}^2 \rangle}{\ell^2} = \frac{1}{2\nu}\left(1 + \frac{s^2 k^2}{4\omega_T \Omega}\right) \simeq \frac{1}{2\nu}\left(1 + \frac{\pi^2 \sqrt{10}}{9}\right) \simeq \frac{2.23}{\nu},$$

$$(50)$$

where $\ell = (1/\pi n_v)^{1/2}$ is the radius of the Wigner-Seitz cell around a given vortex and we have used Eq. (19). The "1" term arises from the inertial mode, with equal contributions from the transverse and longitudinal displacements; the final term arises solely from the soft Tkachenko mode contribution to the transverse displacements. In order to take into account approximately the mode structure at larger wave vector, we include the kinetic term $-\nabla^2 n/4m$ in the pressure. In the nonrotating weakly interacting gas, this term modifies the linear spectrum sk into the full Bogoliubov spectrum, $E_k = [(sk)^2 + (k^2/2m)^2]^{1/2}$. Inclusion of such a term is equivalent to replacing s^2 by $s^2 + k^2/4m^2$ in the mode spectrum, with the effect of stiffening the Tkachenko mode at high wave vector. Then at $\Omega/ms^2 \sim 40$, as expected under typical experimental conditions at $\nu \sim 10$, the contribution of the Tkachenko modes to $\langle \epsilon^2 \rangle$ is reduced from $1.73/\nu$ in Eq. (50) to $\sim 0.72/\nu$, a result consistent with that reported in Ref. [11], $m\Omega\langle \vec{\epsilon}^2 \rangle \simeq 0.66/\nu$. Then with the contribution of the inertial modes included,

$$\frac{\langle \vec{\epsilon}^2 \rangle}{\ell^2} \simeq \frac{1.22}{\nu}.$$

$$(51)$$

Taking the Lindemann criterion for melting in two dimensions put forth in Ref. [10], $\langle \vec{\epsilon}^2 \rangle/\ell^2 \simeq 0.07$, we find melting of the lattice at filling $\nu \sim 17$; were we to take only the Tkachenko mode contribution, the melting would be at $\nu \sim 10$.

B. Phase correlations

We next determine the effect of the vortex excitations on the correlations of the order parameter in the superfluid and on the condensate fraction. The condensate density n_0 is most conveniently found, à la Onsager-Penrose, as the limit of the single-particle density matrix for large separation:

$$\langle \psi(r)\psi^\dagger(r') \rangle \to n_0, \quad |\vec{r} - \vec{r}'| \to \infty, \qquad (52)$$

where $\psi(r)$ is the single-particle annihilation operator [33]. We write $\psi(r)$ in terms of the density and phase operators, viz., $\psi(r) = \sqrt{n(r)}e^{i\Phi(r)}$, and expand the long-wavelength structure to second order in small fluctuations of n about $\bar{n}$, its equilibrium value, and $e^{i\Phi(r)}$ about unity. Then in terms of equal-time correlation functions,

$$\langle \psi(r)\psi^\dagger(r') \rangle \simeq \bar{n}\langle e^{i\Phi(r)}e^{-i\Phi(r')} \rangle + \frac{1}{4\bar{n}}\{\langle \delta n(r)\delta n(r') \rangle$$

$$- \langle [\delta n(r)]^2 \rangle\} + \frac{1}{2}\langle \delta n(r)(e^{i\Phi(r)} - e^{-i\Phi(r')})$$

$$+ (e^{i\Phi(r)} - e^{-i\Phi(r')})\delta n(r') \rangle, \qquad (53)$$

where $\delta n(r) = n(r) - \bar{n}$ and $\bar{n}$ denotes the average density. The first term on the right is the U(1)-invariant correlation of the

order parameter, given, for Gaussianly distributed Fourier components of the phase, by

$$\langle e^{i\Phi(r)}e^{-i\Phi(r')} \rangle = e^{-(1/2)\langle [\Phi(r) - \Phi(r')]^2 \rangle}. \qquad (54)$$

Equation (33) implies that as $|\vec{r} - \vec{r}'| \to \infty$, $\langle \delta n(r)\delta n(r') \rangle$ and also the final bracket in Eq. (53) vanish. Thus the density of particles in the condensate is given by

$$n_0 = \lim_{|r-r'| \to \infty}\left[\bar{n}e^{-(1/2)\langle [\Phi(\vec{r}) - \Phi(\vec{r}')]^2 \rangle} - \frac{1}{4\bar{n}}\langle \delta n(r)^2 \rangle\right]. \quad (55)$$

When the phase fluctuations are convergent, and vanishing at large separation, we can expand to second order to find the usual expression for the condensate depletion in the Bogoliubov approximation (see Sec. IV):

$$n' = \bar{n}\langle \Phi^2 \rangle + \frac{1}{4\bar{n}}\langle \delta n(r)^2 \rangle. \qquad (56)$$

To determine the phase-phase and the density-phase correlations, we note that the divergence of Eq. (9) implies that the Fourier components of the phase obey $\Phi = -(im/k)(v_L - 2\Omega\epsilon_T)$. Thus

$$\langle \Phi\Phi \rangle(k,z) = \frac{m^2}{k^2}[\langle v_L v_L \rangle(k,z) - 2\Omega[\langle \epsilon_T v_L \rangle(k,z) + \langle v_L \epsilon_T \rangle(k,z)]$$

$$+ 4\Omega^2\langle \epsilon_T \epsilon_T \rangle(k,z)], \qquad (57)$$

which, with Eqs. (38), (41), and (44), becomes,

$$\langle \Phi\Phi \rangle(k,z) = \frac{m^2}{k^2}\left[\frac{z^2}{n^2 k^2}\left(\langle nn \rangle - \frac{nk^2}{mz^2}\right) - 4\Omega^2\frac{z^2 + s^2 k^2}{z^2 - s^2 k^2}\langle \epsilon_T \epsilon_T \rangle\right]$$

$$= \frac{ms^2}{nD}\left(z^2 - 4\Omega^2 - \frac{4(C_1 + C_2)}{nm}k^2\right). \qquad (58)$$

Similarly,

$$\langle \Phi n \rangle(k,z) = -i\frac{mz}{nk^2}\left[\langle nn \rangle(k,z) - \frac{4\Omega^2 n^2 k^2}{z^2 - s^2 k^2}\langle \epsilon_T \epsilon_T \rangle\right]$$

$$= -iz\frac{n}{ms^2}\langle \Phi\Phi \rangle(k,z). \qquad (59)$$

Note that the phase fluctuations are more divergent in the infrared limit by a factor $1/k^2$ than the transverse displacement fluctuations.

The relative phase correlations are given by

$$\langle [\Phi(r) - \Phi(r')]^2 \rangle = \frac{ms^2}{n}\int \frac{d^2k}{(2\pi)^2 Z}\frac{1 - \cos\vec{k}\cdot\vec{R}}{\omega_T}[1 + 2f(\omega_T)],$$

$$(60)$$

plus finite terms, where $\vec{R} = \vec{r} - \vec{r}'$. In the stiff limit ($s^2 k^2 \gg \Omega^2$), the relative phase fluctuations are convergent at zero temperature, but, as discussed in Ref. [23], the finite temperature contribution of the Tkachenko modes washes out the phase correlations logarithmically, and the system is in fact in a fragmented condensate phase [34]. One finds similar behavior at zero temperature in the soft limit. The zero tem-

perature part of the integral (60) is $\sim \ln(k_D R)$. Thus from Eq. (54), we find the correlation of the phase factors,

$$\langle e^{i\phi(r)} e^{-i\phi(r')} \rangle \sim (k_D R)^{-\eta} \sim N_v(R)^{-\eta/2}, \qquad (61)$$

where again $N_v(R)$ is the number of vortices within radius R, and

$$\eta = \frac{1}{\nu} \left(\frac{ms^2 \bar{n}}{8C_2} \right)^{1/2}. \qquad (62)$$

From the limits $\Omega \ll ms^2$ to $\Omega \gg ms^2$, the range of η is

$$\frac{1}{\nu} \left(\frac{ms^2}{\Omega} \right)^{1/2} \geq \eta \geq \frac{\pi^2 \sqrt{10}}{9\nu}. \qquad (63)$$

The phase correlations fall off algebraically at large R, as expected for a two-dimensional system [35]. As the phase correlations decrease, the condensate fraction also falls, as $(n_0/n) \sim N_v^{-\eta/2}$.

The falloff of the phase correlations begins to become important for $(\eta/2)\ln N_v \gtrsim 1$, which translates effectively into the condition in the quantum Hall limit that $\nu \lesssim 1.7 \ln N_v$ or $N \lesssim \nu e^{0.6\nu}$; for $N=10^6$, the condition is that $N_v \gtrsim 5 \times 10^4$; for $N=10^4$, $N_v \gtrsim 10^3$; and for $N=10^3$, one needs only $N_v \gtrsim 10^2$ to find loss of phase coherence across the system.

One may ask whether the falloff is significant by the time the lattice will have melted. The most divergent terms in the phase fluctuations are induced by the transverse displacement fluctuations in Eq. (58), so that in a system of transverse radius $R_\perp$, one has roughly,

$$\lim_{|\vec{r}-\vec{r}'| \to R_\perp} \frac{1}{2}\langle [\Phi(r) - \Phi(r')]^2 \rangle \simeq \frac{8}{\ell^4} \langle k^{-2} \rangle \langle \epsilon_T^2 \rangle, \qquad (64)$$

where the average $\langle k^{-2} \rangle \sim (1/4\pi n_v)\ln N_v$ is taken with respect to the weight in the transverse displacement correlation function. Thus

$$\lim_{|\vec{r}-\vec{r}'| \to R_\perp} \frac{1}{2}\langle [\Phi(r) - \Phi(r')]^2 \rangle \sim 2\frac{\langle \epsilon_T^2 \rangle}{\ell^2} \ln N_v. \qquad (65)$$

The Lindemann criterion, $\langle \epsilon_T^2 \rangle / \ell^2 \sim 0.07$, implies that for $N_v \gtrsim 10^3$ the right side of Eq. (65) exceeds unity at the melting point, setting the scale for number of vortices for which loss of long-range order of the condensate prior to melting becomes important.

At finite temperature, the phase correlation integral (60) is logarithmically singular in the infrared for all R, indicating that in two dimensions, the system is, as expected, no longer Bose condensed. The situation is the same as in the stiff Thomas-Fermi limit, where the phase correlations in two dimensions diverge in the infrared, but in three dimensions fall algebraically with separation [23].

IV. THE SINGLE-PARTICLE GREEN'S FUNCTION AND CONDENSATE DEPLETION

We now determine the structure of the single-particle excitations in terms of the modes of the lattice by constructing the long-wavelength behavior of the single-particle Green's function,

$$G(rt, r't') = -i\langle \mathcal{T}[(\psi(rt) - \langle \psi(rt) \rangle)(\psi^\dagger(r't') - \langle \psi^\dagger(r't') \rangle)] \rangle \qquad (66)$$

($\mathcal{T}$ denotes the time ordered product), in terms of the correlation functions calculated in the preceding section. Again expanding in small fluctuations about equilibrium, we have

$$G(k,z) = \frac{1}{4\bar{n}}\langle nn \rangle(k,z) + \frac{1}{2\bar{n}}[\langle ne^{-i\Phi} \rangle(k,z) + \langle e^{i\Phi} n \rangle(k,z)] + \bar{n}\langle e^{i\Phi} e^{-i\Phi} \rangle(k,z). \qquad (67)$$

To second order in the fluctuations of the density and phase,

$$G(k,\omega) = \frac{1}{4\bar{n}}\langle nn \rangle(k,z) + \frac{i}{2}[\langle \Phi n \rangle(k,z) - \langle n\Phi \rangle(k,z)] + \bar{n}\langle \Phi\Phi \rangle(k,z). \qquad (68)$$

To illustrate this method of calculating G, we first consider a weakly interacting nonrotating system, for which Eqs. (33), (27), (59), and (58) imply

$$\langle nn \rangle(k,z) = \frac{nk^2/m}{z^2 - E_k^2}, \qquad (69)$$

$$\langle \Phi n \rangle(k,z) = -\langle n\Phi \rangle(k,z) = -i\frac{mz}{nk^2}\langle nn \rangle(k,z), \qquad (70)$$

and

$$\langle \Phi\Phi \rangle(k,z) = \frac{m^2}{k^2}\langle v_L v_L \rangle = \left(\frac{mz}{nk^2} \right)^2 \langle nn \rangle(k,z) - \frac{m}{nk^2}, \qquad (71)$$

where $E_k = [gnk^2/m + (k^2/2m)^2]^{1/2}$ is the Bogoliubov single-particle energy. (The term $\sim k^4$ in E_k^2 appears only if one includes the kinetic term, $-\nabla^2 n/4m$ in the pressure.) Substitution of these correlation functions into Eq. (68) yields the usual Bogoliubov result:

$$G(k,z) = \frac{z + gn + k^2/2m}{z^2 - E_k^2}. \qquad (72)$$

In the presence of rotation, we use Eqs. (58) and (59) to write

$$G(k,z) = \frac{1}{D}\left\{ (z + ms^2)\left(z^2 - 4\Omega^2 - 4\frac{(C_1 + C_2)k^2}{nm} \right) + \frac{k^2}{4m}\left(z^2 - 2\frac{C_2}{nm}k^2 \right) \right\}. \qquad (73)$$

The contribution of the modes to the density n_k' of particles of momentum k excited out of the condensate, to leading orders in k^2, is then

$$n_k' = \frac{ms^2}{\omega_T}\left(f(\omega_T) + \frac{1}{2}\right) + \frac{k^2}{8m\Omega}\left[\left[1 + \left(\frac{ms^2}{\Omega}\right)^2\right]f(\omega_I)\right.$$
$$\left. + \frac{1}{2}\left(\frac{ms^2}{\Omega} - 1\right)^2\right]. \tag{74}$$

The contribution of the inertial mode (the final terms) is always finite in two dimensions. However, the contribution of the Tkachenko mode in the soft limit (the first term) leads to a logarithmic divergence of n' in two dimensions at zero temperature [11], and a quadratic divergence at finite temperature, thus destroying the condensation in an infinite system.

It is instructive, finally, to calculate the effect of the lattice modes on the superfluid mass density ρ_s, which is related to the transverse velocity autocorrelation function by [36]

$$\rho_n = \rho - \rho_s = -(mn)^2\lim_{k\to 0}\langle v_T v_T\rangle(k, \omega = 0). \tag{75}$$

Since $v_T = -2\Omega\epsilon_L$, we find from Eq. (40) that

$$\rho_n = mn, \tag{76}$$

that is, the superfluid mass density vanishes. As discussed in Ref. [23] the lattice excitations replenish the sum rule (75), a reflection of the fact that the moment of inertia of a rapidly rotating superfluid is effectively the classical value [37]. The vanishing of ρ_s is consistent with the behavior of $G(k \to 0, 0)$, which according to Josephson's sum rule should approach $-m^2 n_0/\rho_s k^2$ [36], while in fact $G(k \to 0, 0) \to -(\Omega n/C_2)(2m^2/k^4)$.

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