Normal modes of a vortex in a trapped Bose-Einstein condensate

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A hydrodynamic description is used to study the normal modes of a vortex in a zero-temperature Bose-Einstein condensate. In the Thomas-Fermi limit, the circulating superfluid velocity far from the vortex core provides a small perturbation that splits the originally degenerate normal modes of a vortex-free condensate. The relative frequency shifts are small in all cases considered (they vanish for the lowest dipole mode with $|m|=1$), suggesting that the vortex is stable. The Bogoliubov equations serve to verify the existence of helical waves, similar to those of a vortex line in an unbounded weakly interacting Bose gas. In the large-condensate (small-core) limit, the condensate wave function reduces to that of a straight vortex in an unbounded condensate; the corresponding Bogoliubov equations have no bound-state solutions that are uniform along the symmetry axis and that decay exponentially far from the vortex core. [S1050-2947(98)04010-4]

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I. INTRODUCTION

Recent experimental demonstrations of Bose-Einstein condensation in dilute low-temperature trapped alkali-metal gases [1–3] rapidly stimulated detailed theoretical [4–7] and experimental [8,9] studies of the lowest-lying normal modes. These latter experiments confirmed the essential applicability of the Bogoliubov approximation [10], which assumes that most of the particles remain in the self-consistent condensate for temperatures well below the Bose-Einstein condensation temperature. Although Bogoliubov's original work described only a uniform system, the formalism was soon extended to a vortex line [11,12] and other more general nonuniform configurations [13].

In a harmonic trap with anisotropic frequencies ω_α ($\alpha = x$, y, and z), an ideal-Bose-gas condensate would have characteristic dimensions $d_\alpha = \sqrt{\hbar/M\omega_\alpha}$, where M is the atomic mass. As the number N of particles grows, however, the repulsive interactions become important and the condensate expands from the noninteracting mean oscillator radius $d_0 = (d_x d_y d_z)^{1/3}$ to a larger mean radius R_0. For $R_0 \gg d_0$, the kinetic energy of a particle in the condensate is negligible relative to the trap potential energy and the repulsive interparticle potential energy, leading to the Thomas-Fermi (TF) approximation for the condensate density [14]. This limit occurs when the condensate number N_0 is much larger than the ratio d_0/a, where a is the effective s-wave interatomic scattering length (typically [15] d_0 is approximately equal to a few micrometers, a is approximately equal to a few nanometers, and the experimental value $N_0 \gtrsim 10^6$ amply satisfies the TF criterion that $N_0 \gg 10^3$). In the TF regime, the expansion ratio is given by $R_0/d_0 \approx 15(N_0 a/d_0)^{1/5} \gg 1$ [14].

The normal modes of a nonuniform condensate can be described with either the pair of quantum-mechanical quasiparticle amplitudes u and v that obey the coupled Bogoliubov equations [12,13,16] or the equivalent hydrodynamic amplitudes for the fluctuations n' in the particle density and Φ' in the velocity potential [5,17,18]. In the TF limit, the latter description simplifies considerably for the low-lying normal modes, allowing accurate predictions of lowest

eigenfrequencies of an axisymmetric trapped condensate [5,8,9].

In contrast, the creation and detection of a vortex in a trapped Bose condensate remains relatively unexplored. The condensate density for a vortex line on the axis of an axisymmetric harmonic potential has been evaluated both numerically [19,20] and with the TF approximation [21,22]. Using the Bogoliubov equations in the TF limit, Sinha [21] studied the normal modes of a condensate with a vortex, showing that the presence of the vortex causes only a small shift of order $(d_0/R_0)^2 \ll 1$ for a subset of the eigenfrequencies. In addition, a detailed numerical study has been carried out [23] for the specific trap configuration of Ref. [1] for $Na/d_0 \lesssim 50$ (for the distinct but related example of an annular condensate, see Ref. [24] and references therein). Finally, Zambelli and Stringari [25] have used sum rules to calculate the splitting of the lowest (quadrupole) normal modes caused by the presence of a vortex.

The present work uses the equivalent hydrodynamic picture to study the low-lying normal modes of a quantized vortex line on the symmetry axis of an axisymmetric harmonic trap. In the TF limit, the vortex core radius $\xi \sim d_0^2/R_0$ is small compared to both the mean oscillator length d_0 and the mean dimension R_0 of the condensate. The resulting hydrodynamic equations then yield a perturbative treatment of the normal-mode eigenfrequencies and eigenfunctions for the density fluctuations, in which the vortex's circulating velocity splits the degenerate unperturbed eigenfrequencies with azimuthal angular momentum $\pm m$ by a relative amount of order $\xi/R_0 \sim (d_0/R_0)^2 \ll 1$, largely confirming Sinha's conclusion [21] based on the Bogoliubov equations (for $|m|=1$, however, we find additional terms, which ensure that the shift vanishes for the lowest dipole mode). As emphasized by Zambelli and Stringari [25], this splitting provides a sensitive test of the existence of a vortex.

Since the vortex-free trapped condensate is stable (both experimentally and theoretically), these results suggest that the corresponding trapped vortex is also stable at zero temperature. In the presence of weak dissipation, however, a slightly off-center vortex presumably will slowly spiral outward and eventually disappear, as noted by Hess [26] and by

Packard and Sanders [27] for superfluid ^{4}He and by Rokhsar [22] for trapped condensates.

Section II reviews the basic hydrodynamic formalism, focusing on a condensate containing a q-fold quantized vortex in an axisymmetric trap. For the normal modes proportional to $e^{im\phi}$, a general variational principle demonstrates that the frequencies are manifestly non-negative (and hence stable) for $m=0$ and for $m^2 \geq 4q^2$ (Sec. V gives a more detailed treatment for the remaining cases, but only in the TF limit). For a large condensate (the TF limit), Sec. III makes use of the small parameter ξ/R_0 to perform a perturbation expansion of the hydrodynamic equations, yielding a splitting of the originally degenerate modes with $\pm m$. In Sec. IV the relative frequency shift of all low-lying normal modes is evaluated analytically for a spherical trap, whereas only a subset of the modes allows a closed-form expression for a strictly cylindrical trap and for a general axisymmetric but anisotropic trap. Section V contains additional considerations on the stability of a vortex line, based on the full Bogoliubov equations, including a demonstration that normal modes with $k=0$ have non-negative eigenfrequencies in the TF limit (and thus are stable).

II. BASIC FORMALISM

Consider a trapped condensate characterized by the equilibrium condensate wave function Ψ. It satisfies the Gross-Pitaevskii (GP) equation [11,12]

$$(T+V_{tr}+g|\Psi|^2-\mu)\Psi=0, \qquad (1)$$

where $T=-\hbar^2\nabla^2/2M$ is the kinetic-energy operator, V_{tr} is the external trap potential, $g=4\pi a\hbar^2/M$ is the effective interparticle interaction strength (expressed in terms of the s-wave scattering length a, here taken as positive), and μ is the chemical potential. It is convenient to write $\Psi=|\Psi|e^{iS}$, where $n_0=|\Psi|^2$ is the static condensate particle density and $\mathbf{v}_0=(\hbar/M)\nabla S$ is the static condensate velocity. These quantities satisfy the static conservation law $\nabla\cdot(n_0\mathbf{v}_0)=0$, which is the imaginary part of the GP equation. Correspondingly, the real part

$$\mu=|\Psi|^{-1}T|\Psi|+\tfrac{1}{2}Mv_0^2+V_{tr}+g|\Psi|^2 \qquad (2)$$

generalizes Bernoulli's equation to include the quantum (kinetic-energy) contribution $|\Psi|^{-1}T|\Psi|$.

Hydrodynamics relies on the local density and velocity as the relevant variables. In the present context, the linearized hydrodynamic equations determine the harmonic fluctuations in the particle density $n'e^{-i\omega t}$ and the velocity potential $\Phi'e^{-i\omega t}$. The corresponding (complex) normal-mode amplitudes satisfy the coupled equations [5,17]

$$i\omega n'=\nabla\cdot(\mathbf{v}_0 n')+\nabla\cdot(n_0\nabla\Phi'), \qquad (3a)$$

$$i\omega\Phi'=\mathbf{v}_0\cdot\nabla\Phi'+\frac{g}{M}n'-\frac{\hbar^2}{4M^2 n_0}\nabla\cdot\left[n_0\nabla\left(\frac{n'}{n_0}\right)\right], \qquad (3b)$$

both of which involve the (real) static condensate density n_0 and (real) condensate velocity $\mathbf{v}_0$. These very general equations can be derived from either the linearized quantum-field

equations or the conservation of particles and Bernoulli's equation for irrotational isentropic compressible flow [17,18]. The last term in Eq. (3b) corresponds to the kinetic-energy (quantum) pressure and is omitted in the TF limit of a large condensate. Although the resulting formalism is frequently known simply as the ''hydrodynamic approach,'' we prefer to use the term ''hydrodynamic equations'' for the more complete Eqs. (3).

The appropriate normalization of these amplitudes follows from that of the Bogoliubov amplitudes, which satisfy the condition [13] $\int dV(|u|^2-|v|^2)=1$. A straightforward comparison of the linearized quantum-field equations in the presence of condensate flow [17] shows that

$$n'=\Psi^* u-\Psi v, \qquad (4a)$$

$$\Phi'=\frac{\hbar}{2Mi|\Psi|^2}(\Psi^* u+\Psi v). \qquad (4b)$$

As a result, we have

$$|u|^2-|v|^2=\frac{iM}{\hbar}(n'^*\Phi'-\Phi'^* n'), \qquad (5)$$

so that the normalization of the hydrodynamic amplitudes requires

$$\int dV\, i(n'^*\Phi'-\Phi'^* n')=\frac{\hbar}{M}. \qquad (6)$$

Multiply Eq. (3a) by Φ'^*, multiply Eq. (3b) by n'^*, integrate the results over all space, and subtract the first from the second. A simple integration by parts yields

$$\omega\int dV\, i(n'^*\Phi'-\Phi'^* n')$$

$$=\int dV\Bigg[(n'^*\mathbf{v}_0\cdot\nabla\Phi'+n'\mathbf{v}_0\cdot\nabla\Phi'^*)+n_0|\nabla\Phi'|^2$$

$$+\frac{g}{M}|n'|^2+\frac{\hbar^2}{4M^2}n_0\left|\nabla\left(\frac{n'}{n_0}\right)\right|^2\Bigg]. \qquad (7)$$

In this way, the normal-mode frequency is expressed as the ratio of two manifestly real integrals, so that ω itself must be real [assuming that the normalization integral Eq. (6) is nonzero]. In addition, variation of Eq. (7) with respect to each of the two hydrodynamic variables n'^* and Φ'^* reproduces the dynamical equations (3), so that Eq. (7) serves as a variational principle for the normal-mode eigenfrequencies. Finally, Eq. (6) shows that the left-hand side here is just $\hbar\omega/M$ for a properly normalized set of solutions.

We consider a quantized vortex with q quanta of circulation on the symmetry axis of an axisymmetric trap with

$$V_{tr}=\tfrac{1}{2}M(\omega_\perp^2\rho^2+\omega_z^2 z^2)=\tfrac{1}{2}M\omega_\perp^2(\rho^2+\lambda^2 z^2), \qquad (8)$$

where (ρ,ϕ,z) are the usual cylindrical polar coordinates, and we define the anisotropy parameter $\lambda=\omega_z/\omega_\perp$. The appropriate solution of the GP equation (1) has the form $\Psi_q=e^{iq\phi}|\Psi_q(\rho,z)|$, so that the condensate velocity can be written $\mathbf{v}_0=V\hat{\phi}$, where

$$V = \frac{\hbar q}{M\rho}. \tag{9}$$

In this case, the linearized hydrodynamic equations (3a) and (3b) have single-valued solutions of the form

$$n'(\mathbf{r}), \Phi'(\mathbf{r}) \propto \exp(im\phi), \tag{10}$$

where m is an integer and the corresponding amplitudes depend only on ρ and z. Substitution into Eq. (7) yields

$$\frac{\hbar\omega}{M} = \int dV \left[\frac{mq}{\rho^2} \frac{\hbar}{M} i(n'^* \Phi' - n' \Phi'^*) \right.$$
$$+ \frac{m^2}{\rho^2} \left(n_q |\Phi'|^2 + \frac{\hbar^2}{4M^2 n_q} |n'|^2 \right)$$
$$\left. + n_q |\boldsymbol{\nabla}\Phi'|^2 + \frac{g}{M} |n'|^2 + \frac{\hbar^2}{4M^2} n_q \left| \boldsymbol{\nabla}\left(\frac{n'}{n_q}\right) \right|^2 \right], \tag{11}$$

where $n_q = |\Psi_q|^2$ is the (axisymmetric) condensate density for a q-fold vortex.

Simple manipulation yields the equivalent form

$$\frac{\hbar\omega}{M} = \int dV \left[\frac{1}{\rho^2 n_q} \left| m \frac{\hbar}{2M} n' + 2qin_q\Phi' \right|^2 \right.$$
$$+ \frac{m^2 - 4q^2}{\rho^2} n_q |\Phi'|^2 + n_q |\boldsymbol{\nabla}\Phi'|^2 + \frac{g}{M} |n'|^2$$
$$\left. + \frac{\hbar^2}{4M^2} n_q \left| \boldsymbol{\nabla}\left(\frac{n'}{n_q}\right) \right|^2 \right]. \tag{12}$$

Here the right-hand side is manifestly non-negative for $m = 0$ and for $m^2 \geq 4q^2$, so that all such hydrodynamic modes of a general q-fold quantized vortex necessarily have nonnegative frequencies. As discussed in Sec. V, the quantum-mechanical ''grand-canonical'' Hamiltonian in the Bogoliubov approximation reduces to a set of uncoupled harmonic oscillators for the different quasiparticle modes and is thus bounded from below as long as none of the eigenfrequencies is negative. Consequently, the vortex is stable with respect to modes with $m = 0$ and $m^2 \geq 4q^2$. For the remaining modes with $0 < m^2 < 4q^2$, Eq. (12) contains terms of both signs that can in principle produce an instability (in the usual case of a singly quantized vortex with $|q| = 1$, this condition affects only modes with $|m| = 1$).

Rokhsar [22] has argued that a multiply quantized vortex is indeed unstable because of the formation of a bound state of radius $\sim \xi$ inside the vortex core (for a singly quantized vortex, he finds such a bound state only for relatively small condensates with $Na/d_0 \lesssim 5$, where d_0 is the mean oscillator length for the trap; a variational study of the Bogoliubov equations confirms this qualitative conclusion). In the TF limit, the core radius ξ becomes small and the details of the trap potential become irrelevant for phenomena on the scale of the vortex core and the problem reduces to that of a long vortex in an unbounded condensate. For this case, the radial eigenfunctions for normal modes that are independent of z do not decay exponentially on the length scale ξ [28], pre-

cluding such bound states as solutions of the relevant Bogoliubov equations (see Sec. V for a more detailed discussion).

Although the variational equation (12) makes no use of the TF approximation, it does assume that the hydrodynamic variables provide the appropriate physical description. For a uniform condensate in the Bogoliubov approximation, the hydrodynamic picture holds only for long-wavelength modes with $k\xi \lesssim 1$ when the quasiparticle creation operator is an essentially equal admixture of a particle and a hole creation operator [29,30]. For $k\xi \gtrsim 1$, in contrast, the quasiparticle creation operator becomes a nearly pure particle creation operator.

Correspondingly, a nonuniform trapped condensate has no hydrodynamic modes unless it is sufficiently large that $\xi \lesssim R_0$, since the system otherwise approximates an ideal noninteracting gas. Even for such a large condensate, however, the only hydrodynamic modes are those with $\hbar\omega \lesssim \mu$, where μ is the chemical potential of the condensate [5]. This latter condition becomes progressively more restrictive as N_0 decreases and the condensate approaches the ideal-gas limit.

III. PERTURBATION ANALYSIS OF THE NORMAL MODES

We return to the hydrodynamic equations (3). In the present case of modes proportional to $e^{im\phi}$, they reduce to the coupled equations

$$-i\left(\omega - \frac{mV}{\rho}\right)n' - \frac{m^2}{\rho^2} n_q \Phi' + \boldsymbol{\nabla}\cdot(n_q \boldsymbol{\nabla}\Phi') = 0, \tag{13a}$$

$$-i\left(\omega - \frac{mV}{\rho}\right)\Phi' + \frac{g}{M}n' + \frac{\hbar^2 m^2}{4M^2 n_q \rho^2}n'$$
$$- \frac{\hbar^2}{4M^2 n_q} \boldsymbol{\nabla}\cdot\left[n_q \boldsymbol{\nabla}\left(\frac{n'}{n_q}\right)\right] = 0, \tag{13b}$$

where n_q is the condensate density for the q-fold quantized vortex.

A. Stationary condensate

Any practical solution of these equations requires a physically motivated approximation and we here use the TF limit of a large condensate, when the kinetic energy in the GP equation (1) is negligible compared to the remaining terms. In this limit, the equilibrium condensate density $n_0 = |\Psi|^2$ is given by

$$g|\Psi|^2 = \mu - \tfrac{1}{2}Mv_0^2 - V_{\text{tr}} \tag{14}$$

wherever the right-hand side is positive and zero elsewhere; the chemical potential is determined by the normalization condition

$$N \approx N_0 = \int dV |\Psi|^2 \tag{15}$$

(the present zero-temperature approximation neglects the difference between the total number of particles N and the number N_0 in the condensate).

In the simplest case of an axisymmetric trap potential (8) and a stationary condensate with $\mathbf{v}_0 = 0$ [14], Eq. (14) gives the condensate density

$$g n_0 = \mu_0 - V_{\mathrm{tr}}, \qquad (16)$$

where μ_0 is the chemical potential for a stationary condensate. The condensate density varies quadratically and can be rewritten in terms of the central density $n_0(0) = \mu_0/g$ to give

$$\frac{n_0(\rho,z)}{n_0(0)} = \left(1 - \frac{\rho^2}{R_\perp^2} - \frac{z^2}{R_z^2} \right) \Theta \left(1 - \frac{\rho^2}{R_\perp^2} - \frac{z^2}{R_z^2} \right), \qquad (17)$$

where $\Theta(x)$ is the unit positive step function. The condensate is ellipsoidal with radial and axial dimensions $R_\perp$ and R_z given by

$$R_\alpha^2 = \frac{2\mu_0}{M\omega_\alpha^2} = \frac{2\mu_0}{\hbar\omega_\alpha} d_\alpha^2. \qquad (18)$$

In the TF limit, each of these dimensions is much larger than the corresponding oscillator lengths $d_\alpha = \sqrt{\hbar/M\omega_\alpha}$, so that $\mu_0 \gg \hbar\omega_\alpha$. Note that the aspect ratio is given by $R_z/R_\perp = 1/\lambda$, with a cigar-shaped (disk-shaped) condensate corresponding to the limit $\lambda \ll 1$ ($\lambda \gg 1$).

The normalization condition (15) readily yields

$$\frac{Na}{d_0} = \frac{1}{15} \left(\frac{2\mu_0}{\hbar\omega_0} \right)^{5/2} = \frac{1}{15} \left(\frac{R_0}{d_0} \right)^5 \gg 1, \qquad (19)$$

where $R_0 = (R_\perp^2 R_z)^{1/3}$, $d_0 = (d_\perp^2 d_z)^{1/3}$, and $\omega_0 = (\omega_\perp^2 \omega_z)^{1/3}$ are appropriate "geometric" means. It is conventional to define the coherence length ξ in terms of the central density

$$\xi^2 = \frac{1}{8\pi n_0(0)a}. \qquad (20)$$

It satisfies the simple relation

$$\xi^2 = \frac{\hbar^2}{2\mu_0 M} = \frac{d_0^4}{R_0^2}, \qquad (21)$$

implying the set of TF inequalities $\xi \ll d_0 \ll R_0$.

B. Rotating condensate with a vortex

In the presence of a quantized vortex with q quanta of circulation on the trap's symmetry axis, Eq. (14) yields the corresponding condensate density

$$g n_q = \mu_q - V_{\mathrm{tr}} - \tfrac{1}{2} M V^2, \qquad (22)$$

where μ_q is the chemical potential for a q-fold vortex and $V = \hbar q/M\rho$. This expression can be rewritten as

$$\frac{n_q(\rho,z)}{n_0(0)} = \left(\frac{\mu_q}{\mu_0} - \frac{\rho^2}{R_\perp^2} - \frac{z^2}{R_z^2} - \frac{q^2\xi^2}{\rho^2} \right)$$

$$\times \Theta \left(\frac{\mu_q}{\mu_0} - \frac{\rho^2}{R_\perp^2} - \frac{z^2}{R_z^2} - \frac{q^2\xi^2}{\rho^2} \right), \qquad (23)$$

using Eq. (21). The centrifugal barrier arising from the vortex alters the shape from ellipsoidal to toroidal; in the TF

approximation, it creates a flared hollow core with radius approximately equal to $|q|\xi$ in the plane $z = 0$ [20–22]. The normalization condition (15) can be expanded to show that the fractional change in the chemical potential ($\mu_q - \mu_0)/\mu_0$ for fixed V_{tr} and N is of order $q^2 d_0^4/R_0^4 \ln(R_0^2/|q|d_0^2) \ll 1$ [21].

Equation (13b) can be simplified considerably in the TF limit when the kinetic energy is treated as small [5]. Specifically, Eq. (13b) contains the density fluctuation n' in two physically distinct ways. The last two terms involving $\hbar^2/M^2 n_q$ reflect the quantum kinetic-energy contribution to Bernoulli's equation; in contrast, the preceding one involving g/M reflects the repulsive interparticle interaction energy. In the present case, the kinetic-energy term for $\rho \gtrsim \xi$ is smaller than the interparticle potential energy term by a factor of order $\hbar^2/gMn_0(0)R_0^2 \approx \xi^2/R_0^2 = d_0^4/R_0^4$. As will be seen below, the effect of interest here is of order $\xi/R_0 = d_0^2/R_0^2$, so that the kinetic-energy term in Eq. (13b) can be omitted entirely for $\rho \gtrsim \xi$, giving

$$-i \left(\omega - \frac{m\hbar q}{M\rho^2} \right) \Phi' + \frac{g}{M} n' = 0. \qquad (24)$$

A combination of Eqs. (13a) and (24) yields the single eigenvalue equation

$$\nabla \cdot \left(\frac{g n_q}{M} \nabla \Phi' \right) - \frac{g n_q}{M} \frac{m^2}{\rho^2} \Phi' + \left(\omega - \frac{m\hbar q}{M\rho^2} \right)^2 \Phi' = 0. \qquad (25)$$

This second-order partial differential equation determines the frequencies and normal modes of an axisymmetric trapped condensate containing a vortex line through terms of order $d_\perp^2/R_\perp^2$. A combination of Eqs. (18) and (19) shows that this small parameter has the dependence

$$\frac{d_\perp^2}{R_\perp^2} = \left(\frac{d_\perp}{15Na\lambda} \right)^{2/5} \qquad (26)$$

on the condensate number N and the asymmetry parameter λ.

It is now convenient to introduce dimensionless units, with $\omega' \equiv \omega/\omega_\perp$, $\rho' \equiv \rho/R_\perp$, and $z' \equiv z/R_z = z\lambda/R_\perp$; in particular, the dimensionless ratio $m\hbar q/\omega M\rho^2 = (mq d_\perp^2/R_\perp^2)\rho'^{-2}$ is seen to be of order $d_\perp^2/R_\perp^2$ and thus small. In addition, the actual condensate density n_q differs from that for a vortex-free condensate n_0 by corrections of order $\xi^2/R_0^2 = d_0^4/R_0^4$, which is negligible compared to the effect of the superfluid flow. Omitting the primes on the dimensionless variables, we therefore find

$$\frac{1}{2n_0(0)} \left[\frac{1}{\rho} \frac{\partial}{\partial\rho} \left(\rho n_0 \frac{\partial\Phi'}{\partial\rho} \right) - \frac{m^2 n_0 \Phi'}{\rho^2} \right.$$

$$\left. + \lambda^2 \frac{\partial}{\partial z} \left(n_0 \frac{\partial\Phi'}{\partial z} \right) \right] + \left(\omega^2 - \frac{2\omega mq}{\rho^2} \frac{d_\perp^2}{R_\perp^2} \right) \Phi' = 0, \qquad (27)$$

where terms of order $d_\perp^4/R_\perp^4$ have been omitted and the TF condensate density n_0 is taken from Eq. (17).

Equation (27) for the velocity potential Φ' contains an explicit term of order $d_\perp^2/R_\perp^2$ arising from the circulating condensate velocity. In fact, however, we seek solutions in a class of functions for which the particle-density fluctuation n' is finite on the vortex axis (at $\rho=0$). To incorporate this boundary condition, it is natural to rewrite Eq. (27) in terms of the function n'. Equation (24) shows that $n' \propto (1-mqd_\perp^2/\omega\rho^2R_\perp^2)\Phi'$, or equivalently,

$$\Phi' \propto \left(1 - \frac{mqd_\perp^2}{\omega\rho^2R_\perp^2}\right)^{-1} n' \approx \left(1 + \frac{mqd_\perp^2}{\omega\rho^2R_\perp^2}\right)n'.$$

Then, omitting terms of order $d_\perp^4/R_\perp^4$, we can replace Eq. (27) for Φ' by the following equation for n':

$$\frac{1}{2n_0(0)}\left[\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho n_0\frac{\partial n'}{\partial\rho}\right) - \frac{m^2n_0n'}{\rho^2} + \lambda^2\frac{\partial}{\partial z}\left(n_0\frac{\partial n'}{\partial z}\right)\right]$$
$$+ \omega^2 n' + \frac{2mqd_\perp^2}{\omega\rho^2R_\perp^2}\left[1 - \omega^2 + \frac{n_0}{n_0(0)\rho}\left(\frac{1}{\rho} - \frac{\partial}{\partial\rho}\right)\right]n' = 0,$$
$$(28)$$

where $n_0/n_0(0) = 1 - \rho^2 - z^2$ in dimensionless units.

The normal modes of a large vortex-free ($q=0$) axisymmetric condensate satisfy the equation [31,32]

$$\frac{1}{2n_0(0)}\left[\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho n_0\frac{\partial n_0'}{\partial\rho}\right) - \frac{m^2n_0n_0'}{\rho^2}\right.$$
$$\left. + \lambda^2\frac{\partial}{\partial z}\left(n_0\frac{\partial n_0'}{\partial z}\right)\right] + (\omega^0)^2 n_0' = 0, \qquad (29)$$

where ω^0 is the frequency of a particular unperturbed normal mode. These low-lying TF normal modes have been studied analytically for a spherical [5] and a cylindrical trap [33]; for a general axisymmetric trap, some of them can be found analytically [5,31,32], but, in general, numerical techniques are necessary [34].

A comparison of Eqs. (28) and (29) immediately yields a simple perturbation expression for the frequency shift of any particular normal mode of an axisymmetric condensate induced by a q-fold quantized vortex

$$\omega^2 - (\omega^0)^2$$
$$= \frac{2mqd_\perp^2}{\omega^0R_\perp^2}\left\langle\frac{(\omega^0)^2-1}{\rho^2} + \frac{n_0}{n_0(0)\rho^3}\left(\frac{\partial}{\partial\rho} - \frac{1}{\rho}\right)\right\rangle, \quad (30)$$

where the angular bracket denotes a matrix element evaluated with the appropriate eigenfunction of the vortex-free condensate. Note that this result describes only modes with $m \neq 0$ because the convective operator $\mathbf{v}_0\cdot\boldsymbol{\nabla}=(V/\rho)\partial/\partial\phi$ vanishes for an axisymmetric state with $m=0$. To the accuracy considered here, the region far from the vortex core makes the main contribution to the frequency shift and the result is independent of the detailed core structure.

One should mention that if we take Eq. (27) for Φ', instead of Eq. (28) for n', as a basis for perturbation analysis, we obtain equivalent results for the frequency shift only for $|m|\geq 2$. The point is that the unperturbed solutions for Φ_0'

and n_0' both behave like $\rho^{|m|}$ near the vortex line and, as follows from Eq. (24), n' will be finite at $\rho=0$ for such solutions only if $|m|\geq 2$. However, if we treat Eq. (28) for the particle-density fluctuation n' perturbatively, the boundary condition for n' is satisfied automatically for any value of m. As a specific example that illustrates this distinction, Eqs. (58) and (59) show that the exact lowest dipole modes of a vortex (those with $|m|=1$) have $n' \propto \rho$ and $\Phi' \propto \rho^{-1}$ for $\rho \to 0$.

IV. FREQUENCY SHIFTS FOR SPECIFIC CASES

It is convenient to write the fractional shift in the squared frequency as

$$\frac{\omega^2-(\omega^0)^2}{(\omega^0)^2} \equiv q\Delta \text{ sgn } m \frac{d_\perp^2}{R_\perp^2}, \qquad (31)$$

where sgn $m = m/|m|$ and

$$\Delta = \frac{2|m|}{(\omega^0)^3}\left\langle\frac{(\omega^0)^2-1}{\rho^2} + \frac{n_0}{n_0(0)\rho^3}\left(\frac{\partial}{\partial\rho} - \frac{1}{\rho}\right)\right\rangle. \quad (32)$$

Since the matrix element is independent of the sign of m, the vortex, generally speaking, splits the two previously degenerate modes with $\pm|m|$ into two levels, according to the sign of m. In particular, the (small) splitting for $|m|>0$ can be characterized by the expression

$$\frac{\omega_{|m|} - \omega_{-|m|}}{\omega_{|m|}^0} \approx q\Delta\frac{d_\perp^2}{R_\perp^2} = q\Delta\left(\frac{d_\perp}{15Na\lambda}\right)^{2/5}, \qquad (33)$$

where the last form follows from Eq. (26). The remaining calculation depends on the shape of the harmonic trap and we consider several specific cases.

A. Spherical trap

For a spherical trap, each unperturbed eigenfunction is a product of a spherical harmonic and a radial polynomial [5]

$$n_0' \propto Y_{lm}(\theta,\phi)r^l P_n^{(l+1/2,0)}(1-2r^2),$$

where $P_n^{(l+1/2,0)}(x)$ are Jacobi polynomials, $r^2 = \rho^2 + z^2$, $\rho = r\sin\theta$, and n is the radial quantum number. The corresponding dimensionless unperturbed eigenfrequency is given by [5] $(\omega_{nl}^0)^2 = l + n(2n + 2l + 3)$.

To calculate the matrix element, one performs an angular and radial integration. The angular averaging of the corresponding operators gives rise to the result

$$\left\langle\frac{1}{\sin^2\theta}\right\rangle_\Omega = \frac{2l+1}{2|m|}, \qquad (34)$$

$$\left\langle \frac{\cos\theta}{\sin^3\theta}\frac{\partial}{\partial\theta} - \frac{1}{\sin^4\theta}\right\rangle_\Omega$$

$$= \begin{cases} -\dfrac{(2l+1)}{4|m|}, & |m|>1 \\[2mm] -\dfrac{(2l+1)}{4}\left(1+\dfrac{l(l+1)}{2}\right), & |m|=1. \end{cases} \qquad (35)$$

The remaining radial integral can be evaluated using the integrals for Jacobi polynomials [35]

$$\int_{-1}^{1}(1-x)^\alpha[P_n^{(\alpha,0)}(x)]^2 = \frac{2^{\alpha+1}}{\alpha+2n+1}, \qquad (36)$$

$$\int_{-1}^{1}(1-x)^{\alpha-1}[P_n^{(\alpha,0)}(x)]^2 = \frac{2^\alpha}{\alpha}, \qquad (37)$$

$$\int_{-1}^{1}(1-x)^{\alpha-2}[P_n^{(\alpha,0)}(x)]^2$$
$$= \frac{2^{\alpha-1}}{\alpha}\frac{(\alpha^2+2n\alpha+\alpha+2n^2+2n)}{(\alpha^2-1)}. \qquad (38)$$

As a result, the general frequency shift for the spherical mode with quantum numbers (nlm) is given by the expression

$$\Delta_{nlm}^{\text{sph}} = \begin{cases} \dfrac{4n+2l+3}{\sqrt{l+n(2n+2l+3)}}, & |m|>1 \\[4mm] \dfrac{4n+2l+3}{\sqrt{l+n(2n+2l+3)}} - \dfrac{(4n+2l+3)l(l+1)[(1+2n)(l+3/2)+2n^2]}{4(l-1/2)(l+3/2)[l+n(2n+2l+3)]^{3/2}}, & |m|=1. \end{cases} \qquad (39)$$

Equations (33) and (39) show that the vortex splits some (but not all) of the previously degenerate $2l+1$ modes with given (nl) and different m. The $m=0$ mode remains unshifted in this approximation, the $|m|=1$ mode is split by an amount $\Delta_{nl1}^{\text{sph}}$, and the remaining modes with $2\le|m|\le l$ are split by a different amount $\Delta_{nlm}^{\text{sph}}$ that is independent of the particular value of m. For $|m|=l=1$, $n=0$ (the lowest dipole mode), one can see that the frequency shift and splitting vanish, as expected, because the vortex-free condensate and that with the vortex both oscillate at the bare trap frequency $\omega_\perp$.

B. Cylindrical trap

The cylindrical case is obtained from the general trap potential by setting $\omega_z=0$, but it is also necessary to use the radius $R_\perp$ as the characteristic length for both the radial and axial coordinates. The unperturbed condensate density is now given simply by $n_0(\rho)=n_0(0)(1-\rho^2)$ and the perturbed eigenfunctions can be written in the form $n'(\rho)e^{i(m\phi+kz)}$, where n' obeys the eigenvalue equation

$$\left[\frac{1}{2}(1-\rho^2)\left(\frac{1}{\rho}\frac{d}{d\rho}\rho\frac{d}{d\rho}-\frac{m^2}{\rho^2}-k^2R_\perp^2\right)-\rho\frac{d}{d\rho}+\omega^2\right]n'(\rho)$$
$$+\frac{2mqd_\perp^2}{\omega\rho^2R_\perp^2}\left[1-\omega^2+\frac{1-\rho^2}{\rho}\left(\frac{1}{\rho}-\frac{d}{d\rho}\right)\right]n'(\rho)=0. \qquad (40)$$

The term with $k^2R_\perp^2$ precludes an exact solution [33], but the frequency shift can again be found perturbatively for wavelengths long compared to $R_\perp$, when we find

$$\omega^2-(\omega^0)^2\approx\frac{2mqd_\perp^2}{\omega^0R_\perp^2}\left\langle\frac{(\omega^0)^2-1}{\rho^2}+\frac{1-\rho^2}{\rho^3}\left(\frac{d}{d\rho}-\frac{1}{\rho}\right)\right\rangle$$
$$+\frac{1}{2}k^2R_\perp^2\langle 1-\rho^2\rangle. \qquad (41)$$

Here the angular brackets denote a matrix element evaluated with the unperturbed eigenfunctions, which have the explicit form $n_0'\propto\rho^{|m|}P_n^{(|m|,0)}(1-2\rho^2)$, and the corresponding eigenfrequency is given by [31,33] $(\omega_{nm}^0)^2=|m|+2n(n+|m|+1)$.

The fractional shift in the squared eigenfrequencies is readily evaluated with the same radial integrals used for the spherical case

$$\frac{\omega_{nm}^2-(\omega_{nm}^0)^2}{(\omega_{nm}^0)^2} = \frac{k^2R_\perp^2}{2(|m|+2n)(|m|+2n+2)}+q\,\text{sgn}\,m\,\frac{d_\perp^2}{R_\perp^2}$$
$$\times\begin{cases}\dfrac{(4n+2|m|+2)}{\sqrt{|m|+2n(n+|m|+1)}}, & |m|>1 \\[4mm]\dfrac{4n(n+1)(n+2)}{[1+2n(n+2)]^{3/2}}, & |m|=1,\end{cases} \qquad (42)$$

where the second term is the contribution of the vortex and the first is that of the traveling wave. The vortex again has only a small effect, which implies that the configuration is stable, assuming that perturbation theory is valid. For $|m|=1$, $n=0$, $k=0$ (dipole mode), the frequency shift is equal to zero.

C. General axisymmetric trap

The normal modes of a general axisymmetric trap in the TF limit have been classified completely and reduced to the diagonalization of a sequence of increasingly large matrices [31,32]. We shall use the cylindrical coordinates of Ref. [32], which turn out to be more convenient for the present purpose. The unnormalized solutions of Eq. (29) have the form

$n_0'^+(\rho,z)=\rho^{|m|}B_{nm}^+$ and $n_0'^-(\rho,z)=\rho^{|m|}zB_{nm}^-$ and are, respectively, even and odd under the transformation $z\rightarrow -z$ (our notation differs somewhat from that of Ref. [32]). Here B_{nm}^+ and B_{nm}^- are even polynomials with typical terms $\rho^{2n_1}z^{2n_2}$, where n_1, n_2, and n are non-negative integers with $n_1+n_2\lesssim n$. For any given n and m, there are $n+1$ even-parity normal modes and, separately, $n+1$ odd-parity normal modes, labeled by $j=0,1,\ldots,n$, with frequencies $\omega_{jnm}^{0\pm}$ determined by diagonalizing $(n+1)$-dimensional matrices given in Ref. [32]. For $n=0$, the even $(+)$ and odd $(-)$ modes have dimensionless frequencies $\omega_{j0m}^{0+}=\sqrt{|m|}$ and $\omega_{j0m}^{0-}=\sqrt{|m|+\lambda^2}$, respectively, with $B_{0m}^+=B_{0m}^-=1$ and $j=0$. For general values of the asymmetry parameter $\lambda=\omega_z/\omega_\perp$, it is straightforward to determine the corresponding solutions for $n=1$ [32], but the higher-order modes require numerical analysis (for $\lambda\rightarrow 0$, Ref. [31] shows that all the unperturbed frequencies agree with those for a cylindrical trap).

Equations (31) and (32) determine the vortex-induced frequency shift $\Delta_{jnm}^{\pm}$ of any particular normal mode. The matrix element should be evaluated for $j=0,1,2,\ldots,n$ (the $n+1$ distinct eigenfunctions for a given $nm\pm$). For the simplest case of $n=0$, we find

$$\Delta_{00m}^+=\begin{cases}\dfrac{2|m|+3}{\sqrt{|m|}}, & |m|>1 \\[2mm] 0, & |m|=1,\end{cases} \tag{43}$$

$$\Delta_{00m}^-=\begin{cases}\dfrac{2|m|+5}{\sqrt{|m|+\lambda^2}}, & |m|>1 \\[2mm] \dfrac{7\lambda^2}{(1+\lambda^2)^{3/2}}, & |m|=1.\end{cases} \tag{44}$$

These expressions are equivalent to those found in Ref. [21] apart from the case $|m|=1$, where our treatment of the singular terms gives additional terms, ensuring that the lowest dipole mode is unshifted because $\Delta_{001}^+=0$.

As a check, we note that Δ_{00m}^+ is just Δ_{nlm}^{sph} from Eq. (39) for a spherical condensate with $n=0$ and $l=|m|$ and that Δ_{00m}^- for $\lambda=1$ is just Δ_{nlm}^{sph} for a spherical condensate with $n=0$ and $l=|m|+1$. In addition, Eq. (33) with our values for $\Delta_{002}^+=7/\sqrt{2}$ and $\Delta_{001}^-=7\lambda^2/(1+\lambda^2)^{3/2}$ yields frequency splittings that agree with those found independently from sum rules by Zambelli and Stringari [25] for the lowest quadrupole modes with $|m|=2$ and $|m|=1$ in the TF limit.

For the next case of $n=1$, Figs. 1 and 2 show the corresponding dimensionless eigenfrequencies $\omega_{j1m}^{0\pm}$ and fractional frequency shifts and splittings $\Delta_{j1m}^{\pm}$ for the even and odd modes $j=0,1$ and $|m|=1,2$ as a function of the asymmetry parameter λ at fixed $d_\perp^2/R_\perp^2$. It is interesting that the curves for $\Delta_{011}^{\pm}$ and $\Delta_{111}^{\pm}$ are nonmonotonic functions of λ and reach a maximum near $\lambda=1$, which is a spherical trap (this behavior is also seen in Δ_{001}^- above). In contrast to the situation for the unperturbed frequencies, the frequency shifts $\Delta^{\pm}$ for a long cigar-shaped axisymmetric trap (the limit $\lambda\rightarrow 0$) differ from those for a cylindrical trap because of the different angular integrations. To check these results,
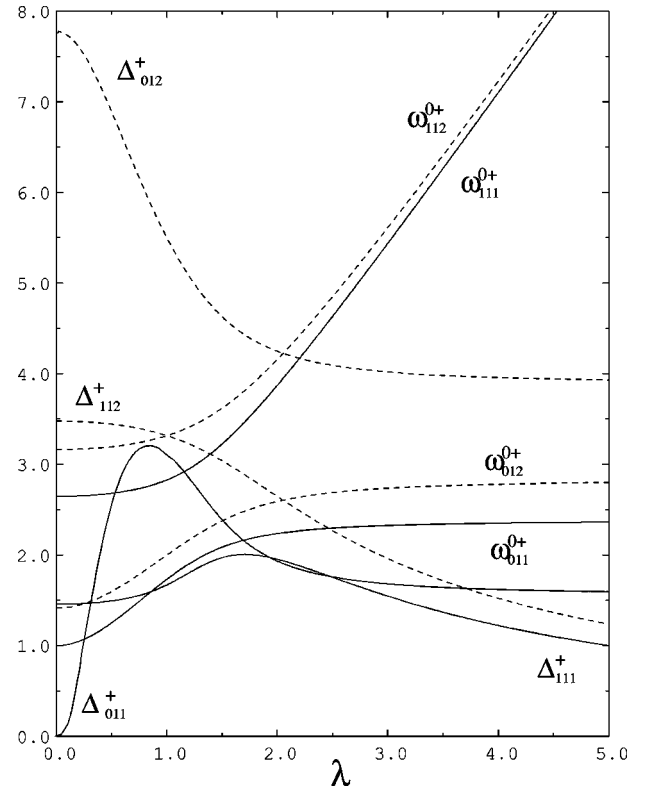

FIG. 1. λ dependence (asymmetry parameter) of the unperturbed dimensionless frequency ω_{j1m}^{0+} and the fractional frequency shift Δ_{j1m}^+ for even modes with $n=1$ and $j=0,1$ (note that $\lambda\ll 1$ represents a cigar-shaped condensate). Solid lines denote $m=1$ and dashed lines denote $m=2$.

the particular case of $\lambda=1$ (a spherical trap) was studied analytically for $n=1$, confirming that these normal modes are indeed consistent with those found in Ref. [5].

V. BOGOLIUBOV EQUATIONS AND VORTEX STABILITY

In the Bogoliubov approximation, the grand-canonical Hamiltonian operator $\hat{K}=\hat{H}-\mu\hat{N}$ reduces to a pure condensate part K_0 that depends only on the condensate wave function Ψ and a noncondensate part $\hat{K}'$ [13,17]. The latter involves the solutions of the Bogoliubov equations

$$\mathcal{L}u_j-g\Psi^2 v_j=E_j u_j, \tag{45}$$

$$\mathcal{L}v_j-g(\Psi^*)^2 u_j=-E_j v_j, \tag{46}$$

where $\mathcal{L}=T+V_{tr}+2g|\Psi|^2-\mu$. Here u_j and v_j are a complete set of coupled amplitudes that obey the normalization condition $\int dV(|u_j|^2-|v_j|^2)=1$ and E_j is the associated eigenvalue. The ground-state occupation of the jth mode is simply $N_j'=\int dV|v_j|^2$ and the noncondensate part (a second-quantized operator) reduces to

$$\hat{K}'=-\sum_j{}' E_j N_j'+\sum_j{}' E_j \beta_j^\dagger \beta_j, \tag{47}$$

where the primed sum means to omit the condensate mode and $\beta_j^\dagger$ and β_j are quasiparticle creation and annihilation operators that obey boson commutation relations $[\beta_j,\beta_k^\dagger]$
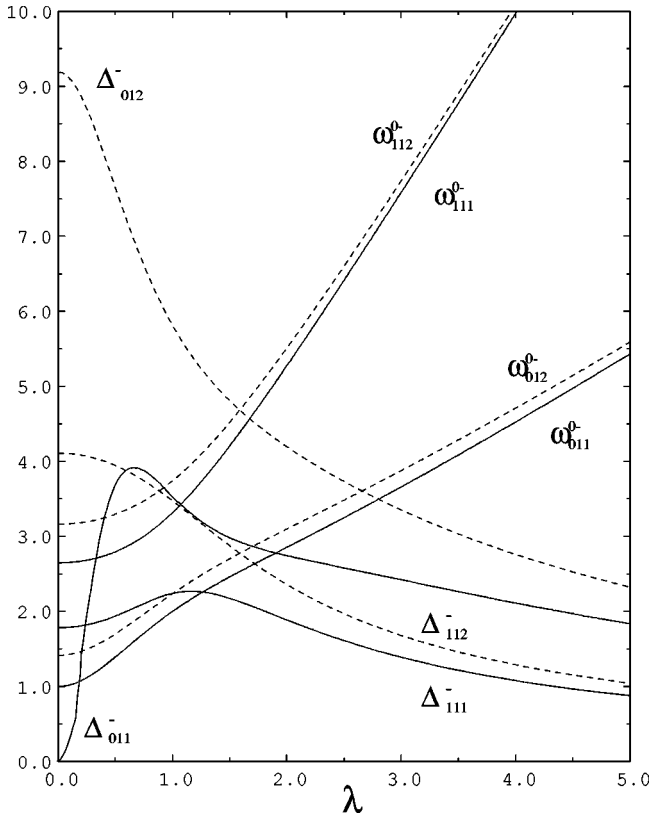
FIG. 2. λ dependence (asymmetry parameter) of the unperturbed dimensionless frequency ω_{j1m}^{0-} and the fractional frequency shift Δ_{j1m}^{-} for odd modes with $n=1$ and $j=0,1$ (note that $\lambda \ll 1$ represents a cigar-shaped condensate). Solid lines denote $m=1$ and dashed lines denote $m=2$.

$=\delta_{jk}$. The ground state of $\hat{K}'$ is the vacuum $|0\rangle$, defined by $\beta_j|0\rangle=0$ for all $j\neq 0$, so that the first term in Eq. (47) is the ground-state expectation value of $\hat{K}'$; the remaining term involves the quasiparticle number operator $\beta_j^{\dagger}\beta_j$ and the spectrum of $\hat{K}'$ is bounded from below as long as $E_j>0$ for all j. If, however, any of the properly normalized solutions has a negative eigenvalue, the system can lower its ground-state thermodynamic potential $\Omega=\langle\hat{K}\rangle$ arbitrarily by creating more and more quasiparticles for that particular normal mode. This behavior indicates that the original ground state Ψ is unstable and must be reconstructed to form a new stable ground state [36].

A. Axial vortex waves

For a condensate containing a q-fold quantized vortex in an axisymmetric trap (studied in Sec. II), the condensate wave function has the form $\Psi(\mathbf{r})=e^{iq\phi}|\Psi_q(\rho,z)|$. In the presence of a general condensate velocity $\mathbf{v}_0=(\hbar/M)\boldsymbol{\nabla}S$, it is convenient to make explicit the phase of the condensate wave function through factors $e^{\pm iS}$ [17] and the Bogoliubov amplitudes here can be written as

$$\begin{pmatrix} u(\mathbf{r}) \\ v(\mathbf{r}) \end{pmatrix} = e^{im\phi+ikz}\begin{pmatrix} e^{iq\phi}\tilde{u}_m(\rho,z) \\ e^{-iq\phi}\tilde{v}_m(\rho,z) \end{pmatrix}. \quad (48)$$

This form represents a helical wave with wave number k and angular momentum m relative to the condensate; we assume

that the factor e^{ikz} varies rapidly compared to the amplitudes $\tilde{u}$ and $\tilde{v}$, with the condensate containing many wavelengths ($kR_z\gg 1$). The corresponding amplitudes obey the coupled equations [12]

$$\left[-\frac{\hbar^2}{2M}\nabla^2 + \frac{\hbar^2}{2M}\left(\frac{(m+q)^2}{\rho^2} + k^2 - 2ik\frac{\partial}{\partial z}\right) \right.$$
$$\left. + V_{\text{tr}} + 2g|\Psi_q|^2 - \mu \right]\tilde{u}_m - g|\Psi_q|^2\tilde{v}_m = E\tilde{u}_m, \quad (49a)$$

$$\left[-\frac{\hbar^2}{2M}\nabla^2 + \frac{\hbar^2}{2M}\left(\frac{(m-q)^2}{\rho^2} + k^2 - 2ik\frac{\partial}{\partial z}\right) \right.$$
$$\left. + V_{\text{tr}} + 2g|\Psi_q|^2 - \mu \right]\tilde{v}_m - g|\Psi_q|^2\tilde{u}_m = -E\tilde{v}_m, \quad (49b)$$

where $\nabla^2 = \partial^2/\partial\rho^2 + \rho^{-1}\partial/\partial\rho + \partial^2/\partial z^2$ is expressed in cylindrical polar coordinates. The different centrifugal barriers imply that the two amplitudes behave differently near the axis of symmetry, with $\tilde{u}_m\propto\rho^{|m+q|}$ and $\tilde{v}_m\propto\rho^{|m-q|}$ as $\rho\to 0$.

As noted in Sec. II, these Bogoliubov amplitudes are linearly related to the corresponding hydrodynamic variables n' and Φ'. In an axisymmetric trap with a q-fold vortex on the symmetry axis, their angular dependence is simply $e^{im\phi}$ and the corresponding amplitudes are given by Eqs. (4) [17,18,37]:

$$n'_{jm} = |\Psi_q|(\tilde{u}_{jm} - \tilde{v}_{jm}), \quad (50a)$$

$$\Phi'_{jm} = \frac{\hbar}{2Mi|\Psi_q|}(\tilde{u}_{jm} + \tilde{v}_{jm}), \quad (50b)$$

where j denotes the remaining set of quantum numbers and $|\Psi_q|\propto\rho^{|q|}$ for $\rho\to 0$. In the simplest case of a vortex-free condensate ($q=0$), these amplitudes reproduce the behavior proportional to $\rho^{|m|}$ found in Refs. [31,32], but the situation is more complicated for $q\neq 0$. The boundary condition that $\tilde{u}$ and $\tilde{v}$ remain bounded near the origin (discussed below in Sec. V B and in Ref. [28]) implies a corresponding behavior for n' and Φ'. Although Φ' has an apparent singularity for small ρ from the factor $|\Psi_q|^{-1}$, physical quantities such as the hydrodynamic current involve an additional factor $n_q=|\Psi_q|^2$ [17]. As shown in Sec. III, the detailed core structure has negligible effect on the normal-mode eigenfrequencies in the TF limit (a large condensate), but it would be significant for a smaller condensate with $Na/d_0\lesssim 10$, when the TF relation in Eq. (19) implies that $d_0/R_0\gtrsim 0.4$ is no longer small [23].

The general Bogoliubov equations are difficult to solve. In the present case of a harmonic trap potential V_{tr}, however, they have three exact solutions. For these special ''dipole'' modes, any condensate described by a wave function Ψ that satisfies the GP equation (1) undergoes a rigid oscillation along the three principal axes of the trap with the bare oscillator frequencies [37]. Define the raising and lowering operators

$$a_\alpha^\dagger \equiv \frac{1}{\sqrt{2}}\left(\frac{x_\alpha}{d_\alpha} - d_\alpha \frac{\partial}{\partial x_\alpha}\right), \tag{51a}$$

$$a_\alpha \equiv \frac{1}{\sqrt{2}}\left(\frac{x_\alpha}{d_\alpha} + d_\alpha \frac{\partial}{\partial x_\alpha}\right), \tag{51b}$$

where $d_\alpha \equiv \sqrt{\hbar/m\omega_\alpha}$ and $\alpha = x$, y, and z. As shown in Ref. [37], the state

$$\mathcal{U}_\alpha(\mathbf{r}) \equiv \begin{pmatrix} u_\alpha(\mathbf{r}) \\ v_\alpha(\mathbf{r}) \end{pmatrix} = \begin{pmatrix} a_\alpha^\dagger \Psi(\mathbf{r}) \\ a_\alpha \Psi^*(\mathbf{r}) \end{pmatrix} \tag{52}$$

satisfies the coupled Bogoliubov equations with an eigenvalue $E_\alpha = \hbar\omega_\alpha$.

For an axisymmetric trap potential $V_{\mathrm{tr}} = \frac{1}{2}M\omega_\perp^2 \rho^2 + \frac{1}{2}M\omega_z^2 z^2$, the two states $\mathcal{U}_x$ and $\mathcal{U}_y$ are degenerate, each with energy $\hbar\omega_\perp$. In terms of the new operators [38]

$$a_\pm = \frac{1}{\sqrt{2}}(a_x \mp i a_y), \tag{53}$$

$$a_\pm^\dagger = \frac{1}{\sqrt{2}}(a_x^\dagger \pm i a_y^\dagger), \tag{54}$$

the (conserved) angular momentum about the symmetry axis can be rewritten as $L_z = \hbar(a_+^\dagger a_+ - a_-^\dagger a_-)$. As a result, $a_+^\dagger$ ($a_-^\dagger$) adds a unit of positive (negative) angular momentum. Correspondingly, we have two distinct solutions of the Bogoliubov equations

$$\mathcal{U}_\pm(\mathbf{r}) = \begin{pmatrix} u_\pm(\mathbf{r}) \\ v_\pm(\mathbf{r}) \end{pmatrix} = \mathcal{U}_x(\mathbf{r}) \pm i \mathcal{U}_y(\mathbf{r}) = \begin{pmatrix} a_\pm^\dagger \Psi(\mathbf{r}) \\ a_\mp \Psi^*(\mathbf{r}) \end{pmatrix}, \tag{55}$$

each with energy $\hbar\omega_\perp$. In cylindrical polar coordinates, the explicit form of these amplitudes is readily determined; for a q-fold quantized vortex, we find

$$u_+(\mathbf{r}) = \frac{1}{2}e^{i(q+1)\phi}\left(\frac{\rho}{d_\perp} - d_\perp \frac{\partial}{\partial\rho} + \frac{qd_\perp}{\rho}\right)|\Psi_q(\rho,z)|, \tag{56a}$$

$$v_+(\mathbf{r}) = \frac{1}{2}e^{i(-q+1)\phi}\left(\frac{\rho}{d_\perp} + d_\perp \frac{\partial}{\partial\rho} + \frac{qd_\perp}{\rho}\right)|\Psi_q(\rho,z)|, \tag{56b}$$

$$u_-(\mathbf{r}) = \frac{1}{2}e^{i(q-1)\phi}\left(\frac{\rho}{d_\perp} - d_\perp \frac{\partial}{\partial\rho} - \frac{qd_\perp}{\rho}\right)|\Psi_q(\rho,z)|, \tag{57a}$$

$$v_-(\mathbf{r}) = \frac{1}{2}e^{i(-q-1)\phi}\left(\frac{\rho}{d_\perp} + d_\perp \frac{\partial}{\partial\rho} - \frac{qd_\perp}{\rho}\right)|\Psi_q(\rho,z)|, \tag{57b}$$

and it is clear that these solutions are properly orthogonal, with $\int dV(u_-^* u_+ - v_-^* v_+) = 0$. A comparison with Eq. (48) shows that $\mathcal{U}_\pm$ has $m = \pm 1$ and $k = 0$. Pitaevskii [12] constructed analogous zero-frequency solutions in the case of a vortex line in an unbounded fluid (his solutions emerge in the limit $\omega_\perp \to 0$, so that $d_\perp \to \infty$).

It is worth noting that the fluctuations in the particle density n' and the velocity potential Φ' for these lowest dipole-mode solutions (56) and (57) have simple explicit expressions

$$n' = -\frac{d_\perp}{2} e^{\pm i\phi} \frac{\partial n_q}{\partial\rho}, \tag{58}$$

$$\Phi' = \frac{\hbar}{2Mi} e^{\pm i\phi}\left(\frac{\rho}{d_\perp} \pm \frac{qd_\perp}{\rho}\right). \tag{59}$$

The contrasting behavior of n' and Φ' near the vortex axis helps motivate the discussion in Sec. III B.

The form of Eq. (49) suggests an approximation for small k and $m = \pm 1$, using $\mathcal{U}_\pm$ as the unperturbed eigenfunctions. Standard first-order perturbation theory gives the corrected eigenvalue

$$E_\pm(k) \approx \hbar\omega_\perp + \frac{\hbar^2 k^2}{2M} \frac{\int dV(|u_\pm|^2 + |v_\pm|^2)}{\int dV(|u_\pm|^2 - |v_\pm|^2)}, \tag{60}$$

where the additional term involving $\partial/\partial z$ yields a surface contribution that vanishes. Use of Eqs. (56) and (57) yields the explicit expressions

$$\int dV(|u_\pm|^2 + |v_\pm|^2)$$
$$= 2\int dV\left[\left(\frac{\rho}{d_\perp} \pm q \frac{d_\perp}{\rho}\right)^2 |\Psi_q|^2 + d_\perp^2 \left(\frac{\partial|\Psi_q|}{\partial\rho}\right)^2\right], \tag{61a}$$

$$\int dV(|u_\pm|^2 - |v_\pm|^2)$$
$$= 2\int dV\left(\rho \pm q \frac{d_\perp^2}{\rho}\right)\left(-\frac{\partial|\Psi_q|^2}{\partial\rho}\right). \tag{61b}$$

Equation (61b) is easily evaluated with an integration by parts and the normalization condition Eq. (15). In the TF limit of a large condensate with $d_\perp \ll R_\perp$, the dominant contribution to Eq. (61a) comes far from the vortex core, so that we can replace $|\Psi_q|$ from Eq. (23) by $|\Psi_0|$ from Eq. (17). Neglecting corrections of order $d_\perp^2/R_\perp^2$, we find the compact result

$$E_\pm(k) \approx \hbar\omega_\perp + \frac{\hbar^2 k^2}{2M}\left(\frac{R_\perp^2}{7d_\perp^2} \pm q\right)$$
$$= \hbar\omega_\perp[1 + \tfrac{1}{2}(\tfrac{1}{7}k^2 R_\perp^2 \pm q k^2 d_\perp^2)]. \tag{62}$$

As in Eq. (42), the correction is indeed small if $|q|$ is not too large and $kR_\perp \lesssim 1$ (since we also require $kR_z \gg 1$, this latter condition implies a cigar-shaped condensate with small asymmetry parameter $\lambda \ll 1$). Consequently, a q-fold quantized vortex has long-wavelength propagating axial helical modes with wave number k and angular distortion $m = \pm 1$. In this perturbative limit, the frequencies of these two modes exceed the bare dipole frequency $\omega_\perp$, as expected for an internal distortion. The separate parabolic branches are split by an amount

$$E_+(k) - E_-(k) \approx q \frac{\hbar^2 k^2}{M}. \qquad (63)$$

In his original work on a singly quantized vortex in an unbounded dilute Bose gas, Pitaevskii [12] developed an approximate description of vortex waves for shorter wavelengths, assuming only that $k\xi \ll 1$. To logarithmic accuracy, the dispersion relation has the classical form $\omega_k = (\hbar k^2/2M)\ln(1/k\xi)$ [39]. This method is easily generalized to the present case of a q-fold quantized vortex on the symmetry axis of an axisymmetric trap in the TF limit. We use the coherence length ξ as the length scale and rewrite the exact condensate density as $\psi_q^2 \equiv |\Psi_q|^2/n_0(0)$. The coupled Bogoliubov equations (49) for $m = -1$ become (here the coordinates are dimensionless)

$$\left[-\nabla^2 + \frac{(q-1)^2}{\rho^2} + \kappa^2 - 2i\kappa \frac{\partial}{\partial z} + \frac{\xi^4}{d_\perp^4}(\rho^2 + \lambda^2 z^2) + 2\psi_q^2 - 1 \right]$$
$$\times \tilde{u}_{-1} - \psi_q^2 \tilde{v}_{-1} = \epsilon \tilde{u}_{-1}, \qquad (64a)$$

$$\left[-\nabla^2 + \frac{(q+1)^2}{\rho^2} + \kappa^2 - 2i\kappa \frac{\partial}{\partial z} + \frac{\xi^4}{d_\perp^4}(\rho^2 + \lambda^2 z^2) + 2\psi_q^2 - 1 \right]$$
$$\times \tilde{v}_{-1} - \psi_q^2 \tilde{u}_{-1} = -\epsilon \tilde{v}_{-1}, \qquad (64b)$$

where we have set $\mu_q \approx \mu_0 = g n_0(0)$ and introduced the abbreviations $\epsilon = (2E/\hbar\omega_\perp)(\xi^2/d_\perp^2) = E/\mu_0 \ll 1$ and $\kappa = k\xi \ll 1$.

The amplitudes vary slowly along the axis with a characteristic length R_z, so that the various derivatives with respect to z can be omitted, along with the trap potential (which is here of order $\xi^4/d_\perp^4 \ll 1$), in which case these equations (64) take the approximate form

$$\left(-\frac{1}{\rho} \frac{d}{d\rho} \rho \frac{d}{d\rho} + \frac{(q-1)^2}{\rho^2} + \kappa^2 + 2\psi_q^2 - 1 \right) \tilde{u}_{-1}$$
$$- \psi_q^2 \tilde{v}_{-1} = \epsilon \tilde{u}_{-1}, \qquad (65a)$$

$$\left(-\frac{1}{\rho} \frac{d}{d\rho} \rho \frac{d}{d\rho} + \frac{(q+1)^2}{\rho^2} + \kappa^2 + 2\psi_q^2 - 1 \right) \tilde{v}_{-1}$$
$$- \psi_q^2 \tilde{u}_{-1} = -\epsilon \tilde{v}_{-1}. \qquad (65b)$$

In the same approximation, the condensate wave function satisfies the GP equation

$$\left(-\frac{1}{\rho} \frac{d}{d\rho} \rho \frac{d}{d\rho} + \frac{q^2}{\rho^2} + \psi_q^2 - 1 \right) \psi_q = 0, \qquad (66)$$

whose asymptotic solution for $\rho \gg 1$ is $\psi_q \approx 1 - q^2/2\rho^2$. This set of equations is now precisely the same as that obtained by Pitaevskii [12], apart from the appearance of the quantum number q instead of $q = 1$. Exactly the same arguments then show that the dispersion relation is given by $E_k \approx q(\hbar^2 k^2/2M)\ln(1/k\xi)$, where we assume $k\xi \ll 1 \ll kd_\perp$. This condition implies that E_k is significantly greater than the lowest dipole energy $E_\perp = \hbar\omega_\perp = \hbar^2/Md_\perp^2$.

B. Bound core states

Rokhsar [22] has argued that the mere existence of a bound state localized in the vortex core implies an instability, with the original axisymmetric configuration undergoing a displacive transition of the vortex core. In the present context, such an instability would arise only from the presence of one or more negative-energy eigenvalues of the Bogoliubov equations, causing the original ground state of Eq. (47) to collapse. For such a situation, our preceding perturbative approach would fail completely, requiring a more powerful approach.

In their original solution for a long vortex line in unbounded superfluid ^{4}He, Ginzburg and Pitaevskii [40] remarked that multiply quantized vortices are energetically unfavorable. It is also notable that no clear evidence for doubly quantized vortices in superfluid ^{4}He II has ever been reported [41]. Thus we consider only the case $q = 1$ and seek solutions with relative angular symmetry proportional to $e^{im\phi}$ for the full Bogoliubov equations with a singly quantized vortex in an axisymmetric trap. In the TF limit, we neglect small corrections of order $\xi^2/R_0^2 \sim \xi^4/d_0^4$ and these equations then reduce to those for a vortex line in an unbounded condensate [here they are direct generalizations of Eqs. (65), obtained for general m by setting $q = 1$]. For $q = 1$, the result of Eq. (12) shows that the only possible unstable solution corresponds to the choice $|m| = 1$. Then, since the amplitude $\tilde{u}_{-1}$ remains finite at the origin, we anticipate that the choice $m = -1$ will yield a positive normalization for $|\tilde{u}_{-1}|^2 - |\tilde{v}_{-1}|^2$.

These coupled Bogoliubov radial equations (65) are equivalent to a single fourth-order ordinary differential equation. Near the origin, there are four linearly independent solutions and it is straightforward to show [28] that only two are finite as $\rho \to 0$. In a matrix notation, they can be written $(\tilde{u}_{-1}^{(1)}, \tilde{v}_{-1}^{(1)})$ and $(\tilde{u}_{-1}^{(2)}, \tilde{v}_{-1}^{(2)})$. Each is a power series in ρ^2, with leading behavior (omitting constant factors)

$$\begin{pmatrix} \tilde{u}^{(1)}(\rho) \\ \tilde{v}^{(1)}(\rho) \end{pmatrix} \approx \begin{pmatrix} \rho^0 \\ \rho^4 \end{pmatrix}, \quad \begin{pmatrix} \tilde{u}^{(2)}(\rho) \\ \tilde{v}^{(2)}(\rho) \end{pmatrix} \approx \begin{pmatrix} \rho^6 \\ \rho^2 \end{pmatrix}, \qquad (67)$$

where we have suppressed the subscript -1. The general solution is a linear combination, with $\tilde{u} \approx c_1 + O(\rho^2)$ and $\tilde{v} \approx c_2 \rho^2 + O(\rho^4)$, where c_1 and c_2 are constants. Note that the linear combinations $\tilde{u} \pm \tilde{v}$ appearing in the hydrodynamic amplitudes [Eqs. (50)] both have the same behavior at the origin. Although this situation apparently produces a singularity in the variational expression in Eq. (12), it is not hard to verify that the divergence in fact cancels out.

Similarly, it is not difficult to verify [28] that the asymptotic solutions of Eqs. (65) have the form proportional to $\rho^{-1/2}\exp(i\zeta\rho)$, where ζ^2 is the solution of a quadratic equation. One solution is negative with $\zeta^2 = -\zeta_{im}^2 = -\sqrt{\epsilon^2+1} - \kappa^2 - 1$; the other is $\zeta_1^2 = \sqrt{\epsilon^2+1} - \kappa^2 - 1$, which can take either sign. To eliminate the growing exponential solution proportional to $\rho^{-1/2}\exp(|\zeta_{im}|\rho)$, it is necessary to adjust the ratio c_1/c_2, leaving a solution for $\tilde{u}$ and $\tilde{v}$ that is bounded both at the origin and at infinity. It has the asymptotic form

$$\tilde{u}, \tilde{v} \sim \rho^{-1/2} [A \exp(i\zeta_1 \rho) + B \exp(-i\zeta_1 \rho)$$

$$+ C \exp(-|\zeta_{\rm im}|\rho)], \qquad (68)$$

where A, B, and C depend on ϵ and κ.

For $\kappa^2 > 0$, it is easy to see that ζ_1 is real if $|\epsilon| > \epsilon_B(\kappa) \equiv \sqrt{\kappa^4 + 2\kappa^2}$, which is the Bogoliubov energy for the wave number κ along the vortex axis. For this range of parameters, it follows that the asymptotic solutions oscillate and there is no bound state. In contrast, if $|\epsilon| < \epsilon_B(\kappa)$, then ζ_1 becomes imaginary and the coefficient of the growing exponential must vanish. If we choose Im $\zeta_1 > 0$, then $B(\kappa, \epsilon)$ must vanish, which in principle determines the dispersion relation of the bound state as a function of the axial wave number. In effect, this procedure yields Pitaevskii's vortex-wave solution [12].

The situation is different if $\kappa = 0$, for then $\epsilon_B(0)$ vanishes and $\zeta_1^2 = \sqrt{\epsilon^2 + 1} - 1$ is non-negative for all real ϵ. Thus the general solution for $\kappa = 0$ necessarily oscillates far from the vortex core for any real positive value of ϵ^2, implying that there is no bound state in this case (the solution with $\epsilon = 0$ is simply the dipole oscillation, here with zero frequency, as found by Pitaevskii [12]). A similar argument holds in the extreme TF limit for general q and m since there are always two linearly independent solutions that remain finite at the origin [28] and the asymptotic behavior for large ρ is independent of the values of q and m (the centrifugal barriers contribute only in higher orders).

This behavior is easily understood from a classical perspective. A long classical hollow-core vortex line in an incompressible fluid has irrotational flow that is describable with a velocity potential. In addition to boundary conditions at the inner surface of the core, the velocity potential satisfies Laplace's equation [42]. Solutions of Laplace's equation cannot exhibit spatial oscillations in all directions, so that a vortex wave propagating along the axis with wave number k necessarily decays exponentially $\sim e^{-k\rho}$ in the radial direction. In the limit $k \to 0$, this exponential decay length k^{-1} diverges and the amplitude can only vanish algebraically as $\rho \to \infty$, which is the behavior found here for a vortex line in an unbounded condensate.

Based on Rokhsar's discussion [22], the numerical study of Dodd *et al.* [23] found one negative-frequency normal-mode solution for a singly quantized vortex (the actual negative frequency should be the negative of that shown in their Fig. 2). In the ideal-gas limit, the anomalous frequency is $-\omega_\perp$ and it increases toward zero from below in the large-condensate limit $N_0 a/d_0 \gg 1$. The present treatment implies that this particular eigenfrequency approaches zero from below in the limit of a large condensate ($\xi/R_0 \ll 1$). The precise dependence on the small parameter and the relation to the critical angular velocity for vortex creation remain for future analysis.

VI. DISCUSSION

We have used a variety of methods to study the stability of an axisymmetric vortex in a large axisymmetric trapped Bose condensate. In this limit, the repulsive interparticle condensate energy dominates the condensate kinetic energy. The resulting Thomas-Fermi approximation holds when the equatorial condensate radius $R_\perp$ is much larger than the corresponding oscillator length $d_\perp = \sqrt{\hbar/M\omega_\perp}$. The hydrodynamic description of the condensate's normal modes simplifies greatly in the TF limit and the presence of a vortex leads to small corrections of relative order $d_\perp^2/R_\perp^2 \ll 1$. A perturbation analysis shows that a vortex splits the degenerate normal modes of the vortex-free condensate (those for $\pm m$) by an amount of this same order, which should provide a clear signal for the existence of a vortex in the trapped condensate. Since the vortex-free condensate is considered stable from both a theoretical and an experimental perspective [4–9], this conclusion indicates that the vortex should also be stable, at least in perturbation theory. To preclude the possibility of nonperturbative effects associated with the possible presence of bound states with a radial range $\sim \xi$ localized in the vortex core [22], we demonstrate that a long vortex line in an unbounded condensate (the extreme TF limit of a trapped condensate) has no such normal modes [23] other than the vortex waves found by Pitaevskii [12].

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