Stationary bound-state scalar configurations supported by rapidly-spinning exotic compact objects

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Some quantum-gravity theories suggest that the absorbing horizon of a classical black hole should be replaced by a reflective surface which is located a microscopic distance above the would-be classical horizon. Instead of an absorbing black hole, the resulting horizonless spacetime describes a reflective exotic compact object. Motivated by this intriguing prediction, in the present paper we explore the physical properties of exotic compact objects which are linearly coupled to stationary bound-state massive scalar field configurations. In particular, solving the Klein-Gordon wave equation for a stationary scalar field of proper mass μ and spheroidal harmonic indices (l, m) in the background of a rapidly-rotating exotic compact object of mass M and angular momentum $J = Ma$, we derive a compact analytical formula for the *discrete* radii $\{r_{\text{c}}(\mu, l, m, M, a; n)\}$ of the exotic compact objects which can support the stationary bound-state massive scalar field configurations. We confirm our analytical results by direct numerical computations.

I. INTRODUCTION

Black holes in classical theories of gravity describe compact spacetime regions which are bounded by event horizons with absorbing boundary conditions. Interestingly, however, some candidate quantum-gravity models [1–13] have recently suggested that quantum effects may prevent the formation of stable black-hole horizons. These models have put forward the intriguing idea that, within the framework of a quantum theory of gravity, horizonless exotic compact objects may serve as alternatives to the familiar classical black-hole spacetimes [1–13].

In a very interesting work, Maggio, Pani, and Ferrari [13] have recently studied numerically the physical properties of spinning exotic compact objects which are characterized by spacetime geometries that modify the familiar Kerr metric only at some microscopic scale around the would-be classical horizon. In particular, the physical model analyzed in [13] assumes that the absorbing horizon of the classical Kerr black-hole spacetime is replaced by a slightly larger quantum membrane with reflective boundary conditions.

The interplay between compact astrophysical objects and fundamental matter fields has attracted much attention over the years from both physicists and mathematicians. In particular, recent analytical [14] and numerical [15] studies of the Einstein-Klein-Gordon field equations have revealed the fact that, within the framework of classical general relativity, spinning Kerr black holes can support stationary spatially regular bound-state matter configurations which are made of massive scalar fields. This fact naturally raises the following physically interesting question: Can the exotic compact objects of the suggested quantum-gravity models [1–13], like the more familiar classical black holes [14, 15], support stationary bound-state massive scalar field configurations in their exterior regions?

In the present paper we shall address this physically intriguing question by solving the Klein-Gordon wave equation for massive scalar fields in the background of a rapidly-rotating exotic compact object. In particular, motivated by the suggested quantum-gravity model recently studied in [13] (see also [1–12]), we shall use *analytical* techniques in order to study the physical properties of exotic compact objects with reflective boundary conditions which are linearly coupled to stationary bound-state massive scalar field configurations. Interestingly, as we shall explicitly show below, one can derive a remarkably compact analytical formula for the *discrete* radii $\{r_{\text{c}}(\mu; n)\}$ [16] of the horizonless rapidly-spinning exotic compact objects which, for a given proper mass μ of the external field, can support the stationary bound-state massive scalar field configurations.

It is important to note that the horizonless spinning configurations that we shall study in the present paper are characterized by the dimensionless relation $M\mu = O(m)$, where M is the mass of the central exotic compact object and m is the azimuthal harmonic index of the massive scalar field mode. This characteristic relation implies, in particular, that for astrophysically realistic black-hole candidates, the relevant massive scalar fields are ultralight: $\mu \sim 10^{-10} - 10^{-19}$eV. In this context, it is worth noting that the physical motivation to consider such ultralight massive scalar fields is manifold and ranges from possible dark matter candidate fields to new fundamental bosonic fields which might appear in suggested extensions of the Standard Model and in fundamental string theories. In particular, such ultralight fields naturally appear in the suggested string axiverse scenario, in theories of exotic dark photons, and in hidden $U(1)$ sectors [17–21].

Before proceeding, it is worth emphasizing that the composed spinning-exotic-compact-object-massive-scalar-field configurations that we shall analyze in the present paper, like the composed spinning-black-hole-massive-scalar-field configurations studied in [14, 15], owe their existence to the intriguing physical phenomenon of superradiant scattering in rotating spacetimes [22, 23]. In particular, the stationary field configurations, whose physical properties will be analyzed below, are characterized by the critical (marginal) frequency ω_{c} for the superradiant scattering phenomenon of bosonic fields in spinning spacetimes.

II. DESCRIPTION OF THE SYSTEM

We shall study analytically the physical properties of massive scalar field configurations which are linearly coupled to a rapidly-spinning exotic compact object. Motivated by the suggested quantum-gravity models [1–13], we shall assume that the spacetime of the exotic compact object is described by a curved geometry that modifies the familiar Kerr metric only at some microscopic scale around the would-be classical horizon. In particular, following the interesting work of Maggio, Pani, and Ferrari [13], we shall consider a reflective spinning compact object of radius r_{c}, mass M, and angular momentum $J = Ma$ whose exterior spacetime metric is described by the curved Kerr line element [24, 25] (we shall use natural units in which $G = c = \hbar = 1$)

$$ds^2 = -\frac{\Delta}{\rho^2}(dt - a\sin^2\theta d\phi)^2 + \frac{\rho^2}{\Delta}dr^2 + \rho^2 d\theta^2 + \frac{\sin^2\theta}{\rho^2}\big[adt - (r^2+a^2)d\phi\big]^2 \quad \text{for} \quad r > r_{\text{c}} \tag{1}$$

with $\Delta \equiv r^2 - 2Mr + a^2$ and $\rho^2 \equiv r^2 + a^2\cos^2\theta$. Here (t, r, θ, ϕ) are the familiar Boyer-Lindquist spacetime coordinates. The radius of the would-be classical horizon is given by

$$r_+ = M + (M^2 - a^2)^{1/2} \ . \tag{2}$$

Following the intriguing quantum-gravity models [1–13], which predict the occurrence of quantum corrections to the curved spacetime only at some microscopic scale around the would-be classical horizon, we shall assume that the radius r_{c} of the reflective quantum membrane is characterized by the strong inequality

$$x_{\text{c}} \equiv \frac{r_{\text{c}} - r_+}{r_+} \ll 1 \ . \tag{3}$$

It is important to emphasize that the assumption made in [13], that the exterior spacetime region of the spinning exotic compact object is described by the Kerr metric (1), is a non-trivial one. As emphasized in [13] (see also Refs. [26–28]), this assumption is expected to be valid in the physically interesting regime $x_{\text{c}} \ll 1$ of small quantum corrections.

The Klein-Gordon wave equation [29, 30]

$$(\nabla^\nu\nabla_\nu - \mu^2)\Psi = 0 \tag{4}$$

governs the dynamics of the scalar field in the curved spacetime of the exotic compact object, where μ is the proper mass of the linearized field [31]. Substituting into (4) the mathematical decomposition [29, 30, 32]

$$\Psi(t, r, \theta, \phi) = \sum_{l,m} e^{im\phi} S_{lm}(\theta; a\sqrt{\mu^2 - \omega^2}) R_{lm}(r; M, a, \mu, \omega) e^{-i\omega t} \tag{5}$$

for the eigenfunction Ψ of the massive scalar field, and using the metric components (1) which characterize the exterior curved spacetime of the exotic compact object, one finds that the radial eigenfunction of the massive scalar field satisfies the ordinary differential equation [29, 30]

$$\Delta\frac{d}{dr}\Big(\Delta\frac{dR_{lm}}{dr}\Big) + \Big\{[\omega(r^2+a^2) - ma]^2 + \Delta[2ma\omega - \mu^2(r^2+a^2) - K_{lm}]\Big\}R_{lm} = 0 \ , \tag{6}$$

where the angular eigenvalues $\{K_{lm}(a\sqrt{\mu^2-\omega^2})\}$ are determined by the characteristic angular differential equation [29, 30, 33–37]

$$\frac{1}{\sin\theta}\frac{d}{\theta}\Big(\sin\theta\frac{dS_{lm}}{d\theta}\Big) + \Big[K_{lm} + a^2(\mu^2-\omega^2) - a^2(\mu^2-\omega^2)\cos^2\theta - \frac{m^2}{\sin^2\theta}\Big]S_{lm} = 0 \tag{7}$$

with the physically motivated boundary conditions of regularity at the two angular poles $\theta = 0$ and $\theta = \pi$. It is worth noting that the characteristic eigenvalues of the spheroidal angular equation (7) can be expanded in the form $K_{lm} + a^2(\mu^2 - \omega^2) = l(l+1) + \sum_{k=1}^{\infty} c_k a^{2k}(\mu^2 - \omega^2)^k$, where the expansion coefficients $\{c_k(l, m)\}$ are given in [35].

The stationary bound-state configurations of the spatially regular massive scalar fields in the curved spacetime of the spinning exotic compact object are characterized by exponentially decaying (normalizable) radial eigenfunctions at spatial infinity [38] [for brevity, we shall henceforth omit the angular harmonic indices (l,m) which characterize the spatially regular massive scalar field mode]:

$$R(r\to\infty)\sim {{1}\over{r}}e^{-\sqrt{\mu^2-\omega^2}r}\ \ \ \ \text{with}\ \ \ \ \omega^2<\mu^2\ . \tag{8}$$

In addition, following the quantum-gravity model studied in [13], we shall assume that the exotic compact object is characterized by a reflecting surface which is located a microscopic distance [see Eq. (3)] above the would-be classical horizon. In particular, we shall consider two types of inner boundary conditions [13]:

$$\begin{cases} R(r=r_{\text{c}})=0 & \text{Dirichlet B. C.}\ ; \\ dR(r=r_{\text{c}})/dr=0 & \text{Neumann B. C.}\ \ . \end{cases} \tag{9}$$

The set of equations (6)-(9) determines the *discrete* spectrum of radii $\{r_{\text{c}}(\mu,l,m,M,a;n)\}$ of the spinning exotic compact objects which can support the spatially regular stationary bound-state massive scalar field configurations. Interestingly, in the next section we shall explicitly prove that, for rapidly-rotating horizonless compact objects, this characteristic discrete spectrum of supporting radii can be determined *analytically*.

III. THE RESONANCE CONDITIONS OF THE COMPOSED SPINNING-EXOTIC-COMPACT-OBJECT-MASSIVE-SCALAR-FIELD CONFIGURATIONS

In the present section we shall analyze the set of equations (6)-(9) which determine the bound-state resonances of the massive scalar fields in the curved background of the spinning horizonless exotic compact object. In particular, below we shall use *analytical* techniques in order to derive remarkably compact resonance conditions for the *discrete* radii $\{r^{\text{Dirichlet}}_{\text{c}}(\mu,l,m,M,a;n)\}$ and $\{r^{\text{Neumann}}_{\text{c}}(\mu,l,m,M,a;n)\}$ of the rapidly-spinning exotic compact objects which can support the spatially regular stationary bound-state massive scalar field configurations.

It is convenient to use the dimensionless physical parameters [29, 30]

$$x\equiv {{r-r_+}\over{r_+}}\ \ \ ;\ \ \ \tau\equiv{{r_+-r_-}\over{r_+}}\ \ \ ;\ \ \ k\equiv 2\omega r_+\ \ \ ;\ \ \ \bar\mu\equiv M\mu\ \ \ ;\ \ \ \omega_{\text{c}}r_+\equiv{{ma}\over{2M}}\ \ \ ;\ \ \ \varpi\equiv{{2M(\omega-\omega_{\text{c}})}\over{\tau}}\ , \tag{10}$$

in terms of which the scalar radial equation (6) takes the form [30, 39]

$$x(x+\tau){{d^2R}\over{dx^2}}+(2x+\tau){{dR}\over{dx}}+UR=0\ , \tag{11}$$

where

$$U(x)={{(\omega r_+x^2+kx+\varpi\tau)^2}\over{x(x+\tau)}}-K+2ma\omega-\mu^2[r^2_+(1+x)^2+a^2]\ . \tag{12}$$

Interestingly, and most importantly for our analysis, the radial differential equation (11) is amenable to an analytical treatment in the spatial region

$$x\geq x_{\text{c}}\gg\tau\times\max(1,\varpi)\ . \tag{13}$$

Note that the strong inequalities (3) and (13) can be satisfied simultaneously in the regime

$$\tau\times\max(1,\varpi)\ll1\ , \tag{14}$$

in which case one may approximate the massive scalar equation (11) by [30, 39]

$$x^2{{d^2R}\over{dx^2}}+2x{{dR}\over{dx}}+\bar UR=0\ , \tag{15}$$

where

$$\bar U(x)=[(m^2/4-\bar\mu^2)x^2+(m^2-2\bar\mu^2)x+(-K+2m^2-2\bar\mu^2)]\cdot[1+O(\tau,\tau\varpi)]\ . \tag{16}$$

It should be emphasized that the $\tau \ll 1$ regime (14) corresponds to rapidly-spinning exotic compact objects. In addition, it is worth noting that the physical significance of the field frequency $\omega = \omega_{\text{c}}$ [which corresponds to $\varpi = 0$, see Eq. (10)] stems from the fact that, for classical spinning black-hole spacetimes, this unique resonant frequency characterizes the composed stationary black-hole-massive-scalar-field configurations studied in [14, 15]. Note, in particular, that in (16) we have used the dimensionless relation $M\omega = \frac{1}{2}m \cdot [1 + O(\tau, \tau\varpi)]$ in the regime (14) of rapidly-spinning exotic compact objects and field frequencies that lie in the vicinity of the critical superradiant frequency $\omega = \omega_{\text{c}}$ (or equivalently, in the vicinity of $\varpi = 0$).

The general mathematical solution of the radial differential equation (15) can be expressed in terms of the familiar Whittaker functions [35, 40]. In particular, defining the dimensionless variables

$$\epsilon \equiv \sqrt{\bar{\mu}^2 - m^2/4} \quad ; \quad \kappa \equiv \frac{m^2/2 - \bar{\mu}^2}{\epsilon} \quad ; \quad \delta^2 \equiv -K - \frac{1}{4} + 2(m^2 - \bar{\mu}^2)\ , \tag{17}$$

one finds [35, 39–41]

$$R(x) = x^{-1}\big[A \cdot W_{\kappa,i\delta}(2\epsilon x) + B \cdot M_{\kappa,i\delta}(2\epsilon x)\big]\ , \tag{18}$$

where $\{A, B\}$ are normalization constants. We shall henceforth assume that [42]

$$\{\delta, \kappa\} \in \mathbb{R}\ . \tag{19}$$

Note that the assumption $\kappa \in \mathbb{R}$ corresponds to bound-state resonances of the massive scalar fields with $\omega^2 < \mu^2$ [see Eqs. (8) and (17)].

The asymptotic spatial behavior of the radial scalar function (18) is given by [43]

$$R(x \to \infty) = A \cdot x^{-1+\kappa} e^{-\epsilon x} + B \cdot \frac{\Gamma(1 + 2i\delta)}{\Gamma(\frac{1}{2} + i\delta - \kappa)} x^{-1-\kappa} e^{\epsilon x}\ . \tag{20}$$

Taking cognizance of the asymptotic boundary condition (8), which characterizes the spatially regular bound-state (normalizable) configurations of the massive scalar fields, one concludes that the coefficient of the exploding exponent in the asymptotic radial expression (20) should vanish:

$$B = 0\ . \tag{21}$$

We therefore find that the stationary bound-state resonances of the massive scalar fields in the horizonless curved spacetime of the rapidly-spinning exotic compact object are characterized by the spatially regular radial eigenfunction

$$R(x) = A \cdot x^{-1} W_{\kappa,i\delta}(2\epsilon x)\ , \tag{22}$$

where $W_{\kappa,\beta}(z)$ is the familiar Whittaker function of the second kind [35].

Taking cognizance of the inner boundary conditions (9) at the reflective surface $x = x_{\text{c}}$ of the exotic compact object, together with the characteristic radial eigenfunction (22) of the stationary bound-state massive scalar field configurations, one obtains the remarkably compact resonance conditions

$$W_{\kappa,i\delta}(2\epsilon x_{\text{c}}) = 0 \quad \text{for} \quad \text{Dirichlet B. C.} \tag{23}$$

and

$$\frac{d}{dx}[x^{-1} W_{\kappa,i\delta}(2\epsilon x)]_{x=x_{\text{c}}} = 0 \quad \text{for} \quad \text{Neumann B. C.} \tag{24}$$

for the composed exotic-compact-object-linearized-massive-scalar-field configurations. The analytically derived resonance conditions (23) and (24) determine the dimensionless *discrete* radii $\{x_{\text{c}}(\mu, l, m, M, a; n)\}$ of the rapidly-spinning exotic compact objects which can support the stationary bound-state spatially regular massive scalar field configurations.

In the next section we shall use analytical techniques in order to prove that the resonance conditions (23) and (24) for the characteristic dimensionless radii of the central exotic compact objects can only be satisfied in the bounded radial regime

$$\epsilon x_{\text{c}} < \kappa + \sqrt{\kappa^2 + \delta^2}\ . \tag{25}$$

IV. UPPER BOUND ON THE RADII OF THE SUPPORTING EXOTIC COMPACT OBJECTS

In the present section we shall use a simple analytical argument in order to derive a remarkably compact upper bound on the characteristic dimensionless radii $\{x_{\text{c}}(\mu, l, m, M, a; n)\}$ of the rapidly-spinning exotic compact objects which can support the stationary bound-state massive scalar field configurations.

It proves useful to define the new radial function

$$R = x^{\gamma}\Phi \quad \text{with} \quad \gamma \geq -1\ , \tag{26}$$

in terms of which the radial differential equation (15) can be written in the form

$$x^2\frac{d^2\Phi}{dx^2} + 2(1+\gamma)x\frac{d\Phi}{dx} + [\bar{U}(x) + \gamma(\gamma+1)]\Phi = 0\ . \tag{27}$$

Taking cognizance of the boundary conditions (8) and (9), one concludes that the eigenfunction $\Phi(x)$, which characterizes the spatially regular stationary bound-state configurations of the massive scalar fields in the background of the exotic compact object, must have (at least) one inflection point, $x = x_{\text{in}}$, in the radial interval

$$x_{\text{in}} \in (x_{\text{c}}, \infty) \tag{28}$$

which is characterized by the functional relations

$$\{\Phi \cdot \frac{d\Phi}{dx} < 0 \quad \text{and} \quad \frac{d^2\Phi}{dx^2} = 0\} \quad \text{for} \quad x = x_{\text{in}}\ . \tag{29}$$

Substituting the characteristic relations (29) into the radial scalar equation (27), one concludes that the composed exotic-compact-object-linearized-massive-scalar-field configurations are characterized by the inequality

$$\bar{U}(x_{\text{in}}) + \gamma(\gamma+1) > 0\ , \tag{30}$$

which implies [see Eq. (16)]

$$\epsilon^2 \cdot x^2_{\text{in}} - 2\epsilon\kappa \cdot x_{\text{in}} - [\delta^2 + \frac{1}{4} + \gamma(\gamma+1)] < 0\ . \tag{31}$$

Taking cognizance of (28) and (31), one finds the upper bound

$$x_{\text{c}} < \frac{\kappa + \sqrt{\kappa^2 + \delta^2 + \frac{1}{4} + \gamma(\gamma+1)}}{\epsilon}\ . \tag{32}$$

The strongest upper bound on the dimensionless radius x_{c} can be obtained by minimizing the r.h.s of (32). In particular, the term $\gamma(\gamma+1)$ is minimized for $\gamma = -1/2$, in which case one finds from (32) the characteristic upper bound

$$x_{\text{c}} < \frac{\kappa + \sqrt{\kappa^2 + \delta^2}}{\epsilon}\ . \tag{33}$$

on the dimensionless radii of the rapidly-spinning exotic compact objects which can support the stationary bound-state massive scalar field configurations.

V. THE CHARACTERISTIC RESONANCE SPECTRA OF THE STATIONARY COMPOSED EXOTIC-COMPACT-OBJECT-LINEARIZED-MASSIVE-SCALAR-FIELD CONFIGURATIONS

Interestingly, as we shall now prove explicitly, the resonance equations (23) and (24), which determine the discrete radii $\{x_{\text{c}}(\mu, l, m, M, a; n)\}$ of the supporting exotic compact objects, can be solved *analytically* in the physically interesting regime

$$x_{\text{c}} \ll 1\ . \tag{34}$$

It is worth emphasizing again that the strong inequality (34) characterizes exotic compact objects whose quantum reflective surfaces are located in the radial vicinity of the would-be classical black-hole horizons [see Eqs. (2) and (3)].

In particular, the small-$x_{\rm c}$ regime (34) corresponds to the physically interesting model of spinning exotic compact objects recently studied numerically in [13] (see also [1–12]).

Using Eqs. 13.1.3, 13.1.33, and 13.5.5 of [35], one can express the (Dirichlet) resonance condition (23) in the form

$$(2\epsilon x_{\rm c})^{2i\delta}={{\Gamma(1+2i\delta)\Gamma({1\over2}-i\delta-\kappa)}\over{\Gamma(1-2i\delta)\Gamma({1\over2}+i\delta-\kappa)}}\ \ \ \ \ \text{for}\ \ \ \ x_{\rm c}\ll1\ . \tag{35}$$

Likewise, using Eqs. 13.1.3, 13.1.33, 13.4.33, and 13.5.5 of [35], one can express the (Neumann) resonance condition (24) in the form

$$(2\epsilon x_{\rm c})^{2i\delta}={{\Gamma(2+2i\delta)\Gamma({1\over2}-i\delta-\kappa)}\over{\Gamma(2-2i\delta)\Gamma({1\over2}+i\delta-\kappa)}}\ \ \ \ \ \text{for}\ \ \ \ x_{\rm c}\ll1\ . \tag{36}$$

From Eqs. (35) and (36) one obtains respectively the analytical formulas [44]

$$x^{\text{Dirichlet}}_{\rm c}(n)={{e^{-\pi n/\delta}}\over{2\epsilon}}\Big[{{\Gamma(1+2i\delta)\Gamma({1\over2}-i\delta-\kappa)}\over{\Gamma(1-2i\delta)\Gamma({1\over2}+i\delta-\kappa)}}\Big]^{1/2i\delta}\ \ \ \ ;\ \ \ \ n\in\mathbb{Z} \tag{37}$$

and

$$x^{\text{Neumann}}_{\rm c}(n)={{e^{-\pi n/\delta}}\over{2\epsilon}}\Big[{{\Gamma(2+2i\delta)\Gamma({1\over2}-i\delta-\kappa)}\over{\Gamma(2-2i\delta)\Gamma({1\over2}+i\delta-\kappa)}}\Big]^{1/2i\delta}\ \ \ \ ;\ \ \ \ n\in\mathbb{Z} \tag{38}$$

for the dimensionless *discrete* radii of the horizonless rapidly-spinning exotic compact objects which can support the stationary bound-state massive scalar field configurations [45].

It is worth noting that, taking cognizance of Eq. 6.1.23 of [35], one deduces that $\Gamma(1+2i\delta)/\Gamma(1-2i\delta)=e^{i\phi_1}$, $\Gamma(2+2i\delta)/\Gamma(2-2i\delta)=e^{i\phi_2}$, and $\Gamma(1/2-i\delta-\kappa)/\Gamma(1/2+i\delta-\kappa)=e^{i\phi_3}$ for $\{\delta,\kappa\}\in\mathbb{R}$ [see Eq. (19)], where $\{\phi_1,\phi_2,\phi_3\}\in\mathbb{R}$. These relations imply that $\{x^{\text{Dirichlet}}_{\rm c}(n),x^{\text{Neumann}}_{\rm c}(n)\}\in\mathbb{R}$.

VI. THE EIKONAL LARGE-MASS $M\mu\gg1$ REGIME

In the present section we shall show that the analytically derived resonance spectra (37) and (38), which characterize the composed exotic-compact-object-linearized-massive-scalar-field configurations, can be further simplified in the asymptotic eikonal regime

$$m\gg1 \tag{39}$$

of large field masses [46].

In the regime (39) one finds the compact asymptotic expression [37, 47]

$$K_{mm}={5\over4}m^2-\bar\mu^2+\sqrt{{3\over4}m^2+\bar\mu^2}+O(1)\ , \tag{40}$$

which implies [see Eq. (17)]

$$\delta=\sqrt{{3\over4}m^2-\bar\mu^2-\sqrt{{3\over4}m^2+\bar\mu^2}}\cdot[1+O(\tau,\tau\varpi)]\ . \tag{41}$$

In addition, in the large-m regime one may use the asymptotic approximations [35]

$${{\Gamma(2i\delta)}\over{\Gamma(-2i\delta)}}=i\Big({{2\delta}\over{e}}\Big)^{4i\delta}\cdot[1+O(m^{-1})] \tag{42}$$

and

$${{\Gamma({1\over2}-i\delta-\kappa)}\over{\Gamma({1\over2}+i\delta-\kappa)}}=e^{2i\delta}(\delta^2+\kappa^2)^{-i\delta}e^{2i\pi\kappa(1-\theta)}[1+O(m^{-1})]\ \ \ \ \text{where}\ \ \ \ \theta\equiv\pi^{-1}\arctan(\delta/\kappa)\ . \tag{43}$$

Substituting (42) and (43) into the resonant spectra (37) and (38), one finds respectively the simplified expressions [48]

$$x_{\text{c}}^{\text{Dirichlet}}(n)=\frac{2e^{\pi[\kappa(1-\theta)-1/4-n]/\delta}}{e\sqrt{\delta^2+\kappa^2}\epsilon}\ \ \ \ ;\ \ \ \ n\in\mathbb{Z} \tag{44}$$

and

$$x_{\text{c}}^{\text{Neumann}}(n)=\frac{2e^{\pi[\kappa(1-\theta)+1/4-n]/\delta}}{e\sqrt{\delta^2+\kappa^2}\epsilon}\ \ \ \ ;\ \ \ \ n\in\mathbb{Z} \tag{45}$$

for the dimensionless discrete radii of the rapidly-spinning exotic compact objects which can support the stationary bound-state configurations of the spatially regular massive scalar fields.

VII. NUMERICAL CONFIRMATION

It is of physical interest to confirm the validity of the compact analytical formulas (37) and (38) for the discrete radii of the exotic compact objects which can support the stationary bound-state massive scalar field configurations. In Table I we present the dimensionless radii $x_{\text{c}}^{\text{analytical}}(n)$ of the horizonless exotic compact objects with reflecting Dirichlet boundary conditions as calculated from the analytically derived formula (37). We also present the corresponding dimensionless radii $x_{\text{c}}^{\text{numerical}}(n)$ of the rapidly-spinning exotic compact objects as obtained from a direct numerical solution of the compact resonance condition (23). The numerically computed roots of the Whittaker function $W_{\kappa,i\delta}(2\epsilon x)$ and its spatial derivative [see the analytically derived resonance equations (23) and (24)] were obtained directly from the WolframAlpha: Computational Knowledge Engine.

In the physically interesting $x_{\text{c}}\ll 1$ regime [49] [see Eq. (3)], one finds a remarkably good agreement between the approximated discrete radii of the horizonless exotic compact objects [as calculated from the analytical formula (37)] and the corresponding exact radii of the rapidly-spinning exotic compact objects [as obtained numerically from the characteristic resonance equation (23)]. It is worth emphasizing the fact that the characteristic dimensionless radii $x_{\text{c}}^{\text{Dirichlet}}(n)$ of the supporting exotic compact objects, as presented in Table I, conform to the analytically derived upper bound (33).

Formula	$x_{\text{c}}^{\text{Dir}}(n=0)$	$x_{\text{c}}^{\text{Dir}}(n=1)$	$x_{\text{c}}^{\text{Dir}}(n=2)$	$x_{\text{c}}^{\text{Dir}}(n=3)$	$x_{\text{c}}^{\text{Dir}}(n=4)$	$x_{\text{c}}^{\text{Dir}}(n=5)$
Analytical [Eq. (37)]	0.07998	0.02449	0.00750	0.00229	0.00070	0.00022
Numerical [Eq. (23)]	0.08690	0.02503	0.00754	0.00230	0.00070	0.00022

TABLE I: Composed stationary exotic-compact-object-massive-scalar-field configurations. We present the dimensionless discrete radii $x_{\text{c}}^{\text{Dirichlet}}(n)$ of the rapidly-spinning exotic compact objects with reflective Dirichlet boundary conditions as calculated from the analytically derived compact formula (37). We also present the corresponding dimensionless radii of the horizonless exotic compact objects as obtained from a direct numerical solution of the characteristic resonance condition (23). The data presented is for massive scalar field configurations with angular harmonic indices $l=m=10$, $M\omega=5$, and $\mu=1.5\omega$ [these field parameters correspond to $\epsilon=5.59$, $\kappa=-1.12$, and $\delta=2.65$, see Eq. (17)]. One finds a remarkably good agreement between the approximated discrete radii of the rapidly-spinning exotic compact objects [as calculated from the analytically derived compact formula (37)] and the corresponding exact radii of the horizonless exotic compact objects [as obtained from a direct numerical solution of the Dirichlet resonance equation (23)]. Note that the dimensionless radii $x_{\text{c}}^{\text{Dirichlet}}(n)$ of the supporting rapidly-spinning exotic compact objects conform to the analytically derived upper bound (33).

In Table II we display the dimensionless discrete radii $x_{\text{c}}^{\text{analytical}}(n)$ of the rapidly-spinning exotic compact objects with reflecting Neumann boundary conditions as calculated from the analytically derived formula (38). We also present the corresponding dimensionless radii $x_{\text{c}}^{\text{numerical}}(n)$ of the horizonless exotic compact objects as obtained numerically from the resonance condition (24). Again, in the physically interesting $x_{\text{c}}\ll 1$ regime [49] [see Eq. (3)], one finds a remarkably good agreement between the approximated discrete radii of the rapidly-spinning exotic compact objects [as calculated from the analytically derived formula (38)] and the corresponding exact radii of the horizonless exotic compact objects [as obtained numerically from the characteristic resonance equation (24)]. It is worth pointing out that the characteristic dimensionless radii $x_{\text{c}}^{\text{Neumann}}(n)$ of the supporting exotic compact objects, as presented in Table II, conform to the analytically derived upper bound (33).

Formula	$x_{\rm c}^{\rm Neu}(n=0)$	$x_{\rm c}^{\rm Neu}(n=1)$	$x_{\rm c}^{\rm Neu}(n=2)$	$x_{\rm c}^{\rm Neu}(n=3)$	$x_{\rm c}^{\rm Neu}(n=4)$	$x_{\rm c}^{\rm Neu}(n=5)$
Analytical [Eq. (38)]	0.13476	0.04125	0.01263	0.00387	0.00118	0.00036
Numerical [Eq. (24)]	0.16106	0.04291	0.01277	0.00388	0.00118	0.00036

TABLE II: Composed stationary exotic-compact-object-massive-scalar-field configurations. We present the dimensionless discrete radii $x_{\rm c}^{\rm Neumann}(n)$ of the horizonless exotic compact objects with reflective Neumann boundary conditions as calculated from the analytically derived formula (38). We also present the corresponding dimensionless radii of the rapidly-spinning exotic compact objects as obtained from a direct numerical solution of the characteristic resonance condition (24). The data presented is for massive scalar field configurations with angular harmonic indices $l=m=10$, $M\omega=5$, and $\mu=1.5\omega$ [these field parameters correspond to $\epsilon=5.59$, $\kappa=-1.12$, and $\delta=2.65$, see Eq. (17)]. One finds a remarkably good agreement between the approximated discrete radii of the horizonless exotic compact objects [as calculated from the analytically derived compact formula (38)] and the corresponding exact radii of the rapidly-spinning exotic compact objects [as obtained from a direct numerical solution of the Neumann resonance equation (24)]. Note that the dimensionless radii $x_{\rm c}^{\rm Neumann}(n)$ of the supporting rapidly-spinning exotic compact objects conform to the analytically derived upper bound (33).

VIII. SUMMARY AND DISCUSSION

Some candidate quantum-gravity models [1–13] have recently put forward the intriguing idea that quantum effects may prevent the formation of stable black-hole horizons. These models have suggested, in particular, that within the framework of a quantum theory of gravity, horizonless exotic compact objects may serve as alternatives to classical black-hole spacetimes.

Motivated by this intriguing prediction, we have raised here the following physically interesting question: Can horizonless exotic compact objects with reflective boundary conditions [13] (see also [1–12]) support spatially regular massive scalar field configurations in their exterior regions? In order to address this intriguing question, in the present paper we have solved *analytically* the Klein-Gordon wave equation for a stationary linearized scalar field of mass μ, proper frequency $\omega_{\rm c}$, and spheroidal harmonic indices (l,m) in the background of a rapidly-spinning [see Eq. (14)] exotic compact object of mass M and angular momentum $J=Ma$. The main physical results derived in the present paper are as follows:

(1) It was proved that the compact upper bound [see Eqs. (17) and (33)]

$$x_{\rm c}<\frac{\kappa+\sqrt{\kappa^2+\delta^2}}{\epsilon} \tag{46}$$

on the dimensionless radius of the spinning exotic compact object provides a necessary condition for the existence of the composed stationary exotic-compact-object-massive-scalar-field configurations.

(2) We have shown that, for a given set (μ,l,m) of the physical parameters which characterize the massive scalar field, there exists a *discrete* spectrum of radii $\{r_{\rm c}(\mu,l,m,M,a;n)\}$ of the rapidly-spinning exotic compact objects which can support the stationary bound-state massive scalar field configurations. In particular, it was proved analytically that the compact resonance conditions [see Eqs. (23) and (24)]

$$W_{\kappa,i\delta}(2\epsilon x_{\rm c}^{\rm Dirichlet})=0 \quad ; \quad \frac{d}{dx}[x^{-1}W_{\kappa,i\delta}(2\epsilon x)]_{x=x_{\rm c}^{\rm Neumann}}=0 \tag{47}$$

determine the characteristic discrete sets of supporting radii which characterize the rapidly-spinning reflective exotic compact objects.

(3) It was explicitly shown that the physical properties of the composed stationary exotic-compact-object-massive-scalar-field configurations can be studied *analytically* in the physically interesting $x_{\rm c}\ll 1$ regime [49] [see Eq. (3)]. In particular, we have used analytical techniques in order to derive the compact formula [see Eqs. (37) and (38)]

$$x_{\rm c}(n)=\frac{e^{-\pi n/\delta}}{2\epsilon}\Big[\nabla\frac{\Gamma(1+2i\delta)\Gamma(\frac{1}{2}-i\delta-\kappa)}{\Gamma(1-2i\delta)\Gamma(\frac{1}{2}+i\delta-\kappa)}\Big]^{1/2i\delta} \quad ; \quad \nabla=\begin{cases}1 & \text{Dirichlet B. C.}\\ \frac{1+2i\delta}{1-2i\delta} & \text{Neumann B. C.}\end{cases} \tag{48}$$

for the discrete spectra of reflecting radii $\{x_{\rm c}(\mu,l,m,M,a;n)\}$ which characterize the rapidly-spinning exotic compact objects that can support the stationary bound-state massive scalar field configurations.

(4) We have explicitly shown that, in the physically interesting $x_{\rm c}\ll 1$ regime [49], the analytically derived formulas (37) and (38) for the discrete radii of the horizonless exotic compact objects that can support the stationary bound-state massive scalar field configurations agree with direct numerical computations of the corresponding radii of the rapidly-spinning exotic compact objects.

(5) It is worth emphasizing again that the composed spinning-exotic-compact-object-massive-scalar-field configurations that we have studied in the present paper, like the composed spinning-black-hole-massive-scalar-field configurations studied recently in [14, 15], owe their existence to the intriguing physical phenomenon of superradiant scattering in rotating spacetimes [22, 23]. In particular, the spatially regular stationary massive scalar field configurations (22) are characterized by the critical (marginal) frequency ω_{c} for the superradiant scattering phenomenon of bosonic (integer-spin) fields in spinning spacetimes [see Eqs. (10) and (14)].

(6) Finally, we would like to stress the fact that, combining the results of the present paper with the results presented in [14, 15], one deduces that both spinning black holes and horizonless spinning exotic compact objects with reflecting surfaces can support spatially regular configurations of stationary massive scalar fields in their exterior regions. It is important to note, however, that for given physical parameters $\{M, a\}$ of the central supporting object, the discrete resonant spectra $\{\mu(M, a)\}$ of the allowed field masses are different. Thus, our analytically derived theoretical results may one day be of practical observational importance since the different resonant spectra of the external stationary massive scalar field configurations may help astronomers to distinguish horizonless spinning exotic compact objects from genuine black holes.

In particular, it is worth pointing out that the regime of existence of the composed spinning-black-hole-massive-scalar-field configurations is characterized, in the rapidly-rotating $a/M \to 1$ limit, by the dimensionless relations $m/2 < M\mu < m/\sqrt{2}$ [14]. On the other hand, the composed spinning-exotic-compact-object-massive-scalar-field configurations exist for all real values of the physical parameters δ and κ [see Eq. (19)]. Taking cognizance of the large-m relation (41), one deduces that the horizonless spinning configurations studied in the present paper have the *larger* regime of existence $m/2 < M\mu < \sqrt{3}m/2$.

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[41] Here $M_{\kappa,\beta}(z)$ and $W_{\kappa,\beta}(z)$ are the Whittaker function of the first kind and the Whittaker function of the second kind [35], respectively.

[42] One may assume, without loss of generality, that $\delta > 0$.

[43] Here we have used Eqs. 13.1.4, 13.1.8, 13.1.32, and 13.1.33 of [35].

[44] Here we have used the relation $1 = e^{-2i\pi n}$, where the integer n is the resonance parameter of the composed exotic-compact-object-massive-scalar-field configurations.

[45] It is worth noting that the analytically derived resonance spectra (37) and (38), which characterize the composed exotic-compact-object-linearized-massive-scalar-field configurations, can be further simplified in the asymptotic regime $\omega^2/\mu^2 \to 1^-$ of marginally-bound massive scalar field configurations [see Eq. (8)]. These marginally-bound resonances correspond to $\kappa \to \infty$ [see Eq. (17)], in which case one may use the asymptotic approximation $\Gamma(1/2 - i\delta - \kappa)/\Gamma(1/2 + i\delta - \kappa) = \kappa^{-2i\delta} e^{2i\pi\kappa}[1 + O(\kappa^{-1})]$ [35]. Substituting this dimensionless ratio into (37) and (38), one finds the simplified resonant spectra $x_{\text{c}}^{\text{Dirichlet}}(n) = \frac{2e^{\pi(\kappa-n)/\delta}}{m^2}\Big[\frac{\Gamma(1+2i\delta)}{\Gamma(1-2i\delta)}\Big]^{1/2i\delta}$ and $x_{\text{c}}^{\text{Neumann}}(n) = \frac{2e^{\pi(\kappa-n)/\delta}}{m^2}\Big[\frac{\Gamma(2+2i\delta)}{\Gamma(2-2i\delta)}\Big]^{1/2i\delta}$ [Here we have used the dimensionless relation $M\omega = \frac{1}{2}m \cdot [1 + O(\tau, \tau\varpi)]$ in the regime (14) of rapidly-spinning exotic compact objects]. Note also that $K_{mm} = m(m+1) + O[a^2(\mu^2 - \omega^2)]$ in the marginally-bound $\omega^2/\mu^2 \to 1^-$ regime [29, 30, 33–37], which implies $\delta_{mm} = (m^2/2 - m - 1/4)^{1/2}$ [see Eq. (17)].

[46] Note that the large-m regime (39) corresponds to the large-frequency and large-mass regimes $M\mu > M\omega \gg 1$ [see Eqs.

(8), (10), and (14)]. Likewise, the large-m regime (39) corresponds to $\delta \gg 1$ and $\kappa \gg 1$ [see Eq. (17)].

[47] It is worth emphasizing that the asymptotic large-m expression (40) for the eigenvalues of the characteristic spheroidal angular equation (7) is valid in the regime $-a^2(\mu^2-\omega^2)<m^2$ [37]. Taking cognizance of the inequality $\omega^2<\mu^2$ [see (8)], which characterizes the bound-state resonances of the massive scalar fields, one immediately concludes that the requirement $-a^2(\mu^2-\omega^2)<m^2$ is trivially satisfied by the stationary spatially regular scalar configurations.

[48] Here we have used Eq. 6.1.15 of [35], which implies $\Gamma(1+2i\delta)/\Gamma(1-2i\delta)=-\Gamma(2i\delta)/\Gamma(-2i\delta)$ and $\Gamma(2+2i\delta)/\Gamma(2-2i\delta)=\Gamma(2i\delta)/\Gamma(-2i\delta)\cdot[1+O(m^{-1})]$ in the asymptotic large-m regime (39).

[49] It is worth emphasizing again that the strong inequality $x_{\text{c}} \ll 1$ characterizes exotic compact objects whose quantum reflective surfaces are located in the radial vicinity of the would-be classical black-hole horizons [see Eq. (3)]. This small-x_{c} regime corresponds to the physically interesting model of spinning exotic compact objects recently studied numerically in [13] (see also [1–12]).