Free expansion of Bose-Einstein Condensates with a Multi-charged Vortex

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Abstract

In this work, we analyse the free expansion of Bose-Einstein condensates containing multi-charged vortices. The atomic cloud is initially confined in a three-dimensional assymetric harmonic trap.We employ both approximate variational solutions and numerical simulations of the Gross-Pitaevskii equation. The data obtained provide a way to determine the presence as well as the multiplicity of vortices based only on the properties of the expanded cloud which can be obtained via time-of-flight measurements. In addition, several features like the evolution of the vortex core size and the asymptotic velocity during free expansion were studied considering the atomic cloud as being released from different harmonic trap configurations.

I. INTRODUCTION

The extension of quantum phenomena into macroscopic scales is responsible for a whole class of effects such as superconductivity, superfluidity, and Bose-Einstein Condensation, which played central roles in the last-century physics. The production of the first Bose-Einstein condensates (BECs), using rubidium [1] and sodium [2] atoms, turned possible the realization of experiments involving macroscopic quantum phenomena with unprecedented level of control of the external parameters.

Vortices in BECs are topological defects characterized by a quantized angular momentum. A conventional method for generation of such defects consists in confining the condensed atomic cloud into a rotating trap. It turns out that, for angular velocities higher than a critical value Ω_c, vortex states become energetically favorable, thus inducing the creation of quantized vortices [3–6]. Experimental realizations of condensed alkali atoms confined by more general time-dependent potentials allowed the observation not only of vortex lattices but also of quantum turbulence [2, 7–10]. Since quantum turbulence is characterized by the presence of a self-interacting tangle of quantized vortices, the correct understanding of dynamics, formation, and stability of vortices have shown to be of paramount importance [9, 11–14] being the subject of many theoretical works [10, 15–19].

The radius of vortex cores are tipicaly of the order of the healing length of the condensate. Such a small size makes *in situ* observations very dificult. The most common method for visualization of vortices in BECs relies on the so called time-flight pictures which can be obtained after releasing the condensed cloud from its trap and letting it expand freely for some time, typically tens of miliseconds [8, 17, 20–22]. To determine the charge multiplicity of of vortices in confined clouds using time-of-flight pictures, it is necessary to stabilish the correct conection between the characteristics of the trapped and expanded clouds.

In this paper, we considered the expansion of a BEC containing a multiply charged vortex at its center. The main calculations are performed by using a variational method which takes into account the presence of non-fundamental vortices. A similar work for single charged vortices in two-dimensional condensates was done by Lundh *et al.[23]* and a more numerical approach was emploied by Dalfovo *et al.* in Ref. [24].

This work is divided as follows:

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The section II present the theoretical methods used in this work. In the section III, we have search for a suitable trial function and analyses this one to determine the radii of condensate. Soon after it, in the sections IV and V, we found the initials condition and the motion equations to the free expansion, and also take the TF-limit to the both.

Finally, the section VI has the discussions and outlooks about our results by the method employed.

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II. VARIATIONAL METHOD

At zero temperature, a Bose gas with scattering length a_s much smaller than the average interparticle distance can be described by the Gross-Pitaevskii equation (GPE)

$$i\hbar\frac{\partial\Psi(\mathbf{r},t)}{\partial t} = \left[-\frac{\hbar^2}{2m}\nabla^2 + V(\mathbf{r}) + U_0\left|\Psi(\mathbf{r},t)\right|^2\right]\Psi(\mathbf{r},t), \tag{1}$$

with the harmonic trap given in cylindrical coordinates given by $V(\mathbf{r}) = \frac{1}{2}m\omega_\rho^2(\rho^2 + \lambda^2 z^2)$, where m is the mass of the particles, λ is an anisotropy parameter, and the coupling constant is given by $U_0 = 4\pi\hbar^2 a_s/m$. Following the variational principle, the Lagrangian density $\mathcal{L}$ which recovers the GPE for a complex field $\Psi(\mathbf{r},t)$ can be written as

$$\begin{aligned}\mathcal{L} = \frac{i\hbar}{2}\left(\Psi^*(\mathbf{r},t)\frac{\partial\Psi(\mathbf{r},t)}{\partial t} - \Psi(\mathbf{r},t)\frac{\partial\Psi^*(\mathbf{r},t)}{\partial t}\right)\\ -\frac{\hbar^2}{2m}\left|\nabla\Psi(\mathbf{r},t)\right|^2 - V(\mathbf{r})\left|\Psi(\mathbf{r},t)\right|^2 - \frac{U_0}{2}\left|\Psi(\mathbf{r},t)\right|^4.\end{aligned} \tag{2}$$

In the variational method, the wave funcion $\Psi(\mathbf{r},t)$ of a condensate containing one central vortex with charge l is approximated by a trial function $\Psi_\ell(\mathbf{r},t)$ which depends on a set of variational parameters $q_i = q_i(t)$ [25, 26]. This function can then be substituted into Lagrange function

$$L = \int \mathcal{L} d^3\mathbf{r}. \tag{3}$$

This way, the time evolution of the parameters q_i follows the Euler-Lagrange equations

$$\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0. \tag{4}$$

Now, we generalize the trial function in Ref. [23] to the case of three-dimensional BECs with multiply charged vortices

$$\Psi_\ell(\rho,\phi,z,t) = \left[\frac{N}{\pi^{\frac{3}{2}}\ell! R_\rho(t)^{2\ell+2} R_z(t)}\right]^{\frac{1}{2}} \rho^\ell e^{i\ell\phi} \psi_G(\rho,z,t) e^{iB_\rho(t)\frac{\rho^2}{2} + iB_z(t)\frac{z^2}{2}}, \tag{5}$$

with the function ψ_G being given by

$$\psi_G(\rho,z,t) = \exp\left(-\frac{\rho^2}{2R_\rho(t)^2}\right) \exp\left(-\frac{z^2}{2R_z(t)^2}\right). \tag{6}$$

If we consider the $\ell = 0$, we recover the vortex-free approximation proposed by Pérez-García *et al.* in Ref. [25]. In our case, however, the parameter R_ρ is no longer the mean square root of ρ. Instead, it is related to $\sqrt{\langle \rho^2 \rangle}$ according to

$$\sqrt{\langle \rho^2 \rangle} = \sqrt{\ell + 1} R_\rho. \tag{7}$$

Here we define the vortex core ξ as the healing length calculated at the center of the condensate disconsidering the presence of a central vortex. It leads us to

$$\xi = \ell \hbar \pi^{\frac{3}{4}} R_\rho \sqrt{\frac{R_z}{2mNU_0}}. \tag{8}$$

III. DYNAMICAL EQUATIONS

By substituting (5) into Eqs. (2) and (3) and then performing the spacial integrations, we obtain the Lagrange function for the variational parameters

$$L = -N\hbar\omega_\rho \left\{ \frac{(\ell+1)}{2}\left[\frac{1}{r_\rho^2} + \left(\dot{\beta}_\rho + \beta_\rho^2 + 1\right) r_\rho^2\right] + \frac{1}{4}\left[\frac{1}{r_z^2} + \left(\dot{\beta}_z + \beta_z^2 + \lambda^2\right) r_z^2\right] + \frac{\gamma(2\ell)!}{2^{2\ell}\sqrt{2\pi}\,(\ell!)^2\, r_\rho^2 r_z} \right\}, \tag{9}$$

where the variational parameters were scaled according to $R_\rho(t) = a_{osc} r_\rho(t)$, $R_z(t) = a_{osc} r_z(t)$, $B_\rho(t) = a_{osc}^{-2}\beta_\rho(t)$, and $B_z(t) = a_{osc}^{-2}\beta_z(t)$. The harmonic oscilator length was defined as $a_{osc} = \sqrt{\hbar/m\omega_\rho}$, whereas the dimensionless interation was defined according to $\gamma = N a_s / a_{osc}$. The Euler-Lagrange equations (4) applied to the Langrangean (9) finally lead us to the equations of motion

$$\left(\dot{\beta}_\rho + \beta_\rho^2 + 1\right) r_\rho = \frac{1}{r_\rho^3} + \frac{\gamma(2\ell)!}{2^{2\ell-1}\sqrt{2\pi}\,(\ell+1)!\ell! r_\rho^3 r_z}, \tag{10}$$

$$\beta_\rho = \frac{\dot{r_\rho}}{r_\rho}, \tag{11}$$

$$\left(\dot{\beta}_z + \beta_z^2 + \lambda^2\right) r_z = \frac{1}{r_z^3} + \frac{\gamma(2\ell)!}{2^{2\ell-1}\sqrt{2\pi}\,(\ell!)^2\, r_\rho^2 r_z^2}, \tag{12}$$

$$\beta_z = \frac{\dot{r_z}}{r_z}. \tag{13}$$

By taking the time-derivative of Eqs. (11) and (13), the parameters β_ρ and β_z can be eliminated in such a way that these four equations can be reduced to the following two:

$$\ddot{r_\rho} + r_\rho = \frac{1}{r_\rho^3} + \frac{\gamma(2\ell)!}{2^{2\ell-1}\sqrt{2\pi}\,(\ell+1)!\ell! r_\rho^3 r_z}, \tag{14}$$

$$\ddot{r_z} + \lambda^2 r_z = \frac{1}{r_z^3} + \frac{\gamma(2\ell)!}{2^{2\ell-1}\sqrt{2\pi}\,(\ell!)^2\, r_\rho^2 r_z^2}. \tag{15}$$

A. Free expansion

By considering the stationary solution for the Eqs. of motion (16) and (17), we obtain the algebraic equations

$$r_{\rho 0}^4 = 1 + \frac{\gamma(2\ell)!}{2^{2\ell-1}\sqrt{2\pi}(\ell+1)!\ell! r_{z0}} \tag{16}$$

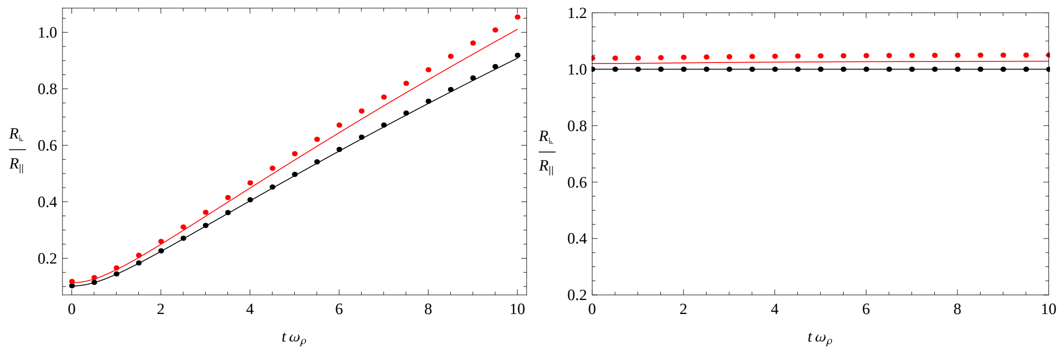
$$\lambda^2 r_{z0}^4 = 1 + \frac{\gamma(2\ell)! r_{z0}}{2^{2\ell-1}\sqrt{2\pi}(\ell!)^2 r_{\rho 0}^2}. \tag{17}$$

The free expansion equations are obtained when the trap terms in Eqs. (14) and (15) are neglected, and thus they become

$$\ddot{r_\rho} = \frac{1}{r_\rho^3} + \frac{(2\ell)!\gamma}{2^{2\ell-1}\,(2\pi)^{\frac{1}{2}}\,(\ell+1)!\ell! r_\rho^3 r_z}, \tag{18}$$

$$\ddot{r_z} = \frac{1}{r_z^3} + \frac{(2\ell)!\gamma}{2^{2\ell-1}\,(2\pi)^{\frac{1}{2}}\,(\ell!)^2 r_\rho^2 r_z^2}. \tag{19}$$

The first and second terms in the r.h.s of (18) and (19) come from the kinectic and interaction terms in (2), respectively. From (18) and (19), we can also observe that the



(a) Prolate shape ($\lambda = 0.1$).

(b) Spherical shape ($\lambda = 1$).

Figure 1: Free expansion from the direct numerical simulation of GPE. Full line is from the variational method, and dots come ($\gamma = 800$). Black color corresponds to $\ell = 0$ and red color to $\ell = 2$.

interaction term is dominant in the long-time limit, while the kinectic terms plays a role only at the first miliseconds of the expansion. This is however not the case when we consider extremely large values of ℓ. In this case, the interaction terms can be neglected thus leading to equations identical to the ones describing the free expansion of an ideal gas

$$\ddot{r_\rho} = \frac{1}{r_\rho^3}, \tag{20}$$

$$\ddot{r_z} = \frac{1}{r_z^3}, \tag{21}$$

which have the simple solution

$$r_j(\tau) = \sqrt{r_j(0)^2 + r_j(0)^{-2}\tau^2}, \tag{22}$$

with $j = \rho, z$.

In order to check precision of our results, we performed direct simulation of the GP equation using the Fourier spectral method in space where the Fourier components of $\Psi(\mathbf{r}, t)$ where computed using fast Fourier transformations. In figure 1, our variational method is compared with the high-precision numerical simulation for expanding spherical as well as prolate condensates.

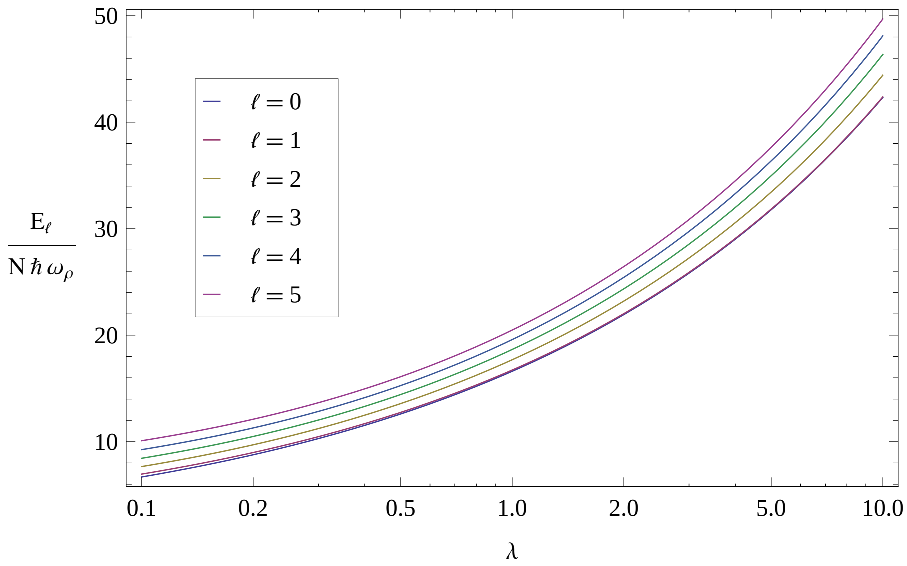


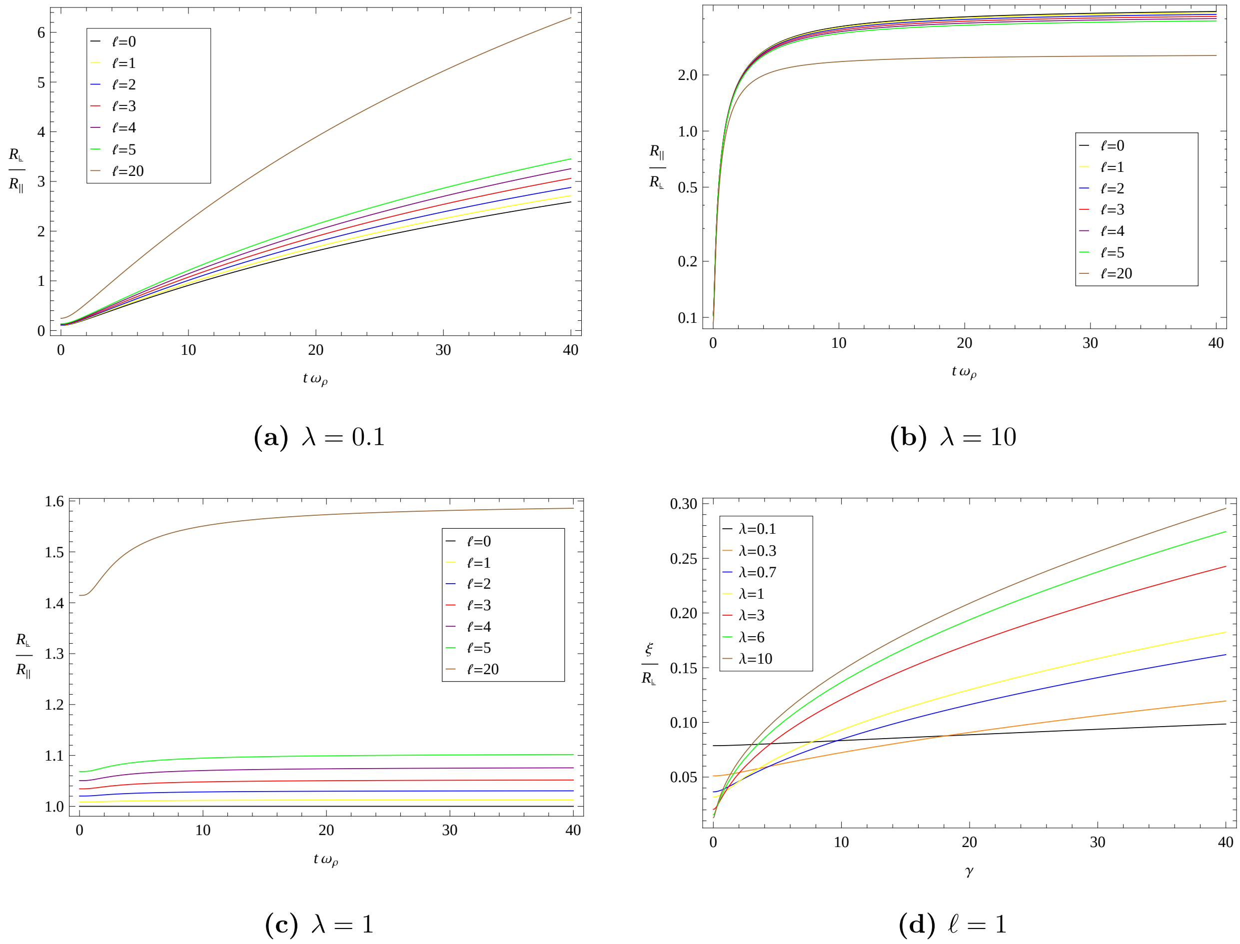
Figure 2: Energy plotted as function of the trap anisotropy for different vortex circulations. The dimensionless interaction parameter is taken as $\gamma = 800$, which corresponds to a condensate with 10^5 atoms and s-wave scatering length $a_s = 100a_0$.

IV. RESULTS AND DISCUSSIONS

The initial condition for the variational parameters where calculated from Eqs. (16) and (17) considering the radial frequency $\omega_\rho = 2\pi \times 207Hz$. The time evolution of the parameters was obtained by numerically solving Eqs. (18) and (19) using the fourth-order adaptative Runge-Kutta method, with expansion times of the order of $10ms$. These values where choosen in order to be consistent with our experiments using ^{87}Rb 87 atoms [colocar referência].

Figure 2 shows the initial energy of the system for a fixed interaction parameter as function of the trap anysopropy measured by the parameter λ. It shows the monotonic growth of the condensate energy with the circulation of the vortex due to the extra kinect energy and the larger volume ocupied by the condensate with increasing ℓ.

The vortex core evolution was analyzed considering its size to be given by Eq. (8). A disadvantage of this method is the constraint (8) between the vortex core and the cloud dimensions, which is valid for static configurations, is considered to be also valid during the entire expansion time. At least in principle, our Ansatz can be improved by introducing an additional parameter which characterizes the core expansion independently.



(a) $\lambda = 0.1$

(b) $\lambda = 10$

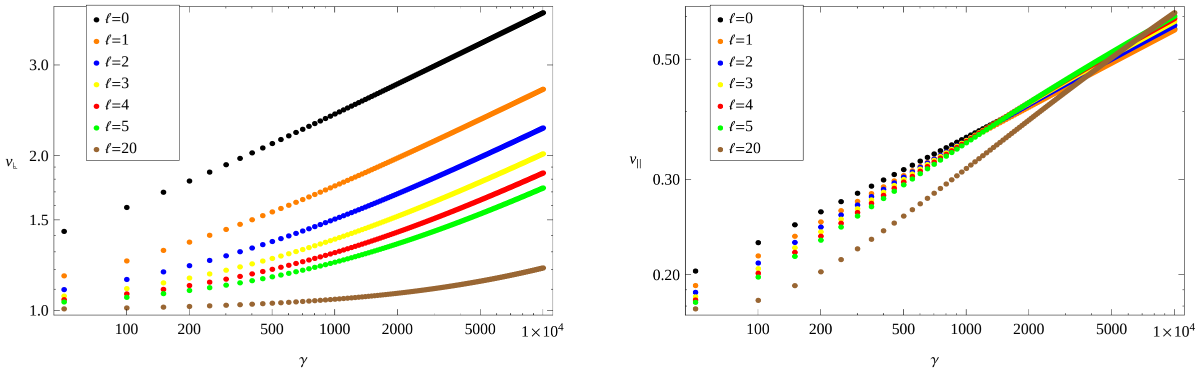
(c) $\lambda = 1$

(d) $\ell = 1$

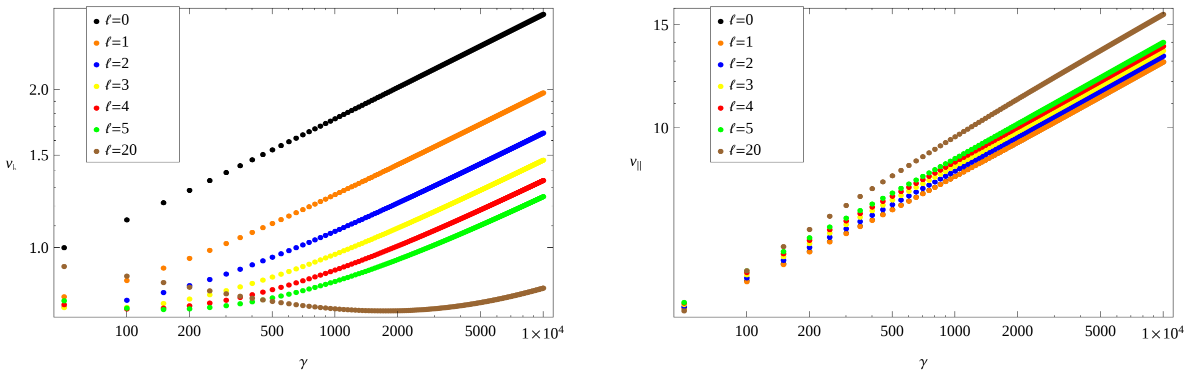
Figure 3: Aspect ratio plotted while the free expansion for the respective trapped shapes: (a) prolate shape, (b) oblate shape and (c) spherical shape. They were calculated for $\gamma = 800$. The plot of (d) represents the ratio of vortex core by radial radius, while the free expansion, for several kind of trap shapes.

In Fig. 3(a)–(c), the time evolution of the aspect ratio for different trap configurations and vortex circulations is depicted. Figure 3(c) shows how the growth of the dimensions of the cloud during the expansion are influentiated by the circulation of the central vortex. The higher the circulation of these vortices, the greater is the anisoproty introduced in the could shape. This effect is further increased during the cloud expansion.

In prolate condensates, Fig. 3a shows that the circulation has the effect of decreasing the time for the aspect ratio invertion due to the larger velocity field in the plane perpendicular to the vortex line before the expansion. In the oposite case, where the initial geometry is oblate, as in Fig. 3b, the circulation increases the time required for aspect ratio invertion.



(a) $\lambda = 0.1$



(b) $\lambda = 10$

Figure 4: The asymptotic behavior of expansion velocities with γ for the prolate and oblate shapes of initial potencial.

In this work, the expansion of the vortex core is analyzed by evolving both the radial and axial radius using eqs. (18) and (19), and then calculating the radius ξ of the vortex according to (8). By taking the asymptotic solutions of eqs. (18) and (19), it was also possible to extract the asymptotic expansion velocities along the radial and axial directions, as shown in Fig. 4. These information supply a method for determining the presence as well as the circulation of central vortice in an assymetric cloud by simply looking at the asymptotic expansion velocities, which would require the repetion of the expreiment considering different expention time. An alternative method would rely on dependence of the expansion dinamics on the circulation ℓ as depicted in Fig. 3d.

Acknowledgments

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