Coupling the vortex dynamics with collective excitations in Bose-Einstein Condensate due modulation of the scattering length

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Abstract

Here we made a study and implementation of suitable phases for a better description of physical systems using the variational method of variable parameters. Generally the condensate phases must be polynomials of degree equal to or greater than two. Thus taking a new appropriate phase was possible to calculate the collective modes of a condensate containing a single vortex along of the axial axis (cylindrical symmetry), as well as its free expansion. As result, a degeneracy was opened on each oscillatory mode already known, monopole (breathing) and quadrupole, being associated to vortex core oscillation in or out phase with respect to the axial radius. The high energetic mode just can be, effectively, seen using the modulation of scattering length.

I. INTRODUCTION

In this work, we are interested in the dynamics of a trapped condensate containing a line vortex at its center, i.e., in obtaining the collective oscillation modes of the system which couples the vortex core oscillation with the oscillation of the condensate dimensions. The interest in this problem came from the fact that these oscillations can be measured in the laboratory by moving the atomic cloud out of its equilibrium configuration by using the Feshbach resonance in order to modulate the scattering length [1–5]. These oscillations are also studied in other physical systems such as: two-species condensates [6], BCS-BEC crossover [7–9], and superfluid Helium [10]. From the theoretical point of view, we are interested on how the size of the vortex core oscillates with respect to the external dimensions of the cloud. The mode with the smallest frequency of oscillation is the quadrupole mode which occurs when the longitudinal and radial sizes of condensate oscillate out phase. The breathing mode requires more energy to be excited since the change in the density of the atomic cloud imposes a greater resistance against deviation from its equilibrium configuration than for quadrupole excitations [11, 12].

In Refs. [13–15], the dynamics of normal modes for a single vortex was studied using hydrodynamics models, which focus on movement of the vortex with respect to the center of mass of the condensate. This concept was also used in the case of multicomponent Bose-Einstein condensates [16].

Preliminary calculations using a variational calculation with a Gaussian Ansatz, which does not take into account the size of the vortex core [3, 6, 16, 17], shows a small shift in the frequencies of the aforementioned modes (Figure 1). This shift has already been obtained via a hydrodynamic approximation in Refs. [12] and [18]. Thus we can expect that the frequency of the monopole (breathing) mode to decrease while the quadrupole frequency increases in the presence of the vortex.

To calculate this in a way more consistent with the physical reality which allows for the coupling between vortex core and the external dimensions of condensate, we could naïvely use a Thomas-Fermi (TF) Ansatz [19]

$$\psi\left(\rho,\varphi,z,t\right)=A\left(t\right)\left[\frac{\rho^{2}}{\rho^{2}+\xi\left(t\right)^{2}}\right]^{\frac{1}{2}}\sqrt{1-\frac{\rho^{2}}{R_{\rho}\left(t\right)^{2}}-\frac{z^{2}}{R_{z}\left(t\right)^{2}}}\exp\left[i\ell\varphi+iB_{\rho}\left(t\right)\frac{\rho^{2}}{2}+iB_{z}\left(t\right)\frac{z^{2}}{2}\right], \tag{1}$$

and then calculate the equations of motion for the three variational parameters. Following these calculations, the equations of motion would be linearized. This procedure leads to imaginary frequencies which are not consistent with the stable configuration where a single vortex resides at center of the condensate. The linearized equations can be written in matrix form according

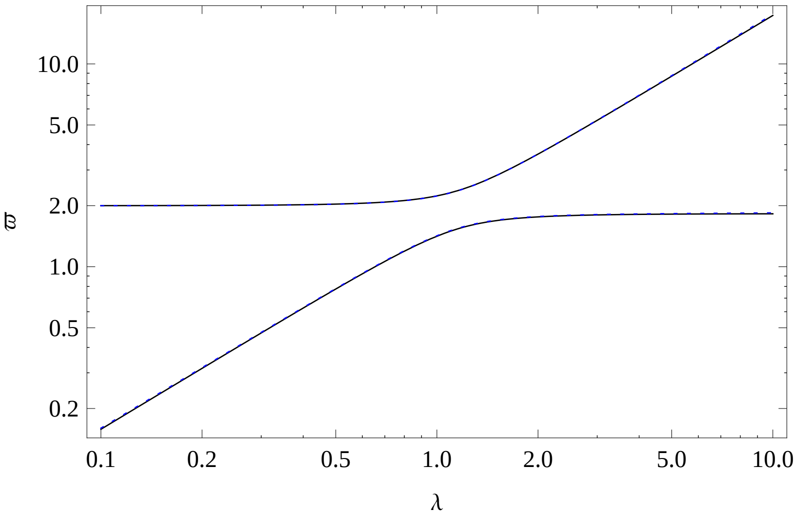


Figure 1: The upper lines relate to the frequencies of the breathing mode as a function of the anisotropy of the harmonic potential (trap), whereas the lower lines represent the frequencies of the quadrupole mode. The solid (black) lines were used to the case of vortex-free TF-profile, and the dotted (blue) lines describe the Gaussian approximation for a profile with a single vortex. This approximation becomes poor above $\lambda \approx 1.5$ because the Gaussian line crosses the TF line for breathing mode, although it is unnoticeable on graph. Necessarily the presence of the vortex must decrease the frequency of breathing mode (does not agree with the region of oblate condensates), and increase the frequency of quadrupole mode (in agreement), being these the shift of the frequencies. Note that ϖ is normalized by the frequency of the radial direction ω_ρ.

to

$$M\ddot{\delta} + V\delta = 0. \tag{2}$$

The solution of (2) is a linear combination of oscillatory modes whose oscillation frequencies obey the equation

$$\prod_n \varpi_n^2 = \det\left(M^{-1}V\right) = \frac{\det V}{\det M}. \tag{3}$$

In order to ensure that all frequencies ϖ_n are real, we must have $\det V/\det M > 0$. We know that $\det V > 0$ since the sign reflects the sign of the parameters which represents the external dimensions of the cloud in the stationary situation. Therefore $\det M$ must also be positive. In the case of Ansatz (1), such conditions are not satisfied since $\det M < 0$, which indicates that there is something wrong with Ansatz (1). In previous works [15, 16, 20, 21], since the authors did not consider the size of the vortex core as a variational parameter, this problem did not appear. Indeed, the problem relies on the fact that the phase of the wave function have to be modified.

In Section II, the necessary requirements for the wave function phase are discussed in order to give support to our variational method. Section III has the calculation based on the new Ansatz where the corresponding equations of motion are obtained. The collective modes with coupling between vortex and atomic cloud are obtained via linearization of the equations of motion, resulting in new collective oscillations (section IV). We showed that this excitation modes can be excited using the scattering length modulation at section V. The free expansion was also calculated to complement a previous work about expansion [21]. Finally, section VII contains the conclusions about our subject of study.

II. WAVE-FUNCTION PHASE

We start with the Lagrangian density,

$$\mathcal{L} = \frac{i\hbar}{2}\left(\psi^*\frac{\partial\psi}{\partial t} - \psi\frac{\partial\psi^*}{\partial t}\right) - \frac{\hbar^2}{2m}\left|\nabla\psi\right|^2 - V\left(\mathbf{r}\right)\left|\psi\right|^2 - \frac{g}{2}\left|\psi\right|^4 \tag{4}$$

whose extremization leads to the Gross-Pitaevskii equation (GPE):

$$i\hbar\frac{\partial\psi}{\partial t} = \left[-\frac{\hbar^2}{2m}\nabla^2 + V\left(\mathbf{r}\right) + g\left|\psi\right|^2\right]\psi, \tag{5}$$

where $V\left(\mathbf{r}\right)$ is an external potential, and g is the coupling constant. The complex field $\psi\left(\mathbf{r},t\right)$ can be written as an amplitude profile with a respective phase, as follows:

$$\psi\left(\mathbf{r},t\right) = f\left(w_l,\mathbf{r}\right)e^{iS(\chi_l,\mathbf{r})}, \tag{6}$$

where

$$S\left(\chi_l,\mathbf{r}\right) = \ell\varphi + \sum_l \chi_l\phi_l(\mathbf{r}). \tag{7}$$

We denoted both, $w_l = w_l\left(t\right)$ and $\chi_l = \chi_l\left(t\right)$, respectively, as the amplitude and phase variational parameters. In principle, $\{\phi_l\left(\mathbf{r}\right)\}$ should be a complete set of functions but in our present approximation, we use only a representative incomplete set of functions. Substituting (6) and (7) into (4), the Lagrangian $L = \int \mathcal{L}d^3\mathbf{r}$ becomes

$$L = -\hbar\sum_l \dot{\chi}_l\int d^3\mathbf{r}f^2\phi_l - \frac{\hbar^2}{2m}\sum_l \chi_l^2\int d^3\mathbf{r}f^2\left|\nabla\phi_l\right|^2 - \int d^3\mathbf{r}\left(\frac{\hbar^2}{2m}\left|\nabla f\right|^2 + Vf^2 + \frac{g}{2}f^4\right). \tag{8}$$

In order to account for the dynamics of all three variational parameter in f we include a variational phase which also contains three variational parameters. Here we chose the following trial function:

$$S(\rho,z,t) = \ell\varphi + B_\rho(t)\frac{\rho^2}{2} + C(t)\frac{\rho^4}{4} + B_z(t)\frac{z^2}{2}. \tag{9}$$

As the current is connected to the density variation, it is desirable that both, amplitude and phase, have the same number of variational parameters. The Ansatz (9) also leads to linearized equations of motion (2) with $\det M > 0$ which is consistent with the stability of the condensate with a single vortex at its center.

III. EQUATIONS OF MOTION

Now we correct the Thomas-Fermi Ansatz according to the discussion in section II. This leads to the following trial function:

$$\begin{aligned}\psi\left(\mathbf{r},t\right) =& \sqrt{\frac{N}{R_\rho(t)^2 R_z(t) A_0(\xi\left(t\right)/R_\rho\left(t\right))}}\left[\frac{\rho^2}{\rho^2+\xi\left(t\right)^2}\right]^{\frac{\ell}{2}}\sqrt{1-\frac{\rho^2}{R_\rho\left(t\right)^2}-\frac{z^2}{R_z\left(t\right)^2}} \\ &\times \exp\left[i\ell\varphi + iB_\rho\left(t\right)\frac{\rho^2}{2} + iC(t)\frac{\rho^4}{4} + iB_z\left(t\right)\frac{z^2}{2}\right],\end{aligned} \tag{10}$$

with

$$\begin{aligned}A_0(\alpha) =& \frac{2\pi^{3/2}\left(\ell\right)!}{15\alpha^{2\ell}\left(\frac{3}{2}+\ell\right)!}\left[\left(3+2\ell\alpha^2\right){}_2F_1\left(\ell,1+\ell;\frac{5}{2}+\ell;-\frac{1}{\alpha^2}\right)\right. \\ &\left.-2\ell\left(1+\alpha^2\right){}_2F_1\left(1+\ell,1+\ell;\frac{5}{2}+\ell;-\frac{1}{\alpha^2}\right)\right],\end{aligned} \tag{11}$$

where, for simplicity we define $\alpha\left(t\right) = \xi\left(t\right)/R_\rho\left(t\right)$, the ${}_pF_q\left(a_1,\ldots,a_p;b_1,\ldots,b_q;x\right)$ are the hypergeometric functions, $\xi(t)$ is the size of the vortex core, $R_\rho(t)$ is the condensate size in radial direction ($\hat{\rho}$), and $R_z(t)$ is the condensate size in axial direction ($\hat{z}$). The wave function (10) has integration domain determined by $1-\frac{\rho^2}{R_\rho^2}-\frac{z^2}{R_z^2}\geq 0$, where the wave function is approximately an inverted parabola (TF-shape), except for the central vortex. The trapping potential shape sets the condensate dimensions. To organize our calculations, we split the

Lagrangian so that it is a sum $L = L_{time} + L_{kin} + L_{pot} + L_{int}$ of the following terms:

$$
\begin{aligned}
L_{time} =& \frac{i\hbar}{2} \int d^3\mathbf{r} \left[\psi^* \left(\mathbf{r}, t\right) \frac{\partial \psi \left(\mathbf{r}, t\right)}{\partial t} - \psi \left(\mathbf{r}, t\right) \frac{\partial \psi^* \left(\mathbf{r}, t\right)}{\partial t} \right] \\
=& - \frac{N\hbar}{2} \left(D_1 \dot{B}_\rho R_\rho^2 + D_2 \dot{B}_z R_z^2 + \frac{1}{2} D_3 \dot{C} R_\rho^4 \right), \qquad (12) \\
L_{kin} =& - \frac{\hbar^2}{2m} \int d^3\mathbf{r} \left[\nabla \psi^* \left(\mathbf{r}, t\right) \right] \left[\nabla \psi \left(\mathbf{r}, t\right) \right] \\
=& - \frac{N\hbar^2}{2m} \left[D_1 B_\rho^2 R_\rho^2 + D_2 B_z^2 R_z^2 + 2 D_3 B_\rho C R_\rho^4 + R_\rho^{-2} \left(\ell^2 D_4 + D_5 \right) + D_6 C^2 R_\rho^6 \right], \qquad (13) \\
L_{pot} =& - \frac{1}{2} m \omega_\rho^2 \int d^3\mathbf{r} \left(\rho^2 + \lambda^2 z^2 \right) \psi^* \left(\mathbf{r}, t\right) \psi \left(\mathbf{r}, t\right) \\
=& - \frac{N}{2} m \omega_\rho^2 \left(D_1 R_\rho^2 + \lambda^2 D_2 R_z^2 \right), \qquad (14) \\
L_{int} =& - \frac{g}{2} \int d^3\mathbf{r} \left[\psi^* \left(\mathbf{r}, t\right) \psi \left(\mathbf{r}, t\right) \right]^2 \\
=& - \frac{N^2 g D_7}{2 R_\rho^2 R_z}, \qquad (15)
\end{aligned}
$$

The functions of α, $D_i(\alpha)$, are:

$$\begin{aligned}
D_1(\alpha) =& A_0(\alpha)^{-1}\frac{2\pi^{3/2}(1+\ell)!}{21\alpha^{2\ell}\left(\frac{5}{2}+\ell\right)!}\left[\left(3+2\ell\alpha^2\right){}_2F_1\left(\ell,2+\ell;\frac{7}{2}+\ell;-\frac{1}{\alpha^2}\right)\right.\\
&\left.-2\ell\left(1+\alpha^2\right){}_2F_1\left(1+\ell,2+\ell;\frac{7}{2}+\ell;-\frac{1}{\alpha^2}\right)\right], \qquad (16)\\
D_2(\alpha) =& A_0(\alpha)^{-1}\frac{\pi^{3/2}(\ell)!}{4\alpha^{2\ell}\left(\frac{7}{2}+\ell\right)!}\left[(7+2\ell){}_2F_1\left(\ell,1+\ell;\frac{7}{2}+\ell;-\frac{1}{\alpha^2}\right)\right.\\
&\left.-(5+2\ell){}_3F_2\left(\ell,1+\ell,\frac{7}{2}+\ell;\frac{5}{2}+\ell,\frac{9}{2}+\ell;-\frac{1}{\alpha^2}\right)\right], \qquad (17)\\
D_3(\alpha) =& A_0(\alpha)^{-1}\frac{2\pi^{3/2}(2+\ell)!}{27\alpha^{2\ell}\left(\frac{7}{2}+\ell\right)!}\left[(3+2\ell){}_2F_1\left(\ell,3+\ell;\frac{9}{2}+\ell;-\frac{1}{\alpha^2}\right)\right.\\
&\left.-2\ell\left(1+\alpha^2\right){}_2F_1\left(1+\ell,3+\ell;\frac{9}{2}+\ell;-\frac{1}{\alpha^2}\right)\right], \qquad (18)\\
D_4(\alpha) =& A_0(\alpha)^{-1}\frac{2\pi^{3/2}(\ell-1)!}{3\alpha^{2\ell}\left(\frac{1}{2}+\ell\right)!}\left[\left(1-2\ell\alpha^2\right){}_2F_1\left(\ell,2+\ell;\frac{3}{2}+\ell;-\frac{1}{\alpha^2}\right)\right.\\
&\left.+2\ell\left(1+\alpha^2\right){}_2F_1\left(1+\ell,2+\ell;\frac{3}{2}+\ell;-\frac{1}{\alpha^2}\right)\right], \qquad (19)\\
D_5(\alpha) =& A_0(\alpha)^{-1}\frac{2\pi^{3/2}(\ell-1)!}{9\alpha^{2\ell}\left(\frac{1}{2}+\ell\right)!}\left[\left(3+2\ell\alpha^2\right){}_2F_1\left(\ell,\ell;\frac{3}{2}+\ell;-\alpha^2\right)\right.\\
&\left.-2\ell\left(1+\alpha^2\right){}_2F_1\left(\ell,1+\ell;\frac{3}{2}+\ell;-\frac{1}{\alpha^2}\right)\right], \qquad (20)\\
D_6(\alpha) =& A_0(\alpha)^{-1}\frac{2\pi^{3/2}(3+\ell)!}{33\alpha^{2\ell}\left(\frac{9}{2}+\ell\right)!}\left[\left(3+2\ell\alpha^2\right){}_2F_1\left(\ell,4+\ell;\frac{11}{2}+\ell;-\frac{1}{\alpha^2}\right)\right.\\
&\left.-2\ell\left(1+\alpha^2\right){}_2F_1\left(1+\ell,4+\ell;\frac{11}{2}+\ell;-\frac{1}{\alpha^2}\right)\right], \qquad (21)\\
D_7(\alpha) =& A_0(\alpha)^{-2}\frac{2\pi^{3/2}(2\ell)!}{\alpha^{4\ell}\left(\frac{7}{2}+\ell\right)!}{}_2F_1\left(2\ell,1+2\ell;\frac{9}{2}+2\ell;-\frac{1}{\alpha^2}\right). \qquad (22)
\end{aligned}$$

For simplicity we can scale the parameters of Lagrangian and the time in order to make them dimensionless,

$$\begin{aligned}
R_\rho(t) \to& a_{osc} r_\rho(t),\\
R_z(t) \to& a_{osc} r_z(t),\\
\xi(t) \to& a_{osc} r_\xi(t),\\
B_\rho(t) \to& a_{osc}^{-2}\beta_\rho(t),\\
B_z(t) \to& a_{osc}^{-2}\beta_z(t),\\
C(t) \to& a_{osc}^{-4}\zeta(t),\\
t \to& \omega_\rho^{-1}\tau,
\end{aligned}$$

where the harmonic oscillator length is $a_{osc} = \sqrt{\hbar/m\omega_\rho}$ and the dimensionless parameter of interaction is $\gamma = Na_s/a_{osc}$. Thus the Lagrangian becomes

$$\begin{aligned}L =& -\frac{N\hbar\omega_\rho}{2}\left[D_1 r_\rho^2\left(\dot{\beta}_\rho + \beta_\rho^2 + 1\right) + D_2 r_z^2\left(\dot{\beta}_z + \beta_z^2 + \lambda^2\right)\right.\\ &\left. + D_3 r_\rho^4\left(\frac{1}{2}\dot{\zeta} + 2\beta_\rho\zeta\right) + \ell^2 r_\rho^{-2}\left(D_4 + D_5\right) + D_6\zeta^2 r_\rho^6 + D_7\frac{4\pi\gamma}{r_\rho^2 r_z}\right].\end{aligned} \tag{23}$$

Taking the Euler-Lagrange equations,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0, \tag{24}$$

for each one of the six variational parameters from Lagrangian (23) leads to the six differential equations:

$$\beta_\rho - \frac{\dot{r_\rho}}{r_\rho} - \frac{D_1'\dot{\alpha}}{2D_1} + \frac{D_3 r_\rho^2\zeta}{D_1} = 0, \tag{25}$$

$$\beta_z - \frac{\dot{r_z}}{r_z} - \frac{D_2'\dot{\alpha}}{2D_2} = 0, \tag{26}$$

$$\zeta - \frac{D_3\dot{r_\rho}}{D_6 r_\rho} - \frac{D_3\dot{\alpha}}{4D_6 r_\rho^2} + \frac{D_3\beta_\rho}{D_6 r_\rho^2} = 0, \tag{27}$$

$$\begin{aligned}&D_1 r_\rho\left(\dot{\beta}_\rho + \beta_\rho^2 + 1\right) + D_3 r_\rho^3\left(\dot{\zeta} + 4\beta_\rho\zeta\right) - \\ &\frac{\ell^2}{r_\rho^3}\left(D_4 + D_5\right) + 3D_6\zeta^2 r_\rho^5 - D_7\frac{4\pi\gamma}{r_\rho^3 r_z} = 0,\end{aligned} \tag{28}$$

$$D_2 r_z\left(\dot{\beta}_z + \beta_z^2 + \lambda^2\right) - D_7\frac{2\pi\gamma}{r_\rho^2 r_z^2} = 0, \tag{29}$$

$$\begin{aligned}&D_1' r_\rho^2\left(\dot{\beta}_\rho + \beta_\rho^2 + 1\right) + D_2' r_z^2\left(\dot{\beta}_z + \beta_z^2 + \lambda^2\right) + D_3' r_\rho^4\left(\frac{1}{2}\dot{\zeta} + 2\beta_\rho\zeta\right) + \\ &\frac{\ell^2}{r_\rho^2}\left(D_4' + D_5'\right) + D_6'\zeta^2 r_\rho^6 - D_7'\frac{4\pi\gamma}{r_\rho^2 r_z} = 0.\end{aligned} \tag{30}$$

Solving the equations for the parameters in the phase, we have:

$$\beta_\rho = \frac{\dot{r_\rho}}{r_\rho} + F_1\dot{\alpha}, \tag{31}$$

$$\beta_z = \frac{\dot{r_z}}{r_z} + F_2\dot{\alpha} \tag{32}$$

$$\zeta = F_3\frac{\dot{\alpha}}{r_\rho^2}; \tag{33}$$

where

$$F_1 = \frac{D_3' D_3 - 2D_1' D_6}{4\left(D_3^2 - D_1 D_6\right)}, \tag{34}$$

$$F_2 = \frac{D_2'}{2D_2}, \tag{35}$$

$$F_3 = \frac{2D_1' D_3 - D_1 D_3'}{4\left(D_3^2 - D_1 D_6\right)}. \tag{36}$$

Replacing (31), (32), and (33) into equations (28), (29), and (30), we reduce our six coupled equations to only three, which are given by:

$$D_1\left(\ddot{r_\rho}+r_\rho\right)+G_1 r_\rho\ddot{\alpha}+G_2 r_\rho\dot{\alpha}^2+G_3\dot{r_\rho}\dot{\alpha}-G_4\frac{\ell^2}{r_\rho^3}-D_7\frac{4\pi\gamma}{r_\rho^3 r_z}=0 \tag{37}$$

$$D_2\left(\ddot{r_z}+\lambda^2 r_z\right)+G_5 r_z\ddot{\alpha}+G_6 r_z\dot{\alpha}^2+G_7\dot{r_z}\dot{\alpha}-D_7\frac{2\pi\gamma}{r_\rho^2 r_z^2}=0 \tag{38}$$

$$\begin{aligned}D_1' r_\rho\left(\ddot{r_\rho}+r_\rho\right)+D_2' r_z\left(\ddot{r_z}+\lambda^2 r_z\right)+\left(G_8 r_\rho^2+G_9 r_z^2\right)\ddot{\alpha}+\left(G_{10} r_\rho^2+G_{11} r_z^2\right)\dot{\alpha}^2&\\+\left(G_{12} r_\rho\dot{r_\rho}+G_{13} r_z\dot{r_z}\right)\dot{\alpha}+G_{14}\frac{\ell^2}{r_\rho^2}+D_7'\frac{4\pi\gamma}{r_\rho^2 r_z}=0,&\end{aligned} \tag{39}$$

with

$$G_1 = D_1F_1 + D_3F_3, \tag{40}$$

$$G_2 = D_1\left(F_1^2+F_1'\right)+D_3\left(4F_1F_3+F_3'\right)+3D_6F_3^2, \tag{41}$$

$$G_3 = 2\left(D_1F_1+D_3F_3\right)=2G_1, \tag{42}$$

$$G_4 = D_4+D_5, \tag{43}$$

$$G_5 = D_2F_2, \tag{44}$$

$$G_6 = D_2\left(F_2^2+F_2'\right), \tag{45}$$

$$G_7 = 2D_2F_2 = 2G_5, \tag{46}$$

$$G_8 = D_1'F_1+\frac{1}{2}D_3'F_3, \tag{47}$$

$$G_9 = D_2'F_2, \tag{48}$$

$$G_{10} = D_1'\left(F_1^2+F_1'\right)+D_3'\left(\frac{1}{2}F_3'+2F_1F_3\right)+D_6'F_3^2, \tag{49}$$

$$G_{11} = D_2'\left(F_2^2+F_2'\right), \tag{50}$$

$$G_{12} = 2D_1'F_1+D_3'F_3, \tag{51}$$

$$G_{13} = 2D_2'F_2 = 2G_9, \tag{52}$$

$$G_{14} = D_4'+D_5'. \tag{53}$$

The terms $D_1 r_\rho$, $D_2\lambda^2 r_z$, $D_1' r_\rho^2$ and $D_2' r_z^2$ come from the trapping term L_{pot}, which can be thrown away in the case of a freely expanding condensate. The parameter γ indicates the elements that give the contribution of the atomic interaction potential, while the fractions proportional to r_ρ^{-2} and r_ρ^{-3} come from the kinetic energy contribution due to the presence of the vortex with charge ℓ, having the effect of adding a quantum pressure. The remaining factors represent the coupling effect between the outer dimensions of the condensate and the vortex core.

Making the velocities ($\dot{r_\rho}$, $\dot{r_z}$,$\dot{\alpha}$) and accelerations ($\ddot{r_\rho}$, $\ddot{r_z}$, $\ddot{\alpha}$) equal to zero leads to the equations for the stationary solution,

$$D_1 r_{\rho 0} = G_4 \frac{\ell^2}{r_{\rho 0}^3} + D_7 \frac{4\pi\gamma}{r_{\rho 0}^3 r_{z0}}, \tag{54}$$

$$D_2 \lambda^2 r_{z0} = D_7 \frac{2\pi\gamma}{r_{\rho 0}^2 r_{z0}^2}, \tag{55}$$

$$D_1' r_{\rho 0}^2 + D_2' \lambda^2 r_{z0}^2 = -G_{14} \frac{\ell^2}{r_{\rho 0}^2} - D_7' \frac{4\pi\gamma}{r_{\rho 0}^2 r_{z0}}, \tag{56}$$

where r_ρ, r_z, and r_ξ take their respective equilibrium values $r_{\rho 0}$,r_{z0}, and $r_{\xi 0}$. We implemented the Newton's method to solve the coupled stationary equations above. The value of the atomic interaction parameter, used from now on in this paper, is $\gamma = 800$. This value is well close to the value used in Rubidium experiments [22].

IV. COLLECTIVE EXCITATIONS

For small deviations from the equilibrium configuration, we assume $r_\rho(t) \to r_{\rho 0} + \delta_\rho(t)$, $r_z(t) \to r_{z0} + \delta_z(t)$, $\alpha(t) \to \alpha_0 + \delta_\alpha(t)$, and neglect all terms of order two or higher in (37)–(39). This leads to the linearization equations

$$\begin{aligned} D_1 \ddot{\delta}_\rho + G_1 \rho_0 \ddot{\delta}_\alpha + \left(D_1 + 3G_4 \frac{\ell^2}{\rho_0^4} + D_7 \frac{12\pi\gamma}{\rho_0^4 z_0} \right) \delta_\rho + \\ \left(D_7 \frac{4\pi\gamma}{\rho_0^3 z_0^2} \right) \delta_z + \left(D_1' \rho_0 - G_4' \frac{\ell^2}{\rho_0^3} - D_7' \frac{4\pi\gamma}{\rho_0^3 z_0} \right) \delta_\alpha = 0 \end{aligned} \tag{57}$$

$$\begin{aligned} D_2 \ddot{\delta}_z + G_5 z_0 \ddot{\delta}_\alpha + \left(D_7 \frac{4\pi\gamma}{\rho_0^3 z_0^2} \right) \delta_\rho + \left(D_2 \lambda^2 + D_7 \frac{4\pi\gamma}{\rho_0^2 z_0^3} \right) \delta_z + \\ \left(D_2' \lambda^2 z_0 - D_7' \frac{2\pi\gamma}{\rho_0^2 z_0^2} \right) \delta_\alpha = 0 \end{aligned} \tag{58}$$

$$\begin{aligned} D_1' \rho_0 \ddot{\delta}_\rho + D_2' z_0 \ddot{\delta}_z + \left(G_8 \rho_0^2 + G_9 z_0^2 \right) \ddot{\delta}_\alpha + \left(2D_1' \rho_0 - 2G_{14} \frac{\ell^2}{\rho_0^3} - D_7' \frac{8\pi\gamma}{\rho_0^3 z_0} \right) \delta_\rho + \\ \left(2D_2' \lambda^2 z_0 - D_7' \frac{4\pi\gamma}{\rho_0^2 z_0^2} \right) \delta_z + \left(D_1'' \rho_0^2 + D_2'' \lambda^2 z_0^2 + G_{14}' \frac{\ell^2}{\rho_0^2} + D_7'' \frac{4\pi\gamma}{\rho_0^2 z_0} \right) \delta_\alpha = 0. \end{aligned} \tag{59}$$

For convenience we are changing to matrix notation (2),

$$\begin{pmatrix} m_{11} & 0 & m_{13} \\ 0 & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{pmatrix} \begin{pmatrix} \ddot{\delta}_\rho \\ \ddot{\delta}_z \\ \ddot{\delta}_\alpha \end{pmatrix} + \begin{pmatrix} v_{11} & v_{12} & v_{13} \\ v_{21} & v_{22} & v_{23} \\ v_{31} & v_{32} & v_{33} \end{pmatrix} \begin{pmatrix} \delta_\rho \\ \delta_z \\ \delta_\alpha \end{pmatrix} = 0. \tag{60}$$

Solving the characteristic equation,

$$\det \left(M^{-1} V - \varpi^2 I \right) = 0, \tag{61}$$

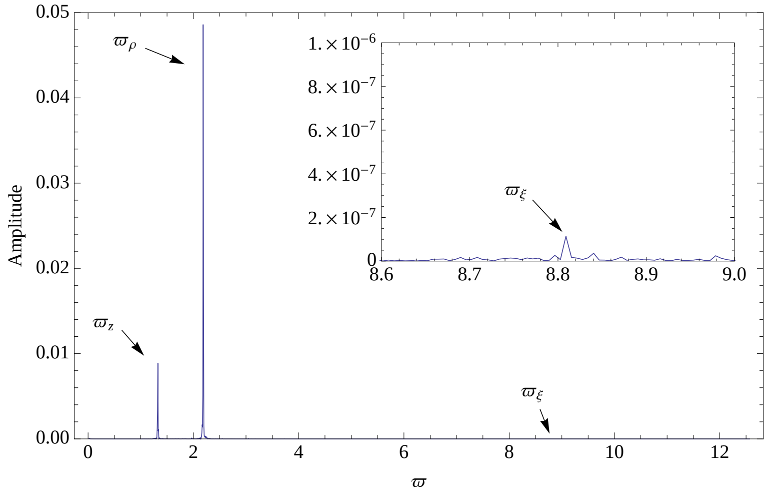
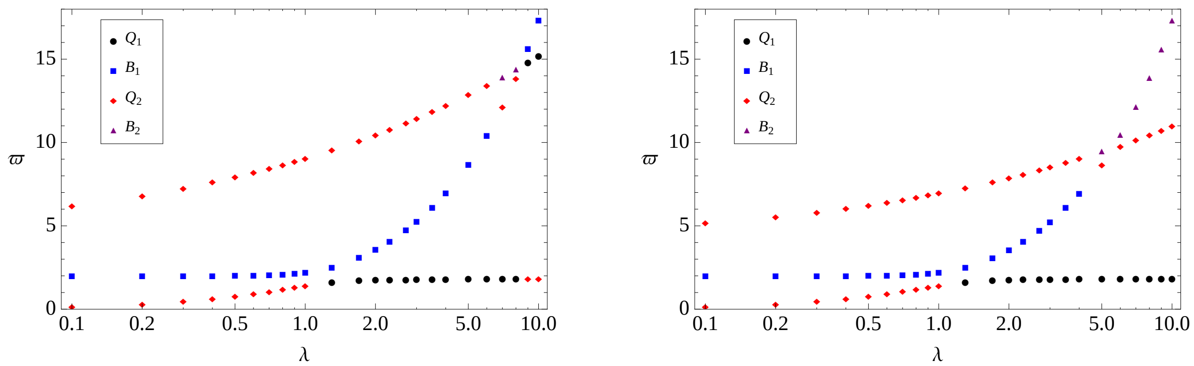


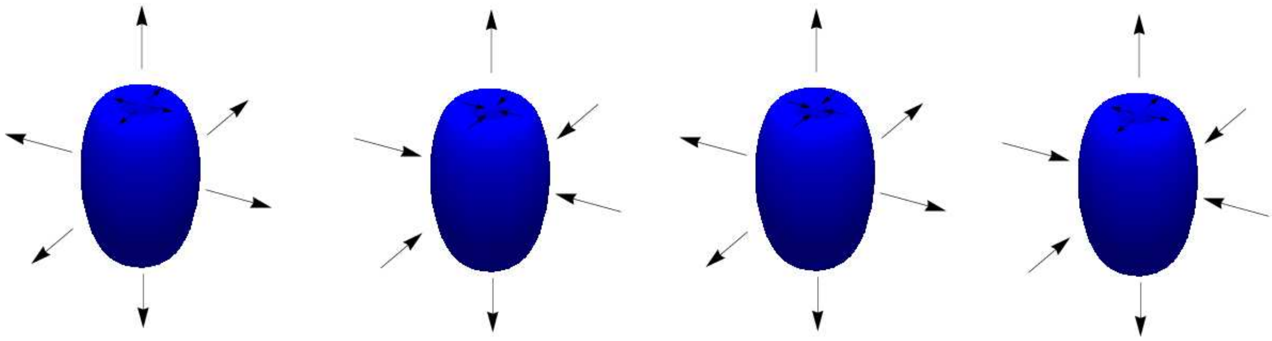
Figure 2: Numerical simulations using [23], where we set $\gamma = 800$, $\tilde{\mu} = 20.74$ and $\lambda = 0.9$, and ϖ_i are the frequencies of the oscillation modes, going from less energetic ϖ_z to more energetic ϖ_ξ. The analytical values are $\varpi_z = 1.317$, $\varpi_\rho = 2.166$, and $\varpi_\xi = 8.874$.



(a) $\ell = 1$ (b) $\ell = 2$

Figure 3: The frequencies of oscillation modes of a condensate containing a single vortex at its center.

results in the frequency of the collective modes of oscillation. Now the determinants $\det M$ and $\det V$ are, both, positive for $\ell = 1$. Meaning that we are in the lower energy state for the case of a central vortex in a Bose-Einstein condensate. Since (61) is a cubic equation of ϖ^2, we have three pair of frequencies $\pm\varpi_i$ $(i = z, \rho, \xi)$ which can not be linked to only one mode of oscillation. We have three frequencies and four modes of oscillation in total, of which only three modes can be observed according to the anisotropy of harmonic potential λ as it is shown in figure 3. About these four modes, two of them represent the monopole oscillations, and the other two represent the quadrupole oscillations for the atomic cloud. The first two (B_1 and Q_1



(a) B_1 (b) Q_1 (c) B_2 (d) Q_2

Figure 4: A picture drawing which represents the collective modes.

in figures 4a and 4b) present the oscillation of the size of vortex core r_ξ is in (out) phase with the radial (axial) oscillation of the condensate r_ρ (r_z), and the last two (B_2 and Q_2 in figures 4c and 4d) present the oscillation of r_ξ in (out) phase with the oscillation of r_z (r_ρ). Note that there is one point ($\lambda = 8$) in figure 3 which looks like degenerate mode however it does not, these curves never intersect. Extrapolating to an ideal situation where $\gamma = 0$, we decouple the equations of motions. Therefore, the ϖ_z (lower frequency) represents only a r_z oscillation, ϖ_ρ (middle frequency) represents only a r_ρ oscillation and ϖ_ξ (upper frequency) represents only a r_ξ oscillation.

The numerical simulations where done in order to validate our results (figure 2), and frequencies values ϖ_i in the variational calculations differ from numerical values by less than 1%.

For $\ell = 1$ (figure 3a), there are two Q_2. It means that the difference between them is on amplitude of oscillation, i.e. the amplitude of vortex core oscillation is two order of magnitude lower at the less energetic mode. Indeed, it occurs when $0.1 \leq \lambda \leq 1.3$. At $\lambda = 7$ is the r_ρ radius which shows itself almost stopped on $\varpi_\rho = 12.144$ (Q_2). The same happens when $\ell = 2$ (figure 3b), the vortex core is almost still for the lower frequency in the same interval of λ, and at $\lambda = 4$ is the most energetic mode ($\varpi_\xi = 9.053$) which shows the axial radius almost stopped.

As greater the vortex is, more easy to excite itself. It means that the energy, which is necessary to excite the mode with frequency ϖ_ξ, will be lower if ℓ is increased. However, these results can only be used when the vortex core r_ξ is smaller and smaller than the condensates radius r_ρ, i.e., we must evaluate the size of the components of condensate (r_ρ, r_z, r_ξ) according to the value of the parameters λ and γ. This is the way to know when this approach works.

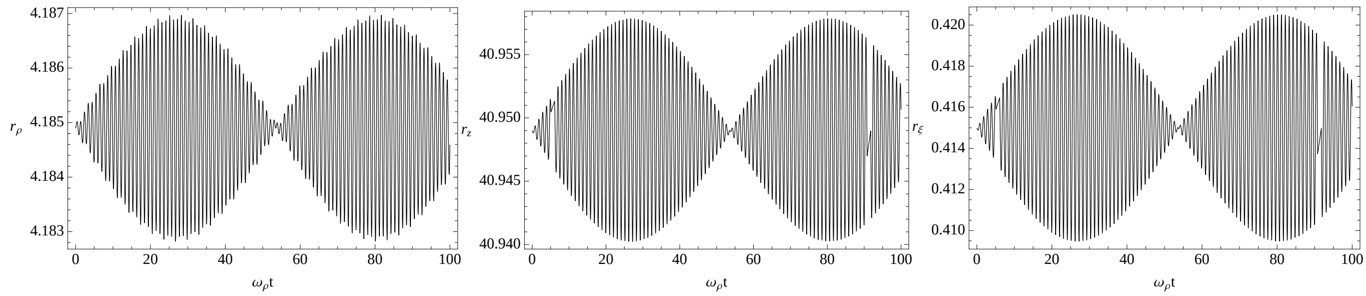


Figure 5: We excited the collective mode Q_2 ($\varpi_\xi = 6.211$) of a condensate with cigar shape ($\lambda = 0.1$, $\gamma_0 = 800$) via scattering modulation with amplitude $\delta\gamma = 0.4$, and frequency $\Omega = 6$.

V. SCATTERING LENGTH MODULATION

One of the mechanisms of excitation of collective modes, which were described in the previous section, is via a modulation of the s-wave scattering length, i.e.

$$a_s(t) = a_0 + \delta a \cos(\Omega t) , \tag{62}$$

this is equivalent to do $\gamma \to \gamma(\tau)$ with the same form of $a_s(t)$,

$$\gamma(\tau) = \gamma_0 + \delta\gamma \cos(\Omega\tau) . \tag{63}$$

Where γ_0 is the average value of the interaction parameter $\gamma(\tau)$, $\delta\gamma$ is the modulation amplitude and Ω is the excitation frequency. Substituting (63) in (57)–(59), and keeping only first order terms (δ_ρ, δ_z, δ_α, and $\delta\gamma$), we obtain a heterogeneous matrix equation

$$M\ddot{\delta} + V\delta = P\cos(\Omega\tau) \tag{64}$$

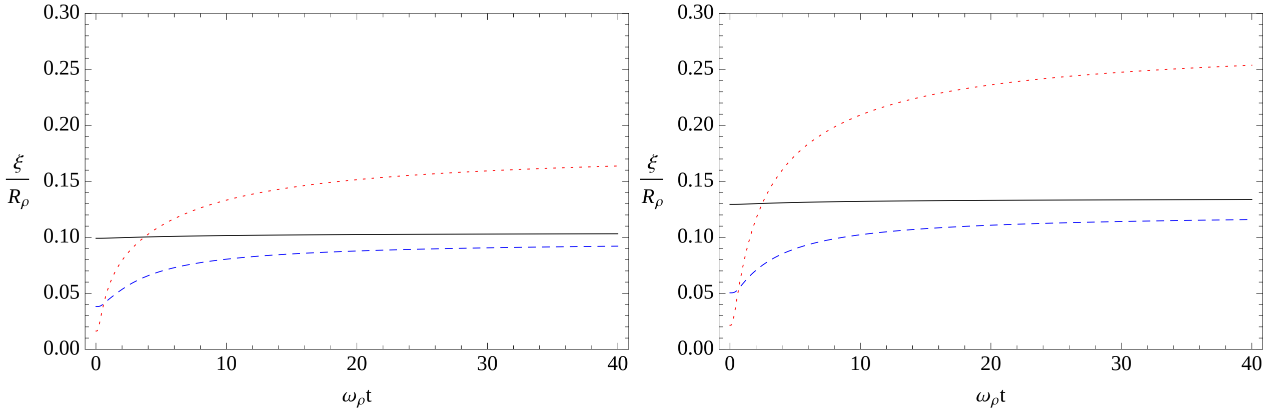
with

$$P = 2\pi\delta\gamma \begin{pmatrix} \frac{2D_7}{r_{\rho 0}^3 r_{z0}} \\ \frac{D_7}{r_{\rho 0}^2 r_{z0}^2} \\ \frac{D_7'}{r_{\rho 0}^2 r_{z0}} \end{pmatrix} . \tag{65}$$

The particular solution of (64) is

$$\delta(\tau) = \frac{M^{-1}P}{(M^{-1}V - \Omega^2)} \cos(\Omega\tau) . \tag{66}$$

In this way we showed that a specific collective mode can be excited using a scattering length modulation with small amplitude $\delta\gamma$, and using a frequency Ω close to the resonance frequency ϖ_i. In Figure 5 is excited the more energetic oscillation to a condensate with anisotropy $\lambda = 0.1$. Where it can be seen beat wave which modulates the oscillations.



(a) $\ell = 1$ (b) $\ell = 2$

Figure 6: This graphic shows the free expansion of the $\alpha(t) = \xi(t)/R_\rho(t)$, being the black line to a prolate condensate ($\lambda = 0.1$), the blue line to the isotropic case ($\lambda = 1$) and the yellow line to a oblate condensate ($\lambda = 8$).

VI. FREE EXPANSION

The interest in free expansion comes from the fact that we can only make measurements of condensate after switching off the trapping potential. For this purpose, as previously mentioned, we use the equations of motion (37)–(39) without the terms arising from the harmonic potential, i.e..,

$$D_1 \ddot{r_\rho} + G_1 r_\rho \ddot{\alpha} + G_2 r_\rho \dot{\alpha}^2 + G_3 \dot{r_\rho} \dot{\alpha} - \frac{G_4}{r_\rho^3} - \frac{4D_7\gamma}{r_\rho^3 r_z} = 0 \tag{67}$$

$$D_2 \ddot{r_z} + G_5 r_z \ddot{\alpha} + G_6 r_z \dot{\alpha}^2 + G_7 \dot{r_z} \dot{\alpha} - \frac{2D_7\gamma}{r_\rho^2 r_z^2} = 0 \tag{68}$$

$$\begin{aligned} D_1' r_\rho \ddot{r_\rho} + D_2' r_z \ddot{r_z} + \left(G_8 r_\rho^2 + G_9 r_z^2\right) \ddot{\alpha} + \left(G_{10} r_\rho^2 + G_{11} r_z^2\right) \dot{\alpha}^2 \\ + \left(G_{12} r_\rho \dot{r_\rho} + G_{13} r_z \dot{r_z}\right) \dot{\alpha} + \frac{G_{14}}{r_\rho^3} + \frac{4D_7'\gamma}{r_\rho^2 r_z} = 0, \end{aligned} \tag{69}$$

whose initial conditions are given by the stationary solution. This result agrees with our preview work [21], where the free expansion of the vortex core is given in figure 7. In the work that preceding the present one [21], the figure 6b could not be calculated.

VII. CONCLUSIONS

In this paper we proposed a change in the phase used the variational method to correct the imaginary frequencies of collective modes when we have a parameter that describes the dynamics of the vortex core. This change was that we had a proper atomic density fluctuation.

Based on the continuity equation, we conclude that we have a variational phase parameter, respectively, for each parameter in density. With that in calculating the frequencies for the vortex with charge $\ell = 1$, they gave them real. Thus it was observed that the vortex opens a degeneracy in the collective modes, which depends on the anisotropy of the trap and the atomic interaction parameter. This degeneration is linked to the oscillation of the vortex core, which needs a lot of energy to dephase its oscillation with respect to the radial component of the atomic cloud.

With modulation of scattering length we show that these new modes can be excited.

We also made the free expansion of this system, which can evaluate the core of the vortex to $\ell \geq 2$, in contrast previous work which only allowed us to calculate the free expansion of this core vortex radius when $\ell = 1$. We maintain, with this Ansatz, the logarithmic ratio of the energy vortex.

Acknowledgments

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