Small-amplitude normal modes of a vortex in a trapped Bose-Einstein condensate

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We consider a rotating axisymmetric trap containing a small Bose-Einstein condensate with a singly quantized vortex near the axis of symmetry. A time-dependent variational Lagrangian analysis yields the small-amplitude dynamics of the vortex and the condensate, directly determining the equations of motion of the coupled normal modes. As found previously from the Bogoliubov equations, there are two rigid dipole modes and one anomalous mode with a negative frequency when seen in the nonrotating laboratory frame.

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I. INTRODUCTION

The achievement of Bose-Einstein condensates in trapped atomic gases [1–3] has stimulated great interest in these systems. An especially intriguing question concerns condensates containing a vortex. The low-energy normal modes are important in characterizing the behavior of such a system. They have been studied extensively [4–10], mostly with the Bogoliubov equations [11] that involve quantum-mechanical normal-mode amplitudes. Consequently, the associated dynamical motion of the vortex core and the condensate center of mass can only be obtained indirectly. Here, we use a variational Lagrangian procedure to provide a more direct and intuitive treatment of the small-amplitude normal modes of a small (weakly interacting) Bose-Einstein condensate containing a singly quantized vortex. As in previous applications [7,12], this approach yields a clear physical picture of the dynamics. We show that it reproduces the three lowest modes as found from the more intricate Bogoliubov analysis, thereby clarifying the physical interpretation. This description is particularly interesting for the anomalous mode that causes the local instability of the vortex in a trap at rest [4,5,9].

The paper is organized as follows. In Sec. II we derive the effective Lagrangian for a slightly off-center vortex in a small condensate and obtain the corresponding equations of motion for the vortex and the condensate. Sections III and IV analyze the motion and discuss the physical interpretation for the two separate modes. Finally, we summarize our results in the Conclusion.

II. EFFECTIVE LAGRANGIAN

We consider a small Bose-Einstein condensate in an axisymmetric harmonic trap with radial and axial frequencies $\omega_\perp$ and ω_z (the ratio $\lambda=\omega_z/\omega_\perp$ characterizes the axial asymmetry). The N-particle condensate contains a singly quantized vortex near the axis of symmetry, and the system rotates with an angular velocity Ω around the symmetry axis. The s-wave scattering length a (assumed positive) characterizes the interparticle interactions. For a general condensate wave function Ψ, the Lagrangian for such a system is given by

$$L(\Psi)=\int dV\bigg[\frac{i}{2}(\Psi^*\partial_t\Psi-\Psi\partial_t\Psi^*)-\Psi^*(H_0-\Omega L_z)\Psi -2\pi\gamma\left|\Psi\right|^4\bigg], \tag{1}$$

where

$$H_0=\frac{1}{2}[-\partial_x^2-\partial_y^2+(x^2+y^2)+\lambda(-\partial_z^2+z^2)] \tag{2}$$

is the Hamiltonian for the noninteracting condensate and $L_z=-i(x\partial_y-y\partial_x)$ is the z component of the angular-momentum operator. All quantities are expressed in dimensionless units (the radial and axial coordinates are scaled with the radial and axial oscillator lengths $d_\perp=\sqrt{\hbar/M\omega_\perp}$ and $d_z=\sqrt{\hbar/M\omega_z}$, and frequencies are scaled with the radial trap frequency $\omega_\perp$). The condensate wave function Ψ is normalized to unity and $\gamma\equiv Na/d_z$ is the small interaction parameter. Variation of this Lagrangian with respect to the condensate wave function Ψ^* leads to the time-dependent Gross-Pitaevskii equation.

In the present application, we use a variational trial wave function with time-dependent parameters. The resulting Lagrangian will contain terms that involve the first time derivatives of these parameters along with the negative of the variational Gross-Pitaevskii Hamiltonian [13,14]. If the time-dependent terms in the Lagrangian are omitted, the two approaches are effectively equivalent. For example, Sec. IV of Ref. [15] uses the time-independent energy functional (namely the Hamiltonian) to study the onset of metastability through the change of sign of the curvature for small transverse displacements. The present work uses the time-dependent Lagrangian functional to determine the actual dynamical motion. From this latter (Lagrangian) perspective, the curvature of the energy functional for small transverse displacements acquires a different and more powerful interpretation, for it determines explicitly the normal-mode frequencies and the corresponding dynamical motion.

To proceed, it is necessary to choose a trial condensate wave function. The condensate wave function is assumed to be unchanged with respect to the noninteracting case along the axis of symmetry, which allows us to use the ground-state Gaussian $\varphi_0(z)=\pi^{-1/4}\exp(-z^2/2)$. The radial wave function is based on the vortex ground state, namely, a twodimensional Gaussian times the vortex factor $x+iy=re^{i\varphi}$. We are interested in the relative motion of the vortex and the condensate, which requires introducing time-dependent parameters for the vortex core's position and the condensate's center of mass. In the present weak-coupling limit, the radius of the vortex core is comparable to the radius of the condensate, so that the displacement $\mathbf{r}_0(t)=[x_0(t),y_0(t)]$ of the vortex, the displacement $\mathbf{r}_1(t)=[x_1(t),y_1(t)]$ of the condensate, and the induced velocity of the condensate $\boldsymbol{\alpha}(t)=[\alpha_x(t),\alpha_y(t)]$ must all be included. Hence we use the following trial function:

$$\Psi_v(x,y,z,t)=\frac{C}{\pi^{3/4}}[(x-x_0)+i(y-y_0)]\times e^{-1/2[(x-x_1)^2+(y-y_1)^2+z^2]}e^{i(\alpha_x x+\alpha_y y)}, \tag{3}$$

where $C^{-2}=1+|\mathbf{r}_1-\mathbf{r}_0|^2$. Since we focus on the dynamical motion of the vortex and the condensate, we ignore the possibility of monopole or quadrupole modes, and the dimensionless Gaussian widths are taken as one. From the time-independent Hamiltonian perspective, the same trial function Eq. (3) served to characterize the stability of the vortex in a small condensate [15], but the present Lagrangian approach also yields the explicit normal modes.

In evaluating the integration in $L(\Psi_v)$, we retain all terms up to second order in the small time-dependent parameters. The resulting effective Lagrangian takes a simpler form when written in terms of new variables $\boldsymbol{\delta}=\boldsymbol{r}_1-\boldsymbol{r}_0$ and $\boldsymbol{\epsilon}=2\boldsymbol{r}_1-\boldsymbol{r}_0$:

$$L_{\rm eff}=-2-\frac{\lambda}{2}-\frac{\gamma}{2\sqrt{2\pi}}+\Omega-\dot{\alpha}_x\epsilon_x-\dot{\alpha}_y\epsilon_y-(\dot{\epsilon}_x-2\dot{\delta}_x)\delta_y+(\dot{\epsilon}_y-2\dot{\delta}_y)\delta_x-\frac{1}{2}\alpha^2+\alpha_x(\delta_y-\Omega\epsilon_y)+\alpha_y(-\delta_x+\Omega\epsilon_x)-\frac{1}{2}\epsilon^2+\Omega\,\boldsymbol{\delta}\cdot\boldsymbol{\epsilon}+\left(-2\Omega+\frac{3}{2}-\frac{\gamma}{\sqrt{2\pi}}\right)\delta^2. \tag{4}$$

The effective Lagrangian has only linear terms in the generalized velocities and hence it yields six coupled first-order differential equations for the displacements and the induced velocity. Elimination of the latter through Lagrange's equations

$$\boldsymbol{\alpha}=\dot{\boldsymbol{\epsilon}}+\hat{\boldsymbol{z}}\times(\Omega\boldsymbol{\epsilon}-\boldsymbol{\delta}) \tag{5}$$

for α_x and α_y leads to two uncoupled pairs of homogeneous equations for the displacements $\boldsymbol{\delta}$ and $\boldsymbol{\epsilon}$:

$$\left(1-\frac{\gamma}{2\sqrt{2\pi}}-\Omega\right)\delta_y(t)+\dot{\delta}_x(t)=0, \tag{6}$$

$$-\left(1-\frac{\gamma}{2\sqrt{2\pi}}-\Omega\right)\delta_x(t)+\dot{\delta}_y(t)=0, \tag{7}$$

$$(\Omega^2-1)\epsilon_x(t)+2\Omega\dot{\epsilon}_y(t)-\ddot{\epsilon}_x(t)=0, \tag{8}$$

$$(\Omega^2-1)\epsilon_y(t)-2\Omega\dot{\epsilon}_x(t)-\ddot{\epsilon}_y(t)=0. \tag{9}$$

Thus the dynamics reduces to two distinct normal modes associated with $\boldsymbol{\delta}$ and $\boldsymbol{\epsilon}$; we consider them separately.

III. ANOMALOUS MODE

The $\boldsymbol{\delta}$ mode with $\boldsymbol{\epsilon}=0$ is the anomalous mode found from the Bogoliubov equations [4,5].

A. Normal-mode amplitude

Since $\boldsymbol{\epsilon}=0$, this mode has $\boldsymbol{r}_0=2\boldsymbol{r}_1$, so that the displacement of the vortex is twice that of the condensate. The corresponding equations of motion have a familiar form,

$$\dot{\delta}_x=\omega_a\delta_y \quad\text{and}\quad \dot{\delta}_y=-\omega_a\delta_x, \tag{10}$$

involving uniform circular motion at the characteristic frequency of the anomalous mode,

$$\omega_a=-1+\frac{\gamma}{2\sqrt{2\pi}}+\Omega. \tag{11}$$

This generic pair of first-order equations has only one solution with a definite sense of rotation, as can be seen by considering the complex variable $\delta_x(t)+i\delta_y(t)\equiv\zeta(t)$ that obeys the differential equation $\dot{\zeta}(t)=-i\omega_a\zeta(t)$. The solution $\zeta(t)=\zeta(0)e^{-i\omega_a t}$ represents a uniform rotation with a sense determined by the sign of the anomalous frequency ω_a. In real form, the solution for the anomalous mode becomes

$$\boldsymbol{\delta}(t)=\frac{1}{\sqrt{2}}\delta_0(\cos\omega_a t,-\sin\omega_a t), \tag{12}$$

where δ_0 is an infinitesimal amplitude and the factor $1/\sqrt{2}$ is chosen to ensure unit normalization.

Before discussing the choice of normalization, it is convenient to focus on the physical displacements of the vortex $\boldsymbol{r}_0=\boldsymbol{\epsilon}-2\boldsymbol{\delta}$ and the condensate $\boldsymbol{r}_1=\boldsymbol{\epsilon}-\boldsymbol{\delta}$. Since $\boldsymbol{\epsilon}$ vanishes for the anomalous mode, we find

$$\boldsymbol{r}_0=\sqrt{2}\delta_0(-\cos\omega_a t,\sin\omega_a t), \tag{13}$$

$$\boldsymbol{r}_1=\frac{1}{2}\sqrt{2}\delta_0(-\cos\omega_a t,\sin\omega_a t). \tag{14}$$

The vortex and the condensate both execute a small-amplitude motion with $\boldsymbol{r}_0$ twice as large as $\boldsymbol{r}_1$. For $\Omega=0$, the anomalous frequency (11) is negative, and the motion is counter clockwise (namely right-circular motion with positive helicity). With increasing rotation frequency Ω, however, the negative anomalous frequency increases towards zero and vanishes at the metastable rotation frequency $\Omega_m = 1-\gamma/(2\sqrt{2\pi})$ [15]. For $\Omega > \Omega_m$, the apparent motion reverses and becomes clockwise (namely left-circular motion in a mathematically negative direction). In the weak-coupling limit, the vortex and condensate move in phase, in contrast to the behavior found in the strong-coupling (Thomas-Fermi) limit [7].

B. Normalization

The present formalism focuses on the dynamical normal modes of the vortex and the condensate. Since this approach uses quite different variables from the quantum-mechanical amplitudes that arise from the Bogoliubov equations, it is valuable to make explicit the connection between the two apparently distinct descriptions. It is convenient to represent the condensate wave function (3) in the hydrodynamic (density-phase) picture $\Psi_v = |\Psi_v| e^{iS}$. Here, the condensate density is $n = |\Psi_v|^2$ and the condensate velocity $\boldsymbol{v} = (\hbar/M)\boldsymbol{\nabla} S$ is related to the phase S. Equivalently, the quantity $\Phi = (\hbar/M)S$ is the velocity potential.

The physically relevant solutions of the equations of motion must correspond to quantum states with positive normalization. In this language, the normalization condition for the dynamic first-order changes of the condensate wave function is [6]

$$\int dV i(n'^{*}\Phi' - \Phi'^{*} n') = \frac{\hbar}{M}, \tag{15}$$

where n' and Φ' are the complex first-order changes of the particle density and the velocity potential, respectively. In our case, the changes in density are obtained by expanding the trial density $n = |\Psi_v|^2 = n_0 + \delta n$, where $n_0 = (r^2/\pi^{3/2})\exp(-r^2)\exp(-z^2)$ is the static density for the condensate with a vortex on the symmetry axis. We find

$$\delta n \approx 2 n_0 \boldsymbol{r}\cdot\left(\boldsymbol{r}_1 - \frac{\boldsymbol{r}_0}{r^2}\right), \tag{16}$$

and use of the normal-mode amplitudes $\boldsymbol{\delta}$ and $\boldsymbol{\epsilon}$ yields the general expression in plane polar coordinates,

$$\delta n = 2 n_0\{\cos\varphi[r(\epsilon_x - \delta_x) - r^{-1}(\epsilon_x - 2\delta_x)] + \sin\varphi[r(\epsilon_y - \delta_y) - r^{-1}(\epsilon_y - 2\delta_y)]\}. \tag{17}$$

This real first-order change can be converted into the complex density variation n' through $\delta n = n' e^{-i\omega t} + n'^{*} e^{i\omega t}$.

For the anomalous mode (12), we obtain

$$n'_a = -\frac{\delta_0}{\sqrt{2}} n_0 \left(r - \frac{2}{r}\right) e^{-i\varphi} \tag{18}$$

(with the time dependence $e^{-i\omega_a t}$ omitted). In the noninteracting limit, the unstable anomalous mode represents a single-particle transition from the vortex condensate with unit angular momentum per particle to the unoccupied axisymmetric Gaussian ground state. As expected from this elementary picture, the azimuthal quantum number (the z component of angular momentum) for this mode is $m = -1$.

In a similar way, the first-order change in the velocity potential follows by expanding the phase of the trial condensate wave function $S = \arctan[(y-y_0)/(x-x_0)] + \boldsymbol{\alpha}\cdot\boldsymbol{r} \approx \varphi + \delta S$, where

$$\delta S = \frac{-y_0 x + x_0 y}{r^2} + \boldsymbol{\alpha}\cdot\boldsymbol{r} \tag{19}$$

is real. Use of Eqs. (5) yields

$$\delta S = \cos\varphi\left[r\dot{\epsilon}_x - \epsilon_y\left(\frac{1}{r} + r\Omega\right) + \delta_y\left(\frac{2}{r} + r\right)\right] + \sin\varphi\left[r\dot{\epsilon}_y + \epsilon_x\left(\frac{1}{r} + r\Omega\right) - \delta_x\left(\frac{2}{r} + r\right)\right], \tag{20}$$

expressed in terms of the normal-mode variables $\boldsymbol{\delta}$ and $\boldsymbol{\epsilon}$. The corresponding complex fluctuation Φ' in the velocity potential follows from $\delta\Phi = (\hbar/M)\delta S = \Phi' e^{-i\omega t} + \Phi'^{*} e^{i\omega t}$. For the anomalous mode (12), we find the complex velocity potential

$$\Phi'_a = \frac{\delta_0 \hbar}{2\sqrt{2} i M}\left(\frac{2}{r} + r\right) e^{-i\varphi}. \tag{21}$$

With the expressions in Eqs. (18) and (21), it is easy to verify that the normalization integral (15) has the proper positive value. The expressions found here for the first-order changes in the density and the velocity potential, n'_a and Φ'_a, coincide with those found from the Bogoliubov approach [5]. If we had omitted either the induced velocity or the displacement of the condensate, we would have found different frequencies, just as in the discussion of vortex stability [15]. The present Lagrangian treatment has the advantage of providing a physical picture of the motion, clarifying the nature of the anomalous mode.

IV. DIPOLE MODE

In a similar manner, we can treat the $\boldsymbol{\epsilon}$ modes. These are dipole modes, since $\boldsymbol{\delta} = 0$ implies that $\boldsymbol{r}_0 = \boldsymbol{r}_1$, and the vortex moves rigidly with the condensate. Equations (8) and (9) show that the x and y motions are uncoupled if $\Omega = 0$. When the system rotates, however, the modes are coupled, and the procedure used in solving Eq. (10) yields two normal modes with frequencies $\omega_+ = 1-\Omega$ and $\omega_- = 1+\Omega$. Our notation $\omega_\pm$ reflects the general construction of the Bogoliubov amplitudes $u_\pm = a^\dagger_\pm \Psi$ and $v_\pm = a_\mp \Psi^*$ for the dipole modes of an axisymmetric trap in the laboratory frame [16]. Here, Ψ is any solution of the Gross-Pitaevskii equation in an axisymmetric trap, and the raising and lowering operators are defined as

$$a^\dagger_\pm = \frac{1}{\sqrt{2}}(a^\dagger_x \pm i a^\dagger_y) \quad \text{and} \quad a_\pm = \frac{1}{\sqrt{2}}(a_x \mp i a_y) \tag{22}$$

with

$$a_\alpha \equiv \frac{1}{\sqrt{2}}\left(x_\alpha + \frac{\partial}{\partial x_\alpha}\right) \quad \text{and} \quad a_\alpha^\dagger \equiv \frac{1}{\sqrt{2}}\left(x_\alpha - \frac{\partial}{\partial x_\alpha}\right), \tag{23}$$

where α represents either the x or y direction. In the laboratory frame, the $+$ mode has positive helicity $\propto e^{i\varphi}$ and rotates in the positive direction with frequency 1 in the present dimensionless units. When the trap rotates, the corresponding frequency becomes $1-\Omega$. Similarly, the $-$ mode has frequency $1+\Omega$ when the trap rotates. It is easy to see that the general solution for the real dipole modes is a linear combination of the two distinct normal modes

$$\boldsymbol{\epsilon}_+(t) = \boldsymbol{\epsilon}_+(0)(\cos\omega_+ t, \sin\omega_+ t), \tag{24}$$

$$\boldsymbol{\epsilon}_-(t) = \boldsymbol{\epsilon}_-(0)(\cos\omega_- t, -\sin\omega_- t), \tag{25}$$

where $\boldsymbol{\epsilon}_+(0), \boldsymbol{\epsilon}_-(0)$ are infinitesimal small amplitudes.

The final expression for the motion of the vortex and the condensate in the dipole modes [note that $\boldsymbol{r}(t) \equiv \boldsymbol{r}_0(t) = \boldsymbol{r}_1(t)$ since $\boldsymbol{\delta}$ vanishes] is

$$\boldsymbol{r}_\pm(t) = r_0(\cos\omega_\pm t, \pm\sin\omega_\pm t), \tag{26}$$

with an infinitesimal amplitude r_0. We again find circular trajectories, and here they are the same for the vortex and for the condensate center of mass, confirming that the modes are indeed the rigid dipole oscillations of the vortex and the condensate. Only the values $|\Omega|<1$ for the external rotation are relevant, since the oscillator binding force cannot balance the centrifugal force for rotation faster than $\omega_\perp$ (which is 1 in our dimensionless units). At $|\Omega|=1$, the noninteracting harmonic-oscillator ground state undergoes a catastrophic alteration arising from the cancellation of the confining potential by the centrifugal potential. This behavior is similar to that of a classical particle moving in a parabolic potential. Within the physical region of the rotation frequency ($|\Omega|<1$), the motion is always counter clockwise for ω_+ and clockwise for ω_-.

We can now check the normalization for the dipole modes. Due to the structure of the dipole modes in Eqs. (24) and (25), the first-order contributions to the density and velocity potential for both modes can be written in the complex form

$$n'_\pm = \boldsymbol{\epsilon}_\pm(0) n_0 \left(r - \frac{1}{r}\right) e^{\pm i\varphi}, \tag{27}$$

$$\Phi'_\pm = \frac{\boldsymbol{\epsilon}_\pm(0)\hbar}{2iM}\left(r \pm \frac{1}{r}\right) e^{\pm i\varphi}, \tag{28}$$

independent of the external rotation frequency Ω. With these expressions, the normalization integral (15) has the correct value for both normal modes with frequencies $\omega_\pm$.

Comparison with the Bogoliubov results [5] shows exact agreement for $\Omega=0$. Without external rotation, the singly quantized positive and negative helicity states $a_\pm^\dagger \varphi_0(x)\varphi_0(y)$ (with φ_0 being a Gaussian) are degenerate with the same excitation frequency $\omega_\pm = 1$. In a frame rotating with positive frequency Ω, the frequency for the two states is split to $1 \mp \Omega$. The occurrence of $\mp$ is analogous to the solution $f(x \mp vt)$ of the one-dimensional wave equation for a pulse $f(x)$ traveling rigidly to the right/left with speed v.

V. CONCLUSION

We have shown that a time-dependent Lagrangian approach provides physical insight into the nature of the low-energy modes of a vortex in a small trapped Bose-Einstein condensate. The anomalous mode is an in-phase circular motion of the vortex and the condensate with different amplitudes. In a rotating frame, the sense of this mode reverses when the external rotation frequency exceeds the metastable frequency [15]. The dipole modes are rigid rotations of the vortex and the condensate together. In terms of the density and phase fluctuations, all these results agree exactly with those from the Bogoliubov approach [5].

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