Numerical Solution of A Linear Klein-Gordon Equation

Noraini Kasron, Mohd Agos Salim Nasir, Siti Salmah Yasiran, Khairil Iskandar Othman
Faculty of Computer and Mathematical Sciences
Universiti Teknologi MARA
40450 Shah Alam, Selangor, Malaysia
quratul_aini88@yahoo.com, masn@tmsk.uitm.edu.my,
sitisalmah@tmsk.uitm.edu.my, khairil@tmsk.uitm.edu.my

Abstract—**A new scheme of a linear inhomogeneous Klein-Gordon equation is developed by utilizing finite difference method incorporated with arithmetic mean averaging of functional values. This study considered the central time central space (CTCS) finite difference scheme incorporated with four points arithmetic mean averaging. In addition, the theoretical aspects of finite difference scheme are also considered such as stability, consistency and convergence. The von Neumann stability analysis method and Miller Norm Lemma are used to analyze the stability of the proposed scheme. The performance analysis shows the proposed scheme is stable, consistent and convergent. These theoretical analyses are verified by a numerical experiment. The comparison results shown the proposed scheme produces better accuracy rather than the standard CTCS scheme.**

Keywords—Klein-Gordon equation; finite difference method; arithmetic mean; stability; consistency; convergence

I. INTRODUCTION

Linear Klein-Gordon equation is one of the most important mathematical models in quantum mechanics and relativistic physics. The Klein-Gordon equation has been widely studied by utilizing the numerical methods for approximating the solutions. Many researchers have been suggested the numerical approaches to solve the Klein-Gordon equation such as Adomian Decomposition Method (ADM) ([1]; [2]; [3]; [4]; [5]; [6]; [7]), Finite Difference Method (FDM) ([8]; [9]; [10]; [11]; [12]), Finite Element Method (FEM) [13], Homotopy Analysis Method (HAM) [14] and Homotopy Perturbation Method (HPM) ([15]; [16]). However, the mean averaging method is not extensively used for approximating the solutions of the problem. Several researchers have been utilized the mean averaging method to solve Goursat problem ([17]; [18]; [19]; [20]), systems of nonlinear equations [21], and diffusion equations ([22]; [23]). But the utilizing of mean averaging method did not receive any attention on the Klein-Gordon equation.

In this study, we develop a new finite difference scheme incorporated with the arithmetic mean method. The central time central space (CTCS) finite difference scheme and four points arithmetic mean formula are considered. The theoretical aspects of finite difference scheme are also studied such as stability, consistency and convergence. The analysis results will be verified by implementing a numerical experiment.

II. KLEIN-GORDON EQUATION AND FINITE DIFFERENCE ARITHMETIC MEAN (FDAM) SCHEME

A. Linear Klein-Gordon Equation

The standard form of the linear Klein-Gordon equation [1] can be expressed as follows:

$$u_{tt}(x,t) - u_{xx}(x,t) + a\,u(x,t) = h(x,t),$$
$$0 < x < L,\ t > 0 \tag{1}$$

subject to the initial conditions

$$u(x,0) = f(x),\quad u_t(x,0) = g(x),\quad 0 \le x \le L \tag{2}$$

where u is a function of the independent variables x and t, a is a constant, u_{tt}, u_{xx} are the second derivative of the independent variables, $h(x,t)$ is a source term (inhomogeneous term), $f(x), g(x)$ are the function of the independent variable.

B. Finite Difference Arithmetic Mean (FDAM) Scheme

We develop a new finite difference scheme incorporated with arithmetic mean averaging of functional values for the linear inhomogeneous Klein-Gordon equation. It can be called as FDAM scheme. This scheme is based on the Taylor series expansions in central time central space (CTCS) and four points arithmetic mean formula as follows:

$$\frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{k^2} - \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{k^2} + a u_{i,j} = \frac{1}{4}\begin{pmatrix} h_{i,j} + h_{i+1,j} \\ + h_{i,j+1} + h_{i+1,j+1} \end{pmatrix} \tag{3}$$

k denotes as the grid size. Here, the functional value in inhomogeneous term is approximated by:

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$$\frac{1}{4}\left(h_{i,j}+h_{i+1,j}+h_{i,j+1}+h_{i+1,j+1}\right) \tag{4}$$

The left hand side is derived based on Taylor series expansions which include forward, backward and central finite difference formulas. The derivations are as follow:

$$\begin{aligned} u_{tt}(x,t) &= \frac{u(x,t+\Delta t)-2u(x,t)+u(x,t-\Delta t)}{(\Delta t)^2} \\ &= \frac{u(x,t+k)-2u(x,t)+u(x,t-k)}{k^2} \end{aligned} \tag{5}$$

and

$$\begin{aligned} u_{xx}(x,t) &= \frac{u(x+\Delta x,t)-2u(x,t)+u(x-\Delta x,t)}{(\Delta x)^2} \\ &= \frac{u(x+k,t)-2u(x,t)+u(x-k,t)}{k^2} \end{aligned} \tag{6}$$

where $k=\Delta t=\Delta x$.

We consider the following linear inhomogeneous Klein-Gordon equation due to it is widely used by the other researchers, such as [1] and [27].

$$\begin{gathered} u_{tt}-u_{xx}-u=-\cos(x)\cos(t), \\ 0 \le x \le 1,\ 0 \le t \le 1, \end{gathered} \tag{7}$$

subject to the initial conditions

$$u(x,0)=\cos(x),\ u_t(x,0)=0. \tag{8}$$

The exact solution of the linear Klein-Gordon equation and its initial conditions in (7) and (8) is as follows [1]:

$$u(x,t)=\cos(x)\cos(t) \tag{9}$$

The approximate CTCS scheme for the linear Klein-Gordon equation in (7) is shown below.

$$\begin{aligned} &U_{i,j+1}+U_{i,j-1}-U_{i+1,j}-U_{i-1,j}-k^2U_{i,j} \\ &\quad =-k^2\cos(x_i)\cos(t_j) \end{aligned} \tag{10}$$

The approximate FDAM scheme for the linear Klein-Gordon equation in (7) is as follows:

$$\begin{aligned} &U_{i,j+1}+U_{i,j-1}-U_{i+1,j}-U_{i-1,j}-k^2U_{i,j} \\ &\quad =-\frac{k^2}{4}\begin{pmatrix} \cos(x_i)\cos(t_j) \\ +\cos(x_{i+1})\cos(t_j) \\ +\cos(x_i)\cos(t_{j+1}) \\ +\cos(x_{i+1})\cos(t_{j+1}) \end{pmatrix} \end{aligned} \tag{11}$$

where $k=\Delta t=\Delta x$.

By shifting the index $i,j \to i+1, j+1$, the approximate FDAM scheme can be written as:

$$\begin{aligned} &U_{i+2,j+1}=U_{i+1,j+2}+U_{i+1,j}-U_{i`,j+1} \\ &\quad -k^2U_{i+1,j+1}+\frac{k^2}{4}\begin{pmatrix} \cos(x_{i+1})\cos(t_{j+1}) \\ +\cos(x_{i+2})\cos(t_{j+1}) \\ +\cos(x_{i+1})\cos(t_{j+2}) \\ +\cos(x_{i+2})\cos(t_{j+2}) \end{pmatrix}=0 \end{aligned} \tag{12}$$

The theoretical aspects of finite difference scheme will be analyzed in the next section.

III. THE THEORETICAL ASPECTS OF FDAM SCHEME

In this section, we will analyze the theoretical aspects of the FDAM scheme which include the stability, consistency and convergence. The stability, consistency and convergence of the FDAM scheme for the linear inhomogeneous Klein-Gordon equation will be analyzed by considering the linear inhomogeneous Klein-Gordon equation in (7).

A. Stability Analysis

The most common method of the stability analysis is von Neumann stability analysis method [24]. It is also called as a Fourier series method. This method describes the errors between the approximate solutions and the exact solutions distributed along grid lines which the errors can be represented as finite Fourier series. The exact and approximate solutions are related as the following equation.

$$U_{i,j}=u_{i,j}+\varepsilon_{i,j} \tag{13}$$

where the exact solution is denoted as $u_{i,j}$, $U_{i,j}$ is the approximate solution and $\varepsilon_{i,j}$ is the error at the mesh point (i,j).

The error form to be

$$\varepsilon_{i,j}=\lambda^j e^{I\theta_m i} \tag{14}$$

where λ is the amplification factor of the error in time, I is the unit imaginary number and $\theta_m=\frac{m\pi}{L}$ with $m=1,2,\ldots,M$ and $M=\frac{L}{\Delta x}$.

The error equation for the linear Klein-Gordon FDAM scheme (11) is:

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$$\varepsilon_{i,j+1}+\varepsilon_{i,j-1}-\varepsilon_{i+1,j}-\varepsilon_{i-1`,j} -k^2\varepsilon_{i,j}+\frac{k^2}{4}\begin{pmatrix}\cos(x_i)\cos(t_j)\\ +\cos(x_{i+1})\cos(t_j)\\ +\cos(x_i)\cos(t_{j+1})\\ +\cos(x_{i+1})\cos(t_{j+1})\end{pmatrix}=0 \quad (15)$$

By substituting (14) into (15) and rearranging the equation, it gives the error equation for the linear Klein-Gordon FDAM scheme as follows:

$$\lambda^2-\begin{pmatrix} e^{I\theta}+e^{-I\theta}+k^2 \\ -\dfrac{k^2}{4\varepsilon_{i,j}}\begin{bmatrix}\cos(x_i)\cos(t_j)\\ +\cos(x_{i+1})\cos(t_j)\\ +\cos(x_i)\cos(t_{j+1})\\ +\cos(x_{i+1})\cos(t_{j+1})\end{bmatrix}\end{pmatrix}\lambda+1=0 \quad (16)$$

In this study, we necessary to introduce Miller Norm Lemma to analyze the stability of the FDAM scheme. The Miller Norm Lemma is shown below [10].

Miller Norm Lemma: Letting $A>0$, the quadratic equation with real coefficients $A\lambda^2+B\lambda+C=0$.

The necessary and sufficient conditions of the roots of the quadratic equation are less than or equal to one. The conditions are

$$A-C\geq 0,\ A+B+C\geq 0,\ A-B+C\geq 0 \quad (17)$$

The performance analysis of the conditions are as follow:

$$A-C=1-1=0 \quad (18)$$

$$\begin{aligned} &A+B+C\\ &=1-\begin{pmatrix} e^{I\theta}+e^{-I\theta}+k^2 \\ -\dfrac{k^2}{4\varepsilon_{i,j}}\begin{bmatrix}\cos(x_i)\cos(t_j)\\ +\cos(x_{i+1})\cos(t_j)\\ +\cos(x_i)\cos(t_{j+1})\\ +\cos(x_{i+1})\cos(t_{j+1})\end{bmatrix}\end{pmatrix}+1\\ &=2-\begin{pmatrix} e^{I\theta}+e^{-I\theta}+k^2 \\ -\dfrac{k^2}{4\varepsilon_{i,j}}\begin{bmatrix}\cos(x_i)\cos(t_j)\\ +\cos(x_{i+1})\cos(t_j)\\ +\cos(x_i)\cos(t_{j+1})\\ +\cos(x_{i+1})\cos(t_{j+1})\end{bmatrix}\end{pmatrix}\\ &\geq 0\end{aligned} \quad (19)$$

$$\begin{aligned} &A-B+C\\ &=1+\begin{pmatrix} e^{I\theta}+e^{-I\theta}+k^2 \\ -\dfrac{k^2}{4\varepsilon_{i,j}}\begin{bmatrix}\cos(x_i)\cos(t_j)\\ +\cos(x_{i+1})\cos(t_j)\\ +\cos(x_i)\cos(t_{j+1})\\ +\cos(x_{i+1})\cos(t_{j+1})\end{bmatrix}\end{pmatrix}+1\\ &=2+\begin{pmatrix} e^{I\theta}+e^{-I\theta}+k^2 \\ -\dfrac{k^2}{4\varepsilon_{i,j}}\begin{bmatrix}\cos(x_i)\cos(t_j)\\ +\cos(x_{i+1})\cos(t_j)\\ +\cos(x_i)\cos(t_{j+1})\\ +\cos(x_{i+1})\cos(t_{j+1})\end{bmatrix}\end{pmatrix}\\ &\geq 0\end{aligned} \quad (20)$$

The Maple 13 code programming is used to illustrate the conditions of the Miller Norm Lemma. This is to prove the conditions of Miller Norm Lemma are absolutely true. The illustrations are shown in the following figures.

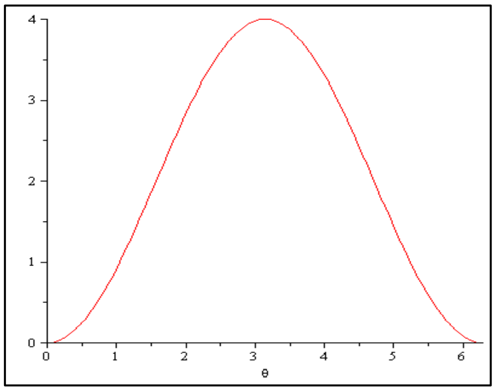

Figure 1. Illustration of the second condition of Miller Norm Lemma - $A+B+C\geq 0$.

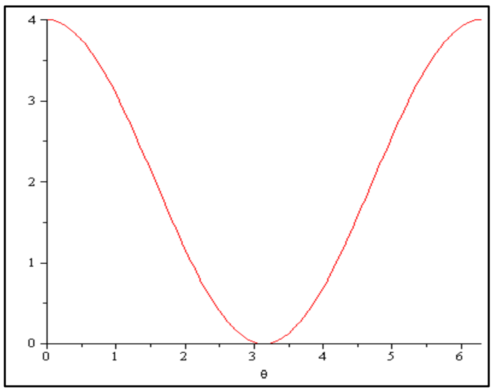

Figure 2. Illustration of the third condition of Miller Norm Lemma - $A-B+C\geq 0$.

3

The illustrations above show the conditions of the Miller Norm Lemma are satisfied. So, it can conclude that the condition of the von Neumann stability analysis method is satisfied which is $|\lambda|\leq 1$. Hence, the FDAM scheme for the linear Klein-Gordon equation is stable. The next analysis is consistency analysis.

B. Consistency Analysis

The consistency analysis will be analyzed by substituting the exact solution of the partial differential equation into the FDAM scheme. The equation (11) can be rewritten as

$$\begin{aligned} &u\left(x_i,t_{j+1}\right)+u\left(x_i,t_{j-1}\right)-u\left(x_{i+1},t_j\right) \\ &-u\left(x_{i-1},t_j\right)-k^2u\left(x_i,t_j\right) \\ &=-\frac{k^2}{4}\begin{pmatrix}\cos\left(x_i\right)\cos\left(t_j\right) \\ +\cos\left(x_{i+1}\right)\cos\left(t_j\right) \\ +\cos\left(x_i\right)\cos\left(t_{j+1}\right) \\ +\cos\left(x_{i+1}\right)\cos\left(t_{j+1}\right)\end{pmatrix}\end{aligned} \quad (21)$$

Every term of the exact solution will be expanded by using Taylor series expansion. For consistency analysis, the discretized equation should approach tend to the partial differential equation as the grid size tends to zero [25].

The exact solution is substituted into (21), it becomes

$$\begin{aligned} &\left(u+ku_t+\frac{k^2}{2}u_{tt}+...\right) \\ &+\left(u-ku_t+\frac{k^2}{2}u_{tt}-...\right) \\ &-\left(u+ku_x+\frac{k^2}{2}u_{xx}+...\right) \\ &-\left(u-ku_x+\frac{k^2}{2}u_{xx}-...\right) \\ &-k^2u=-\frac{k^2}{4}\begin{pmatrix}\cos\left(x\right)\cos\left(t\right) \\ +\cos\left(x+k\right)\cos\left(t\right) \\ +\cos\left(x\right)\cos\left(t+k\right) \\ +\cos\left(x+k\right)\cos\left(t+k\right)\end{pmatrix}\end{aligned} \quad (22)$$

where $u=u\left(x_i,t_j\right)$.

The equation (22) can be simplified by using Maple 13, it becomes

$$u_{tt}-u_{xx}-u=-\cos\left(x\right)\cos\left(t\right) \quad (23)$$

As $k\to 0$, (22) becomes $u_{tt}-u_{xx}-u=-\cos\left(x\right)\cos\left(t\right)$. Thus the condition for consistency is satisfied. Hence, the FDAM scheme for the linear Klein-Gordon equation is consistent. The convergence analysis will be presented in the next section.

C. Convergence Analysis

If the approximate solution approaches the exact solution for each value of the independent variables as grid size tends to zero, then a solution of the finite difference scheme approximates the partial differential equation is said to be convergent scheme. On the other hand, if the approximate solution of the linear partial differential equation properly satisfied the consistency and stability conditions, then it is sufficient to be convergent. This leads to the Lax-Richtmyer Equivalence Theorem [26].

Lax-Richtmyer Equivalence Theorem: A consistent one-step scheme for a well-posed initial value problem for a partial differential equation is convergent if and only if it is stable.

The FDAM scheme (11) is established for the linear Klein-Gordon equation (7). We found that the FDAM scheme is both stable and consistent. Hence, we can conclude that the FDAM scheme is convergent referred to the Lax-Richtmyer Equivalence Theorem.

IV. NUMERICAL EXPERIMENTS

The equation (7) will be solved using the FDAM scheme (12). A computer program is developed by applying (10) and (12) for computing the relative errors of (7) with different values of step size. The following numerical results are presented for selected mesh points.

TABLE 1. RELATIVE ERRORS FOR CTCS SCHEME (10)

k	(x,t)			
	(0.25, 0.25)	*(0.5, 0.5)*	*(0.75, 0.75)*	*(1.00, 1.00)*
0.005	0.063204604	0.27189171	0.58549741	0.86207473
0.025	0.060680424	0.26666824	0.59292822	0.90563177
0.050	0.057528228	0.26015589	0.60036660	0.95671168

TABLE 2. RELATIVE ERRORS FOR FDAM SCHEME (12)

k	(x,t)			
	(0.25, 0.25)	*(0.5, 0.5)*	*(0.75, 0.75)*	*(1.00, 1.00)*
0.005	0.063177700	0.27165216	0.58499248	0.86204649
0.025	0.060547393	0.26547712	0.59019268	0.90491576
0.050	0.057270653	0.25780036	0.59439142	0.95380718

TABLE 3. A COMPARISON OF AVERAGE RELATIVE ERROR BETWEEN SCHEMES (10) AND (12)

k	**CTCS Scheme (10)**	**FDAM Scheme (12)**
0.005	0.33027422	0.33004176
0.025	0.33921285	0.33793823
0.050	0.35016616	0.34732202

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The numerical results indicate that the relative error and the average relative error of the FDAM scheme (12) and the CTCS scheme (10) become smaller as k tends to approach zero for all selected mesh points. Hence, it is clear to conclude the FDAM scheme is more accurate rather than the CTCS scheme.

V. CONCLUSIONS

Several numerical methods are used for solving the Klein-Gordon equation such as ADM, FDM, HPM and HAM. In this paper, a new finite difference scheme for a linear inhomogeneous Klein-Gordon equation is developed based on central time central space (CTCS) incorporated with four points arithmetic mean averaging of functional value. The scheme is also known as FDAM scheme. The theoretical aspects of the finite difference scheme are also studied for the linear FDAM scheme. From the theoretical study, the stability analysis is shown the FDAM scheme is stable by applying the von Neumann stability analysis method and the Miller Norm Lemma. In addition, we have also shown the FDAM scheme is consistent and the Lax-Richtmyer Equivalence Theorem leads to verify the FDAM scheme is convergent. From the comparison, the numerical results observed the FDAM scheme is more accurate and reliable scheme rather than the CTCS scheme for the linear inhomogeneous Klein-Gordon equation.

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