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ON GLOBAL DYNAMICS OF THE MAXWELL-KLEIN-GORDON EQUATIONS

SHIWU YANG AND PIN YU

ABSTRACT. On the three dimensional Euclidean space, for data with finite energy, it is well-known that the Maxwell-Klein-Gordon equations admit global solutions. However, the asymptotic behaviours of the solutions for the data with non-vanishing charge and arbitrary large size are unknown. It is conjectured that the solutions disperse as linear waves and enjoy the so-called peeling properties for pointwise estimates. We provide a gauge independent proof of the conjecture.

The Maxwell-Klein-Gordon (MKG) equations are a nonlinear system modeling the motion of a charged particle moving in an electric-magnetic field. From the physics point of view, the electric-magnetic field fades away and the final state of the particle must be static. The mathematical interpretation of this conjecture is that the MKG equations admit global *decaying* solutions. The global solvability of this system has been well understood since the work of Eardley-Moncrief in 1980's. But very little is known on the aysmptotic behaviours of the solutions for the general large initial data. The aim of our current study is to show that the solutions enjoy the peeling estimates for the massless case.

We begin with a quick introduction to the MKG equations. Let $A = A_\mu dx^\mu$ be a $\mathbb{R}$-valued connection 1-form for a given complex line bundle $\mathbf{L}$ over the Minkowski spacetime $\mathbb{R}^{3+1}$. The curvature 2-form $F = \big(F_{\mu\nu}\big)$ associated to A is simply dA, i.e., $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. In particular, F is a closed 2-form or equivalently $\nabla_{[\alpha} F_{\beta\gamma]} = 0$. The pair of square brackets denote the anti-symmetrization of the three indices α, β and γ. We will use Einstein summation convention throughout the paper. The above equation is also called the Bianchi equation of F. It is also equivalent to

$$\nabla^\mu {}^*F_{\mu\nu} = 0,$$

where ${}^*F_{\mu\nu}$ is the Hodge dual of $F_{\mu\nu}$. We recall that the Hodge dual ${}^*G_{\alpha\beta}$ of a 2-form $G_{\alpha\beta}$ is $\frac{1}{2}\mathscr{E}_{\alpha\beta\gamma\delta}G^{\gamma\delta}$, where $\mathscr{E}_{\alpha\beta\gamma\delta}$ is the volume form of the Minkowski metric $m_{\alpha\beta}$.

A section of the bundle $\mathbf{L}$ can be represented by a $\mathbb{C}$-valued function ϕ. The covariant derivative of ϕ with respect to A is

$$D_\mu \phi = \partial_\mu \phi + \sqrt{-1} A_\mu \cdot \phi.$$

The curvature form measures the non-commutativity of the covariant derivatives

$$[D_\mu, D_\nu]\phi = \sqrt{-1} F_{\mu\nu} \cdot \phi.$$

The massless MKG equations is a system of equations for a connection A on $\mathbf{L}$ and a section ϕ of $\mathbf{L}$:

$$\begin{cases} \nabla^\mu F_{\mu\nu} = -J_\nu, \\ \Box_A \phi = 0, \end{cases} \tag{0.1}$$

where $J_\mu = \Im(\phi \cdot \overline{D_\mu \phi})$ is called the *current* and $\Box_A = D^\mu D_\mu$. It can be derived as the Euler-Lagrange equations for the action

$$\mathcal{L}(A, \phi) = \frac{1}{2}\int_{\mathbb{R}^{3+1}} D^\mu \phi \cdot \overline{D_\mu \phi} + \frac{1}{4}\int_{\mathbb{R}^{3+1}} F^{\mu\nu} F_{\mu\nu}.$$

We use the volume form of the Minkowski metric in the action. The system is a $\mathbf{U}(1)$-gauge theory, namely, if (A, ϕ) is a solution of (0.1), then $(A - d\chi, e^{i\chi}\phi)$ is also a solution for any smooth function χ.

The total charge of the system is given by

$$q_0 = \frac{1}{4\pi}\int_{\mathbb{R}^3} \mathbf{div} E dx = \frac{1}{4\pi}\int_{\mathbb{R}^3} \Im(\overline{D_t\phi} \cdot \phi) = \frac{1}{4\pi}\lim_{r\to\infty}\int_{|\omega|=1} r^2 E_i \frac{x_i}{r} d\omega,$$

where $E_i = F_{0i}$ is the electric field. It is easy to check that the total charge is conserved. This in particular implies that the electric field E has the nontrivial tail $q_0 r^{-3} x$ at any fixed time.

The pioneering works [7] and [8] of Eardley-Moncrief established the celebrated global existence result to the general Yang-Mills-Higgs equations with sufficiently smooth initial data. Around ten years later, by introducing the weighted Sobolev spaces, Klainerman-Machedon systematically studied the bilinear estimates of the null forms. As a consequence, they derived the notable global existence result for data merely bounded in the energy space. The idea of proving bilinear estimates of null form introduced in [11] leads to an revolution on the global well-posedness of PDEs of classical field theory, such as MKG equations, Yang-Mills equations, wave maps, etc., aiming at studying low regularity initial data in order to construct global solutions, see [12] and references therein. For a more recent and comprehensive summary of the progresses along this line, we refer to the work of Oh-Tataru [18]. The common feature of all these works is to construct a local solution with rough data so that the global well-posedness follows from conserved energy quantities. However regarding the global dynamics of the solutions, very little can be obtained through this approach.

The long time dynamics of solutions of MKG equations have only been well understood for sufficiently small initial data or data which are essentially compactly supported. The robust vector field method introduced by Klainerman in [10] has been successfully applied to derive the decay estimates for linear fields in [5] or nonlinear spin fields in [21] with small initial data. If the data are compactly supported, one can also use the conformal compactification method (see e.g. [2]) and this approach requires strong decay of the initial data, which in particular forces the total charge to be vanishing. To tackle the general case with nonzero charge, Shu in [22] proposed a frame work but without details. A complete proof towards this direction was contributed by Lindblad-Sterbenz in [16], also see a recent work [1] of Bieri-Miao-Shahshahani. However all these works are restricted to the small data regime or can be viewed as global stability problems of trivial solutions.

As for the large data problem, by using the conformal compactification method together with Eardley-Moncrief's results, Petrescu in [19] obtained the asymptotic decay properties of solutions to MKG equations with essentially compactly supported data, i.e., the scalar field has compact support and the Maxwell field is electrostatic outside the support. A similar result for Yang-Mills equation on $\mathbb{R}^{3+1}$ was obtained by Georgiev-Schirmer in [9] but with spherically symmetric data bounded in the conformal energy space (in particular charges must be vanishing!). For general initial data, the global asymptotic behaviour is only partially known. A Morawetz type of integrated local energy estimate was obtained by Psarelli in [20] for solutions of massive Maxwell-Klein-Gordon equations with data bounded in the energy space. For massless MKG equations, the first author Yang in [24] derived the stronger inverse polynomial decay of the energy flux through outgoing null cones with data bounded in weighted energy space. Both results allow the existence of nonzero charges. However the decay estimates in [20] are too weak to see the effect of the charges while Yang's work in addition affirmatively answered a conjecture of Shu in [21] that the nonzero charge can only affect the asymptotic behaviours of the solutions outside a forward light cone. Another consequence of the method used in [24] allowed the author to improve the small data results to data merely small on the scalar field while the Maxwell field can be arbitrarily large, see details in [23]. This result can be interpreted as the global nonlinear stability of linear Maxwell fields.

It is of great interests to remove the restriction of essentially compactly supported data without any smallness assumption. This is the final state conjecture of charged scalar fields: the solution should eventually decay as long as it decays suitably initially. In this work, we will propose a sequence of new ideas to handle the long range effect of the large charge and we will prove this conjecture for rapidly decaying data in this paper.

1. Introduction to the main result

Throughout the paper, we use the following conventions:

- The Greek letters $\alpha, \beta, \cdots$ denote indices from 0 to 3. The capital Latin letters $A, B, \cdots$ denote indices from 1 to 2. The little Latin letters $i, j, k, \cdots$ denote indices from 1 to 3.
- (ϕ, F) is a *given* finite energy smooth solution of the MKG equations. It exists globally and remains smooth according to the classical result of Klainerman-Machedon [11].
- The letter f denotes an *arbitrary* section of the bundle $\mathbf{L}$ (it may not be ϕ). The letter G denotes an *arbitrary* 2-form $G_{\mu\nu}$ (it may not be F).
- We define $\psi = r\phi$.

We use two coordinate systems on the Minkowski spacetime $\mathbb{R}^{3+1}$: the Cartesian coordinates $(x^0 = t, x^1, x^2, x^3)$ and the polar coordinates (t, r, ϑ). The optical functions u and v are defined as

$$u = \frac{1}{2}(t - r), \quad v = \frac{1}{2}(t + r), \quad u_+ = 1 + |u|, \quad v_+ = 1 + |v|.$$

A *null frame* is defined by $(e_1, e_2, e_3 = \underline{L}, e_4 = L)$, where $L = \partial_t + \partial_r$, $\underline{L} = \partial_t - \partial_r$ and e_1, e_2 is an orthonormal complement of L and $\underline{L}$.

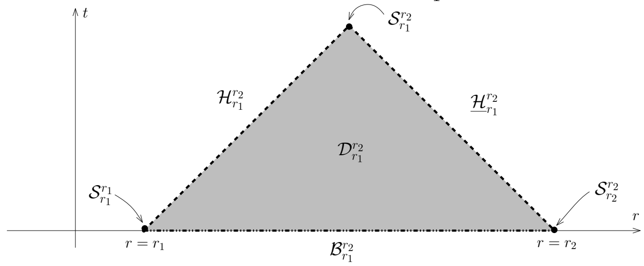
The level sets of u and v define (locally) null foliations of the Minkowski spacetime. Given $r_2 > r_1 > 0$, we define the outgoing (or incoming) null hypersurfaces $\mathcal{H}_{r_1}^{r_2}$ (or $\underline{\mathcal{H}}_{r_2}^{r_1}$) as

$$\mathcal{H}_{r_1}^{r_2} := \big\{(t, r, \vartheta) \,\big|\, t \geq 0, u = -\frac{1}{2}r_1, r_1 \leq r \leq r_2\big\} \quad \text{or} \quad \underline{\mathcal{H}}_{r_2}^{r_1} := \big\{(t, r, \vartheta) \,\big|\, t \geq 0, v = \frac{1}{2}r_2, r_1 \leq r \leq r_2\big\}$$

respectively. On the initial time slice $\big\{t = 0\big\}$ where the Cauchy datum is given, we define

$$\mathcal{B}_{r_1}^{r_2} := \big\{(t, r, \vartheta) \,\big|\, t = 0, r_1 \leq r \leq r_2\big\}.$$

In the limiting case where $r_2 = \infty$, we write $\mathcal{H}_{r_1} = \mathcal{H}_{r_1}^{\infty}$, $\underline{\mathcal{H}}_{r_1} = \underline{\mathcal{H}}_{r_1}^{\infty}$ and $\mathcal{B}_{r_1} = \mathcal{B}_{r_1}^{\infty}$. Three hypersurfaces $\mathcal{H}_{r_1}^{r_2}$, $\underline{\mathcal{H}}_{r_2}^{r_1}$ and $\mathcal{B}_{r_1}^{r_2}$ bound a spacetime region and it is denoted by $\mathcal{D}_{r_1}^{r_2}$. In the following picture, the gray region is $\mathcal{D}_{r_1}^{r_2}$. The truncated light cones $\mathcal{H}_{r_1}^{r_2}$ and $\underline{\mathcal{H}}_{r_2}^{r_1}$ are denoted by the dashed line segments. Their intersection is a 2-sphere of radius $\frac{r_1+r_2}{2}$ and it is the tip of $\mathcal{D}_{r_1}^{r_2}$ in the picture. We denote this sphere by $\mathcal{S}_{r_1}^{r_2}$. The dashed-dotted line segment on the bottom is $\mathcal{B}_{r_1}^{r_2}$.



In the null frame, we have $\nabla_L L = \nabla_L \underline{L} = \nabla_{\underline{L}} L = \nabla_{\underline{L}} \underline{L} = 0$. Moreover, we have

$$\nabla_{e_A} L = \frac{1}{r} e_A, \ \nabla_{e_A} \underline{L} = -\frac{1}{r} e_A, \ \ \nabla_{e_A} e_B = \nabla\!\!\!/_{e_A} e_B + \frac{1}{2r} g\!\!\!/_{AB}(\underline{L} - L),$$

where $\nabla\!\!\!/_{e_A} e_B$ is the projection of $\nabla_{e_A} e_B$ to a 2-sphere $\mathcal{S}_{r_1}^{r_2}$ (or to the span of e_1 and e_2) and $g\!\!\!/_{AB}$ is the restriction of the Minkowski metric to $\mathcal{S}_{r_1}^{r_2}$.

We can decompose $G_{\mu\nu}$ with respect to the null frame:

$$\alpha(G)_A := G(L, e_A), \ \underline{\alpha}(G)_A := G(\underline{L}, e_A), \ \rho(G) := \frac{1}{2} G(\underline{L}, L), \ \sigma(G)_{AB} := G_{AB}.$$

For the special case $G_{\mu\nu} = F_{\mu\nu}$, we write

$$\alpha_A = F(L, e_A), \ \underline{\alpha}_A = F(\underline{L}, e_A), \ \rho := \frac{1}{2} F(\underline{L}, L), \ \sigma_{AB} := F_{AB}.$$

Since σ_{AB} is a 2-form on $\mathcal{S}_{r_1}^{r_2}$, there exists a function σ so that $\sigma_{AB} = \sigma \epsilon\!\!/_{AB}$ where $\epsilon\!\!/_{AB}$ is the volume form on $\mathcal{S}_{r_1}^{r_2}$. For the Hodge dual ${}^*F$ of F, if we denote ${}^*\alpha_A = -\epsilon\!\!/_A{}^B \alpha_B$ (the Hodge dual of α on $\mathcal{S}_{r_1}^{r_2}$), we have

$$\alpha_A({}^*F) = {}^*\alpha_A, \ \underline{\alpha}_A({}^*F) = -{}^*\underline{\alpha}_A, \ \rho({}^*F) = \sigma, \ \sigma({}^*F)_{AB} = -\rho \epsilon\!\!/_{AB}.$$

1.1. **The main theorem.** We consider Cauchy problem to (0.1) with initial data given by

$$\phi_0(x) = \phi(0, x), \ \phi_1(x) = \partial_t \phi(0, x), \ E_i^{(\mathrm{ini})}(x) = E_i(0, x), \ B_i^{(\mathrm{ini})}(x) = B_i(0, x).$$

The initial data set $(\phi_0, \phi_1, \ E^{(\mathrm{ini})}, \ B^{(\mathrm{ini})})$ is said to be *admissible* if it satisfies the compatibility condition

$$\mathbf{div}(E^{(\mathrm{ini})}) = \Im(\phi_0 \cdot \overline{\phi_1}), \quad \mathbf{div}(B^{(\mathrm{ini})}) = 0, \tag{1.1}$$

To impose precise assumptions on the initial data, split the electric field $E^{(\mathrm{ini})}$ into the divergence free part E^{df} and the curl free part E^{cf}, that is,

$$\mathbf{div}(E^{df}) = 0, \quad \mathbf{curl}(E^{cf}) = 0, \quad E^{(\mathrm{ini})} = E^{df} + E^{cf}.$$

From the above constraint equation, E^{cf} is uniquely determined by $\Im(\phi_0 \cdot \overline{\phi_1})$. In particular we can freely assign $(\phi_0, \phi_0, E^{df}, B^{(\mathrm{ini})})$ as long as E^{cf}, $B^{(\mathrm{ini})}$ are divergence free on the initial hypersurface $\{t = 0\}$. We require this part of data decay rapidly and belong to certain weighted Sobolev space. However, since E^{cf} satisfies an elliptic equation on $\mathbb{R}^3$, it has a nontrivial tail $\frac{q_0 x}{r^3}$ even with (ϕ_0, ϕ_1) compactly supported. To describe the asymptotic behaviour of the solutions, we need to precisely capture the asymptotic behaviour of the solution contributed by the charge. By formally expanding the Green's function for Laplacian:

$$|x - y|^{-1} = |x|^{-1} + |x|^{-3} x \cdot y + \sum_{i,j=1}^{3} \frac{1}{2} |x|^{-3} (3|x|^{-2} x_i x_j - \delta_{ij}) y_i y_j + o(|y|^2),$$

we can define a potential function $V(x)$ as

$$V(x) = |x|^{-1} \frac{1}{4\pi} \int_{\mathbb{R}^3} (1 + |x|^{-2} x \cdot y + \frac{1}{2} |x|^{-2} (3|x|^{-2} (x \cdot y)^2 - |y|^2)) \Im(\phi_0 \cdot \bar{\phi}_1) dy, \quad |x| > 0.$$

The potential is well defined if the initial data (ϕ_0, ϕ_1) of the scalar field decay rapidly. With the potential $V(x)$, we can define the general charge 2-form $F[q_0]$ with components

$$F[q_0]_{0i} = E_i[q_0] = \partial_i V(x), \quad F[q_0]_{ij} = 0.$$

It is straightforward to check that $F[q_0]$ satisfies the linear Maxwell equation on the region away from the axis $\{x = 0\}$. Moreover, there is a constant C, depending only on ϕ_0 and ϕ_1, so that

$$|\rho(F[q_0])| \leq C r^{-2}, \quad |\underline{\alpha}(F[q_0])| = |\alpha(F[q_0])| \leq C r^{-3}, \quad |\sigma(F[q_0])| = 0. \tag{1.2}$$

We remark that most commonly one uses $F[q_0]=\frac{q_0}{r^2}dt\wedge dr$ to denote the charge part near special infinity and it is a special case of the above construction.

Let ε_0 be a small positive constant (say 10^{-2}). We assume that the initial data is bounded in the following gauge invariant weighted Sobolev norm

$$\begin{aligned} C_0:=\sum_{k\leq 2}\int_{\mathbb{R}^3}\Big[(1+r)^{2k+6+8\varepsilon_0}\big(|D^{k+1}\phi_0|^2+|D^k\phi_1|^2+|\nabla^k\big(E^{(\mathrm{ini})}-E[q_0]\mathbf{1}_{|x|\geq 1}\big)|^2 \\ +|\nabla^k B^{(\mathrm{ini})}|^2\big)+r^{4+8\varepsilon_0}|\phi_0|^2\Big]dx. \end{aligned} \tag{1.3}$$

The main theorem of the paper is as follows:

Theorem 1.1 (Main result). *Consider the Cauchy problem to the massless MKG equation* (0.1) *with admissible initial data* $(\phi_0,\phi_1,E_i^{(ini)},B_i^{(ini)})$ *bounded in the above weighted norm* (1.3). *Then there is a global in time solution* (ϕ,F) *satisfying the following pointwise peeling estimates*

$$\begin{aligned} &|\phi|\leq Cu_+^{-1}v_+^{-1},\quad |D_L(r\phi)|\leq Cu_+^{-1}v_+^{-2},\quad |\alpha(\mathring{F})|\leq Cu_+^{-1}v_+^{-3},\\ &|\rho(\mathring{F})|+|\sigma(\mathring{F})|+|\not{D}\phi|\leq Cu_+^{-2}v_+^{-2},\quad |\underline{\alpha}(\mathring{F})|+|D_{\underline{L}}\phi|\leq Cu_+^{-3}v_+^{-1}. \end{aligned} \tag{1.4}$$

for some constant C *depending only on* C_0, *where* $\mathring{F}=F-F[q_0]\mathbf{1}_{\{1+t\leq|x|\}}$ *with* $\mathbf{1}_{\{1+t\leq|x|\}}$ *the characteristic function of the exterior region* $\{(t,x)|t+1\leq|x|\}$.

We give several remarks.

Remark 1.2. *There is no restriction on the size or on the support of the data. In particular, the charge* q_0 *can be large. Besides the above pointwise estimates, uniform energy estimates as well as weighted energy estimates can also be derived in the course of the proof.*

Remark 1.3. *The peeling estimates* (1.4) *for the chargeless part of the solution together with the trivial bound* (1.2) *of the charge part describe the asymptotic behaviour of the full solution in the exterior region. Moreover the estimate implies that the nontrivial charge can only affect the asymptotic behaviour of the solution in the exterior region. This confirms the conjecture of Shu in* [21].

Remark 1.4. *There is a heuristic explanation of the construction of the charge part* $F[q_0]$ *from the dipole expansion perspective: if we expand the Maxwell field* F *in a Taylor series near spatial infinity* $r=\infty$ *as*

$$F=F_2+F_3+F_4+F_5+\cdots,$$

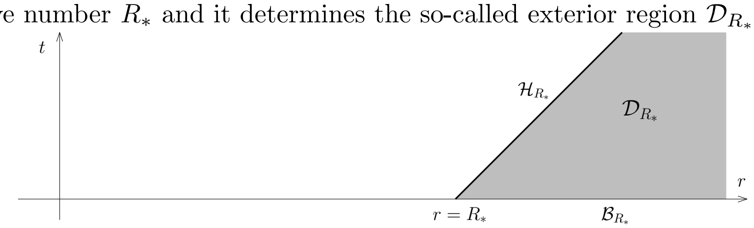
where $F_k=O(r^{-k})$. *The formal expansion of the Green function gives the* $F[q_0]=F_2+F_3+F_4$. *In this work we require that the perturbation starts from* F_5. *Indeed, the main reason for doing this is to make* $F-F[q_0]$ *decay sufficiently fast initially so that the chargeless part is bounded in the weighted Sobolev norm defined in* (1.3).

Remark 1.5. *Regarding the dependence of the constant* C *on the size of the initial data, our proof can easily imply that* C *depends exponentially on the zeroth order weighted energy (without derivative of the initial data) but polynomially on the higher order weighted energies. Simply from the charge part, it seems that exponential dependence on the zeroth order energy can not be improved. However from the point of view of the bilinear estimates in* [11], *we conjecture that the dependence on higher order energy should be linear.*

1.2. **An outline of the proof: difficulties, ideas and novelties.** The proof uses almost all the existing techniques and results for Maxwell-Klein-Gordon equations: the vector field method, the conformal compactification, the conformal analogues in the vector field method and the low regularity existence results of Klainerman-Machedon. Besides these, we will also introduce new commutation vector fields, new null forms and study some new structure of the nonlinearities. In the rest of the section, we will first sketch the proof in three steps. Then, we will present the difficulties in each step and provide heuristic ideas to handle these difficulties. Finally, we will summarize some new aspects of the proof.

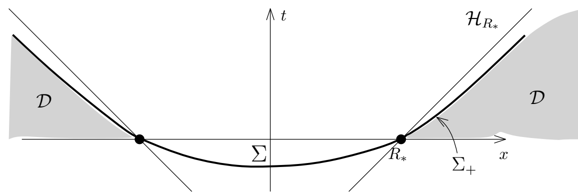
1.2.1. *The structure of the proof.* The proof consists of three steps:

Step 1 We take a positive number R_* and it determines the so-called exterior region $\mathcal{D}_{R_*}$ (grey part).



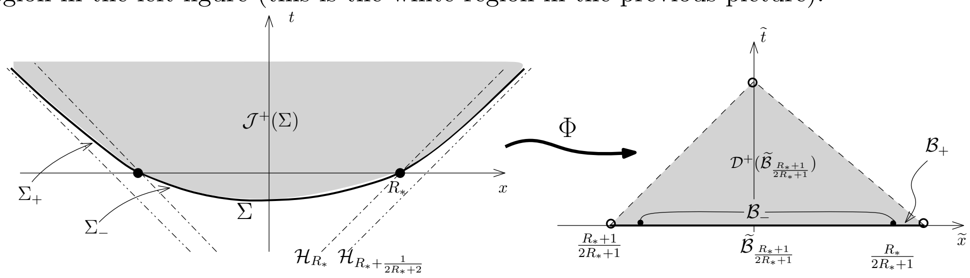
For large R_*, by restricting data on the region where $r\geq R_*$, i.e., $\mathcal{B}_{R_*}$ (as the bottom of the grey region), we can assume the chargeless part of the restricted data is small. Since the grey region is the domain of dependence of $\mathcal{B}_{R_*}$, the solution in $\mathcal{D}_{R_*}$ is completely determined by the restricted data on $\mathcal{B}_{R_*}$. We therefore study the long time behaviour of solutions of MKG equations in the grey region $\mathcal{D}_{R_*}$ with data small in the chargeless part. We emphasize that this is not a small data problem as the charge part of the solution is large and is independent of the radius R_*.

Step 2 This step connects the first step to the third. First of all, we will carefully choose a hyperboloid in $\mathcal{D}_{R_*}$ (on which we have precise control on the solution from the previous step). This hypersurface is denoted by Σ_+ in the next picture.



The solution restricted on this hyperboloid can be viewed as initial datum for the solution in the interior region which is unknown so far. This step is devoted to showing that the solution obtained from the previous step is sufficient regular on Σ_+ so that we can conduct the next step.

Step 3 In this last step, we will study the asymptotics of the solution in the causal future $\mathcal{J}^+(\Sigma)$ which is the grey region in the left figure (this is the white region in the previous picture).



The hypersurface Σ consists of two parts: Σ_- and Σ_+. Since Σ_- is finite, the solution on Σ_- can be well controlled by the data on the compact region $\{t = 0, |x| \leq R_*\}$. This indeed follows from the classical result of Eardley-Moncrief or the result of Klainerman-Machedon. Together with Step 2, the restriction of the solution on Σ will be well-understood in terms of the initial data.

Then we will perform a conformal transformation Ψ to map $\mathcal{J}^+(\Sigma)$ to a backward finite light cone (the grey cone on the right hand side of the picture). The hypersurface Σ will be mapped to the bottom of the cone. By multiplying conformal factors appropriately, the global dynamics of solutions to MKG equations defined on the left of the picture is then reduced to understanding the solution to MKG equations defined on the right of the picture. The estimates from Step 2 provide a bound of the H^2-norm of the solution on the bottom of the cone on the right hand side of the picture. This allows us to use the classical theory of Klainerman-Machedon to bound the solution on the cone up to two derivatives hence the L^∞ norm of the solution. Finally, we undo the conformal transformation by rewriting the solution on the left hand side in terms of the solutions on the right hand side. The conformal factors then give the decay estimates of the solution in $\mathcal{J}^+(\Sigma)$. Together with the decay estimates from Step 1, we can derive the peeling estimates in the main theorem.

1.2.2. *Difficulties in the proof.* We list several difficulties which did not appear in previous works on MKG equations. We would like to emphasize that the first difficulty (the largeness of charge) listed below is related to all the rest. The remaining difficulties arise in course of the resolution of the first one. We also want to point out that the most difficult part of the proof is Step 1.

(1) The large nonzero charge.

Although the energy norm of the chargeless part of the data in Step 1 is small, the charge q_0 can be large. First of all, the traditional conformal compactification method used in [3] by Christodoulou-Bruhat requires strong decay of the data which forces the charge to be vanishing. Secondly, the presence of nonzero charge may cause a logarithmic divergence in the energy estimates, see a more thorough discussion in the work [16] of Lindblad-Sterbenz for the purely small data case. The error term caused by the charge can in fact be absorbed if the charge is sufficiently small. We overcome this large charge difficulty by using the method developed by Yang in [24].

We would also like to compare this charge difficulty with the massive case of recent work [15] of Klainerman-Wang-Yang, in which they studied the massive MKG equations with small initial data. Their method also applies to the case with arbitrary large charge. However due to the existence of mass which gives control of the scalar field itself, the effect of nonzero charge can be easily controlled (see more detailed discussion in the next subsection). The main difficulty there, however, lies in the inconsistent asymptotic behaviours of Maxwell fields and solutions of Klein-Gordon equations.

(2) The sharp peeling estimates.

Since in Step 3 we have to compactify $\mathcal{J}^+(\Sigma)$, the estimates for the solution obtained from Step 1 on Σ must be sufficiently regular so that the solution on its conformal compactification are bounded in the right Sobolev spaces. In particular it requires to obtain the sharp decay estimates such as $D_L(r\phi) = O(r^{-2})$ and $\alpha = O(r^{-3})$ along outgoing light cones. So far as we know, even for the small data regime (with small charge), these estimates are unknown.

(3) New commutators to prove the necessary sharp peeling estimates.

The idea to obtain the above sharp peeling estimates is straightforward: we need to put more r-weights in the usual energy estimates so that the r-weights will be converted into extra decay via Sobolev inequality. We will use the conformal Morawetz vector field K as commutators. This vector field is of order 2 in terms of weights r and is used traditionally only as multipliers. In such a way, the structure of the nonlinear terms after commutation becomes the primary concern and we will show that it has some new null structure.

(4) The hidden null structure of the MKG equations related to commutators.

This is related to point (2) above. When one commutes vector fields with the MKG equations, although it may generate many error terms, one needs to at least make sure some of the fundamental structures remain unchanged. Very often, these structures are important in the analytic perspective and more precisely they should be phrased in such a way that they fit into the energy estimates. We will show that there is a new null structure of the nonlinear terms which is invariant after commuting correct vector fields. Also, there is another important type structure, which we will call it *reduced structure*, also remains unchanged after commutations.

(5) The choice of conformal compactifications.

The presence of nonzero charge prevents us to use the usual Penrose type compactification for the entire spacetime (see [3]): the ρ-component of the Maxwell field behaves as $\frac{q_0}{r^2}$ which cannot be compactified near the spatial infinity. However this effect of charge does not propagate from the spatial infinity to the future null infinity so that we can indeed perform a conformal transformation inside a null cone to avoid spatial infinity.

(6) Precise energy estimates on the hypersurface Σ in Step 2.

Since Σ is a hyperboloid in Minkowski spacetime, the energy estimates on Σ, especially those needed in the Klainerman-Machedon theory after the compactification, are not straightforward. Nevertheless, this part is less serious compared to all the previous ones and can be derived by using the classical energy estimates in a geometric way.

1.2.3. *Key ideas and novelties of the proof.* In the subsection, we will list all the ideas and new features of the proof in order to deal with the difficulties mentioned in the previous subsection.

(1) The reduced structure and converting spatial decay against the logarithmic growth.

We first explain the reduced structure of the nonlinearity. Let $F = dA$. We may think of A as ϕ. Thus, the Maxwell equations are reduced to the form

$$\Box A = \phi \cdot D\phi.$$

While most commonly, a nonlinear wave equation with quadratic interaction looks like

$$\Box\phi = \nabla\phi \cdot \nabla\phi.$$

The MKG equations is one derivative less in the nonlinearities. This is the reduced structure.

In terms of energy estimates, the reduced structure will be reflected in the following formula:

$$\int_{\mathcal{H}_{r_1}^{\infty}} |D_L\phi|^2 \leq C_1\mathring{\varepsilon} r_1^{-\gamma_0} + C_2 \int_{r_1}^{\infty} \int_{\mathcal{H}_s^{\infty}} \frac{|q_0|}{r^2} |\phi||D_L\phi|.$$

The left hand side is a classical energy flux term through outgoing null cones $\{u = r_1\}$. The first term on the right hand side is coming from the data and the exponent $-\gamma_0$ reflects the decay of the data near spatial infinity. The second term on the right hand side contains a ϕ without any derivative acting on it. We remark that the $\frac{q_0}{r^2}$ factor is arising from the charge. Heuristically for waves, a $\frac{1}{r}$ factor can be regard as D_L-derivative so that we should think of the second term as $\frac{1}{r}|D_L\phi|^2$ thus we see that there is a logarithmic growth when we integrate. We remark here that for the massive case in [15], since solution of massive Klein-Gordon equation decays as quickly as its derivatives, i.e., one can regard ϕ as $D_L\phi$, the above error term can be easily absorbed by using Gronwall's inequality.

We use an idea introduced by Yang in [24] to handle this logarithmic loss. The precise statement is summarized and proved in Lemma 3.1. Morally speaking, to obtain the estimates for the energy flux, we can afford a loss in r instead of in time:

$$\int_{\mathcal{H}_{r_1}^{\infty}} |D_L\phi|^2 \leq C\mathring{\varepsilon} \cdot r_1^{-\gamma_0+\varepsilon_0}.$$

In other words, the decay rate near null infinity changes from γ_0 to $\gamma_0 - \varepsilon_0$.

(2) The ε_0-reductive argument for higher order energy estimates.

The argument is designed to make a better use of the reduced structure of the nonlinearity when we do higher order energy estimates. Although the charge vanishes when taking derivatives, the above type error term arises from the connection field A and has the same structure as described previously. We design an ansatz which allows higher order derivative to lose more decay. To be more precise, we will lose $2(k+1)\varepsilon_0$-decay for the k-th order derivatives. We will use the following example to illustrate how the argument works. In the course of deriving energy estimates for the first order derivatives, schematically, the nonlinear terms look like $\int |\nabla\phi||\nabla^2\phi|$. On the other hand, $|\nabla\phi|$ is already controlled when one derives estimates for the solution itself without commuting vector fields with equations, thus the estimates on $\nabla\phi$ only lose $2\varepsilon_0$ decay. Compared to the $4\varepsilon_0$ loss in the first order derivative case, we indeed have a gain in decay for the nonlinear terms. This gain will play an essential role in closing the estimates.

(3) Morawetz vector field as commutator and new commutation formulas.

Traditionally, the Morawetz vector field K is only used as multipliers in the energy estimates. In this work, we will commute K with the equation. The extra weights compared to the classical commutators such as rotations and scaling provide an extra decay factor for the solutions near null infinity. This extra decay factor is indispensable when we perform the conformal compactification. We would also like to remark that, since K is the image of ∂_t under the inversion map, commuting K with the equation can be regarded as the usual commutation of ∂_t after the conformal transformation. Thus, this idea should be viewed as a vector field method version of conformal transformations.

More precisely, for $Z \in \mathcal{Z} = \{T, \Omega_{12}, \Omega_{23}, \Omega_{31}, S, K\}$, where T is the time translation, Ω_{ij} are rotations and S is scaling, for $\mathbf{Div}$ (the principle part of the Maxwell equations) and $\Box_A$, we have the following two formulas

$$[r^2\mathbf{Div}\,, \mathcal{L}_Z]G = 0, \quad [r^2\Box_A, D_Z + \frac{Z(r)}{r}]\phi = r^2 Q(\phi, F; Z) \tag{1.5}$$

for any closed 2-form G and complex scalar field ϕ. We emphasize that the formula holds for K and $Q(\phi, F; Z)$ is quadratic in ϕ and $F = dA$. We also remark that to our knowledge these commutator formulas are new.

(4) A new null form.

The quadratic form $Q(\phi, F; Z)$ is indeed a null form. Take $Z = S$ for example. It can be shown that

$$\begin{aligned}|Q(\phi,F;Z)| \lesssim &\big(\frac{r}{|u|}|\rho| + |\underline{\alpha}|\big)|D_L(r\phi)| + \big(\frac{r}{|u|}|\alpha| + \frac{|u|}{r}|\underline{\alpha}| + |\sigma|\big)|\not{D}(r\phi)| \\ &+ \big(|\alpha| + \frac{|u|}{r}|\rho|\big)|D_{\underline{L}}(r\phi)| + \big(|\rho| + |\sigma|\big)|\phi| + \text{cubic terms}.\end{aligned}$$

Similar estimates hold for other vector fields in $\mathcal{Z}$. We remark that rather than ϕ itself, the derivatives of $r\phi$ appear naturally in the above null structure estimate.

The most remarkable property of $Q(\phi, F; Z)$ is that it has an iterative structure. This is crucial when we commutate multiple derivatives with equations. More precisely, if we define $\widehat{D}_Z = D_Z + \frac{Z(r)}{r}$, we can show that

$$\big[\widehat{D}_Y, [\widehat{D}_X, r^2\Box_A]\big]\phi = -r^2 Q(\phi, F; [Y,X]) - r^2 Q(\phi, \mathcal{L}_Y F; X) + 2r^2 F_{Y\mu}F_X{}^{\mu}\phi.$$

The right hand side after commutating two derivatives can still be expressed in terms of Q and it still satisfies the null structure. This is one of the keys in the proof.

We remark that to our knowledge this null structure is also new.

(5) The algebraic structure of J.

We have seen that $r\phi$ appears naturally in the null form estimates. We would like to point out another perspective. We mentioned previously that $D_L(r\phi) = O(\frac{1}{r^2})$. We can also show that the best decay estimates for $D_L\phi$ is still $O(\frac{1}{r^2})$ instead of $O(\frac{1}{r^3})$. From this point of view, we may consider $r\phi$ to be "better" than ϕ itself. On the other hand, for the Maxwell equation, instead of commuting with the operator $\mathbf{Div}$, we commute with $r^2\mathbf{Div}$. It thus requires to analyze $r^2 \cdot J$, where the charge density J has components $J_\mu = \Im(\phi \cdot \overline{D_\mu \phi})$. The special algebraic form implies

$$r^2 \cdot J_\mu[\phi] = \Im\big((r\phi) \cdot \overline{D_\mu(r\phi)}\big) = J_\mu[r\phi].$$

Therefore, we only have to deal with the "better" field $r\phi$ rather than ϕ itself. This special cancellation from the algebraic structure is crucial to obtain the sharp peeling estimates and to close the energy estimates.

(6) The conformal compactification.

Since the trace of the energy momentum tensor for MKG equations are not zero, this field theory is not conformal. However, for special conformal transformations, it can still be conformally invariant, e.g., if $\Box\Lambda = 0$ where Λ is the conformal factor. The inversion map restricted in the forward light cone is such a conformal map in $\mathbb{R}^{3+1}$ (not in other dimensions).

On the other hand, there is another important observation: although the presence of a nonzero charge does not allow compactification around the spatial infinity, this effect indeed does not appear on the null infinity. This was first pointed out by Shu in [21]. The following computation for $F[q_0]$ justifies this observation: on a outgoing light cone $\mathcal{H}_u$ defined by $r - t = 2u$, the conformal energy flux passing through this light cone (this is the basic energy quantity needed after the conformal transformation) is given by

$$\mathcal{E}\big[F[q_0]\big] \approx \int_{\mathcal{H}_u} |u|^4 |\rho|^2.$$

Since $|\rho| \approx \frac{q_0}{r^2}$ (as $F[q_0]$ has the leading term $q_0 dt \wedge dr$) and u is a constant on $\mathcal{H}_u$, the above energy flux is finite. On the other hand, if one considers conformal energy on a constant time slice, the factor u^4 would be replace by r^2u^2 (near spatial infinity) so that the contribution of the charge part of the field would be divergent. This is why we choose inversions as the conformal mappings.

(7) r^p-weighted energy estimates.

We use the r^p-weighted energy estimates which was first introduced by Dafermos-Rodnianski in [6] for the study of decay of linear waves on black hole spacetimes. The method has also been used in the first author's works on MKG equations, see [24, 23], where $p < 2$. The new point in the current work is that we have to deal with the end point case $p = 2$ to get the sharp peeling estimates.

1.3. **Further discussions and future plans.** It is instructive to make a comparison to the works [14], [13] of Klainerman-Nicolò to prove higher peeling estimates near Minkowski spacetime in an exterior region and the work [17] of Luk-Oh for proving global nonlinear stability of dispersive solutions to Einstein equations. Indeed, for a given initial datum of the vacuum Einstein field equations, one can work in the region $r \geq R_*$ and can assume that the datum is small provided R_* is sufficiently large. The mass m for the Einstein equations plays a similar role as the charge q_0 for the Maxwell-Klein-Gordon equations: they all represent a slow decay tail representing a static solution at spatial infinity (which is the Schwarzschild solutions in the Einstein equations' case). The proof of Klainerman and Nicolò indeed does not use the smallness of the mass m and this is similar to our case where we do not assume q_0 is small. For vacuum Einstein field equations, the mass m comes in through the ρ-component of the curvature:

$$\rho = \frac{m}{r^3} + \mathring{\rho},$$

where $\mathring{\rho}$ decays as $\frac{1}{r^4}$. However, for MKG equations, the charge q_0 comes in through the ρ-component of the Maxwell field:

$$\rho = \frac{q_0}{r^2} + \mathring{\rho}.$$

The r^{-3} decay is sufficient to apply Gronwall's inequality in the Einstein equations' case while for MKG equations we have to find a new way to compensate the logarithmic loss as we mentioned before.

Alternatively, for this large mass issue for Einstein equations, Luk-Oh in [17] choose a special gauge condition so that such mass problem does not appear. Since our approach in this paper is gauge invariant and the charge is inherited in the connection field A, the charge difficulty is essentially different from the mass problem for Einstein field equations.

For Einstein field equations coupled with other fields, say a scalar field, the coupling field may bring a slower decay tail. We believe that our method in the exterior region can also be applied to these cases to derive sharp peeling estimates.

Finally, there will be two forthcoming papers as a continuation for the project initiated in this work.

(A) We have assumed that the chargeless part of the initial data are bounded in the weighted Sobolev space with weights r^{6+}. Our companion paper [26] will be devoted to reducing the weights to be r^{1+}. We note that the norm with weights r is scaling invariant while the norm with weights r^2

corresponds to the conformal energy. In particular the slower decay of the data in [26] prevents the use of conformal Killing vector fields as commutators. However the approach developed in this paper still works for such data in the exterior region $\{t+1 \leq |x|\}$. Then the asymptotic behaviour of the solutions in the interior region $\{t+1 \geq |x|\}$ can be reduced to study the MKG equations on a compact region with data blowing up on the boundary (due to the fact that the data are below the conformal energy space). This may be of independent interest for studying the formation of singularities of MKG equations. However, in order to implement this new observation, the proof becomes more technical and lengthy. Since the current work consists of all the essential new ideas, by consideration of the length of the paper, we plan to address the sharp results in our subsequent work [26]. In other words, the current paper mainly concerns the solution in the exterior region while the the main difficulty in our companion paper [26] lies in understanding the solution in the interior region. Moreover our subsequent paper will also study the higher order energy estimates and prove that there is no weak turbulence phenomenon (i.e., basic energy is conserved but higher order Sobolev norms grow) for MKG equations.

(B) In our third paper [25], we will address the decay of solutions of Yang-Mills equations on $\mathbb{R}^{3+1}$ by using the approach and ideas developed in this paper as well as those in our subsequent paper [26].

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2. Preparations

2.1. The null decompositions of equations.

Recall from the main theorem that the chargeless part $\mathring{F}$ of the solution is defined as

$$\mathring{F} = F - F[q_0]\mathbf{1}_{\{t+1\leq |x|\}}.$$

It is straightforward to see that $\mathring{F}$ satisfies the same equations as F:

$$\nabla^\mu \mathring{F}_{\mu\nu} = -J_\nu \tag{2.1}$$

in the exterior region $\{t+1 \leq |x|\}$. In terms of the null components, we can rewrite this equation as

$$\begin{cases} L(r^2\mathring{\rho}) + \mathbf{d}\!\!\!/\mathbf{iv}\,(r^2\mathring{\alpha}) = r^2 J_L, \quad \underline{L}(r^2\mathring{\rho}) - \mathbf{d}\!\!\!/\mathbf{iv}\,(r^2\underline{\mathring{\alpha}}) = -r^2 J_{\underline{L}},\\ L(r^2\mathring{\sigma}) + \mathbf{d}\!\!\!/\mathbf{iv}\,(r^2\,{}^*\mathring{\alpha}) = 0, \quad \underline{L}(r^2\mathring{\sigma}) + \mathbf{d}\!\!\!/\mathbf{iv}\,(r^2\,{}^*\underline{\mathring{\alpha}}) = 0,\\ \nabla\!\!\!\!/_{\underline{L}}(r\mathring{\alpha})_A - \nabla\!\!\!\!/_A(r\mathring{\rho}) - {}^*\nabla\!\!\!\!/_A(r\mathring{\sigma}) = rJ_A,\\ \nabla\!\!\!\!/_L(r\underline{\mathring{\alpha}})_A + \nabla\!\!\!\!/_A(r\mathring{\rho}) - {}^*\nabla\!\!\!\!/_A(r\mathring{\sigma}) = rJ_A. \end{cases} \tag{2.2}$$

Here for simplicity, $(\mathring{\alpha}, \underline{\mathring{\alpha}}, \mathring{\rho}, \mathring{\sigma})$ are the null components associated to the 2-form $\mathring{F}$. For any complex scalar field f, the covariant wave operator $\Box_A$ can be expressed in null frames:

$$r\Box_A f = -D_{\underline{L}} D_L(rf) + D\!\!\!\!/^2(rf) + i\rho\cdot(rf) = -D_L D_{\underline{L}}(rf) + D\!\!\!\!/^2(rf) - i\rho\cdot(rf), \tag{2.3}$$

where $D\!\!\!\!/^2(rf) = \sum_{A,B=1}^2 m^{AB} D_{e_A} D_{e_B}(rf)$.

2.2. Commutator vector fields and null structures.

We shall use the following set of vector fields as commutators:

$$\mathcal{Z} = \big\{T, \Omega_{12}, \Omega_{23}, \Omega_{31}, S, K\big\},$$

where $K = v^2 L + u^2 \underline{L}$ is the Morawetz vector field, $S = vL + u\underline{L}$ is the scaling vector field, $\Omega_{ij} = x_i\partial_j - x_j\partial_i$ are the rotation vector fields and $T = \partial_t$ is the time translation. For vector fields in $\mathcal{Z}$, we define their discrepancy index as

$$\xi(T) = -1, \quad \xi(\Omega_{ij}) = \xi(S) = 0, \quad \xi(K) = 1.$$

In the energy estimates, it involves the the deformation tensor of these vector fields:

$${}^{(Z)}\pi_{\mu\nu} = \frac{1}{2}\mathcal{L}_Z m_{\mu\nu} = \frac{1}{2}(\nabla_\mu Z_\nu + \nabla_\nu Z_\mu),$$

where $\mathcal{L}_Z m$ is the Lie derivative of the Minkowski metric. By computation, we have

$${}^{(K)}\pi_{\mu\nu} = t\cdot m_{\mu\nu}, \quad {}^{(S)}\pi_{\mu\nu} = m_{\mu\nu}, \quad {}^{(\Omega_{ij})}\pi_{\mu\nu} = 0, \quad {}^{(T)}\pi_{\mu\nu} = 0,$$

where $m_{\mu\nu}$ is the flat metric of the Minkowski spacetime. We also remark that the set $\mathcal{Z}$ is closed under the Lie bracket: the only non-vanishing $[Z_1, Z_2]$'s for $Z_1, Z_2 \in \mathcal{Z}$ are

$$[T, S] = T, \quad [T, K] = 2S, \quad [S, K] = K.$$

For $Z \in \mathcal{Z}$, we define the modified covariant derivative acting on complex scalar field associated to the 1-form A as follows:

$$\widehat{D}_Z = D_Z + \frac{Z(r)}{r}.$$

This is the conjugate of D_Z by the function r, i.e., $\widehat{D}_Z f = r^{-1} D_Z(rf)$.

Lemma 2.1 (Commutator formula)**.** *For any closed 2-form G and any complex scalar field f, we have*

$$[r^2 \mathbf{Div}\,, \mathcal{L}_Z]G = 0, \tag{2.4}$$

$$[r^2 \Box_A, \widehat{D}_Z]f = 2\sqrt{-1} r^2 F_{\mu\nu} Z^\nu D^\mu f + \sqrt{-1} r^2 \nabla^\mu \big(Z^\nu F_{\mu\nu}\big) f \tag{2.5}$$

for all $Z \in \mathcal{Z}$.

Remark 2.2. *To our knowledge, this set of commutator formulas are new and it is one of the key ingredients to the proof.*

Proof. We first show the following formula

$$[\Box_A, D_Z + \frac{Z(r)}{r}]f = \frac{2Z(r)}{r}\Box_A f + 2\sqrt{-1} F_{\mu\nu} Z^\nu D^\mu f + \sqrt{-1}\nabla^\mu\big(Z^\nu F_{\mu\nu}\big) f. \tag{2.6}$$

By commuting derivatives, we have

$$[\Box_A, D_Z]f = \Box Z_\mu D^\mu f + 2\,{}^{(Z)}\pi_{\mu\nu} D^\mu D^\nu f + 2\sqrt{-1} F_{\mu\nu} Z^\nu D^\mu f + \sqrt{-1}\nabla^\mu\big(Z^\nu F_{\mu\nu}\big) f.$$

For any function f_1, we have

$$[\Box_A, f_1]f = \Box f_1 \cdot f + 2\nabla^\mu f_1 D_\mu f,$$

where f_1 will be $\frac{Z(r)}{r}$.

For $Z \in \mathcal{Z}$, if $Z \neq K$ or S, we have ${}^{(Z)}\pi_{\mu\nu} = 0$ and $f = 0$, therefore, (2.6) holds.

For K, we have $f_1 = t$, $[\Box_A, f_1]f = 2\nabla^\mu f_1 D_\mu f$ and $\Box K = -T$. Hence,

$$[\Box_A, D_K + \frac{K(r)}{r}]f = -T_\mu D^\mu f + 2t\Box_A f + 2\sqrt{-1} F_{\mu\nu} Z^\nu D^\mu f + \sqrt{-1}\nabla^\mu\big(Z^\nu F_{\mu\nu}\big) f + 2\nabla^\mu t D_\mu f.$$

The first term and the last term on the right hand side cancel. This proves the case for $Z = K$.

For S, we have $f_1 = 1$ and the proof follows exactly in the same manner. Thus formula (2.5) holds.

We turn to the proof of (2.4). By commuting the derivatives, we have

$$[\mathbf{Div}\,, \mathcal{L}_Z]G_\nu = \Box Z^\mu\, G_{\mu\nu} + \nabla_\nu \nabla^\mu Z^\delta\, G_{\mu\delta} + 2\,{}^{(Z)}\pi^{\mu\delta}\nabla_\delta G_{\mu\nu}.$$

If $Z \in \mathcal{Z}$ but $Z \neq K$ or S, then $[r^2, \mathcal{L}_Z] = 0$. The above formula shows that $[\mathbf{Div}\,, \mathcal{L}_Z] = 0$. Hence, (2.4) holds.

For K, the above formula implies

$$[\mathbf{Div}\,, \mathcal{L}_K]G_\nu = -2T^\mu\, G_{\mu\nu} + \nabla_\nu \nabla^\mu K^\delta\, G_{\mu\delta} + 2t\nabla^\mu G_{\mu\nu}.$$

In the Cartesian coordinates, one can check immediately that

$$\nabla_\nu \nabla^\mu K^\delta\, G_{\mu\delta} = 2G(\partial_\nu, \partial_t).$$

Therefore, we obtain

$$[\mathbf{Div}\,, \mathcal{L}_K]G = 2t\,\mathbf{Div}\, G.$$

Finally, we have

$$\begin{aligned}
\mathcal{L}_K\big(r^2 \mathbf{Div}\, G\big) &= K(r^2)\mathbf{Div}\, G + r^2 \mathcal{L}_K\big(\mathbf{Div}\, G\big)\\
&= 2tr^2 \mathbf{Div}\, G + r^2 \mathbf{Div}\,\big(\mathcal{L}_K G\big) - r^2[\mathbf{Div}\,, \mathcal{L}_K]G\\
&= r^2 \mathbf{Div}\,\big(\mathcal{L}_K G\big).
\end{aligned}$$

For $Z = S$, recall that ${}^{(S)}\pi = m$. The computation in this case is straightforward. This yields (2.4). $\square$

Motivated by the formula (2.5), we introduce the following commutator null form.

Definition 2.3. *For any closed 2-form G and any complex scalar field f, we define for any vector field Z the quadratic form*

$$Q(f, G; Z) = 2\sqrt{-1} G_{\mu\nu} Z^\nu D^\mu f + \sqrt{-1}\nabla^\mu\big(Z^\nu G_{\mu\nu}\big) f.$$

We then can write (2.5) as

$$[r^2 \Box_A, \widehat{D}_Z]f = r^2 Q(f, F; Z). \tag{2.7}$$

To avoid to many constants, in the sequel we use the convention that $B \lesssim K$ means that there is a constant C, depending only on the charge q_0 such that $B \leq CK$. The next proposition manifests the null structure of the quadratic form $Q(f, G; Z)$:

Proposition 2.4 (Pointwise estimate of null form)**.** *For all $Z \in \mathcal{Z}$, $r \geq 1$ and $|u| \geq 1$, we have*

$$
\begin{aligned}
|u|^{-\xi(Z)}|Q(f,G;Z)| \lesssim &\big(\frac{r}{|u|}|\rho| + |\underline{\alpha}|\big)|D_L(rf)| + \big(\frac{r}{|u|}|\alpha| + \frac{|u|}{r}|\underline{\alpha}| + |\sigma|\big)|\not{D}(rf)| \\
&+ \big(|\alpha| + \frac{|u|}{r}|\rho|\big)|D_{\underline{L}}(rf)| + \big(|\rho| + |\sigma|\big)|f| + \big(|u||J_{\underline{L}}| + \frac{r^2}{|u|}|J_L| + r|\not{J}|\big)|f|
\end{aligned}
\tag{2.8}
$$

for all G and f. The current J is associated to G, i.e., $J_\nu = \nabla^\mu G_{\mu\nu}$. Similarly, the null components α, ρ, σ and $\underline{\alpha}$ are all defined with respect to G.

Proof. We show bound $Q(f,G;Z)$ for each $Z \in \mathcal{Z}$ one by one. We have

$$
\frac{Q(f,G;Z)}{\sqrt{-1}} = \underbrace{2r^{-1}\, G_{\mu\nu}\, Z^\nu D^\mu(rf)}_{\mathbf{I}_1} - \underbrace{(Z^\nu J_\nu)\cdot f}_{\mathbf{I}_2} - \underbrace{\big(2r^{-1}\nabla^\mu r G_{\mu\nu} Z^\nu - \nabla^\mu Z^\nu G_{\mu\nu}\big) f}_{\mathbf{I}_3}.
$$

For $Z = T$, we have

$$
\mathbf{I}_1 = -\frac{1}{r}(\alpha + \underline{\alpha})\cdot \not{D}(rf) + \frac{1}{r}\rho\big(D_{\underline{L}}(rf) - D_L(rf)\big),\ \mathbf{I}_2 = \frac{1}{2}(J_L + J_{\underline{L}})f,\ \mathbf{I}_3 = -r^{-1}\rho f.
$$

Therefore, we have

$$
|Q(f,G;T)| \lesssim \frac{|\not{D}(rf)|}{r}\big(|\alpha| + |\underline{\alpha}|\big) + \frac{|\rho|}{r}\big(|f| + |D_L(rf)| + |D_{\underline{L}}(rf)|\big) + \big(|J_L| + |J_{\underline{L}}|\big)|f|.
$$

For $Z = \Omega_{ij}$, we have

$$
\begin{aligned}
&\mathbf{I}_1 \lesssim |D_L(rf)||\underline{\alpha}| + |D_{\underline{L}}(rf)||\alpha| + |\sigma||\not{D}(rf)|, \quad I_2 \leq r|\not{J}|f, \\
&\mathbf{I}_3 = \Big(\frac{1}{r}\big(G_{L\Omega_{ij}} - G_{\underline{L}\Omega_{ij}}\big) + \nabla_{\underline{L}}\Omega_{ij}^A G_{LA} + \nabla_L \Omega_{ij}^A G_{\underline{L}A} - \nabla^A \Omega_{ij}^B G_{AB}\Big) f = -\nabla^A\Omega_{ij}^B G_{AB} f.
\end{aligned}
$$

Therefore, we have

$$
|Q(f,G;\Omega_{ij})| \lesssim |D_L(rf)||\underline{\alpha}| + |D_{\underline{L}}(rf)||\alpha| + |\sigma||f| + r|\not{J}||f| + |\sigma||\not{D}(rf)|. \tag{2.9}
$$

For $Z = S$, we have

$$
\begin{aligned}
&\mathbf{I}_1 = 2\frac{u}{r}\rho D_{\underline{L}}(rf) - 2\frac{v}{r}\rho D_L(rf) - 2\frac{v}{r}\alpha\cdot\not{D}(rf) - 2\frac{u}{r}\underline{\alpha}\cdot\not{D}(rf), \\
&\mathbf{I}_2 = -vJ_L f - uJ_{\underline{L}} f, \quad \mathbf{I}_3 = -2\frac{u+v}{r}\rho f.
\end{aligned}
$$

Therefore, we have

$$
|Q(f,G;S)| \lesssim r^{-1}|u|\big(|\rho||D_{\underline{L}}(rf)| + |\underline{\alpha}||\not{D}(rf)|\big) + \big(|\rho||D_L(rf)| + |\alpha||\not{D}(rf)|\big) + |\rho||f| + r|J_L||f| + |u||J_{\underline{L}}||f|.
$$

For $Z = K$, we have

$$
\begin{aligned}
&\mathbf{I}_1 = -2\frac{u^2}{r}\rho D_{\underline{L}}(rf) + 2\frac{v^2}{r}\rho D_L(rf) + 2\frac{v^2}{r}\alpha\cdot\not{D}(rf) + 2\frac{u^2}{r}\underline{\alpha}\cdot\not{D}(rf), \\
&\mathbf{I}_2 = -v^2 J_L f - u^2 J_{\underline{L}} f, \quad \mathbf{I}_3 = -4\frac{uv}{r}\rho f.
\end{aligned}
$$

Therefore, we have

$$
\begin{aligned}
|Q(f,G;K)| \lesssim &r^{-1}u^2\big(|\rho||D_{\underline{L}}(rf)| + |\underline{\alpha}||\not{D}(rf)|\big) + r\big(|\rho||D_L(rf)| + |\alpha||\not{D}(rf)|\big) \\
&+ |u||\rho||f| + r^2|J_L||f| + u^2|J_{\underline{L}}||f|.
\end{aligned}
\tag{2.10}
$$

The lemma is an immediate consequence of the above estimates. $\square$

To analyze the higher order energy estimates of the solution, we will commute the equations with the vector fields twice. From the commutation formula (2.7), we have the following identity:

$$
\begin{aligned}
&r^2\Box_A \widehat{D}_{Z_1}\widehat{D}_{Z_2} f \\
&= [r^2\Box_A, \widehat{D}_{Z_1}]\widehat{D}_{Z_2} f + [r^2\Box_A, \widehat{D}_{Z_2}]\widehat{D}_{Z_1} f + \big[\widehat{D}_{Z_1}, [r^2\Box_A, \widehat{D}_{Z_2}]\big] f + \widehat{D}_{Z_1}\widehat{D}_{Z_2}\big(r^2\Box_A f\big) \\
&= r^2 Q(\widehat{D}_{Z_1} f, F; Z_2) + r^2 Q(\widehat{D}_{Z_2} f, F; Z_1) + \big[\widehat{D}_{Z_1}, [r^2\Box_A, \widehat{D}_{Z_2}]\big] f + \widehat{D}_{Z_1}\widehat{D}_{Z_2}\big(r^2\Box_A f\big).
\end{aligned}
\tag{2.11}
$$

Note that for solution of MKG equations the last term vanishes. In particular to derive the equation for the second order derivative of the solution, we need to estimate the double commutator.

Proposition 2.5. *For all $X, Y \in \mathcal{Z}$, we have*

$$
\big[\widehat{D}_Y, [r^2\Box_A, \widehat{D}_X]\big] f = r^2 Q(f, F; [Y,X]) + r^2 Q(f, \mathcal{L}_Y F; X) - 2r^2 F_{Y\mu} F_X{}^\mu f. \tag{2.12}
$$

Proof. First from Lemma 2.1, we can write

$$[r^2\Box_A, \widehat{D}_X]f = r^2(2\sqrt{-1}X^\nu F_{\mu\nu}D^\mu f + \sqrt{-1}\nabla^\mu(F_{\mu\nu}X^\nu)f).$$

Then for any two vector fields X and Y, direction computation implies that

$$\begin{aligned}
&\big[\widehat{D}_Y, [\widehat{D}_X, r^2\Box_A]\big]f \\
&= -\nabla_Y\big(2\sqrt{-1}r^2X^\nu F_{\mu\nu}\big)D^\mu f - \nabla_Y\big(\sqrt{-1}r^2\nabla^\mu(F_{\mu\nu}X^\nu)\big)f \\
&\quad + 2\sqrt{-1}r^2X^\nu F_{\mu\nu}\nabla^\mu\big(\frac{Y(r)}{r}\big)f - 2\sqrt{-1}r^2X^\nu F_{\mu\nu}[D_Y, D^\mu]f \\
&= -\Big(\underbrace{\nabla_Y\big(2\sqrt{-1}r^2X^\nu F_{\mu\nu}\big)D^\mu f}_{\mathbf{I}_1} + \underbrace{\nabla_Y\big(\sqrt{-1}r^2\nabla^\mu(F_{\mu\nu}X^\nu)\big)f}_{\mathbf{I}_2}\Big) \\
&\quad + 2\sqrt{-1}r^2X^\nu F_{\mu\nu}\nabla^\mu\big(\frac{Y(r)}{r}\big)f + 2\sqrt{-1}r^2X^\nu\nabla^\mu Y^\delta F_{\mu\nu}D_\delta f + 2r^2F_{Y\mu}F_X{}^\mu f.
\end{aligned}$$

Now for the term $\mathbf{I}_1$, we can compute that

$$\begin{aligned}
\mathbf{I}_1 &= 2\sqrt{-1}Y(r^2)X^\nu F_{\mu\nu}D^\mu f + 2\sqrt{-1}r^2\nabla_Y X^\nu F_{\mu\nu}D^\mu f + 2\sqrt{-1}r^2X^\nu\nabla_Y F_{\mu\nu}D^\mu f \\
&= 2\sqrt{-1}Y(r^2)X^\nu F_{\mu\nu}D^\mu f + 2\sqrt{-1}r^2\big(\mathcal{L}_Y X^\nu + \nabla_X Y^\nu\big)F_{\mu\nu}D^\mu f \\
&\quad + 2\sqrt{-1}r^2X^\nu\big(\mathcal{L}_Y F_{\mu\nu} - \nabla_\mu Y^\delta F_{\delta\nu} - \nabla_\nu Y^\delta F_{\mu\delta}\big)D^\mu f \\
&= \underbrace{2\sqrt{-1}Y(r^2)X^\nu F_{\mu\nu}D^\mu f}_{\mathbf{I}_{11}} + \underbrace{2\sqrt{-1}r^2\mathcal{L}_Y X^\nu F_{\mu\nu}D^\mu f}_{\mathbf{I}_{12}} \\
&\quad + \underbrace{2\sqrt{-1}r^2X^\nu\mathcal{L}_Y F_{\mu\nu}D^\mu f}_{\mathbf{I}_{13}} - 2\sqrt{-1}r^2X^\nu\nabla_\mu Y^\delta F_{\delta\nu}D^\mu f.
\end{aligned}$$

As for the term $\mathbf{I}_2$, we can further show that

$$\begin{aligned}
\mathbf{I}_2 &= \sqrt{-1}Y(r^2)\nabla^\mu(F_{\mu\nu}X^\nu)f + \sqrt{-1}r^2\nabla^\mu(F_{\mu\nu}\nabla_Y X^\nu)f \\
&\quad + \sqrt{-1}r^2\nabla^\mu(\nabla_Y F_{\mu\nu}X^\nu)f + \sqrt{-1}r^2[\nabla_Y, \nabla^\mu](F_{\mu\nu}X^\nu)f \\
&= \sqrt{-1}Y(r^2)\nabla^\mu(F_{\mu\nu}X^\nu)f + \sqrt{-1}r^2\nabla^\mu\big(F_{\mu\nu}(\mathcal{L}_Y X^\nu + \nabla_X Y^\nu)\big)f - \sqrt{-1}r^2\nabla^\mu Y^\delta F_{\mu\nu}\nabla_\delta X^\nu f \\
&\quad + \sqrt{-1}r^2\nabla^\mu\Big(\big(\mathcal{L}_Y F_{\mu\nu} - \nabla_\mu Y^\delta F_{\delta\nu} - \nabla_\nu Y^\delta F_{\mu\delta}\big)X^\nu\Big)f - \sqrt{-1}r^2\nabla^\mu Y^\delta\nabla_\delta F_{\mu\nu}X^\nu f \\
&= \underbrace{\sqrt{-1}Y(r^2)\nabla^\mu(F_{\mu\nu}X^\nu)f}_{\mathbf{I}_{21}} + \underbrace{\sqrt{-1}r^2\nabla^\mu\big(F_{\mu\nu}\mathcal{L}_Y X^\nu\big)f}_{\mathbf{I}_{22}} + \underbrace{\sqrt{-1}r^2\nabla^\mu\big(\mathcal{L}_Y F_{\mu\nu}X^\nu\big)f}_{\mathbf{I}_{23}} \\
&\quad - \sqrt{-1}r^2\nabla^\mu\big(\nabla_\mu Y^\delta F_{\delta\nu}X^\nu\big)f - \sqrt{-1}r^2\nabla^\mu Y^\delta\nabla_\delta F_{\mu\nu}X^\nu f - \sqrt{-1}r^2\nabla^\mu Y^\delta F_{\mu\nu}\nabla_\delta X^\nu f
\end{aligned}$$

We notice that the $\mathbf{I}_{1i} + \mathbf{I}_{2i}$'s can be expressed in terms of the quadratic form Q. We therefore derive that

$$\begin{aligned}
&\big[\widehat{D}_Y, [\widehat{D}_X, r^2\Box_A]\big]f \\
&= -Y(r^2)Q(f, F; X) - r^2Q(f, F; [Y, X]) - r^2Q(f, \mathcal{L}_Y F; X) + 2r^2F_{Y\mu}F_X{}^\mu f \\
&\quad + 2\sqrt{-1}r^2X^\nu F_{\mu\nu}\nabla^\mu\big(\frac{Y(r)}{r}\big)f + 4\sqrt{-1}r^2X^\nu\, {}^{(Y)}\pi^{\delta\mu}F_{\mu\nu}D_\delta f \\
&\quad + \sqrt{-1}r^2\nabla^\mu\big(\nabla_\mu Y^\delta F_{\delta\nu}X^\nu\big)f + \sqrt{-1}r^2\nabla^\mu Y^\delta\nabla_\delta F_{\mu\nu}X^\nu f + \sqrt{-1}r^2\nabla^\mu Y^\delta F_{\mu\nu}\nabla_\delta X^\nu f \\
&= -Y(r^2)Q(f, F; X) - r^2Q(f, F; [Y, X]) - r^2Q(f, \mathcal{L}_Y F; X) + 2r^2F_{Y\mu}F_X{}^\mu f \\
&\quad + 2\sqrt{-1}r^2X^\nu F_{\mu\nu}\nabla^\mu\big(\frac{Y(r)}{r}\big)f + 4\sqrt{-1}r^2X^\nu\, {}^{(Y)}\pi^{\delta\mu}F_{\mu\nu}D_\delta f \\
&\quad + \sqrt{-1}r^2\Box Y^\delta F_{\delta\nu}X^\nu f + 2\sqrt{-1}r^2\, {}^{(Y)}\pi^{\delta\mu}\left(\nabla_\delta F_{\mu\nu}X^\nu f + F_{\mu\nu}\nabla_\delta X^\nu f\right).
\end{aligned}$$

Note that the last two terms can be written as

$${}^{(Y)}\pi^{\delta\mu}(\nabla_\delta F_{\mu\nu}X^\nu f + F_{\mu\nu}\nabla_\delta X^\nu f) = {}^{(Y)}\pi^{\delta\mu}\nabla_\delta\big(F_{\mu\nu}X^\nu\big)f.$$

We now simplify the previous identity by checking vector fields $Y \in \mathcal{Z}$. We basically have two cases: when $Y = K$, S or $Y = T$, Ω_{ij}. For the latter situation, we notice that Y is Killing and

$$Y(r) = 0, \quad {}^{(Y)}\pi = 0, \quad \Box Y^\delta = 0.$$

Therefore we conclude from the previous identity that

$$\big[\widehat{D}_Y, [\widehat{D}_X, r^2\Box_A]\big]f = -r^2Q(f, F; [Y, X]) - r^2Q(f, \mathcal{L}_Y F; X) + 2r^2F_{Y\mu}F_X{}^\mu f.$$

Now for the first case when $Y = K$ or S, note that we can write these two vector fields in a uniform way

$$Y = u^p \underline{L} + v^p L, \quad p = 1, 2,$$

in which $p = 1$ corresponds to the scaling vector field S while $p = 2$ stands for the conformal Killing vector field K. We then can compute that

$$^{(Y)}\pi = t^{p-1} m, \quad r^{-1} Y(r) = t^{p-1}, \quad \Box Y^\delta \partial_\delta = p(p-1)\partial_t, \quad Y(r^2) = 2rY(r) = 2r^2 t^{p-1}.$$

We therefore can show that

$$\begin{aligned}
&4X^\nu\, {}^{(K)}\pi^{\delta\mu} F_{\mu\nu} D_\delta f + 2X^\nu F_{\mu\nu} \nabla^\mu \big(\frac{K(r)}{r}\big) f + \Box K^\delta F_{\delta\nu} X^\nu f + 2^{(K)}\pi^{\delta\mu} \nabla_\delta \big(F_{\mu\nu} X^\nu\big) f \\
&= 4X^\nu t^{p-1} m^{\delta\mu} F_{\mu\nu} D_\delta f + 2X^\nu F_{\mu\nu} \nabla^\mu (t^{p-1}) f + p(p-1) F_{0\nu} X^\nu f + 2t^{p-1} m^{\delta\mu} \nabla_\delta \big(F_{\mu\nu} X^\nu\big) f \\
&= 4X^\nu t^{p-1} F_{\mu\nu} D^\mu f + (p-2)(p-1) F_{0\nu} X^\nu f + 2t^{p-1} \nabla^\mu \big(F_{\mu\nu} X^\nu\big) f \\
&= -\sqrt{-1} r^{-2} Y(r^2) Q(f, F; X).
\end{aligned}$$

The last step follows by the definition of $Q(f, F; X)$ and the fact that $p = 1$ or 2. In particular we have shown that estimate (2.12) holds for all X, $Y \in \mathcal{Z}$. $\Box$

We are now ready to commute vector fields with MKG equations (0.1). First of all, recall that we have defined the discrepancy index ξ for $Z \in \mathcal{Z}$, that is, the value of T, Ω_{ij}, S and K are -1, 0, 0 and 1 respectively. Let $\mathbf{k} = (k_0, k_1, k_2)$ be a triplet of nonnegative integers. The number k_0, k_1 and k_2 denote the number of index -1, 0 and 1 vector fields respectively. For a given $\mathbf{k}$, we define the **discrepancy index** $\xi(\mathbf{k})$ as

$$\xi(\mathbf{k}) = k_2 - k_0.$$

We also define $|\mathbf{k}| = k_0 + k_1 + k_2$. For derivatives on forms, for example the Maxwell field F or the charge density J, we take the Lie derivative $\mathcal{L}$. For any given tensor field $\mathcal{T}$, we use the expression $\mathcal{L}_Z^{\mathbf{k}} \mathcal{T}$ to denote the following $\mathbf{k}$-derivatives on forms for $Z \in \mathcal{Z}$:

$$\mathcal{L}_Z^{\mathbf{k}} \mathcal{T} = \mathcal{L}_{Z_1} \mathcal{L}_{Z_2} \cdots \mathcal{L}_{Z_{|k|}} \mathcal{T},$$

where there are exactly k_0 degree -1 vector fields, exactly k_1 degree 0 vector fields and exactly k_2 degree 1 vector fields in the collection $\{Z_i | 1 \leq i \leq |\mathbf{k}|\}$. In the sequel we only consider situations where $|k| \leq 2$. It corresponds to commute at most two derivatives with the Maxwell-Klein-Gordon equations (0.1).

As for derivatives on the complex scalar fields, we use the modified covariant derivative $\widehat{D}$. Define

$$\widehat{D}_Z^{\mathbf{k}} f = \widehat{D}_{Z_1} \widehat{D}_{Z_2} \cdots \widehat{D}_{Z_{|\mathbf{k}|}} f.$$

For the solution ϕ, we also define shorthand notations $\phi^{(\mathbf{k})} = \widehat{D}_Z^{\mathbf{k}} \phi$ and $\psi^{(\mathbf{k})} = r \widehat{D}_Z^{\mathbf{k}} \phi$. The previous commutator calculations allow us to derive the wave equations for $\phi^{(\mathbf{k})}$. Define

$$N^{(\mathbf{k})} = \Box_A \phi^{(\mathbf{k})}.$$

In particular we have $N^{(\mathbf{0})} = 0$. By definition of Q, we see that $N^{(\mathbf{1})} = Q(\phi, F; Z)$. For the second order derivative $\phi^{(\mathbf{k})} = \widehat{D}_{Z_1} \widehat{D}_{Z_2} \phi$, Proposition 2.5 together with the identity (2.11) implies that

$$N^{(\mathbf{2})} = Q(\widehat{D}_{Z_1}\phi, F; Z_2) + Q(\widehat{D}_{Z_2}\phi, F; Z_1) + Q(\phi, F; [Z_1, Z_2]) + Q(\phi, \mathcal{L}_{Z_1} F; Z_2) - 2F_{Z_1\mu} F_{Z_2}{}^\mu \phi. \quad (2.13)$$

We now turn to the Maxwell part. We first explain our notations. For $r \neq 0$, we shall use the following shorthand notations to derivatives of the chargeless part of F:

$$\alpha^{(\mathbf{k})} = \alpha(\mathcal{L}_Z^{\mathbf{k}} \mathring{F}), \quad \underline{\alpha}^{(\mathbf{k})} = \underline{\alpha}(\mathcal{L}_Z^{\mathbf{k}} \mathring{F}), \quad \rho^{(\mathbf{k})} = \rho(\mathcal{L}_Z^{\mathbf{k}} \mathring{F}), \quad \sigma^{(\mathbf{k})} = \sigma(\mathcal{L}_Z^{\mathbf{k}} \mathring{F}).$$

We notice that $\rho^{(\mathbf{0})} \neq \rho$ for $q_0 \neq 0$. In most of the cases in this paper, only the total number of derivatives in $\mathcal{L}_Z^{\mathbf{k}}$ is important. The exact form of $\mathbf{k}$ is usually irrelevant unless it is emphasized. Therefore, we will use shorthand notations $(\mathbf{1})$ and $(\mathbf{2})$ rather than writing down the explicit expression of $\mathbf{k}$, e.g., for $\alpha(\mathcal{L}_T \mathcal{L}_\Omega \mathring{F})$ we simply write it as $\alpha^{(\mathbf{2})}$.

For a given $\mathbf{k}$, we also define

$$J^{(\mathbf{k})} = \mathcal{L}_Z^{\mathbf{k}} (r^2 J). \quad (2.14)$$

We remark that $J^{(\mathbf{0})} = r^2 J$ which is **not** the current J. The null components of $J^{(\mathbf{k})}$ are denoted by $J_L^{(\mathbf{k})}$, $J_{\underline{L}}^{(\mathbf{k})}$ and $\not{J}^{(\mathbf{k})}$. More precisely, we define

$$J_L^{(\mathbf{k})} = -\frac{1}{2} m(\mathcal{L}_Z^{\mathbf{k}}(r^2 J), \underline{L}), \quad J_{\underline{L}}^{(\mathbf{k})} = -\frac{1}{2} m(\mathcal{L}_Z^{\mathbf{k}}(r^2 J), L), \quad \not{J}_A^{(\mathbf{k})} = m(\mathcal{L}_Z^{\mathbf{k}}(r^2 J), e_A) \text{ for } A = 1, 2.$$

In view of (2.1), (2.2) and (2.4), we can commute $\mathcal{L}_Z^{\mathbf{k}}$ to derive

$$\begin{cases} L(r^2\rho^{(\mathbf{k})}) + \mathbf{d}\!\!/\mathbf{iv}\,(r^2\alpha^{(\mathbf{k})}) = J_L^{(\mathbf{k})}, \quad \underline{L}(r^2\rho^{(\mathbf{k})}) - \mathbf{d}\!\!/\mathbf{iv}\,(r^2\underline{\alpha}^{(\mathbf{k})}) = -J_{\underline{L}}^{(\mathbf{k})}, \\ L(r^2\sigma^{(\mathbf{k})}) + \mathbf{d}\!\!/\mathbf{iv}\,(r^2\,{}^*\alpha^{(\mathbf{k})}) = 0, \quad \underline{L}(r^2\sigma^{(\mathbf{k})}) + \mathbf{d}\!\!/\mathbf{iv}\,(r^2\,{}^*\underline{\alpha}^{(\mathbf{k})}) = 0, \\ \nabla\!\!\!/_{\underline{L}}(r\alpha^{(\mathbf{k})})_A - \nabla\!\!\!/_A(r\rho^{(\mathbf{k})}) - {}^*\nabla\!\!\!/_A(r\sigma^{(\mathbf{k})}) = r^{-1}J\!\!\!/_A^{(\mathbf{k})}, \\ \nabla\!\!\!/_L(r\underline{\alpha}^{(\mathbf{k})})_A + \nabla\!\!\!/_A(r\rho^{(\mathbf{k})}) - {}^*\nabla\!\!\!/_A(r\sigma^{(\mathbf{k})}) = r^{-1}J\!\!\!/_A^{(\mathbf{k})}. \end{cases} \tag{2.15}$$

2.3. **Multiplier vector fields and energy quantities.** One can associate the so-called energy momentum 2-tensor $T[G,f]_{\alpha\beta}$ to a closed 2-form G and any complex scalar field f:

$$T[G,f]_{\alpha\beta} = \underbrace{G_{\alpha\mu}G_\beta{}^\mu - \frac{1}{4}m_{\alpha\beta}G_{\mu\nu}G^{\mu\nu}}_{T[G]_{\alpha\beta}} + \underbrace{\Re\big(\overline{D_\alpha f}D_\beta f\big) - \frac{1}{2}m_{\alpha\beta}\overline{D^\mu f}D_\mu f}_{T[f]_{\alpha\beta}}.$$

Given a smooth $\mathbb{R}-$valued function χ and a vector field Y^μ, for any (multiplier) vector field X, we define the associated current as:

$$^{(X)}\widetilde{J}[G,f]_\mu = T[G,f]_{\mu\nu}X^\nu - \frac{1}{2}\nabla_\mu\chi\cdot|f|^2 + \frac{1}{2}\chi\cdot\nabla_\mu\big(|f|^2\big) + Y_\mu. \tag{2.16}$$

It can be computed that the space-time divergence of $^{(X)}\widetilde{J}[G,f]$ is given by the following formula:

$$\begin{aligned}\mathbf{Div}\,\big({}^{(X)}\widetilde{J}[G,f]\big) &= \underbrace{T[G,f]_{\mu\nu}\,{}^{(X)}\pi^{\mu\nu} + \chi\overline{D^\mu f}D_\mu f - \frac{1}{2}\Box\chi\cdot|f|^2 + \mathbf{Div}\,Y}_{\mathbf{D}_1} \\ &\quad + \underbrace{\Re\big(\overline{\Box_A f}(D_X f + \chi f)\big) + \nabla^\mu G_{\mu\nu}\cdot G^{\delta\nu}X_\delta + X^\mu F_{\mu\nu}J[f]^\nu}_{\mathbf{D}_2},\end{aligned} \tag{2.17}$$

where the current $J_\mu[f] = \Im(f\cdot\overline{D_\mu f})$.

In this paper, we will use two types of vector fields as multipliers. In particular the multiplier X will be chosen as $X = \partial_t$ or $X = r^pL$ $(0 \le p \le 2)$. Their deformation tensors are recorded in the following table:

	π_{LL}	$\pi_{L\underline{L}}$	$\pi_{\underline{L}\underline{L}}$	π_{LA}	$\pi_{\underline{L}A}$	π_{AB}
$X=\partial_t$	0	0	0	0	0	0
$X=r^pL$	0	$-pr^{p-1}$	$2pr^{p-1}$	0	0	$r^{p-1}\not{g}_{AB}$

To define energy quantities, we first clarify the measure over different regions or hypersurfaces. In the sequel, the variable ϑ denotes for a coordinate on the unit sphere $\mathbf{S}^2$. We have

$$\int_{\mathcal{H}_{r_1}^{r_2}}\cdot = \int_{\frac{r_1}{2}}^{\frac{r_2}{2}}\int_{\mathbf{S}^2}\cdot\ r^2dvd\vartheta, \qquad \int_{\underline{\mathcal{H}}_{r_2}^{r_1}}\cdot = \int_{-\frac{r_2}{2}}^{-\frac{r_1}{2}}\int_{\mathbf{S}^2}\cdot\ r^2dud\vartheta,$$
$$\int_{\mathcal{B}_{r_1}^{r_2}}\cdot = \int_{r_1}^{r_2}\int_{\mathbf{S}^2}\cdot\ r^2drd\vartheta, \qquad \int_{\mathcal{D}_{r_1}^{r_2}}\cdot = \frac{1}{2}\int\int\int_{\mathbf{S}^2}\cdot\ r^2dudvd\vartheta.$$

Given G and f, the energy through $\mathcal{B}_{r_1}^{r_2}$ and the energy flux through $\mathcal{H}_{r_1}^{r_2}$ or $\underline{\mathcal{H}}_{r_2}^{r_1}$ are defined as

$$\mathcal{E}[G,f](\mathcal{B}_{r_1}^{r_2}) := \int_{\mathcal{B}_{r_1}^{r_2}}|\alpha(G)|^2 + |\underline{\alpha}(G)|^2 + |\rho(G)|^2 + |\sigma(G)|^2 + |Df|^2,$$
$$\mathcal{F}[G,f](\mathcal{H}_{r_1}^{r_2}) := \int_{\mathcal{H}_{r_1}^{r_2}}|\alpha(G)|^2 + |\rho(G)|^2 + |\sigma(G)|^2 + |D_Lf|^2 + |D\!\!\!/f|^2,$$
$$\underline{\mathcal{F}}[G,f](\underline{\mathcal{H}}_{r_2}^{r_1}) := \int_{\underline{\mathcal{H}}_{r_2}^{r_1}}|\underline{\alpha}(G)|^2 + |\rho(G)|^2 + |\sigma(G)|^2 + |D_{\underline{L}}f|^2 + |D\!\!\!/f|^2.$$

One can take $X = \partial_t$, $\chi = 0$, $Y = 0$ and then integrate (2.17) over $\mathcal{D}_{r_1}^{r_2}$. This leads to the classical energy identity:

Lemma 2.6 (Classical energy identity)**.** *For all closed 2-form G and any complex scalar field f and all $0 < r_1 < r_2$, we have*

$$\mathcal{F}[G,f](\mathcal{H}_{r_1}^{r_2}) + \underline{\mathcal{F}}[G,f](\underline{\mathcal{H}}_{r_2}^{r_1}) = \mathcal{E}[G,f](\mathcal{B}_{r_1}^{r_2}) - \int_{D_{r_1}^{r_2}}\Re\big(\overline{\Box_A f}\cdot D_{\partial_t}f\big) + \nabla^\mu G_{\mu\nu}\cdot G_0{}^\nu + F_{0\mu}J[f]^\mu. \tag{2.18}$$

If we choose $X = r^pL$, $\chi = r^{p-1}$ and $Y = \frac{p}{2}r^{p-2}|f|^2L$, this leads to the r-weighted energy identity

Lemma 2.7 (r-weighted energy identity)**.** *For all closed 2-form G and complex scalar field f, we have*

$$
\begin{aligned}
&\int_{\mathcal{B}_{r_1}^{r_2}} r^{p-2}\big(|D_L(rf)|^2+|\not{D}(rf)|^2\big)+r^p\big(|\alpha(G)|^2+|\rho(G)|^2+|\sigma(G)|^2\big)\\
=&\int_{\mathcal{H}_{r_1}^{r_2}} r^{p-2}\big(|D_L(rf)|^2+r^2|\alpha(G)|^2\big)+\int_{\underline{\mathcal{H}}_{r_2}^{r_1}} r^{p-2}\big(|\not{D}(rf)|^2+r^2|\rho(G)|^2+r^2|\sigma(G)|^2\big)\\
&+\frac{1}{2}\int_{\mathcal{D}_{r_1}^{r_2}} r^{p-3}\Big(p\big(|D_L(rf)|^2+r^2|\alpha(G)|^2\big)+(2-p)(|\not{D}(rf)|^2+r^2|\rho(G)|^2+r^2|\sigma(G)|^2)\Big)\\
&+\underbrace{\int_{\mathcal{D}_{r_1}^{r_2}} r^{p-1}\Re\big(\overline{\Box_A f}\cdot D_L(rf)\big)+r^p\nabla^\mu G_{\mu\nu}\cdot G_L{}^\nu+r^pF_{L\mu}J[f]^\mu}_{r\text{-weighted error term }\mathbf{Err_p}}
\end{aligned}
\tag{2.19}
$$

for all $0<r_1<r_2$ and $p\in[0,2]$.

One can find the detailed proof in [24]. For reader's interest, we provide the proof here.

Proof. The identity (2.19) is equivalent to the following one:

$$
\begin{aligned}
&\underbrace{\int_{r_1}^{r_2}\int_{\mathbf{S}^2} r^p\big(|D_L(rf)|^2+|\not{D}(rf)|^2\big)+r^{p+2}\big(|\alpha(G)|^2+|\rho(G)|^2+|\sigma(G)|^2\big)drd\vartheta}_{L_1}\\
=&\underbrace{\int_{\frac{r_1}{2}}^{\frac{r_2}{2}}\int_{\mathbf{S}^2} r^p\big(|D_L(rf)|^2+r^2|\alpha(G)|^2\big)dvd\vartheta}_{R_1}+\underbrace{\int_{-\frac{r_2}{2}}^{-\frac{r_1}{2}}\int_{\mathbf{S}^2} r^p\big(|\not{D}(rf)|^2+r^2|\rho(G)|^2+r^2|\sigma(G)|^2\big)dud\vartheta}_{R_2}\\
&+\int_u\int_\vartheta\int_v r^{p-1}\Big(p\big(|D_L(rf)|^2+r^2|\alpha(G)|^2\big)+(2-p)(|\not{D}(rf)|^2+r^2|\rho(G)|^2+r^2|\sigma(G)|^2)\Big)dvd\vartheta du\\
&+\int_{\mathcal{D}_{r_1}^{r_2}} r^{p-1}\Re\big(\overline{\Box_A f}\cdot D_L(rf)\big)+r^p\nabla^\mu G_{\mu\nu}\cdot G_L{}^\delta+r^pF_{L\mu}J[f]^\mu.
\end{aligned}
\tag{2.20}
$$

We take $X=r^pL$, $\chi=r^{p-1}$ and $Y=\frac{p}{2}r^{p-2}|f|^2L$ in (2.17). We can compute that

$$
\begin{aligned}
T[G,f]_{\mu\nu}\,{}^{(X)}\pi^{\mu\nu}=&-\frac{p-2}{2}r^{p-1}\big(\rho(G)^2+\sigma(G)^2\big)-\frac{p}{2}r^{p-1}|\not{D}f|^2\\
&+\frac{p}{2}r^{p-1}\big(|D_Lf|^2+|\alpha(G)|^2\big)+r^{p-1}D_LfD_{\underline{L}}f,\\
\chi\overline{D^\mu f}D_\mu f=&-r^{p-1}D_LfD_{\underline{L}}f+r^{p-1}|\not{D}f|^2,\quad -\frac{1}{2}\Box\chi\cdot|f|^2=-\frac{p(p-1)}{2}r^{p-3}|f|^2,\\
\mathbf{Div}\,Y=&\frac{p^2}{2}r^{p-3}|f|^2+pr^{p-2}\Re(\overline{D_Lf}\cdot\phi).
\end{aligned}
$$

Since $r^2|D_Lf|^2=|D_L(rf)|^2-L(r|f|^2)$, we obtain

$$
\begin{aligned}
\mathbf{D}_1&=\frac{2-p}{2}r^{p-3}\big(r^2\rho(G)^2+r^2\sigma(G)^2+|\not{D}(rf)|^2\big)+\frac{p}{2}r^{p-3}\big(|\alpha(G)|^2+|D_L(rf)|^2\big),\\
\mathbf{D}_2&=r^{p-1}\Re\big(\overline{\Box_A f}\cdot D_L(rf)\big)+r^p\nabla^\mu G_{\mu\nu}\cdot G_L{}^\nu+r^pF_{L\mu}J[f]^\mu,
\end{aligned}
\tag{2.21}
$$

where $\mathbf{D}_i$'s are defined in (2.17). Now we turn to the boundary integrals. On $\mathcal{B}_{r_1}^{r_2}$, the normal n^μ is ∂_t, we have

$$
\begin{aligned}
{}^{(X)}\widetilde{J}[G,f]^\mu n_\mu=&\frac{1}{2}r^{p-2}\big(r^2\alpha(G)^2+r^2\rho(G)^2+r^2\sigma(G)^2+|D_L(rf)|^2+|\not{D}(rf)|^2\big)\\
&-\frac{1}{2}\big((p+1)r^{p-2}|f|^2+r^{p-1}\partial_r(|f|^2)\big).
\end{aligned}
$$

Therefore we derive that

$$
\begin{aligned}
\int_{\mathcal{B}_{r_1}^{r_2}}{}^{(X)}\widetilde{J}[G,f]^\mu n_\mu=&\frac{1}{2}\underbrace{\int_{\mathcal{B}_{r_1}^{r_2}} r^2\alpha(G)^2+r^2\rho(G)^2+r^2\sigma(G)^2+|D_L(rf)|^2+|\not{D}(rf)|^2}_{L_1\ in\ (2.20)}\\
&-\frac{1}{2}\int_{r_1}^{r_2}\int_{\mathbf{S}^2}\underbrace{(p+1)r^p|f|^2+r^{p+1}\partial_r(|f|^2)}_{=\partial_r(r^{p+1}|f|^2)}d\vartheta dr\\
=&\frac{1}{2}L_1+\frac{1}{2}\int_{\mathcal{S}_{r_1}^{r_1}} r^{p-1}|f|^2-\frac{1}{2}\int_{\mathcal{S}_{r_2}^{r_2}} r^{p-1}|f|^2.
\end{aligned}
\tag{2.22}
$$

On $\mathcal{H}_{r_1}^{r_2}$, the normal n^μ is L. Hence,

$$^{(X)}\widetilde{J}[G,f]^\mu n_\mu = r^{p-2}\big(r^2\alpha(G)^2 + |D_L(rf)|^2 + |\not{D}(rf)|^2\big) - \frac{1}{2}\big((p+1)r^{p-2}|f|^2 + r^{p-1}L(|f|^2)\big).$$

Therefore, we have

$$\begin{aligned}\int_{\mathcal{H}_{r_1}^{r_2}} {}^{(X)}\widetilde{J}[G,f]^\mu n_\mu &= \underbrace{\int_{\frac{r_1}{2}}^{\frac{r_2}{2}}\int_{\mathbf{S}^2} r^p\big(|D_L(rf)|^2 + r^2|\alpha(G)|^2\big)dvd\vartheta}_{R_1\ in\ (2.20)} \\ &\quad - \frac{1}{2}\int_{\frac{r_1}{2}}^{\frac{r_2}{2}}\int_{\mathbf{S}^2} \underbrace{(p+1)r^p|f|^2 + r^{p+1}L(|f|^2)}_{=L(r^{p+1}|f|^2)} d\vartheta dv \\ &= R_1 + \frac{1}{2}\int_{\mathcal{S}_{r_1}^{r_1}} r^{p-1}|f|^2 - \frac{1}{2}\int_{\mathcal{S}_{r_1}^{r_2}} r^{p-1}|f|^2.\end{aligned} \tag{2.23}$$

On $\underline{\mathcal{H}}_{r_1}^{r_2}$, the normal n^μ is $\underline{L}$. Hence,

$$^{(X)}\widetilde{J}[G,f]^\mu n_\mu = r^{p-2}\big(r^2\rho(G)^2 + r^2\sigma(G)^2 + |\not{D}(rf)|^2\big) + \frac{1}{2}\big(-(p+1)r^{p-2}|f|^2 + r^{p-1}\underline{L}(|f|^2)\big).$$

Therefore, we have

$$\begin{aligned}\int_{\underline{\mathcal{H}}_{r_1}^{r_2}} {}^{(X)}\widetilde{J}[G,f]^\mu n_\mu &= \underbrace{\int_{-\frac{r_2}{2}}^{-\frac{r_1}{2}}\int_{\mathbf{S}^2} r^p\big(|\not{D}(rf)|^2 + r^2|\rho(G)|^2 + r^2|\sigma(G)|^2\big)dud\vartheta}_{R_2\ in\ (2.20)} \\ &\quad + \frac{1}{2}\int_{-\frac{r_2}{2}}^{-\frac{r_1}{2}}\int_{\mathbf{S}^2} \underbrace{-(p+1)r^p|f|^2 + r^{p+1}\underline{L}(|f|^2)}_{=\underline{L}(r^{p+1}|f|^2)} d\vartheta dv \\ &= R_2 + \frac{1}{2}\int_{\mathcal{S}_{r_1}^{r_2}} r^{p-1}|f|^2 - \frac{1}{2}\int_{\mathcal{S}_{r_2}^{r_2}} r^{p-1}|f|^2.\end{aligned} \tag{2.24}$$

By combining(2.21)-(2.24) we can the use Stokes formula to complete the proof. □

To end this section, we introduce energy norms. For all $r_1 > 0$, $p \in [0,2]$, $\mathbf{k} \le 2$ and a given small $\delta > 0$, we define the standard energy norms

$$\begin{aligned}\mathcal{E}^{(\mathbf{k})}(\phi; r_1) &= \mathcal{F}[0,\phi^{(\mathbf{k})}](\mathcal{H}_{r_1}) + \sup_{r_2\ge r_1} \underline{\mathcal{F}}[0,\phi^{(\mathbf{k})}](\underline{\mathcal{H}}_{r_2}^{r_1}), \\ \mathcal{E}^{(\mathbf{k})}(\mathring{F}; r_1) &= \mathcal{F}[\mathcal{L}_Z^{\mathbf{k}}(\mathring{F}),0](\mathcal{H}_{r_1}) + \sup_{r_2\ge r_1} \underline{\mathcal{F}}[\mathcal{L}_Z^{\mathbf{k}}(\mathring{F}),0](\underline{\mathcal{H}}_{r_2}^{r_1}),\end{aligned}$$

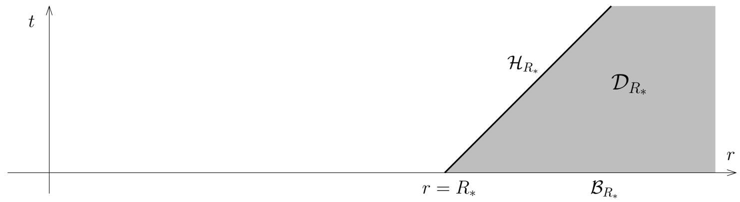
and the r^p-weighted energy norms

$$\begin{aligned}\mathcal{E}^{(\mathbf{k})}(\phi; p; r_1) &= \int_{\mathcal{H}_{r_1}} r^{p-2}|D_L\psi^{(\mathbf{k})}|^2 + \sup_{r_2\ge r_1}\int_{\underline{\mathcal{H}}_{r_2}^{r_1}} r^{p-2}|\not{D}\psi^{(\mathbf{k})}|^2 \\ &\quad + \int_{\mathcal{D}_{r_1}} r^{p-3}\Big(p|D_L\psi^{(\mathbf{k})}|^2 + (2-p)|\not{D}\psi^{(\mathbf{k})}|^2\Big), \\ \mathcal{E}^{(\mathbf{k})}(\mathring{F}; p; r_1) &= \int_{\mathcal{H}_{r_1}} r^p|\alpha^{(\mathbf{k})}|^2 + \sup_{r_2\ge r_1}\int_{\underline{\mathcal{H}}_{r_2}^{r_1}} r^p\big(|\rho^{(\mathbf{k})}|^2 + |\sigma^{(\mathbf{k})}|^2\big) \\ &\quad + \int_{\mathcal{D}_{r_1}} r^{p-1}\Big(p|\alpha^{(\mathbf{k})}|^2 + (2-p)\big(|\rho^{(\mathbf{k})}|^2 + |\sigma^{(\mathbf{k})}|^2\big)\Big).\end{aligned}$$

3. The analysis in the exterior region 0: set-up and zeroth order energy estimates

We emphasize again that, till the end of the paper, (ϕ, F) is a given solution of (0.1) associated to a given finite energy smooth initial datum. According to the result of Klainerman-Machedon [11], the solution exits globally.

3.1. **The exterior region.** We take a positive number R_* and require that $R_* \ge 1$. The number R_* should be understood as a large number and its size will be determined later on (solely by the initial datum). It determines the so-called exterior region $\mathcal{D}_{R_*}$. It is the grey region in the following picture.



The boundary of the exterior region consists of two pieces: the outgoing null hypersurface $\mathcal{H}_{r_1}^{r_2}$ and its bottom $\mathcal{B}_{R_*}$. The exterior region is also the domain of dependence of $\mathcal{B}_{R_*}$.

According to (1.3), the following number is the initial energy for ϕ and $\mathring{F}$ on $\mathcal{B}_{R_*}$:

$$\mathring{\mathcal{E}}_{\geq R_*} = \sum_{k=0}^{2}\int_{r\geq R_*}\int_{\mathbf{S}^2}\Big[r^{2k+6+8\varepsilon_0}\big(|D^k D\phi_0|^2+|D^k\phi_1|^2+|\nabla^k\mathring{F}|^2\big)+r^{4+8\varepsilon_0}|\phi_0|^2\Big]r^2drd\vartheta. \tag{3.1}$$

Since we will eventually take a large R_*, we can assume that for an given small positive number $\mathring{\varepsilon}<1$ one has

$$\mathring{\mathcal{E}}_{\geq R_*}\leq\mathring{\varepsilon}.$$

Before we proceed to the energy estimates, we prove a key technical lemma. The lemma is indispensable to the estimate on terms with critical decay (coming from the charge term) of the current J.

Lemma 3.1. *(Key technical lemma) Let C_0, C_1, C_2, γ_0 and ε_0 be positive numbers. The constant ε_0 is small, say $\varepsilon_0 = 0.001$ and $\gamma_0 > 100\varepsilon_0$. Let f be an arbitrary scalar field satisfying the following two conditions:*

1). For all $r_1\geq R_*$, we have

$$\int_{\mathcal{B}_{r_1}} r^{-2}|f|^2\leq C_0\mathring{\varepsilon}r_1^{-\gamma_0}. \tag{3.2}$$

2). For all $r_2>r_1\geq R_*$, we have

$$\int_{\mathcal{H}_{r_1}^{r_2}}|D_Lf|^2\leq C_1\mathring{\varepsilon}r_1^{-\gamma_0}+C_2\int_{\mathcal{D}_{r_1}^{r_2}}\frac{1}{r^2}|f||D_Lf|. \tag{3.3}$$

Then there exists a constant C depending only on C_0, C_1, C_2 and ε_0 such that

$$\int_{\mathcal{H}_{r_1}^{r_2}}|D_Lf|^2\leq C\mathring{\varepsilon}\cdot r_1^{-\gamma_0+\varepsilon_0}. \tag{3.4}$$

Remark 3.2. *As we have mentioned in the introduction that the error term caused by the charge may lead to a logarithmic growth by using the standard Gronwall's inequality. The importance of this lemma is to avoid this log-loss with the price of losing a bit of decay. This technique was introduced by the first author in* [24] *to derive the energy flux decay. For completeness we summarize it as a Lemma which will also be used to obtain higher order energy estimates.*

Proof. Recall that $u_+ = 1+|u|$. By virtue of Cauchy-Schwarz inequality, we have

$$\mathbf{I} := \int_{\mathcal{D}_{r_1}^{r_2}}\frac{1}{r^2}|f||D_Lf|\lesssim \underbrace{\int_u\int_v\int_\vartheta u_+^{-1}r^2|D_Lf|^2dudvd\vartheta}_{\mathbf{I}_1}+\underbrace{\int_u\int_v\int_\vartheta u_+r^{-2}|f|^2dudvd\vartheta}_{\mathbf{I}_2}.$$

We first deal with $\mathbf{I}_2$. In view of the case $\gamma=4$ in (A.11) of Appendix A, we have

$$\begin{aligned}\mathbf{I_2}&\lesssim\int_{\frac{r_1}{2}}^{\frac{r_2}{2}}u_+\Big(\int_{\mathcal{H}_{2u}^{r_2}}r^{-4}|f|^2\Big)du\lesssim\int_{\frac{r_1}{2}}^{\frac{r_2}{2}}u_+\Big(u_+^{-3}\int_{\mathcal{S}_{2u}^{2u}}|f|^2+u_+^{-2}\int_{\mathcal{H}_{2u}^{r_2}}\big|D_Lf\big|^2\Big)du\\&=\underbrace{\int_{\frac{r_1}{2}}^{\frac{r_2}{2}}u_+^{-2}\Big(\int_{\mathcal{S}_{2u}^{2u}}|f|^2\Big)du}_{\approx\int_{\mathcal{B}_{r_1}^{r_2}}|f|^2\lesssim C_0r_1^{-\gamma_0}\mathring{\varepsilon}}+\int_{\mathcal{D}_{r_1}^{r_2}}u_+^{-1}\big|D_Lf\big|^2.\end{aligned}$$

Here the implicit constant is a universal constant. In particular there exists a universal constant C such that

$$\begin{aligned}\int_{\mathcal{H}_{r_1}^{r_2}}|D_Lf|^2&\leq CC_1r_1^{-\gamma_0}\mathring{\varepsilon}+CC_2\int_{\mathcal{D}_{r_1}^{r_2}}\big(1+|u|\big)^{-1}|D_Lf|^2\\&=CC_1r_1^{-\gamma_0}\mathring{\varepsilon}+CC_2\int_{r_1}^{r_2}\frac{1}{s}\Big(\int_{\mathcal{H}_s^{r_2}}|D_Lf|^2\Big)ds.\end{aligned}$$

We now apply Gronwall's inequality in Lemma A.1 (by setting $f(s)=\int_{\mathcal{H}_s^{r_2}}|D_L f|^2$) to conclude that

$$\int_{\mathcal{H}_{r_1}^{r_2}}|D_L f|^2\leq C(C_1+C_2)\mathring{\varepsilon}\cdot r_1^{-\gamma_0}\left(r_2 r_1^{-1}\right)^{CC_2}.$$

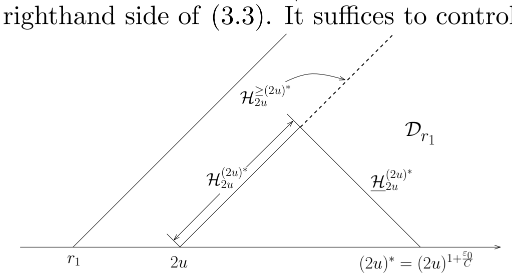
For a given r_1, define $r_1^*:=r_1^{1+\frac{\varepsilon_0}{2CC_2}}$. Then for all $r_2\leq r_1^*$, we have

$$\int_{\mathcal{H}_{r_1}^{r_2}}|D_L f|^2\lesssim_{C_1,C_2}\mathring{\varepsilon}\cdot r_1^{-\gamma_0+\frac{\varepsilon_0}{2}},\quad r_2\leq r_1^*=r_1^{1+\frac{\varepsilon_0}{2CC_2}}. \tag{3.5}$$

Here the implicit constant depends only on C_1 and C_2. We now study the case $r_2>r_1^*$ in a different way. In fact, we take $r_2=\infty$ and we have

$$\begin{aligned}\mathbf{I}=\int_{\mathcal{D}_{r_1}}\frac{1}{r^2}|f||D_L f|&\leq\int_{\mathcal{D}_{r_1}}u_+^{-1-\frac{\varepsilon_0}{2CC_2}}|D_L f|^2+u_+^{1+\frac{\varepsilon_0}{2CC_2}}r^{-4}|f|^2\\&=\underbrace{\int_{\frac{r_1}{2}}u_+^{-1-\frac{\varepsilon_0}{2CC_2}}\int_{\mathcal{H}_{2u}}|D_L f|^2du}_{\mathbf{II}_1}+\underbrace{\int_{\frac{r_1}{2}}^{\infty}u_+^{1+\frac{\varepsilon_0}{2CC_2}}\Big(\overbrace{\int_v\int_\vartheta r^{-2}|f|^2dvd\vartheta}^{\mathbf{II'}_2(u)=\int_{\mathcal{H}_{2u}}r^{-4}|f|^2}\Big)du}_{\mathbf{II}_2},\end{aligned}\tag{3.6}$$

where C is the constant in the definition of r_1^*. Because $u_+^{-1-\frac{\varepsilon_0}{2CC_2}}$ is integrable in u, Gronwall's inequality enables us to bound $\mathbf{II}_1$ by the righthand side of (3.3). It suffices to control $\mathbf{II}_2$.



The cone $\mathcal{H}_{2u}$ is the union of $\mathcal{H}_{2u}^{(2u)^*}$ and $\mathcal{H}_{2u}^{\geq(2u)^*}$ which is the cone emanated from the sphere $\mathcal{S}_{2u}^{(2u)^*}$ ($(2u)^*=(2u)^{1+\frac{\varepsilon_0}{2CC_2}}$). In the picture, $\mathcal{H}_{2u}^{\geq(2u)^*}$ is denoted by the dashed line. Thus, we have

$$\mathbf{II'}_2(u)=\underbrace{\int_{\mathcal{H}_{2u}^{(2u)^*}}r^{-4}|f|^2}_{\mathbf{A}}+\underbrace{\int_{\mathcal{H}_{2u}^{\geq(2u)^*}}r^{-4}|f|^2}_{\mathbf{B}}.$$

For the term $\mathbf{A}$, we can apply the $\gamma=4$ case of (A.11) and we obtain

$$\int_{\mathcal{H}_{2u}^{(2u)^*}}\frac{1}{r^4}|f|^2\lesssim\underbrace{u_+^{-3}\int_{\mathcal{S}_{2u}^{2u}}|f|^2}_{\mathbf{A}_1}+\underbrace{u_+^{-2}\int_{\mathcal{H}_{2u}^{(2u)^*}}\left|D_L f\right|^2}_{\mathbf{A}_2,\text{ use (3.5)}}\lesssim_{C_1,C_2}u_+^{-3}\int_{\mathcal{S}_{2u}^{2u}}|f|^2+\mathring{\varepsilon}\cdot u_+^{-\gamma_0-2+\frac{\varepsilon_0}{2}}.$$

So the contribution of $\mathbf{A}$ in $\mathbf{II}_2$ is bounded by (we can always assume that $CC_2\geq 1$)

$$\begin{aligned}\int_{\frac{r_1}{2}}^{\infty}u_+^{1+\frac{\varepsilon_0}{2CC_2}}\mathbf{A}\,du&\lesssim_{C_1,C_2}\underbrace{\int_{\frac{r_1}{2}}^{\infty}u_+^{-2+\frac{\varepsilon_0}{2CC_2}}\int_{\mathcal{S}_{2u}^{2u}}|f|^2du}_{\text{use (3.2) and Lemma A.2}}+\mathring{\varepsilon}\int_{\frac{r_1}{2}}^{\infty}u_+^{-\gamma_0-1+\varepsilon_0+\frac{\varepsilon_0}{2CC_2}}du\\&\lesssim_{C_0,C_1,C_2}\mathring{\varepsilon}\cdot u^{-\gamma_0+\varepsilon_0}.\end{aligned}$$

For the term $\mathbf{B}$, we can apply Lemma A.9 with $\gamma=4$ and $r_2=\infty$ to obtain that

$$\mathbf{B}\lesssim\underbrace{\left((2u)^*\right)^{-3}\int_{\mathcal{S}_{2u}^{(2u)^*}}|f|^2}_{\mathbf{B}_1}+\underbrace{(u^*)^{-2}\int_{\mathcal{H}_{2u}^{\geq(2u)^*}}\left|D_L f\right|^2}_{\mathbf{B}_2}. \tag{3.7}$$

The contribution of $\mathbf{B}_2$ in $\mathbf{II}_2$ is bounded by

$$\int_{\frac{r_1}{2}}^{\infty}u_+^{1+\frac{\varepsilon_0}{2CC_2}}\mathbf{B}_2du\lesssim\int_{\frac{r_1}{2}}^{\infty}u_+^{-1-\frac{\varepsilon_0}{2CC_2}}\int_{\mathcal{H}_{2u}}|D_L\psi|^2du.$$

Therefore, this is the same expression as $\mathbf{II_1}$ and we can ignore this term.

For $\mathbf{B}_1$, up to a universal constant, according to Lemma A.9 for $r_1=2u$ and $r_2=(2u)^*$, we have

$$\mathbf{B}_1\leq|u|^{-3}\int_{\mathcal{S}_{2u}^{2u}}|\psi|^2+\int_{\mathcal{H}_{2u}^{(2u)^*}}\frac{1}{r^2}|D_L\psi|^2.$$

Now the first term on the right hand side is $\mathcal{A}_1$ and the second term is $\mathcal{A}_2$ which have already been estimated. In particular we have bounded all the terms and thus complete the proof. $\square$

3.2. **Zeroth order energy estimates.** We prove the zeroth order energy estimate.

Proposition 3.3. *For $r_1 \geq R_*$ and $1 \leq p \leq 2$, we have*

$$\begin{aligned} \mathcal{E}^{(0)}(\phi; r_1) + \mathcal{E}^{(0)}(\mathring{F}; r_1) &\leq 2\mathring{\varepsilon} \cdot r_1^{-6-6\varepsilon_0}, \\ \mathcal{E}^{(0)}(\phi; p; r_1) + \mathcal{E}^{(0)}(\mathring{F}; p; r_1) &\leq 2\mathring{\varepsilon} \cdot r_1^{p-6-6\varepsilon_0}. \end{aligned} \tag{3.8}$$

Proof. The first estimate is an immediate consequence of the basic energy identity (2.18). We prove the second estimate for the endpoint case $p = 2$ (This is the only case which has applications in the current work. Indeed, for $p < 2$, the proof is exactly the same and one may also see [24]). We set $G = \mathring{F}$, $f = \phi$ and $rf = \psi = r\phi$ in Lemma 2.7. Thus, (2.19) yields

$$\begin{aligned} &\int_{\mathcal{H}_{r_1}^{r_2}} |D_L\psi|^2 + r^2|\mathring{\alpha}|^2 + \int_{\underline{\mathcal{H}}_{r_2}^{r_1}} |\not{D}\psi|^2 + r^2|\mathring{\rho}|^2 + r^2|\mathring{\sigma}|^2 + \int_{\mathcal{D}_{r_1}^{r_2}} r^{-1}\big(|D_L\psi|^2 + r^2|\mathring{\alpha}|^2\big) + \mathbf{Err_p} \\ &= \int_{\mathcal{B}_{r_1}^{r_2}} |D_L\psi|^2 + |\not{D}\psi|^2 + r^2\big(|\mathring{\alpha}|^2 + |\mathring{\rho}|^2 + |\mathring{\sigma}|^2\big) \leq \mathring{\varepsilon} \cdot r_1^{-4-8\varepsilon_0}. \end{aligned} \tag{3.9}$$

It suffices to bound the term $\mathbf{Err_p}$ of (2.19). It is straightforward to see that the integrand of $\mathbf{Err_p}$ is $q_0 r^{p-2} J_L$. Hence,

$$\big|\mathbf{Err_p}\big| \stackrel{p=2}{=} \big|q_0 \int_{\mathcal{D}_{r_1}^{r_2}} J_L\big| = \big|q_0 \int_{\mathcal{D}_{r_1}^{r_2}} \Im(\overline{D_L\phi} \cdot \phi)\big| = \big|q_0 \int_{\mathcal{D}_{r_1}^{r_2}} r^{-2}\Im(\overline{D_L\psi} \cdot \psi)\big|.$$

In particular, (3.9) implies

$$\int_{\mathcal{H}_{r_1}^{r_2}} |D_L\psi|^2 \lesssim \mathring{\varepsilon} \cdot r_1^{-4-8\varepsilon_0} + \int_{\mathcal{D}_{r_1}^{r_2}} r^{-2}|\psi||D_L\psi|$$

We now can use Lemma 3.1 (with $\gamma = -4 - 8\varepsilon_0$) and we obtain that

$$\int_{\mathcal{H}_{r_1}^{r_2}} |D_L\psi|^2 \lesssim \mathring{\varepsilon} \cdot r_1^{-4-7\varepsilon_0}.$$

We may always assume that $\mathbf{R}_*$ is large enough so that by afford a factor $r_1^{-\varepsilon_0}$ we beat all the constants from Lemma 3.1. This completes the proof. $\square$

4. The analysis in the exterior region 1: bootstrap ansatz and decay estimates

4.1. **Bootstrap ansatz.** We make two sets of ansatz on the exterior region $\mathcal{D}_{R_*}$. The first set is on the energy quantities:

$$\boxed{\begin{aligned} \mathcal{E}^{(\mathbf{k})}(\mathring{F}; r_1) + \mathcal{E}^{(\mathbf{k})}(\phi; r_1) &\leq 4\mathring{\varepsilon} r_1^{-6+2\xi(\mathbf{k})-(6-2|\mathbf{k}|)\varepsilon_0}, \\ \mathcal{E}^{(\mathbf{k})}(\mathring{F}; p; r_1) + \mathcal{E}^{(\mathbf{k})}(\phi; p; r_1) &\leq 4\mathring{\varepsilon} r_1^{p-6+2\xi(\mathbf{k})-(6-2|\mathbf{k}|)\varepsilon_0}, \end{aligned} \quad r_1 \geq R_*,\ |\mathbf{k}| = 1, 2 \text{ and } p \in [0, 2].} \tag{$\mathbf{B}$}$$

The second set is on the current terms:

$$\boxed{\begin{aligned} &\text{For all } r_1 \geq R_*,\ |\mathbf{k}| \leq 1, \text{ we assume} \\ &\int_{\mathcal{H}_{r_1}} \frac{|J_L^{(\mathbf{k})}|^2}{r_1^2} + \int_{\mathcal{H}_{r_1}} \frac{|\not{J}^{(\mathbf{k})}|^2}{r^2} + \sup_{r_2 \geq r_1} \int_{\underline{\mathcal{H}}_{r_2}^{r_1}} \frac{|\not{J}^{(\mathbf{k})}|^2}{r^2} + \sup_{r_2 \geq r_1} r_1^{\frac{3}{2}} \int_{\underline{\mathcal{H}}_{r_2}^{r_1}} \frac{|J_{\underline{L}}^{(\mathbf{k})}|^2}{r^{\frac{7}{2}}} \leq 4\mathring{\varepsilon}^2 r_1^{-8+2\xi(\mathbf{k})-4\varepsilon_0} \\ &\text{and for } |\mathbf{k}| = 2, \text{ we assume} \\ &\int_{\mathcal{H}_{r_1}} \frac{|J_L^{(\mathbf{k})}|^2}{r_1^2} + \int_{\mathcal{H}_{r_1}} \frac{|\not{J}^{(\mathbf{k})}|^2}{r^2} + \sup_{r_2 \geq r_1} r_1^{\frac{3}{2}} \int_{\underline{\mathcal{H}}_{r_2}^{r_1}} \frac{|J_{\underline{L}}^{(\mathbf{k})}|^2}{r^{\frac{7}{2}}} \leq 4\mathring{\varepsilon}^2 r_1^{-8+2\xi(\mathbf{k})-4\varepsilon_0}. \end{aligned}} \tag{$\mathbf{C}$}$$

We will show that if $\mathring{\varepsilon}$ is sufficiently small (by setting R_* to be sufficiently large), the constant 4 in the ansatz can be improved to be 2. In the sequel, the bootstrap argument should be understood dynamically (as one does in solving the Cauchy problem): we assume that the solution is defined in the region where $0 \leq t \leq T_*$ where T_* is a fixed positive number. Therefore, for sufficiently small T_*, ($\mathbf{B}$) holds. The bootstrap argument will show that one can indeed replace the constant 4 by 2 and this is independent of T_*. Therefore, by letting $T_* \to \infty$, we obtain estimates on the entire spacetime.

Based on these ansatz, we will first derive pointwise estimates on $\mathring{F}$ and ϕ.

4.2. **Pointwise decay estimates of the Maxwell field.** We use (**B**) to bound $\mathring{\alpha}$, $\mathring{\rho}$, $\mathring{\sigma}$ and $\underline{\mathring{\alpha}}$.

Proposition 4.1. *We have the following decay estimates:*

$$|\mathring{\alpha}| \lesssim \sqrt{\mathring{\varepsilon}} r^{-3} u_+^{-1-\varepsilon_0}, \quad |\mathring{\rho}| + |\mathring{\sigma}| \lesssim \sqrt{\mathring{\varepsilon}} r^{-2} u_+^{-2-\varepsilon_0}, \quad |\underline{\mathring{\alpha}}| \lesssim \sqrt{\mathring{\varepsilon}} r^{-1} u_+^{-3-\varepsilon_0}.$$

Proof. **Step 1.** L^∞ **estimate of** $\underline{\mathring{\alpha}}$**.**

In view of (A.6), Lemma A.8, the last equation in (2.15) and the fact that $\underline{L} = 2T - L$, we have

$$\begin{aligned}\int_{\underline{\mathcal{H}}_{r_2}^{r_1}} |\mathcal{L}_{\underline{L}}\big(\mathcal{L}_\Omega \underline{\mathring{\alpha}}\big)|^2 &\leq \int_{\underline{\mathcal{H}}_{r_2}^{r_1}} |\mathcal{L}_T\big(\mathcal{L}_\Omega \underline{\mathring{\alpha}}\big)|^2 + \int_{\underline{\mathcal{H}}_{r_2}^{r_1}} |\mathcal{L}_L\big(\mathcal{L}_\Omega \underline{\mathring{\alpha}}\big)|^2 \\ &= \int_{\underline{\mathcal{H}}_{r_2}^{r_1}} |\underline{\alpha}^{(\mathbf{2})}|^2 + \int_{\underline{\mathcal{H}}_{r_2}^{r_1}} |-\nabla\!\!\!/\,\rho^{(\mathbf{1})} + {}^*\nabla\!\!\!/\,\sigma^{(\mathbf{1})} + r^{-2} J\!\!\!/^{(\mathbf{1})} + \frac{1}{r}\underline{\alpha}^{(\mathbf{1})}|^2.\end{aligned}$$

We remark that in this case $\xi(\mathbf{2}) = -1$ and $\xi(\mathbf{1}) = 0$. By (**B**), we then have

$$\int_{\underline{\mathcal{H}}_{r_2}^{r_1}} |\mathcal{L}_{\underline{L}}\big(\mathcal{L}_\Omega \underline{\mathring{\alpha}}\big)|^2 \lesssim \mathring{\varepsilon} r_1^{-8-2\varepsilon_0} + \mathring{\varepsilon} r_1^{-6-4\varepsilon_0} + \int_{\underline{\mathcal{H}}_{r_2}^{r_1}} \frac{|J\!\!\!/^{(\mathbf{1})}|^2}{r^4}.$$

We can use the first term of (**C**) to bound the last term in the above inequality. Recall that for forms Ξ, we have $r^2|\nabla\!\!\!/\,\Xi|^2 \lesssim |\mathcal{L}_\Omega \Xi|^2 + |\Xi|^2$. Therefore, (**B**) together (**C**) imply that

$$\int_{\underline{\mathcal{H}}_{r_2}^{r_1}} |\mathcal{L}_{\underline{L}}\big(\mathcal{L}_\Omega \underline{\mathring{\alpha}}\big)|^2 \lesssim \mathring{\varepsilon} r_1^{-6-4\varepsilon_0}.$$

By (**B**), we also have

$$\int_{\underline{\mathcal{H}}_{r_2}^{r_1}} |\mathcal{L}_\Omega\big(\mathcal{L}_\Omega \underline{\mathring{\alpha}}\big)|^2 \lesssim \mathring{\varepsilon} r_1^{-6-2\varepsilon_0}.$$

We then can apply (A.2) to derive

$$\|\mathcal{L}_\Omega \underline{\mathring{\alpha}}\|_{L^4(\mathcal{S}_{r_1}^{r_2})} \lesssim r_2^{-\frac{1}{2}} \sqrt{\mathring{\varepsilon}} r_1^{-3-\varepsilon_0}.$$

We can repeat the above argument by switching $\mathcal{L}_\Omega \underline{\mathring{\alpha}}$ to $\underline{\mathring{\alpha}}$ and we obtain

$$\|\underline{\mathring{\alpha}}\|_{L^4(\mathcal{S}_{r_1}^{r_2})} \lesssim r_2^{-\frac{1}{2}} \sqrt{\mathring{\varepsilon}} r_1^{-3-2\varepsilon_0}.$$

Compared to the L^4 bound of $\mathcal{L}_\Omega \underline{\mathring{\alpha}}$, this bound gains an extra $r^{-\varepsilon_0}$ because we use one less derivative in this case. This is clear from the bootstrap ansatz (**B**). We then apply the Sobolev inequality (A.1) on $\mathcal{S}_{r_2}^{r_1}$. In view of the fact that $\frac{r_1+r_2}{2} \approx r_2$ and $|u| \approx r_1$ on $\mathcal{S}_{r_2}^{r_1}$, we obtain

$$|\underline{\mathring{\alpha}}| \lesssim \sqrt{\mathring{\varepsilon}} r^{-1} u_+^{-3-\varepsilon_0}. \tag{4.1}$$

Step 2. L^∞ **estimate of** $\mathring{\rho}$ **and** $\mathring{\sigma}$**.** We only derive the bound on $\mathring{\rho}$ since $\mathring{\sigma}$ can be bounded exactly in the same manner. First of all, for $\mathbf{l} = (0,1,0)$ and $\mathbf{k} = (0,2,0)$, we have

$$\underline{L}\big(\mathcal{L}_\Omega(r\mathring{\rho})\big) = r^{-1}\underline{L}(r^2\rho^{(\mathbf{1})}) - \rho^{(\mathbf{1})}.$$

Thus by using the null equation for $\mathring{\rho}$ as well as the bootstrap assumptions we can show that

$$\begin{aligned}\int_{\underline{\mathcal{H}}_{r_2}^{r_1}} |\underline{L}\big(\mathcal{L}_\Omega(r\mathring{\rho})\big)|^2 &\leq \int_{\underline{\mathcal{H}}_{r_2}^{r_1}} r^{-2}|\mathcal{L}_{\underline{L}}(r^2\rho^{(\mathbf{l})})|^2 + |\rho^{(\mathbf{l})}|^2 \overset{(2.15)}{=} \int_{\underline{\mathcal{H}}_{r_2}^{r_1}} r^{-2}|\mathbf{d}\!\!\!/\mathbf{iv}\,(r^2\underline{\alpha}^{(\mathbf{l})}) - J_{\underline{L}}^{(\mathbf{l})}|^2 + |\rho^{(\mathbf{l})}|^2 \\ &\leq \int_{\underline{\mathcal{H}}_{r_2}^{r_1}} |\underline{\alpha}^{(\mathbf{k})})|^2 + |\rho^{(\mathbf{l})}|^2 + r^{-2}|J_{\underline{L}}^{(\mathbf{l})}|^2 \overset{(\mathbf{B}),(\mathbf{C})}{\lesssim} \mathring{\varepsilon} r_1^{-6-2\varepsilon_0}.\end{aligned}$$

By the $p = 2$ case of (**B**), we also have

$$\int_{\underline{\mathcal{H}}_{r_2}^{r_1}} r^2|\mathcal{L}_\Omega \mathring{\rho}|^2 + \int_{\underline{\mathcal{H}}_{r_2}^{r_1}} r^2|\mathcal{L}_\Omega\big(\mathcal{L}_\Omega \mathring{\rho}\big)|^2 \lesssim \mathring{\varepsilon} r_1^{-4-2\varepsilon_0}.$$

Therefore, we obtain that

$$\int_{\underline{\mathcal{H}}_{r_2}^{r_1}} |\mathcal{L}_\Omega(r\mathring{\rho})|^2 + \int_{\underline{\mathcal{H}}_{r_2}^{r_1}} \Big|\mathcal{L}_{\underline{L}}\Big(\mathcal{L}_\Omega(r\mathring{\rho})\Big)\Big|^2 + \int_{\underline{\mathcal{H}}_{r_2}^{r_1}} \Big|\mathcal{L}_\Omega\Big(\mathcal{L}_\Omega(r\mathring{\rho})\Big)\Big|^2 \lesssim \mathring{\varepsilon} r_1^{-4-2\varepsilon_0}.$$

According to (A.2), the above energy estimate implies that

$$\|\mathcal{L}_\Omega(r\mathring{\rho})\|_{L^4(\mathcal{S}_{r_1}^{r_2})} \lesssim r_2^{-\frac{1}{2}} \sqrt{\mathring{\varepsilon}} r_1^{-2-\varepsilon_0}.$$

Similarly, we have

$$\|r\mathring{\rho}\|_{L^4(\mathcal{S}_{r_1}^{r_2})} \lesssim r_2^{-\frac{1}{2}} \sqrt{\mathring{\varepsilon}} r_1^{-2-2\varepsilon_0}.$$

We notice that this is a similar bound but with an extra $r_1^{-\varepsilon_0}$ due to one less derivative compared to the previous case. We then apply (A.1) on $\mathcal{S}_{r_2}^{r_1}$ and conclude that

$$|\mathring{\rho}| \lesssim \sqrt{\mathring{\varepsilon}} r^{-2} u_+^{-2-\varepsilon_0}. \tag{4.2}$$

Remark 4.2. *By using the flux on $\mathcal{H}_{r_1}^{r_2}$, the same argument yields:*

$$|\mathring{\alpha}| \lesssim \sqrt{\mathring{\varepsilon}} r^{-2} u_+^{-2-\varepsilon_0}.$$

This is not optimal and we will obtain a better decay in the next step.

Step 3. L^∞ estimate of $\mathring{\alpha}$. The sharp decay of $\mathring{\alpha}$ relies on the commutator K and the r^p-weighted energy estimate. Note that for an arbitrary two form G, we have

$$\alpha(\mathcal{L}_K G)_A = v^{-1}\nabla_L(v^3\alpha(G))_A + u^2\nabla_{\underline{L}}\alpha(G)_A + u\alpha(G)_A.$$

Therefore, we have

$$v\alpha(\mathcal{L}_K G)_A = \nabla_L\big(v^3\alpha(G)\big)_A + u^2\nabla_{\underline{L}}\big(r\alpha(G)\big)_A + (u^2+uv)\alpha(G)_A.$$

If we take $G = \mathcal{L}_\Omega \mathring{F}$, in view of the third equation in (2.15), we also have

$$\nabla_L(v^3\mathcal{L}_\Omega\mathring{\alpha}) = v\alpha^{(0,1,1)} - (u^2+uv)\alpha^{(0,1,0)} - u^2\big[\nabla\!\!\!/(r\rho^{(0,1,0)}) + {}^*\nabla\!\!\!/(r\sigma^{(0,1,0)}) - r^{-1}J\!\!\!/^{(0,1,0)}\big]. \tag{4.3}$$

By virtue of the bootstrap assumptions (**B**), (**C**) and $|u|\lesssim r$, especially the r^p-weighted energy norms, we have

$$\begin{aligned}\int_{\mathcal{H}_{r_1}} |\nabla_L(v^3\mathcal{L}_\Omega\mathring{\alpha})|^2 &\lesssim \int_{\mathcal{H}_{r_1}} v^2|\alpha^{(0,1,1)}|^2 + |u|^2v^2|\alpha^{(0,1,0)}|^2 + |u|^4\big(|\rho^{(0,2,0)}|^2 + |\sigma^{(0,2,0)}|^2\big) + \frac{|u|^4}{r^2}|J\!\!\!/^{(0,1,0)}|^2\\ &\lesssim \mathring{\varepsilon} r_1^{-2-2\varepsilon_0}.\end{aligned}$$

In view of $v = u + r$, we have

$$\|\nabla_L(rv^2\mathcal{L}_\Omega\mathring{\alpha})\|_{L^2(\mathcal{H}_{r_1})} \lesssim \sqrt{\mathring{\varepsilon}} r_1^{-1-\varepsilon_0}. \tag{4.4}$$

This estimate can be used to get a sharp decay estimates for $\|\mathcal{L}_\Omega\mathring{\alpha}\|_{L^2(\mathcal{S}_{r_1}^{r_2})}$. In fact, we have

$$\begin{aligned}\|v^2\mathcal{L}_\Omega\mathring{\alpha}\|^2_{L^2(\mathcal{S}_{r_1}^{r_2})} - \|v^2\mathcal{L}_\Omega\mathring{\alpha}\|^2_{L^2(\mathcal{S}_{r_1}^{r_1})} &= \int_{\frac{r_2}{2}}^{\frac{r_1}{2}}\int_{\mathbf{S}^2} L\big(\big|rv^2\mathcal{L}_\Omega\mathring{\alpha}\big|^2\big)d\vartheta dv\\ &\lesssim \int_{\frac{r_2}{2}}^{\frac{r_1}{2}}\int_{\mathbf{S}^2} |\nabla_L\big(rv^2\mathcal{L}_\Omega\mathring{\alpha}\big)||r\mathcal{L}_\Omega\mathring{\alpha}|r^2 d\vartheta dv\\ &\leq \|\nabla_L(rv^2\mathcal{L}_\Omega\mathring{\alpha})\|_{L^2(\mathcal{H}_{r_1})}\|r\mathcal{L}_\Omega\mathring{\alpha}\|_{L^2(\mathcal{H}_{r_1})}\end{aligned}$$

Thus,

$$\begin{aligned}\|v^2\mathcal{L}_\Omega\mathring{\alpha}\|^2_{L^2(\mathcal{S}_{r_1}^{r_2})} &\lesssim \|v^2\mathcal{L}_\Omega\mathring{\alpha}\|^2_{L^2(\mathcal{S}_{r_1}^{r_1})} + \|\nabla_L(rv^2\mathcal{L}_\Omega\mathring{\alpha})\|_{L^2(\mathcal{H}_{r_1})}\|r\mathcal{L}_\Omega\mathring{\alpha}\|_{L^2(\mathcal{H}_{r_1})}\\ &\lesssim \mathring{\varepsilon} r_1^{-3-3\varepsilon_0}\end{aligned}$$

As a result, we obtain

$$\|\mathcal{L}_\Omega\mathring{\alpha}\|_{L^2(\mathcal{S}_{r_1}^{r_2})} \lesssim \sqrt{\mathring{\varepsilon}} r_2^{-2} r_1^{-\frac{3}{2}-\frac{3}{2}\varepsilon_0}. \tag{4.5}$$

One can also bound $\|\mathcal{L}_\Omega\mathring{\alpha}\|_{L^4(\mathcal{S}_{r_1}^{r_2})}$. We take $\Xi = r\mathcal{L}_\Omega\mathring{\alpha}$ in (A.2) and we obtain

$$\begin{aligned}r_2^3\|\mathcal{L}_\Omega\mathring{\alpha}\|^2_{L^4(\mathcal{S}_{r_1}^{r_2})} &\lesssim \int_{\mathcal{H}_{r_2}^{r_1}} |r\mathcal{L}_\Omega\alpha|^2 + \int_{\mathcal{H}_{r_2}^{r_1}} \frac{1}{r^2}|\mathcal{L}_L(r^2\mathcal{L}_\Omega\mathring{\alpha})|^2 + \int_{\mathcal{H}_{r_2}^{r_1}} r^2|\mathcal{L}_\Omega(\mathcal{L}_\Omega\mathring{\alpha})|^2\\ &\lesssim \int_{\mathcal{H}_{r_2}^{r_1}} r^2|\alpha^{(0,1,0)}|^2 + \underbrace{\int_{\mathcal{H}_{r_2}^{r_1}} \frac{1}{r^2}|\mathcal{L}_L(r^2\mathcal{L}_\Omega\mathring{\alpha})|^2}_{\textit{bounded in (4.4)}} + \int_{\mathcal{H}_{r_2}^{r_1}} r^2|\alpha^{(0,2,0)}|^2\\ &\lesssim \mathring{\varepsilon} r_1^{-4-2\varepsilon_0}.\end{aligned}$$

In other words, we have

$$\|\mathcal{L}_\Omega\mathring{\alpha}\|_{L^4(\mathcal{S}_{r_1}^{r_2})} \lesssim \sqrt{\mathring{\varepsilon}} r_2^{-\frac{3}{2}} r_1^{-2-\varepsilon_0}. \tag{4.6}$$

For $q \in [2,4]$, by interpolating (4.5) and (4.6), we have

$$\|\mathcal{L}_\Omega\mathring{\alpha}\|_{L^q(\mathcal{S}_{r_1}^{r_2})} \lesssim \sqrt{\mathring{\varepsilon}} r_2^{-\left(1+\frac{2}{q}\right)} r_1^{-\left(\frac{5}{2}-\frac{2}{q}+(\frac{1}{2}+\frac{2}{q})\varepsilon_0\right)},\ 2\leq q\leq 4. \tag{4.7}$$

We now try to improve decay in r_2 in (4.7) for $2 < q < \frac{9}{4}$. For this purpose, we choose γ so that

$$\gamma + \frac{2}{q} = 3.$$

Therefore, we have

$$\begin{aligned}\|r^\gamma \mathcal{L}_\Omega \mathring{\alpha}\|^q_{L^q(\mathcal{S}^{r_2}_{r_1})} - \|r^\gamma \mathcal{L}_\Omega \mathring{\alpha}\|^q_{L^q(\mathcal{S}^{r_1}_{r_1})} &= \int_{\frac{r_2}{2}}^{\frac{r_1}{2}} \int_{\mathbf{S}^2} L\big(|r^3 \mathcal{L}_\Omega \mathring{\alpha}|^q\big) d\vartheta dv \\ &\lesssim \int_{\frac{r_2}{2}}^{\frac{r_1}{2}} \int_{\mathbf{S}^2} |\nabla_L\big(rv^2 \mathcal{L}_\Omega \mathring{\alpha}\big)||r^3 \mathcal{L}_\Omega \mathring{\alpha}|^{q-1} d\vartheta dv.\end{aligned}$$

According to Cauchy-Schwarz inequality, we have

$$\|r^\gamma \mathcal{L}_\Omega \mathring{\alpha}\|^q_{L^q(\mathcal{S}^{r_2}_{r_1})} \lesssim \|r^\gamma \mathcal{L}_\Omega \mathring{\alpha}\|^q_{L^q(\mathcal{S}^{r_1}_{r_1})} + \|\nabla_L\big(rv^2 \mathcal{L}_\Omega \mathring{\alpha}\big)\|_{L^2(\mathcal{H}_{r_1})} \underbrace{\|r^{3q-5}|\mathcal{L}_\Omega \mathring{\alpha}|^{q-1}\|_{L^2(\mathcal{H}_{r_1})}}_{\mathbf{I}}. \tag{4.8}$$

To bound $\mathbf{I}$, since $q < \frac{9}{4}$, we proceed as follows

$$\begin{aligned}\mathbf{I} &= \Big(\int_{\mathcal{H}_{r_1}} r^{6q-10} |\mathcal{L}_\Omega \mathring{\alpha}|^{2q-2}\Big)^{\frac{1}{2}} = \Big(\int_{r_1}^{r_2} r^{6q-10} \|\mathcal{L}_\Omega \mathring{\alpha}\|^{2q-2}_{L^{2q-2}(\mathcal{S}^r_{r_1})} dr\Big)^{\frac{1}{2}} \\ &\overset{(4.7)}{\lesssim} \Big(\int_{r_1}^{r_2} r^{6q-10} \cdot \mathring{\varepsilon}^{q-1} \cdot r^{-2q} r_1^{-\big(5q-7+(q+1)\varepsilon_0\big)} dr\Big)^{\frac{1}{2}} \lesssim \mathring{\varepsilon}^{\frac{q-1}{2}} r_1^{-\frac{1}{2}\big(q+2+(q+1)\varepsilon_0\big)}.\end{aligned}$$

In view of (6.18) and (4.4), we have

$$\|r^\gamma \mathcal{L}_\Omega \mathring{\alpha}\|^q_{L^q(\mathcal{S}^{r_2}_{r_1})} \lesssim \mathring{\varepsilon}^{\frac{q}{2}} r_1^{-q(1+\varepsilon_0)} + \mathring{\varepsilon}^{\frac{q}{2}} r_1^{-\frac{1}{2}\big(q+4+(q+3)\varepsilon_0\big)}$$

Therefore, we have

$$\|\mathcal{L}_\Omega \mathring{\alpha}\|_{L^q(\mathcal{S}^{r_2}_{r_1})} \lesssim \sqrt{\mathring{\varepsilon}} r_2^{\frac{2}{q}-3} r_1^{-(1+\varepsilon_0)}, \text{ for } 2 < q < \frac{9}{4}. \tag{4.9}$$

We remark that, compared to (4.7), the decay in r_2 has been improved. Similary, we also have

$$\|\mathring{\alpha}\|_{L^q(\mathcal{S}^{r_2}_{r_1})} \lesssim \sqrt{\mathring{\varepsilon}} r_2^{\frac{2}{q}-3} r_1^{-(1+\varepsilon_0)}, \text{ for } 2 < q < \frac{9}{4}. \tag{4.10}$$

We can fix a $q \in (2, \frac{9}{4})$ (say $q = \frac{17}{8}$) and apply (A.1). Therefore, (4.9) and (4.10) together yield

$$|\mathring{\alpha}| \lesssim \sqrt{\mathring{\varepsilon}} r^{-3} u_+^{-1-\varepsilon_0}.$$

This completes the proof. □

4.3. **Pointwise decay estimates of the scalar field.** We start with the decay estimate of ϕ on the initial slice $\mathcal{B}_{R_*}$. By (A.2) and (A.1), we have

$$\|\phi\|_{L^4(\mathcal{S}^{r_1}_{r_1})} \lesssim \sqrt{\mathring{\varepsilon}} r_1^{-\frac{5}{2}-4\varepsilon_0}, \quad \|D_\Omega \phi\|_{L^4(\mathcal{S}^{r_1}_{r_1})} \lesssim \sqrt{\mathring{\varepsilon}} r_1^{-\frac{5}{2}-4\varepsilon_0}.$$

By (A.1), we have

$$\|\phi\|_{L^\infty(\mathcal{S}^{r_1}_{r_1})} \lesssim \sqrt{\mathring{\varepsilon}} r_1^{-3-4\varepsilon_0}.$$

Proposition 4.3. *For the solution (ϕ, F) of the MKG equations on the exterior region $\{t + R_* \leq |x|\}$, the scalar field verifies the following decay estimates:*

$$\begin{aligned}|\phi| \lesssim \sqrt{\mathring{\varepsilon}} r^{-1} u_+^{-\frac{5}{2}-2\varepsilon_0}, &\quad |D_{\underline{L}}\phi| \lesssim \sqrt{\mathring{\varepsilon}} r^{-1} u_+^{-3-\varepsilon_0}, \\ |\not{D}\phi| \lesssim \sqrt{\mathring{\varepsilon}} r^{-2} u_+^{-2-\varepsilon_0}, &\quad |D_L \psi| \lesssim \sqrt{\mathring{\varepsilon}} r^{-2} u_+^{-1-\varepsilon_0}.\end{aligned}$$

Proof. **Step 1. L^∞ estimate of ϕ.** For $k \leq 2$, by Lemma A.5 and (**B**) we have

$$\|D^k_\Omega \phi\|^2_{L^2(\mathcal{S}^{r_2}_{r_1})} \lesssim \|D^k_\Omega \phi\|^2_{L^2(\mathcal{S}^{r_1}_{r_1})} + \frac{1}{r_1} \int_{\mathcal{H}_{r_1}} |D_L D^k_\Omega \psi|^2 \overset{(\mathbf{B})}{\lesssim} \mathring{\varepsilon} r_1^{-5-2\varepsilon_0}. \tag{4.11}$$

We now use (A.1) to conclude that

$$\|\phi\|_{L^\infty(\mathcal{S}^{r_2}_{r_1})} \lesssim \sqrt{\mathring{\varepsilon}} r^{-1} u_+^{-\frac{5}{2}-\varepsilon_0}.$$

Here note that $u_+ = 1 + \frac{1}{2}|t - r| = 1 + \frac{1}{2} r_1$. We can indeed improve the estimates by gaining a $r_1^{-\varepsilon_0}$. First of all, notice that in (4.11), for $k \leq 1$, we have

$$\|D^k_\Omega \phi\|^2_{L^2(\mathcal{S}^{r_2}_{r_1})} \lesssim \mathring{\varepsilon} r_1^{-5-4\varepsilon_0}.$$

To save one derivative, we can use the second equation in (A.3) to derive that

$$\|D_\Omega \phi\|^2_{L^4(\mathcal{S}^{r_2}_{r_1})} \lesssim \mathring{\varepsilon} r_2^{-1} r_1^{-4-4\varepsilon_0}.$$

Thus, by (A.1) again, we have

$$\|\phi\|_{L^\infty(\mathcal{S}^{r_2}_{r_1})} \lesssim \sqrt{\mathring{\varepsilon}} r^{-1} r_1^{-\frac{5}{2}-2\varepsilon_0} \lesssim \sqrt{\mathring{\varepsilon}} r^{-1} u_+^{-\frac{5}{2}-2\varepsilon_0}. \tag{4.12}$$

Step 2. L^∞ estimate of $D_{\underline{L}}\phi$. We first bound $\int_{\underline{\mathcal{H}}_{r_2}^{r_1}} |D_{\underline{L}}(D_\Omega D_{\underline{L}}\phi)|^2$. It can be split into:

$$\int_{\underline{\mathcal{H}}_{r_2}^{r_1}} |D_{\underline{L}}(D_\Omega D_{\underline{L}}\phi)|^2 \leq \underbrace{\int_{\underline{\mathcal{H}}_{r_2}^{r_1}} |D_T(D_\Omega D_{\underline{L}}\phi)|^2}_{\mathbf{I}_1} + \underbrace{\int_{\underline{\mathcal{H}}_{r_2}^{r_1}} |D_L(D_\Omega D_{\underline{L}}\phi)|^2}_{\mathbf{I}_2}.$$

To bound $\mathbf{I}_1$, we first commute derivatives to derive

$$\begin{aligned} D_T D_\Omega D_{\underline{L}}\phi &= D_T\big([D_\Omega, D_{\underline{L}}]\phi\big) + [D_T, D_{\underline{L}}]D_\Omega\phi + D_{\underline{L}}D_T D_\Omega\phi \\ &= \sqrt{-1}\mathcal{L}_T F_{\Omega\underline{L}}\phi + \sqrt{-1}F_{\Omega\underline{L}}D_T\phi + \sqrt{-1}F_{T\underline{L}}D_\Omega\phi + D_{\underline{L}}D_T D_\Omega\phi. \end{aligned}$$

We therefore can bound that

$$|D_T D_\Omega D_{\underline{L}}\phi| \leq r|\underline{\alpha}^{(\mathbf{1})}||\phi| + r|\underline{\alpha}||\phi^{(\mathbf{1})}| + r|\rho||\not{D}\phi| + |D_{\underline{L}}\phi^{(\mathbf{2})}|,$$

where the discrepancy indices of the $(\mathbf{1})$ and $(\mathbf{2})$ are all equal to -1 and we note that α, $\underline{\alpha}$ and ρ are the curvature components for the full Maxwell field F. Therefore, we can split $\mathbf{I}_1$ into four terms:

$$\mathbf{I}_1 \leq \int_{\underline{\mathcal{H}}_{r_2}^{r_1}} r^2|\underline{\alpha}^{(\mathbf{1})}|^2|\phi|^2 + \int_{\underline{\mathcal{H}}_{r_2}^{r_1}} r^2|\underline{\alpha}|^2|\phi^{(\mathbf{1})}|^2 + \int_{\underline{\mathcal{H}}_{r_2}^{r_1}} r^2|\rho|^2|\not{D}\phi|^2 + \int_{\underline{\mathcal{H}}_{r_2}^{r_1}} |D_{\underline{L}}\phi^{(\mathbf{2})}|^2.$$

Recall that the full Maxwell field F splits into the chargeless part $\mathring{F}$ which has been bounded in Proposition 4.1 and the charge part $F[q_0]$ satisfying the trivial bound (1.2). Since $F[q_0]$ is stationary, we note that $\underline{\alpha}^{(\mathbf{1})} = \mathring{\underline{\alpha}}^{(\mathbf{1})}$. Therefore we can use (4.12) to bound ϕ in the first term, use $|\underline{\alpha}| \lesssim r^{-1}u_+^{-2}$ for the second term, use $|\rho| \lesssim r^{-2}$ in the third term and the bootstrap assumption $(\mathbf{B})$ to bound the last term. In particular we can show that

$$\begin{aligned} \mathbf{I}_1 &\lesssim \int_{\underline{\mathcal{H}}_{r_2}^{r_1}} |\mathring{\underline{\alpha}}^{(\mathbf{1})}|^2 r_1^{-5-4\varepsilon_0} + u_+^{-4}|\phi^{(\mathbf{1})}|^2 + r^{-2}|\not{D}\phi|^2 + |D_{\underline{L}}\phi^{(\mathbf{2})}|^2 \\ &\lesssim \mathring{\varepsilon} r_1^{-8-2\varepsilon_0} + r_1^{-4}\int_{\underline{\mathcal{H}}_{r_2}^{r_1}} |\phi^{(\mathbf{1})}|^2. \end{aligned}$$

Since $\phi^{(\mathbf{1})} = D_T\phi$, according to (A.5), for $k \leq 1$ we have

$$\|D_T D_\Omega^k\phi\|^2_{L^2(\mathcal{S}_{r_1}^{r_2})} \lesssim \|D_T D_\Omega^k\phi\|^2_{L^2(\mathcal{S}_{r_1}^{r_1})} + \frac{1}{r_1}\int_{\mathcal{H}_{r_1}} |D_L D_T D_\Omega^k\psi|^2 \overset{(\mathbf{B})}{\lesssim} \mathring{\varepsilon} r_1^{-7-2\varepsilon_0}. \tag{4.13}$$

We now use the case $k = 0$ to conclude that

$$\int_{\underline{\mathcal{H}}_{r_2}^{r_1}} |\phi^{(\mathbf{1})}|^2 \lesssim \int_{-\frac{r_2}{2}}^{-\frac{r_1}{2}} \Big(\int_{\mathcal{S}_{-2u}^{r_2}} \mathring{\varepsilon} u_+^{-4}|D_T\phi|^2\Big)du \lesssim \mathring{\varepsilon}\int_{-\frac{r_2}{2}}^{-\frac{r_1}{2}} \mathring{\varepsilon} u_+^{-11-2\varepsilon_0}du \lesssim \mathring{\varepsilon}^2 r_1^{-10-2\varepsilon_0}.$$

Here we keep in mind that $u_+ = 1 + \frac{1}{2}r_1$. In particular we derive that

$$\mathbf{I}_1 \lesssim \mathring{\varepsilon} r_1^{-8-2\varepsilon_0}.$$

Now we turn to the estimate of $\mathbf{I}_2$. By using the null equations for ϕ, we first can write that

$$\begin{aligned} D_L D_\Omega D_{\underline{L}}\phi &= D_L\big([D_\Omega, D_{\underline{L}}]\phi\big) + r^{-1}D_L D_{\underline{L}}\big(rD_\Omega\phi\big) + r^{-1}(D_L D_\Omega\phi - D_{\underline{L}}D_\Omega\phi) \\ &\overset{(2.3)}{=} \sqrt{-1}D_L\big(F_{\Omega\underline{L}}\phi\big) - \Box_A D_\Omega\phi + \not{D}^2 D_\Omega\phi - \sqrt{-1}\rho\cdot D_\Omega\phi + \frac{1}{r}(D_L D_\Omega\phi - D_{\underline{L}}D_\Omega\phi) \\ &= \sqrt{-1}\mathcal{L}_L F_{\Omega\underline{L}}\cdot\phi - Q(\phi, F; \Omega) + \Big(2\sqrt{-1}F_{\Omega\underline{L}}D_T\phi + \frac{2}{r}D_T D_\Omega\phi\Big) \\ &\quad + \Big(\not{D}^2\big(D_\Omega\phi\big) - \sqrt{-1}\rho\cdot\big(D_\Omega\phi\big) - \frac{2}{r}D_{\underline{L}}D_\Omega\phi - \sqrt{-1}F_{\Omega\underline{L}}D_{\underline{L}}\phi\Big). \end{aligned}$$

For the integral of the last term, we use the pointwise bounds:

$$|\rho| \lesssim r^{-2}, \quad |F_{\Omega\underline{L}}| = r|\underline{\alpha}| \lesssim r^{-2} + \sqrt{\mathring{\varepsilon}}u_+^{-3-\varepsilon_0} \lesssim u_+^{-2}.$$

We therefore can bound that

$$\begin{aligned} &\int_{\underline{\mathcal{H}}_{r_2}^{r_1}} |\not{D}^2\big(D_\Omega\phi\big) - \sqrt{-1}\rho\cdot\big(D_\Omega\phi\big) - \frac{2}{r}D_{\underline{L}}D_\Omega\phi - \sqrt{-1}F_{\Omega\underline{L}}D_{\underline{L}}\phi|^2 \\ &\lesssim \int_{\underline{\mathcal{H}}_{r_2}^{r_1}} r^{-2}|\not{D}D_\Omega^2\phi|^2 + r^{-4}|D_\Omega\phi|^2 + r^{-2}|D_{\underline{L}}D_\Omega\phi|^2 + u_+^{-4}|D_{\underline{L}}\phi|^2 \\ &\lesssim \mathring{\varepsilon} r_1^{-8-2\varepsilon_0}. \end{aligned}$$

For the third term in the previous identity, by using the above estimate (4.13), we can show that

$$\int_{\underline{\mathcal{H}}_{r_2}^{r_1}} |2\sqrt{-1}F_{\Omega\underline{L}}D_T\phi + \frac{2}{r}D_T D_\Omega\phi|^2 \lesssim \int_{\underline{\mathcal{H}}_{r_2}^{r_1}} r^{-2}|D_T D_\Omega\phi|^2 + u_+^{-4}|D_T\phi|^2 \lesssim \mathring{\varepsilon} r_1^{-8-2\varepsilon_0}.$$

For the first term $\sqrt{-1}\mathcal{L}_L F_{\Omega\underline{L}}\cdot\phi$, we use the null equation (2.15) to show that

$$\int_{\underline{\mathcal{H}}_{r_2}^{r_1}} |\sqrt{-1}\mathcal{L}_L F_{\Omega\underline{L}}\cdot\phi|^2 \lesssim \int_{\underline{\mathcal{H}}_{r_2}^{r_1}} \big(|\rho^{(\mathbf{1})}|^2+|\sigma^{(\mathbf{1})}|^2+r^2|\not{J}|^2+r^2|\underline{\alpha}|^2\big)|\phi|^2.$$

Now recall that $|\not{J}|=|\phi||\not{D}\phi|$ and we have the bounds $|\mathcal{L}_\Omega F[q_0]|\lesssim r^{-3}$. Then by using the bootstrap assumptions on $\mathring{F}$ as well as the pointwise bound for ϕ, we indeed can show that

$$\int_{\underline{\mathcal{H}}_{r_2}^{r_1}} |\sqrt{-1}\mathcal{L}_L F_{\Omega\underline{L}}\cdot\phi|^2 \lesssim \int_{\underline{\mathcal{H}}_{r_2}^{r_1}} \big(|\mathring{\rho}^{(\mathbf{1})}|^2+|\mathring{\sigma}^{(\mathbf{1})}|^2+u_+^{-5}|\not{D}\phi|^2+r^2|\mathring{\underline{\alpha}}|^2+r^{-4}\big)|\phi|^2 \lesssim \mathring{\varepsilon} r_1^{-8-2\varepsilon_0}.$$

Finally for the quadratic term $Q(\phi,F;\Omega)$, we use the bound (2.9) in the proof for Proposition 2.4 to show that

$$\begin{aligned}\int_{\underline{\mathcal{H}}_{r_2}^{r_1}} |Q(\phi,F;\Omega)|^2 &\lesssim \int_{\underline{\mathcal{H}}_{r_2}^{r_1}} |D_L\psi|^2|\underline{\alpha}|^2+|D_{\underline{L}}\psi|^2|\alpha|^2+|\sigma|^2|\phi|^2+r^2|\not{J}|^2|\phi|^2+|\sigma|^2|\not{D}\psi|^2\\
&\lesssim \int_{\underline{\mathcal{H}}_{r_2}^{r_1}} |D_L\psi|^2 r^{-2}u_+^{-4}+|D_{\underline{L}}\psi|^2 r^{-6}+r^{-4}u_+^{-2}(|\phi|^2+r^2|\not{D}\phi|^2)+u_+^{-10}|\not{D}\phi|^2\\
&\lesssim \int_{\underline{\mathcal{H}}_{r_2}^{r_1}} |D_T\phi|^2u_+^{-4}+|D_{\underline{L}}\phi|^2u_+^{-4}+r^{-4}u_+^{-2}|\phi|^2+u_+^{-4}|\not{D}\phi|^2\\
&\lesssim \mathring{\varepsilon} r_1^{-8-2\varepsilon_0}.\end{aligned}$$

Here we have used the fact that $L=2T-\underline{L}$ to bound $D_L\psi$ and estimate (4.13) to bound the integral of ϕ as well as $D_T\phi$. Combining the above estimate, we have shown that

$$\int_{\underline{\mathcal{H}}_{r_2}^{r_1}} |D_{\underline{L}}\big(D_\Omega D_{\underline{L}}\phi\big)|^2 \lesssim \mathring{\varepsilon} r_1^{-8-2\varepsilon_0}. \tag{4.14}$$

The next object is to derive estimate for $\int_{\underline{\mathcal{H}}_{r_2}^{r_1}}|D_\Omega\big(D_\Omega D_{\underline{L}}\phi\big)|^2$. First for Ω, Ω' being angular momentum vector fields, recall the following commutation formula:

$$D_\Omega\big(D_{\Omega'}D_{\underline{L}}\phi\big)=\sqrt{-1}\big(\mathcal{L}_\Omega F_{\Omega'\underline{L}}\phi+F([\Omega,\Omega'],\underline{L})\phi+F_{\Omega'\underline{L}}D_\Omega\phi+F_{\Omega\underline{L}}D_{\Omega'}\phi\big)+D_{\underline{L}}D_\Omega D_{\Omega'}\phi$$

For the first four terms, we can bound the full Maxwell field by the pointwise bound according to Proposition 4.1 together with the property of the charge 2-form $F[q_0]$. More precisely we can show that

$$\begin{aligned}&\int_{\underline{\mathcal{H}}_{r_2}^{r_1}} |\sqrt{-1}\big(\mathcal{L}_\Omega F_{\Omega'\underline{L}}\phi+F([\Omega,\Omega'],\underline{L})\phi+F_{\Omega'\underline{L}}D_\Omega\phi+F_{\Omega\underline{L}}D_{\Omega'}\phi\big)|^2\\
&\lesssim \int_{\underline{\mathcal{H}}_{r_2}^{r_1}} (|\mathcal{L}_\Omega\mathring{\underline{\alpha}}|^2+|\mathring{\underline{\alpha}}|^2)u_+^{-5}+r^{-4}|\phi|^2+u_+^{-4}r^2|\not{D}\phi|^2 \lesssim \mathring{\varepsilon} r_1^{-8-2\varepsilon_0}.\end{aligned}$$

Then by using the ansatz (**B**), we can derive that

$$\int_{\underline{\mathcal{H}}_{r_2}^{r_1}} |D_\Omega\big(D_\Omega D_{\underline{L}}\phi\big)|^2 \lesssim \mathring{\varepsilon} r_1^{-6-2\varepsilon_0}.$$

Similarly, we also have

$$\int_{\underline{\mathcal{H}}_{r_2}^{r_1}} |D_\Omega D_{\underline{L}}\phi|^2 \lesssim \mathring{\varepsilon} r_1^{-6-4\varepsilon_0}.$$

Then using the Sobolev inequality (A.3), we derive that

$$\|D_\Omega D_{\underline{L}}\phi\|_{L^4(\mathcal{S}_{r_1}^{r_2})} \lesssim r_2^{-\frac{1}{2}}\sqrt{\mathring{\varepsilon}}r_1^{-3-\varepsilon_0}.$$

We then repeat the same argument for $D_{\underline{L}}\phi$ to derive

$$\|D_{\underline{L}}\phi\|_{L^4(\mathcal{S}_{r_1}^{r_2})} \lesssim r_2^{-\frac{1}{2}}\sqrt{\mathring{\varepsilon}}r_1^{-3-2\varepsilon_0}.$$

Finally, by virtue of (A.1) and the fact that $u_+=1+\frac{1}{2}r_1$, we obtain that

$$\|D_{\underline{L}}\phi\|_{L^\infty(\mathcal{S}_{r_1}^{r_2})} \lesssim \sqrt{\mathring{\varepsilon}}r_1^{-1}u_+^{-3-\varepsilon_0}.$$

Step 3. L^∞ estimate of $\not{D}\phi$. By the bootstrap ansatz (**B**), we have

$$\int_{\underline{\mathcal{H}}_{r_2}^{r_1}} |D_{\underline{L}}D_{\Omega'}\big(D_\Omega\phi\big)|^2 \lesssim \mathring{\varepsilon} r_1^{-6-2\varepsilon_0},$$

We now use the r^p-weighted energy estimate with $p=2$ of the bootstrap assumption (**B**) to show that

$$\int_{\underline{\mathcal{H}}_{r_2}^{r_1}} |D_{\Omega''}\big(D_{\Omega'}D_\Omega\phi\big)|^2 \lesssim \int_{\underline{\mathcal{H}}_{r_2}^{r_1}} |\not{\nabla}\big(D_{\Omega'}D_\Omega\psi\big)|^2 \lesssim \mathring{\varepsilon} r_1^{-4-2\varepsilon_0},$$

Therefore by using the Sobolev embedding, we have

$$\|D_{\Omega'} D_\Omega \phi\|_{L^4(\mathcal{S}_{r_1}^{r_2})} \lesssim r_2^{-\frac{1}{2}} \sqrt{\mathring{\varepsilon}} r_1^{-2-\varepsilon_0}.$$

Similarly, we can also obtain

$$\|D_\Omega \phi\|_{L^4(\mathcal{S}_{r_1}^{r_2})} \lesssim r_2^{-\frac{1}{2}} \sqrt{\mathring{\varepsilon}} r_1^{-2-2\varepsilon_0}.$$

Therefore, (A.1) implies that, for all angular momentum vector field Ω, we have

$$\|D_\Omega \phi\|_{L^\infty(\mathcal{S}_{r_1}^{r_2})} \lesssim \sqrt{\mathring{\varepsilon}} r_2^{-1} u_+^{-2-\varepsilon_0}.$$

Considering that $|D_\Omega \phi| = r|\mathbb{D}\!\!\!/ \phi|$, the above estimate implies that

$$\|\mathbb{D}\!\!\!/ \phi\|_{L^\infty(\mathcal{S}_{r_1}^{r_2})} \lesssim \sqrt{\mathring{\varepsilon}} r^{-2} u_+^{-2-\varepsilon_0}.$$

Here note that on the sphere $\mathcal{S}_{r_1}^{r_2}$ it holds the relation $r = \frac{r_1+r_2}{2}$.

Step 4. L^∞ estimate of $D_L(r\phi)$. The idea is to use the highest weight commutator $K = v^2 L + u^2 \underline{L}$. According to the bootstrap ansatz, we have

$$\sum_{k\leq 1} \int_{\underline{\mathcal{H}}_{r_2}^{r_1}} r_1^2 |D_{\underline{L}} D_\Omega^k \big(\widehat{D}_K \phi\big)|^2 + r^2 |\mathbb{D}\!\!\!/ D_\Omega^k \big(\widehat{D}_K \phi\big)|^2 \lesssim \mathring{\varepsilon} r_1^{-2-2\varepsilon_0}.$$

Here we may note that $D_\Omega = \widehat{D}_\Omega$. In particular we conclude that

$$\int_{\underline{\mathcal{H}}_{r_2}^{r_1}} |D_\Omega D_{\Omega'} \big(\widehat{D}_K \phi\big)|^2 + |D_\Omega \big(\widehat{D}_K \phi\big)|^2 \lesssim \sqrt{\mathring{\varepsilon}} r_1^{-2-2\varepsilon_0}.$$

Therefore the Sobolev embedding implies that

$$|\widehat{D}_K \phi| \lesssim \sqrt{\mathring{\varepsilon}} r^{-1} u_+^{-1-\varepsilon_0}.$$

On the other hand, we have

$$v^2 D_L(r\phi) = D_K(r\phi) - u^2 D_{\underline{L}}(r\phi) = r\widehat{D}_K \phi - r u^2 D_{\underline{L}} \phi + u^2 \phi.$$

Then by using the bounds for ϕ and $D_{\underline{L}}\phi$, we derive that

$$v^2 |D_L(r\phi)| \lesssim \sqrt{\mathring{\varepsilon}} (1+|u|)^{-1-\varepsilon_0}.$$

This completes the proof. $\square$

5. The analysis in the exterior region 2: energy estimates

5.1. Energy estimates on Maxwell field. For an multi-index $\mathbf{k}$ with $1 \leq |\mathbf{k}| \leq 2$, we can take $G = \mathcal{L}_Z^{\mathbf{k}} \mathring{F}$ and $f = 0$ in (2.18) and (2.19) to deduce:

$$\begin{aligned}
\mathcal{E}^{(\mathbf{k})}(\mathring{F}; r_1) &\leq \mathcal{E}[\mathcal{L}_Z^{\mathbf{k}} \mathring{F}](\mathcal{B}_{r_1}) + \int_{\mathcal{D}_{r_1}} r^{-2} \big| J^{(\mathbf{k})}{}_\nu \cdot \mathcal{L}_Z^{\mathbf{k}} \mathring{F}_0{}^\nu \big| \\
&\leq \mathring{\varepsilon} r_1^{-6+2\xi(\mathbf{k})-8\varepsilon_0} + C \int_{\mathcal{D}_{r_1}} \underbrace{\frac{|J^{(\mathbf{k})}{}_L||\rho^{(\mathbf{k})}|}{r^2}}_{\mathbf{I}_1} + \underbrace{\frac{|J^{(\mathbf{k})}{}_{\underline{L}}||\rho^{(\mathbf{k})}|}{r^2}}_{\mathbf{I}_2} + \underbrace{\frac{|J\!\!\!/^{(\mathbf{k})}||\alpha^{(\mathbf{k})}|}{r^2}}_{\mathbf{I}_3} + \underbrace{\frac{|J\!\!\!/^{(\mathbf{k})}||\underline{\alpha}^{(\mathbf{k})}|}{r^2}}_{\mathbf{I}_4},
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{E}^{(\mathbf{k})}(\mathring{F}; p=2; r_1) &\leq \int_{\mathcal{B}_{r_1}} r^2 \big(|\alpha^{(\mathbf{k})}|^2 + |\rho^{(\mathbf{k})}|^2 + |\sigma^{(\mathbf{k})}|^2\big) + \int_{\mathcal{D}_{r_1}} \big| J^{(\mathbf{k})}{}_\nu \cdot \mathcal{L}_Z^{\mathbf{k}} \mathring{F}_L{}^\nu \big|, \\
&\leq \mathring{\varepsilon} r_1^{-4+2\xi(\mathbf{k})-8\varepsilon_0} + C \int_{\mathcal{D}_{r_1}} \underbrace{|J^{(\mathbf{k})}{}_L||\rho^{(\mathbf{k})}|}_{\mathbf{I}_5} + \underbrace{|J\!\!\!/^{(\mathbf{k})}||\alpha^{(\mathbf{k})}|}_{\mathbf{I}_6},
\end{aligned}$$

where C is a universal constant. In this section, the constant C may change but they all denote universal constants. We now bound the $\mathbf{I}_i$'s one by one.

For $\mathbf{I}_1$ and $\mathbf{I}_5$, we have

$$\begin{aligned}
\int_{\mathcal{D}_{r_1}} \mathbf{I}_5 &\lesssim \Big(\int_{\mathcal{D}_{r_1}} |J^{(\mathbf{k})}{}_L|^2 \Big)^{\frac{1}{2}} \Big(\int_{\mathcal{D}_{r_1}} |\rho^{(\mathbf{k})}|^2 \Big)^{\frac{1}{2}} \\
&\lesssim \Big(\int_{r \geq r_1} \big(\int_{\mathcal{H}_r} |J^{(\mathbf{k})}{}_L|^2 \big) dr \Big)^{\frac{1}{2}} \Big(\int_{r_1}^{\infty} \big(\int_{\mathcal{H}_r} |\rho^{(\mathbf{k})}|^2 \big) dr \Big)^{\frac{1}{2}} \\
&\lesssim \mathring{\varepsilon} r_1^{-\frac{5}{2}+\xi(\mathbf{k})-2\varepsilon_0} \cdot \mathring{\varepsilon}^{\frac{1}{2}} r_1^{-\frac{5}{2}+\xi(\mathbf{k})-(3-|\mathbf{k}|)\varepsilon_0} \lesssim \mathring{\varepsilon}^{\frac{3}{2}} r_1^{-4+2\xi(\mathbf{k})-(6-2|\mathbf{k}|)\varepsilon_0}.
\end{aligned}$$

The last step follows from the bootstrap assumption (**C**) as well as the bootstrap assumption (**B**). Similarly,

$$\int_{\mathcal{D}_{r_1}} \mathbf{I}_1 \lesssim \mathring{\varepsilon}^{\frac{3}{2}} r_1^{-6+2\xi(\mathbf{k})-(6-2|\mathbf{k}|)\varepsilon_0}.$$

For $\mathbf{I}_2$, we have

$$\begin{aligned}
\int_{\mathcal{D}_{r_1}} \mathbf{I}_2 &\lesssim \Big(\int_{\mathcal{D}_{r_1}} \frac{|J^{(\mathbf{k})}{}_{\underline{L}}|^2}{r^{\frac{19}{4}}}\Big)^{\frac{1}{2}} \Big(\int_{\mathcal{D}_{r_1}} r^{\frac{3}{4}}|\rho^{(\mathbf{k})}|^2\Big)^{\frac{1}{2}} \\
&= \Big(\int_{\frac{r_1}{2}}^{\infty} \frac{1}{v^{\frac{5}{4}}}\big(\int_{\underline{\mathcal{H}}_{2v}^{r_1}} \frac{|J^{(\mathbf{k})}{}_{\underline{L}}|^2}{r^{\frac{7}{2}}}\big)dv\Big)^{\frac{1}{2}} \Big(\int_{\frac{r_1}{2}}^{\infty} \frac{1}{v^{\frac{5}{4}}}\big(\int_{\underline{\mathcal{H}}_{2v}^{r_1}} r^2|\rho^{(\mathbf{k})}|^2\big)dv\Big)^{\frac{1}{2}} \\
&\lesssim \mathring{\varepsilon} r_1^{-\frac{39}{8}+\xi(\mathbf{k})-2\varepsilon_0} \cdot \mathring{\varepsilon}^{\frac{1}{2}} r_1^{-\frac{17}{8}+\xi(\mathbf{k})-(3-|\mathbf{k}|)\varepsilon_0} \lesssim \mathring{\varepsilon}^{\frac{3}{2}} r_1^{-6+2\xi(\mathbf{k})-(6-2|\mathbf{k}|)\varepsilon_0}.
\end{aligned}$$

We remark that in the last step we have used the bootstrap assumption (**B**) since $\int_{\underline{\mathcal{H}}_r^{r_1}} r^2|\rho^{(\mathbf{k})}|^2$ appears in the r^p-weighted energy. Another key point is that $v^{-\frac{5}{4}}$ is integrable on $[\frac{r_1}{2}, \infty)$.

For $\mathbf{I}_3$ and $\mathbf{I}_6$, we have

$$\begin{aligned}
\int_{\mathcal{D}_{r_1}} \mathbf{I}_6 &\lesssim \Big(\int_{\mathcal{D}_{r_1}} \frac{|\not{J}^{(\mathbf{k})}|^2}{r^2}\Big)^{\frac{1}{2}} \Big(\int_{\mathcal{D}_{r_1}} r^2|\alpha^{(\mathbf{k})}|^2\Big)^{\frac{1}{2}} \\
&= \Big(\int_{r_1}^{\infty} \big(\int_{\mathcal{H}_{r_2}} \frac{|\not{J}^{(\mathbf{k})}|^2}{r^2}\big)dr_2\Big)^{\frac{1}{2}} \Big(\int_{r_1}^{\infty} \big(\int_{\mathcal{H}_r} r^2|\alpha^{(\mathbf{k})}|^2\big)dr\Big)^{\frac{1}{2}} \\
&\lesssim \mathring{\varepsilon} r_1^{-\frac{7}{2}+\xi(\mathbf{k})-2\varepsilon_0} \cdot \mathring{\varepsilon}^{\frac{1}{2}} r_1^{-\frac{3}{2}+\xi(\mathbf{k})-(3-|\mathbf{k}|)\varepsilon_0} \lesssim \mathring{\varepsilon}^{\frac{3}{2}} r_1^{-6+2\xi(\mathbf{k})-(4-2|\mathbf{k}|)\varepsilon_0}.
\end{aligned}$$

Similarly, we have

$$\int_{\mathcal{D}_{r_1}} \mathbf{I}_3 \lesssim \mathring{\varepsilon}^{\frac{3}{2}} r_1^{-6+2\xi(\mathbf{k})-(6-2|\mathbf{k}|)\varepsilon_0}.$$

For $\mathbf{I}_4$, we have

$$\begin{aligned}
\int_{\mathcal{D}_{r_3}} \mathbf{I}_4 &\lesssim \Big(\int_{\mathcal{D}_{r_1}} \frac{|\not{J}^{(\mathbf{k})}|^2}{r^2}\Big)^{\frac{1}{2}} \Big(\int_{\mathcal{D}_{r_1}} \frac{|\underline{\alpha}^{(\mathbf{k})}|^2}{r^2}\Big)^{\frac{1}{2}} \\
&\lesssim \Big(\int_{r_1}^{\infty} \big(\int_{\mathcal{H}_{r_2}} \frac{|\not{J}^{(\mathbf{k})}|^2}{r^2}\big)dr_2\Big)^{\frac{1}{2}} \Big(\int_{\frac{r_1}{2}}^{\infty} \frac{1}{v^2}\big(\int_{\underline{\mathcal{H}}_{2v}^{r_1}} |\underline{\alpha}^{(\mathbf{k})}|^2\big)dv\Big)^{\frac{1}{2}} \\
&\lesssim \mathring{\varepsilon} r_1^{-\frac{7}{2}+\xi(\mathbf{k})-2\varepsilon_0} \cdot \mathring{\varepsilon}^{\frac{1}{2}} r_1^{-\frac{7}{2}+\xi(\mathbf{k})-(3-|\mathbf{k}|)\varepsilon_0} \lesssim \mathring{\varepsilon}^{\frac{3}{2}} r_1^{-6+2\xi(\mathbf{k})-(6-2|\mathbf{k}|)\varepsilon_0}.
\end{aligned}$$

As a conclusion and by our convention on the implicit constant, we derive that

$$\begin{aligned}
\mathcal{E}^{(\mathbf{k})}(\mathring{F}; r_1) &\leq \mathring{\varepsilon} r_1^{-6+2\xi(\mathbf{k})-(6-2|\mathbf{k}|)\varepsilon_0}\big(1 + C\mathring{\varepsilon}^{\frac{1}{2}}\big), \\
\mathcal{E}^{(\mathbf{k})}(\mathring{F}; p = 2; r_1) &\leq \mathring{\varepsilon} r_1^{-6+2\xi(\mathbf{k})-(6-2|\mathbf{k}|)\varepsilon_0}\big(1 + C\mathring{\varepsilon}^{\frac{1}{2}}\big)
\end{aligned}$$

for some universal constant C.

For sufficiently small $\mathring{\varepsilon}$, we then has closed the bootstrap argument for the Maxwell fields in (**B**):

$$\mathcal{E}^{(\mathbf{k})}(\mathring{F}; r_1) \leq 2\mathring{\varepsilon} r_1^{-6+2\xi(\mathbf{k})-(6-2|\mathbf{k}|)\varepsilon_0}, \quad \mathcal{E}^{(\mathbf{k})}(\mathring{F}; p = 2; r_1) \leq 2\mathring{\varepsilon} r_1^{-6+2\xi(\mathbf{k})-(6-2|\mathbf{k}|)\varepsilon_0}. \tag{5.1}$$

5.2. **Energy estimates on scalar field.** For all multi-index $\mathbf{k}$ such that $1 \leq |\mathbf{k}| \leq 2$, we take $f = \widehat{D}_Z^{\mathbf{k}}\phi$ and $G = 0$ in (2.18) and (2.19). Let $\psi^{(\mathbf{k})} = r\phi^{(\mathbf{k})}$. We deduce the following energy estimates

$$\begin{aligned}
\mathcal{E}^{(\mathbf{k})}(\phi; r_1) &\leq \mathcal{E}[\phi^{(\mathbf{k})}](\mathcal{B}_{r_1}) + \int_{\mathcal{D}_{r_1}} \big|\Box_A\phi^{(\mathbf{k})} \cdot D_{\partial_t}\phi^{(\mathbf{k})}\big| + |F_{0\mu}J[\phi^{(\mathbf{k})}]^\mu| \\
&\leq \mathring{\varepsilon} r_1^{-6+2\xi(\mathbf{k})-8\varepsilon_0} + \underbrace{\int_{\mathcal{D}_{r_1}} |\Box_A\phi^{(\mathbf{k})}|\big(|D_L\phi^{(\mathbf{k})}| + |D_{\underline{L}}\phi^{(\mathbf{k})}|\big)}_{\mathbf{R}_1} \\
&\quad + \underbrace{\int_{\mathcal{D}_{r_1}} (|\alpha| + |\underline{\alpha}|)|\not{D}\phi^{(\mathbf{k})}||\phi^{(\mathbf{k})}|}_{\mathbf{S}_1} + \underbrace{\int_{\mathcal{D}_{r_1}} |\rho|\big(|D_L\phi^{(\mathbf{k})}| + |D_{\underline{L}}\phi^{(\mathbf{k})}|\big)|\phi^{(\mathbf{k})}|}_{\mathbf{T}_1}
\end{aligned} \tag{5.2}$$

as well as the r-weighted energy estimates

$$
\begin{aligned}
\mathcal{E}^{(\mathbf{k})}(\phi;p=2;r_1) &\leq \int_{\mathcal{B}_{r_1}} |D_L\psi^{(\mathbf{k})}|^2 + |\not{D}\psi^{(\mathbf{k})}|^2 + \int_{\mathcal{D}_{r_1}} r\big|\Box_A\phi^{(\mathbf{k})}\cdot D_L\psi^{(\mathbf{k})}\big| + r^2\big|F_{L\mu}J[\phi^{(\mathbf{k})}]^\mu\big| \\
&\leq \mathring{\varepsilon} r_1^{-4+2\xi(\mathbf{k})-8\varepsilon_0} + \underbrace{\int_{\mathcal{D}_{r_1}} r\big|\Box_A\phi^{(\mathbf{k})}\big||D_L\psi^{(\mathbf{k})}|}_{\mathbf{R}_2} + \underbrace{\int_{\mathcal{D}_{r_1}} r^2|\alpha||\not{D}\phi^{(\mathbf{k})}||\phi^{(\mathbf{k})}|}_{\mathbf{S}_2} \\
&\quad + \underbrace{\int_{\mathcal{D}_{r_1}} |\rho||D_L\psi^{(\mathbf{k})}||\psi^{(\mathbf{k})}|}_{\mathbf{T}_2}.
\end{aligned}
\tag{5.3}
$$

We remark that for the term $\mathbf{T}_2$, we have used the following structure of current term:

$$
r^2 J[\phi^{(\mathbf{k})}] = r^2\Im(\phi^{(\mathbf{k})}\cdot\overline{D\phi^{(\mathbf{k})}}) = \Im(\psi^{(\mathbf{k})}\cdot\overline{D\psi^{(\mathbf{k})}}) = J[\psi^{(\mathbf{k})}].
$$

This will be crucial for the estimate of $\mathbf{T}_2$. We first bound the $\mathbf{S}_i$'s which rely on the following lemma:

Lemma 5.1. *Under the bootstrap ansatz, for $\gamma_2 \geq 0$, $\gamma_1 > 1$, we have*

$$
\int_{\mathcal{D}_{r_1}} \frac{|\phi^{(\mathbf{k})}|^2}{r^{\gamma_1}|u|^{\gamma_2}} \lesssim \mathring{\varepsilon} r_1^{-3-\gamma_1-\gamma_2+2\xi(\mathbf{k})-(6-2|\mathbf{k}|)\varepsilon_0}.
$$

Proof. Let $\mathcal{S}_{u,v}$ be the intersection of $\mathcal{H}_u$ and $\underline{\mathcal{H}}_v$. By (A.5), we then have

$$
\begin{aligned}
\int_{\mathcal{D}_{r_1}} \frac{|\phi^{(\mathbf{k})}|^2}{r^{\gamma_1}|u|^{\gamma_2}} &= \int_u\int_v \frac{\int_{\mathcal{S}_{u,v}}|\phi^{(\mathbf{k})}|^2}{r^{\gamma_1}|u|^{\gamma_2}} \lesssim \int_u\int_v \frac{\int_{\mathcal{S}_{u,u}}|\phi^{(\mathbf{k})}|^2 + |u|^{-1}\int_{\mathcal{H}_u}|D_L\psi^{(\mathbf{k})}|^2}{r^{\gamma_1}|u|^{\gamma_2}} \\
&\lesssim \int_u \frac{\int_{\mathcal{S}_{u,u}}|\phi^{(\mathbf{k})}|^2}{|u|^{\gamma_1+\gamma_2-1}} + \int_u \frac{\int_{\mathcal{H}_u}|D_L\psi^{(\mathbf{k})}|^2}{|u|^{\gamma_1+\gamma_2}}.
\end{aligned}
$$

The first term is from the initial data and it is bounded by $\mathring{\varepsilon} r_1^{-3-\gamma_1-\gamma_2+2\xi(\mathbf{k})-8\varepsilon_0}$. We can control the second term by the bootstrap ansatz and it is bounded by $C\mathring{\varepsilon} r_1^{-3-\gamma_1-\gamma_2+2\xi(\mathbf{k})-(6-2|\mathbf{k}|)\varepsilon_0}$. This completes the proof. □

For $\mathbf{S}_1$, according to Proposition 4.1 and the decay properties of the charge part $\alpha(F[q_0])$, $\underline{\alpha}(F[q_0])$, we in particular have the following bounds

$$
|\alpha| \lesssim \sqrt{\mathring{\varepsilon}} r^{-3}u_+^{-1-\varepsilon_0} + r^{-3} \lesssim r^{-3}, \quad |\underline{\alpha}| \lesssim \sqrt{\mathring{\varepsilon}} r^{-1}u_+^{-3-\varepsilon_0} + r^{-3} \lesssim r^{-1}u_+^{-2}.
$$

We have used the fact that $\mathring{\varepsilon}$ is sufficiently small. Therefore we can show that

$$
\begin{aligned}
\mathbf{S}_1 &\lesssim \int_{\mathcal{D}_{r_1}} \frac{|\not{D}\phi^{(\mathbf{k})}||\phi^{(\mathbf{k})}|}{r|u|^2} \lesssim \Big(\int_{\mathcal{D}_{r_1}} \frac{|\not{D}\phi^{(\mathbf{k})}|^2}{|u|}\Big)^{\frac12}\Big(\int_{D_{r_1}} \frac{|\phi^{(\mathbf{k})}|^2}{r^2|u|^3}\Big)^{\frac12} \\
&= \Big(\int_{|u|\geq\frac{r_1}{2}} \frac{\int_{\mathcal{H}_u}|\not{D}\phi^{(\mathbf{k})}|^2}{|u|}du\Big)^{\frac12}\Big(\int_{D_{r_1}} \frac{|\phi^{(\mathbf{k})}|^2}{r^2|u|^3}\Big)^{\frac12}.
\end{aligned}
$$

We use the bootstrap ansatz to bound the first term and use Lemma 5.1 to bound the second term. Therefore, we obtain

$$
\mathbf{S}_1 \lesssim \mathring{\varepsilon} r_1^{-6.5+2\xi(\mathbf{k})-(6-2|\mathbf{k}|)\varepsilon_0}.
$$

We can also derive in the same manner that

$$
\mathbf{S}_2 \lesssim \mathring{\varepsilon} r_1^{-4.5+2\xi(\mathbf{k})-(6-2|\mathbf{k}|)\varepsilon_0}.
$$

By our convention the implicit constant is independent of R_*. Since $r_1 \geq R_*$, by choosing R_* sufficiently large, we derive the following estimates

$$
\begin{aligned}
\mathcal{E}^{(\mathbf{k})}(\phi;r_1) &\leq \frac54\mathring{\varepsilon} r_1^{-6+2\xi(\mathbf{k})-8\varepsilon_0} + \mathbf{R}_1 + \mathbf{T}_1, \\
\mathcal{E}^{(\mathbf{k})}(\phi;p=2;r_1) &\leq \frac54\mathring{\varepsilon} r_1^{-4+2\xi(\mathbf{k})-8\varepsilon_0} + \mathbf{R}_2 + \mathbf{T}_2
\end{aligned}
\tag{5.4}
$$

with $\mathbf{R}_i$, $\mathbf{T}_i$ defined in (5.2) and (5.3).

5.2.1. *Energy estimates on one derivatives of the scalar field.* We consider the case where $|\mathbf{k}| = 1$. The multi-index $\mathbf{k}$ then represents a vector field $Z \in \Gamma$. In view of (2.8) and the pointwise bounds in Proposition 4.1 and Proposition 4.3, we have

$$|u|^{-\xi(Z)}|Q(\phi, F; Z)| \lesssim \frac{1}{r|u|}|D_L\psi| + \frac{1}{r^2|u|}|\not{D}\psi| + \frac{|u|}{r^3}|D_{\underline{L}}\psi| + \frac{1}{r^2}|\phi|.$$

Thus, we have

$$\begin{aligned} r^2|\Box_A\phi^{(\mathbf{1})}|^2 &\lesssim r^2|Q(\phi, F; Z)|^2 \\ &\lesssim |u|^{2\xi(\mathbf{1})-2}|D_L\psi|^2 + |u|^{2\xi(\mathbf{1})-2}|\not{D}\phi|^2 + \frac{|u|^{2\xi(\mathbf{1})+2}}{r^2}|D_{\underline{L}}\phi|^2 + \frac{|u|^{2\xi(\mathbf{1})}}{r^2}|\phi|^2. \end{aligned} \tag{5.5}$$

Since $r^{-2} \lesssim |u|^{-2}$, according to the bounds on the zeroth order energy estimates, we have

$$\int_{\mathcal{D}_{r_1}} |u|^{2\xi(\mathbf{1})-2}|D_L\psi|^2 \leq \int_u |u|^{2\xi(\mathbf{1})-2}\Big(\int_{\mathcal{H}_u} |D_L\psi|^2\Big)du \lesssim \mathring{\varepsilon} r_1^{2\xi(\mathbf{1})-5-6\varepsilon_0},$$

$$\int_{\mathcal{D}_{r_1}} |u|^{2\xi(\mathbf{1})-2}|\not{D}\phi|^2 \lesssim \int_u |u|^{2\xi(\mathbf{1})-4-2\varepsilon_0}\Big(\int_{\mathcal{H}_u} |\not{D}\phi|^2\Big)du \lesssim \mathring{\varepsilon} r_1^{2\xi(\mathbf{1})-7-6\varepsilon_0},$$

and

$$\int_{\mathcal{D}_{r_1}} \frac{|u|^{2\xi(\mathbf{1})+2}}{r^2}|D_{\underline{L}}\phi|^2 \lesssim \int_v |u|^{2\xi(\mathbf{1})}|v|^{-2}\Big(\int_{\underline{\mathcal{H}}_v} |D_{\underline{L}}\phi|^2\Big)du \lesssim \mathring{\varepsilon} r_1^{2\xi(\mathbf{1})-5-6\varepsilon_0}.$$

By Lemma 5.1, we also have

$$\int_{\mathcal{D}_{r_1}} \frac{|u|^{2\xi(\mathbf{1})}}{r^2}|\phi|^2 \lesssim \mathring{\varepsilon} r_1^{2\xi(\mathbf{1})-5-6\varepsilon_0}.$$

Thus, we have

$$\int_{\mathcal{D}_{r_1}} r^2|Q(\phi, F; Z)|^2 \lesssim r_1^{2\xi(Z)-5-6\varepsilon_0}. \tag{5.6}$$

Let $(\mathbf{2})$ denotes two vector fields Z_1 and Z_2. If we replace ϕ by $\widehat{D}_{Z_1}$ in the proof between (5.5) and (5.6), we obtain

$$\int_{\mathcal{D}_{r_1}} r^2|Q(\widehat{D}_{Z_1}\phi, F; Z_2)|^2 \lesssim r_1^{2\xi(\mathbf{2})-5-6\varepsilon_0}. \tag{5.7}$$

Similarly, we have

$$\begin{aligned} &\int_{\mathcal{D}_{r_1}} r^{-2}\big(|D_L\phi^{(\mathbf{1})}| + |D_{\underline{L}}\phi^{(\mathbf{1})}|\big)^2 \\ &\lesssim \int_{|u|\geq r_1} u_+^{-2}\Big(\int_{\mathcal{H}_u} |D_L\phi^{(\mathbf{1})}|^2\Big)du + \int_v v^{-2}\Big(\int_{\underline{\mathcal{H}}_v} |D_{\underline{L}}\phi^{(\mathbf{1})}|^2\Big)dv \lesssim \mathring{\varepsilon} r_1^{2\xi(\mathbf{1})-5-4\varepsilon_0}. \end{aligned}$$

Therefore, we can bound $\mathbf{R}_1$ as follows

$$\mathbf{R}_1 \leq \Big(\int_{\mathcal{D}_{r_1}} r^2|\Box_A\phi^{(\mathbf{1})}|^2\Big)^{\frac{1}{2}}\Big(\int_{\mathcal{D}_{r_1}} r^{-2}\big(|D_L\phi^{(\mathbf{1})}| + |D_{\underline{L}}\phi^{(\mathbf{1})}|\big)^2\Big)\Big)^{\frac{1}{2}} \lesssim \mathring{\varepsilon} r_1^{-6+2\xi(\mathbf{1})-5\varepsilon_0}.$$

One can also proceed exactly in the same manner to prove that

$$\mathbf{R}_2 \lesssim \Big(\int_{\mathcal{D}_{r_1}} r^2|\Box_A\phi^{(\mathbf{1})}|^2\Big)^{\frac{1}{2}}\Big(\int_{\mathcal{D}_{r_1}} |D_L\psi^{(\mathbf{1})}|^2\Big)^{\frac{1}{2}} \lesssim \mathring{\varepsilon} r_1^{-4+2\xi(\mathbf{1})-5\varepsilon_0}$$

by using the r-weighted energy estimates. Therefore, for sufficiently large R_*, since $r_1 \geq R_*$, we have

$$\begin{aligned} \mathcal{E}^{(\mathbf{1})}(\phi; r_1) &\lesssim \mathring{\varepsilon} r_1^{-6+2\xi(\mathbf{1})-5\varepsilon_0} + \mathbf{T}_1 \\ \mathcal{E}^{(\mathbf{1})}(\phi; p = 2; r_1) &\lesssim \mathring{\varepsilon} r_1^{-4+2\xi(\mathbf{1})-5\varepsilon_0} + \mathbf{T}_2. \end{aligned} \tag{5.8}$$

At this stage, we need to first control $\mathbf{T}_2$ in the second equation. In view of the definition of $\mathcal{E}^{(\mathbf{k})}(\phi; p = 2; r_1)$ and the fact that $|\rho| \lesssim r^{-2}$, the second inequality gives

$$\int_{\mathcal{H}_{r_1}} |D_L\psi^{(\mathbf{1})}|^2 \lesssim \mathring{\varepsilon} r_1^{-4+2\xi(\mathbf{1})-5\varepsilon_0} + \int_{\mathcal{D}_{r_1}} \frac{|D_L\psi^{(\mathbf{1})}||\psi^{(\mathbf{1})}|}{r^2}.$$

When we apply Lemma 3.1 in this case, we change ε_0 to $\frac{1}{2}\varepsilon_0$. This leads to

$$\int_{\mathcal{H}_{r_1}} |D_L\psi^{(\mathbf{1})}|^2 \lesssim \mathring{\varepsilon} r_1^{-4+2\xi(\mathbf{1})-4.5\varepsilon_0}.$$

The gain of $r^{-0.5\varepsilon_0}$ can be used to improve the estimates in Lemma 5.1. This gives

$$\int_{\mathcal{D}_{r_1}} \frac{|\psi^{(\mathbf{1})}|^2}{r^4} = \int_{\mathcal{D}_{r_1}} \frac{|\phi^{(\mathbf{1})}|^2}{r^2} \lesssim \mathring{\varepsilon} r_1^{-5+2\xi(\mathbf{1})-4.5\varepsilon_0}. \tag{5.9}$$

Hence,

$$\mathbf{T}_2 \lesssim \Big(\int_{\mathcal{D}_{r_1}} |D_L\psi^{(\mathbf{1})}|^2\Big)^{\frac12}\Big(\int_{\mathcal{D}_{r_1}} \frac{|\psi^{(\mathbf{1})}|^2}{r^4}\Big)^{\frac12} \lesssim \mathring{\varepsilon} r_1^{-4+2\xi(\mathbf{1})-4.5\varepsilon_0}.$$

This improved estimate (5.9) also allows us to bound $\mathbf{T}_1$ as follows:

$$\mathbf{T}_1 \lesssim \big(\underbrace{\int_{\mathcal{D}_{r_1}} r^{-2}|D_L\phi^{(\mathbf{1})}|^2 + \int_{\mathcal{D}_{r_1}} r^{-2}|D_{\underline{L}}\phi^{(\mathbf{1})}|^2}_{\lesssim r_1^{-7+2\xi(\mathbf{1})-4\varepsilon_0} \text{ by } (\mathbf{B})}\big)^{\frac12}\big(\underbrace{\int_{\mathcal{D}_{r_1}} \frac{|\phi^{(\mathbf{1})}|^2}{r^2}}_{\lesssim r_1^{-5+2\xi(\mathbf{1})-4.5\varepsilon_0}}\big)^{\frac12}$$
$$\lesssim \mathring{\varepsilon} r_1^{-6+2\xi(\mathbf{1})-4.25\varepsilon_0}.$$

Thus, the estimate (5.8) implies

$$\mathcal{E}^{(\mathbf{1})}(\phi; r_1) \lesssim \mathring{\varepsilon} r_1^{-6+2\xi(\mathbf{1})-4.25\varepsilon_0}, \quad \mathcal{E}^{(\mathbf{1})}(\phi; p=2; r_1) \lesssim \mathring{\varepsilon} r_1^{-4+2\xi(\mathbf{1})-4.5\varepsilon_0}.$$

For sufficiently large R_*, we then have closed the bootstrap argument for first order energy quantities on scalar field in $(\mathbf{B})$:

$$\mathcal{E}^{(\mathbf{1})} \le 2\mathring{\varepsilon} r_1^{-6+2\xi(\mathbf{1})-4\varepsilon_0}, \quad \mathcal{E}^{(\mathbf{1})}(\phi; p=2; r_1) \le 2\mathring{\varepsilon} r_1^{-4+2\xi(\mathbf{1})-4\varepsilon_0}. \tag{5.10}$$

5.2.2. *Energy estimates on second derivatives of the scalar field.* We now fix a $\mathbf{k}$ so that $|\mathbf{k}| = 2$ and the first objective is to bound the $\mathbf{R}_1$ and $\mathbf{R}_2$ term in (5.4). For this purpose, we first recall that, for $(\mathbf{2})$ representing $\widehat{D}_{Z_1}\widehat{D}_{Z_2}$, we have

$$\Box_A\phi^{(\mathbf{2})} = Q(\widehat{D}_{Z_1}\phi, F; Z_2) + Q(\widehat{D}_{Z_2}\phi, F; Z_1) + Q(\phi, F; [Z_1, Z_2]) + Q(\phi, \mathcal{L}_{Z_1}F; Z_2) - 2F_{Z_1\mu}F_{Z_2}{}^{\mu}\phi.$$

For $\mathbf{R}_1$, according to the above expression, we split it into three parts:

$$\mathbf{R}_1 \lesssim \int_{\mathcal{D}_{r_1}} \underbrace{\Big(\big|Q(\widehat{D}_{Z_1}\phi, F; Z_2)\big| + \big|Q(\widehat{D}_{Z_2}\phi, F; Z_1)\big| + \big|Q(\phi, F; [Z_1, Z_2])\big|\Big)\big(|D_L\phi^{(\mathbf{2})}| + |D_{\underline{L}}\phi^{(\mathbf{2})}|\big)}_{\mathbf{R}_{11}}$$
$$+ \int_{\mathcal{D}_{r_1}} \underbrace{\big|Q(\phi, \mathcal{L}_{Z_1}F; Z_2)\big|\big(|D_L\phi^{(\mathbf{2})}| + |D_{\underline{L}}\phi^{(\mathbf{2})}|\big)}_{\mathbf{R}_{12}} + \int_{\mathcal{D}_{r_1}} \underbrace{\big|F_{Z_1\mu}F_{Z_2}{}^{\mu}\phi\big|\big(|D_L\phi^{(\mathbf{2})}| + |D_{\underline{L}}\phi^{(\mathbf{2})}|\big)}_{\mathbf{R}_{13}}$$

All the three Q-terms in $\mathbf{R}_{11}$ can be schematically written as either $Q(\phi^{(\mathbf{1})}, F; Z)$ or $Q(\phi^{(\mathbf{0})}, F; Z)$ due to the observation that the linear span of $\mathcal{Z}$ is closed under commutations. These terms resemble the terms in $\mathbf{R}_1$ in Section 5.2.1. Thanks to (5.7), they can be bounded exactly in the same manner:

$$\mathbf{R}_1 \lesssim \Big(\int_{\mathcal{D}_{r_1}} r^2\big|Q(\widehat{D}_{Z_1}\phi, F; Z_2)\big|^2 + r^2\big|Q(\widehat{D}_{Z_2}\phi, F; Z_1)\big|^2 + r^2\big|Q(\phi, F; [Z_1, Z_2])\big|^2\Big)^{\frac12}$$
$$\times \Big(\int_{\mathcal{D}_{r_1}} r^{-2}\big(|D_L\phi^{(\mathbf{2})}| + |D_{\underline{L}}\phi^{(\mathbf{2})}|\big)^2\Big)\Big)^{\frac12} \lesssim \mathring{\varepsilon} r_1^{-6+2\xi(\mathbf{2})-3\varepsilon_0}.$$

Here we remark that compared with the estimate of $\mathbf{R}_1$ in the last subsection we lose a decay power of ε_0 is due to the weaker decay of second order energy estimates in the bootstrap assumption.

For $\mathbf{R}_{12}$, we use $(\mathbf{1})$ to denote the vector field Z_1, according to (2.8) and the pointwise bounds on the scalar field, we have

$$\begin{aligned}
|u|^{-\xi(Z_2)}|Q(\phi, \mathcal{L}_{Z_1}F; Z_2)| \lesssim{}& \big(\frac{r}{|u|}|\rho(\mathcal{L}_{Z_1}F)| + |\underline{\alpha}(\mathcal{L}_{Z_1}F)|\big)|D_L\psi| \\
&+ \big(\frac{r}{|u|}|\alpha(\mathcal{L}_{Z_1}F)| + \frac{|u|}{r}|\underline{\alpha}(\mathcal{L}_{Z_1}F)| + |\sigma(\mathcal{L}_{Z_1}F)|\big)|\displaystyle{\not}D\psi| \\
&+ \big(|\alpha(\mathcal{L}_{Z_1}F)| + \frac{|u|}{r}|\rho(\mathcal{L}_{Z_1}F)|\big)|D_{\underline{L}}\psi| + \big(|\rho(\mathcal{L}_{Z_1}F)| + |\sigma(\mathcal{L}_{Z_1}F)|\big)|\phi| \\
&+ \big(\frac{|u|}{r^2}|J(\mathcal{L}_{Z_1}F)_{\underline{L}}| + \frac{1}{|u|}|J(\mathcal{L}_{Z_1}F)_L| + \frac{1}{r}|\displaystyle{\not}J(\mathcal{L}_{Z_1}F)|\big)|\phi|.
\end{aligned}$$

Since $F = \mathring{F} + F[q_0]$ and $F[q_0]$ solves the linear Maxwell equations, accroding to (2.4), we have

$$J(\mathcal{L}_{Z_1}F)_{\underline{L}} = J^{(\mathbf{1})}_{\underline{L}}, \ J(\mathcal{L}_{Z_1}F)_L = J^{(\mathbf{1})}_L, \ \displaystyle{\not}J(\mathcal{L}_{Z_1}F) = \displaystyle{\not}J^{(\mathbf{1})}.$$

Therefore, according to the pointwise decay for the scalar field, we have

$$
\begin{aligned}
&|u|^{-\xi(Z_2)}|Q(\phi,\mathcal{L}_{Z_1}F;Z_2)|\\
&\lesssim \big(\frac{r}{|u|}|\not{D}\psi|+|D_{\underline{L}}\psi|\big)|\alpha(\mathcal{L}_{Z_1}F)|+\big(\frac{r}{|u|}|D_L\psi|+\frac{|u|}{r}|D_{\underline{L}}\psi|+|\phi|\big)|\rho(\mathcal{L}_{Z_1}F)|+\big(|\not{D}\psi|+|\phi|\big)|\sigma(\mathcal{L}_{Z_1}F)|\\
&\quad+\big(|D_L\psi|+\frac{|u|}{r}|\not{D}\psi|\big)|\underline{\alpha}(\mathcal{L}_{Z_1}F)|+\big(\frac{|u|}{r^2}|J_{\underline{L}}^{(\mathbf{1})}|+\frac{1}{|u|}|J_L^{(\mathbf{1})}|+\frac{1}{r}|\not{J}^{(\mathbf{1})}|\big)|\phi|\\
&\lesssim \underbrace{\frac{\sqrt{\mathring{\varepsilon}}}{|u|^{3+\varepsilon_0}}|\alpha(\mathcal{L}_{Z_1}F)|}_{\mathbf{A}_0}+\underbrace{\frac{\sqrt{\mathring{\varepsilon}}}{r|u|^{2+\varepsilon_0}}|\rho(\mathcal{L}_{Z_1}F)|+\frac{\sqrt{\mathring{\varepsilon}}}{r|u|^{2+\varepsilon_0}}|\sigma(\mathcal{L}_{Z_1}F)|}_{\mathbf{A}_1}+\underbrace{\frac{\sqrt{\mathring{\varepsilon}}}{r^2|u|^{1+\varepsilon_0}}|\underline{\alpha}(\mathcal{L}_{Z_1}F)|}_{\mathbf{A}_2}\\
&\quad+\underbrace{\frac{\sqrt{\mathring{\varepsilon}}}{r|u|^{\frac{7}{2}+2\varepsilon_0}}|J_L^{(\mathbf{1})}|+\frac{\sqrt{\mathring{\varepsilon}}}{r^2|u|^{\frac{5}{2}+2\varepsilon_0}}|\not{J}^{(\mathbf{1})}|}_{\mathbf{A}_3}+\underbrace{\frac{\sqrt{\mathring{\varepsilon}}}{r^3|u|^{\frac{3}{2}+2\varepsilon_0}}|J_{\underline{L}}^{(\mathbf{1})}|}_{\mathbf{A}_4}.
\end{aligned}
$$

On the other hand, according to Lemma A.8, we have

$$\big|\alpha(\mathcal{L}_{Z_1}F[q_0])\big|\leq\big|\mathcal{L}_{Z_1}\big(\alpha(F[q_0])\big)\big|+r^{\xi(Z_1)}\big|\alpha(F[q_0])\big|\lesssim r^{-3+\xi(Z_1)}.$$

Hence,

$$\big|\alpha(\mathcal{L}_{Z_1}F)\big|\leq\big|\alpha(\mathcal{L}_{Z_1}\mathring{F})\big|+\big|\alpha(\mathcal{L}_{Z_1}F[q_0])\big|\leq|\alpha^{(\mathbf{1})}|+r^{-3+\xi(Z_1)}.$$

Similarly, since we have

$$\big|\underline{\alpha}(\mathcal{L}_{Z_1}F[q_0])\big|\lesssim r^{-3+\xi(Z_1)},\ \big|\rho(\mathcal{L}_{Z_1}F[q_0])\big|\lesssim r^{-3+\xi(Z_1)},\ \sigma(\mathcal{L}_{Z_1}F[q_0])=0,$$

We notice that the estimate on $\rho(\mathcal{L}_{Z_1}F[q_0])$ is as good as the other components. This is due to the fact that $\mathcal{L}_Z\big(\frac{1}{r^2}dt\wedge dr\big)=0$ for all $Z\in\mathcal{Z}$. We conclude that

$$
\begin{aligned}
&\big|\alpha(\mathcal{L}_{Z_1}F)\lesssim|\alpha^{(\mathbf{1})}|+r^{-3+\xi(Z_1)},\ \big|\underline{\alpha}(\mathcal{L}_{Z_1}F)\lesssim|\underline{\alpha}^{(\mathbf{1})}|+r^{-3+\xi(Z_1)},\\
&\big|\rho(\mathcal{L}_{Z_1}F)\lesssim|\rho^{(\mathbf{1})}|+r^{-3+\xi(Z_1)},\ \big|\sigma(\mathcal{L}_{Z_1}F)\lesssim|\sigma^{(\mathbf{1})}|.
\end{aligned}\tag{5.11}
$$

We notice that for $Z_1=K$ we lose decay in r. For the α component, we can improve the decay in r:

Lemma 5.2.

$$\big|\alpha(\mathcal{L}_KF[q_0])\big|\lesssim r^{-3}|u|.\tag{5.12}$$

Proof. We recall the definition for $F[q_0]$:

$$F[q_0]_{0i}=\partial_iV(x),\quad F[q_0]_{ij}=0,\mathrm{for} i,j=1,2,3,$$

where the potential $V(x)$ is given by

$$V(x)=\frac{1}{4\pi}\int_{\mathbb{R}^3}(\underbrace{\frac{1}{r}}_{V_1}+\underbrace{\frac{x\cdot y}{r^3}}_{V_2}+\underbrace{\frac{1}{2}\frac{(3|x|^{-2}(x\cdot y)^2-|y|^2)}{r^3}}_{V_3})\Im(\phi_0\cdot\bar{\phi}_1)dy,\quad |x|>0.$$

The contribution from V_3 is of order r^{-3} so that we can ignore it. The contribution from V_1 gives the charge part $\frac{1}{r^2}dt\wedge dr$ and it will vanish when one takes $\mathcal{L}_K$ derivative. Thus, we consider

$$F^{(2)}[q_0]_{0i}=\partial_i\Big(\frac{1}{4\pi}\int_{\mathbb{R}^3}\frac{x\cdot y}{r^3}\Im\big((\phi_0\cdot\bar{\phi}_1)(y)\big)dy\Big),\ F^{(2)}[q_0]_{ij}=0.$$

Thus, we have

$$\alpha(F^{(2)}[q_0])_A=\frac{1}{4\pi}\int_{\mathbb{R}^3}\frac{e_A\cdot y}{r^3}\Im\big((\phi_0\cdot\bar{\phi}_1)(y)\big)dy.$$

By virtue of the formula for $\mathcal{L}_K$ in Lemma A.8, we obtain

$$\alpha(\mathcal{L}_KF^{(2)}[q_0])_A=\frac{1}{4\pi}\int_{\mathbb{R}^3}\frac{(r-t)e_A\cdot y}{r^3}\Im\big((\phi_0\cdot\bar{\phi}_1)(y)\big)dy.$$

This completes the proof of the lemma. $\square$

As a corollary, we have

$$\big|\alpha(\mathcal{L}_{Z_1}F)\lesssim|\alpha^{(\mathbf{1})}|+r^{-3}|u|^{\xi(Z_1)}.\tag{5.13}$$

Lemma 5.3. *We have the following spacetime estimates:*

$$\|rQ(\phi,\mathcal{L}_{Z_1}F;Z_2)\|_{L^2(\mathcal{D}_{r_1})}\lesssim\sqrt{\mathring{\varepsilon}}r_1^{\xi(\mathbf{2})-3-\varepsilon_0}.\tag{5.14}$$

Proof. With the help of (5.11) and (5.12), we can bound the terms $\int_{\mathcal{D}_{r_1}} r^2|u|^{2\xi(Z_2)}|\mathbf{A}_i|^2$ one by one. This will prove the lemma:

$$\begin{aligned}\int_{\mathcal{D}_{r_1}} r^2|u|^{2\xi(Z_2)}|\mathbf{A}_0|^2 &\lesssim \mathring{\varepsilon}\int_{\mathcal{D}_{r_1}} |u|^{2\xi(Z_2)-6-2\varepsilon_0}\big(r^2|\alpha^{(\mathbf{1})}|^2+r^{-4}|u|^{2\xi(Z_1)}\big)\\ &\lesssim \mathring{\varepsilon}\int_{r_1}^{\infty} |u|^{2\xi(Z_2)-6-2\varepsilon_0}\Big(\int_{\mathcal{H}_{r_2}}\big(r^2|\alpha^{(\mathbf{1})}|^2+r^{-4}|u|^{2\xi(Z_1)}\big)\Big)dr_2.\end{aligned}$$

For $\mathbf{A}_1$, we have

$$\begin{aligned}\int_{\mathcal{D}_{r_1}} r^2|u|^{2\xi(Z_2)}|\mathbf{A}_1|^2 &\lesssim \mathring{\varepsilon}\int_{\mathcal{D}_{r_1}} |u|^{2\xi(Z_2)-4-2\varepsilon_0}\big(|\rho^{(\mathbf{1})}|^2+|\sigma^{(\mathbf{1})}|^2+r^{-6+2\xi(Z_1)}\big)\\ &\lesssim \mathring{\varepsilon}\int_{r_1}^{\infty} |u|^{2\xi(Z_2)-4-2\varepsilon_0}\Big(\int_{\mathcal{H}_{r_2}}\big(|\rho^{(\mathbf{1})}|^2+|\sigma^{(\mathbf{1})}|^2+r^{-6+2\xi(Z_1)}\big)\Big)dr_2.\end{aligned}$$

For $\mathbf{A}_2$, we have

$$\begin{aligned}\int_{\mathcal{D}_{r_1}} r^2|u|^{2\xi(Z_2)}|\mathbf{A}_2|^2 &\lesssim \mathring{\varepsilon}\int_{\mathcal{D}_{r_1}} |u|^{2\xi(Z_2)-2-2\varepsilon_0}r^{-2}\big(|\underline{\alpha}^{(\mathbf{1})}|^2+r^{-6+2\xi(Z_1)}\big)\\ &\lesssim \mathring{\varepsilon}\int_{\frac{r_1}{2}}^{\infty} r_1^{2\xi(Z_2)-4-2\varepsilon_0}v^{-2}\Big(\int_{\underline{\mathcal{H}}_{2v}^{r_1}}\big(|\underline{\alpha}^{(\mathbf{1})}|^2+r^{-6+2\xi(Z_1)}\big)\Big)dr_2.\end{aligned}$$

For $\mathbf{A}_3$, based on the ansatz ($\mathbf{C}$), we can proceed in the same manner to obtain

$$\begin{aligned}\int_{\mathcal{D}_{r_1}} r^2|u|^{2\xi(Z_2)}|\mathbf{A}_3|^2 &\lesssim \mathring{\varepsilon}\int_{\mathcal{D}_{r_1}} |u|^{2\xi(Z_2)-7-4\varepsilon_0}|J_L^{(\mathbf{1})}|^2+|u|^{2\xi(Z_2)-5-4\varepsilon_0}\frac{|\not{J}^{(\mathbf{1})}|^2}{r^2}\\ &= \mathring{\varepsilon}\int_{r_1}^{\infty}\Big(|u|^{2\xi(Z_2)-7-4\varepsilon_0}\int_{\mathcal{H}_{r_2}}|J_L^{(\mathbf{1})}|^2+|u|^{2\xi(Z_2)-5-4\varepsilon_0}\int_{\mathcal{H}_{r_2}}\frac{|\not{J}^{(\mathbf{1})}|^2}{r^2}\Big)dr_2.\end{aligned}$$

All the terms on the righthand sides of the above four inequalities now can be integrated. They are all bounded by $\mathring{\varepsilon}r_1^{2\xi(\mathbf{2})-6-2\varepsilon_0}$.

For $\mathbf{A}_4$, let the vector field Z represent the index ($\mathbf{1}$), we have

$$J^{(\mathbf{1})} = \mathcal{L}_Z(r^2J) = \mathcal{L}_Z(\Im(\overline{\psi}\cdot D\psi)).$$

Since

$$\mathcal{L}_Z\big(\overline{\psi}\cdot D\psi\big)_\mu = \overline{D_Z\psi}\cdot D_\mu\psi + (D_\mu\log(r))\overline{\psi}\cdot D_Z\psi + r\overline{\psi}\cdot D_\mu\big(\widehat{D}_Z\phi\big)+iF_{Z\mu}|\psi|^2,$$

we have

$$|J_\mu^{(\mathbf{1})}|^2 \lesssim r^4|\phi|^2|D_\mu\big(\widehat{D}_Z\phi\big)|^2+\big(|D_\mu\psi|^2+|\phi|^2\big)|D_Z\psi|^2+r^4|F_{Z\mu}|^2|\phi|^4. \tag{5.15}$$

In particular, we have

$$\int_{\mathcal{D}_{r_1}} r^2|u|^{2\xi(Z_2)}|\mathbf{A}_4|^2 \lesssim \mathring{\varepsilon}\int_{\mathcal{D}_{r_1}} |u|^{2\xi(Z_2)-3-4\varepsilon_0}\big(|\phi|^2|D_{\underline{L}}\big(\widehat{D}_{Z_1}\phi\big)|^2+\frac{\big(|D_{\underline{L}}\psi|^2+|\phi|^2\big)|D_{Z_1}\psi|^2}{r^4}+|F_{Z_1\underline{L}}|^2|\phi|^4\big).$$

In view of the pointwise bounds, we can then use the following crude bound for $D_{Z_1}\psi$ and $F_{Z_1\underline{L}}$:

$$|D_{Z_1}\psi| \lesssim |u|^{\xi(Z_1)-1-\varepsilon_0},\quad |F_{Z_1\underline{L}}| \lesssim r^{\xi(Z_1)-1}|u|^{-1-\varepsilon_0}.$$

Therefore, we obtain

$$\begin{aligned}\int_{\mathcal{D}_{r_1}} r^2|u|^{2\xi(Z_2)}|\mathbf{A}_4|^2 &\lesssim \mathring{\varepsilon}\int_{\mathcal{D}_{r_1}} |u|^{2\xi(Z_2)-5-4\varepsilon_0}\big(\frac{|D_{\underline{L}}\big(\widehat{D}_{Z_1}\phi\big)|^2+|D_{\underline{L}}\phi|^2}{r^2}+r^{-4}|u|^{-1}\mathring{\varepsilon}^2\big)\\ &\lesssim \mathring{\varepsilon}^2 r_1^{2\xi(\mathbf{2})-6-2\varepsilon_0}\end{aligned}$$

where we bound $D_{\underline{L}}(\widehat{D}_{Z_1}\phi)$ and $D_{\underline{L}}\phi$ on $\underline{\mathcal{H}}_{r_2}$ as before.

We complete the proof by putting the estimates of the $\mathbf{A}_i$'s all together. □

The term $\mathbf{R}_{12}$ can be easily bounded by the lemma:

$$\begin{aligned}\int_{\mathcal{D}_{r_1}}\mathbf{R}_{12} &\lesssim \|rQ(\phi,\mathcal{L}_{Z_1}F;Z_2)\|_{L^2(\mathcal{D}_{r_1})}\Big(\int_{\mathcal{D}_{r_1}} r^{-2}(|D_L\phi^{(\mathbf{2})}|^2+|D_{\underline{L}}\phi^{(\mathbf{2})}|^2)\Big)^{\frac{1}{2}}\\ &\lesssim \mathring{\varepsilon}r_1^{2\xi(\mathbf{2})-6.5-2\varepsilon_0}.\end{aligned}$$

To bound $\mathbf{R}_{13}$, we need the following lemma:

Lemma 5.4. *We have the following estimates:*

$$\|rF_{Z_1\mu}F_{Z_2}{}^{\mu}\|_{L^2(\mathcal{D}_{r_1})} \lesssim \sqrt{\mathring{\varepsilon}}r_1^{\xi(\mathbf{2})-2.5-2\varepsilon_0}. \tag{5.16}$$

Proof. According to the different choices of Z_1 and Z_2, we have
Case 1 $(Z_1, Z_2) = (\Omega, \Omega')$. We have

$$r|F_{\Omega\mu}F_{\Omega}{}^{\mu}\phi| \lesssim r^3|\phi|\big(|\alpha||\underline{\alpha}| + |\sigma|^2\big) \lesssim \frac{\sqrt{\mathring{\varepsilon}}}{r^4|u|^{\frac{5}{2}+2\varepsilon_0}}.$$

Case 2 $Z_1 = \Omega$ and $Z_2 = v^{1+\xi(Z_2)}L + u^{1+\xi(Z_2)}\underline{L}$. Thus,

$$r|F_{\Omega\mu}F_{Z_2}{}^{\mu}\phi| \lesssim r^2|\phi|\big(|\sigma| + |\rho|\big)\big(v^{1+\xi(Z_2)}|\alpha| + u^{1+\xi(Z_2)}|\underline{\alpha}|\big) \lesssim \frac{\sqrt{\mathring{\varepsilon}}}{r^{3-\xi(Z_2)}|u|^{\frac{5}{2}+2\varepsilon_0}}$$

Case 3 $Z_1 = v^{1+\xi(Z_1)}L + u^{1+\xi(Z_1)}\underline{L}$ and $Z_2 = v^{1+\xi(Z_2)}L + u^{1+\xi(Z_2)}\underline{L}$. We have

$$\begin{aligned}|F_{Z_1\mu}F_{Z_2}{}^{\mu}| &\lesssim v^{2+\xi(Z_1)+\xi(Z_2)}|\alpha|^2 + |u|^{2+\xi(Z_1)+\xi(Z_2)}|\underline{\alpha}|^2 \\ &\quad + \big(|u|^{1+\xi(Z_1)}v^{1+\xi(Z_2)} + |u|^{1+\xi(Z_2)}v^{1+\xi(Z_1)}\big)\big(|\alpha||\underline{\alpha}| + |\rho|^2\big).\end{aligned}$$

Thus, we have

$$r|F_{Z_1\mu}F_{Z_2}{}^{\mu}\phi| \lesssim \sqrt{\mathring{\varepsilon}}|u|^{-\frac{5}{2}-2\varepsilon_0}\big(r^{-4+\xi(\mathbf{2})} + |u|^{-4+\xi(\mathbf{2})}r^{-2}\mathring{\varepsilon} + |u|^{1+\xi(Z_1)}r^{-3+\xi(Z_2)} + |u|^{1+\xi(Z_2)}r^{-3+\xi(Z_1)}\big).$$

Then we can simply integrate the above pointwise bounds to conclude. □

This lemma leads to the estimate of $\mathbf{R}_{13}$:

$$\begin{aligned}\int_{\mathcal{D}_{r_1}} \mathbf{R}_{13} &\lesssim \|rF_{Z_1\mu}F_{Z_2}{}^{\mu}\|_{L^2(\mathcal{D}_{r_1})}\Big(\int_{\mathcal{D}_{r_1}} r^{-2}(|D_L\phi^{(\mathbf{2})}|^2 + |D_{\underline{L}}\phi^{(\mathbf{2})}|^2)\Big)^{\frac{1}{2}} \\ &\lesssim \mathring{\varepsilon}r_1^{2\xi(\mathbf{2})-6-1.5\varepsilon_0}.\end{aligned}$$

Finally, from the estimates of $\mathbf{R}_{11}, \mathbf{R}_{12}$ and $\mathbf{R}_{13}$, we conclude that

$$\int_{\mathcal{D}_{r_1}} \mathbf{R}_1 \lesssim \mathring{\varepsilon}r_1^{-6+2\xi(\mathbf{2})-1.5\varepsilon_0}.$$

Based on (5.14) and (5.16), one can also proceed exactly in the same manner to prove that

$$\int_{\mathcal{D}_{r_1}} \mathbf{R}_2 \lesssim \mathring{\varepsilon}r_1^{-4+2\xi(\mathbf{1})-1.5\varepsilon_0}.$$

Therefore, for sufficiently large R_*, since $r_1 \geq R_*$, we have

$$\begin{aligned}\mathcal{E}^{(\mathbf{2})}(\phi; r_1) &\lesssim \mathring{\varepsilon}r_1^{-6+2\xi(\mathbf{2})-1.5\varepsilon_0} + \int_{\mathcal{D}_{r_1}} \underbrace{|\rho|\big(|D_L\phi^{(\mathbf{2})}| + |D_{\underline{L}}\phi^{(\mathbf{2})}|\big)|\phi^{(\mathbf{2})}|}_{\mathbf{T}_1}, \\ \mathcal{E}^{(\mathbf{2})}(\phi; p=2; r_1) &\lesssim \mathring{\varepsilon}r_1^{-4+2\xi(\mathbf{2})-1.5\varepsilon_0} + \int_{\mathcal{D}_{r_1}} \underbrace{|\rho||D_L\psi^{(\mathbf{2})}||\psi^{(\mathbf{2})}|}_{\mathbf{T}_2}.\end{aligned}$$

For $\mathbf{T}_1$ and $\mathbf{T}_2$ we can the proceed exactly in the same manner as in the previous subsection. Finally, for sufficiently large R_*, we can close the bootstrap argument for second order energy quantities on scalar field in ($\mathbf{B}$):

$$\mathcal{E}^{(\mathbf{2})} \leq 2\mathring{\varepsilon}r_1^{-6+2\xi(\mathbf{1})-2\varepsilon_0}, \quad \mathcal{E}^{(\mathbf{2})}(\phi; p=2; r_1) \leq 2\mathring{\varepsilon}r_1^{-4+2\xi(\mathbf{1})-2\varepsilon_0}. \tag{5.17}$$

5.2.3. *The estimates on the current terms.* We now recover the estimates for the current terms in ($\mathbf{C}$).

▶ <u>Zeroth order estimates</u>

For $\mathbf{k} = (\mathbf{0})$, $J^{(\mathbf{0})} = r^2 J = \Im(\overline{\psi} \cdot D\psi)$, according to the pointwise bound on ϕ, we have

$$|J_L^{(\mathbf{0})}| \lesssim \mathring{\varepsilon}r^{-2}|u|^{-\frac{7}{2}-3\varepsilon_0}, \quad |\not{J}^{(\mathbf{0})}| \lesssim \mathring{\varepsilon}r^{-1}|u|^{-\frac{9}{2}-3\varepsilon_0}, \quad |J_{\underline{L}}^{(\mathbf{0})}| \lesssim \mathring{\varepsilon}|u|^{-\frac{11}{2}-3\varepsilon_0}.$$

We then can directly integrate these bounds and we obtain

$$r_1^{-2}\int_{\mathcal{H}_{r_1}} |J_L^{(\mathbf{0})}|^2 + \int_{\mathcal{H}_{r_1}} \frac{|\not{J}^{(\mathbf{0})}|^2}{r^2} + \sup_{r_2\geq r_1}\int_{\underline{\mathcal{H}}_{r_2}^{r_1}} \frac{|\not{J}^{(\mathbf{0})}|^2}{r^2} + \sup_{r_2\geq r_1} r_1^{\frac{3}{2}}\int_{\underline{\mathcal{H}}_{r_2}^{r_1}} \frac{|J_{\underline{L}}^{(\mathbf{0})}|^2}{r^{\frac{7}{2}}} \lesssim \mathring{\varepsilon}^2 r_1^{-10+2\xi(\mathbf{k})-6\varepsilon_0}.$$

Thus, for sufficiently large R_*, it immediately closes ($\mathbf{C}$) for $\mathbf{k} = (\mathbf{0})$.

▶ <u>First order estimates</u>

For $\mathbf{k} = (\mathbf{1})$, we use a vector field Z to represent this index. Notice that we have

$$\begin{aligned}J_\mu^{(\mathbf{1})} &= \Im\Big(\overline{D_Z\psi} \cdot D_\mu\psi + \overline{\psi} \cdot D_\mu(r\widehat{D}_Z\phi) + iF_{Z\mu}|\psi|^2\Big) \\ &= \Im\Big(\overline{\psi^{(\mathbf{1})}} \cdot D_\mu\psi + \overline{\psi} \cdot D_\mu\psi^{(\mathbf{1})}\Big) + F_{Z\mu}|\psi|^2.\end{aligned}$$

♦ For $J_L^{(\mathbf{1})}$, we have

$$\begin{aligned}|J_L^{(\mathbf{1})}| &\lesssim |\psi^{(\mathbf{1})}||D_L\psi| + |\psi||D_L\psi^{(\mathbf{1})}| + |F_{ZL}||\psi|^2 \\ &\lesssim \underbrace{\sqrt{\mathring{\varepsilon}}r^{-1}|u|^{-1-\varepsilon_0}|\phi^{(\mathbf{1})}|}_{\mathbf{I}_{L1}} + \underbrace{\sqrt{\mathring{\varepsilon}}|u|^{-\frac{5}{2}-2\varepsilon_0}|D_L\psi^{(\mathbf{1})}|}_{\mathbf{I}_{L2}} + \underbrace{\mathring{\varepsilon}|u|^{-5-4\varepsilon_0}|F_{ZL}|}_{\mathbf{I}_{L3}}.\end{aligned}$$

For $k \leq 2$, by Lemma A.5, we have

$$\|\phi^{(\mathbf{k})}\|^2_{L^2(\mathcal{S}^{r_2}_{r_1})} \lesssim \|\phi^{(\mathbf{k})}\|^2_{L^2(\mathcal{S}^{r_1}_{r_1})} + \frac{1}{r_1}\int_{\mathcal{H}_{r_1}} |D_L\psi^{(\mathbf{k})}|^2 \lesssim \mathring{\varepsilon} r_1^{-5+2\xi(\mathbf{k})-2\varepsilon_0}. \tag{5.18}$$

For $\mathbf{I}_{L1}$, we have

$$r_1^{-2}\int_{\mathcal{H}_{r_1}} |\mathbf{I}_{L1}|^2 \lesssim \mathring{\varepsilon} r_1^{-2}|u|^{-2-2\varepsilon_0}\int_{r_1}^{r_2} r^{-2}\big(\int_{\mathcal{S}^r_{r_1}} |\phi^{(\mathbf{1})}|^2\big)dr \lesssim \mathring{\varepsilon}^2 r_1^{-10+2\xi(\mathbf{1})-4\varepsilon_0}.$$

For $\mathbf{I}_{L2}$, we have

$$r_1^{-2}\int_{\mathcal{H}_{r_1}} |\mathbf{I}_{L2}|^2 \lesssim \mathring{\varepsilon} r_1^{-2}|u|^{-5-4\varepsilon_0}\int_{\mathcal{H}_{r_1}} |D_L\psi^{(\mathbf{1})}|^2 \lesssim \mathring{\varepsilon}^2 r_1^{-11+2\xi(\mathbf{1})-6\varepsilon_0}.$$

For $\mathbf{I}_{L3}$, we have two cases:

$$\mathbf{I}_{L3} \leq \begin{cases}\mathring{\varepsilon}|u|^{-5-4\varepsilon_0}r|\alpha| \lesssim \mathring{\varepsilon}|u|^{-5-4\varepsilon_0}r^{-2}, \text{ if } Z=\Omega; \\ \mathring{\varepsilon}|u|^{-5-4\varepsilon_0}|u|^{1+\xi(\mathbf{1})}|\rho| \lesssim \mathring{\varepsilon}|u|^{-4+\xi(\mathbf{1})-4\varepsilon_0}r^{-2}, \text{ if } Z = v^{1+\xi(\mathbf{1})}L + u^{1+\xi(\mathbf{1})}\underline{L}.\end{cases}$$

In both cases, we can simply directly integrate the pointwise bounds on $\mathcal{H}_{r_1}$. Therefore, the contribution from $\mathbf{I}_{L3}$ is also bounded by $\mathring{\varepsilon}^2 r_1^{-11+\xi(\mathbf{1})-8\varepsilon_0}$. Hence, we conclude that

$$r_1^{-2}\int_{\mathcal{H}_{r_1}} |J_L^{(\mathbf{1})}|^2 \lesssim \mathring{\varepsilon}^2 r_1^{-10+2\xi(\mathbf{1})-4\varepsilon_0}.$$

♦ For $\not{J}^{(\mathbf{1})}$, we have

$$\begin{aligned}|\not{J}^{(\mathbf{1})}| &\lesssim |\psi^{(\mathbf{1})}||\not{D}\psi| + |\psi||\not{D}\psi^{(\mathbf{1})}| + |F_{ZA}||\psi|^2 \\ &\lesssim \underbrace{\sqrt{\mathring{\varepsilon}}|u|^{-2-\varepsilon_0}|\phi^{(\mathbf{1})}|}_{\not{\mathbf{I}}_1} + \underbrace{\sqrt{\mathring{\varepsilon}}r|u|^{-\frac{5}{2}-2\varepsilon_0}|\not{D}\phi^{(\mathbf{1})}|}_{\not{\mathbf{I}}_2} + \underbrace{\mathring{\varepsilon}|u|^{-5-4\varepsilon_0}|F_{ZA}|}_{\not{\mathbf{I}}_3}.\end{aligned}$$

For $\not{\mathbf{I}}_1$, according to (5.18), we have

$$\int_{\mathcal{H}_{r_1}} \frac{|\not{\mathbf{I}}_1|^2}{r^2} + \int_{\underline{\mathcal{H}}^{r_2}_{r_1}} \frac{|\not{\mathbf{I}}_1|^2}{r^2} \lesssim \mathring{\varepsilon}\int_{r_1}^{r_2} r^{-2}|u|^{-4-2\varepsilon_0}\big(\int_{\mathcal{S}^r_{r_1}} |\phi^{(\mathbf{1})}|^2\big)dr \lesssim \mathring{\varepsilon}^2 r_1^{-10+2\xi(\mathbf{1})-2\varepsilon_0}.$$

For $\not{\mathbf{I}}_2$, we have

$$\int_{\mathcal{H}_{r_1}} \frac{|\not{\mathbf{I}}_2|^2}{r^2} + \int_{\underline{\mathcal{H}}^{r_2}_{r_1}} \frac{|\not{\mathbf{I}}_2|^2}{r^2} \lesssim \mathring{\varepsilon}|u|^{-5-4\varepsilon_0}\Big(\int_{\mathcal{H}_{r_1}} |\not{D}\phi^{(\mathbf{1})}|^2 + \int_{\underline{\mathcal{H}}^{r_2}_{r_1}} |\not{D}\phi^{(\mathbf{1})}|^2\Big) \lesssim \mathring{\varepsilon}^2 r_1^{-11+2\xi(\mathbf{1})-8\varepsilon_0}.$$

For $\not{\mathbf{I}}_3$, we have two cases:

$$|\not{\mathbf{I}}_3| \leq \begin{cases} \mathring{\varepsilon}|u|^{-5-4\varepsilon_0}r|\mathring{\sigma}| \lesssim \mathring{\varepsilon}^{\frac{3}{2}}|u|^{-7-6\varepsilon_0}r^{-1}, \text{ if } Z=\Omega; \\ \lesssim \mathring{\varepsilon}|u|^{-5-4\varepsilon_0}\big(r^{-2+\xi(\mathbf{1})} + \sqrt{\varepsilon_0}r^{-1}|u|^{-2+\xi(\mathbf{1})-\varepsilon_0}\big), \text{ if } Z = v^{1+\xi(\mathbf{1})}L + u^{1+\xi(\mathbf{1})}\underline{L}.\end{cases}$$

In both cases, the contribution of $\not{I}_3$ can be estimated directly by integrating the above bounds and it is bounded by $\mathring{\varepsilon}^2 r_1^{-13+2\xi(\mathbf{1})-8\varepsilon_0}$. Hence, we conclude that

$$\int_{\mathcal{H}_{r_1}} \frac{|\not{J}^{(\mathbf{1})}|^2}{r^2} + \sup_{r_2\geq r_1}\int_{\underline{\mathcal{H}}^{r_2}_{r_1}} \frac{|\not{J}^{(\mathbf{1})}|^2}{r^2} \lesssim \mathring{\varepsilon}^2 r_1^{-10+2\xi(\mathbf{1})-2\varepsilon_0}.$$

♦ For $J_{\underline{L}}^{(\mathbf{1})}$, we have

$$\begin{aligned}|J_{\underline{L}}^{(\mathbf{1})}| &\lesssim |\psi^{(\mathbf{1})}||D_{\underline{L}}\psi| + |\psi||D_{\underline{L}}\psi^{(\mathbf{1})}| + |F_{Z\underline{L}}||\psi|^2 \\ &\lesssim \underbrace{\sqrt{\mathring{\varepsilon}}r|u|^{-\frac{5}{2}-2\varepsilon_0}|\phi^{(\mathbf{1})}|}_{\mathbf{I}_{\underline{L}1}} + \underbrace{\sqrt{\mathring{\varepsilon}}|u|^{-\frac{5}{2}-2\varepsilon_0}|D_{\underline{L}}\psi^{(\mathbf{1})}|}_{\mathbf{I}_{\underline{L}2}} + \underbrace{\mathring{\varepsilon}|u|^{-5-4\varepsilon_0}|F_{Z\underline{L}}|}_{\mathbf{I}_{\underline{L}3}}.\end{aligned}$$

For $\mathbf{I}_{\underline{L}1}$, according to (5.18), we have

$$r_1^{\frac{3}{2}}\int_{\underline{\mathcal{H}}^{r_2}_{r_1}} \frac{|\mathbf{I}_{\underline{L}1}|^2}{r^{\frac{7}{2}}} \lesssim \mathring{\varepsilon} r_1^{\frac{3}{2}}\int_{r_1}^{r_2} r^{-\frac{3}{2}}r_1^{-5-4\varepsilon_0}\big(\int_{\mathcal{S}^r_{r_1}} |\phi^{(\mathbf{1})}|^2\big)dr \lesssim \mathring{\varepsilon}^2 r_1^{-10+2\xi(\mathbf{1})-6\varepsilon_0}.$$

For $\mathbf{I}_{\underline{L}2}$, we first notice that $|D_{\underline{L}}\psi^{(\mathbf{1})}| \lesssim r|D_{\underline{L}}\phi^{(\mathbf{1})}| + |\phi^{(\mathbf{1})}|$. The contribution from $|\phi^{(\mathbf{1})}|$ can be ignored since it has been already treated in $\mathbf{I}_{\underline{L}1}$. Thus, modulo this term, we have

$$r_1^{\frac{3}{2}} \int_{\underline{\mathcal{H}}_{r_1}^{r_2}} \frac{|\mathbf{I}_{\underline{L}2}|^2}{r^{\frac{7}{2}}} \lesssim \mathring{\varepsilon} r_1^{\frac{3}{2}} \int_{\underline{\mathcal{H}}_{r_1}^{r_2}} r^{-\frac{3}{2}}|u|^{-5-4\varepsilon_0}|D_{\underline{L}}\phi^{(\mathbf{1})}|^2 \lesssim \mathring{\varepsilon}^2 r_1^{-11+2\xi(\mathbf{1})-6\varepsilon_0}.$$

For $\mathbf{I}_{\underline{L}3}$, we have two cases:

$$|\mathbf{I}_{\underline{L}3}| \leq \begin{cases} \mathring{\varepsilon}|u|^{-5-4\varepsilon_0} r|\underline{\alpha}| \lesssim \mathring{\varepsilon}|u|^{-5-4\varepsilon_0} r^{-2} + \mathring{\varepsilon}^{\frac{3}{2}}|u|^{-8-5\varepsilon_0}, \text{ if } Z = \Omega; \\ \mathring{\varepsilon}|u|^{-5-4\varepsilon_0}|v|^{1+\xi(\mathbf{1})}|\rho| \lesssim |u|^{-5-4\varepsilon_0} r^{-1+\xi(\mathbf{1})}, \text{ if } Z = v^{1+\xi(\mathbf{1})}L + u^{1+\xi(\mathbf{1})}\underline{L}. \end{cases}$$

In both cases, we can simply directly integrate the pointwise bounds on $\underline{\mathcal{H}}_{r_1}^{r_2}$ to obtain bound $r_1^{\frac{3}{2}} \int_{\underline{\mathcal{H}}_{r_1}^{r_2}}$ by $\mathring{\varepsilon}^2 r_1^{-11+2\xi(\mathbf{1})-2\varepsilon_0}$. Hence, we conclude that

$$\sup_{r_2 \geq r_1} r_1^{\frac{3}{2}} \int_{\underline{\mathcal{H}}_{r_1}^{r_2}} \frac{|J_{\underline{L}}^{(\mathbf{1})}|^2}{r^{\frac{7}{2}}} \lesssim \mathring{\varepsilon}^2 r_1^{-10+2\xi(\mathbf{1})-2\varepsilon_0}.$$

Putting all the estimates together, we obtain that

$$r_1^{-2} \int_{\mathcal{H}_{r_1}} |J_L^{(\mathbf{1})}|^2 + \int_{\mathcal{H}_{r_1}} \frac{|\not{J}^{(\mathbf{1})}|^2}{r^2} + \sup_{r_2 \geq r_1} \int_{\underline{\mathcal{H}}_{r_2}^{r_1}} \frac{|\not{J}^{(\mathbf{1})}|^2}{r^2} + \sup_{r_2 \geq r_1} r_1^{\frac{3}{2}} \int_{\underline{\mathcal{H}}_{r_2}^{r_1}} \frac{|J_{\underline{L}}^{(\mathbf{1})}|^2}{r^{\frac{7}{2}}} \lesssim \mathring{\varepsilon}^2 r_1^{-10+2\xi(\mathbf{k})-2\varepsilon_0}.$$

Thus, for sufficiently large R_*, it immediately closes $(\mathbf{C})$ for $\mathbf{k} = (\mathbf{1})$.

► Second order estimates

For $\mathbf{k} = (\mathbf{2})$, we assume that the vector fields Z_1 and Z_2 represent this index. We also use $(\mathbf{1})$ to denote Z_1 and $(\mathbf{1}')$ to denote Z_2. We first derive the a formula for $J^{(\mathbf{2})} = \mathcal{L}_{Z_1}\mathcal{L}_{Z_2}(\Im(\overline{\psi} \cdot D\psi))$. Indeed, we have

$$\begin{aligned} \mathcal{L}_{Z_1}\mathcal{L}_{Z_2}(\overline{\psi} \cdot D\psi)_\mu &= \overline{\psi} \cdot D_\mu D_{Z_1} D_{Z_2}\psi + \overline{D_{Z_1} D_{Z_2}\psi} \cdot D_\mu\psi + \overline{D_{Z_2}\psi} \cdot D_\mu D_{Z_1}\psi + \overline{D_{Z_1}\psi} \cdot D_\mu D_{Z_2}\psi \\ &\quad + 2i\Re\big(\overline{D_{Z_2}\psi} \cdot \psi\big) F_{Z_1\mu} + i\big(\mathcal{L}_{Z_1}F\big)_{Z_2\mu}|\psi|^2 + iF_{[Z_2,Z_1]\mu}|\psi|^2 + iF_{Z_2\mu}Z_1(|\psi|^2). \end{aligned}$$

Hence,

$$\begin{aligned} J_\mu^{(\mathbf{2})} &= \Im\Big(\overline{\psi} \cdot D_\mu\psi^{(\mathbf{2})} + \overline{\psi^{(\mathbf{2})}} \cdot D_\mu\psi + \overline{\psi^{(\mathbf{1}')}} \cdot D_\mu\psi^{(\mathbf{1})} + \overline{\psi^{(\mathbf{1})}} \cdot D_\mu\psi^{(\mathbf{1}')}\Big) \\ &\quad + 2\Re\big(\overline{\psi^{(\mathbf{1}')}} \cdot \psi\big) F_{Z_1\mu} + 2\Re\big(\overline{\psi^{(\mathbf{1})}} \cdot \psi\big) F_{Z_2\mu} + \big(\mathcal{L}_{Z_1}F\big)_{Z_2\mu}|\psi|^2 + F_{[Z_2,Z_1]\mu}|\psi|^2. \end{aligned}$$

In view of the symmetry of the indices $(\mathbf{1})$ and $(\mathbf{1}')$, we may drop the terms with similar structures and bound $J_\mu^{(\mathbf{2})}$ as follows:

$$|J_L^{(\mathbf{2})}| \lesssim |\psi||D_\mu\psi^{(\mathbf{2})}| + |\psi^{(\mathbf{2})}||D_\mu\psi| + |\psi^{(\mathbf{1}')}||D_\mu\psi^{(\mathbf{1})}| + |\psi||\psi^{(\mathbf{1}')}||F_{Z_1\mu}| + \big(|\big(\mathcal{L}_{Z_1}F\big)_{Z_2\mu}| + |F_{[Z_2,Z_1]\mu}|\big)|\psi|^2.$$

In order to bound $|J_L^{(\mathbf{2})}|$ in an efficient way, we first bound $\psi^{(\mathbf{1}')}$ in L^∞. We have

$$|\psi^{(\mathbf{1}')}| \leq \begin{cases} r^2|\not{D}\varphi|, & \text{if } Z = \Omega; \\ |v^{1+\xi(\mathbf{1}')}||D_L\psi| + |u^{1+\xi(\mathbf{1}')}|\big(|rD_{\underline{L}}\phi| + |\phi|\big), & \text{if } Z = v^{1+\xi(\mathbf{1})}L + u^{1+\xi(\mathbf{1})}\underline{L}. \end{cases}$$

By virtue of the pointwise bounds on $D\phi$, we have

$$|\psi^{(\mathbf{1}')}| \lesssim \sqrt{\mathring{\varepsilon}}|u|^{-\frac{3}{2}+\xi(\mathbf{1}')-2\varepsilon_0}.$$

We can now bound ϕ and $\psi^{(\mathbf{1}')}$ in $J_\mu^{(\mathbf{2})}$ to derive

$$\begin{aligned} |J_\mu^{(\mathbf{2})}| &\lesssim \Big(\sqrt{\mathring{\varepsilon}}|u|^{-\frac{5}{2}-2\varepsilon_0}|D_\mu\psi^{(\mathbf{2})}| + \sqrt{\mathring{\varepsilon}}|u|^{-\frac{3}{2}+\xi(\mathbf{1}')-2\varepsilon_0}|D_\mu\psi^{(\mathbf{1})}|\Big) + |\psi^{(\mathbf{2})}||D_\mu\psi| \\ &\quad + \Big(\mathring{\varepsilon}|u|^{-4+\xi(\mathbf{1}')-4\varepsilon_0}|F_{Z_1\mu}| + \mathring{\varepsilon}|u|^{-5-4\varepsilon_0}\big(|\big(\mathcal{L}_{Z_1}F\big)_{Z_2\mu}| + |F_{[Z_2,Z_1]\mu}|\big)\Big). \end{aligned} \tag{5.19}$$

◆ For $J_L^{(\mathbf{2})}$, we can use the pointwise bound for $D_L\psi$ in (5.19) (where $\mu = L$) and we obtain

$$\begin{aligned} |J_L^{(\mathbf{2})}| &\lesssim \underbrace{\Big(\sqrt{\mathring{\varepsilon}}|u|^{-\frac{5}{2}-2\varepsilon_0}|D_L\psi^{(\mathbf{2})}| + \sqrt{\mathring{\varepsilon}}|u|^{-\frac{3}{2}+\xi(\mathbf{1}')-2\varepsilon_0}|D_L\psi^{(\mathbf{1})}|\Big)}_{\mathbf{II}_{L2}} + \underbrace{\sqrt{\mathring{\varepsilon}}r^{-2}|u|^{-1-\varepsilon_0}|\psi^{(\mathbf{2})}|}_{\mathbf{II}_{L1}} \\ &\quad + \underbrace{\Big(\mathring{\varepsilon}|u|^{-4+\xi(\mathbf{1}')-4\varepsilon_0}|F_{Z_1L}| + \mathring{\varepsilon}|u|^{-5-4\varepsilon_0}|F_{[Z_2,Z_1]L}|\Big)}_{\mathbf{II}_{L3}} + \underbrace{\mathring{\varepsilon}|u|^{-5-4\varepsilon_0}|\big(\mathcal{L}_{Z_1}F\big)_{Z_2L}|}_{\mathbf{II}_{L4}}. \end{aligned}$$

For $\mathbf{II}_{L1}$, in view of (5.18), we have

$$r_1^{-2}\int_{\mathcal{H}_{r_1}}|\mathbf{II}_{L1}|^2\lesssim\mathring{\varepsilon}r_1^{-2}|u|^{-2-2\varepsilon_0}\int_{r_1}^{r_2}r^{-2}\big(\int_{\mathcal{S}_{r_1}^r}|\phi^{(\mathbf{2})}|^2\big)dr\lesssim\mathring{\varepsilon}^2r_1^{-10+2\xi(\mathbf{2})-4\varepsilon_0}.$$

For $\mathbf{II}_{L2}$, by the r^p-weighted energy estimates, we have

$$r_1^{-2}\int_{\mathcal{H}_{r_1}}|\mathbf{II}_{L2}|^2\lesssim\mathring{\varepsilon}r_1^{-2}\int_{\mathcal{H}_{r_1}}|u|^{-5-4\varepsilon_0}|D_L\psi^{(\mathbf{2})}|^2+|u|^{-3+2\xi(\mathbf{1}')-4\varepsilon_0}|D_L\psi^{(\mathbf{1})}|^2\lesssim\mathring{\varepsilon}^2r_1^{-9+2\xi(\mathbf{2})-6\varepsilon_0}.$$

For $\mathbf{II}_{L3}$, we will need the following two inequalities:

$$|F_{Z_1L}|\lesssim r^{-2}|u|^{1+\xi(\mathbf{1})},\quad |F_{[Z_1,Z_2]L}|\lesssim r^{-2}|u|^{1+\xi(\mathbf{2})}.$$

The first one can be checked by a direction computation. For the second, we notice that the only non-vanishing $[Z_1,Z_2]$'s for $Z_1,Z_2\in\mathcal{Z}$ are $[T,S]=T$, $[T,K]=2S$ and $[S,K]=K$. For those vector fields, it is clear that $\xi([Z_1,Z_2])=\xi(Z_1)+\xi(Z_2)$. Therefore, the second inequality follows from the first one. In particular, we have

$$|\mathbf{II}_{L3}|\lesssim\mathring{\varepsilon}r^{-2}|u|^{-3+\xi(\mathbf{2})-4\varepsilon_0}.$$

We can integrate this pointwise bound on $\mathcal{H}_{r_1}$ to bound $r_1^{-2}\int_{\mathcal{H}_{r_1}}|\mathbf{II}_{L3}|^2$ by $\lesssim\mathring{\varepsilon}^2r_1^{-9+2\xi(\mathbf{2})-8\varepsilon_0}$.
For $\mathbf{II}_{L4}$, we have two cases

$$|\big(\mathcal{L}_{Z_1}F\big)_{\Omega L}|\leq\begin{cases}r|\alpha^{(\mathbf{1})}|+r^{-2}|u|^{\xi(\mathbf{1})}, & \text{if } Z_2=\Omega;\\ |u|^{1+\xi(\mathbf{1}')}\big(|\rho^{(\mathbf{1})}|+r^{-3}|u|^{\xi(\mathbf{1})}\big), & \text{if } Z_2=v^{1+\xi(\mathbf{1}')}L+u^{1+\xi(\mathbf{1}')}\underline{L}.\end{cases}$$

For the first case, we have

$$r_1^{-2}\int_{\mathcal{H}_{r_1}}|\mathbf{II}_{L4}|^2\lesssim r_1^{-2}\mathring{\varepsilon}^2|u|^{-10-4\varepsilon_0}\int_{\mathcal{H}_{r_1}}r^2|\alpha^{(\mathbf{1})}|^2+r^{-4}|u|^{2\xi(\mathbf{1})}\lesssim\mathring{\varepsilon}^2r_1^{-13+2\xi(\mathbf{1})-4\varepsilon_0}.$$

For the second case, we have

$$r_1^{-2}\int_{\mathcal{H}_{r_1}}|\mathbf{II}_{L4}|^2\lesssim r_1^{-2}\mathring{\varepsilon}^2|u|^{-8+2\xi(\mathbf{1}')-4\varepsilon_0}\int_{\mathcal{H}_{r_1}}|\rho^{(\mathbf{1})}|^2+r^{-6}|u|^{2\xi(\mathbf{1})}\lesssim\mathring{\varepsilon}^2r_1^{-13+2\xi(\mathbf{2})-4\varepsilon_0}.$$

Hence, $r_1^{-2}\int_{\mathcal{H}_{r_1}}|\mathbf{II}_{L4}|^2$ is bounded by $\mathring{\varepsilon}^2r_1^{-13+2\xi(\mathbf{2})-4\varepsilon_0}$ Together with previous estimates, we obtain

$$r_1^{-2}\int_{\mathcal{H}_{r_1}}|J_L^{(\mathbf{2})}|^2\lesssim\mathring{\varepsilon}^2r_1^{-9+2\xi(\mathbf{2})-6\varepsilon_0}.$$

- For $\not{J}^{(\mathbf{2})}$, we bound $\not{D}\psi$ in (5.19) in L^∞ (where $\mu=e_A$) and we obtain

$$|J_\mu^{(\mathbf{2})}|\lesssim\underbrace{\Big(\sqrt{\mathring{\varepsilon}}r|u|^{-\frac{5}{2}-2\varepsilon_0}|\not{D}\phi^{(\mathbf{2})}|+\sqrt{\mathring{\varepsilon}}r|u|^{-\frac{3}{2}+\xi(\mathbf{1}')-2\varepsilon_0}|\not{D}\phi^{(\mathbf{1})}|\Big)}_{\not{\mathbf{II}}_2}+\underbrace{\sqrt{\mathring{\varepsilon}}|u|^{-2-\varepsilon_0}|\phi^{(\mathbf{2})}|}_{\not{\mathbf{II}}_1}$$
$$+\underbrace{\Big(\mathring{\varepsilon}|u|^{-4+\xi(\mathbf{1}')-4\varepsilon_0}|F_{Z_1A}|+\mathring{\varepsilon}|u|^{-5-4\varepsilon_0}|F_{[Z_2,Z_1]A}|\Big)}_{\not{\mathbf{II}}_3}+\underbrace{\mathring{\varepsilon}|u|^{-5-4\varepsilon_0}|\big(\mathcal{L}_{Z_1}F\big)_{Z_2A}|}_{\not{\mathbf{II}}_4}.$$

For $\not{\mathbf{II}}_1$, according to (5.18), we have

$$\int_{\mathcal{H}_{r_1}}\frac{|\not{\mathbf{II}}_1|^2}{r^2}\lesssim\mathring{\varepsilon}\int_{r_1}^{r_2}r^{-2}|u|^{-4-2\varepsilon_0}\big(\int_{\mathcal{S}_{r_1}^r}|\phi^{(\mathbf{2})}|^2\big)dr\lesssim\mathring{\varepsilon}^2r_1^{-10+2\xi(\mathbf{2})-2\varepsilon_0}.$$

For $\not{\mathbf{II}}_2$, we have

$$\int_{\mathcal{H}_{r_1}}\frac{|\not{\mathbf{II}}_2|^2}{r^2}\lesssim\mathring{\varepsilon}|u|^{-5-4\varepsilon_0}\int_{\mathcal{H}_{r_1}}|\not{D}\phi^{(\mathbf{2})}|^2+\mathring{\varepsilon}|u|^{-5-4\varepsilon_0}\int_{\underline{\mathcal{H}}_{r_1}^{r_2}}|\not{D}\phi^{(\mathbf{2})}|^2$$
$$+\mathring{\varepsilon}|u|^{-3+2\xi(\mathbf{1}')-4\varepsilon_0}\int_{\mathcal{H}_{r_1}}|\not{D}\phi^{(\mathbf{1})}|^2+\mathring{\varepsilon}|u|^{-3+2\xi(\mathbf{1}')-4\varepsilon_0}\int_{\underline{\mathcal{H}}_{r_1}^{r_2}}|\not{D}\phi^{(\mathbf{1})}|^2\lesssim\mathring{\varepsilon}^2r_1^{-9+2\xi(\mathbf{2})-8\varepsilon_0}.$$

For $\not{\mathbf{II}}_3$, since

$$|F_{Z_1A}|\lesssim\sqrt{\mathring{\varepsilon}}r^{-1}|u|^{-2+\xi(\mathbf{1})-\varepsilon_0}+r^{-2+\xi(\mathbf{1})},\quad |F_{[Z_1,Z_2]A}|\lesssim\sqrt{\mathring{\varepsilon}}r^{-1}|u|^{-2+\xi(\mathbf{2})-\varepsilon_0}+r^{-2+\xi(\mathbf{2})},$$

we can integrate these pointwise bounds to derive

$$\int_{\mathcal{H}_{r_1}}\frac{|\not{\mathbf{II}}_3|^2}{r^2}\lesssim\mathring{\varepsilon}^2r_1^{-11+2\xi(\mathbf{2})-8\varepsilon_0}.$$

For $\not{\mathbf{I}}_4$, we have two cases

$$|(\mathcal{L}_{Z_1}F)_{\Omega A}| \leq \begin{cases} r|\sigma^{(\mathbf{1})}|, & \text{if } Z_2 = \Omega; \\ r^{1+\xi(\mathbf{1}')}|\alpha^{(\mathbf{1})}| + |u|^{1+\xi(\mathbf{1}')}|\underline{\alpha}^{(\mathbf{1})}| + r^{-2+\xi(\mathbf{1}')}|u|^{\xi(\mathbf{1})}, & \text{if } Z_2 = v^{1+\xi(\mathbf{1}')}L + u^{1+\xi(\mathbf{1}')}\underline{L}. \end{cases}$$

We claim that in both cases we all have

$$\int_{\mathcal{H}_{r_1}} \frac{|\not{\mathbf{I}}_4|^2}{r^2} \lesssim \mathring{\varepsilon}^2 r_1^{-16+2\xi(\mathbf{2})-8\varepsilon_0}.$$

The proof for the first case is straightforward. For the second case, we need the following bound on $\alpha^{(\mathbf{1})}$:

$$\|\underline{\alpha}^{(\mathbf{1})}\|_{L^2(\mathcal{S}_{r_1}^{r_2})} \lesssim \sqrt{\mathring{\varepsilon}} r_1^{-3+\xi(\mathbf{1})-\varepsilon_0}. \tag{5.20}$$

In fact,

$$\begin{aligned} \|\underline{\alpha}^{(\mathbf{1})}\|^2_{L^2(\mathcal{S}_{r_1}^{r_2})} - \|\underline{\alpha}^{(\mathbf{1})}\|^2_{L^2(\mathcal{S}_{r_1}^{r_1})} &= \int_{\frac{r_2}{2}}^{\frac{r_1}{2}} \int_{\mathbf{S}^2} L\big(|r\underline{\alpha}^{(\mathbf{1})}|^2\big) d\vartheta dv \lesssim \int_{\frac{r_2}{2}}^{\frac{r_1}{2}} \int_{\mathbf{S}^2} |\nabla\!\!\!/_L\big(r\underline{\alpha}^{(\mathbf{1})}\big)||r\underline{\alpha}^{(\mathbf{1})}| d\vartheta dv \\ &\lesssim \|\nabla\!\!\!/_L(r\underline{\alpha}^{(\mathbf{1})})\|_{L^2(\mathcal{H}_{r_1}^{r_2})} \Big(\int_{\frac{r_2}{2}}^{\frac{r_1}{2}} \frac{1}{r^2} \big(\int_{\mathcal{S}_{r_1}^{r}} |\underline{\alpha}^{(\mathbf{1})}|^2 \big) dr \Big)^{\frac{1}{2}}. \end{aligned}$$

To bound $\|\nabla\!\!\!/_L(r\underline{\alpha}^{(\mathbf{1})})\|_{L^2(\mathcal{H}_{r_1}^{r_2})}$, according to (2.15) and the facts that $r|\nabla\!\!\!/\rho(\mathcal{L}_{Z_1}\mathring{F})| \simeq |\rho(\mathcal{L}_\Omega \mathcal{L}_{Z_1}\mathring{F})|$ and $r|\nabla\!\!\!/\sigma(\mathcal{L}_{Z_1}\mathring{F})| \simeq |\sigma(\mathcal{L}_\Omega \mathcal{L}_{Z_1}\mathring{F})|$, we have

$$\|\nabla\!\!\!/_L(r\underline{\alpha}^{(\mathbf{1})})\|^2_{L^2(\mathcal{H}_{r_1}^{r_2})} \leq \int_{\mathcal{H}_{r_1}^{r_2}} |\rho(\mathcal{L}_\Omega \mathcal{L}_{Z_1}\mathring{F})|^2 + |\sigma(\mathcal{L}_\Omega \mathcal{L}_{Z_1}\mathring{F})|^2 + \frac{|\not{J}^{(\mathbf{1})}|^2}{r^2} \lesssim \mathring{\varepsilon} r_1^{-6+2\xi(\mathbf{1})-4\varepsilon_0}.$$

Similar to the proof of Lemma A.7, we use Gronwall's inequality to obtain (5.20). Thus,

$$\int_{\mathcal{H}_{r_1}} \frac{|\not{\mathbf{I}}_4|^2}{r^2} \lesssim \mathring{\varepsilon}^2 \int_{\mathcal{H}_{r_1}} |u|^{-10-8\varepsilon_0} \Big(r^{2\xi(\mathbf{1}')}|\alpha^{(\mathbf{1})}|^2 + |u|^{2+2\xi(\mathbf{1}')} \frac{|\underline{\alpha}^{(\mathbf{1})}|^2}{r^2} + r^{-6+2\xi(\mathbf{1}')}|u|^{2\xi(\mathbf{1})} \Big)$$

The first term can be bounded by the r^p-weighted energy estimates. The last term can bounded directly. For the second term, we use (5.20) to get its L^2 bound on $\mathcal{S}_{r_1}^{r_2}$ then integrate over r. This proves the estimate for $\not{\mathbf{I}}_4$. Together with other estimates, we obtain

$$\int_{\mathcal{H}_{r_1}} \frac{|\not{\mathbf{I}}|^2}{r^2} \lesssim \mathring{\varepsilon}^2 r_1^{-9+2\xi(\mathbf{2})-6\varepsilon_0}.$$

♦ For $J_{\underline{L}}^{(\mathbf{2})}$, we can use the pointwise bound for $D_L\psi$ in (5.19) (where $\mu = L$) and we obtain

$$\begin{aligned} |J_\mu^{(\mathbf{2})}| \lesssim\ & \underbrace{\Big(\sqrt{\mathring{\varepsilon}}|u|^{-\frac{5}{2}-2\varepsilon_0}|D_{\underline{L}}\psi^{(\mathbf{2})}| + \sqrt{\mathring{\varepsilon}}|u|^{-\frac{3}{2}+\xi(\mathbf{1}')-2\varepsilon_0}|D_{\underline{L}}\psi^{(\mathbf{1})}|\Big)}_{\mathbf{II}_{\underline{L}2}} + \underbrace{\sqrt{\mathring{\varepsilon}}|u|^{-3-\varepsilon_0}|\psi^{(\mathbf{2})}|}_{\mathbf{II}_{\underline{L}1}} \\ &+ \underbrace{\Big(\mathring{\varepsilon}|u|^{-4+\xi(\mathbf{1}')-4\varepsilon_0}|F_{Z_1\underline{L}}| + \mathring{\varepsilon}|u|^{-5-4\varepsilon_0}|F_{[Z_2,Z_1]\underline{L}}|\Big)}_{\mathbf{II}_{\underline{L}3}} + \underbrace{\mathring{\varepsilon}|u|^{-5-4\varepsilon_0}|(\mathcal{L}_{Z_1}F)_{Z_2\underline{L}}|}_{\mathbf{II}_{\underline{L}4}}. \end{aligned}$$

For $\mathbf{II}_{\underline{L}1}$, according to (5.18), we have

$$r_1^{\frac{3}{2}} \int_{\underline{\mathcal{H}}_{r_1}^{r_2}} \frac{|\mathbf{II}_{\underline{L}1}|^2}{r^{\frac{7}{2}}} \lesssim \mathring{\varepsilon} r_1^{\frac{3}{2}} \int_{r_1}^{r_2} r^{-\frac{3}{2}} r_1^{-6-2\varepsilon_0} \big(\int_{\mathcal{S}_{r_1}^{r}} |\phi^{(\mathbf{2})}|^2 \big) dr \lesssim \mathring{\varepsilon}^2 r_1^{-11+2\xi(\mathbf{2})-6\varepsilon_0}.$$

For $\mathbf{II}_{\underline{L}2}$, we first notice that $|D_{\underline{L}}\psi^{(\mathbf{k})}| \lesssim r|D_{\underline{L}}\phi^{(\mathbf{k})}| + |\phi^{(\mathbf{k})}|$. The contribution from $|\phi^{(\mathbf{1})}|$ and $|\phi^{(\mathbf{2})}|$ can be ignored since they have been already treated in $\mathbf{I}_{\underline{L}1}$ and $\mathbf{II}_{\underline{L}1}$. Thus, modulo those terms, we have

$$r_1^{\frac{3}{2}} \int_{\underline{\mathcal{H}}_{r_1}^{r_2}} \frac{|\mathbf{II}_{\underline{L}2}|^2}{r^{\frac{7}{2}}} \lesssim \mathring{\varepsilon} r_1^{\frac{3}{2}} \int_{\underline{\mathcal{H}}_{r_1}^{r_2}} r^{-\frac{3}{2}} \big(|u|^{-5-4\varepsilon_0}|D_{\underline{L}}\phi^{(\mathbf{2})}|^2 + |u|^{-3+2\xi(\mathbf{1}')-4\varepsilon_0}|D_{\underline{L}}\phi^{(\mathbf{1})}|^2 \big) \lesssim \mathring{\varepsilon}^2 r_1^{-9+2\xi(\mathbf{1})-6\varepsilon_0}.$$

For $\mathbf{II}_{\underline{L}3}$, we notice that

$$|F_{Z_1\underline{L}}| \lesssim \begin{cases} \sqrt{\mathring{\varepsilon}}|u|^{-3-\varepsilon_0} + r^{-2}, & \text{if } Z_1 = \Omega; \\ r^{-1+\xi(\mathbf{1})}, & \text{if } Z_1 = v^{1+\xi(\mathbf{1})}L + u^{1+\xi(\mathbf{1})}\underline{L}. \end{cases}.$$

Since Ω can not be a commutator $[Z_1, Z_2]$, we have

$$|F_{[Z_1,Z_2]\underline{L}}| \lesssim r^{-1}|u|^{1+\xi(\mathbf{2})}.$$

We remark that the $\xi(\mathbf{2})$ in the above formula is at most 1. Thus, we can directly integrate those pointwise bounds and we obtain

$$r_1^{\frac{3}{2}}\int_{\underline{\mathcal{H}}_{r_1}^{r_2}}\frac{|\mathbf{II}_{\underline{L}}|^2}{r^{\frac{7}{2}}}\lesssim\mathring{\varepsilon}^2 r_1^{-9+2\xi(\mathbf{1})-2\varepsilon_0}.$$

For $\mathbf{II}_{\underline{L}4}$, we have two cases

- If $Z_2=\Omega$, by (5.11), we have
$$|\big(\mathcal{L}_{Z_1}F\big)_{\Omega\underline{L}}|\lesssim r|\underline{\alpha}^{(\mathbf{1})}|+r^{-2+\xi(\mathbf{1})}.$$
Thus,
$$r_1^{\frac{3}{2}}\int_{\underline{\mathcal{H}}_{r_1}^{r_2}}\frac{|\mathbf{II}_{\underline{L}4}|^2}{r^{\frac{7}{2}}}\lesssim\mathring{\varepsilon}^2 r_1^{\frac{3}{2}}\int_{\underline{\mathcal{H}}_{r_1}^{r_2}}r^{-\frac{3}{2}}|u|^{-10-8\varepsilon_0}|\underline{\alpha}^{(\mathbf{1})}|^2+r^{-\frac{13}{2}+2\xi(\mathbf{1})}|u|^{-10-8\varepsilon_0}.$$
- If $Z_2=v^{1+\xi(\mathbf{1}')}L+u^{1+\xi(\mathbf{1}')}\underline{L}$, by (5.11), we have
$$|\big(\mathcal{L}_{Z_1}F\big)_{Z_2\underline{L}}|\lesssim r^{1+\xi(\mathbf{1}')}|\rho^{(\mathbf{1})}|+r^{-2+\xi(2)}.$$
Thus,
$$r_1^{\frac{3}{2}}\int_{\underline{\mathcal{H}}_{r_1}^{r_2}}\frac{|\mathbf{II}_{\underline{L}4}|^2}{r^{\frac{7}{2}}}\lesssim\mathring{\varepsilon}^2 r_1^{\frac{3}{2}}\int_{\underline{\mathcal{H}}_{r_1}^{r_2}}r^{2\xi(\mathbf{1}')-\frac{7}{2}}|u|^{-10-8\varepsilon_0}r^2|\rho^{(\mathbf{1})}|^2+r^{-\frac{15}{2}+2\xi(\mathbf{2})}|u|^{-10-8\varepsilon_0}.$$

In both cases, we have

$$r_1^{\frac{3}{2}}\int_{\underline{\mathcal{H}}_{r_1}^{r_2}}\frac{|\mathbf{II}_{\underline{L}4}|^2}{r^{\frac{7}{2}}}\lesssim\mathring{\varepsilon}^2 r_1^{-12+2\xi(\mathbf{1})-4\varepsilon_0}.$$

Hence, we conclude that

$$\sup_{r_2\geq r_1} r_1^{\frac{3}{2}}\int_{\underline{\mathcal{H}}_{r_1}^{r_2}}\frac{|J_{\underline{L}}^{(\mathbf{2})}|^2}{r^{\frac{9}{2}}}\lesssim\mathring{\varepsilon}^2 r_1^{-10+2\xi(\mathbf{1})-2\varepsilon_0}.$$

Putting all the estimates together, we obtain that

$$r_1^{-2}\int_{\mathcal{H}_{r_1}}|J_L^{(\mathbf{2})}|^2+\int_{\mathcal{H}_{r_1}}\frac{|\not{J}^{(\mathbf{2})}|^2}{r^2}+\sup_{r_2\geq r_1} r_1^{\frac{3}{2}}\int_{\underline{\mathcal{H}}_{r_2}^{r_1}}\frac{|J_{\underline{L}}^{(\mathbf{2})}|^2}{r^{\frac{7}{2}}}\lesssim\mathring{\varepsilon}^2 r_1^{-9+2\xi(\mathbf{k})-2\varepsilon_0}.$$

Thus, for sufficiently large R_*, it immediately closes (**C**) for $\mathbf{k}=(\mathbf{2})$.

6. The analysis in the interior region

6.1. **The conformal theory of Maxwell-Klein-Gordon equations.** We review the conformal theory of the Maxwell-Klein-Gordon equations. We refer the readers to Chapter 4 of the the booklet [4] of Christdoloulou for more details. Let $\mathbf{L}$ be a line bundle over a four dimension Lorentzian manifold (M,g) with a given hermitian metric h. Let D_A be a $\mathbf{U}(1)$-connection compatible with h where A is the corresponding connection 1-form and let ϕ be a section of $\mathbf{L}$. The action for Maxwell-Klein-Gordon equations is

$$\mathcal{L}(A,\phi;g,h)=\int_M\underbrace{\frac{1}{2}g^{\mu\nu}h\big((D_A)_{\partial_\mu}\phi,(D_A)_{\partial_\nu}\phi\big)}_{L_s(A,\phi;g,h)}+\underbrace{\frac{1}{4}F^{\mu\nu}F_{\mu\nu}}_{L_m(A,\phi;g,h)}\,d\mathrm{vol}_g.$$

Let Λ,λ be two positive smooth functions on M. We conformally change the metrics g and h by the following rules:

$$\widetilde{g}=\Lambda^2 g,\quad \widetilde{h}=\lambda^2 h.$$

The $\widetilde{D_A}\phi=D_A\phi+(d\log\gamma)\phi$ is a connection compatible with $\widetilde{h}$ (A is viewed as a given 1-form which gives a connection compatible with $\widetilde{h}$ via this formula). We remark that this is not a gauge transformation. The action in the new conformal settings is related to the old one by the following formulae:

$$\begin{aligned}L_s(A,\phi;g,h)&=\frac{\Lambda^2}{\lambda^2}\Big(L_s(A,\phi;\widetilde{g},\widetilde{h})+\frac{1}{2}|\phi|_{\widetilde{h}}^2\gamma^{-1}\Box_{\widetilde{g}}\gamma-\frac{1}{2}\mathrm{Div}_{\widetilde{g}}\big(|\phi|_{\widetilde{h}}^2\cdot\mathrm{grad}_{\widetilde{g}}(\log\gamma)\big)\Big),\\ L_m(A,\phi;g,h)&=\Lambda^4 L_m(A,\phi;\widetilde{g},\widetilde{h}).\end{aligned}$$

We also notice that $d\mathrm{vol}_g=\Lambda^{-4}d\mathrm{vol}_{\widetilde{g}}$. By setting $\lambda=\Lambda^{-1}$, we obtain

$$\mathcal{L}(A,\phi;g,h)=\mathcal{L}(A,\phi;\widetilde{g},\widetilde{h})+\int_M\frac{1}{2}|\phi|_{\widetilde{h}}^2\gamma^{-1}\Box_{\widetilde{g}}\gamma-\frac{1}{2}\mathrm{Div}_{\widetilde{g}}\big(|\phi|_{\widetilde{h}}^2\cdot\mathrm{grad}_{\widetilde{g}}(\log\gamma)\big)\Big).$$

We now impose a condition on γ: $\Box_{\widetilde{g}}\gamma=0$. In applications, we will take $\widetilde{g}$ to be the Minkowski metric and $\gamma(t,x)=\big((t+C)^2-|x|^2\big)^{-1}$ where C is a constant so that this condition is always satisfied. Thus, we conclude that the difference between the two actions $\mathcal{L}(A,\phi;g,h)$ and $\mathcal{L}(A,\phi;\widetilde{g},\widetilde{h})$ is simply a divergence term. In particular, this implies that the two actions give the same Euler-Lagrange equations. Therefore (A,ϕ) being a solution of (0.1) with (g,h) is equivalent to being a solution of (0.1) with $(\widetilde{g},\widetilde{h})$. We point out that the two sets of (identical) solutions need to be measured in different metrics.

We study a special case. Let $\Phi:(U,m)\to(\widetilde{U},\widetilde{m})$ be a conformal mapping between two domains of Minkowski space. We assume that

$$\Phi^*\widetilde{m}_{\mu\nu}=\Lambda^{-2}m_{\mu\nu}.$$

where $\Lambda(t,x)=(t+R_*+1)^2-|x|^2$ is a smooth function on U. By setting $\widetilde{g}=m$ and $g=\Phi^*\widetilde{m}$, we apply the previous constructions. This yields the following lemma:

Lemma 6.1. *Let $\widetilde{\mathbf{L}}$ be a complex line bundle on $\widetilde{U}$ and $\Phi:U\to\widetilde{U}$ be a conformal diffeomorphism described above. If $(\widetilde{\phi},\widetilde{A})$ is a solution of* (0.1) *for $\widetilde{\mathbf{L}}$, then $(\phi,A):=(\Phi^*\widetilde{\phi},\Phi^*\widetilde{A})$ is solution of* (0.1) *for $\mathbf{L}=\Phi^*\widetilde{\mathbf{L}}$ with respect to $(m,\Lambda^{-2}\Phi^*\widetilde{h})$. In particular, one takes $h=\Phi^*\widetilde{h}$ on $\mathbf{L}$ (this is always the case since we usually identify scalar fields with a $\mathbb{C}-$valued functions by fixing a unit section in $\mathbf{L}$ or $\widetilde{\mathbf{L}}$). We can reformulate the statement as follows: If $(\widetilde{\phi},\widetilde{A})$ (ϕ is a complex function) a solution of* (0.1) *on $\widetilde{U}$, then $(\Lambda^{-1}\Phi^*\widetilde{\phi},\Phi^*\widetilde{A})$ is also a solution of* (0.1) *on U.*

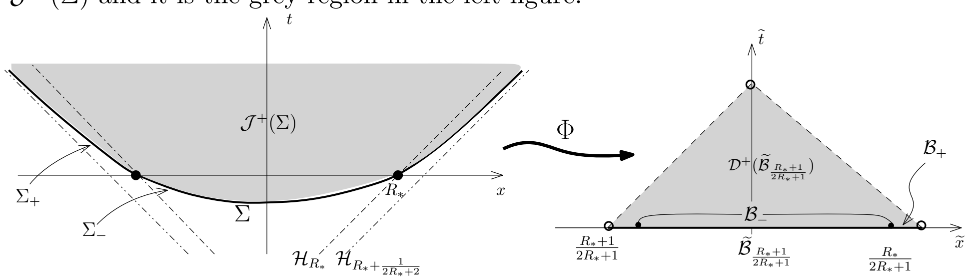
We can also reverse the direction:

Corollary 6.2. *If (ϕ,A) is a solution of* (0.1) *on U, then $\left((\Phi^{-1})^*(\Lambda\cdot\phi),(\Phi^{-1})^*A\right)$ is also a solution of* (0.1) *on $\widetilde{U}$.*

6.2. **The conformal picture.** From now on until the end of the paper, the radius R_* is fixed. And in the sequel we allow the implicit constant depends also on R_* and the size of the data C_0. More precisely, $B\lesssim P$ means that there is a constant C depending only on C_0 such that $B\leq CP$. Here we point out that the radius R_* as well as the charge q_0 only depends on the size of the data C_0. We define a hyperboloid Σ in the Minkowski space by the following equation:

$$\Sigma=\Big\{(t,x)\Big|-\big(t+R_*+1-\frac{2R_*+1}{2(R_*+1)}\big)^2+|x|^2=-\Big(\frac{2R_*+1}{2(R_*+1)}\Big)^2\Big\}.$$

Geometrically, Σ (drawn as the bold black curve in the left figure) is the unique hyperboloid passing through $\mathcal{S}_{R_*}^{R_*}$ and asymptotic to $\mathcal{H}_{R_*+\frac{1}{2R_*+2}}$ (the dash-dot-dot line in the left figure). We denote its causal future by $\mathcal{J}^+(\Sigma)$ and it is the grey region in the left figure.



We now define a map $\Phi:\mathcal{J}^+(\Sigma)\to\mathbf{R}^{3+1}$. To distinguish the domain and the target of the map, we will use $\widetilde{t}$ and $\widetilde{x}_i$'s as coordinate system on the target Minkowski space $(\mathbb{R}^{3+1},\widetilde{m}_{\alpha\beta})$ where $\widetilde{m}_{\alpha\beta}$ is the Minkowski metric on the target. The map Φ is given by the following formula:

$$\Phi:(t,x)\mapsto(\widetilde{t},\widetilde{x})=\Big(-\frac{t+R_*+1}{(t+R_*+1)^2-|x|^2}+\frac{R_*+1}{2R_*+1},\frac{x}{(t+R_*+1)^2-|x|^2}\Big)$$

Geometrically, it is the composition of a time translation with the standard inversion map centered at $(-R_*-1,0,0,0)$. It is straightforward to see that the image of Σ is given by $\widetilde{t}=0$ and $|\widetilde{x}|<\frac{R_*+1}{2R_*+1}$. It is a 3-dimensional open ball denoted by $\widetilde{\mathcal{B}}_{\frac{R_*+1}{2R_*+1}}$ (see the bold straight line segment in the right figure). We also define $\Sigma_\pm=\Sigma\cap\{\pm t\geq 0\}$ and their images under Φ are denoted by $\mathcal{B}_\pm$. We remark that the Σ_+ are entirely inside the exterior region where we have already obtained good control on the solutions. The image of $\mathcal{J}^+(\Sigma)$ is indeed the future domain of dependence of $\widetilde{\mathcal{B}}_{\frac{R_*+1}{2R_*+1}}$ and denoted by $\mathcal{D}^+(\widetilde{\mathcal{B}}_{\frac{R_*+1}{2R_*+1}})$. It is depicted as the grey region in the right figure. As a result, we obtain a diffeomorphism:

$$\Phi:\mathcal{J}^+(\Sigma)\to\mathcal{D}^+(\widetilde{\mathcal{B}}_{\frac{R_*+1}{2R_*+1}})$$

With the naturally induced flat metrics, Φ is indeed a comformal map:

$$\Phi^*\widetilde{m}_{\mu\nu}=\frac{1}{\Lambda(t,x)^2}m_{\mu\nu}\quad\text{with}\quad\Lambda(t,x)=(t+R_*+1)^2-|x|^2.$$

We define functions on the domain of definition on the target of Φ

$$u_* = u + \frac{R_*+1}{2},\ v_* = v + \frac{R_*+1}{2},\ \widetilde{u} = \frac{1}{2}(\widetilde{t} - \frac{R_*+1}{2R_*+1} - \widetilde{r}),$$

$$\widetilde{v} = \frac{1}{2}(\widetilde{t} - \frac{R_*+1}{2R_*+1} + \widetilde{r}),\quad \widetilde{\Lambda}(\widetilde{t},\widetilde{x}) = \big(\widetilde{t} - \frac{R_*+1}{2R_*+1}\big)^2 - |\widetilde{x}|^2.$$

It is straightforward to check that

$$\Lambda = 4u_*v_*,\ \ \widetilde{\Lambda} = 4\widetilde{u}\widetilde{v},\ \ \Phi^*(\widetilde{u}) = -\frac{1}{4u_*},\ \ \Phi^*(\widetilde{v}) = -\frac{1}{4v_*},\ \ \big(\Phi^{-1}\big)^*\widetilde{\Lambda} = \Lambda^{-1}.$$

We can also define two principal null vector fields on the target:

$$\widetilde{L} = \partial_{\widetilde{t}} + \partial_{\widetilde{r}},\ \ \underline{\widetilde{L}} = \partial_{\widetilde{t}} - \partial_{\widetilde{r}}.$$

We can compute the tangent map Φ_* as follows:

$$\Phi_*L = 4\widetilde{v}^2\widetilde{L},\ \ \Phi_*\underline{L} = 4\widetilde{u}^2\underline{\widetilde{L}},\quad \Phi_*(x_i\partial_{x_j} - x_j\partial_{x_i}) = \widetilde{x}_i\partial_{\widetilde{x}_j} - \widetilde{x}_j\partial_{\widetilde{x}_i}.$$

Now define $\widetilde{e}_1$ and $\widetilde{e}_2$ via the following formula:

$$\Phi_*e_A = \widetilde{\Lambda}(\widetilde{t},\widetilde{x})\widetilde{e}_A.$$

Thus, $(\widetilde{e}_1,\widetilde{e}_2,\widetilde{L},\underline{\widetilde{L}})$ consists of a null frame for the target space $\mathcal{D}^+(\widetilde{\mathcal{B}}_{\frac{R_*+1}{2R_*+1}})$. The following set of formulae gives the image of $\mathcal{Z}$ under Φ_*:

$$\Phi_*T = 2\widetilde{v}^2\widetilde{L} + 2\widetilde{u}^2\underline{\widetilde{L}},\ \ \Phi_*(x_i\partial_{x_j} - x_j\partial_{x_i}) = \widetilde{x}_i\partial_{\widetilde{x}_j} - \widetilde{x}_j\partial_{\widetilde{x}_i},\ \ \Phi_*K = \frac{1}{2}\partial_{\widetilde{t}}.$$

The next objective is to define the fields on the target of Φ corresponding to $(G = dA, f)$ (on the domain of definition). The correspondence is given by the following formulae: [1]

$$\widetilde{f}(\widetilde{t},\widetilde{x}) := \Big(\big(\Phi^{-1}\big)^*(\Lambda\cdot f)\Big)(\widetilde{t},\widetilde{x}) = \Lambda(t,x)f(t,x),$$

$$\widetilde{A}(\widetilde{t},\widetilde{x}) := \Big(\big(\Phi^{-1}\big)^*A\Big)(\widetilde{t},\widetilde{x})\ \ (\text{hence } \widetilde{\Omega}(\widetilde{t},\widetilde{x}) := \Big(\big(\Phi^{-1}\big)^*\Omega\Big)(\widetilde{t},\widetilde{x})).$$

In view of the conformal theory presented at the beginning of the section, if we take $(G,f) = (F,\phi)$ the solution of (0.1), the pair $(\widetilde{\phi},\widetilde{F})$ is a solution of (0.1) on $\mathcal{D}^+(\widetilde{\mathcal{B}}_{\frac{R_*+1}{2R_*+1}})$. In the rest of this subsection (Section 6.2), we will use the following shorthand notations for the null components of G and $\widetilde{G}$:

$$\alpha = \alpha(G),\ \ \rho = \rho(G),\ \ \sigma = \sigma(G),\ \ \underline{\alpha} = \underline{\alpha}(G),$$

$$\widetilde{\alpha} = \alpha(\widetilde{G}),\ \ \widetilde{\rho} = \rho(\widetilde{G}),\ \ \widetilde{\sigma} = \sigma(\widetilde{G}),\ \ \underline{\widetilde{\alpha}} = \underline{\alpha}(\widetilde{G}).$$

Since Φ_* and Φ^* behave well in the functorial way, the following formulae are immediate consequences of the previous computations:

$$\begin{gathered}\widetilde{\alpha}_A(\widetilde{t},\widetilde{x}) = 16u_*v_*^3\alpha_A(t,x),\ \ \widetilde{\rho}(\widetilde{t},\widetilde{x}) = 16u_*^2v_*^2\rho(t,x),\\ \widetilde{\sigma}(\widetilde{t},\widetilde{x}) = 16u_*^2v_*^2\sigma(t,x),\ \ \underline{\widetilde{\alpha}}_A(\widetilde{t},\widetilde{x}) = 16u_*^3v_*\underline{\alpha}_A(t,x),\\ \widetilde{D}_{\widetilde{L}}\widetilde{\phi}(\widetilde{t},\widetilde{x}) = 4v_*^2D_L(\Lambda\phi)(t,x),\ \ \widetilde{D}_{\widetilde{e}_A}\widetilde{\phi} = 4u_*v_*D_{e_A}(\Lambda\phi)(t,x),\ \ \widetilde{D}_{\underline{\widetilde{L}}}\widetilde{\phi}(\widetilde{t},\widetilde{x}) = 4u_*^2D_{\underline{L}}(\Lambda\phi)(t,x).\end{gathered} \tag{6.1}$$

The correspondence also behaves well with respect to taking derivatives for $Z \in \mathcal{Z}$: [2]

$$\begin{gathered}\widetilde{\mathcal{L}_ZG} = \mathcal{L}_{\Phi_*Z}\widetilde{G},\ \ \forall Z\in\mathcal{Z},\quad \widetilde{\widetilde{D}_{\Omega_{ij}}f} = \widetilde{D}_{\Phi_*\Omega_{ij}}\widetilde{f},\\ \widetilde{\widetilde{D}_Tf} = \widetilde{D}_{\Phi_*T}\widetilde{f} + \big(\widetilde{t} - \frac{R_*+1}{2R_*+1}\big)\widetilde{f},\ \ \widetilde{\widetilde{D}_Kf} = \widetilde{D}_{\Phi_*K}\widetilde{f} + (R_*+1)^2\big(\widetilde{t} - \frac{R_*^2}{(R_*+1)(2R_*+1)}\big)\widetilde{f}.\end{gathered} \tag{6.2}$$

We will study the energy flux through the space-like hypersurface Σ. For this purpose, we need to study the geometry of Σ. It is more convenient to use a new coordinate system to characterize Σ. We define

$$U = \sqrt{\big(t + R_* + \frac{1}{2R_*+2}\big)^2 - r^2},\ \ V = \sqrt{\big(t + R_* + \frac{1}{2R_*+2}\big)^2 + r^2}.$$

The new coordinates system is (U,V,ϑ) where $\vartheta \in \mathbf{S}^2$ is the standard spherical coordinates. According to the definition, Σ is defined by $U = \frac{2R_*+1}{2(R_*+1)}$. Thus, (V,ϑ) should be regarded as local coordinate system on Σ. To simplify notations, we also define

$$t_* = t + R_* + \frac{1}{2R_*+2}.$$

[1] In the sequel of the section, when we write (t,x) and $(\widetilde{t},\widetilde{x})$, it is always understood as $\Phi : (t,x) \mapsto (\widetilde{t},\widetilde{x})$.

[2] We also use $\widetilde{D}$ to denote the covariant derivatives corresponding to $\widetilde{A}$ on the target. This should not be confused with the $\widetilde{D}$ on the domain of definition of Φ.

In the new coordinate system, the volume form of the Minkowski metric m can be written as

$$d\mathrm{vol}_m = r^2 dt dr d\vartheta = \frac{rUV}{2t_*} dU dV d\vartheta. \tag{6.3}$$

The hypersurface Σ can also be viewed as the graph of the function g over $\mathbb{R}^3$, where g is defined as

$$t = g(x) = \sqrt{|x|^2 + \Big(\frac{2R_* + 1}{2(R_* + 1)}\Big)^2} - \frac{2R_*^2 + 2R_* + 1}{2(R_* + 1)}.$$

Therefore the surface measure on Σ_+ is given by (using r and ϑ as coordinate function):

$$d\mu_\Sigma = \sqrt{1 + |\nabla g|^2} dx = \sqrt{\frac{2r^2 + \big(\frac{2R_*+1}{2(R_*+1)}\big)^2}{r^2 + \big(\frac{2R_*+1}{2(R_*+1)}\big)^2}} r^2 dr d\vartheta.$$

In view of the defining equation of Σ, one can express it in terms of (V, ϑ) on Σ:

$$d\mu_\Sigma = \frac{rV^2}{t_*} dV d\vartheta. \tag{6.4}$$

The following lemma gives a formula to compute the contraction of a vector field with the spacetime volume form on Σ. It will play a key role in the computations of the local energy density on Σ.

Lemma 6.3. *Let J be a smooth vector field on $\mathbb{R}^{3+1}$ and $\iota : \Sigma \hookrightarrow \mathbb{R}^{3+1}$ be the canonical embedding. We use $i_J d\mathrm{vol}_m$ to denote the contraction of J and the spacetime volume form. Then we have*

$$\iota^*\big(i_J d\mathrm{vol}_m\big) = -\frac{rV}{4t_*}\big((t_* - r)J_{\underline{L}} + (t_* + r)J_L\big) dV d\vartheta.$$

Proof. Since we have already derived the formulae for volume/surface measures in terms of (U, V, ϑ), it just remains to relate L and $\underline{L}$ to ∂_U and ∂_V. This is recorded in the following formulae:

$$L = \frac{t_* - r}{U}\partial_U + \frac{t_* + r}{V}\partial_V, \quad \underline{L} = \frac{t_* + r}{U}\partial_U + \frac{t_* - r}{V}\partial_V. \tag{6.5}$$

□

We turn to the study of energy quantities. Given fields $(\widetilde{f}, \widetilde{G})$ on $\widetilde{\mathcal{B}}_{\frac{R_*+1}{2R_*+1}}$, the standard energy is defined as

$$\mathcal{E}[\widetilde{f}, \widetilde{G}](\widetilde{\mathcal{B}}_{\frac{R_*+1}{2R_*+1}}) = \int_0^{\frac{R_*+1}{2R_*+1}} \int_{\mathbf{S}^2} \Big(|\widetilde{\alpha}|^2 + |\widetilde{\rho}|^2 + |\widetilde{\sigma}|^2 + |\widetilde{\underline{\alpha}}|^2 + |\widetilde{D}_{\widetilde{L}}\widetilde{\phi}|^2 + \sum_{A=1}^2 |\widetilde{D}_{\widetilde{e}_A}\widetilde{\phi}|^2 + |\widetilde{D}_{\widetilde{\underline{L}}}\widetilde{\phi}|^2\Big) \widetilde{r}^2 d\widetilde{r} d\widetilde{\vartheta}.$$

In view of our analysis in the exterior region, the more relevant part of the energy is as follows:

$$\mathcal{E}[\widetilde{f}, \widetilde{G}](\widetilde{\mathcal{B}}_+) = \int_{\frac{R_*}{2R_*+1}}^{\frac{R_*+1}{2R_*+1}} \int_{\mathbf{S}^2} \Big(|\widetilde{\alpha}|^2 + |\widetilde{\rho}|^2 + |\widetilde{\sigma}|^2 + |\widetilde{\underline{\alpha}}|^2 + |\widetilde{D}_{\widetilde{L}}\widetilde{\phi}|^2 + \sum_{A=1}^2 |\widetilde{D}_{\widetilde{e}_A}\widetilde{\phi}|^2 + |\widetilde{D}_{\widetilde{\underline{L}}}\widetilde{\phi}|^2\Big) \widetilde{r}^2 d\widetilde{r} d\widetilde{\vartheta}.$$

The main objective is to rewrite this energy in terms of (f, G) on Σ_+:

Proposition 6.4. *Given the conformal correspondence $(f, G) \mapsto (\widetilde{f}, \widetilde{G})$, we have*

$$\mathcal{E}[\widetilde{f}, \widetilde{G}](\widetilde{\mathcal{B}}_+) \lesssim \int_{\Sigma_+} r^2|\widetilde{\alpha}|^2 + |\widetilde{\rho}|^2 + |\widetilde{\sigma}|^2 + \frac{|\widetilde{\underline{\alpha}}|^2}{r^2} + |D_L\Psi|^2 + |D_L f|^2 + |\not{D} f|^2 + \frac{|D_{\underline{L}} f|^2}{r^2} + \frac{|f|^2}{r^2}. \tag{6.6}$$

where we use the hypersurface measure $d\mu_\Sigma$ for the integration and $\Psi = rf$.

Proof. We start to express the volume form $\widetilde{r}^2 d\widetilde{r} d\widetilde{\vartheta}$ in terms of $dV d\vartheta$. We notice that $\big(\Phi^{-1}\big)^*(U) = \frac{2R_*+1}{2R_*+2}$ on Σ_+. In terms of V and U, we have

$$\widetilde{r} = \frac{\sqrt{\frac{V^2 - U^2}{2}}}{U^2 + \big(\frac{2R_*+1}{2R_*+2}\big)^2 + \frac{2R_*+1}{R_*+1}\sqrt{\frac{V^2+U^2}{2}}} = \frac{\sqrt{V^2 - U^2}}{2^{\frac{3}{2}}U^2 + 2U\sqrt{V^2 + U^2}}.$$

Thus,

$$d\widetilde{r} = \frac{2^{\frac{3}{2}}U^2\sqrt{V^2 + 4U^3}V}{\big(2^{\frac{3}{2}}U^2 + 2U\sqrt{V^2 + U^2}\big)^2\sqrt{V^2 - U^2}\sqrt{V^2 + U^2}} dV.$$

Since $U \approx 1$ and $V \approx r$ on Σ_+ (provided $R_* \geq 1$ which always holds). Thus, $\widetilde{r} \approx 1$ and we have

$$\widetilde{r}^2 d\widetilde{r} d\widetilde{\vartheta} \approx r^{-2} dV d\vartheta. \tag{6.7}$$

In view of (6.4) and the facts that $t_* \approx r \approx v_*$ on Σ_+, we conclude that

$$\widetilde{r}^2 d\widetilde{r} d\widetilde{\vartheta} \approx v_*^{-4} d\mu_{\Sigma_+}. \tag{6.8}$$

Thus, combing with (6.1), the above equation yields

$$\begin{aligned}\mathcal{E}[\widetilde{f},\widetilde{G}](\widetilde{\mathcal{B}}_+) = \int_{\Sigma_+} & u_*^2 v_*^2 |\widetilde{\alpha}|^2 + u_*^4\big(|\widetilde{\rho}|^2 + |\widetilde{\sigma}|^2\big) + u_*^6 v_*^{-2}|\underline{\widetilde{\alpha}}|^2 + |D_L(\Lambda f)|^2 \\ & + u_*^2 v_*^{-2}|\not{D}(\Lambda f)|^2 + u_*^4 v_*^{-4}|D_{\underline{L}}(\Lambda f)|^2.\end{aligned} \tag{6.9}$$

The above integration is understood over the measure $d\mu_{\Sigma_+}$. On the other hand, since $Lu_* = \underline{L}v_* = 1$, $Lv_* = \underline{L}u_* = 0$ and $\Lambda = 4u_*v_*$, we can easily obtain that

$$\begin{aligned}|D_L(\Lambda f)|^2 = |4u_* D_L(v_* f)|^2 = |4u_* D_L\big((u_*+r)f\big)|^2 \lesssim u_*^4|D_L f|^2 + u_*^2|D_L\Psi|^2,\\ |\not{D}(\Lambda f)|^2 \approx u_*^2v_*^2|\not{D}f|^2, \quad |D_{\underline{L}}(\Lambda f)|^2 \lesssim v_*^2u_*^2|D_{\underline{L}}f|^2 + |v_*|^2|f|^2.\end{aligned}$$

Since $\frac{1}{2} - \frac{1}{4(R_*+1)} \leq u_* \leq \frac{1}{2}$, thus $u_* \approx 1$. The above estimate together with (6.9) completes the proof of the proposition. □

We can further more eliminate the term $\frac{|f|^2}{r^2}$ in (6.6):

Corollary 6.5. *Given the conformal correspondence $(f,G) \mapsto (\widetilde{f},\widetilde{G})$, we have*

$$\mathcal{E}[\widetilde{f},\widetilde{G}](\widetilde{\mathcal{B}}_+) \lesssim R_*^{-1}\int_{\mathcal{S}_{R_*}^{R_*}} |f|^2 + \int_{\Sigma_+} r^2|\widetilde{\alpha}|^2 + |\widetilde{\rho}|^2 + |\widetilde{\sigma}|^2 + \frac{|\underline{\widetilde{\alpha}}|^2}{r^2} + |D_L\Psi|^2 + |D_L f|^2 + |\not{D}f|^2 + \frac{|D_{\underline{L}}f|^2}{r^2}. \tag{6.10}$$

Proof. According to (6.4), by Newton-Leibniz formula, we have

$$\begin{aligned}\int_{\Sigma_+}\frac{|f|^2}{r^2} &\approx \int_{R_*}^{\infty}\int_{\mathbf{S}^2}|f|^2 dV d\vartheta = R_*^{-1}\int_{\mathcal{S}_{R_*}^{R_*}}|f|^2 - \lim_{V_0\to\infty}\Big(V_0^{-1}\int_{\Sigma_+\cap V=V_0}|f|^2 + 2\int_{\Sigma_+}V_0^{-1}\Re(\overline{D_{\partial_V}f}\cdot f)\Big)\\ &\lesssim R_*^{-1}\int_{\mathcal{S}_{R_*}^{R_*}}|f|^2 + \int_{\Sigma_+}V^{-1}|D_{\partial_V}f||f| \lesssim R_*^{-1}\int_{\mathcal{S}_{R_*}^{R_*}}|f|^2 + \int_{\Sigma_+}|D_{\partial_V}f|^2) + \frac{1}{2}\int_{\Sigma_+}\frac{|f|^2}{r^2}.\end{aligned}$$

Thus,

$$\int_{\Sigma_+}\frac{|f|^2}{r^2} \lesssim R_*^{-1}\int_{\mathcal{S}_{R_*}^{R_*}}|f|^2 + \int_{\Sigma_+}|D_{\partial_V}f|^2.$$

According to (6.5), on Σ_+, we have

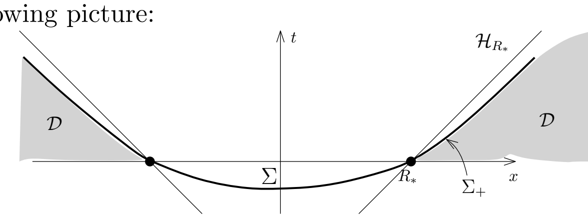
$$|D_{\partial_V}f|^2 = |\frac{V}{4t_*r}\big((t_*+r)L - (t_*-r)\underline{L}\big)f|^2 \lesssim |D_L f|^2 + \frac{|D_{\underline{L}}f|^2}{r^2}.$$

Therefore,

$$\int_{\Sigma_+}\frac{|f|^2}{r^2} \lesssim R_*^{-1}\int_{\mathcal{S}_{R_*}^{R_*}}|f|^2 + \int_{\Sigma_+}|D_L f|^2 + \frac{|D_{\underline{L}}f|^2}{r^2}.$$

In view of (6.6), this completes the proof. □

To bound the righthand side of (6.6), we need standard energy estimate and r^p-weighted energy estimates on Σ_+. We take the integration domain $\mathcal{D}$ to be the spacetime slab bounded by Σ_+ and $\mathcal{B}_{R_*}$, see the grey region in the following picture:



Lemma 6.6. *For all G and f, $1 \leq p \leq 2$, we have*

$$\begin{aligned}&\int_{\Sigma_+}\frac{(t_*+r)\big(|D_L f|^2 + |\alpha|^2\big) + 2t_*\big(|\not{D}f|^2 + |\rho|^2 + |\sigma|^2\big) + (t_*-r)\big(|D_{\underline{L}}f|^2 + |\underline{\alpha}|^2\big)}{8V}\\ &= \mathcal{E}[G,f](\mathcal{B}_{R_*}) - \int_{\mathcal{D}}\Re\big(\overline{\Box_A f}\cdot D_{\partial_t}f\big) + \nabla^\mu G_{\mu\nu}\cdot G_0{}^\nu + F_{0\mu}J[f]^\mu,\end{aligned} \tag{6.11}$$

and

$$
\begin{aligned}
&\int_{\mathcal{B}_{R_*}} r^{p-2}\big(|D_L\Psi|^2+|\not{D}\Psi|^2\big)+r^p\big(|\alpha(G)|^2+|\rho(G)|^2+|\sigma(G)|^2\big)\\
=&\int_{\Sigma_+}\frac{r^p\big(|D_L\Psi|^2+r^2|\alpha|^2\big)(t_*+r)+r^p\big(|\not{D}\Psi|^2+r^2(|\rho|^2+|\sigma|^2)\big)(t_*-r)}{4r^2V}\\
&+\int_{\mathcal{D}} r^{p-3}\Big(p(|D_L\Psi|^2+r^2|\alpha(G)|^2)+(2-p)(|\not{D}\Psi|^2+r^2|\rho(G)|^2+r^2|\sigma(G)|^2)\Big)\\
&+\int_{\mathcal{D}} r^{p-1}\Re\big(\overline{\Box_A f}\cdot D_L\Psi\big)+r^p\nabla^\mu G_{\mu\nu}\cdot G_L{}^\delta+r^pF_{L\mu}J[f]^\mu.
\end{aligned}\tag{6.12}
$$

Here $\Psi=rf$.

Proof. We first derive r^p-weighted energy identity. The following computations are similar to those in the proof of Lemma 2.7. We take $X=r^pL$, $\chi=r^{p-1}$ and $Y=\frac{p}{2}r^{p-2}|f|^2L$ in (2.17) and then integrate on $\mathcal{D}$. The expressions of $\mathbf{D_1}$ and $\mathbf{D_2}$ (in (2.17)) are given in (2.21). According to Stokes formula, we have

$$
\int_{\mathcal{B}_{R_*}}\widetilde{J}[G,f]^\mu n_\mu+\int_{\Sigma_+}\iota^*\big(i_X d\mathrm{vol}_m\big)=\int_{\mathcal{D}}\mathbf{D_1}+\mathbf{D_2}
$$

On $\mathcal{B}_{R_*}$, the normal n^μ is ∂_t, we have

$$
{}^{(X)}\widetilde{J}[G,f]^\mu n_\mu=\frac12 r^{p-2}\big(r^2\alpha(G)^2+r^2\rho(G)^2+r^2\sigma(G)^2+|D_L\Psi|^2+|\not{D}\Psi|^2\big)-\frac12\big((p+1)r^{p-2}|f|^2+r^{p-1}\partial_r(|f|^2)\big).
$$

Therefore, we have

$$
\begin{aligned}
\int_{\mathcal{B}_{R_*}}{}^{(X)}\widetilde{J}[G,f]^\mu n_\mu&=\frac12\int_{\mathcal{B}^{r_2}_{r_1}} r^2\alpha(G)^2+r^2\rho(G)^2+r^2\sigma(G)^2+|D_L\Psi|^2+|\not{D}\Psi|^2\\
&\quad-\frac12\int_{r_1}^{r_2}\int_{\mathbf{S}^2}\underbrace{(p+1)r^p|f|^2+r^{p+1}\partial_r(|f|^2)}_{=\partial_r(r^{p+1}|f|^2)}d\vartheta dr\\
&=\frac12 L_1+\frac12\int_{\mathcal{S}^{r_1}_{r_1}} r^{p-1}|f|^2.
\end{aligned}
$$

On the other hand, in view of Lemma 6.3, we need to compute ${}^{(X)}\widetilde{J}[G,f]_L$ and ${}^{(X)}\widetilde{J}[G,f]_{\underline{L}}$. In fact, we have

$$
\begin{aligned}
r^2\cdot{}^{(X)}\widetilde{J}[G,f]_L&=r^p\big(|D_L\Psi|^2+r^2|\alpha|^2\big)-\frac12 L(r^{p+1}|f|^2),\\
r^2\cdot{}^{(X)}\widetilde{J}[G,f]_{\underline{L}}&=r^p\big(|\not{D}\Psi|^2+r^2(|\rho|^2+|\sigma|^2)\big)+\frac12\underline{L}(r^{p+1}|f|^2).
\end{aligned}
$$

Therefore, by Lemma 6.3 and replacing L and $\underline{L}$ in the above formulae by (6.5), we obtain that

$$
\begin{aligned}
\iota^*\big(i_{\widetilde{J}}d\mathrm{vol}_m\big)=&-\frac{r^p\big(|D_L\Psi|^2+r^2|\alpha|^2\big)(t_*+r)+r^p\big(|\not{D}\Psi|^2+r^2(|\rho|^2+|\sigma|^2)\big)(t_*-r)}{4r^2V}d\mu_\Sigma\\
&+\frac12\partial_V(r^{p+1}|f|^2)dVd\vartheta
\end{aligned}
$$

Therefore, we have

$$
\begin{aligned}
\int_{\Sigma_+}\iota^*\big(i_X d\mathrm{vol}_m\big)=&-\int_{\Sigma_+}\frac{r^p\big(|D_L\Psi|^2+r^2|\alpha|^2\big)(t_*+r)+r^p\big(|\not{D}\Psi|^2+r^2(|\rho|^2+|\sigma|^2)\big)(t_*-r)}{4r^2V}\\
&-\frac12\int_{\mathcal{S}^{r_1}_{r_1}} r^{p-1}|f|^2.
\end{aligned}\tag{6.13}
$$

The r^p-weighted energy identity follows immediately.

For the basic energy identity, we simply take $X=\partial_t$, $\chi=0$ and $Y=0$ in (2.17). The identity easily follows if we observe that

$$
\iota^*\big(i_{\widetilde{J}}d\mathrm{vol}_m\big)=-\frac{(t_*+r)|D_Lf|^2+2t_*|\not{D}f|^2+(t_*-r)|D_{\underline{L}}f|^2}{8V}d\mu_\Sigma.
$$

□

6.3. **The energy estimates on the conformal conformal compactification.** We first apply the theory of the previous section to the static solution $(f,G)=(0,F[q_0])$. By the definition of the charge field $F[q_0]$, according to (6.9), we can bound that

$$\mathcal{E}\big[F[q_0]\big](\widetilde{\mathcal{B}}_+)\lesssim\int_{\Sigma_+}u_*^4|r^{-2}|^2+u_+^2|r^{-3}|^2\lesssim 1. \tag{6.14}$$

This is due to the fact that $|\rho(F[q_0])|$ decays like r^{-2} while all the other components decay at least r^{-3}. We also note that on Σ_+, $u_*\approx 1$.

Remark 6.7. *Despite the simplicity, the computation in* (6.14) *is of great conceptual importance. Indeed, if one considers conformal energy on a constant time slice, the factor u_*^4 would be replace by r^2 (near spatial infinity) so that the contribution of the charge part of the field would be divergent. However,* (6.14) *shows that the charge part behaves very well near null infinity. This is the reason we choose the inversion to compactify the spacetime over the usually Penrose compactification.*

The main purpose of the current section is to obtain (the L^2 bounds up to two derivatives) of $(\widetilde{\phi},\widetilde{F})$ on $\mathcal{B}_+$. In view of (6.10), it is reasonable to define the following quantity:

$$\mathcal{E}_+[f,G]=\int_{\Sigma_+}r^2|\widetilde{\alpha}|^2+|\widetilde{\rho}|^2+|\widetilde{\sigma}|^2+\frac{|\widetilde{\underline{\alpha}}|^2}{r^2}+|D_L\Psi|^2+|D_Lf|^2+|\not{D}f|^2+\frac{|D_{\underline{L}}f|^2}{r^2}. \tag{6.15}$$

In what follows, we take the (f,G) to be $(\phi^{(\mathbf{k})},\mathcal{L}_Z^{(\mathbf{k})}\mathring{F})$ for $|\mathbf{k}|\leq 2$. In this specific set-up, we first simplify the estimates in Lemma 6.6.

We start with identity (6.11). Because $t_*\approx V\approx r$ and $|t_*-r|\approx 1$, its lefthand side is approximately

$$\int_{\Sigma_+}|\alpha^{(\mathbf{k})}|^2+|\rho^{(\mathbf{k})}|^2+|\sigma^{(\mathbf{k})}|^2+|D_L\phi^{(\mathbf{k})}|^2+|\not{D}\phi^{(\mathbf{k})}|^2+\frac{|\underline{\alpha}^{(\mathbf{k})}|^2+|D_{\underline{L}}\phi^{(\mathbf{k})}|^2}{r^2}.$$

The first term of the righthand side is coming from the data hence bounded by $\mathring{\varepsilon}R_*^{-6-2\varepsilon_0}$. The second one is precisely the error terms that we have controlled in the exterior region (indeed, $\mathcal{D}\subset\mathcal{D}_{R_*}$), hence also bounded by $\mathring{\varepsilon}R_*^{-6-2\varepsilon_0}$. Therefore, via (6.6), we arrive at the following estimates

$$\int_{\Sigma_+}|\alpha^{(\mathbf{k})}|^2+|\rho^{(\mathbf{k})}|^2+|\sigma^{(\mathbf{k})}|^2+|D_L\phi^{(\mathbf{k})}|^2+|\not{D}\phi^{(\mathbf{k})}|^2+\frac{|\underline{\alpha}^{(\mathbf{k})}|^2+|D_{\underline{L}}\phi^{(\mathbf{k})}|^2}{r}\lesssim\mathring{\varepsilon}R_*^{-6-2\varepsilon_0}. \tag{6.16}$$

We use the $p=2$ case of (6.12). The first term of the left hand side is coming from the data hence bounded by $\mathring{\varepsilon}R_*^{-4-2\varepsilon_0}$. The second and third terms of the righthand side are precisely the error terms that we have controlled in the exterior region hence also bounded by $\mathring{\varepsilon}R_*^{-4-2\varepsilon_0}$. Finally, we use $t_*\approx V\approx r$ and $|t_*-r|\approx 1$ for the first term on the righthand side. Therefore, we arrive at the following estimate:

$$\int_{\Sigma_+}|D_L\psi^{(\mathbf{k})}|^2+r^2|\alpha|^2+r\big(|\not{D}\phi^{(\mathbf{k})}|^2+|\rho|^2+|\sigma|^2\big)\lesssim_{R_*}\mathring{\varepsilon}R_*^{-4-2\varepsilon_0}. \tag{6.17}$$

As a conclusion, we obtain that

$$\mathcal{E}_+[\phi^{(\mathbf{k})},\mathcal{L}_Z^{(\mathbf{k})}\mathring{F}]\lesssim 1. \tag{6.18}$$

Here we recall that the implicit constant here depends only on the size of the initial data C_0 defined before the main theorem.

Remark 6.8. *From this point till the end of the paper, we will ignore the dependence on R_* for the universal constants since R_* is already fixed.*

We now have all the preparations to bound the H^2 norms of $(\widetilde{\phi},\widetilde{F})$ on $\mathcal{B}_+$.

We take $(f,G)=(\phi,F)$ in (6.10). The first term on the righthand of (6.10) is bounded by the initial datum. Therefore, by taking $(\mathbf{k})=(0)$ in (6.18), the estimate (6.10) gives

$$\mathcal{E}(\widetilde{\phi},\widetilde{\mathring{F}})(\mathcal{B}_+)\lesssim 1.$$

On the other hand, we know that $\widetilde{F}=\widetilde{\mathring{F}}+\widetilde{F_{q_0}}$ and we have already shown that $\mathcal{E}(\widetilde{F}[q_0])\lesssim\mathcal{E}_{\text{initial}}$. Hence, we have

$$\mathcal{E}(\widetilde{\phi},\widetilde{F})(\mathcal{B}_+)\lesssim 1. \tag{6.19}$$

We now assume $|\mathbf{k}|=1$ and we take $(f,G)=(\widehat{D}_Z\phi,\mathcal{L}_ZF)$ where Z corresponds to the index $(\mathbf{k})$. If $Z=\Omega_{ij}$, the commutator formula (6.2) has no error term for Ω_{ij}'s. We then can repeat the above procedure. Therefore, we have

$$\mathcal{E}(\widetilde{D}_{\widetilde{\Omega_{ij}}}\widetilde{\phi},\mathcal{L}_{\widetilde{\Omega_{ij}}}\widetilde{F})(\mathcal{B}_+)\lesssim 1,$$

where $\widetilde{\Omega_{ij}}=\widetilde{x}_i\partial_{\widetilde{x_j}}-\widetilde{x}_j\partial_{\widetilde{x}_i}$.

If $Z = T$, according to (6.2), we have $\widetilde{D}_{\Phi_* T}\widetilde{\phi} = \widetilde{\widehat{D}_T\phi} - \big(\widetilde{t} - \frac{R_*+1}{2R_*+1}\big)\widetilde{\phi}$. Similar to the previous case, $\mathcal{E}(\widetilde{\widehat{D}_T\phi}, \mathcal{L}_{\Phi_* T}\widetilde{F} = \widetilde{\mathcal{L}_T \mathring{F}})(\mathcal{B}_+)$ are bounded by (6.10) and (6.18). Since $\widetilde{t}$ and its derivatives on $\mathcal{B}_+$ is bounded, in view of the L^∞ estimates on ϕ (which implies the $\widetilde{\phi}$ is bounded in L^2) and (6.19), the energy contributed by $\big(\widetilde{t} - \frac{R_*+1}{2R_*+1}\big)\widetilde{\phi}$ is also bounded. Thus, we conclude that

$$\mathcal{E}(\widetilde{D}_{\widetilde{v}^2\widetilde{L}+\widetilde{u}^2\widetilde{\underline{L}}}\widetilde{\phi}, \mathcal{L}_{\widetilde{v}^2\widetilde{L}+\widetilde{u}^2\widetilde{\underline{L}}}\widetilde{F})(\mathcal{B}_+) \lesssim 1. \tag{6.20}$$

If $Z = K$, according to(6.2), we have $\widetilde{D}_{\Phi_* K}\widetilde{\phi} = \widetilde{\widehat{D}_K\phi} - (R_*+1)^2\big(\widetilde{t} - \frac{R_*^2}{(R_*+1)(2R_*+1)}\big)\widetilde{\phi}$. Similarly, since $\widetilde{t}$ and its derivatives on $\mathcal{B}_+$ are bounded, we can argue exactly in the same manner that

$$\mathcal{E}(\widetilde{D}_{\partial_{\widetilde{t}}}\widetilde{\phi}, \mathcal{L}_{\partial_{\widetilde{t}}}\widetilde{F})(\mathcal{B}_+) \lesssim 1. \tag{6.21}$$

On the other hand, on $\mathcal{B}_+$, both $\widetilde{r}$ and its inverse are bounded (as well as their derivatives). We also have

$$\widetilde{v}^2\widetilde{L} + \widetilde{u}^2\widetilde{\underline{L}} = \frac{1}{2}\Big(\widetilde{r}^2 + \big(\frac{R_*+1}{2R_*+1}\big)\Big)\partial_{\widetilde{t}} + \frac{R_*+1}{2R_*+1}\widetilde{r}\partial_{\widetilde{r}}.$$

Therefore, (6.20) and (6.21) together with all the previous estimates imply that

$$\mathcal{E}(\widetilde{\phi}, \widetilde{F})(\mathcal{B}_+) + \mathcal{E}(\widetilde{D}\widetilde{\phi}, \nabla\widetilde{F})(\mathcal{B}_+) \lesssim 1. \tag{6.22}$$

We now assume $|\mathbf{k}| = 2$ and we take $(f, G) = (\phi^{(\mathbf{2})}, \mathcal{L}_Z^{(\mathbf{2})}F)$. Since (6.2) has no error term for Ω_{ij}'s, it is immediate to see that

$$\mathcal{E}(\widetilde{D}_{\widetilde{\Omega_{ij}}}\widetilde{D}_{\widetilde{\Omega'_{ij}}}\widetilde{\phi}, \mathcal{L}_{\widetilde{\Omega_{ij}}}\mathcal{L}_{\widetilde{\Omega'_{ij}}}\widetilde{F})(\mathcal{B}_+) \lesssim 1.$$

We now consider the case $(f, G) = (\widehat{D}_T\widehat{D}_T\phi, \mathcal{L}_T\mathcal{L}_T F)$. On $\widetilde{t} = 0$ (or $\mathcal{B}_+$), we have

$$\widetilde{D}_{\Phi_* T}\widetilde{D}_{\Phi_* T}\widetilde{\phi} = \widetilde{\widehat{D}_T\widehat{D}_T\phi} + \frac{2R_*+2}{2R_*+1}\widetilde{D}_{\Phi_* T}\widetilde{\phi} + \Big(2\big(\frac{R_*+1}{2R_*+1}\big)^2 + r^2\Big)\widetilde{\phi}.$$

We can bound all the terms on the righthand side and we obtain

$$\mathcal{E}(\widetilde{D}_{\widetilde{v}^2\widetilde{L}+\widetilde{u}^2\widetilde{\underline{L}}}\widetilde{D}_{\widetilde{v}^2\widetilde{L}+\widetilde{u}^2\widetilde{\underline{L}}}\widetilde{\phi}, \mathcal{L}_{\widetilde{v}^2\widetilde{L}+\widetilde{u}^2\widetilde{\underline{L}}}\mathcal{L}_{\widetilde{v}^2\widetilde{L}+\widetilde{u}^2\widetilde{\underline{L}}}\widetilde{F})(\mathcal{B}_+) \lesssim 1. \tag{6.23}$$

We now consider the case $(f, G) = (\widehat{D}_K\widehat{D}_K\phi, \mathcal{L}_K\mathcal{L}_K F)$. On $\mathcal{B}_+$, we have

$$\frac{1}{4}\widetilde{D}_{\widetilde{T}}\widetilde{D}_{\widetilde{T}}\widetilde{\phi} = \widetilde{\widehat{D}_T\widehat{D}_T\phi} + \frac{R_*^2(R_*+1)}{2R_*+1}\widetilde{D}_{\widetilde{T}}\widetilde{\phi} + \Big(\frac{1}{2}(R_*+1)^2 + \frac{(R_*+1)^2R_*^4}{(2R_*+1)^2}\Big)\widetilde{\phi}.$$

This implies

$$\mathcal{E}(\widetilde{D}_{\partial_{\widetilde{t}}}\widetilde{D}_{\partial_{\widetilde{t}}}\widetilde{\phi}, \mathcal{L}_{\partial_{\widetilde{t}}}\mathcal{L}_{\partial_{\widetilde{t}}}\widetilde{F})(\mathcal{B}_+) \lesssim 1. \tag{6.24}$$

Similar to (6.22), by combining (6.23), (6.24) and all the previous estimates, we finally obtain that

$$\mathcal{E}(\widetilde{\phi}, \widetilde{F})(\mathcal{B}_+) + \mathcal{E}(\widetilde{D}\widetilde{\phi}, \nabla\widetilde{F})(\mathcal{B}_+) + \mathcal{E}(\widetilde{D}^2\widetilde{\phi}, \nabla^2\widetilde{F})(\mathcal{B}_+) \lesssim 1. \tag{6.25}$$

Finally, to obtain the H^2 bound of $(\widetilde{\phi}, \widetilde{F})$ on the entire ball $\mathcal{B}_{\frac{R_*+1}{2R_*+1}}$, it remains to bound the contribution from $\mathcal{B}_{\frac{R_*+1}{2R_*+1}} - \mathcal{B}_+ = \Phi(\Sigma_-)$. On the other hand, according to the theory of Klainerman-Machedon [11], since Σ_- is bounded, the solution is also bounded up to two derivatives in L^2 norms. This implies immediately that the H^2 energy of $(\widetilde{\phi}, \widetilde{F})$ on $\mathcal{B}_{\frac{R_*+1}{2R_*+1}} - \mathcal{B}_+$ is bounded. Finally, we obtain that

$$\mathcal{E}(\widetilde{\phi}, \widetilde{F})(\mathcal{B}_{\frac{R_*+1}{2R_*+1}}) + \mathcal{E}(\widetilde{D}\widetilde{\phi}, \nabla\widetilde{F})(\mathcal{B}_{\frac{R_*+1}{2R_*+1}}) + \mathcal{E}(\widetilde{D}^2\widetilde{\phi}, \nabla^2\widetilde{F})(\mathcal{B}_{\frac{R_*+1}{2R_*+1}}) \lesssim 1. \tag{6.26}$$

Once again, by the Main Theorem proved in Klainerman-Machedon [11], we have uniform H^2 control of $\widetilde{\phi}$ and $\widetilde{F}$. According to Sobolev inequality on $\mathcal{B}_{\frac{R_*+1}{2R_*+1}}$, we conclude that there exists a constant C, so that we have the pointwise bound

$$|\widetilde{\phi}| + |\widetilde{D}\widetilde{\phi}| + |\widetilde{F}| \lesssim 1.$$

Finally, we use the formulae in (6.1) and this provides the peeling estimates for the null components of $D\phi$ and F in the interior region. Together with the pointwise estimates derived in the exterior region, this finishes the proof of the main theorem.

Appendix A. Tool kit

We collect some frequently used inequalities and calculations in this appendix.

A.1. **Gronwall type inequalities.**

Lemma A.1 (A variant of Gronwall's inequality). *Let $f(t) \in C^0\big([0,T]\big)$ and C_1 and C_2 are two positive constants. If we have*

$$f(t) \leq C_1 + C_2 \int_t^T s^{-1} f(s) ds,$$

for all $t \leq T$, then we have

$$f(t) \leq C_1 + \frac{C_1}{C_2}\Big(\big(\frac{T}{t}\big)^{C_2} - 1\Big) \quad and \quad \int_t^T s^{-1} f(s) ds \leq \frac{C_1}{C_2}\Big(\big(\frac{T}{t}\big)^{C_2} - 1\Big).$$

Lemma A.2. *Let $f(t) \in C^1\big([1,+\infty)\big)$ be a positive function. If for all $r_1 \geq 1$, there are positive constants C and p so that*

$$\int_{r_1}^{\infty} f(s) ds \leq C r_1^{-p}.$$

Then, for all $q < p$, we have

$$\int_{r_1}^{\infty} s^q f(s) ds \leq \frac{Cp}{p-q} r_1^{q-p}.$$

The proofs are straightforward and we omit the details.

A.2. **Sobolev inequalities.** We first recall the standard Sobolev inequalities on spheres.

Lemma A.3 (Sobolev inequalities on spheres). *Let Ξ be a (p_0, q_0)-tensor field ($p_0 + q_0 \leq 3$ in the current work) on $\mathbf{S}_r$ (the standard sphere of radius r). Then, for $p > 2$, we have*

$$\begin{aligned}
\|\Xi\|_{L^\infty(\mathbf{S}_r)} &\lesssim_p r^{-\frac{2}{p}} \big(\|\Xi\|_{L^p(\mathbf{S}_r)} + r\|\nabla\!\!\!/\,\Xi\|_{L^p(\mathbf{S}_r)}\big),\\
\|\Xi\|_{L^4(\mathbf{S}_r)} &\lesssim r^{-\frac{1}{2}} \big(\|\Xi\|_{L^2(\mathbf{S}_r)} + r\|\nabla\!\!\!/\,\Xi\|_{L^2(\mathbf{S}_r)}\big),
\end{aligned}$$

where $\nabla\!\!\!/$ is the covariant derivative on $\mathbf{S}_r$.

Remark A.4. *We use $\mathcal{L}_\Omega \Xi$ to denote all possible Lie derivatives with respect to the rotation vector fields Ω_{12}, Ω_{23} and Ω_{31}. It is straightforward to check that*

$$|\mathcal{L}_\Omega \Xi|^2 \approx |\Xi|^2 + r^2 |\nabla\!\!\!/\,\Xi|^2.$$

Thus, the above Sobolev inequalities can also be stated as

$$\begin{aligned}
\|\Xi\|_{L^\infty(\mathbf{S}_r)} &\lesssim_p r^{-\frac{2}{p}} \big(\|\Xi\|_{L^p(\mathbf{S}_r)} + \|\mathcal{L}_\Omega\Xi\|_{L^p(\mathbf{S}_r)}\big),\\
\|\Xi\|_{L^4(\mathbf{S}_r)} &\lesssim r^{-\frac{1}{2}} \big(\|\Xi\|_{L^2(\mathbf{S}_r)} + \|\mathcal{L}_\Omega\Xi\|_{L^2(\mathbf{S}_r)}\big).
\end{aligned} \tag{A.1}$$

Lemma A.5 (A Sobolev inequality on parameterized hypersurfaces). *Let Ξ be a tensor field on a parameterized hypersurface with parameters $(s,\vartheta) \in [a,b] \times \mathbf{S}^2$ (equipped with the product measure). Then, we have*

$$\sup_{s\in[a,b]} \|\Xi(s,\vartheta)\|_{L^4(\mathbf{S}_\vartheta)} \lesssim \|\Xi(s,\vartheta)\|_{L^2(\mathbf{S}_\vartheta)} + \|\partial_s\Xi(s,\vartheta)\|_{L^2(\mathbf{S}_\vartheta)} + \|\partial_\vartheta\Xi(s,\vartheta)\|_{L^2(\mathbf{S}_\vartheta)}$$

Proof. It suffices to prove the inequality for any fixed $s_0 \in [a,b]$. According to Sobolev inequality on $\mathbf{S}^2$, we have

$$\|\Xi(s_0,\vartheta)\|_{L^4(\mathbf{S}^2)} \lesssim \|\Xi(s_0,\vartheta)\|_{W^{\frac{1}{2},2}(\mathbf{S}^2)}.$$

Regarding $s = s_0$ as a codimension 1 hypersurface in $[a,b] \times \mathbf{S}^2$, the standard trace theorem yields

$$\|\Xi(s_0,\vartheta)\|_{W^{\frac{1}{2},2}(\mathbf{S}^2)} \lesssim \|\Xi(s,\vartheta)\|_{W^{1,2}([a,b]\times\mathbf{S}^2)}.$$

This gives the proof of the inequality. □

The following corollary of the above inequality will be extremely useful when one derives pointwise decay:

Corollary A.6. *For a smooth tensor field Ξ on the incoming null hypersurface $\mathcal{H}_{r_2}^{r_1}$, $\underline{\mathcal{H}}_{r_1}^{r_2}$ or $\mathcal{B}_{r_1}$, we have*

$$\begin{aligned}
r_2 \|\Xi\|^2_{L^4(\mathcal{S}_{r_1}^{r_2})} &\lesssim \int_{\underline{\mathcal{H}}_{r_2}^{r_1}} |\Xi|^2 + \int_{\underline{\mathcal{H}}_{r_2}^{r_1}} |\mathcal{L}_{\underline{L}}\Xi|^2 + \int_{\underline{\mathcal{H}}_{r_2}^{r_1}} |\mathcal{L}_\Omega\Xi|^2,\\
r_2 \|\Xi\|^2_{L^4(\mathcal{S}_{r_1}^{r_2})} &\lesssim \int_{\mathcal{H}_{r_2}^{r_1}} |\Xi|^2 + \int_{\mathcal{H}_{r_2}^{r_1}} |\mathcal{L}_L\Xi|^2 + \int_{\mathcal{H}_{r_2}^{r_1}} |\mathcal{L}_\Omega\Xi|^2,\\
r_1 \|\Xi\|^2_{L^4(\mathcal{S}_{r_1}^{r_1})} &\lesssim \int_{\mathcal{B}_{r_1}} |\Xi|^2 + \int_{\mathcal{B}_{r_1}} |\mathcal{L}_{\partial_r}\Xi|^2 + \int_{\mathcal{B}_{r_1}} |\mathcal{L}_\Omega\Xi|^2.
\end{aligned} \tag{A.2}$$

For the a scalar field f (as a section of a line bundle $\mathbf{L}$) and a given connection A, we have

$$
\begin{aligned}
r_2\|f\|^2_{L^4(\mathcal{S}^{r_2}_{r_1})} &\lesssim \int_{\underline{\mathcal{H}}^{r_1}_{r_2}} |f|^2 + \int_{\underline{\mathcal{H}}^{r_1}_{r_2}} |D_{\underline{L}} f|^2 + \int_{\underline{\mathcal{H}}^{r_1}_{r_2}} |D_\Omega f|^2,\\
r_2\|f\|^2_{L^4(\mathcal{S}^{r_2}_{r_1})} &\lesssim \int_{\mathcal{H}^{r_1}_{r_2}} |f|^2 + \int_{\mathcal{H}^{r_1}_{r_2}} |D_L f|^2 + \int_{\mathcal{H}^{r_1}_{r_2}} |D_\Omega f|^2,\\
r_1\|f\|^2_{L^4(\mathcal{S}^{r_1}_{r_1})} &\lesssim \int_{\mathcal{B}_{r_1}} |f|^2 + \int_{\mathcal{B}_{r_1}} |D_{\partial_r} f|^2 + \int_{\mathcal{B}_{r_1}} |D_\Omega f|^2.
\end{aligned}
\tag{A.3}
$$

Proof. We will apply the previous lemma by taking $a = \frac{-r_2}{2}$, $b = -\frac{r_1}{2}$ and using $s = u$ and ϑ to parameterize $\underline{\mathcal{H}}^{r_1}_{r_2}$. Since we have

$$
\int_{\underline{\mathcal{H}}^{r_1}_{r_2}} |\mathcal{L}_L \Xi|^2 = \int_b^a |\partial_u (r\Xi)|^2 du d\vartheta - \int_b^a |\Xi|^2 du d\vartheta \quad \text{and} \quad \int_{\underline{\mathcal{H}}^{r_1}_{r_2}} |\mathcal{L}_\Omega \Xi|^2 = \int_b^a |\partial_\vartheta (r\Xi)|^2 du d\vartheta,
$$

we obtain that (notice that $r \geq 1$)

$$
\|r\Xi\|_{W^{1,2}([a,b]\times \mathbf{S}^2)} \lesssim \int_{\underline{\mathcal{H}}^{r_1}_{r_2}} |\Xi|^2 + \int_{\underline{\mathcal{H}}^{r_1}_{r_2}} |\mathcal{L}_L \Xi|^2 + \int_{\underline{\mathcal{H}}^{r_1}_{r_2}} |\mathcal{L}_\Omega \Xi|^2.
$$

According to the previous lemma, we obtain that

$$
\sup_{u\in[a,b]} \Big(\int_{\mathbf{S}^2} r^4 |\Xi(s,\vartheta)|^4 d\vartheta\Big)^{\frac12} \lesssim \int_{\underline{\mathcal{H}}^{r_1}_{r_2}} |\Xi|^2 + \int_{\underline{\mathcal{H}}^{r_1}_{r_2}} |\mathcal{L}_L \Xi|^2 + \int_{\underline{\mathcal{H}}^{r_1}_{r_2}} |\mathcal{L}_\Omega \Xi|^2.
$$

By setting $u = b$, this completes the proof the corollary.

For the second inequality, we take $\Xi_\varepsilon = \sqrt{|f|^2 + \varepsilon^2}$ and apply the first inequality. Since $|\underline{L}(\Xi_\varepsilon)| = \Big|\frac{(D_{\underline{L}} f, f)}{\sqrt{|f|^2+\varepsilon^2}}\Big|$, we have $|\underline{L}(\Xi_\varepsilon)| \leq |D_{\underline{L}} f|$ uniformly in ε. Similarly, $|\Omega(\Xi_\varepsilon)| \leq |D_\Omega f|$. The first inequality then gives

$$
r_2 \|\sqrt{|f|^2+\varepsilon^2}\|^2_{L^4(\mathcal{S}^{r_2}_{r_1})} \lesssim \int_{\underline{\mathcal{H}}^{r_1}_{r_2}} \big(|f|^2 + \varepsilon^2\big) + \int_{\underline{\mathcal{H}}^{r_1}_{r_2}} |D_{\underline{L}} f|^2 + \int_{\underline{\mathcal{H}}^{r_1}_{r_2}} |D_\Omega f|^2. \tag{A.4}
$$

By passing to the limit $\varepsilon \to 0$, we obtain the second inequality. □

The following lemma is also useful to deal with lower order terms.

Lemma A.7. *For a scalar field f on an outcoming null hypersurface $\mathcal{H}_{r_1}$, for all $r_2 \geq r_1$ or on an incoming null hypersurface $\underline{\mathcal{H}}^{r_1}_{r_2}$, we have*

$$
\begin{aligned}
\|f\|^2_{L^2(\mathcal{S}^{r_2}_{r_1})} &\lesssim \|f\|^2_{L^2(\mathcal{S}^{r_1}_{r_1})} + \frac{1}{r_1}\int_{\mathcal{H}_{r_1}} |D_L(rf)|^2,\\
\|f\|^2_{L^2(\mathcal{S}^{r_2}_{r_1})} &\lesssim \|f\|^2_{L^2(\mathcal{S}^{r_2}_{r_2})} + \frac{1}{r_1}\int_{\underline{\mathcal{H}}^{r_1}_{r_2}} |D_{\underline{L}}(rf)|^2.
\end{aligned}
\tag{A.5}
$$

Proof. Indeed, we have

$$
\begin{aligned}
\|f\|^2_{L^2(\mathcal{S}^{r_2}_{r_1})} - \|f\|^2_{L^2(\mathcal{S}^{r_1}_{r_1})} &= \int_{\frac{r_2}{2}}^{\frac{r_1}{2}} \int_{\mathbf{S}^2} L(|rf|^2) d\vartheta dv \leq 2\int_{\frac{r_2}{2}}^{\frac{r_1}{2}} \int_{\mathbf{S}^2} |D_L(rf)||rf| d\vartheta dv\\
&\leq 2\int_{\frac{r_2}{2}}^{\frac{r_1}{2}} \int_{\mathbf{S}^2} |D_L(rf)|^2 r^2 d\vartheta dv + 2\int_{\frac{r_2}{2}}^{\frac{r_1}{2}} \frac{1}{r^2}\int_{\mathbf{S}^2} |rf|^2 d\vartheta dv.
\end{aligned}
$$

The Gronwall's inequality then completes the proof. □

A.3. **Geometric Calculations.** We frequently compare Lie derivative $\mathcal{L}_{\underline{L}}$ and covariant derivative $\nabla\!\!\!\!/$. Indeed, we have

$$
\nabla\!\!\!\!/_{\underline{L}} X_A - \mathcal{L}_{\underline{L}} X_A = \frac{2}{r} X_A, \quad \nabla\!\!\!\!/_L X_A - \mathcal{L}_L X_A = -\frac{2}{r} X_A \tag{A.6}
$$

and for vector fields from $\mathcal{Z}$, we have

$$
\begin{aligned}
&\mathcal{L}_T L = 0,\ \mathcal{L}_T \underline{L} = 0,\ \mathcal{L}_T e_A = 0,\ \ \mathcal{L}_{\Omega_{ij}} L = 0,\ \mathcal{L}_{\Omega_{ij}} \underline{L} = 0,\ \mathcal{L}_{\Omega_{ij}} e_A \perp e_A\ \mathcal{L}_{\Omega_{ij}} e_A \perp L,\ \mathcal{L}_{\Omega_{ij}} e_A \perp \underline{L},\\
&\mathcal{L}_K L = -2vL,\ \mathcal{L}_K \underline{L} = -2u\underline{L},\ \mathcal{L}_K e_A = -te_A,\ \mathcal{L}_S L = -L,\ \mathcal{L}_S \underline{L} = -\underline{L},\ \mathcal{L}_K e_A = -\frac{t}{r} e_A.
\end{aligned}
$$

For a tensor field Ξ, we frequently take Lie derivatives along Z or decompose it in null frames. The next lemma record the commutators of these two operations. The proof is a straightforward computation.

For a 2-form G, it is straightforward to check that

$$
\begin{aligned}
&\mathcal{L}_Z \alpha(G)_A = \alpha(\mathcal{L}_Z G)_A + G(\mathcal{L}_Z L, e_A),\ \ \mathcal{L}_Z \underline{\alpha}(G)_A = \underline{\alpha}(\mathcal{L}_Z G)_A + G(\mathcal{L}_Z \underline{L}, e_A),\\
&\mathcal{L}_Z \rho(G) = \rho(\mathcal{L}_Z G) + \frac12 G(\mathcal{L}_Z \underline{L}, L) + \frac12 G(\underline{L}, \mathcal{L}_Z L),\ \ \mathcal{L}_Z \sigma(G) = \sigma(\mathcal{L}_Z G) + G(\mathcal{L}_Z e_1, e_2) + G(e_1, \mathcal{L}_Z e_2).
\end{aligned}
$$

Based on these formulas, we have

Lemma A.8. *For $Z \in \mathcal{Z}$, if $Z \notin \{S, K\}$, we have*

$$\mathcal{L}_Z\alpha(G)_A = \alpha(\mathcal{L}_Z G)_A, \quad \mathcal{L}_Z\underline{\alpha}(G)_A = \underline{\alpha}(\mathcal{L}_Z G)_A, \quad \mathcal{L}_Z\rho(G) = \rho(\mathcal{L}_Z G), \quad \mathcal{L}_Z\sigma(G) = \sigma(\mathcal{L}_Z G).$$

Otherwise, we have

$$\mathcal{L}_S\alpha(G)_A = \alpha(\mathcal{L}_S G)_A - \alpha(G)_A, \quad \mathcal{L}_S\underline{\alpha}(G)_A = \underline{\alpha}(\mathcal{L}_S G)_A - \underline{\alpha}(G)_A,$$
$$\mathcal{L}_S\rho(G) = \rho(\mathcal{L}_S G) - 2\rho(G), \quad \mathcal{L}_S\sigma(G) = \sigma(\mathcal{L}_S G) - 2\frac{t}{r}\sigma(G).$$

and

$$\mathcal{L}_K\alpha(G)_A = \alpha(\mathcal{L}_K G)_A - 2v\alpha(G)_A, \quad \mathcal{L}_K\underline{\alpha}(G)_A = \underline{\alpha}(\mathcal{L}_K G)_A - 2u\underline{\alpha}(G)_A,$$
$$\mathcal{L}_K\rho(G) = \rho(\mathcal{L}_K G) - 2t\rho(G), \quad \mathcal{L}_K\sigma(G) = \sigma(\mathcal{L}_K G) - 2t\sigma(G).$$

Finally, we collect some calculation on integrated quantities on hypersurfaces. For $\gamma \neq 3$, we define

$$\mathbf{E}^{/}_\gamma = \int_{\mathcal{H}_{r_1}^{r_2}} \frac{1}{r^\gamma}|f|^2, \quad \mathbf{E}^{\setminus}_\gamma \int_{\underline{\mathcal{H}}_{r_1}^{r_2}} \frac{1}{r^\gamma}|f|^2, \quad \mathbf{E}^{-}_\gamma = \int_{\mathcal{B}_{r_1}^{r_2}} \frac{1}{r^\gamma}|f|^2. \tag{A.7}$$

We have

$$\mathbf{E}^{/}_\gamma = \underbrace{\frac{1}{3-\gamma}(\frac{r_1+r_2}{2})^{1-\gamma}\int_{\mathcal{S}_{r_1}^{r_2}}|f|^2 - \frac{1}{3-\gamma}r_1^{1-\gamma}\int_{\mathcal{S}_{r_1}^{r_1}}|f|^2}_{\mathbf{E}^{/}_{\gamma,0}} - \frac{2}{3-\gamma}\int_{\mathcal{H}_{r_1}^{r_2}} r^{1-\gamma}\Re(\overline{D_L f}\cdot f). \tag{A.8}$$

Similarly, we have

$$\mathbf{E}^{\setminus}_\gamma = \underbrace{\frac{1}{3-\gamma}r_2^{1-\gamma}\int_{\mathcal{S}_{r_2}^{r_2}}|f|^2 - \frac{1}{3-\gamma}(\frac{r_1+r_2}{2})^{1-\gamma}\int_{\mathcal{S}_{r_1}^{r_2}}|f|^2}_{\mathbf{E}^{\setminus}_{\gamma,0}} + \frac{2}{3-\gamma}\int_{\underline{\mathcal{H}}_{r_1}^{r_2}} r^{1-\gamma}\Re(\overline{D_{\underline{L}} f}\cdot f), \tag{A.9}$$

and

$$\mathbf{E}^{-}_\gamma = \underbrace{\frac{1}{3-\gamma}r_2^{1-\gamma}\int_{\mathcal{S}_{r_2}^{r_2}}|f|^2 - \frac{1}{3-\gamma}r_1^{1-\gamma}\int_{\mathcal{S}_{r_1}^{r_1}}|f|^2}_{\mathbf{E}^{\setminus}_{-,0}} + \frac{2}{3-\gamma}\int_{\mathcal{B}_{r_1}^{r_2}} r^{1-\gamma}\Re(\overline{D_{\partial_r} f}\cdot f), \tag{A.10}$$

As an application, we prove the following Hardy type inequality:

Lemma A.9. *For $\gamma > 3$, we have*

$$\int_{\mathcal{H}_{r_1}^{r_2}} r^{-\gamma}|f|^2 + \frac{1}{r_2^{\gamma-1}}\int_{\mathcal{S}_{r_1}^{r_2}}|f|^2 \lesssim_\gamma r_1^{-\gamma+1}\int_{\mathcal{S}_{r_1}^{r_1}}|f|^2 + r_1^{-\gamma+2}\int_{\mathcal{H}_{r_1}^{r_2}}\big|D_L f\big|^2. \tag{A.11}$$

Proof. In view of (A.8), by discarding the first term on the righthand side, we have

$$\begin{aligned}\int_{\mathcal{H}_{r_1}^{r_2}} \frac{1}{r^\gamma}|f|^2 + \frac{1}{r_2^{\gamma-1}}\int_{\mathcal{S}_{r_1}^{r_2}}|f|^2 &\lesssim_\gamma r_1^{-(\gamma-1)}\int_{\mathcal{S}_{r_1}^{r_1}}|f|^2 + 2\int_{\mathcal{H}_{r_1}^{r_2}} r^{-(\gamma-1)}\big|D_L(rf)\big|\big|f\big| \\ &\leq C_\gamma r_1^{-3}\int_{\mathcal{S}_{r_1}^{r_1}}|f|^2 + C_\gamma\int_{\mathcal{H}_{r_1}^{r_2}} r^{-\gamma+2}\big|D_L f\big|^2 + \frac{1}{2}\int_{\mathcal{H}_{r_1}^{r_2}}\frac{1}{r^\gamma}|f|^2\end{aligned}$$

Thus,

$$\int_{\mathcal{H}_{r_1}^{r_2}} r^{-\gamma}|f|^2 + \frac{1}{r_2^{\gamma-1}}\int_{\mathcal{S}_{r_1}^{r_2}}|f|^2 \lesssim r_1^{-\gamma+1}\int_{\mathcal{S}_{r_1}^{r_1}}|f|^2 + \int_{\mathcal{H}_{r_1}^{r_2}} r^{-\gamma+2}\big|D_L f\big|^2.$$

This completes the proof. □

References

[1] L. Bieri, S. Miao, and S. Shahshahani. Asymptotic properties of solutions of the Maxwell Klein Gordon equation with small data. *Comm. Anal. Geom.*, 25(1):25–96, 2017.

[2] Y. Choquet-Bruhat and D. Christodoulou. Existence of global solutions of the Yang-Mills, Higgs and spinor field equations in $3+1$ dimensions. *Ann. Sci. École Norm. Sup. (4)*, 14(4):481–506 (1982), 1981.

[3] Y. Choquet-Bruhat and D. Christodoulou. Existence of global solutions of the Yang-Mills, Higgs and spinor field equations in $3+1$ dimensions. *Ann. Sci. École Norm. Sup. (4)*, 14(4):481–506 (1982), 1981.

[4] D. Christodoulou. *Mathematical problems of general relativity. I.* Zurich Lectures in Advanced Mathematics. European Mathematical Society (EMS), Zürich, 2008.

[5] D. Christodoulou and S. Klainerman. Asymptotic properties of linear field equations in Minkowski space. *Comm. Pure Appl. Math.*, 43(2):137–199, 1990.

[6] M. Dafermos and I. Rodnianski. A new physical-space approach to decay for the wave equation with applications to black hole spacetimes. In *XVIth International Congress on Mathematical Physics*, pages 421–432. World Sci. Publ., Hackensack, NJ, 2010.

[7] D. Eardley and V. Moncrief. The global existence of Yang-Mills-Higgs fields in 4-dimensional Minkowski space. I. Local existence and smoothness properties. *Comm. Math. Phys.*, 83(2):171–191, 1982.
[8] D. Eardley and V. Moncrief. The global existence of Yang-Mills-Higgs fields in 4-dimensional Minkowski space. II. Completion of proof. *Comm. Math. Phys.*, 83(2):193–212, 1982.
[9] V. Georgiev and P. Schirmer. The asymptotic behavior of Yang-Mills fields in the large. *Comm. Math. Phys.*, 148(3):425–444, 1992.
[10] S. Klainerman. Uniform decay estimates and the lorentz invariance of the classical wave equation. *Comm. Pure Appl. Math.*, 38(3):321–332, 1985.
[11] S. Klainerman and M. Machedon. On the Maxwell-Klein-Gordon equation with finite energy. *Duke Math. J.*, 74(1):19–44, 1994.
[12] S. Klainerman and M. Machedon. Finite energy solutions of the Yang-Mills equations in $\mathbb{R}^{3+1}$. *Ann. of Math. (2)*, 142(1):39–119, 1995.
[13] S. Klainerman and F. Nicolò. *The evolution problem in general relativity*, volume 25 of *Progress in Mathematical Physics*. Birkhäuser Boston, Inc., Boston, MA, 2003.
[14] S. Klainerman and F. Nicolò. Peeling properties of asymptotically flat solutions to the Einstein vacuum equations. *Classical Quantum Gravity*, 20(14):3215–3257, 2003.
[15] S. Klainerman, Q. Wang, and S. Yang. Global solution for massive Maxwell-Klein-Gordon equations. 2018. arXiv:1801.10380.
[16] H. Lindblad and J. Sterbenz. Global stability for charged-scalar fields on Minkowski space. *IMRP Int. Math. Res. Pap.*, pages Art. ID 52976, 109, 2006.
[17] J. Luk and S. Oh. Global nonlinear stability of large dispersive solutions to the Einstein equations. preprint.
[18] S. Oh and D. Tataru. Global well-posedness and scattering of the $(4+1)$-dimensional Maxwell-Klein-Gordon equation. *Invent. Math.*, 205(3):781–877, 2016.
[19] D. Petrescu. Time decay of solutions of coupled Maxwell-Klein-Gordon equations. *Comm. Math. Phys.*, 179(1):11–23, 1996.
[20] M. Psarelli. Time decay of Maxwell-Klein-Gordon equations in 4-dimensional Minkowski space. *Comm. Partial Differential Equations*, 24(1-2):273–282, 1999.
[21] W. Shu. Asymptotic properties of the solutions of linear and nonlinear spin field equations in Minkowski space. *Comm. Math. Phys.*, 140(3):449–480, 1991.
[22] W. Shu. Global existence of Maxwell-Higgs fields. In *Nonlinear hyperbolic equations and field theory (Lake Como, 1990)*, volume 253 of *Pitman Res. Notes Math. Ser.*, pages 214–227. Longman Sci. Tech., Harlow, 1992.
[23] S. Yang. Decay of solutions of Maxwell-Klein-Gordon equations with arbitrary Maxwell field. *Anal. PDE*, 9(8):1829–1902, 2016.
[24] S. Yang. On the global behavior of solutions of the Maxwell–Klein–Gordon equations. *Adv. Math.*, 326:490–520, 2018.
[25] S. Yang and P. Yu. On global dynamics of solutions of Yang-Mills equtions. preprint.
[26] S. Yang and P. Yu. On the opitmal decaying solutions of the Maxwell-Klein-Gordon equations. preprint.

Beijing International Center for Mathematical Research, Peking University, Beijing, China
E-mail address: shiwuyang@math.pku.edu.cn

Department of Mathematics and Yau Mathematical Sciences Center, Tsinghua University, Beijing, China
E-mail address: yupin@mail.tsinghua.edu.cn