Solitary Waves for Nonlinear Klein-Gordon-Maxwell and Schrödinger-Maxwell Equations

Teresa D'Aprile* Dimitri Mugnai†

Abstract

In this paper we study the existence of radially symmetric solitary waves for nonlinear Klein-Gordon equations and nonlinear Schrödinger equations coupled with Maxwell equations. The method relies on a variational approach and the solutions are obtained as mountain-pass critical points for the associated energy functional.

1 Introduction

This paper has been motivated by the search of nontrivial solutions for the following nonlinear equations of the Klein-Gordon type:

$$\frac{\partial^2\psi}{\partial t^2} - \Delta\psi + m^2\psi - |\psi|^{p-2}\psi = 0, \quad x \in \mathbb{R}^3, \tag{1.1}$$

or of the Schrödinger type:

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\Delta\psi - |\psi|^{p-2}\psi, \quad x \in \mathbb{R}^3, \tag{1.2}$$

where $\hbar > 0$, $m > 0$, $p > 2$, $\psi : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{C}$.

In recent years many papers have been devoted to find standing waves of (1.1) or (1.2), i.e. solutions of the form

$$\psi(x,t) = e^{i\omega t}u(x), \quad \omega \in \mathbb{R}.$$

With this *Ansatz* the nonlinear Klein-Gordon equation, as well as the nonlinear Schrödinger equation, is reduced to a semilinear elliptic equation and existence theorems have been established whether u is radially symmetric and real (see [8], [9]), or u is non-radially symmetric and complex (see [12], [14]). In this paper we want to investigate the existence of nonlinear Klein-Gordon or Schrödinger fields interacting with an electromagnetic field $\mathbf{E} - \mathbf{H}$; such a problem has been extensively pursued in the case of assigned electromagnetic fields (see [3], [4], [11]). Following the ideas already introduced in [5], [6], [7], [10], [13], [15], we do not assume that the electromagnetic field is assigned. Then we have to study a system of equations whose unknowns are the wave function $\psi = \psi(x,t)$ and the gauge potentials $\mathbf{A}$, Φ,

$$\mathbf{A} : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}^3, \quad \Phi : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}$$

*Dipartimento Interuniversitario di Matematica, via E. Orabona 4, 70125 BARI (Italy), e-mail: daprile@dm.uniba.it

†Dipartimento di Matematica e Informatica, via Vanvitelli 1, 06123 PERUGIA (Italy), e-mail: mugnai@dipmat.unipg.it

which are related to $\mathbf{E}-\mathbf{H}$ by the Maxwell equations

$$\begin{aligned}\mathbf{E} &= -\left(\nabla\Phi+\frac{\partial\mathbf{A}}{\partial t}\right)\\ \mathbf{H} &= \nabla\times\mathbf{A}.\end{aligned}$$

Let us first consider equation (1.1). The Lagrangian density related to (1.1) is given by

$$\mathcal{L}_{KG}=\frac{1}{2}\left[\left|\frac{\partial\psi}{\partial t}\right|^2-|\nabla\psi|^2-m^2|\psi|^2\right]+\frac{1}{p}|\psi|^p.$$

The interaction of ψ with the electromagnetic field is described by the minimal coupling rule, that is the formal substitution

$$\frac{\partial}{\partial t}\longmapsto\frac{\partial}{\partial t}+ie\Phi,\quad \nabla\longmapsto\nabla-ie\mathbf{A},$$

where e is the electric charge. Then the Lagrangian density becomes:

$$\mathcal{L}_{KGM}=\frac{1}{2}\left[\left|\frac{\partial\psi}{\partial t}+ie\psi\Phi\right|^2-|\nabla\psi-ie\mathbf{A}\psi|^2-m^2|\psi|^2\right]+\frac{1}{p}|\psi|^p.$$

If we set

$$\psi(x,t)=u(x,t)e^{iS(x,t)},$$

where u, $S:\mathbb{R}^3\times\mathbb{R}\to\mathbb{R}$, the Lagrangian density takes the form

$$\mathcal{L}_{KGM}=\frac{1}{2}\left\{u_t^2-|\nabla u|^2-\left[|\nabla S-e\mathbf{A}|^2-(S_t+e\Phi)^2+m^2\right]u^2\right\}+\frac{1}{p}|u|^p.$$

Now we consider the Lagrangian density of the electromagnetic field $\mathbf{E}-\mathbf{H}$

$$\mathcal{L}_0=\frac{1}{2}(|\mathbf{E}|^2-|\mathbf{H}|^2)=\frac{1}{2}|\mathbf{A}_t+\nabla\Phi|^2-\frac{1}{2}|\nabla\times\mathbf{A}|^2. \tag{1.3}$$

The total action is given by

$$\mathcal{S}=\int\int\mathcal{L}_{KGM}+\mathcal{L}_0.$$

Making the variation of $\mathcal{S}$ with respect to u, S, Φ and $\mathbf{A}$ respectively, we get

$$u_{tt}-\Delta u+[|\nabla S-e\mathbf{A}|^2-(S_t+e\Phi)^2+m^2]u-|u|^{p-2}u=0, \tag{1.4}$$

$$\frac{\partial}{\partial t}\left[(S_t+e\Phi)u^2\right]-\operatorname{div}[(\nabla S-e\mathbf{A})u^2]=0, \tag{1.5}$$

$$\operatorname{div}(\mathbf{A}_t+\nabla\Phi)=e(S_t+e\Phi)u^2, \tag{1.6}$$

$$\nabla\times(\nabla\times\mathbf{A})+\frac{\partial}{\partial t}(\mathbf{A}_t+\nabla\Phi)=e(\nabla S-e\mathbf{A})u^2. \tag{1.7}$$

We are interested in finding standing (or *solitary*) waves of (1.4)-(1.7), that is solutions having the form

$$u=u(x),\quad S=\omega t,\quad \mathbf{A}=0,\quad \Phi=\Phi(x),\quad \omega\in\mathbb{R}.$$

Then the equations (1.5) and (1.7) are identically satisfied, while (1.4) and (1.6) become

$$-\Delta u+[m^2-(\omega+e\Phi)^2]u-|u|^{p-2}u=0, \tag{1.8}$$

$$-\Delta\Phi + e^2u^2\Phi = -e\omega u^2. \tag{1.9}$$

In [6] the authors proved the existence of infinitely many symmetric solutions (u_n, Φ_n) of (1.8)-(1.9) under the assumption $4 < p < 6$, by using an equivariant version of the mountain pass theorem (see [1], [2]).

The object of the first part of this paper is to extend this result as follows.

Theorem A. *If $m > \omega > 0$ and $2 < p < 6$ the system $(1.8) - (1.9)$ has infinitely many radially symmetric solutions (u_n, Φ_n), $u_n \not\equiv 0$ and $\Phi_n \not\equiv 0$, with $u_n \in H^1(\mathbb{R}^3)$, $\Phi_n \in L^6(\mathbb{R}^3)$ and $|\nabla\Phi_n| \in L^2(\mathbb{R}^3)$.*

In the second part of the paper we study the Schrödinger equation for a particle in a electromagnetic field.

Consider the Lagrangian associated to (1.2):

$$\mathcal{L}_S = \frac{1}{2}\left[i\hbar\frac{\partial\psi}{\partial t}\overline{\psi} - \frac{\hbar^2}{2m}|\nabla\psi|^2\right] + \frac{1}{p}|\psi|^p.$$

By using the formal substitution

$$\frac{\partial}{\partial t} \longmapsto \frac{\partial}{\partial t} + i\frac{e}{\hbar}\Phi, \quad \nabla \longmapsto \nabla - i\frac{e}{\hbar}\mathbf{A},$$

we obtain

$$\mathcal{L}_{SM} = \frac{1}{2}\left[i\hbar\frac{\partial\psi}{\partial t}\overline{\psi} - e\Phi|\psi|^2 - \frac{\hbar^2}{2m}\left|\nabla\psi - i\frac{e}{\hbar}\mathbf{A}\psi\right|^2\right] + \frac{1}{p}|\psi|^p.$$

Now take

$$\psi(x,t) = u(x,t)e^{iS(x,t)/\hbar}.$$

With this *Ansatz* the Lagrangian $\mathcal{L}_{SM}$ becomes

$$\mathcal{L}_{SM} = \frac{1}{2}\left[i\hbar u u_t - \frac{\hbar^2}{2m}|\nabla u|^2 - \left(S_t + e\Phi + \frac{1}{2m}|\nabla S - e\mathbf{A}|^2\right)u^2\right] + \frac{1}{p}|\psi|^p.$$

Proceeding as in [5], we consider the total action $\mathcal{S} = \int\int[\mathcal{L}_{SM} + \frac{1}{8\pi}(|\mathbf{E}|^2 - |\mathbf{H}|^2)]$ of the system "particle-electromagnetic field". Then the Euler-Lagrange equations associated to the functional $\mathcal{S} = \mathcal{S}(u, S, \Phi, \mathbf{A})$ give rise to the following system of equations:

$$-\frac{\hbar^2}{2m}\Delta u + \left(S_t + e\Phi + \frac{1}{2m}|\nabla S - e\mathbf{A}|^2\right)u - |u|^{p-2}u = 0, \tag{1.10}$$

$$\frac{\partial}{\partial t}u^2 + \frac{1}{m}\mathrm{div}[(\nabla S - e\mathbf{A})u^2] = 0, \tag{1.11}$$

$$eu^2 = -\frac{1}{4\pi}\mathrm{div}\left(\frac{\partial\mathbf{A}}{\partial t} + \nabla\Phi\right), \tag{1.12}$$

$$\frac{e}{2m}(\nabla S - e\mathbf{A})u^2 = \frac{1}{4\pi}\left[\frac{\partial}{\partial t}\left(\frac{\partial\mathbf{A}}{\partial t} + \nabla\Phi\right) + \nabla\times(\nabla\times\mathbf{A})\right]. \tag{1.13}$$

If we look for solitary wave solutions in the electrostatic case, i.e.

$$u = u(x), \quad S = \omega t, \quad \Phi = \Phi(x), \quad \mathbf{A} = 0, \quad \omega \in \mathbb{R},$$

then (1.11) and (1.13) are identically satisfied, while (1.10) and (1.12) become

$$-\frac{\hbar^2}{2m}\Delta u + e\Phi u - |u|^{p-2}u + \omega u = 0, \tag{1.14}$$

$$-\Delta\Phi = 4\pi e u^2. \tag{1.15}$$

The existence of solutions of (1.14)-(1.15) was already studied for $4 < p < 6$: in [5] existence of infinitely many radial solutions was proved, while in [12] existence of a non radially symmetric solution was established. In the second part of the paper we prove the following result.

Theorem B. *Let $\omega > 0$ and $3 < p < 6$. Then the system $(1.14) - (1.15)$ has at least a radially symmetric solution (u, Φ), $u \neq 0$ and $\Phi \neq 0$, with $u \in H^1(\mathbb{R}^3)$, $\Phi \in L^6(\mathbb{R}^3)$ and $|\nabla\Phi| \in L^2(\mathbb{R}^3)$.*

2 Nonlinear Klein-Gordon Equations coupled with Maxwell Equations

In this section we will prove Theorem A. For sake of simplicity, assume $e = 1$ so that (1.8)-(1.9) give rise to the following system in $\mathbb{R}^3$:

$$-\Delta u + [m^2 - (\omega + \Phi)^2]u - |u|^{p-2}u = 0, \tag{2.16}$$

$$-\Delta\Phi + u^2\Phi = -\omega u^2. \tag{2.17}$$

We will always make the assumptions:

a) $m > \omega > 0$,

b) $2 < p < 6$.

We note that $q = 6$ is the critical exponent for the Sobolev embedding $H^1(\mathbb{R}^3) \subset L^q(\mathbb{R}^3)$.

It is clear that (2.16)-(2.17) are the Euler-Lagrange equations of the functional $F : H^1 \times D^{1,2} \to \mathbb{R}$ defined as

$$F(u, \Phi) = \frac{1}{2}\int_{\mathbb{R}^3} \left(|\nabla u|^2 - |\nabla\Phi|^2 + [m^2 - (\omega + \Phi)^2]u^2\right) dx - \frac{1}{p}\int_{\mathbb{R}^3} |u|^p \, dx.$$

Here $H^1 \equiv H^1(\mathbb{R}^3)$ denotes the usual Sobolev space endowed with the norm

$$\|u\|_{H^1} \equiv \left(\int_{\mathbb{R}^3} \left(|\nabla u|^2 + |u|^2\right) dx\right)^{1/2} \tag{2.18}$$

and $D^{1,2} \equiv D^{1,2}(\mathbb{R}^3)$ is the completion of $C_0^\infty(\mathbb{R}^3, \mathbb{R})$ with respect to the norm

$$\|u\|_{D^{1,2}} \equiv \left(\int_{\mathbb{R}^3} |\nabla u|^2 \, dx\right)^{1/2}. \tag{2.19}$$

The following two propositions hold.

Proposition 2.1. *The functional F belongs to $C^1(H^1 \times D^{1,2}, \mathbb{R})$ and its critical points are the solutions of* $(2.16) - (2.17)$.

(For the proof we refer to [6]).

Proposition 2.2. *For every $u \in H^1$, there exists a unique $\Phi = \Phi[u] \in D^{1,2}$ which solves* (2.17). *Furthermore*

(i) $\Phi[u] \leq 0$*;*

(ii) $\Phi[u] \geq -\omega$ *in the set* $\{x \,|\, u(x) \neq 0\}$*;*

(iii) *if* u *is radially symmetric, then* $\Phi[u]$ *is radial too.*

Proof. Fixed $u \in H^1$, consider the following bilinear form on $D^{1,2}$:

$$a(\phi, \psi) = \int_{\mathbb{R}^3} \left(\nabla\psi\nabla\psi + u^2\phi\psi \right) \, dx.$$

Obviously $a(\phi, \phi) \geq \|\phi\|^2_{D^{1,2}}$. Observe that, since $H^1(\mathbb{R}^3) \subset L^3(\mathbb{R}^3)$, then $u^2 \in L^{3/2}(\mathbb{R}^3)$. On the other hand $D^{1,2}$ is continuously embedded in $L^6(\mathbb{R}^3)$, hence, by Hölder's inequality,

$$a(\phi, \psi) \leq \|\phi\|_{D^{1,2}}\|\psi\|_{D^{1,2}} + \|u^2\|_{L^{3/2}}\|\phi\|_{L^6}\|\psi\|_{L^6} \leq (1 + C\|u\|^2_{L^3})\|\phi\|_{D^{1,2}}\|\psi\|_{D^{1,2}}$$

for some positive constant C, given by Sobolev inequality (see [19]). Therefore a defines an inner product, equivalent to the standard inner product in $D^{1,2}$.

Moreover $H^1(\mathbb{R}^3) \subset L^{12/5}(\mathbb{R}^3)$, and then

$$\left| \int_{\mathbb{R}^3} u^2\psi \, dx \right| \leq \|u^2\|_{L^{6/5}}\|\psi\|_{L^6} \leq c\|u\|^2_{L^{12/5}}\|\psi\|_{D^{1,2}}. \tag{2.20}$$

Therefore the linear map $\psi \in D^{1,2} \mapsto \int_{\mathbb{R}^3} u^2\psi \, dx$ is continuous. By Lax-Milgram's Lemma we get the existence of a unique $\Phi \in D^{1,2}$ such that

$$\int_{\mathbb{R}^3} \left(\nabla\Phi\nabla\psi + u^2\Phi\psi \right) \, dx = -\omega \int_{\mathbb{R}^3} u^2\psi dx \quad \forall\, \psi \in D^{1,2},$$

i.e. Φ is the unique solution of (2.17). Furthermore Φ achieves the minimum

$$\inf_{\phi \in D^{1,2}} \int_{\mathbb{R}^3} \left(\frac{1}{2}\Big(|\nabla\phi|^2 + u^2|\phi|^2\Big) + \omega u^2\phi \right) \, dx = \int_{\mathbb{R}^3} \left(\frac{1}{2}\Big(|\nabla\Phi|^2 + u^2|\Phi|^2\Big) + \omega u^2\Phi \right) \, dx.$$

Note that also $-|\Phi|$ achieves such a minimum; then, by uniqueness, $\Phi = -|\Phi| \leq 0$. Now let $O(3)$ denote the group of rotations in $\mathbb{R}^3$. Then for every $g \in O(3)$ and $f : \mathbb{R}^3 \to \mathbb{R}$, set $T_g(f)(x) = f(gx)$. Note that T_g does not change the norms in H^1, $D^{1,2}$ and L^p. In Lemma 4.2 of [6] it was proved that $T_g\Phi[u] = \Phi[T_g u]$. In this way, if u is radial, we get $T_g\Phi[u] = \Phi[u]$.

Finally, following the same idea of [16], fixed $u \in H^1$, if we multiply (2.17) by $(\omega + \Phi[u])^- \equiv -\min\{\omega + \Phi[u], 0\}$, which is an admissible test function, since $\omega > 0$, we get

$$-\int_{\Phi[u] < -\omega} |D\Phi[u]|^2 \, dx - \int_{\Phi[u] < -\omega} (\omega + \Phi[u])^2 u^2 \, dx = 0,$$

so that $\Phi[u] \geq -\omega$ when $u \neq 0$. $\square$

Remark 2.1. The result **(ii)** of Proposition 2.2 can be strengthened in some cases. Indeed, take $\overline{u}$ in $H^1(\mathbb{R}^3) \cap C^\infty$ radially symmetric such that

$$\overline{u} > 0 \text{ in } B(0, R),\ \overline{u} \equiv 0 \text{ in } \mathbb{R}^3 \setminus B(0, R)$$

for some $R > 0$. Then there results

$$-\omega \leq \Phi[\overline{u}](x) \leq 0 \quad \forall\, x \in \mathbb{R}^3.$$

In fact, since $\Phi[\overline{u}]$ solves (2.17), by standard regularity results for elliptic equations, $\overline{u} \in C^\infty$ implies $\Phi[\overline{u}] \in C^\infty$. By Proposition 2.2, $\Phi[\overline{u}]$ is radial; moreover $\Phi[\overline{u}]$ is harmonic outside $B(0,R)$. Since $\Phi[\overline{u}] \in D^{1,2}$, then

$$\Phi[\overline{u}](x) = -\frac{c}{|x|}, \quad |x| \geq R,$$

for some $c > 0$. Setting $\tilde{\Phi}(r) = \Phi[\overline{u}](x)$ for $|x| = r$, it results $\tilde{\Phi}'(R) > 0$ and $\tilde{\Phi}(r) > \tilde{\Phi}(R)$ for every $r > R$. Therefore the minimum of $\Phi[\overline{u}]$ is achieved in $B(0,R)$. Let $\overline{x}$ be a minimum point for $\Phi[\overline{u}]$. Then (2.17) implies

$$\Phi[\overline{u}](\overline{x}) = \frac{-\omega\overline{u}^2(\overline{x}) + \Delta\Phi[\overline{u}](\overline{x})}{\overline{u}^2(\overline{x})} \geq -\omega.$$

In view of proposition 2.2 we can define the map

$$\Phi : H^1 \longrightarrow D^{1,2}$$

which maps each $u \in H^1$ in the unique solution of (2.17). From standard arguments it results $\Phi \in C^1(H^1, D^{1,2})$ and from the very definition of Φ we get

$$F'_\phi(u, \Phi[u]) = 0 \qquad \forall\, u \in H^1. \tag{2.21}$$

Now let us consider the functional

$$J : H^1 \longrightarrow \mathbb{R}, \qquad J(u) := F(u, \Phi[u]).$$

By proposition 2.1, $J \in C^1(H^1, \mathbb{R})$ and, by (2.21),

$$J'(u) = F'_u(u, \Phi[u]).$$

By definition of F, we obtain

$$J(u) = \frac{1}{2}\int_{\mathbb{R}^3} \left(|\nabla u|^2 - |\nabla\Phi[u]|^2 + [m^2 - \omega^2]u^2 - u^2\Phi[u]^2\right) dx - \omega\int_{\mathbb{R}^3} u^2\Phi[u] - \frac{1}{p}\int_{\mathbb{R}^3} |u|^p\, dx.$$

Multiplying both members of (2.17) by $\Phi[u]$ and integrating by parts, we obtain

$$\int_{\mathbb{R}^3} |\nabla\Phi[u]|^2\, dx + \int_{\mathbb{R}^3} |u|^2|\Phi[u]|^2\, dx = -\omega\int_{\mathbb{R}^3} |u|^2\Phi[u]\, dx. \tag{2.22}$$

Using (2.22) the functional J may be written as

$$J(u) = \frac{1}{2}\int_{\mathbb{R}^3} \left(|\nabla u|^2 + [m^2 - \omega^2]u^2 - \omega u^2\Phi[u]\right) dx - \frac{1}{p}\int_{\mathbb{R}^3} |u|^p\, dx. \tag{2.23}$$

The next lemma states a relationship between the critical points of the functionals F and J (the proof can be found in [6]).

Lemma 2.1. *The following statements are equivalent:*

i) $(u, \Phi) \in H^1 \times D^{1,2}$ *is a critical point of* F,

ii) u *is a critical point of* J *and* $\Phi = \Phi[u]$.

Then, in order to get solutions of (2.16)-(2.17), we look for critical points of J.

Theorem 2.1. *Assume hypotheses a) and b). Then the functional J has infinitely many critical points $u_n \in H^1$ having a radial symmetry.*

Proof. Our aim is to apply the equavariant version of the Mountain-Pass Theorem (see [1], Theorem 2.13, or [17], Theorem 9.12). Since J is invariant under the group of translations, there is clearly a lack of compactness. In order to overcome this difficulty, we consider radially symmetric functions. More precisely we introduce the subspace

$$H^1_r = \left\{ u \in H^1 \,\middle|\, u(x) = u(|x|) \right\}.$$

We divide the remaining part of the proof in three steps.

Step 1. Any critical point $u \in H^1_r$ of $J_{\big| H^1_r}$ is also a critical point of J.

The proof can be found in [6].

Step 2. The functional $J_{\big| H^1_r}$ satisfies the Palais-Smale condition, i.e.

$$\textit{any sequence } \{u_n\}_n \subset H^1_r \textit{ such that } J(u_n) \textit{ is bounded and } J'_{|H^1_r}(u_n) \to 0$$

contains a convergent subsequence.

We remark that in [6] the authors proved the Palais-Smale condition for the functional J for $4 < p < 6$.

For the sake of simplicity, from now on we set $\Omega = m^2 - \omega^2 > 0$. Let $\{u_n\}_n \subset H^1_r$ be such that

$$|J(u_n)| \leq M, \quad J'_{|H^1_r}(u_n) \to 0$$

for some constant $M > 0$. Then, using the form of J given in (2.23),

$$\begin{aligned} pJ(u_n) - J'(u_n)u_n &= \left(\frac{p}{2} - 1\right) \int_{\mathbb{R}^3} \left(|\nabla u_n|^2 + \Omega |u_n|^2\right) dx - \omega \left(\frac{p}{2} - 1\right) \int_{\mathbb{R}^3} u_n^2 \Phi[u_n]\, dx \\ &\geq \left(\frac{p}{2} - 1\right) \int_{\mathbb{R}^3} \left(|\nabla u_n|^2 + \Omega |u_n|^2\right) dx \end{aligned} \tag{2.24}$$

by Proposition 2.2. Moreover

$$\left(\frac{p}{2} - 1\right) \int_{\mathbb{R}^3} \left(|\nabla u_n|^2 + \Omega |u_n|^2\right) dx \geq c_1 \|u_n\|^2 \tag{2.25}$$

and by assumption

$$pJ(u_n) - J'(u_n)u_n \leq pM + c_2 \|u_n\| \tag{2.26}$$

for some positive constants c_1 and c_2.

Combining (2.24), (2.25), (2.26), we deduce that $\{u_n\}_n$ is bounded in H^1_r.

On the other hand, using equation (2.17), and proceeding as in (2.20), we get

$$\begin{aligned} \int_{\mathbb{R}^3} \nabla\Phi[u_n]|^2 dx \leq \int_{\mathbb{R}^3} |\nabla\Phi[u_n]|^2 dx + \int_{\mathbb{R}^3} |u_n|^2 |\Phi[u_n]|^2\, dx = -\omega \int_{\mathbb{R}^3} u_n^2 \Phi[u_n] dx \\ \leq c\omega \|u_n\|^2_{L^{12/5}} \|\Phi[u_n]\|_{D^{1,2}}, \end{aligned}$$

which implies that $\{\Phi[u_n]\}_n$ is bounded in $D^{1,2}$.

Then, up to a subsequence,

$$u_n \rightharpoonup u \qquad \text{in } H^1_r$$

$$\Phi[u_n] \rightharpoonup \phi \qquad \text{in } D^{1,2}.$$

If $L : H^1_r \to (H^1_r)'$ is defined as

$$L(u) = -\Delta u + \Omega u,$$

then

$$L(u_n) = \omega u_n \Phi[u_n] + |u_n|^{p-2} u_n + \varepsilon_n,$$

where $\varepsilon_n \to 0$ in $(H^1_r)'$, that is

$$u_n = L^{-1}(\omega u_n \Phi[u_n]) + L^{-1}(|u_n|^{p-2} u_n) + L^{-1}(\varepsilon_n).$$

Now note that $\{u_n \Phi[u_n]\}$ is bounded in $L^{3/2}_r$; in fact, by Hölder's inequality,

$$\|u_n \Phi[u_n]\|_{L^{3/2}_r} \leq \|u_n\|_{L^2_r} \|\Phi[u_n]\|_{L^6_r} \leq c\|u_n\|_{L^2_r} \|\Phi[u_n]\|_{D^{1,2}}.$$

Moreover $\{|u_n|^{p-2}u_n\}$ is bounded in $L^{p'}_r$ (where $\frac{1}{p} + \frac{1}{p'} = 1$). The immersions $H^1_r \hookrightarrow L^3_r$ and $H^1_r \hookrightarrow L^p_r$ are compact (see [8] or [18]) and thus, by duality, $L^{3/2}_r$ and $L^{p'}_r$ are compactly embedded in $(H^1_r)'$. Then by standard arguments $L^{-1}(\omega u_n \Phi[u_n])$ and $L^{-1}(|u_n|^{p-2}u_n)$ strongly converge in H^1_r. Then we conclude

$$u_n \to u \qquad \text{in } H^1_r.$$

Step 3. The functional $J_{|H^1_r}$ satisfies the geometrical hypothesis of the equivariant version of the Mountain Pass Theorem.

First of all we observe that $J(0) = 0$. Moreover by Proposition 2.2 and (2.23)

$$J(u) \geq \frac{1}{2}\int_{\mathbb{R}^3} |\nabla u|^2\, dx + \frac{\Omega}{2}\int_{\mathbb{R}^3} |u|^2\, dx - \frac{1}{p}\int_{\mathbb{R}^3} |u|^p\, dx.$$

The hypothesis $2 < p < 6$ and the continuous embedding $H^1 \subset L^p$ imply that there exists $\rho > 0$ small enough such that

$$\inf_{\|u\|_{H^1} = \rho} J(u) > 0.$$

Since J is even, the thesis of step 3 will follow if we prove that for every finite dimensional subset V of H^1_r it results

$$\lim_{\substack{u \in V, \\ \|u\|_{H^1} \to +\infty}} J(u) = -\infty. \tag{2.27}$$

Let V be an m-dimensional subspace of H^1_r. For every $u \in V$, since by Proposition 2.2 $\Phi[u] \geq -\omega$ where $u \neq 0$, we get

$$J(u) \leq \frac{1}{2}\int_{\mathbb{R}^3} \left(|\nabla u|^2 + \Omega|u|^2 + \omega^2 u^2\right)\, dx - \frac{1}{p}\int_{\mathbb{R}^3} |u|^p \leq c\|u\|^2_{H^1} - \frac{1}{p}\|u\|^p_{L^p}$$

and (2.27) follows, since all norms in V are equivalent. □

Proof of Theorem A. Lemma 2.1 + Theorem 2.1.

Remark 2.2. In view of Remark 2.1 the existence of one nontrivial critical point for the functional J follows from the classical mountain pass theorem: more precisely, taken $\overline{u} \in H^1_r \cap C^\infty$ as in Remark 2.1, since $\|\Phi[\overline{u}]\|_\infty \leq \omega$, there results

$$J(t\overline{u}) \leq \frac{t^2}{2}\int_{\mathbb{R}^3} \left(|\nabla \overline{u}|^2 + \Omega|\overline{u}|^2 + \omega^2\overline{u}^2\right)\, dx - \frac{t^p}{p}\int_{\mathbb{R}^3} |\overline{u}|^p \to -\infty \text{ as } t \to +\infty.$$

3 Nonlinear Schrödinger Equations coupled with Maxwell Equations

For sake of simplicity assume $\hbar = m = e = 1$ in (1.14)-(1.15). Then we are reduced to study the following system in $\mathbb{R}^3$:

$$-\frac{1}{2}\Delta u + \Phi u + \omega u - |u|^{p-2}u = 0, \tag{3.28}$$

$$-\Delta \Phi = 4\pi u^2. \tag{3.29}$$

We will assume

a') $\omega > 0$

b') $3 < p < 6$.

Of course, (3.28)-(3.29) are the Euler-Lagrange equations of the functional $\mathcal{F} : H^1 \times D^{1,2} \to \mathbb{R}$ defined as

$$\mathcal{F}(u,\Phi) = \frac{1}{4}\int_{\mathbb{R}^3} |\nabla u|^2\,dx - \frac{1}{16\pi}\int_{\mathbb{R}^3} |\nabla \Phi|^2\,dx + \frac{1}{2}\int_{\mathbb{R}^3} \left(\Phi u^2 + \omega u^2\right)\,dx - \frac{1}{p}\int_{\mathbb{R}^3} |u|^p\,dx,$$

where H^1 and $D^{1,2}$ are defined as in the previous section.

It is easy to prove the analogous of Proposition 2.1, i.e. that $\mathcal{F} \in C^1(H^1 \times D^{1,2}, \mathbb{R})$ and that its critical points are solutions of (3.28)-(3.29).

Moreover we have the following proposition.

Proposition 3.1. *For every $u \in H^1$ there exists a unique $\Phi = \Phi[u] \in D^{1,2}$ solution of* (3.29)*, such that*

- $\Phi[u] \geq 0$*;*
- $\Phi[tu] = t^2\Phi[u]$ *for every $u \in H^1$ and $t \in \mathbb{R}$.*

Proof. Let us consider the linear map $\phi \in D^{1,2} \mapsto \int_{\mathbb{R}^3} u^2\phi\,dx$, which is continuous by (2.20). By Lax-Milgram's Lemma we get the existence of a unique $\Phi \in D^{1,2}$ such that

$$\int_{\mathbb{R}^3} \nabla\Phi\nabla\phi\,dx = 4\pi\int_{\mathbb{R}^3} u^2\phi\,dx \quad \forall \phi \in D^{1,2},$$

i.e. Φ is the unique solution of (3.29). Furthermore Φ achieves the minimum

$$\inf_{\phi \in D^{1,2}} \left\{\frac{1}{2}\int_{\mathbb{R}^3} |\nabla\phi|^2 - 4\pi\int_{\mathbb{R}^3} u^2\phi\,dx\right\} = \frac{1}{2}\int_{\mathbb{R}^3} |\nabla\Phi|^2 dx - 4\pi\int_{\mathbb{R}^3} u^2\Phi\,dx.$$

Note that also $|\Phi|$ achieves such minimum, then, by uniqueness, $\Phi = |\Phi| \geq 0$.

Finally,

$$-\Delta\Phi[tu] = 4\pi t^2 u^2 = -t^2\Delta\Phi[u] = -\Delta(t^2\Phi[u]),$$

thus, by uniqueness, $\Phi[tu] = t^2\Phi[u]$. □

Proceeding as in the previous section we can define the map

$$\Phi : H^1 \to D^{1,2},$$

which maps each $u \in H^1$ in the unique solution of (3.29). As before, $\Phi \in C^1(H^1, D^{1,2})$ and

$$\mathcal{F}'_\Phi(u, \Phi[u]) = 0 \quad \forall u \in H^1.$$

Now consider the functional $\mathcal{J} : H^1 \to \mathbb{R}$ defined by

$$\mathcal{J}(u) = \mathcal{F}(u, \Phi[u]).$$

$\mathcal{J}$ belongs to $C^1(H^1, \mathbb{R})$ and satisfies $\mathcal{J}'(u) = \mathcal{F}_u(u, \Phi[u])$. Using the definition of $\mathcal{F}$ and equation (3.29), we obtain

$$\mathcal{J}(u) = \frac{1}{4}\int_{\mathbb{R}^3} |\nabla u|^2\, dx + \frac{\omega}{2}\int_{\mathbb{R}^3} |u|^2\, dx + \frac{1}{4}\int_{\mathbb{R}^3} |u|^2 \Phi[u]\, dx - \frac{1}{p}\int_{\mathbb{R}^3} |u|^p\, dx.$$

As before one can prove the following lemma.

Lemma 3.1. *The following statements are equivalent:*

i) $(u, \Phi) \in H^1 \times D^{1,2}$ *is a critical point of* $\mathcal{F}$*,*

ii) u *is a critical point of* $\mathcal{J}$ *and* $\Phi = \Phi[u]$.

Now we are ready to prove the existence result for equations (3.28)-(3.29).

Theorem 3.1. *Assume hypotheses a') and b'). Then the functional* $\mathcal{J}$ *has a nontrivial critical point* $u \in H^1$ *having a radial symmetry.*

Proof. Let H^1_r be defined as in theorem 2.1.

Step 1. Any critical point $u \in H^1_r$ *of* $\mathcal{J}_{|H^1_r}$ *is also a critical point of* $\mathcal{J}$*.*

The proof is as in theorem 2.1.

Step 2. The functional $\mathcal{J}_{|H^1_r}$ *satisfies the Palais-Smale condition.*

Let $\{u_n\}_n \subset H^1_r$ be such that

$$|\mathcal{J}(u_n)| \le M, \quad \mathcal{J}'_{|H^1_r}(u_n) \to 0$$

for some constant $M > 0$. Then

$$p\mathcal{J}(u_n) - \mathcal{J}'(u_n)u_n = \left(\frac{p}{4} - \frac{1}{2}\right)\int_{\mathbb{R}^3} |\nabla u_n|^2 + \left(\frac{p}{2} - 1\right)\omega \int_{\mathbb{R}^3} |u_n|^2\, dx$$

$$+ \left(\frac{p}{4} - \frac{1}{2}\right)\int_{\mathbb{R}^3} |u_n|^2 \Phi[u_n]\, dx \ge \left(\frac{p}{4} - \frac{1}{2}\right)\int_{\mathbb{R}^3} \left(|\nabla u_n|^2 + \omega |u_n|^2\right) dx$$

by Proposition 3.1. Moreover

$$\left(\frac{p}{4} - \frac{1}{2}\right)\int_{\mathbb{R}^3} \left(|\nabla u|^2 + \omega |u|^2\right) dx \ge c_1 \|u_n\|^2$$

and by assumption

$$p\mathcal{J}(u_n) - \mathcal{J}'(u_n)u_n \leq pM + c_2\|u_n\|_{H^1}$$

for some positive constants c_1 and c_2.

We have thus proved that $\{u_n\}_n$ is bounded in H^1_r.

On the other hand, $\|\Phi[u_n]\|^2_{D^{1,2}} = 4\pi \int_{\mathbb{R}^3} u^2\Phi[u_n]\,dx$, and then, using inequality (2.20), we easily deduce that $\{\Phi[u_n]\}_n$ is bounded in $D^{1,2}$.

The remaining part of the proof follows as in Step 2 of Theorem 2.1, after replacing L with $\mathcal{L} : H^1_r \to (H^1_r)'$ defined as $\mathcal{L}(u) = -\frac{1}{2}\Delta u + \omega u$.

Step 3. The functional $\mathcal{J}_{|H^1_r}$ satisfies the three geometrical hypothesis of the mountain pass theorem.

By Proposition 3.1 it results

$$\mathcal{J}(u) \geq \frac{1}{4}\int_{\mathbb{R}^3} |\nabla u|^2\,dx + \frac{\omega}{2}\int_{\mathbb{R}^3} |u|^2\,dx - \frac{1}{p}\int_{\mathbb{R}^3} |u|^p\,dx.$$

Then, using the continuous embedding $H^1 \subset L^p$, we deduce that $\mathcal{J}$ has a strict local minimum in 0.

We introduce the following notation: if $u : \mathbb{R}^3 \to \mathbb{R}$, we set

$$u_{\lambda,\alpha,\beta}(x) = \lambda^\beta u(\lambda^\alpha x), \quad \lambda > 0,\ \alpha,\ \beta \in \mathbb{R}.$$

Now fix $u \in H^1_r$. We want to show that

$$\Phi[u_{\lambda,\alpha,\beta}] = (\Phi[u])_{\lambda,\alpha,2(\beta-\alpha)}. \tag{3.30}$$

In fact

$$\begin{aligned}-\Delta\Phi[u_{\lambda,\alpha,\beta}](x) &= 4\pi u^2_{\lambda,\alpha,\beta}(x) = 4\pi\lambda^{2\beta}u^2(\lambda^\alpha x)\\ &= -\lambda^{2\beta}(\Delta\Phi[u])(\lambda^\alpha x) = -\Delta((\Phi[u])_{\lambda,\alpha,2(\beta-\alpha)})(x).\end{aligned}$$

By uniqueness (see Proposition 3.1), (3.30) follows.

Now take $u \not\equiv 0$ in H^1_r and evaluate

$$\begin{aligned}\mathcal{J}(u_{\lambda,\alpha,\beta}) &= \frac{\lambda^{2\beta-\alpha}}{4}\int_{\mathbb{R}^3} |\nabla u|^2\,dx + \frac{\omega}{2}\lambda^{2\beta-3\alpha}\int_{\mathbb{R}^3} u^2\,dx\\ &+\frac{\lambda^{4\beta-5\alpha}}{4}\int_{\mathbb{R}^3} u^2\Phi[u]\,dx - \frac{\lambda^{\beta p-3\alpha}}{p}\int_{\mathbb{R}^3} |u|^p\,dx.\end{aligned}$$

We want to prove that $\mathcal{J}(u_{\lambda,\alpha,\beta}) < \mathcal{J}(0)$ for some suitable choice of λ, α and β.

For example assume

$$\begin{cases} \beta p - 3\alpha < 0, \\ \beta p - 3\alpha < 2\beta - \alpha, \\ \beta p - 3\alpha < 2\beta - 3\alpha, \\ \beta p - 3\alpha < 4\beta - 5\alpha, \end{cases} \tag{3.31}$$

then it is clear that $\mathcal{J}(u_{\lambda,\alpha,\beta}) \to -\infty$ as $\lambda \to 0$.

So we look for a couple (α, β) which satisfies (3.31). From the third inequality we get $\beta < 0$. Combining the second and the fourth ones, we derive

$$4 - p < \frac{2\alpha}{\beta} < p - 2. \tag{3.32}$$

Such an inequality is satisfied by taking $\beta = 2\alpha$, which also satisfies the first inequality in (3.31).

In a similar way one can prove that if

$$\begin{cases} \beta p - 3\alpha > 0, \\ \beta p - 3\alpha > 2\beta - \alpha, \\ \beta p - 3\alpha > 2\beta - 3\alpha, \\ \beta p - 3\alpha > 4\beta - 5\alpha, \end{cases} \tag{3.33}$$

then $\mathcal{J}(u_{\lambda,\alpha,\beta}) \to -\infty$ as $\lambda \to +\infty$ with the same choice $\beta = 2\alpha$.

Remark 3.1. The condition $p > 3$ appears exactly in solving systems (3.31) or (3.33). More precisely if $p \in (2, 3]$ there is no couple (α, β) which satisfies the inequality (3.32); then, since the systems (3.31) or (3.33) have no solutions, we cannot conclude $\mathcal{J}(u_{\lambda,\alpha,\beta}) \to -\infty$.

Proof of Theorem B Lemma 3.1 + Theorem 3.1.

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