Free expansion of Bose-Einstein Condensates with a Multi-charged Vortex

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Abstract

We based on variational method to derive analytical expressions to describe the free expansion of a multi-charged condensate with a central vortex. The evolution of the vortex core size and the asymptotic velocity during free expansion was studied with the cloud released from different harmonic trap configurations. We found that an oblate condensate shape magnifying the effects of the vorticity in the expansion of the cloud. Besides that, the asymptotic velocity field of the cloud dimensions gave direct information about the interaction and circulation , being also an important parameter to track.

I. INTRODUCTION

The remarkable discovery in BEC's physics was the nucleation of vortices inside the Bose-Einstein condensate of Alkali atoms, with many experimental evidence[1–5]. These vortices have quantized angular momentum by number of particles, and they are nucleation when the system gains a rotation, i.e., angular momentum. If the rotation of system overcomes a critical velocity of rotation, Ω_c, it triggers the emergence of quantized vortices[6–9]. Thus the dynamics of vortex formation, the stability dynamics and the free expansion dynamics are the subject of several studies[10–12]. The vortex dynamics have proved of paramount importance for the understanding of the quantum turbulence[3, 13], and the free expansion is the main experimental method of measurement[2, 14–17]. There are many theoretical works about vortex latices and vortex in rotating traps [4, 16, 18–21], where the starting point is the Gross-Pitaeviskii equation (GPE) [22, 23].

In this paper we considered a muli-charged vortex at the center of the condensate, which is the center of the harmonic trap. Here, we used the variational method to study the effects that arise while the free expansion of condensate, as consequence of non-fundamental vortices. A similar works was done by E. Lundh *et al*[24] to the case where the vortex has only one quantum of circulation, which the same work can be found with quite numeric character in Micheli *et al*[25].

The section II present the theoretical methods used in this work. In the section III, we have search for a suitable trial function and analyses this one to determine the radii of condensate. Soon after it, in the sections IV and V, we found the initials condition and the motion equations to the free expansion, and also take the TF-limit to the both.

Finally, the section VI has the discussions and outlooks about our results by the method employed.

II. THEORETICAL METHOD

At absolute zero temperature in the absense of thermal cloud the system is exactly described by GPE:

$$i\hbar\frac{\partial\Psi(\vec{r},t)}{\partial t} = \left[-\frac{\hbar^2}{2m}\nabla^2 + V(\vec{r}) + U_0\left|\Psi(\vec{r},t)\right|^2\right]\Psi(\vec{r},t). \quad (1)$$

The harmonic trap is described in cylindrical coordinates as $V(\vec{r}) = \frac{1}{2}m\omega_\rho^2(\rho^2 + \lambda^2 z^2)$, where λ is the anisotropic parameter, and the constant of mean-field interaction is $U_0 = 4\pi\hbar^2 a_s/m$. Following the variational principle we can write the Lagrangian density $\mathcal{L}$ which recoves the GPE for a complex field $\Psi(\vec{r},t)$, thus $\mathcal{L}$ is writed as

$$\mathcal{L} = \frac{i\hbar}{2}\left(\Psi^*(\vec{r},t)\frac{\partial\Psi(\vec{r},t)}{\partial t} - \Psi(\vec{r},t)\frac{\partial\Psi^*(\vec{r},t)}{\partial t}\right)$$
$$-\frac{\hbar^2}{2m}\left|\nabla\Psi(\vec{r},t)\right|^2 - V(\vec{r})\left|\Psi(\vec{r},t)\right|^2 - \frac{U_0}{2}\left|\Psi(\vec{r},t)\right|^4. \tag{2}$$

Where $\Psi(\vec{r},t)$ can be aprproximated by a trial function $\Psi_\ell(\vec{r},t)$ which depends on a set of variational parameters $q_i = q_i(t)$[26, 27]. This function can them be substituted into

$$L = \int \mathcal{L} d^3r, \tag{3}$$

in such a way that their time evolution is given by

$$\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = 0. \tag{4}$$

In that way, the next step is to find a suitable trial function.

III. 2D CASE

Assuming, in cylindrical coordenates, a condensate with multi-carged vortex centred long at z-axis so that this we can be approached it to a 2D system since the effect of free expansion in this direction can be discarded without loss of generality. Here we interestied in test three kind of ansatz, all of them based in TF-approach. To find the fist one, we used a wave function with this form:

$$\psi_\ell(\rho,\phi) = f_\ell(\rho)e^{i\ell\phi}, \tag{5}$$

where $f_\ell(\rho)$ is the amplitude of the wave function, and $e^{i\ell\phi}$ is a phase which carries the information of the velocity field, where ℓ is integer and means the quantum number of circulation. Therefore, by introducing 5 in time-independent GPE we have a centrifugal term, which comes from phase $e^{i\ell\phi}$. This term is write as $\left(\frac{\hbar^2\ell^2}{2m\rho^r}\right)f$. To find a suitable ansatz for multi-charged vortex, we considered the interval of $\rho \ll \xi$, and we have neglected potential terms from GPE. By considering only kinetic terms with the additional centrifugal term, we need to solve

$$-\frac{\hbar^2}{2m}\left[\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial f}{\partial\rho}\right)\right]+\frac{\hbar^2\ell^2}{2m\rho^2}f=0, \tag{6}$$

this equation give us the solution for $f(\rho)$ when this is smaller than cloud, which means the vortex solution. For $\rho > \xi$, we considered the TF-approach, thus our amplitude $f(\rho)$ is

$$f(\rho)=\sqrt{\frac{2\nu}{\pi R^2 A_1^{(1)}}}\left(\frac{\rho^2}{\rho^2+R^2\ell^2\alpha^2}\right)^{\frac{1}{2}}\psi_{TF}(\rho), \tag{7}$$

where $\alpha = \xi/R$, $\psi_{TF}(\rho) = \sqrt{1-(\rho/R)^2}$ for $0 < \rho \leq R$ otherwise $\psi_{TF}(\rho) = 0$ and $A_1^{(1)}$ will be showed after because all anstaz result in the same motion equations distinguish only by the A's terms, ν is the number of atoms per unit lengh. A similar TF-approach can be extract from time-independent GPE, this deduction is in the appendix A. Thus the fist trial function is given by:

$$\Psi_\ell^{(1)}(\rho,\phi,t)=f(\rho,t)e^{i\ell\phi}\exp\left[iB^{(1)}(t)\frac{\rho^2}{2}\right], \tag{8}$$

where R, ξ, α and $A_1^{(1)}$ are time-dependents.

The next ansatz was chosen as a simplification of (8), which is write as:

$$\Psi_\ell^{(2)}(\rho,\phi,t)=\sqrt{\frac{2\nu}{\pi R(t)^{2\ell+2}A_1^{(2)}(t)}}\rho^\ell e^{i\ell\phi}\psi_{TF}(\rho,t)\exp\left[iB^{(2)}(t)\frac{\rho^2}{2}\right], \tag{9}$$

where the vortex core is calculed as being inversaly proportional at the healing lengh of the densite $\left|\Psi_0^{(2)}(\rho=0,t)\right|^2$, given by

$$\xi_\ell(t)=\ell\xi(t)=\frac{\ell R(t)}{4\sqrt{\nu a_s}}. \tag{10}$$

Note that, here the parameter α is time-independent. A similiar trial function was used by E. Lundh *et al* in [24] where they used a gaussian function in the place of $\psi_{TF}(\rho,t)$ with $\ell = 1$.

So the last one is the most commun used as a TF-approach of the vortex state

$$\Psi_\ell^{(3)}(\rho,\phi,t)=\sqrt{\frac{2\nu}{\pi R(t)^2 A_1^{(3)}(t)}}e^{i\ell\phi}\sqrt{1-\left[\frac{\ell R(t)\alpha(t)}{\rho}\right]^2-\left(\frac{\rho}{R(t)}\right)^2}\exp\left[iB^{(3)}(t)\frac{\rho^2}{2}\right], \tag{11}$$

where $\Psi_{\ell}^{(3)}(\rho,\phi,t)$ must be positive. The parameters $B^{(i)}(t)$ are responsible for the function curvature, in other words, it's from these parameters that will appears the acceleration in motion equation by the variational method[26]. Remembering the TF approximation is equivalent to neglict the kinect energy part from time-independent GPE, so we will consider that the cuvature of these, $\psi_{TF}(\rho,t)$ and $\sqrt{1-[\xi(t)/\rho]^2-[\rho/R(t)]^2}$, vary slowly which mean drop all terms that is due to the grandient of this part in the Lagrangian calculation. Another thing I should point out is the limit of integration of our calculation to $\Psi_{\ell}^{(3)}(\rho,\phi,t)$ was calculated in the approach of $\xi \ll R$. In general, when one use ψ_{TF} the parameter R is the radius of condensate ($\rho_{rms} = R$), however, in 9 and 11 we must calculate the $\rho_{rms} = \langle\rho^2\rangle = R\sqrt{A_2/A_1}$, because the size vortex core of these trial functions affects the scale of the parameters of the wave function. The procedure is the same when one uses a Gaussian trial function. Rescaling the parameters to become dimenssionless as: $\tau = \omega t$, $R(t) = (\hbar/m\omega_\rho)^{\frac{1}{2}}\, u(t)$ and $B_\rho^{(i)}(t) = (\hbar/m\omega_\rho)^{-1}\, \beta_\rho^{(i)}(t)$. In this way, the Lagrangian is given by:

$$L = -\nu\hbar\omega\left\{\frac{u(t)^2 A_2^{(i)}(t)}{2A_1^{(i)}(t)}\left(\dot{\beta}_\rho^{(i)}(t)+\beta_\rho^{(i)}(t)^2+1\right)+\frac{\ell^2 A_4^{(i)}(t)}{2u(t)^2 A_1^{(i)}(t)}+\frac{\nu a_s}{u(t)^2}\left(\frac{16A_3^{(i)}(t)}{A_1^{(i)}(t)^2}\right)\right\}, \tag{12}$$

where the A's are

$$\begin{aligned}
A_1^{(1)} &= 1+2\ell^2\alpha^2+2\ell^2\alpha^2\left(1+2\ell^2\alpha^2\right)\log\left(\tfrac{\ell^2\alpha^2}{1+\ell^2\alpha^2}\right)\\
A_2^{(1)} &= 1-\ell^2\alpha^2-2\ell^4\alpha^4-2\ell^4\alpha^4\left(1+\ell^2\alpha^2\right)\log\left(\tfrac{\ell^2\alpha^2}{1+\ell^2\alpha^2}\right)\\
A_3^{(1)} &= \tfrac{1}{6}+2\ell^2\alpha^2+2\ell^4\alpha^4+\ell^2\alpha^2\left(1+3\ell^2\alpha^2+2\ell^4\alpha^4\right)\log\left(\tfrac{\ell^2\alpha^2}{1+\ell^2\alpha^2}\right)\\
A_4^{(1)} &= 2\left(1+\alpha^2\right)\left\{-1-\left(1+\ell^2\alpha^2\right)\log\left(\tfrac{\ell^2\alpha^2}{1+\ell^2\alpha^2}\right)\right\}
\end{aligned} \tag{13}$$

$$\begin{aligned}
A_1^{(2)} &= 2\left[(\ell+1)(\ell+2)\right]^{-1} \\
A_2^{(2)} &= 2\left[(\ell+2)(\ell+3)\right]^{-1} \\
A_3^{(2)} &= \left[2(\ell+1)(2\ell+1)(\ell+3)\right]^{-1} \\
A_4^{(2)} &= 4\left[\ell(\ell+1)\right]^{-1}
\end{aligned} \tag{14}$$

$$\begin{aligned}
A_1^{(3)} &= \sqrt{1-4\ell^2\alpha^2} - 4\ell^2\alpha^2 \mathrm{arctanh}\sqrt{1-4\ell^2\alpha^2} \\
A_2^{(3)} &= \tfrac{1}{3}\left(1-4\ell^2\alpha^2\right)^{\frac{3}{2}} \\
A_3^{(3)} &= \tfrac{1}{6}\left(1+8\ell^2\alpha^2\right)\sqrt{1-4\ell^2\alpha^2} - 2\ell^2\alpha^2 \mathrm{arctanh}\sqrt{1-4\ell^2\alpha^2} \\
A_4^{(3)} &= -4\sqrt{1-4\ell^2\alpha^2} + 4\mathrm{arctanh}\sqrt{1-4\ell^2\alpha^2}
\end{aligned} \tag{15}$$

calculating the Euler-Lagrange equations, we have to $i = 1, 3$

$$\begin{gathered}
\beta_\rho = \frac{\dot{u}}{u} + \left(\frac{A_2'}{A_2} - \frac{A_1'}{A_1}\right)\frac{\dot{\alpha}}{2} \\
u\left(A_2' A_1 - A_2 A_1'\right)\left(\dot{\beta} + \beta^2 + 1\right) = \tfrac{1}{u^3}\left[32\nu a_s\left(2A_3\tfrac{A_1'}{A_1} - A_3'\right) - \ell^2\left(A_4' A_1 - A_4 A_1'\right)\right] \\
u\left(\dot{\beta}_\rho + \beta_\rho^2 + 1\right) = \left[\ell^2 A_4 + 32\nu a_s\left(\frac{A_3}{A_1}\right)\right]\frac{A_2^{-1}}{u^3}
\end{gathered} \tag{16}$$

where the line denotes the derivative with repect to α, and the motion equations to $i = 2$ are

$$\begin{gathered}
\beta_\rho = \frac{\dot{u}}{u} \\
u\left(\dot{\beta}_\rho + \beta_\rho^2 + 1\right) = \left[\ell^2 A_4 + 32\nu a_s\left(\frac{A_3}{A_1}\right)\right]\frac{A_2^{-1}}{u^3}.
\end{gathered} \tag{17}$$

As the framework doesn't have z-dependence thus $\dot{\alpha}$ must be zero, which is $\alpha(t) = \alpha_0 =$

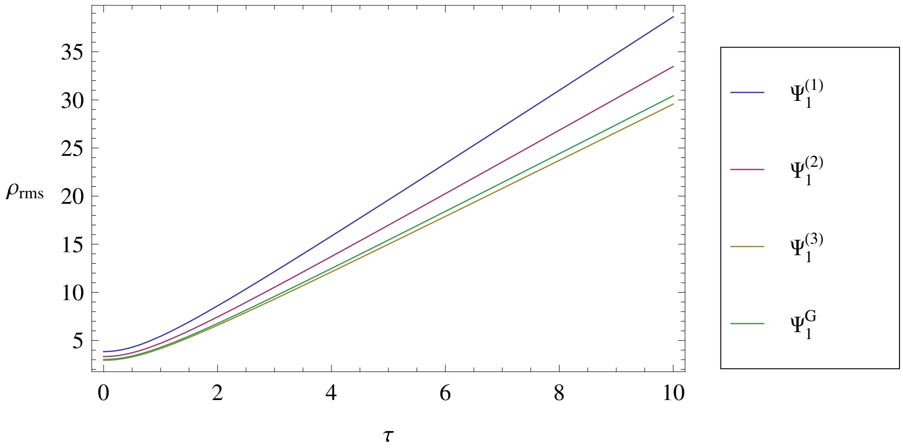


Figure 1: The free expansion of each kind of trial function for $\ell = 1$ and $\nu a_s = 40$. This value of interaction parameter was chosen to be closed of 3D parameter $\gamma = 800$.

constant. Therefore $\beta_\rho^{(1,2,3)} = \dot{u}/u$, resulting in these equations

$$(A_2' A_1 - A_2 A_1')(\ddot{u} + u) = \frac{1}{u^3}\left[32\nu a_s\left(2A_3\frac{A_1'}{A_1} - A_3'\right) - \ell^2\left(A_4' A_1 - A_4 A_1'\right)\right] \quad (i = 1, 3)$$

$$\ddot{u} + u = \left[\ell^2 A_4 + 32\nu a_s\left(\frac{A_3}{A_1}\right)\right]\frac{A_2^{-1}}{u^3} \quad (i = 1, 2, 3) \tag{18}$$

Taking $\ddot{u} = 0$, we have the stationary solution which is

$$32\nu a_s\left[3A_2A_3\frac{A_1'}{A_1} - A_3' A_2 - A_2' A_3\right] = \ell^2\left[A_2' A_1 A_4 + A_4' A_1 A_2 - 2A_1' A_2 A_4\right] \quad (i = 1, 3)$$

$$u_0^4 = \left[\ell^2 A_4 + 32\nu a_s\left(\frac{A_3}{A_1}\right)\right] A_2^{-1} \quad (i = 1, 2, 3) \tag{19}$$

While the expansion the equations 18 become only one that is in agreement with [24], which has an analitical solution given by

$$u(\tau) = u_0\sqrt{1 + \tau}. \tag{20}$$

Considering the $\Psi_\ell^{(3)}$, obviously, give us the exact solution; the $\Psi_0^{(2)}$ and $\Psi_0^{(3)}$ are totaly in agreement with each other, and the another ones exceed in 3% for gaussian and 32% for $\Psi_0^{(1)}$. In the other hand, for $\ell > 0$ the gaussian wave function ($\Psi_\ell^{(G)}$) showed better, when we are interesting in ρ_{rms}, exceeding in 3% ($\ell = 1$) and 4% ($\ell = 2$) the exact solution.

Doing the limit of $\ell \gg 1$ the $\Psi_\ell^{(G)}$ became itself closer and closer of $\Psi_\ell^{(3)}$, however, $\Psi_\ell^{(1)}$ and $\Psi_\ell^{(2)}$ turned most close of each other but they are away from the exact which means the condensate wave function is most like a gaussian when it has a large vortex at center. Thus, the vortex reduces the effect of the interaction potencial. If the point of interesting is the α, $\alpha^{(2)}$ is 1% above $\alpha^{(3)}$, and $\alpha^{(1)}$ exceeds in 8%, for $\ell = 1$. The gaussian doesn't have this parameter. The relation $\xi_\ell = \ell\xi$ works just for $\Psi_\ell^{(2)}$ and $\Psi_\ell^{(3)}$. Hereafter, for simplicity, we will procedure with$\Psi_\ell^{(2)}$ using a gaussian wave function rather than ψ_{TF} in 3D case.

IV. 3D GAUSSIAN TRIAL FUNCTION

As mentioned before, our 3D wave function is the generalized trial function of 2D case from [24], which is

$$\Psi_\ell(\rho,\phi,z,t) = \left[\frac{N}{\pi^{\frac{3}{2}}\ell! R_\rho(t)^{2\ell+2}R_z(t)}\right]^{\frac{1}{2}} \rho^\ell e^{i\ell\phi}\psi_G(\rho,z,t)e^{iB_\rho(t)\frac{\rho^2}{2}+iB_z(t)\frac{z^2}{2}}, \tag{21}$$

with

$$\psi_G(\rho,z,t) = \exp\left(-\frac{\rho^2}{2R_\rho(t)^2}\right)\exp\left(-\frac{z^2}{2R_z(t)^2}\right). \tag{22}$$

If $\ell = 0$, we recover the solution for a condensate without vortex of V.M. Pérez-García *et al*[26]. Thus, the ansatz describe since a fundamental state until condensate with multi-charged vortex at the center of the atomic cloud. We must take care with parameters R_ρ and R_z, because they are not the radii of our condensate for every values of ℓ. In the case, where ℓ is zero, R_ρ and R_z are the radii in fact, however, when we have $\ell > 0$, the parameter R_ρ is the width of Gaussian at least a scale factor, and the peak of that is at the point $R_\rho\sqrt{\ell}$. It's occurs due the absence of a parameter which separate these two information, radius of vortex core and radius of the condensate. Thereby, absence of these two parameters, we have all of information about vortex core and cloud radius in the only one parameter, R_ρ. For that reason, we will define the radius as:

$$\begin{aligned} R_\perp^{(\ell)}(t) &= \sqrt{\langle\rho^2\rangle} = R_\rho^{(\ell)}(t)\sqrt{\ell+1} \\ R_{||}^{(\ell)}(t) &= R_z^{(\ell)}(t). \end{aligned} \tag{23}$$

The ℓ, above of R, is the index about what case we are referring, i.e., what kind of condensate within circulation. And the vortex core, ξ, may be defined as: the healing length of a condensate without vortex calculated at center of density in TF-regime, i.e.,

$$\xi^{(\ell)} = \ell\hbar R_\rho^{(\ell)}\sqrt{\frac{\pi R_z^{(\ell)}}{5mNU_0}}, \tag{24}$$

using the parameters, R_ρ and R_z, of them respective number of circulation.

Now we substited (22)in (2) and integrate its over all space therefore the Lagrangian is

$$L = N\hbar\omega_\rho\left\{\frac{(\ell+1)}{2}\left[\frac{1}{r_\rho^2}+\left(\dot{\beta}_\rho+\beta_\rho^2+1\right)r_\rho^2\right]+\frac{1}{4}\left[\frac{1}{r_z^2}+\left(\dot{\beta}_z+\beta_z^2+\lambda^2\right)r_z^2\right]+\frac{\gamma(2\ell)!}{2^{2\ell}\sqrt{2\pi}\left(\ell!\right)^2 r_\rho^2 r_z}\right\}, \tag{25}$$

where varational parameters were scaled as $R_\rho(t) = a_{osc}r_\rho(t)$, $R_z(t) = a_{osc}r_z(t)$, $B_\rho(t) = a_{osc}^{-2}\beta_\rho(t)$ and $B_z(t) = a_{osc}^{-2}\beta_z(t)$, with the harmonic oscillator length being $a_{osc} = \sqrt{\hbar/m\omega_\rho}$. And the intecation parameter becomes dimensionless being write as $\gamma = Na_s/a_{osc}$. The equation of motion are extracted from (25) using (4) thus these are given by:

$$\begin{aligned}
\left(\dot{\beta}_\rho+\beta_\rho^2+1\right)r_\rho &= \frac{1}{r_\rho^3}+\frac{\gamma(2\ell)!}{2^{2\ell-1}\sqrt{2\pi}\,(\ell+1)!\ell! r_\rho^3 r_z}\\
\beta_\rho &= \frac{\dot{r_\rho}}{r_\rho}\\
\left(\dot{\beta}_z+\beta_z^2+\lambda^2\right)r_z &= \frac{1}{r_z^3}+\frac{\gamma(2\ell)!}{2^{2\ell-1}\sqrt{2\pi}\left(\ell!\right)^2 r_\rho^2 r_z^2}\\
\beta_z &= \frac{\dot{r_z}}{r_z}.
\end{aligned} \tag{26}$$

These four equations can be reduced to only two, which are:

$$\begin{aligned}
\ddot{r_\rho}+r_\rho &= \frac{1}{r_\rho^3}+\frac{\gamma(2\ell)!}{2^{2\ell-1}\sqrt{2\pi}\,(\ell+1)!\ell! r_\rho^3 r_z}\\
\ddot{r_z}+\lambda^2 r_z &= \frac{1}{r_z^3}+\frac{\gamma(2\ell)!}{2^{2\ell-1}\sqrt{2\pi}\left(\ell!\right)^2 r_\rho^2 r_z^2}.
\end{aligned} \tag{27}$$

A. Stationary case

By calculate the functional energy using (5) we obtain each contribution which the circulation carries. Thus, the energy of condensate with ℓ quanta of circulation is:

$$E_\ell = N\hbar\omega_\rho\left[\frac{(\ell+1)}{2}\left(\frac{1}{r_\rho^2}+r_\rho^2\right)+\frac{1}{4}\left(\frac{1}{r_z^2}+\lambda^2 r_z^2\right)+\frac{\gamma(2\ell)!}{2^{2\ell}\sqrt{2\pi}\left(\ell!\right)^2 r_\rho^2 r_z}\right]. \tag{28}$$

Looking at the kinetic terms, proportional to r_ρ^{-2} and r_z^{-2}, we may observe the kinetic energy increase linearity with circulation. The same behavior happens with energy due the harmonic potential, r_ρ^2 and r_z^2. Although, the interaction energy has a distinct behavior, this contribution in the energy of condensate is lower while ℓ increases, i.e, the interaction energy decreases asymptotically as a function of ℓ, and goes to zero in the limit of the high circulation, i.e. big vortex. Through the minimization of (28) in respect by the parameters r_ρ and r_z, we obtain the stationary solution for the trapped condensate, this procedure is equivalent to turns zero the acceleration and velocity in (27). Hence, it results in two coupled equations which are:

$$
\begin{aligned}
r_{\rho 0}^4 &= 1 + \frac{\gamma(2\ell)!}{2^{2\ell-1}\sqrt{2\pi}(\ell+1)!\ell! r_{z0}} \\
\lambda^2 r_{z0}^4 &= 1 + \frac{\gamma(2\ell)! r_{z0}}{2^{2\ell-1}\sqrt{2\pi}(\ell!)^2 r_{\rho 0}^2}.
\end{aligned}
\tag{29}
$$

Here, we might validate the limits of trapped conditions, so: if ℓ values 0 or 1, the limit of Thomas-Fermi regime may be described taking large γ. This is equivalent we neglect the kinetic term in both equations (29). Thus the equations (29) turns:

$$
\begin{aligned}
r_{\rho 0} &= (\ell+1)^{-\frac{3}{10}} \left[\frac{\gamma\lambda(2\ell)!}{2^{2\ell-1}\sqrt{2\pi}\,(\ell!)^2}\right]^{\frac{1}{5}} \\
r_{z0} &= \left[\frac{\gamma(2\ell)!}{2^{2\ell-1}\sqrt{2\pi}\,(\ell!)^2\, r_\rho^2 \lambda^2}\right]^{\frac{1}{3}}.
\end{aligned}
\tag{30}
$$

Now, the case where $\ell \geq 2$, the kinetic term is the order of the interaction term to the $r_{\rho 0}$ equation at (29). Therefore, the $r_{\rho 0}$ equation keeps the same one in the Thomas-Fermi regime, for as much as, the r_{z0} equation have the kinetic term neglected, then (29) becomes:

$$
\begin{aligned}
r_{\rho 0}^4 &= 1 + (\ell+1)^{-3} \left[\frac{\gamma\lambda(2\ell)! r_\rho}{2^{2\ell-1}\sqrt{2\pi}\,(\ell!)^2}\right]^{\frac{2}{3}} \\
r_{z0} &= \left[\frac{\gamma(2\ell)!}{2^{2\ell-1}\sqrt{2\pi}\,(\ell!)^2\, r_\rho^2 \lambda^2}\right]^{\frac{1}{3}}.
\end{aligned}
\tag{31}
$$

B. Free expansion

The free expansion equations are obtained when the trap term from (27) is threw out, and thus they are changed to

$$
\begin{aligned}
\ddot{r_\rho} &= \frac{1}{r_\rho^3} + \frac{(2\ell)!\gamma}{2^{2\ell-1}\,(2\pi)^{\frac{1}{2}}\,(\ell+1)!\ell!r_\rho^3 r_z} \\
\ddot{r_z} &= \frac{1}{r_z^3} + \frac{(2\ell)!\gamma}{2^{2\ell-1}\,(2\pi)^{\frac{1}{2}}\,(\ell!)^2 r_\rho^2 r_z^2}.
\end{aligned}
\tag{32}
$$

It've already known that the asymptotical behavior of expansion's rate is justified by the interaction term is greater than kinetic, thus the kinetic term just affects its beahivor in a short time, i.e., at the first miliseconds. After it the other term dominate. Although when ℓ becomes large enough that it makes the interaction terms goes to zero. Therefore the equations (32) will just have the kinetic terms, which is the same caso of a Bose-Einstein condensate in the limit of ideal gas. Thus the free expansion equations becomes

$$
\begin{aligned}
\ddot{r_\rho} &= \frac{1}{r_\rho^3} \\
\ddot{r_z} &= \frac{1}{r_z^3},
\end{aligned}
\tag{33}
$$

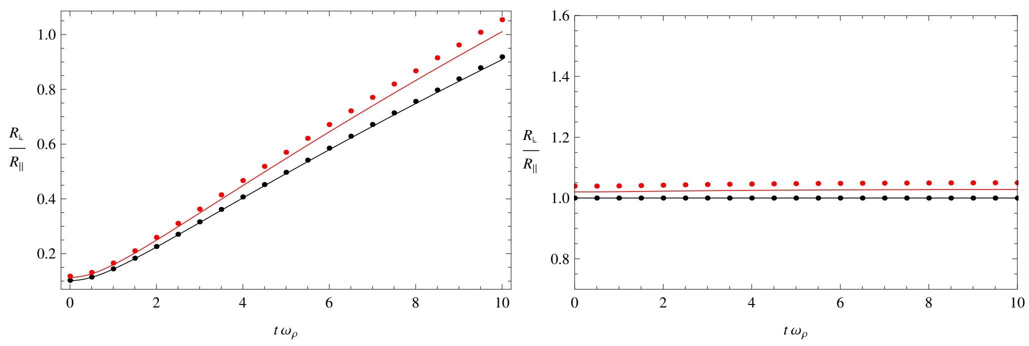
and the initial conditions in that case are $r_\rho(0) = r_{\rho 0} = 1$ and $r_z(0) = r_{z0} = \lambda^{-\frac{1}{2}}$. Solving them we have

$$
r_j(\tau) = \sqrt{r_j(0)^2 + r_j(0)^{-2}\tau^2}, \tag{34}
$$

where $j = \rho, z$. This last result is also similar to 2D case which was done in the section III, nevertheless it carries the interaction term together proportional at r^{-3}. To validate these results, we made a direct simulation of the GPE which is ploted at Fig.2.

V. RESULTS AND DISCUSSIONS

We solved eq. (32), starting with the equilibrium configuration inside the trap, to obtain the evolution of the parameters of the condensate cloud during the free expansion. We used the frequency ($\omega_\rho = 2\pi \times 207Hz$) and expansion time (8ms) according with our experiments with Rubidium 87 alkaline atoms.



(a) Prolate shape ($\lambda = 0.1$).

(b) Spherical shape ($\lambda = 1$).

Figure 2: Free expansion. Full line is from the variational method, and dots come from the direct numerical simulation of GPE in TF-regime ($\gamma = 800$).

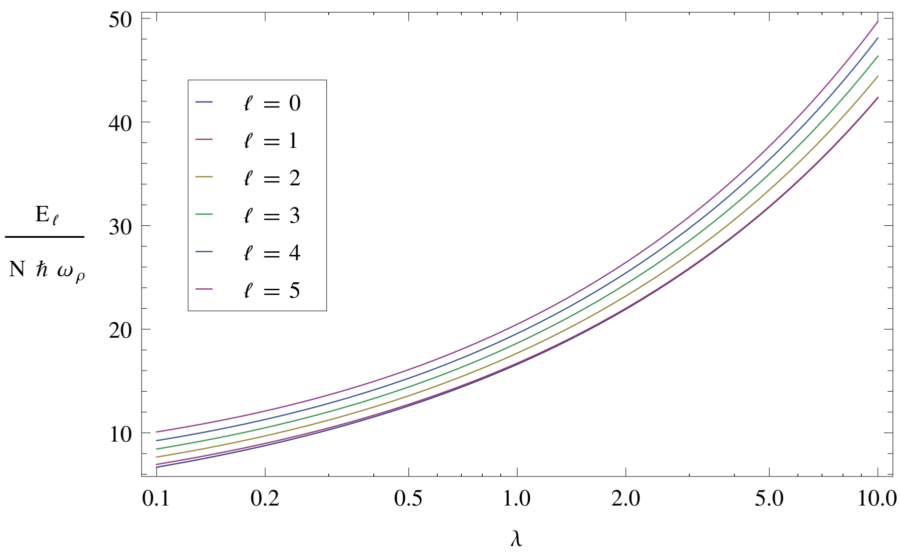
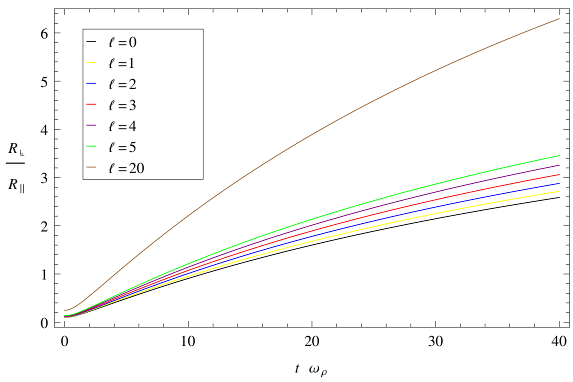


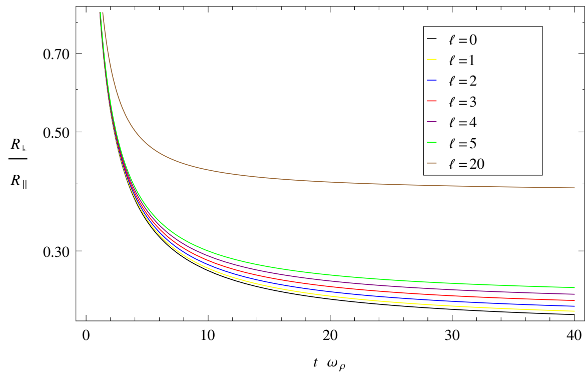
Figure 3: Energy plotted as function of the trap anisotropy by fixing interaction parameter as 800.

To calculate the initial condition we use (32) with the acceleration equal to zero or minimize the energy function (28) in relation of the widths r_i (reference of the appropriate equations showed in the previous section).

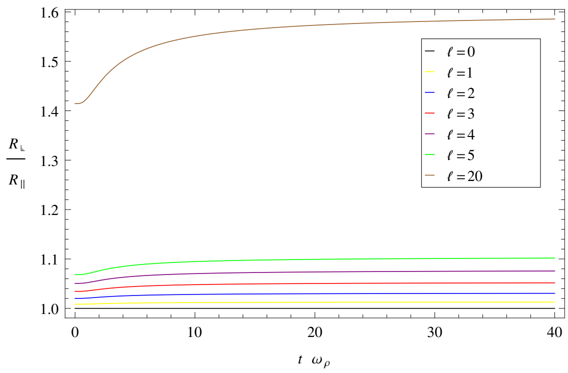
The graphic 3 shows the initial energy of the system for a fixed interaction parameter as function of the trap geometry (prolate–isotropic-oblate / γ (dimensionless) $= 800$ that correspond to real values in the experiment $a_s = 100a_0$. Number atoms in condensate $=$



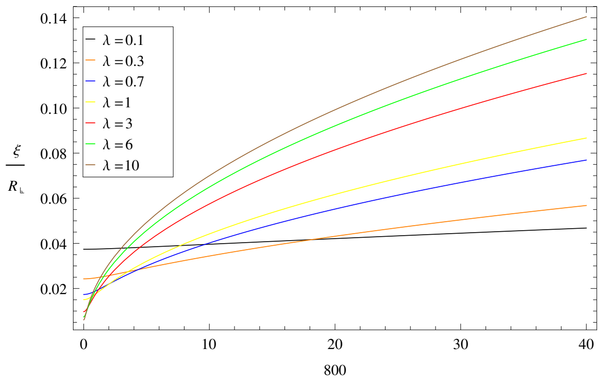
(a) $\lambda = 0.1$



(b) $\lambda = 10$



(c) $\lambda = 1$

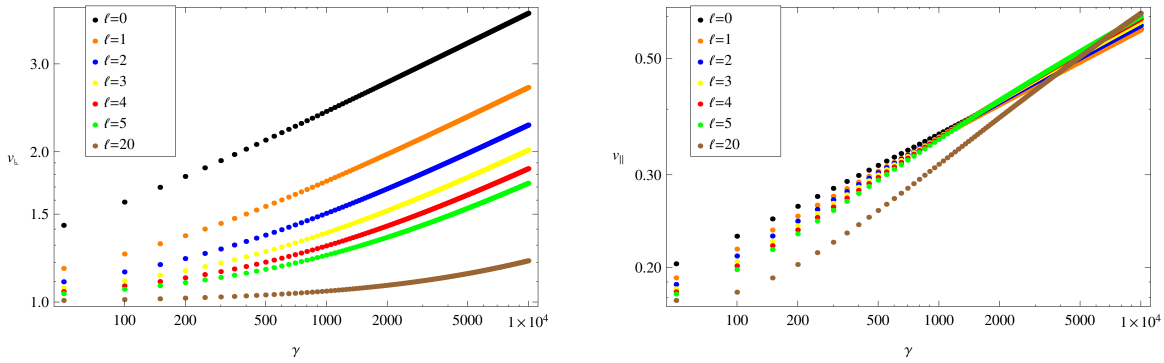


(d) $\ell = 1$

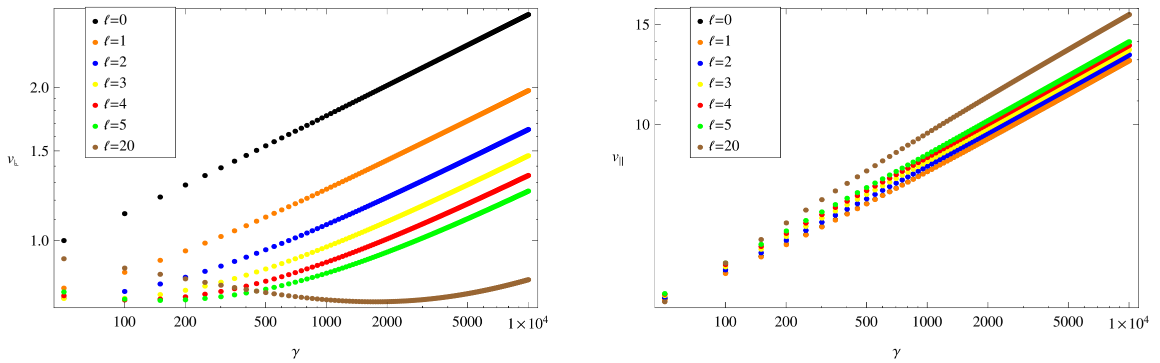
Figure 4: Aspect ratio plotted while the free expansion for the respective trapped shapes: (a) prolate shape, (b) oblate shape and (c) spherical shape. They were calculated for $\gamma = 800$. The plot of (d) represents the ratio of vortex core by radial radius, while the free expansion, for several kind of trap shapes.

10^5) for different circulations. The energy of the $\ell = 1$ state differs very little from the fundamental $\ell = 0$ state, as it was expected in the Thomas Fermi regime (TF). About the higher order circulation states, this graphic shows an important physical effect that comes from the vortex circulation. The latter brings nor only an extra kinetic energy, but also affect the interaction energy since it spread the condensate volume due to the centrifugal effect.

In the following we will analyze the evolution of the core during the expansion. An important remark, however, is that our ansatz not describe the correct behave of the vortex core, since it will be constrained with the cloud dimensions. To improve such ansatz an



(a) $\lambda = 0.1$



(b) $\lambda = 10$

Figure 5: The asymptotic behavior of expansion velocities with γ for the prolate and oblate shapes of initial potencial.

additional parameter should be used to characterize the core expansion independently. Since we are in the TF regime, we used the relation between the vortex core and condensate central density to describe such evolution ((24)). We represent the core equilibrium dimension together the approach proposed, based on the density at $\rho = z = \ell = 0$. It is clear that the two sizes coincide to $\ell = 1$ but for higher order of angular momentum ℓ the ansatz start to fail, mostly when we are close the oblate configuration, where the interaction energy is bigger – In this case our new characterization is even better – TF approach .

In the expansion (Figure 4), we compared the time to the aspect ratio inversion for different circulations, starting with each type of magnetic trap configuration.

The influences of the higher circulation are diferent for each intial geometry of the condensate. For a spherical trap the dimensions of the cloud is changed according with number of circulation, being that in absence of vortex the condensate´s format is really spherical in the other hand the vortex turns the aspect ratio greater than 1, because this one reduces the axial radius and grows the radial radius. In the prolate condensate, the circulation has an effect of decreasing the time for aspect ratio invertion which can be interpreted as the presence of the vortex has a significant addiction in the velocity field on the perpendicular direction (Figure 4a). It is just significant when the perpendicular direction at vortex is the most confined. Thus in the oposite case, when the intial geometry is oblate, the circulation increases the time for aspect ratio invertion caused by fact the axial radius smaller and smaller in the absence of centred atoms in the cloud (Figure 4b); even though the invertion time was lower in oblate than prolate geometry.

Using the value for the core ξ and the solution for the evolution of the extension of the cloud given by eq. (32), we analyze how the vortex expand in relation to the cloud size. This analyze is particularly important since it contributes to improve the information that we extract from the absorption image technique made in time of flight; for example, in the classification and quantification of the number of vortex in the expanded cloud. The graphic of Figure 4d shows the evolution of such ration in time for a cloud with unitary circulation and starting with different trap anisotropy. The curves can be explained basing on the variation of the central density due to the trap configuration. For instance, the initial core size diminished with the increase of λ since the central density increases. Otherwise the variation of the velocity of expansion in the axial direction has direct consequence in the dilution of the density in the vortex position, dictating its rate of expansion .

Comparing the graphs of the aspect ratios we notice a big difference in the inversion time between Figure 4a and Figure 4b. Thus, we think of a way to make a measurement for the prolate case which we can distinguish each condensate, ie, to recognize and differentiate the circulation of the vortex core condensate. The Figure 5 presents the asymptotic velocity for both initial geometry (prolate and oblate) as a function of the parameter of interaction, with these graphics we hoped to find a critical phenomenon which, obviously, doesn't exist at this framework.

Appendix A: New TF-approach

Using $\Psi(\vec{r},t) = \psi(\rho,z)e^{i\ell\phi}e^{-i\mu t/\hbar}$ in (1), we have separated the centrifugal term in the time-independent GPE,

$$\mu\psi(\rho,z) = \left[-\frac{\hbar^2}{2m}\nabla^2_{\rho,z} + \frac{\ell^2\hbar^2}{2m\rho^2} + V(\vec{r}) + U_0 n(\rho,z)\right]\psi(\rho,z). \tag{A1}$$

Now we write the centrifugal terms in terms of $U_0 n(\rho,z)$, to do this we use the realation of the healing lengh ξ,

$$\frac{\hbar^2}{2m\xi^2} = U_0 n(\rho,z), \tag{A2}$$

thus the equation (A1) is wroten as

$$\mu\psi(\rho,z) = \left[-\frac{\hbar^2}{2m}\nabla^2_{\rho,z} + V(\vec{r}) + \left(1 + \frac{\xi_\ell^2}{\rho^2}\right)U_0 n(\rho,z)\right]\psi(\rho,z), \tag{A3}$$

where $\xi_\ell = \ell\xi$. Making the TF-limit, the kinect term throw out. Hence, in this way we obtain

$$n(\rho,z) = \left(\frac{\rho^2}{\rho^2 + \xi_\ell^2}\right)\frac{\mu - V(\vec{r})}{U_0}. \tag{A4}$$

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