

# Coupling vortex dynamics with collective excitations in Bose-Einstein condensates

R. P. Teles, V. S. Bagnato, and F. E. A. dos Santos

*Instituto de Física de São Carlos, USP, Caixa Postal 369,*

*13560-970 São Carlos, São Paulo, Brazil*

(Received 10 June 2013)

## Abstract

Here we analyze the collective excitations as well as the expansion of a trapped Bose-Einstein condensate with a vortex line at its center. To this end, we propose a variational method where the variational parameters have to be carefully chosen in order to produce reliable results. Our variational calculations agree with numerical simulations of the Gross-Pitaevskii equation. The system considered here turns out to exhibit four collective modes of which only three can be observed at a time depending **[AU: Please check]** of the trap anisotropy. We also demonstrate that these collective modes can be excited using well established experimental methods such as modulation of the s-wave scattering length.

PACS numbers: 03.75.Kk

## I. INTRODUCTION

In this work, we are interested in the dynamics of a trapped Bose-Einstein condensate (BEC) containing a line vortex at its center. Here we are particularly interested in obtaining the collective oscillation modes of the system which couples the vortex-core oscillations with the oscillations of the condensate external dimensions. The interest in this problem is motivated by the fact that these oscillations can be measured in the laboratory by moving the atomic cloud out of its equilibrium configuration by using the Feshbach resonance in order to modulate the scattering length [1–5]. These oscillations are also studied in other physical systems such as two-species condensates [6], BCS-BEC crossover [7–9], and superfluid Helium [10]. From the theoretical point of view, we are interested in how the size of the vortex core oscillates with respect to the external dimensions of the cloud. The mode with the smallest oscillation frequency is the quadrupole mode which occurs when the longitudinal and radial sizes of the condensate oscillate out of phase. The breathing mode requires more energy to be excited since the change in the density of the atomic cloud imposes a greater resistance against deviation from its equilibrium configuration than in the case of quadrupole excitations [11, 12].

The frequency shifts of quadrupole oscillations due to the presence of a singly charged vortex have been analytically explored for positive scattering lengths using the sum-rule approach [13] as well as the effects of lower-dimensional geometry on the frequency splitting of quadrupole oscillations [14]. In Refs. [15–17], the dynamics of normal modes for a single vortex has been studied using hydrodynamic models, which focus on the vortex motion with respect to the center-of-mass of the condensate. This concept was also used in the case of multicomponent Bose-Einstein condensates [18] as well as in the description of the dynamics of single perturbed vortex lines [19].

Preliminary calculations using a variational calculation with a Gaussian Ansatz, which does not take into account the independent variation of the vortex-core size [3, 6, 18, 20], shows a small shift in the frequencies of the aforementioned modes (Figure 1). This shift has already been obtained via a hydrodynamic approximation in Refs. [12] and [21]. Thus we can expect the frequency of the monopole (breathing) mode to decrease while the quadrupole frequency increases in the presence of the vortex.

To calculate the dynamics of a vortex with charge  $\ell$  in a more consistent way with the physical reality, which allows for the coupling between vortex core and the external dimensions

Figure 1: (Color online) Oscillation frequencies from Gaussian Ansatz without taking into account the independent variation of the vortex-core size. Upper lines correspond to the frequencies of the breathing mode as a function of the harmonic trap anisotropy, whereas lower lines represent the frequencies of the quadrupole mode. Solid (black) lines correspond to a vortex-free Gaussian profile, while dotted (blue) lines describe a profile with a singly charged vortex. Note that  $\varpi$  is normalized by the frequency of the radial direction  $\omega_\rho$ .

of condensate, we could naively use a Thomas-Fermi (TF) Ansatz [22],

$$\psi(\rho, \varphi, z, t) = A(t) \left[ \frac{\rho^2}{\rho^2 + \xi(t)^2} \right]^{\frac{\ell}{2}} \sqrt{1 - \frac{\rho^2}{R_\rho(t)^2} - \frac{z^2}{R_z(t)^2}} \exp \left[ i\ell\varphi + iB_\rho(t) \frac{\rho^2}{2} + iB_z(t) \frac{z^2}{2} \right], \quad (1)$$

where  $R_\rho(t)$  and  $R_z(t)$  are the respective condensate sizes in radial ( $\hat{\rho}$ ) and axial ( $\hat{z}$ ) directions, and  $\xi(t)$  is the size of the vortex core. The variational parameters  $B_\rho(t)$  and  $B_z(t)$  specify the variations of the velocity field  $\delta\mathbf{v} = B_\rho(t)\rho\hat{\rho} + B_z(t)z\hat{z}$ . The next step is to calculate the equations of motion for the five variational parameters ( $\xi, R_\rho, R_z, B_\rho, B_z$ ). Following these calculations, the equations of motion would be linearized. For the Ansatz (1), this procedure leads to imaginary frequencies which are not consistent with the stable configuration where a singly charged ( $\ell = 1$ ) vortex resides at the center of the condensate. The linearized equations of motion can be written in a matrix form according to

$$M\ddot{\delta} + V\delta = 0, \quad (2)$$

where  $\delta$  is the vector with components given by deviations of the variational parameters from their equilibrium values. The solution of (2) is a linear combination of oscillatory modes whose oscillation frequencies obey the equation

$$\prod_n \varpi_n^2 = \det(M^{-1}V) = \frac{\det V}{\det M}. \quad (3)$$

In order to ensure that all frequencies  $\varpi_n$  are real, we must have  $\det V / \det M > 0$ . We know that  $\det V > 0$  since its sign reflects the sign of the variational parameters, which represents the external dimensions of the cloud in the stationary situation. Therefore,  $\det M$  must also be positive. In the case of Ansatz (1) with  $\ell = 1$ , such conditions are not satisfied since  $\det M < 0$ , which indicates that there is something wrong with Ansatz (1). In previous works [17, 18, 23, 24], since the authors did not consider the size of the vortex core as a variational parameter, this problem did not appear. More specifically, the imaginary frequency problem arises for excitation modes where the vortex core and the external size of the cloud oscillate out of phase. This kind of motion generates a velocity field which changes the sign of its radial

component as the distance to the vortex line is increased. Since the **A**nsatz (1) describes only linear variations of this component, it is natural to expect non-physical results in this case.

In **Sec. II**, the necessary requirements for the wave-function phase are discussed in order to give support to our variational method. Section III has the calculation based on the new **A**nsatz and the corresponding equations of motion are obtained. The collective modes considering the coupling between vortex and atomic cloud are obtained via linearization of the equations of motion, thus resulting in new collective oscillations (**sSec. IV**). In **sSec. V**, we showed that such excitation modes can be excited using the scattering-length modulation. The free expansion was also calculated in order to complement a previous work [24]. Finally, **sSec. VII** contains the conclusions on our subject of study.

## II. WAVE-FUNCTION PHASE

We start with the Lagrangian density,

$$\mathcal{L} = \frac{i\hbar}{2} \left( \psi^* \frac{\partial \psi}{\partial t} - \psi \frac{\partial \psi^*}{\partial t} \right) - \frac{\hbar^2}{2m} |\nabla \psi|^2 - V(\mathbf{r}) |\psi|^2 - \frac{g}{2} |\psi|^4, \quad (4)$$

whose extremization leads to the Gross-Pitaevskii equation (GPE):

$$i\hbar \frac{\partial \psi}{\partial t} = \left[ -\frac{\hbar^2}{2m} \nabla^2 + V(\mathbf{r}) + g |\psi|^2 \right] \psi, \quad (5)$$

where  $V(\mathbf{r}) = \frac{1}{2} m \omega_\rho^2 (\rho^2 + \lambda^2 z^2)$  is an external potential, the trap anisotropy is  $\lambda = \omega_z / \omega_\rho$ , and  $g$  is the coupling constant. The complex field  $\psi(\mathbf{r}, t)$  can be written as an amplitude profile multiplied by a respective phase, as follows:

$$\psi(\mathbf{r}, t) = f(w_l, \mathbf{r}) e^{iS(\chi_l, \mathbf{r})}, \quad (6)$$

where

$$S(\chi_l, \mathbf{r}) = \ell \varphi + \sum_l \chi_l \phi_l(\mathbf{r}). \quad (7)$$

We denoted both,  $w_l = w_l(t)$  and  $\chi_l = \chi_l(t)$ , respectively, as the amplitude and phase variational parameters. In principle,  $\{\phi_l(\mathbf{r})\}$  should be a complete set of functions but, in our present approximation, we use only a representative incomplete set of functions. Substituting (6) and (7) into (4), the Lagrangian  ~~$L = \int \mathcal{L} d^3\mathbf{r}$~~   $L = \int \mathcal{L} d^3\mathbf{r}$  becomes

$$L = -\hbar \sum_l \dot{\chi}_l \int d^3\mathbf{r} f^2 \phi_l - \frac{\hbar^2}{2m} \sum_l \chi_l^2 \int d^3\mathbf{r} f^2 |\nabla \phi_l|^2 - \int d^3\mathbf{r} \left( \frac{\hbar^2}{2m} |\nabla f|^2 + V f^2 + \frac{g}{2} f^4 \right). \quad (8)$$

In order to account for the dynamics of all three variational parameters ( $R_\rho$ ,  $R_z$ , and  $\xi$ ) in  $f$  we include a variational phase which also contains three variational parameters. This way we chose the following trial function:

$$S(\rho, z, t) = \ell\varphi + B_\rho(t)\frac{\rho^2}{2} + C(t)\frac{\rho^4}{4} + B_z(t)\frac{z^2}{2}. \quad (9)$$

This allows the radial component of the velocity field to change its sign for different distances to the symmetry axis of the cloud. As the superfluid current is connected to the density variation, it is also desirable that both, amplitude and phase, have the same number of variational parameters. It is worth noticing that, in principle, any other ansatz allowing for such behavior of the velocity field would be equally valid. We chose to add an extra  $\rho^4$  term to the wave-function phase due to its simplicity. The Ansatz (9) also leads to linearized equations of motion (2) with  $\det M > 0$ , which is consistent with the stability of the condensate with a singly charged vortex at its center.

### III. EQUATIONS OF MOTION

Now we correct the Thomas-Fermi Ansatz according to the discussion in Section II. This leads to the following trial function:

$$\begin{aligned} \psi(\mathbf{r}, t) = & \sqrt{\frac{N}{R_\rho(t)^2 R_z(t) A_0 [\xi(t)/R_\rho(t)]}} \left[ \frac{\rho^2}{\rho^2 + \xi(t)^2} \right]^{\frac{\ell}{2}} \sqrt{1 - \frac{\rho^2}{R_\rho(t)^2} - \frac{z^2}{R_z(t)^2}} \\ & \times \exp \left[ i\ell\varphi + iB_\rho(t)\frac{\rho^2}{2} + iC(t)\frac{\rho^4}{4} + iB_z(t)\frac{z^2}{2} \right], \end{aligned} \quad (10)$$

with

$$\begin{aligned} A_0(\alpha) = & \frac{2\pi^{3/2}(\ell)!}{15\alpha^{2\ell}(\frac{3}{2} + \ell)!} \left[ (3 + 2\ell\alpha^2) {}_2F_1 \left( \ell, 1 + \ell; \frac{5}{2} + \ell; -\frac{1}{\alpha^2} \right) \right. \\ & \left. - 2\ell(1 + \alpha^2) {}_2F_1 \left( 1 + \ell, 1 + \ell; \frac{5}{2} + \ell; -\frac{1}{\alpha^2} \right) \right], \end{aligned} \quad (11)$$

where, for simplicity, we define  $\alpha(t) = \xi(t)/R_\rho(t)$ ,  ${}_pF_q(a_1, \dots, a_p; b_1, \dots, b_q; x)$  are the hypergeometric functions,  $\xi(t)$  is the size of the vortex core,  $R_\rho(t)$  is the condensate size in radial direction ( $\hat{\rho}$ ), and  $R_z(t)$  is the condensate size in axial direction ( $\hat{z}$ ). The wave function (10) has an integration domain defined by  $1 - \frac{\rho^2}{R_\rho^2} - \frac{z^2}{R_z^2} \geq 0$ , where the wave function is approximately an inverted parabola (TF-shape), except for the central vortex. The trapping potential shape sets the condensate dimensions. To organize our calculations, we split the Lagrangian so that

it is a sum  $L = L_{\text{time}} + L_{\text{kin}} + L_{\text{pot}} + L_{\text{int}}$  of the following terms:

$$\begin{aligned} L_{\text{time}} &= \frac{i\hbar}{2} \int d^3\mathbf{r} \left[ \psi^*(\mathbf{r}, t) \frac{\partial \psi(\mathbf{r}, t)}{\partial t} - \psi(\mathbf{r}, t) \frac{\partial \psi^*(\mathbf{r}, t)}{\partial t} \right] \\ &= -\frac{N\hbar}{2} \left( D_1 \dot{B}_\rho R_\rho^2 + D_2 \dot{B}_z R_z^2 + \frac{1}{2} D_3 \dot{C} R_\rho^4 \right), \end{aligned} \quad (12)$$

$$\begin{aligned} L_{\text{kin}} &= -\frac{\hbar^2}{2m} \int d^3\mathbf{r} [\nabla \psi^*(\mathbf{r}, t)] [\nabla \psi(\mathbf{r}, t)] \\ &= -\frac{N\hbar^2}{2m} [D_1 B_\rho^2 R_\rho^2 + D_2 B_z^2 R_z^2 + 2D_3 B_\rho C R_\rho^4 + R_\rho^{-2} (\ell^2 D_4 + D_5) + D_6 C^2 R_\rho^6], \end{aligned} \quad (13)$$

$$\begin{aligned} L_{\text{pot}} &= -\frac{1}{2} m \omega_\rho^2 \int d^3\mathbf{r} (\rho^2 + \lambda^2 z^2) \psi^*(\mathbf{r}, t) \psi(\mathbf{r}, t) \\ &= -\frac{N}{2} m \omega_\rho^2 (D_1 R_\rho^2 + \lambda^2 D_2 R_z^2), \end{aligned} \quad (14)$$

$$\begin{aligned} L_{\text{int}} &= -\frac{g}{2} \int d^3\mathbf{r} [\psi^*(\mathbf{r}, t) \psi(\mathbf{r}, t)]^2 \\ &= -\frac{N^2 g D_7}{2 R_\rho^2 R_z}, \end{aligned} \quad (15)$$

with the functions  $D_i(\alpha)$  given by

$$D_1(\alpha) = A_0(\alpha)^{-1} \frac{2\pi^{3/2}(1+\ell)!}{21\alpha^{2\ell}(\frac{5}{2}+\ell)!} \left[ (3+2\ell\alpha^2) {}_2F_1\left(\ell, 2+\ell; \frac{7}{2}+\ell; -\frac{1}{\alpha^2}\right) - 2\ell(1+\alpha^2) {}_2F_1\left(1+\ell, 2+\ell; \frac{7}{2}+\ell; -\frac{1}{\alpha^2}\right) \right], \quad (16)$$

$$D_2(\alpha) = A_0(\alpha)^{-1} \frac{\pi^{3/2}(\ell)!}{4\alpha^{2\ell}(\frac{7}{2}+\ell)!} \left[ (7+2\ell) {}_2F_1\left(\ell, 1+\ell; \frac{7}{2}+\ell; -\frac{1}{\alpha^2}\right) - (5+2\ell) {}_3F_2\left(\ell, 1+\ell, \frac{7}{2}+\ell; \frac{5}{2}+\ell, \frac{9}{2}+\ell; -\frac{1}{\alpha^2}\right) \right], \quad (17)$$

$$D_3(\alpha) = A_0(\alpha)^{-1} \frac{2\pi^{3/2}(2+\ell)!}{27\alpha^{2\ell}(\frac{7}{2}+\ell)!} \left[ (3+2\ell) {}_2F_1\left(\ell, 3+\ell; \frac{9}{2}+\ell; -\frac{1}{\alpha^2}\right) - 2\ell(1+\alpha^2) {}_2F_1\left(1+\ell, 3+\ell; \frac{9}{2}+\ell; -\frac{1}{\alpha^2}\right) \right], \quad (18)$$

$$D_4(\alpha) = A_0(\alpha)^{-1} \frac{2\pi^{3/2}(\ell-1)!}{3\alpha^{2\ell}(\frac{1}{2}+\ell)!} \left[ (1-2\ell\alpha^2) {}_2F_1\left(\ell, 2+\ell; \frac{3}{2}+\ell; -\frac{1}{\alpha^2}\right) + 2\ell(1+\alpha^2) {}_2F_1\left(1+\ell, 2+\ell; \frac{3}{2}+\ell; -\frac{1}{\alpha^2}\right) \right], \quad (19)$$

$$D_5(\alpha) = A_0(\alpha)^{-1} \frac{2\pi^{3/2}(\ell-1)!}{9\alpha^{2\ell}(\frac{1}{2}+\ell)!} \left[ (3+2\ell\alpha^2) {}_2F_1\left(\ell, \ell; \frac{3}{2}+\ell; -\alpha^2\right) - 2\ell(1+\alpha^2) {}_2F_1\left(\ell, 1+\ell; \frac{3}{2}+\ell; -\frac{1}{\alpha^2}\right) \right], \quad (20)$$

$$D_6(\alpha) = A_0(\alpha)^{-1} \frac{2\pi^{3/2}(3+\ell)!}{33\alpha^{2\ell}(\frac{9}{2}+\ell)!} \left[ (3+2\ell\alpha^2) {}_2F_1\left(\ell, 4+\ell; \frac{11}{2}+\ell; -\frac{1}{\alpha^2}\right) - 2\ell(1+\alpha^2) {}_2F_1\left(1+\ell, 4+\ell; \frac{11}{2}+\ell; -\frac{1}{\alpha^2}\right) \right], \quad (21)$$

$$D_7(\alpha) = A_0(\alpha)^{-2} \frac{2\pi^{3/2}(2\ell)!}{\alpha^{4\ell}(\frac{7}{2}+\ell)!} {}_2F_1\left(2\ell, 1+2\ell; \frac{9}{2}+2\ell; -\frac{1}{\alpha^2}\right). \quad (22)$$

For simplicity we can scale the variational parameters of the Lagrangian as well as the time in order to make them dimensionless,

$$R_\rho(t) \rightarrow a_{\text{osc}} r_\rho(t),$$

$$R_z(t) \rightarrow a_{\text{osc}} r_z(t),$$

$$\xi(t) \rightarrow a_{\text{osc}} r_\xi(t),$$

$$B_\rho(t) \rightarrow a_{\text{osc}}^{-2} \beta_\rho(t),$$

$$B_z(t) \rightarrow a_{\text{osc}}^{-2} \beta_z(t),$$

$$C(t) \rightarrow a_{\text{osc}}^{-4} \zeta(t),$$

$$t \rightarrow \omega_\rho^{-1} \tau,$$

where the harmonic-oscillator length is  $a_{\text{osc}} = \sqrt{\hbar/m\omega_\rho} a_{\text{osc}} = \sqrt{\hbar/m\omega_\rho}$  and the dimensionless interaction parameter is  $\gamma = Na_s/a_{\text{osc}} \gamma = Na_s/a_{\text{osc}}$ . Thus the Lagrangian becomes **[COMP:**

**Please match size of square brackets in Eq. (23)]**

$$L = -\frac{N\hbar\omega_\rho}{2} \left[ D_1 r_\rho^2 \left( \dot{\beta}_\rho + \beta_\rho^2 + 1 \right) + D_2 r_z^2 \left( \dot{\beta}_z + \beta_z^2 + \lambda^2 \right) + D_3 r_\rho^4 \left( \frac{1}{2} \dot{\zeta} + 2\beta_\rho \zeta \right) + \ell^2 r_\rho^{-2} (D_4 + D_5) + D_6 \zeta^2 r_\rho^6 + D_7 \frac{4\pi\gamma}{r_\rho^2 r_z} \right]. \quad (23)$$

The Euler-Lagrange equations,

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_i} \right) - \frac{\partial L}{\partial q_i} = 0, \quad (24)$$

for each one of the six variational parameters from Lagrangian (23) lead to the six differential equations:

$$\beta_\rho - \frac{\dot{r}_\rho}{r_\rho} - \frac{D_1' \dot{\alpha}}{2D_1} + \frac{D_3 r_\rho^2 \zeta}{D_1} = 0, \quad (25)$$

$$\beta_z - \frac{\dot{r}_z}{r_z} - \frac{D_2' \dot{\alpha}}{2D_2} = 0, \quad (26)$$

$$\zeta - \frac{D_3 \dot{r}_\rho}{D_6 r_\rho} - \frac{D_3 \dot{\alpha}}{4D_6 r_\rho^2} + \frac{D_3 \beta_\rho}{D_6 r_\rho^2} = 0, \quad (27)$$

$$D_1 r_\rho \left( \dot{\beta}_\rho + \beta_\rho^2 + 1 \right) + D_3 r_\rho^3 \left( \dot{\zeta} + 4\beta_\rho \zeta \right) - \frac{\ell^2}{r_\rho^3} (D_4 + D_5) + 3D_6 \zeta^2 r_\rho^5 - D_7 \frac{4\pi\gamma}{r_\rho^3 r_z} = 0, \quad (28)$$

$$D_2 r_z \left( \dot{\beta}_z + \beta_z^2 + \lambda^2 \right) - D_7 \frac{2\pi\gamma}{r_\rho^2 r_z^2} = 0, \quad (29)$$

$$D_1' r_\rho^2 \left( \dot{\beta}_\rho + \beta_\rho^2 + 1 \right) + D_2' r_z^2 \left( \dot{\beta}_z + \beta_z^2 + \lambda^2 \right) + D_3' r_\rho^4 \left( \frac{1}{2} \dot{\zeta} + 2\beta_\rho \zeta \right) + \frac{\ell^2}{r_\rho^2} (D_4' + D_5') + D_6' \zeta^2 r_\rho^6 - D_7' \frac{4\pi\gamma}{r_\rho^2 r_z} = 0. \quad (30)$$

Solving these equations for the parameters in the wave-function phase, we have:

$$\beta_\rho = \frac{\dot{r}_\rho}{r_\rho} + F_1 \dot{\alpha}, \quad (31)$$

$$\beta_z = \frac{\dot{r}_z}{r_z} + F_2 \dot{\alpha}, \quad (32)$$

$$\zeta = F_3 \frac{\dot{\alpha}}{r_\rho^2}, \quad (33)$$

where

$$F_1 = \frac{D_3' D_3 - 2D_1' D_6}{4(D_3'^2 - D_1' D_6)}, \quad (34)$$

$$F_2 = \frac{D_2'}{2D_2}, \quad (35)$$

$$F_3 = \frac{2D_1' D_3 - D_1 D_3'}{4(D_3'^2 - D_1' D_6)}. \quad (36)$$



Replacing (31), (32), and (33) into [Equations.](#) (28), (29), and (30), we reduce our six coupled equations to only three, which are given by:

$$D_1 (\ddot{r}_\rho + r_\rho) + G_1 r_\rho \ddot{\alpha} + G_2 r_\rho \dot{\alpha}^2 + G_3 \dot{r}_\rho \dot{\alpha} - G_4 \frac{\ell^2}{r_\rho^3} - D_7 \frac{4\pi\gamma}{r_\rho^3 r_z} = 0, \quad (37)$$

$$D_2 (\ddot{r}_z + \lambda^2 r_z) + G_5 r_z \ddot{\alpha} + G_6 r_z \dot{\alpha}^2 + G_7 \dot{r}_z \dot{\alpha} - D_7 \frac{2\pi\gamma}{r_\rho^2 r_z^2} = 0, \quad (38)$$

$$\begin{aligned} D'_1 r_\rho (\ddot{r}_\rho + r_\rho) + D'_2 r_z (\ddot{r}_z + \lambda^2 r_z) + (G_8 r_\rho^2 + G_9 r_z^2) \ddot{\alpha} + (G_{10} r_\rho^2 + G_{11} r_z^2) \dot{\alpha}^2 \\ + (G_{12} r_\rho \dot{r}_\rho + G_{13} r_z \dot{r}_z) \dot{\alpha} + G_{14} \frac{\ell^2}{r_\rho^2} + D'_7 \frac{4\pi\gamma}{r_\rho^2 r_z} = 0, \end{aligned} \quad (39)$$

with

$$G_1 = D_1 F_1 + D_3 F_3, \quad (40)$$

$$G_2 = D_1 (F_1^2 + F_1') + D_3 (4F_1 F_3 + F_3') + 3D_6 F_3^2, \quad (41)$$

$$G_3 = 2(D_1 F_1 + D_3 F_3) = 2G_1, \quad (42)$$

$$G_4 = D_4 + D_5, \quad (43)$$

$$G_5 = D_2 F_2, \quad (44)$$

$$G_6 = D_2 (F_2^2 + F_2'), \quad (45)$$

$$G_7 = 2D_2 F_2 = 2G_5, \quad (46)$$

$$G_8 = D'_1 F_1 + \frac{1}{2} D'_3 F_3, \quad (47)$$

$$G_9 = D'_2 F_2, \quad (48)$$

$$G_{10} = D'_1 (F_1^2 + F_1') + D'_3 \left( \frac{1}{2} F_3' + 2F_1 F_3 \right) + D'_6 F_3^2, \quad (49)$$

$$G_{11} = D'_2 (F_2^2 + F_2'), \quad (50)$$

$$G_{12} = 2D'_1 F_1 + D'_3 F_3, \quad (51)$$

$$G_{13} = 2D'_2 F_2 = 2G_9, \quad (52)$$

$$G_{14} = D'_4 + D'_5. \quad (53)$$

The terms  $D_1 r_\rho$ ,  $D_2 \lambda^2 r_z$ ,  $D'_1 r_\rho^2$ , and  $D'_2 r_z^2$  come from the trapping term  $\cancel{L}_{\text{pot}} L_{\text{pot}}$ , which can be neglected in the case of a freely expanding condensate. The parameter  $\gamma$  indicates the terms generated by the atomic interaction potential, while the fractions proportional to  $r_\rho^{-2}$  and  $r_\rho^{-3}$  come from the kinetic-energy contribution due to the presence of the vortex with charge  $\ell$ . The remaining factors represent the coupling between the outer dimensions of the condensate and the vortex core.

Making the velocities  $(\dot{r}_\rho, \dot{r}_z, \dot{\alpha})$  and accelerations  $(\ddot{r}_\rho, \ddot{r}_z, \ddot{\alpha})$  equal to zero leads to the

equations for the stationary solution:

$$D_1 r_{\rho 0} = G_4 \frac{\ell^2}{r_{\rho 0}^3} + D_7 \frac{4\pi\gamma}{r_{\rho 0}^3 r_{z0}}, \quad (54)$$

$$D_2 \lambda^2 r_{z0} = D_7 \frac{2\pi\gamma}{r_{\rho 0}^2 r_{z0}^2}, \quad (55)$$

$$D'_1 r_{\rho 0}^2 + D'_2 \lambda^2 r_{z0}^2 = -G_{14} \frac{\ell^2}{r_{\rho 0}^2} - D'_7 \frac{4\pi\gamma}{r_{\rho 0}^2 r_{z0}}, \quad (56)$$

where  $r_\rho$ ,  $r_z$ , and  $r_\xi$  take their respective equilibrium values  $r_{\rho 0}$ ,  $r_{z0}$ , and  $r_{\xi 0}$ . We apply ~~the~~ Newton's method to solve the coupled stationary equations (54)–(56). The value of the atomic interaction parameter used from now on in this paper is  $\gamma = 800$ , which is close to the value used in ~~R~~rubidium experiments [25].

#### IV. COLLECTIVE EXCITATIONS

For small deviations from the equilibrium configuration, we assume  $r_\rho(t) \rightarrow r_{\rho 0} + \delta\rho(t)$ ,  $r_z(t) \rightarrow r_{z0} + \delta z(t)$ , and  $\alpha(t) \rightarrow \alpha_0 + \delta\alpha(t)$ , and neglect all terms of order ~~two~~2 or higher in (37)–(39). This leads to the linearized matrix equation

$$\begin{pmatrix} D_1 & 0 & G_1 r_{\rho 0} \\ 0 & D_2 & G_5 r_{z0} \\ D'_1 r_{\rho 0} & D'_2 r_{z0} & G_8 r_{\rho 0}^2 + G_9 r_{z0}^2 \end{pmatrix} \begin{pmatrix} \ddot{\delta\rho} \\ \ddot{\delta z} \\ \ddot{\delta\alpha} \end{pmatrix} + \begin{pmatrix} D_1 + 3G_4 \frac{\ell^2}{r_{\rho 0}^4} + D_7 \frac{12\pi\gamma}{r_{\rho 0}^4 r_{z0}} & D_7 \frac{4\pi\gamma}{r_{\rho 0}^3 r_{z0}^2} & D'_1 r_{\rho 0} - G'_4 \frac{\ell^2}{r_{\rho 0}^3} - D'_7 \frac{4\pi\gamma}{r_{\rho 0}^3 r_{z0}} \\ D_7 \frac{4\pi\gamma}{r_{\rho 0}^3 r_{z0}^2} & D_2 \lambda^2 + D_7 \frac{4\pi\gamma}{r_{\rho 0}^2 r_{z0}^3} & D'_2 \lambda^2 r_{z0} - D'_7 \frac{2\pi\gamma}{r_{\rho 0}^2 r_{z0}^2} \\ 2D'_1 r_{\rho 0} - 2G_{14} \frac{\ell^2}{r_{\rho 0}^3} - D'_7 \frac{8\pi\gamma}{r_{\rho 0}^3 r_{z0}} & 2D'_2 \lambda^2 r_{z0} - D'_7 \frac{4\pi\gamma}{r_{\rho 0}^2 r_{z0}^2} & D''_1 r_{\rho 0}^2 + D''_2 \lambda^2 r_{z0}^2 + G'_{14} \frac{\ell^2}{r_{\rho 0}^2} + D''_7 \frac{4\pi\gamma}{r_{\rho 0}^2 r_{z0}} \end{pmatrix} \begin{pmatrix} \delta\rho \\ \delta z \\ \delta\alpha \end{pmatrix} = 0, \quad (57)$$

which defines the matrices  $M$  and  $V$ , appearing in Eq. (2). Solving the characteristic equation,

$$\det(M^{-1}V - \varpi^2 I) = 0, \quad (58)$$

results in the frequencies of the collective modes of oscillation. Now the determinants  $\det M$  and  $\det V$  are both positive for  $\ell = 1$ . This means that a trapped condensate with a central singly~~=~~-charged vortex is described by a stable state. Since (58) is a cubic equation of  $\varpi^2$ , we have three pairss of frequencies  $\pm\varpi_n$  ( $n = z, \rho, \xi$ ). There are three frequencies  $\varpi_n$  and four modes of oscillation in total, of which only three modes can be simultaneously observed depending on the anisotropy  $\lambda$  of harmonic potential as shown in ~~f~~Fig. 2. Among these four modes, two of them represent monopole oscillations, while the other two represent quadrupole

(a)

$\ell\ell=$

12

Figure 2: (Color online) Oscillation frequencies as a function of trap anisotropy in a condensate containing a singly (a) and doubly (b) charged vortex at its center. Solid (black) line is  $\varpi_\xi$ , dashed (red) line is  $\varpi_\rho$ , and dotted (blue) line is  $\varpi_z$ .

(b)

$\mathbb{R}_{Q_1}$

(c)

$\mathbb{R}_{Q_2}$

Figure 3: (Color online) Schematic representation of collective modes.  $B_1$  mode has all components oscillating in phase.  $B_2$  mode has  $r_\xi$  oscillating out of phase with  $r_\rho$  and  $r_z$ .  $Q_1$  mode has  $r_z$  oscillation out of phase with  $r_\xi$  and  $r_\rho$ .  $Q_2$  mode has  $r_\rho$  oscillation out of phase with  $r_\xi$  and  $r_z$ .

oscillations of the atomic cloud. The  $B_1$  mode [COMP: Please change "3a" to "3(a)" (add left paren)]([Fig. 3a]) is characterized by having all condensate components  $r_i$  ( $i = z, \rho, \xi$ ) oscillating in phase; however,  $B_2$  mode [COMP: Please change "3c" to "3(c)"]([Fig. 3c]) presents  $r_\xi$  oscillating out of phase with  $r_\rho$  and  $r_z$ . The  $Q_1$  mode [COMP: Please change "3b" to "3(b)"]([Fig. 3b]) shows that  $r_z$  oscillation is out of phase with  $r_\xi$  and  $r_\rho$ , which are in phase with each other. However, in  $Q_2$  mode [COMP: Please change "3d" to "3(d)"]([Fig. 3d]) the oscillations of  $r_z$  and  $r_\xi$  are in phase with each other, [AU: Please check]withbeing the  $r_\rho$  oscillation being out of phase with them. Extrapolating to an ideal situation where  $\gamma = 0$ , the equations of motions (37)–(39) can be decoupled. This way, the  $\varpi_z$  (lower frequency) represents only a  $r_z$  oscillation,  $\varpi_\rho$  (middle frequency) represents only a  $r_\rho$  oscillation, and  $\varpi_\xi$  (upper frequency) represents only a  $r_\xi$  oscillation.

Figure 4: Fourier-transformed temporal evolution of  $\langle \rho^2 \rangle$  obtained from a numerical simulation of the GPE. We set  $\gamma = 800$ ,  $\ell = 1$ ,  $\tilde{\mu} = 20.74$ , and  $\lambda = 0.9$ .  $\varpi_n$  are the frequencies of the oscillation modes from less energetic ( $\varpi_z$ ) to more energetic ( $\varpi_\xi$ ). The analytical values are  $\varpi_z = 1.317$ ,  $\varpi_\rho = 2.166$ , and  $\varpi_\xi = 8.874$ .

In order to validate our results (fFig. 4), numerical simulations were performed using a direct simulation of GPE based on the Fourier spectral method, where the Fourier components of  $\psi(\mathbf{r}, t)$  were computed using fast Fourier transformations [26]. As the initial condition, we considered a small perturbation to the equilibrium configuration. By Fourier transforming the expectation value  $\langle \rho^2 \rangle$ , it was possible to reproduce the excitation spectrum. Frequency values  $\varpi_n$  in the variational calculations differ from numerical values by less than 1%.

In [COMP: Please change "2a" to "2(a)"]fFig. 2a, for  $0.1 \leq \lambda \leq 1$ , exist two  $Q_2$ -like modes. The difference between them comes from the fact that vortex-core oscillation amplitude is two orders of magnitude lower at the less energetic mode. The same happens when  $\ell = 2$  [COMP: Please change "2b" to "2(b)"] (fFig. 2b), i.e., the vortex core is almost still for the lower frequency in the same interval of  $\lambda$ .

The solid lines in fFig. 2 correspond to the mode with largest amplitude for the vortex-core oscillations. As can be seen, the excitation frequency  $\varpi_\xi$  of this mode lowers as the vortex circulation increases. It means that the energy necessary to excite it will be lower if  $\ell$  is increased. However, we must point out that our results apply only for the cases where  $r_\xi \ll r_\rho$ .

## V. SCATTERING-LENGTH MODULATION

One of the mechanisms used for exciting collective modes is via modulation of the s-wave scattering length. This technique has been already applied to excite the lowest-lying quadrupole mode in a Lithium experiment [1]. Therefore, we consider the time-dependent scattering length:

$$a_s(t) = a_0 + \delta a \cos(\Omega t). \quad (59)$$

This is equivalent to making  $\gamma \rightarrow \gamma(\tau)$ , thus giving:

$$\gamma(\tau) = \gamma_0 + \delta\gamma \cos(\Omega\tau). \quad (60)$$

Where  $\gamma_0$  is the average value of the interaction parameter  $\gamma(\tau)$ ,  $\delta\gamma$  is the modulation amplitude, and  $\Omega$  is the excitation frequency. Substituting (60) into (57) and keeping only first-order terms ( $\delta\rho$ ,  $\delta z$ ,  $\delta\alpha$ , and  $\delta\gamma$ ), we obtain a nonhomogeneous linear equation,

$$M\ddot{\delta} + V\delta = P \cos(\Omega\tau), \quad (61)$$

with

$$P = 2\pi\delta\gamma \begin{pmatrix} \frac{2D_\tau}{r_{\rho 0}^3 r_{z 0}} \\ \frac{D_\tau}{r_{\rho 0}^2 r_{z 0}^2} \\ \frac{D'_\tau}{r_{\rho 0}^2 r_{z 0}} \end{pmatrix}. \quad (62)$$

Figure 5: Numerical solution of Eqs. (37)–(39) with a time-dependent interaction  $\gamma(\tau)$  (a), (b), and (c). (d) is the excitation spectrum obtained from variational calculation, where  $\varpi_\xi \approx 6.13$  is close to the value calculated in Eq. (58). We excited the collective mode  $Q_2$  ( $\varpi_\xi = 6.21$ ) of a condensate with a cigar shape ( $\lambda = 0.1$ ,  $\gamma_0 = 800$ ) via scattering-length modulation with amplitude  $\delta\gamma = 0.4$  and frequency  $\Omega = 6$ .

$$(61)$$

$$\ell =$$

$$12$$

Figure 6: (Color online) Ratio between vortex-core size and radial cloud size for different trap anisotropies while in free expansion. The solid (black) line corresponds to a prolate condensate ( $\lambda = 0.1$ ), the dashed (blue) line to the isotropic case ( $\lambda = 1$ ), and the dotted (red) line to an oblate condensate ( $\lambda = 8$ ).

A particular solution of (61) is

$$\delta_\gamma(\tau) = (M^{-1}V - \Omega^2)^{-1} M^{-1}P \cos(\Omega\tau). \quad (63)$$

Projecting the vector  $\delta_\gamma(\tau)$  in the base  $\delta_n$  ( $n = z, \rho, \xi$ ) of the eigenvectors of the homogenous equation associated to Eq. (61), we obtain

$$\langle \delta_n | \delta_\gamma(\tau) \rangle = \frac{\langle \delta_n | M^{-1}P \rangle}{\varpi_n^2 - \Omega^2} \cos(\Omega\tau). \quad (64)$$

Since the scalar product  $|\langle \delta_n | M^{-1}P \rangle|$  is always positive, it shows that specific collective modes can be excited using a scattering-length modulation with small amplitude  $\delta\gamma$  and frequency  $\Omega$  close to the one of the resonance frequency  $\varpi_n$ . In Fig. 5, we see the results from a numerical solution of Eqs. (37)–(39) considering a time-dependent interaction  $\gamma(\tau)$  according to Eq. (60). There we can see the beat behavior corresponding to a superposition of the frequencies  $\Omega = 6$  and  $\varpi_\xi = 6.13$ .

## VI. FREE EXPANSION

The time-of-flight pictures constitute the most common method to measure vortices in BEC. This method consists in switching off the magneto-optical trap and letting the atomic cloud expand freely for some time, typically ten milliseconds, and then taking a picture of the expanded cloud [5, 27–31]. For this purpose, we use the equations of motion (37)–(39) without

the terms arising from the harmonic potential, i.e.,

$$D_1 \ddot{r}_\rho + G_1 r_\rho \ddot{\alpha} + G_2 r_\rho \dot{\alpha}^2 + G_3 \dot{r}_\rho \dot{\alpha} - \frac{G_4}{r_\rho^3} - \frac{4D_7 \gamma}{r_\rho^3 r_z} = 0, \quad (65)$$

$$D_2 \ddot{r}_z + G_5 r_z \ddot{\alpha} + G_6 r_z \dot{\alpha}^2 + G_7 \dot{r}_z \dot{\alpha} - \frac{2D_7 \gamma}{r_\rho^2 r_z^2} = 0, \quad (66)$$

$$\begin{aligned} D'_1 r_\rho \ddot{r}_\rho + D'_2 r_z \ddot{r}_z + (G_8 r_\rho^2 + G_9 r_z^2) \ddot{\alpha} + (G_{10} r_\rho^2 + G_{11} r_z^2) \dot{\alpha}^2 \\ + (G_{12} r_\rho \dot{r}_\rho + G_{13} r_z \dot{r}_z) \dot{\alpha} + \frac{G_{14}}{r_\rho^3} + \frac{4D'_7 \gamma}{r_\rho^2 r_z} = 0, \end{aligned} \quad (67)$$

whose initial conditions are given by the stationary equations (54)–(56).

In Fig. 6, we have the ratio between vortex-core size ( $r_\xi$ ) and radial cloud size ( $r_\rho$ ) during free expansion for three different initial trap configurations. In general, the vortex core expands faster than the condensate at early times, going to the same rate of expansion at large times. The prolate condensate ( $\lambda = 0.1$ ) has an almost constant ~~the~~ ratio  $r_\xi/r_\rho$  during the entire expansion. For the isotropic ( $\lambda = 1$ ) and oblate ( $\lambda = 10$ ) cases, this ratio increases rapidly in the initial stage of the expansion until it converges to a constant value. These results agree with our previous work [24], where **[COMP: Please change "6b" to "6(b)"]** Fig. 6b could not be calculated since the authors considered the healing length as an approximation to the vortex-core radius which is only valid for  $\ell = 1$ . Such an agreement indicates the fact that, indeed, the vortex-core size is always close to the instantaneous healing length during the expansion. Moreover, since the radial component of the velocity field always points outwards from the cloud, the extra  $\rho^4$  term in the wave-function phase is not necessary for a consistent description of the system.

## VII. CONCLUSIONS

In this paper, we proposed a modification in the wave-function phase commonly used with the variational method which corrects the imaginary frequencies of collective modes when we have a parameter describing non-physical vortex-core dynamics with  $\ell = 1$ .

Here we consider variational phase parameters corresponding to each parameter in wave-function amplitude, respectively. This way, we were able to describe the dynamics of both vortex core and the external dimensions of the condensate, which agrees with the numerical simulations of the GPE. Although we observe four modes of oscillation in total, only three of them can be simultaneously observed depending on the trap anisotropy. We also demonstrate that these oscillation modes can be excited by modulating the ~~ss~~-wave scattering length using the same experimental techniques as in Ref. [1].

Finally, we analyzed the time-of-flight dynamics of the vortex core with different circulations in order to complement the results in Ref. [24].

## ACKNOWLEDGMENTS

We acknowledge ~~the~~ financial support ~~of~~ from the National Council for the Improvement of Higher Education (CAPES) and from the State of São Paulo Foundation for Research Support (FAPESP).

- 
- [1] S. E. Pollack, D. Dries, R. G. Hulet, K. M. F. Magalhães, E. A. L. Henn, E. R. F. Ramos, M. A. Caracanhas, and V. S. Bagnato, Phys. Rev. A **81**, 053627 (2010) [Issn: 1050-2947; Coden: PLRAAN] [DOI: 10.1103/PhysRevA.81.053627] .
  - [2] S. Stringari, Phys. Rev. Lett. **77**, 2360 (1996) [Issn: 0031-9007; Coden: PRLTAO] [DOI: 10.1103/PhysRevLett.77.2360] .
  - [3] V. M. Pérez-García, H. Michinel, J. I. Cirac, M. Lewenstein, and P. Zoller, Phys. Rev. Lett. **77**, 5320 (1996) [Issn: 0031-9007; Coden: PRLTAO] [DOI: 10.1103/PhysRevLett.77.5320] .
  - [4] F. Dalfovo, S. Giorgini, L. P. Pitaevskii, and S. Stringari, Rev. Mod. Phys. **71**, 463 (1999) [Issn: 0034-6861; Coden: RMPHAT] [DOI: 10.1103/RevModPhys.71.463] .
  - [5] P. W. Courteille, V. S. Bagnato, and V. I. Yukalov, Laser Phys. **11**, 659 (2001) .
  - [6] T. Busch, J. I. Cirac, V. M. Pérez-García, and P. Zoller, Phys. Rev. A **56**, 2978 (1997) [Issn: 1050-2947; Coden: PLRAAN] [DOI: 10.1103/PhysRevA.56.2978] .
  - [7] Z. Zhang and W. V. Liu, Phys. Rev. A **83**, 023617 (2011) [Issn: 1050-2947; Coden: PLRAAN] [DOI: 10.1103/PhysRevA.83.023617] .
  - [8] H. Heiselberg, Phys. Rev. Lett. **93**, 040402 (2004) [Issn: 0031-9007; Coden: PRLTAO] [DOI: 10.1103/PhysRevLett.93.040402] .
  - [9] A. Altmeyer, S. Riedl, C. Kohstall, M. J. Wright, R. Geursen, M. Bartenstein, C. Chin, J. H. Denschlag, and R. Grimm, Phys. Rev. Lett. **98**, 040401 (2007) [Issn: 0031-9007; Coden: PRLTAO] [DOI: 10.1103/PhysRevLett.98.040401] .
  - [10] M. Človečko, E. Gažo, M. Kupka, and P. Skyba, Phys. Rev. Lett. **100**, 155301 (2008) [Issn: 0031-9007; Coden: PRLTAO] [DOI: 10.1103/PhysRevLett.100.155301] .
  - [11] C. J. Pethick and H. Smith, *Bose-Einstein Condensation in Dilute Gases*, 2nd ed. (Cambridge University Press, Cambridge, [UK](#), 2008) .
  - [12] L. P. Pitaevskii and S. Stringari, *Bose-Einstein Condensation*, ~~first~~<sup>1st</sup> ed.~~ition~~ (Oxford University Press, [New York](#)~~ine~~, 2003) .
  - [13] F. Zambelli and S. Stringari, Phys. Rev. Lett. **81**, 1754 (1998) [Issn: 0031-9007; Coden: PRLTAO] [DOI: 10.1103/PhysRevLett.81.1754] .
  - [14] A. Banerjee and B. Tanatar, Phys. Rev. A **72**, 053620 (2005) [Issn: 1050-2947; Coden: PLRAAN] [DOI: 10.1103/PhysRevA.72.053620] .
  - [15] A. A. Svidzinsky and A. L. Fetter, Phys. Rev. A **62**, 063617 (2000) [Issn: 1050-2947; Coden: PLRAAN] [DOI: 10.1103/PhysRevA.62.063617] .
  - [16] A. A. Svidzinsky and A. L. Fetter, Phys. Rev. Lett. **84**, 5919 (2000) [Issn: 0031-9007; Coden: PRLTAO] [DOI: 10.1103/PhysRevLett.84.5919] .



- PRLTAO] [DOI: 10.1103/PhysRevLett.84.5919] .
- [17] M. Linn and A. L. Fetter, Phys. Rev. A **61**, 063603 (2000) [Issn: 1050-2947; Coden: PLRAAN] [DOI: 10.1103/PhysRevA.61.063603] .
- [18] V. M. Pérez-García and J. J. García-Ripoll, Phys. Rev. A **62**, 033601 (2000) [Issn: 1050-2947; Coden: PLRAAN] [DOI: 10.1103/PhysRevA.62.033601] .
- [19] L. Koens and A. M. Martin, Phys. Rev. A **86**, 013605 (2012) [Issn: 1050-2947; Coden: PLRAAN] [DOI: 10.1103/PhysRevA.86.013605] .
- [20] V. M. Pérez-García, H. Michinel, J. I. Cirac, M. Lewenstein, and P. Zoller, Phys. Rev. A **56**, 1424 (1997) [Issn: 1050-2947; Coden: PLRAAN] [DOI: 10.1103/PhysRevA.56.1424] .
- [21] A. A. Svidzinsky and A. L. Fetter, Phys. Rev. A **58**, 3168 (1998) [Issn: 1050-2947; Coden: PLRAAN] [DOI: 10.1103/PhysRevA.58.3168] .
- [22] D. H. J. O'Dell and C. Eberlein, Phys. Rev. A **75**, 013604 (2007) [Issn: 1050-2947; Coden: PLRAAN] [DOI: 10.1103/PhysRevA.75.013604] .
- [23] F. Dalfovo and M. Modugno, Phys. Rev. A **61**, 023605 (2000) [Issn: 1050-2947; Coden: PLRAAN] [DOI: 10.1103/PhysRevA.61.023605] .
- [24] R. P. Teles, F. E. A. dos Santos, M. A. Caracanhas, and V. S. Bagnato, Phys. Rev. A **87**, 033622 (2013) [Issn: 1050-2947; Coden: PLRAAN] [DOI: 10.1103/PhysRevA.87.033622] .
- [25] E. A. de Lima Henn, Ph.D. thesis, Instituto de Física de São Carlos, Universidade de São Paulo, São Carlos, ~~(2008)~~ .
- [26] G. R. Dennis, J. J. Hope, and M. T. Johnsson, Comput. Phys. Commun. **184**, 201 (2013) [Issn: 0010-4655; Coden: CPHCBZ] [DOI: 10.1016/j.cpc.2012.08.016] .
- [27] W. Ketterle, MIT Phys.~~ics~~ Annu.~~al~~ **2001**, 44 (2001) [AU: Please check [27]].
- [28] E. A. L. Henn, J. A. Seman, E. R. F. Ramos, M. Caracanhas, P. Castilho, E. P. Olímpio, G. Roati, D. V. Magalhães, K. M. F. Magalhães, and V. S. Bagnato, Phys. Rev. A **79**, 043618 (2009) [Issn: 1050-2947; Coden: PLRAAN] [DOI: 10.1103/PhysRevA.79.043618] .
- [29] E. A. L. Henn, J. A. Seman, G. Roati, K. M. F. Magalhães, and V. S. Bagnato, Phys. Rev. Lett. **103**, 045301 (2009) [Issn: 0031-9007; Coden: PRLTAO] [DOI: 10.1103/PhysRevLett.103.045301] .
- [30] F. Chevy, K. W. Madison, and J. Dalibard, Phys. Rev. Lett. **85**, 2223 (200~~20~~) [Issn: 0031-9007; Coden: PRLTAO] [DOI: 10.1103/PhysRevLett.85.2223] .
- [31] B. P. Anderson, P. C. Haljan, C. E. Wieman, and E. A. Cornell, Phys. Rev. Lett. **85**, 2857 (2000) [Issn: 0031-9007; Coden: PRLTAO] [DOI: 10.1103/PhysRevLett.85.2857] .