On The Probability Density of Relativistic Spinless Particles

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In this paper, we shall derive a new conserved current for Klein-Gordon equation. The first component of this current is non-negative and reduces to $|\phi|^2$ in non-relativistic limit. Therefore, it can be interpreted as a suitable probability density for spinless particles. In addition, this current is time-like and so prevents faster than light particle propagation. We will see the probability density has a considerable deviation from $|\phi|^2$ providing the uncertainty in momentum is much greater than m_0c.

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Introduction.—The Born's interpretation of $|\psi|^2$ as position probability density is one of the most fundamental axioms of quantum mechanics as it provides a link between the mathematical formalism and empirical results [1]. This axiom has been incredibly successful in predicting position probability density in non-relativistic quantum mechanics. Although, in the relativistic regime, the quadratic relation between position probability density and wave function has been confirmed by recent high accuracy single photon multi-slit experiments [2–6], a satisfactory mathematical expression for position probability density of relativistic bosons has not yet been found. In the simplest case, finding a well-defined position probability density for the free spinless particles is a long-standing problem (see, e.g., [7, 8]): The time component of the well-known Klein-Gordon conserved current, $J^{\mu}_{KG}{=}i(\phi^*\partial^\mu\phi - \phi\partial^\mu\phi^*)$, may be negative on some regions of space-time and can not be interpreted as position probability density [9]. One may suggest to use the $|\phi|^2$ as probability density, similar to the non-relativistic theory[10, 11]; In which case it is easy to see that the Klein-Gordon equation,

$$\Box\phi + m^2\phi = 0, \tag{1}$$

leads to the following continuity equation for $|\phi|^2$ [10]:

$$\partial_t\rho_B + \nabla.\boldsymbol{J}_B = 0, \tag{2}$$

where

$$\rho_B = |\phi(x)|^2 = N\int \tilde{\phi}(p)\tilde{\phi}^*(k)e^{i(p-k).x}\ d^4p\ d^4k, \tag{3}$$

$$\boldsymbol{J}_B = N\int \tilde{\phi}(p)\tilde{\phi}^*(k)\ e^{i(p-k).x}\ \boldsymbol{u}(p,k)\ d^4p\ d^4k, \tag{4}$$

$$\boldsymbol{u}(p,k) = \frac{\boldsymbol{p}+\boldsymbol{k}}{p_0+k_0}, \tag{5}$$

and $\tilde{\phi}(p)$ is Fourier-transformation of wave function. It should be noted, despite the fact that $|\phi|^2$ is non-negative and conserved, it cannot be considered as position probability density: due to the Lorentz length contraction, the probability density can not be a scalar [12]. In other words, $J^\mu_B = (\rho_B, \boldsymbol{J}_B)$ is not a four-vector and therefore cannot be interpreted as a relativistic probability current density[11]. In addition, Born's probability density leads to faster than light particle propagation [13, 14]. In principle, a correct probability current must satisfy the following conditions:

I) Lorentz transformation: $J^{\mu'} = \Lambda^{\mu'}_{\mu} J^\mu$,

II) probability conservation: $\partial_\mu J^\mu = 0$,

III) future-orientation: $J^0 \geq 0$,

IV) causal propagation: $J^\mu J_\mu \geq 0$.

The last condition is necessary as it ensures the causal propagation of particles. In fact, there are several other currents which have been suggested for the Klein-Gordon equation[15, 16] all of which do not satisfy, at least, one of the above conditions. The aim of this letter is to propose a proper expression for relativistic probability current that satisfies all of the aforementioned conditions.

Position Distribution.— According to equations (3) and (4), we suggest the following expression as the relativistic probability current[17]:

$$J^\mu{=}\int \tilde{\phi}(p)\ \tilde{\phi}^*(k)\ e^{i(p-k).x}\ u^\mu(p,k)\ d^4p\ d^4k, \tag{6}$$

where $u^\mu(p,k)$ is an unknown function that must be determined by theoretical constrains. In this regard, the condition (I) implies that the $u^\mu(p,k)$ is a four-vector. The general form of a four-vector made by p and k is given by

$$u^\mu(p,k) = \alpha(p^\mu + k^\mu) + \beta(p^\mu - k^\nu), \tag{7}$$

where α and β are scalar coefficients. Next, the conservation condition (II) leads to $\beta = 0$. Also, in principle, the coefficient α should be determined using conditions (III) and (IV). This procedure in $(1+1)$-dimensions is straightforward and results in (see appendix)

$$\alpha(p,k) = \frac{\xi}{\sqrt{(p+k)^2}}, \tag{8}$$

where $\xi = \frac{1}{2}(\frac{k^0}{|k^0|} + \frac{p^0}{|p^0|})$. In addition, there is another way to fix the coefficient α by demanding u^μ to be real and $u^\mu u_\mu = c^2$. The obtained alpha from these conditions is in agreement with the previous derivation. So finally we get the $u^\mu(p,k)$ as follows:

$$u^\mu(p,k) = \xi \frac{p^\mu + k^\mu}{\sqrt{(p+k)^2}}. \tag{9}$$

Equation (9) can be rewritten as $u^\mu = |\xi|\gamma(1, \boldsymbol{u})$, in which $\boldsymbol{u}$ is the velocity-vector defined in equation(5) and $\gamma = 1/\sqrt{1-\boldsymbol{u}^2}$ is the corresponding Lorentz coefficient. In fact, the expression (6) is the simplest covariant generalization of the equations (3) and (4). The only difference between this expression and the Born probability current, J_B^μ, is the factor $|\xi|\gamma$:

$$\rho(x) = N \int |\xi|\gamma\tilde{\phi}(p)\tilde{\phi}^*(k)\ e^{i(p-k).x}\ d^4p\ d^4k, \tag{10}$$

$$\boldsymbol{J}(x) = N \int |\xi|\gamma\boldsymbol{u}\ \tilde{\phi}(p)\tilde{\phi}^*(k)e^{i(p-k).x}\ d^4p\ d^4k, \tag{11}$$

The factor γ comes naturally in accordance with Lorentz contraction and the factor $|\xi|$ prohibits the occurrence of *Zitterbewegung* behavior[18, 19]. In appendix, it is shown that, for massive particles in $(1+1)$-dimensions, equations (10) and (11) can be rewritten in position representation as follows:

$$\rho = |\mathcal{D}^+\phi_+|^2 + |\mathcal{D}^-\phi_+|^2 + |\mathcal{D}^+\phi_-|^2 + |\mathcal{D}^-\phi_-|^2, \tag{12}$$

$$J = \left(|\mathcal{D}^+\phi_+|^2 - |\mathcal{D}^-\phi_+|^2 + |\mathcal{D}^+\phi_-|^2 - |\mathcal{D}^-\phi_-|^2\right) c, \tag{13}$$

where $\phi_\pm$ are positive and negative frequency components of ϕ and $\mathcal{D}^\pm$ are pseudo-differential operators which are defined as follows:

$$\mathcal{D}^\pm = \sqrt{\frac{1}{2}(\sqrt{1-\lambda_c^2\frac{d^2}{dx^2}} \mp i\lambda_c\frac{d}{dx})}, \tag{14}$$

in which $\lambda \equiv \hbar/mc$ is Compton wave length. From equations (12) and (13) It is clear that, the probability density is unambiguously positive definite and $|J/\rho| \leq c$.

Notice that, the expressions (12) and (13) can be also used for probabilistic interpretation of the square root Klein-Gordon (also known as spinless Salpeter) equation [10, 20–24]:

$$i\hbar\frac{\partial\phi}{\partial t} = \sqrt{\nabla^2 + m^2}\phi. \tag{15}$$

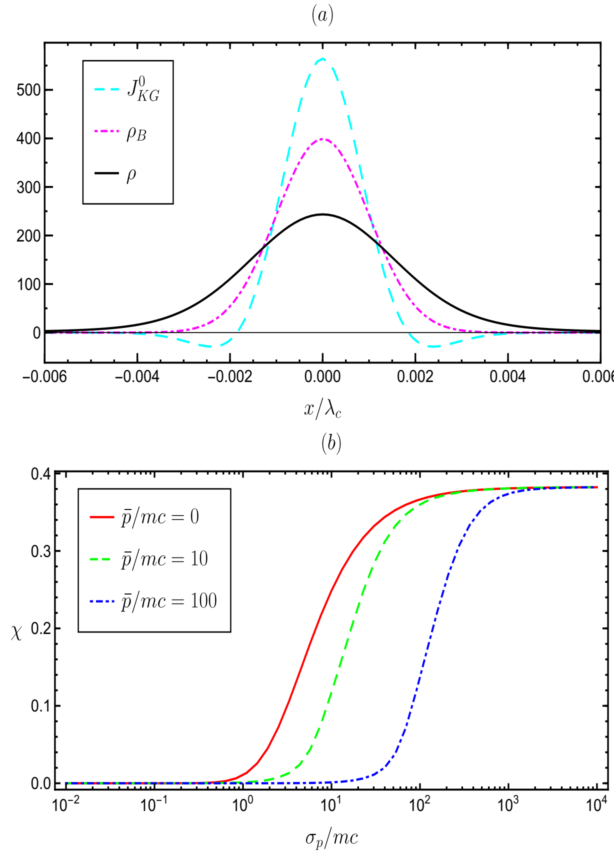

FIG. 1. (a) The first component of Klein-Gordon current J^0_{KG} (dashed line), the Born probability density ρ_B (dash-dotted line) and the relativistic probability density ρ (solid line) referring to the Gaussian wave function (19) with $\sigma_p/mc = 1000$ and $\bar{p}/mc = 0$. (b) Represents the χ for Gaussian wave function (19).

Of course in this case, the wave function only has the positive energy part and so the equations (12) and (13) reduce to the following simpler forms:

$$\rho = |\mathcal{D}^+\phi|^2 + |\mathcal{D}^-\phi|^2, \tag{16}$$

$$J = \left(|\mathcal{D}^+\phi|^2 - |\mathcal{D}^-\phi|^2\right) c. \tag{17}$$

It is easy to see, in non-relativistic regime ($c \to \infty$), the equations (16) and (17) reduce to non-relativistic probability density, $|\phi|^2$, and conventional Schrödinger probability current, $(\hbar/m)\Im(\phi^*\partial_x\phi)$, respectively.

For comparing the relativistic probability density (16) with $|\phi|^2$, In Figure 1, we plot χ (the measure of deviation from Born probability) defined as:

$$\chi = \int_{-\infty}^{\infty} \left|\rho - |\phi|^2\right|\ dx, \tag{18}$$

for this Gaussian wave function

$$\tilde{\phi}(p) = Ne^{-(p-\bar{p})^2/\sigma_p^2}. \tag{19}$$

From figure (1-b) it is clear that, when momentum uncertainty is small compared to m_0c, relativistic probability density deviation from Born probability density is negligible, even assuming that the group velocity of wave packet is comparable with velocity of light.

Finlay, it should be noted, although the expression for probability density in terms of wave function, equation (16), is non-local, there is no inconsistency with special relativity. In fact, this non-locality is essential to introduce a self-consistent relativistic probability density; since the relativistic wave function can propagate outside the light cone, a local relation between wave function and probability density, for instance $\rho = |\phi|^2$, leads to faster than light particles propagation[13, 14].

Momentum Distribution.— Since the position probability density deviates from $|\phi(x)|^2$ in relativistic regime, one may raise the question of whether momentum probability distribution deviates from $|\tilde{\phi}(p)|^2$ as well? To answer this question, we note that, Based on the quantum theory of measurement, each physical measurement can be described as a position measurement: In principle, the variables that account for the outcome of an experiment are ultimately particle positions [25–28]. This fact has been made clear by John Bell[25]:

"In physics the only observations we must consider are position observations, if only the positions of instrument pointers. ... If you make axioms, rather than definitions and theorems, about the "measurement" of anything else, then you commit redundancy and risk inconsistency."

In this regard, It is shown in non-relativistic quantum mechanics that the Born's rule for any observable can be derived considering the Born's rule on position of particles [25–27]. Here we aim to propose a derivation of relativistic momentum probability density from the relativistic position probability density (16). The given argument is based on the Feynman's method for initially confined systems, namely the Time-of-flight measurements [26, 29–31]. Suppose the wave function is initially confined to a volume V centered around the origin $\boldsymbol{x}_0 = 0$ and is negligible elsewhere. After allowing the wave function to freely propagate for a considerable amount of time, a measurement on the position $\boldsymbol{x}$ of the particle is effected. The probability of the particle's momentum lying inside the element d^3p around the point $\boldsymbol{p}$ at $t = 0$ is equal to probability of finding the particle's position in the element d^3x around the point $\boldsymbol{x} = \boldsymbol{v}t$ provided the limit $t \to \infty$ is taken in order to discard the effect of uncertainty in initial position. So we have:

$$g(\boldsymbol{p})d^3p = \lim_{t\to\infty} [\rho(\boldsymbol{x}, t)d^3x]_{\boldsymbol{x}=\boldsymbol{v}t}, \tag{20}$$

where $g(\boldsymbol{p})$ represents the momentum probability density and $\boldsymbol{v} = \boldsymbol{p}/E$. using the relativistic position probability density (16), the equation (20) leads to

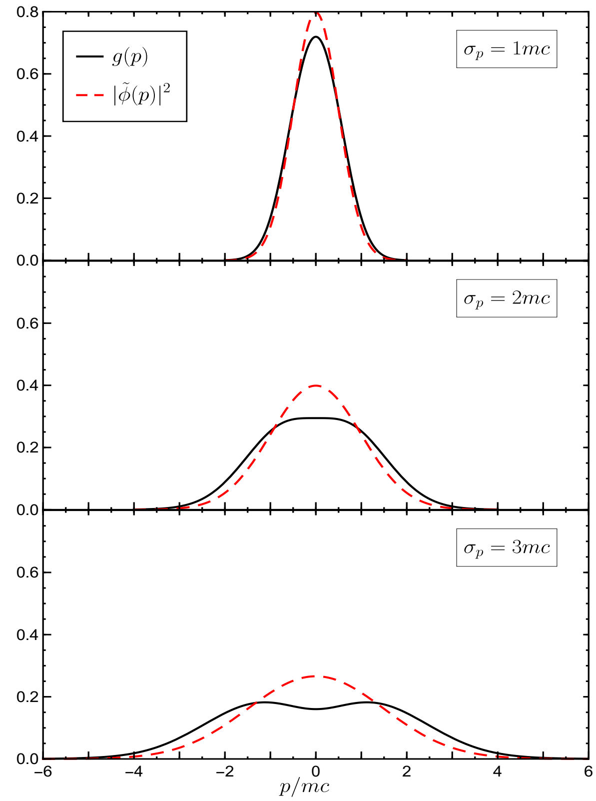

FIG. 2. The plot of the relativistic and non-relativistic momentum probability density referring to the Gaussian wave function (19) with $\bar{p} = 0$.

$$g(\boldsymbol{p}) = \frac{m^2}{E^3} \lim_{t\to\infty} t \left(|\mathcal{D}^+\phi|^2 + |\mathcal{D}^-\phi|^2\right)_{\boldsymbol{x}=\boldsymbol{p}t/E}. \tag{21}$$

In the non relativistic regime ($c \to \infty$) the equation (21) leads to

$$g(\boldsymbol{p}) = \frac{1}{m} \lim_{t\to\infty} [t|\phi(\boldsymbol{x}, t)|^2]_{x=\boldsymbol{p}t/m}. \tag{22}$$

In this case, the Schrödinger equation for an initially confined wave function leads to $\phi(\boldsymbol{p}t/m, t) \sim t^{-1/2}\tilde{\phi}(\boldsymbol{p})$ at $t \to \infty$ and so the equation (22) reduces to the Born rule in momentum space $g(\boldsymbol{p}) = |\tilde{\phi}(\boldsymbol{p})|^2$ [26, 29]. But finding an explicit expression for momentum probability density in the relativistic regime is not straightforward, therefore a numerical calculation of $g(\boldsymbol{p})$ for the Gaussian wave packet (19) is presented in Figure 2. This numerical study indicates that the relativistic momentum probability density deviates significantly from Born rule only when the width of the wave function in momentum space is greater than m_0c.

Conclusion and outlook.— In this letter, in a simple case of single free spinless particle in $(1 + 1)$-dimensions, we have extracted a "well-defined" probability density current. By "well-defined" we mean: the current i) is manifestly covariant, ii) is conserved, iii) has a non-negative first component, iv) does not lead to faster than light particle propagation and v) reduces to Born probability current density in non-relativistic limit. These conditions uniquely give rise to the given probability density current. Therefore probabilistic interpretation of relativistic spinless wave function is possible. Extending this study to $(3+1)$-dimensional interacting particle systems will be the next step. Such systems should be described by quantum filed theory. The state of a system in quantum filed theory is an arbitrary vector in the appropriate Fock space and may well involve a superposition of states of different particle numbers, namely $|\Psi\rangle = \sum_n \int \tilde{\phi}_n(p_1,..,p_n)\,|p_1,..,p_n\rangle$. It evolves according to the appropriate Schrödinger equation; $i\partial_t\,|\Psi\rangle = \mathrm{H}\,|\Psi\rangle$ where H is the Hamiltonian operator in Schrödinger picture. In the presence of interaction this equation leads to a system of coupled integro-differential equations for multi-particle wave functions, ϕ_n, a reccurrent procedure in the literature of Light-front quantization [33]. In future works, we aim to find a probabilistic interpretation for these wave equations in position space.

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APPENDIX

In this appendix, we will derive the equations (8), (12) and (13). Without loss of generality, the wave function can be expanded as a linear combination of plane waves:

$$\phi(x) = \sum_n A_n e^{ip_n.x}. \tag{23}$$

Plugging this into (6) and using (7) yields,

$$J^0 \pm J^1 = \sum_{n,m} A_n A_m^* \alpha(p_n,p_m)(p_n^\pm + p_m^\pm)e^{i(p_n-p_m).x} \tag{24}$$

where $p_n^\pm = p_n^0 \pm p_n^1$. In $(1+1)$-dimension, conditions $J^\mu J_\mu \geq 0$ and $J^0 \geq 0$ lead to

$$J^0 \pm J^1 \geq 0, \tag{25}$$

for arbitrary wave-functions. Therefore we can consider

$$\alpha(p_n,p_m) = \frac{[F^\pm(p_n)]\;[F^\pm(p_m)]^* + [F^\pm(p_n)]^*\;[F^\pm(p_m)]}{p_n^\pm + p_m^\pm}, \tag{26}$$

which leads to the following positive definite expression for $J^0 \pm J^1$:

$$J^0 \pm J^1 = \left|\sum_n F^\pm(p_n) A_n e^{ip_n.x}\right|^2 + \left|\sum_n [F^\pm(p_n)]^* A_n e^{ip_n.x}\right|^2, \tag{27}$$

where $F^\pm(p_n)$ is an unknown function which must be determined. Since the only scalar can be made by p_n is the rest mass, a dimensional analysis leads to $|\alpha(p_n,p_n)| = \frac{1}{2m_0}$; the factor 1/2 is a convention and can be absorbed in normalization constant. Therefore,

$$F^\pm(p_n) = e^{i\lambda^\pm(p_n)}\sqrt{p_n^\pm/2m_0}. \tag{28}$$

Whether one substitutes F^+ or F^- the resulted α is the same. This fact can be used to determine phase of $F^\pm$ as $\lambda^\pm(p_n) = \pm l\pi$, where l is an integer number. Then we have

$$\alpha(p_n,p_m) = \frac{[\sqrt{p_n^\pm}]\;[\sqrt{p_m^\pm}]^* + [\sqrt{p_n^\pm}]^*\;[\sqrt{p_m^\pm}]}{2m_0(p_n^\pm + p_m^\pm)}. \tag{29}$$

A straightforward but tedious calculation shows that, $\alpha(p_n,p_m)$ can be rewritten as equation (8) which ensures that $\alpha(p_n,p_m)$ is a scalar. Also from equation (7), it is clear that

$$\begin{aligned} J^0 = &|\sum_n \sqrt{\frac{p_n^+}{4m_0}} A_n e^{ip_n.x}|^2 + |\sum_n [\sqrt{\frac{p_n^+}{4m_0}}]^* A_n e^{ip_n.x}|^2 \\ &+|\sum_n \sqrt{\frac{p_n^-}{4m_0}} A_n e^{ip_n.x}|^2 + |\sum_n [\sqrt{\frac{p_n^-}{4m_0}}]^* A_n e^{ip_n.x}|^2, \end{aligned} \tag{30}$$

$$\begin{aligned} J^1 = &|\sum_n \sqrt{\frac{p_n^+}{4m_0}} A_n e^{ip_n.x}|^2 + |\sum_n [\sqrt{\frac{p_n^+}{4m_0}}]^* A_n e^{ip_n.x}|^2 \\ &-|\sum_n \sqrt{\frac{p_n^-}{4m_0}} A_n e^{ip_n.x}|^2 - |\sum_n [\sqrt{\frac{p_n^-}{4m_0}}]^* A_n e^{ip_n.x}|^2. \end{aligned} \tag{31}$$

Finally, using the definition of $D^\pm$ operators, (14), equations (30) and (31) reduce to (12) and (13).

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