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Numerical solution of the nonlinear Klein–Gordon equation

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ABSTRACT

A numerical method is developed to solve the nonlinear one-dimensional Klein–Gordon equation by using the cubic B-spline collocation method on the uniform mesh points. We solve the problem for both Dirichlet and Neumann boundary conditions. The convergence and stability of the method are proved. The method is applied on some test examples, and the numerical results have been compared with the exact solutions. The L_2, L_∞ and Root-Mean-Square errors (RMS) in the solutions show the efficiency of the method computationally.

1. Introduction

The nonlinear Klein–Gordon equation arises in various problems in science and engineering. This paper is devoted to the numerical solution of an one-dimensional nonlinear Klein–Gordon equation, which is given in the following form:

$$\frac{\partial^2 u}{\partial t^2} - \mu \frac{\partial^2 u}{\partial x^2} + \chi(x,t,u) = \psi(x,t), \quad a \le x \le b, t > t_0, \tag{1}$$

and subjected to the initial conditions,

$$u(x,t_0) = f(x), \qquad u_t(x,t_0) = g(x), \tag{2}$$

and appropriate boundary conditions:

$$u(a,t) = \alpha_1(t), \qquad u(b,t) = \alpha_2(t), \quad t \ge t_0 \text{ (Dirichlet conditions)}, \tag{3}$$

or,

$$u_x(a,t) = \beta_1(t), \qquad u_x(b,t) = \beta_2(t), \quad t \ge t_0 \text{ (Neumann conditions)}, \tag{4}$$

where $u = u(x,t)$ represents the wave displacement at position x and time t, μ is a known constant and $\chi(x,t,u)$ is the nonlinear force such that $\frac{\partial \chi}{\partial u} \ge 0$.

In the well-known Sine-Gordon equation, the nonlinear force is given by $\chi(x,t,u) = \sin u$. In the physical applications, the nonlinear force $\chi(x,t,u)$ has also other forms [1]. The cases $\chi(x,t,u) = \sin u + \sin 2u$ and $\chi(x,t,u) = \sinh u + \sinh 2u$ are called the double Sine-Gordon equation and the double Sinh-Gordon equation, respectively. The above nonlinear Klein–Gordon equations are Hamiltonian partial differential equations, and for a wide class of force $\chi(x,t,u)$, they have the conserved Hamiltonian quantity (or energy),

$$H = \int \left(\frac{1}{2} u_t^2 + \frac{1}{2} u_x^2 + A(x,t,u) \right) \mathrm{d}x, \tag{5}$$

where $A(x,t,u) = \frac{\partial \chi}{\partial u}$.

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Nonlinear phenomena, which appear in many areas of scientific fields such as solid state physics, plasma physics, fluid dynamics, mathematical biology and chemical kinetics, can be modeled by partial differential equations. The Klein–Gordon equation plays a significant role in many scientific applications such as solid state physics, nonlinear optics and quantum field theory [2] and references therein. Fabian et al. [3] studied the adiabatic dynamics of topological solitons in presence of perturbation terms and the solitons due to Sine-Gordon equation, double Sine-Gordon equation, Sine–Cosine Gordon equation and double Sine–Cosine Gordon equations are studied by them. Dehghan and Shokri [4], developed numerical schemes to solve the one-dimensional nonlinear Klein–Gordon equation with quadratic and cubic nonlinearity using collocation points and approximating the solution using Thin Plate Splines (TPS) radial basis function (RBF) and also [5] proposed a two-dimensional Sine-Gordon equation using the radial basis functions. Several methods are developed to solve the Klein–Gordon-type equations, such as the Weierstrass elliptic function method, the elliptic equation rational expansion method and the extended F-function method (see [6,7] and references therein). Many author by using various methods solved the Klein–Gordon-type equations, such as [2,8–14]. This paper presents a new numerical scheme to solve the nonlinear Klein–Gordon equation using the collocation cubic b-spline method. Kadalbajoo et al. [15] developed B-spline collocation method on a nonuniform mesh for singularly perturbed one-dimensional. Idris et al. [16] applied Galerkin method using the quintic B-spline method to obtain numerical solution of the RLW equation and also Khalifa et al. [17] proposed collocation method with cubic B-splines for solving the MRLW equation. Quintic B-spline techniques provide better accuracy than the finite difference methods [18], least squares quadratic B-spline finite element method used in [19].

In this paper we developed a difference scheme using cubic B-spline function for the solution of nonlinear Klein–Gordon equation. This paper is arranged as follows. In Section 2, first we present a finite difference approximation to discretize the Eq. (1) in time variable then we prove the third order convergence of time discretization process. In Section 3 we applied cubic B-spline collocation method to solve the problem. The uniform convergence of the B-spline method is given in Section 4. In Section 5, numerical experiments are conducted to demonstrate the viability and the efficiency of the proposed method computationally.

2. Temporal discretization

Let us consider a uniform mesh Δ with the grid points $\lambda_{i,j}$ to discretize the region $\Omega = [a, b] \times (t_0, T]$. Each $\lambda_{i,j}$ is the vertices of the grid point (x_i, t_j) where $x_i = a + ih$, $i = 0, 1, 2, \ldots, N$ and $t_j = t_0 + jk$, $k = 0, 1, 2, \ldots$ and h and k are mesh sizes in the space and time directions respectively.

At first we discretize the problem in time variable using the following fourth order finite difference approximation with uniform step size k,

$$u_{tt}^n \cong \frac{\delta_t^2}{k^2(1+\gamma\delta_t^2)}u^n + O(k^4), \tag{6}$$

where $\delta_t^2 u^n = u^{n+1} - 2u^n + u^{n-1}$, $u^n = u(x, t_n)$ and $u^0 = u(x, t_0) = f(x)$. Substituting the above approximation into Eq. (1) and discretizing in time variable we have:

$$\frac{\delta_t^2}{k^2(1+\gamma\delta_t^2)}u^n - \mu u_{xx}^n + \chi(x, t_n, u^n) = \psi(x, t_n) \tag{7}$$

thus we have,

$$\delta_t^2 u^n - \mu k^2(1+\gamma\delta_t^2)u_{xx}^n + k^2(1+\gamma\delta_t^2)\chi(x, t_n, u^n) = k^2(1+\gamma\delta_t^2)\psi(x, t_n). \tag{8}$$

After some simplifications Eq. (8) can be write in the following form

$$-\mu\hat{u}_{xx} + \frac{1}{\gamma k^2}\hat{u} + \chi(\hat{u}) = \hat{\psi}(x), \tag{9}$$

where,

$$\begin{cases} \hat{u} \equiv u^{n+1}, \qquad \hat{u}_{xx} \equiv u_{xx}^{n+1} \\ \chi(\hat{u}) = \chi(x, t_n, u^{n+1}), \\ \hat{\psi}(x) = \dfrac{(1-2\gamma)}{\gamma}\psi^n + (\psi^{n+1} + \psi^{n-1}) + \dfrac{1}{k^2\gamma}(2u^n - u^{n-1}) + \dfrac{\mu(1-2\gamma)}{\gamma}u_{xx}^n + \mu u_{xx}^{n-1} - \dfrac{(1-2\gamma)}{\gamma}\chi^n - \chi^{n-1}, \end{cases}$$

with the boundary conditions,

$$\hat{u}(a) = \alpha_1(t_n), \qquad \hat{u}(b) = \alpha_2(t_n), \tag{10}$$

or

$$\hat{u}_x(a) = \beta_1(t_n), \qquad \hat{u}_x(b) = \beta_2(t_n). \tag{11}$$

Now in each time level we have a nonlinear ordinary differential equation in the form of (9) with the boundary conditions (10) or (11) which we solve by B-spline collocation method. The above presented process (6)–(9) is a three level explicit method, thus in order to start any computations using the above formula we must have the values of u at the nodal points at the zeroth and first level times.

We have an exact formula to compute $u^0 = u(x, t_0) = f(x)$ using the initial conditions. To compute u^1 we may use the initial conditions $u(x, t_0) = f(x)$ and $u_t(x, t_0) = g(x)$, to construct a fourth order approximation at the first time level. Following [20] and using Taylor series for u at $t = t_0 + k$ we have:

$$u^1 = u^0 + ku_t^0 + \frac{k^2}{2!}u_{tt}^0 + \frac{k^3}{3!}u_{ttt}^0 + \frac{k^4}{4!}u_{tttt}^0 + O(k^5). \tag{12}$$

u^0 and u_t^0 are known from initial conditions exactly thus we must compute the other terms. We can obtain all of the successive partial derivatives $u_x, u_{xx}, u_{xxx}, u_{xxxx}, u_t, u_{tx}, u_{txx}, \ldots$ at $t = t_0$ by differentiating the initial conditions with respect to x.

In the limiting case when t tends to t_0, and by assuming the right continuity of u, χ and ψ, and also their partial derivatives up to fourth order at $t = t_0$, by using (1) we have,

$$u_{tt}^0 = \lim_{t\to t_0^+}[\mu u_{xx} - \chi(x, t, u) + \psi(x, t)] = [\mu u_{xx} - \chi(x, t, u) + \psi(x, t)]^0. \tag{13}$$

Differentiating (13) with respect to x successively we obtain,

$$u_{xtt}^0 = [\mu u_{xxx} - \chi_x + \psi_x - \chi_u u_x]^0, \tag{14}$$

$$u_{xxtt}^0 = [\mu u_{xxxx} - \chi_u u_x - \chi_{xx} + \psi_{xx} - \chi_{xu} u_x - u_x(\chi_{xu} + \chi_{uu} u_x)]^0, \tag{15}$$

and similarly differentiating (13) with respect to t successively and using (14) and (15) we obtain,

$$u_{ttt}^0 = [\mu u_{xxt} - \chi_t + \psi_t - \chi_u u_t]^0, \tag{16}$$

$$u_{tttt}^0 = [\mu u_{xxtt} - \chi_u u_{tt} - \chi_{tt} + \psi_{tt} - \chi_{tu} u_t - u_t(\chi_{tu} + \chi_{uu} u_t)]^0. \tag{17}$$

Now substituting (13), (16), (17) and initial conditions into (12) we can obtain a fourth order approximation for u at $t = t_0+k$ in the form of,

$$\begin{aligned} u^1 &= f(x) + kg(x) + \frac{k^2}{2!}[\mu u_{xx} - \chi + \psi]^0 + \frac{k^3}{3!}[\mu u_{xxt} - \chi_t + \psi_t - \chi_u u_t]^0 \\ &\quad + \frac{k^4}{4!}[\mu u_{xxtt} - \chi_u u_{tt} - \chi_{tt} + \psi_{tt} - \chi_{tu} u_t - u_t(\chi_{tu} + \chi_{uu} u_t)]^0 + O(k^5). \end{aligned} \tag{18}$$

So that by using the initial condition $u^0 = f(x)$ and also (18) we are able to start the main scheme (9).

Theorem 1. *The above time discretization process (6)–(9) that we use to discretize equation (1) in time direction is of the third order convergence.*

Proof. Let $u(t_i)$ be the exact solution and u^i the approximate solution of the problem (1) at the ith level of time and also suppose that $\varepsilon_i = u^i - u(t_i)$ be the local truncation error in (9), then using Eq. (6) and replacing $\gamma = \frac{1}{12}$ it can be easily obtained,

$$|\varepsilon_i| \le \vartheta_i k^4, \quad i \ge 2, \tag{19}$$

where ϑ_i is some finite constant independent of k.

Furthermore for $i = 1$ using (12) we have,

$$|\varepsilon_1| \le \vartheta_1 k^5. \tag{20}$$

Let E_{n+1} be the global error in time discretizing process then the global error at $(n+1)$th level is $E_{n+1} = \sum_{i=1}^n \varepsilon_i$, $(k \le T/n)$. Now suppose that $\vartheta = \max\{\vartheta_1, \vartheta_2, \ldots, \vartheta_n\}$, thus with the help of (19) and (20) we have:

$$|E_{n+1}| = \left|\sum_{i=1}^n \varepsilon_i\right| \le \sum_{i=1}^n |\varepsilon_i| \le \vartheta_1 k^5 + \sum_{i=2}^n \vartheta_i k^4 \le n\vartheta k^4 \le \vartheta n(T/n)k^3 = Ck^3,$$

where $C = \vartheta T$, which gives the third order convergence of the method in time direction easily. □

3. B-spline collocation method

In this section we use the B-spline collocation method to solve (9) with the boundary conditions (10) or (11). Let $\Delta^\star = \{a = x_0 < x_1 < \cdots < x_N = b\}$ be the partition in $[a, b]$. B-splines are the unique nonzero splines of smallest compact support with knots at $x_0 < x_1 < \cdots < x_{N-1} < x_N$. We define the cubic B-spline for $i = -1, 0, \ldots, N+1$ by the following relation [21].

$$B_i = \frac{1}{h^3}\begin{cases}(x - x_{i-2})^3, & x \in [x_{i-2}, x_{i-1}],\\ h^3 + 3h^2(x - x_{i-1}) + 3h(x - x_{i-1})^2 - 3(x - x_{i-1})^3, & x \in [x_{i-1}, x_i],\\ h^3 + 3h^2(x_{i+1} - x) + 3h(x_{i+1} - x)^2 - 3(x_{i+1} - x)^3, & x \in [x_i, x_{i+1}],\\ (x_{i+2} - x)^3, & x \in [x_{i+1}, x_{i+2}],\\ 0, & \text{otherwise}.\end{cases} \tag{21}$$

It can be easily seen that the set of functions $\Omega = \{B_{-1}(x), B_0(x), \ldots, B_{N+1}(x)\}$ are linearly independent on $[a, b]$; thus $\Theta = \text{Span}(\Omega)$ is a subspace of $C^2[a, b]$, and Θ is $N + 3$-dimensional. Let us consider that $\hat{S}(x) \in \Theta$ be the B-spline approximation to the exact solution of problem (9), thus we can write $\hat{S}(x)$ in the following form:

$$\hat{S}(x) = \sum_{i=-1}^{N+1} \hat{c}_i B_i(x), \tag{22}$$

where $\hat{c}_i$ are unknown real coefficients and $B_i(x)$ are cubic B-spline functions. Let $\hat{S}(x)$ satisfy the collocation equations plus the boundary conditions, thus we have

$$\begin{aligned} &L\hat{S}(x_i) = \hat{\psi}(x_i), \quad 0 \le i \le N,\\ &\hat{S}(x_0) = \alpha_1(t_n), \qquad \hat{S}(x_N) = \alpha_2(t_n),\end{aligned} \tag{23}$$

or

$$\hat{S}'(x_0) = \beta_1(t_n), \qquad \hat{S}'(x_N) = \beta_2(t_n),$$

where $L\hat{u} \equiv -\mu \hat{u}_{xx} + \frac{1}{\gamma k^2}\hat{u} + \chi(\hat{u})$. Substituting (22) into (23) and using the properties of B-spline functions we can obtain,

$$\frac{-6\mu}{h^2}(\hat{c}_{i-1} - 2\hat{c}_i + \hat{c}_{i+1}) + \frac{1}{\gamma k^2}(\hat{c}_{i-1} + 4\hat{c}_i + \hat{c}_{i+1}) + \chi(x_i, (\hat{c}_{i-1} + 4\hat{c}_i + \hat{c}_{i+1})) = \hat{\psi}(x_i), \quad 0 \le i \le N; \tag{24}$$

simplifying the above relation leads to the following system of $(N - 1)$ nonlinear equations in $(N + 3)$ unknowns:

$$\begin{aligned}&\left(-6\mu + \frac{h^2}{\gamma k^2}\right)\hat{c}_{i-1} + \left(12\mu + \frac{4h^2}{\gamma k^2}\right)\hat{c}_i + \left(-6 + \frac{h^2}{\gamma k^2}\right)\hat{c}_{i+1} + h^2\chi(x_i, (\hat{c}_{i-1} + 4\hat{c}_i + \hat{c}_{i+1})) = h^2\hat{\psi}(x_i),\\ &1 \le i \le N-1.\end{aligned} \tag{25}$$

To obtain a unique solution for $\hat{C} = (\hat{c_{-1}}, \hat{c}_0, \ldots, \hat{c_N}, \hat{c_{N+1}})$ we must use the boundary conditions.

First we obtain boundary formulas for Dirichlet boundary conditions. Using the first boundary condition we have:

$$u(a, t_n) = \hat{S}(a) = \alpha_1(t_n) = \hat{c}_{-1} + 4\hat{c}_0 + \hat{c}_1,$$

now eliminating $\hat{c}_{-1}$ from the above equation and substituting into Eq. (25) for $i = 0$ we have:

$$\left(-6\mu + \frac{h^2}{\gamma k^2}\right)(\alpha_1(t_n) - 4\hat{c}_0 - \hat{c}_1) + \left(12\mu + \frac{4h^2}{\gamma k^2}\right)\hat{c}_0 + \left(-6 + \frac{h^2}{\gamma k^2}\right)\hat{c}_1 + h^2\chi(x_0, \alpha_1(t_n)) = h^2\hat{\psi}(x_0).$$

Simplifying the above equation we obtain,

$$36\mu\hat{c}_0 + h^2\chi_0 = h^2\hat{\psi}_0 + \left(6\mu - \frac{h^2}{\gamma k^2}\right)\alpha_1(t_n), \tag{26}$$

where

$$\chi_0 = \chi(x_0, \alpha_1(t_n)), \qquad \hat{\psi}_0 = \hat{\psi}(x_0).$$

Similarly, eliminating $\hat{c}_{N+1}$ from the last Eq. (25) for $i = N$, we find

$$36\mu\hat{c}_N + h^2\chi_N = h^2\psi_N + \left(6\mu - \frac{h^2}{\gamma k^2}\right)\alpha_2(t_n), \tag{27}$$

where,

$\chi_N = \chi(x_N, \alpha_2(t_n)), \qquad \hat{\psi}_N = \hat{\psi}(x_N).$

Associating (26) and (27) with (25) we obtain a nonlinear $(N+1) \times (N+1)$ system of equations in the following form,

$$\mathbf{A}\hat{\mathbf{C}} + h^2\hat{\mathbf{B}} = h^2\hat{\mathbf{Q}}, \tag{28}$$

where,

$$\mathbf{A} = \begin{pmatrix} 36\mu & 0 & 0 & & \\ -6\mu + \dfrac{h^2}{\gamma k^2} & 12\mu + \dfrac{4h^2}{\gamma k^2} & -6\mu + \dfrac{h^2}{\gamma k^2} & & \\ & \ddots & \ddots & \ddots & \\ & & -6\mu + \dfrac{h^2}{\gamma k^2} & 12\mu + \dfrac{4h^2}{\gamma k^2} & -6\mu + \dfrac{h^2}{\gamma k^2} \\ & & 0 & 0 & 36\mu \end{pmatrix}, \tag{29}$$

$$\hat{\mathbf{B}} = \begin{pmatrix} \chi_0 \\ \chi_1 \\ \vdots \\ \chi_N \end{pmatrix}, \qquad \hat{\mathbf{Q}} = \begin{pmatrix} \hat{\psi}_0 + \left(\dfrac{6\mu}{h^2} - \dfrac{1}{\gamma k^2}\right)\alpha_1(t) \\ \hat{\psi}_1 \\ \vdots \\ \hat{\psi}_{N-1} \\ \hat{\psi}_N + \left(\dfrac{6\mu}{h^2} - \dfrac{1}{\gamma k^2}\right)\alpha_2(t) \end{pmatrix}, \qquad \hat{\mathbf{C}} = \begin{pmatrix} \hat{c}_0 \\ \hat{c}_1 \\ \vdots \\ \hat{c}_N \end{pmatrix}, \tag{30}$$

and, $\chi_i = \chi(x_i, \hat{c}_{i-1} + 4\hat{c}_i + \hat{c}_{i+1})$.

Now we can obtain two boundary formulas for Neumann boundary conditions also. Using first Neumann boundary condition and substituting in the main formula (25) for $i = 0$ we obtain the following boundary formula as first equation,

$$\left(12\mu + \frac{4h^2}{\gamma k^2}\right)\hat{c}_0 + \left(-12\mu + \frac{2h^2}{\gamma k^2}\right)\hat{c}_1 + h^2\chi_0 = h^2\hat{\psi}_0 + \frac{h}{3}\beta_1(t_n)\left(-6\mu + \frac{h^2}{\gamma k^2}\right). \tag{31}$$

Similarly using the second Neumann boundary condition substituting in (25) for $i = N$ we can obtain the following boundary formula as the last equation,

$$\left(-12\mu + \frac{2h^2}{\gamma k^2}\right)\hat{c}_{N-1} + \left(12\mu + \frac{4h^2}{\gamma k^2}\right)\hat{c}_N + h^2\chi_N = h^2\hat{\psi}_N - \frac{h}{3}\beta_2(t_n)\left(-6\mu + \frac{h^2}{\gamma k^2}\right). \tag{32}$$

Associating (31) and (32) with (25) we obtain a nonlinear $(N+1) \times (N+1)$ system of equations in the form of,

$$\mathbf{A}_1\hat{\mathbf{C}} + h^2\hat{\mathbf{B}}_1 = h^2\hat{\mathbf{Q}}_1, \tag{33}$$

where,

$$\mathbf{A}_1 = \begin{pmatrix} 12\mu + \dfrac{4h^2}{\gamma k^2} & -12\mu + \dfrac{2h^2}{\gamma k^2} & 0 & & \\ -6\mu + \dfrac{h^2}{\gamma k^2} & 12\mu + \dfrac{4h^2}{\gamma k^2} & -6\mu + \dfrac{h^2}{\gamma k^2} & & \\ & \ddots & \ddots & \ddots & \\ & & -6\mu + \dfrac{h^2}{\gamma k^2} & 12\mu + \dfrac{4h^2}{\gamma k^2} & -6\mu + \dfrac{h^2}{\gamma k^2} \\ & & 0 & -12\mu + \dfrac{2h^2}{\gamma k^2} & 12\mu + \dfrac{4h^2}{\gamma k^2} \end{pmatrix}, \tag{34}$$

$$\hat{\mathbf{B}}_1 = \begin{pmatrix} \chi_0 \\ \chi_1 \\ \vdots \\ \chi_N \end{pmatrix}, \qquad \hat{\mathbf{Q}}_1 = \begin{pmatrix} \hat{\psi}_0 + \dfrac{1}{3h}\left(-6\mu + \dfrac{h^2}{\gamma k^2}\right)\beta_1(t) \\ \hat{\psi}_1 \\ \vdots \\ \hat{\psi}_{N-1} \\ \hat{\psi}_N - \dfrac{1}{3h}\left(-6\mu + \dfrac{h^2}{\gamma k^2}\right)\beta_2(t) \end{pmatrix}, \qquad \hat{\mathbf{C}} = \begin{pmatrix} \hat{c}_0 \\ \hat{c}_1 \\ \vdots \\ \hat{c}_N \end{pmatrix}, \tag{35}$$

and, $\chi_i = \chi(x_i, \hat{c}_{i-1} + 4\hat{c}_i + \hat{c}_{i+1})$.

4. Convergence analysis

We prove the convergence of our method for Dirichlet boundary conditions then the proof for Neumann conditions is similar. Let $\hat{u}(x)$ be the exact solution of Eq. (9) with boundary conditions (10) and also $\hat{S}(x) = \sum_{i=-1}^{N+1} \hat{c}_i B_i(x)$ be the B-spline collocation approximation to $\hat{u}(x)$. Due to round off errors in computations we assume that $S^*(x) = \sum_{i=-1}^{N+1} c_i^* B_i(x)$ be the computed spline approximation to $\hat{S}(x)$ where $C^* = (c_0^*, c_1^*, \ldots, c_{N-1}^*, c_N^*)$. To estimate the error $\|\hat{u}(x) - \hat{S}(x)\|_\infty$ we must estimate the errors $\|\hat{u}(x) - S^*(x)\|_\infty$ and $\|S^*(x) - \hat{S}(x)\|_\infty$ separately.

Substituting $S^*(x)$ into (28) we have,

$$\mathbf{A}\mathbf{C}^* + h^2\mathbf{B}^* = h^2\mathbf{Q}^*, \tag{36}$$

where

$$\mathbf{B}^* = \begin{pmatrix} \chi_0 \\ \chi_1 \\ \vdots \\ \chi_N \end{pmatrix}, \qquad \mathbf{Q}^* = \begin{pmatrix} \psi_0^* + \left(6\mu - \dfrac{h^2}{\gamma k^2}\right)\alpha_1(t) \\ \psi_1^* \\ \vdots \\ \psi_{N-1}^* \\ \psi_N^* + \left(6\mu - \dfrac{h^2}{\gamma k^2}\right)\alpha_2(t) \end{pmatrix}, \qquad \mathbf{C}^* = \begin{pmatrix} c_0^* \\ c_1^* \\ \vdots \\ c_N^* \end{pmatrix},$$

and $\chi_i = \chi(x_i, c_{i-1}^* + 4c_i^* + c_{i+1}^*)$.

Subtracting (28) and (36) we obtain,

$$\mathbf{A}(\mathbf{C}^* - \hat{\mathbf{C}}) + h^2(\mathbf{B}^* - \hat{\mathbf{B}}) = h^2(\mathbf{Q}^* - \hat{\mathbf{Q}}). \tag{37}$$

First we need to recall the following theorem.

Theorem 2. *Suppose that $f(x) \in C^4[a, b]$ and $|f^{(4)}(x)| \leq L$, $\forall x \in [a, b]$ and $\Delta = \{a = x_0 < x_1 < \cdots < x_N = b\}$ be the equally spaced partition of $[a, b]$ with step size h. If $S(x)$ be the unique spline function interpolate $f(x)$ at knots $x_0, \ldots, x_N \in \Delta$, then there exist a constant λ_j such that,*

$$\|f^{(j)} - S^{(j)}\|_\infty \leq \lambda_j L h^{4-j}, \quad j = 0, 1, 2, 3. \tag{38}$$

Proof. For the proof see [22,23]. □

Now using the above theorem we can obtain a bound on $\|Q^* - \hat{Q}\|_\infty$. Applying (23) we have,

$$\begin{aligned}|\psi^*(x_i) - \hat{\psi}(x_i)| &= |LS^*(x_i) - L\hat{S}(x_i)| \\ &= |-\mu S^{*''}(x_i) + \mu \hat{S}''(x_i) + \frac{1}{\gamma k^2} S^*(x_i) - \frac{1}{\gamma k^2}\hat{S}(x_i) + \chi(x_i, S^*(x_i)) - \chi(x_i, \hat{S}(x_i))|,\end{aligned}$$

and thus,

$$|\psi^*(x_i) - \hat{\psi}(x_i)| \leq |-\mu S^{*''}(x_i) + \mu\hat{S}''(x_i)| + \left|\frac{1}{\gamma k^2} S^*(x_i) - \frac{1}{\gamma k^2}\hat{S}(x_i)\right| + |\chi(x_i, S^*(x_i)) - \chi(x_i, \hat{S}(x_i))|. \tag{39}$$

From the Eq. (39) and using Theorem 2 and [24] (pp. 218), we can find

$$\begin{aligned}\|\mathbf{Q}^* - \hat{\mathbf{Q}}\| &\leq \mu\lambda_2 L h^2 + \left\|\frac{1}{\gamma k^2}\right\| \lambda_0 L h^4 + M(|S^*(x) - \hat{S}(x)|) \\ &\leq \varepsilon\lambda_2 L h^2 + \left\|\frac{1}{\gamma k^2}\right\| \lambda_0 L h^4 + ML\lambda_0 h^4,\end{aligned} \tag{40}$$

where $\|\chi'(z)\| \leq M$, $z \in R^3$. Applying the Eq. (40) we have

$$\|\mathbf{Q}^* - \hat{\mathbf{Q}}\| \leq M_1 h^2, \tag{41}$$

where $M_1 = \mu\lambda_2 L + \|\frac{1}{\gamma k^2}\|\lambda_0 L h^2 + ML\lambda_0 h^2$.

Also consider the nonlinear term in left hand side of (37) we have,

$$h^2(\mathbf{B}^* - \hat{\mathbf{B}}) = h^2 \left(\begin{bmatrix} \chi(x_0, \alpha_1(t)) \\ \vdots \\ \chi(x_i, (\hat{c}_{i-1} + 4\hat{c}_i + \hat{c}_{i+1})) \\ \vdots \\ \chi(x_N, \alpha_2(t)) \end{bmatrix} - \begin{bmatrix} \chi(x_0, \alpha_1(t)) \\ \vdots \\ \chi(x_i, (c^*_{i-1} + 4c^*_i + c^*_{i+1})) \\ \vdots \\ \chi(x_N, \alpha_2(t)) \end{bmatrix} \right).$$

Applying the mean value theorem we can rewrite the above relation in the form of,

$$h^2(\mathbf{B}^* - \hat{\mathbf{B}}) = h^2 \left(\frac{\partial \chi}{\partial u}(\xi_1) \right) \mathbf{J}(\mathbf{C}^* - \hat{\mathbf{C}}), \tag{42}$$

where $\xi_1 \in (0, 1)$ and $\mathbf{J}$ is the following $(N+1)(N+1)$ matrix

$$\mathbf{J} = \begin{pmatrix} 0 & 0 & & & \\ 1 & 4 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & 4 & 1 \\ & & & 0 & 0 \end{pmatrix}.$$

Substituting (37) into (42) we have,

$$\mathbf{R}(\mathbf{C}^* - \hat{\mathbf{C}}) = h^2(\mathbf{Q}^* - \hat{\mathbf{Q}}), \tag{43}$$

where $\mathbf{R} = \mathbf{A} + h^2(\frac{\partial \chi}{\partial u}(\xi_1))\mathbf{J}$. It is obvious that the matrix $\mathbf{R}$ is a strictly diagonally dominant matrix and thus nonsingular, hence we can write:

$$(\mathbf{C}^* - \hat{\mathbf{C}}) = h^2 \mathbf{R}^{-1}(\mathbf{Q}^* - \hat{\mathbf{Q}}). \tag{44}$$

Taking the infinity norm from (44) we have

$$\|(\mathbf{C}^* - \hat{\mathbf{C}})\| \le h^2 \|\mathbf{R}^{-1}\| \|(\mathbf{Q}^* - \hat{\mathbf{Q}})\| \le h^2 \|\mathbf{R}^{-1}\| (h^2 M_1) = h^4 M_1 \|\mathbf{R}^{-1}\|. \tag{45}$$

Now suppose that η_i, $(0 \le i \le N)$ is the summation of the ith row of the matrix $\mathbf{R} = [r_{ij}]_{(N+1)\times(N+1)}$, then we have

$$\eta_0 = \sum_{j=0}^{N} r_{0,j} = 36\mu, \quad i = 0, \tag{46}$$

$$\eta_i = \sum_{j=0}^{N} r_{i,j} = 6h^2 \left(\frac{1}{\gamma k^2} + \frac{\partial \chi}{\partial u} \right), \quad i = 1(1)N-1, \tag{47}$$

$$\eta_N = \sum_{j=0}^{N} r_{N,j} = 36\mu, \quad i = N. \tag{48}$$

From the theory of matrices we have:

$$\sum_{i=0}^{N} r^{-1}_{k,i} \eta_i = 1, \quad k = 0, \dots, N, \tag{49}$$

where $r^{-1}_{k,i}$ are the elements of $\mathbf{R}^{-1}$. Therefore

$$\|\mathbf{R}^{-1}\| = \sum_{i=0}^{N} |r^{-1}_{k,i}| \le \frac{1}{\min\limits_{0 \le i \le N} \eta_i} = \frac{1}{h^2 \sigma_l} \le \frac{1}{h^2 |\sigma_l|}, \tag{50}$$

where l is some index between 0 and N. Substituting (50) into (45) we can find,

$$\|\mathbf{C}^* - \hat{\mathbf{C}}\| \le \frac{M_1 h^4}{h^2 |\sigma_l|} \le M_2 h^2, \tag{51}$$

where $M_2 = \frac{M_1}{|\sigma_l|}$ is some finite constant.

Lemma 1. *The* B-*splines* $\{B_{-1}, B_0, \ldots, B_{N+1}\}$ *satisfy the following inequality:*

$$\left|\sum_{i=-1}^{N+1} B_i(x)\right| \leq 10, \quad (0 \leq x \leq 1). \tag{52}$$

Proof. See Prenter [21] (pp. 284). □

Now observe that we have:

$$S^*(x) - \hat{S}(x) = \sum_{i=-1}^{N+1} (c_i^* - \hat{c}_i) B_i(x), \tag{53}$$

thus taking norm and using (51) and (52) we get,

$$\|S^*(x) - \hat{S}(x)\| = \left\|\sum_{i=-1}^{N+1} (c_i^* - \hat{c}_i) B_i(x)\right\| \leq \left|\sum_{i=-1}^{N+1} B_i(x)\right| \|(\mathbf{C}_i^* - \hat{\mathbf{C}}_i)\| \leq 10 M_2 h^2. \tag{54}$$

Theorem 3. *Let $u^*(x)$ be the exact solution of (9) with Boundary conditions (10) and let $\hat{S}(x)$ be the B-spline collocation approximation to $u^*(x)$ then the method has second order convergence and we have*

$$\|\hat{u}(x) - \hat{S}(x)\| \leq \omega h^2, \tag{55}$$

where $\omega = \lambda_0 L h^2 + 10 M_2$ is some finite constant.

Proof. From Theorem 2 we have:

$$\|\hat{u}(x) - \hat{S}(x)\| \leq \lambda_0 L h^4, \tag{56}$$

and thus from (54) and (56) we find

$$\|\hat{u}(x) - \hat{S}(x)\| \leq \|\hat{u}(x) - S^*(x)\| + \|S^*(x) - \hat{S}(x)\| \leq \lambda_0 L h^4 + 10 M_2 h^2 = \omega h^2,$$

where $\omega = \lambda_0 L h^2 + 10 M_2$. □

Now if we suppose that $u(x, t)$ be the exact solution of (1) and $U(x, t)$ be the numerical approximation to this solution by our numerical process, then we have:

$$\|u(x, t) - U(x, t)\| \leq \rho(k^3 + h^2), \quad (\rho \text{ is a constant}). \tag{57}$$

5. Numerical illustrations

In order to test the viability of our method and to demonstrate its convergence computationally we consider some test problems. We solve these problems for various values of h and k. We compare our results with the results in [4]. The L_2, L_∞ and RMS errors in the solutions are tabulated in Tables 1–5. The graph of the solutions are given in Figs. 1–5.

Example 1. We consider the nonlinear Klein–Gordon equation (1) with $\mu = 1$, $\chi(x, t, u) = u^2$ and $\psi(x, t) = -x\cos(t) + x^2\cos^2(t)$ in the interval $-1 \leq x \leq 1$.

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} + u^2 = -x\cos(t) + x^2\cos^2(t). \tag{58}$$

The initial conditions are given by

$$\begin{cases} u(x, 0) = x, & -1 \leq x \leq 1, \\ u_t(x, 0) = 0, & -1 \leq x \leq 1, \end{cases}$$

with the exact solution $u(x, t) = x\cos(t)$. We extract the boundary functions from the exact solution. The L_2, L_∞ and Root-Mean-Square (RMS) errors in the solutions with $h = 0.04$ and $h = 0.02$, $\gamma = \frac{1}{12}$ and $k = 0.0001$ are tabulated in Table 1 and compared with the results in [4].

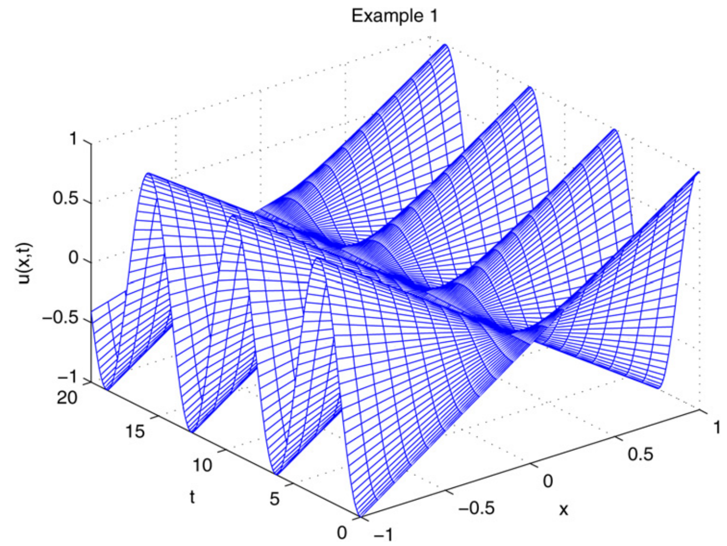

Fig. 1. The graph of the solution with $-1 \le x \le 1$, up to $t = 20$ for Example 1.

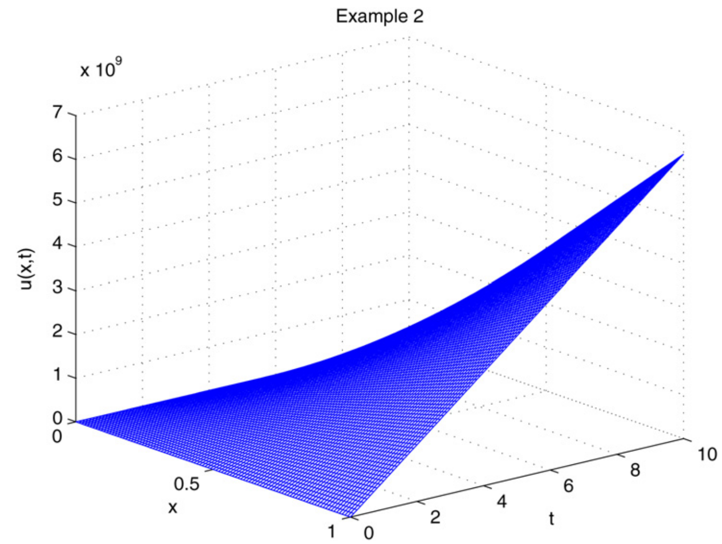

Fig. 2. The graph of the solution with $0 \le x \le 1$, up to $t = 10$ for Example 2.

Table 1
The computational results for Example 1 with $k = 0.0001$ by: Our method $h = 0.04$, Our method $h = 0.02$ and Method in [4], $h = 0.02$.

T	1	3	5	10	20
Our method $h = 0.04$					
L_∞	5.2902×10^{-12}	1.6874×10^{-12}	4.4004×10^{-12}	4.5354×10^{-12}	6.2372×10^{-11}
L_2	1.8060×10^{-11}	4.8007×10^{-12}	1.5271×10^{-11}	2.1344×10^{-11}	7.4654×10^{-10}
RMS	3.6121×10^{-12}	9.6015×10^{-13}	3.0543×10^{-12}	3.1228×10^{-12}	2.5543×10^{-11}
Our method $h = 0.02$					
L_∞	4.7698×10^{-13}	2.8899×10^{-13}	4.3667×10^{-13}	8.4593×10^{-13}	3.1344×10^{-12}
L_2	1.8986×10^{-12}	1.0680×10^{-12}	1.5686×10^{-12}	5.3244×10^{-12}	4.4324×10^{-11}
RMS	2.6850×10^{-13}	1.5105×10^{-13}	2.2184×10^{-13}	5.9738×10^{-13}	2.2332×10^{-12}
T	1	3	5	7	10
Method in [4], $h = 0.02$					
L_∞	1.2540×10^{-5}	1.5554×10^{-5}	3.3792×10^{-5}	3.7753×10^{-5}	1.3086×10^{-5}
L_2	6.5422×10^{-5}	1.1717×10^{-4}	2.2011×10^{-4}	2.5892×10^{-4}	7.9854×10^{-5}
RMS	6.5097×10^{-6}	1.1659×10^{-5}	2.1902×10^{-5}	2.5763×10^{-5}	7.9458×10^{-6}

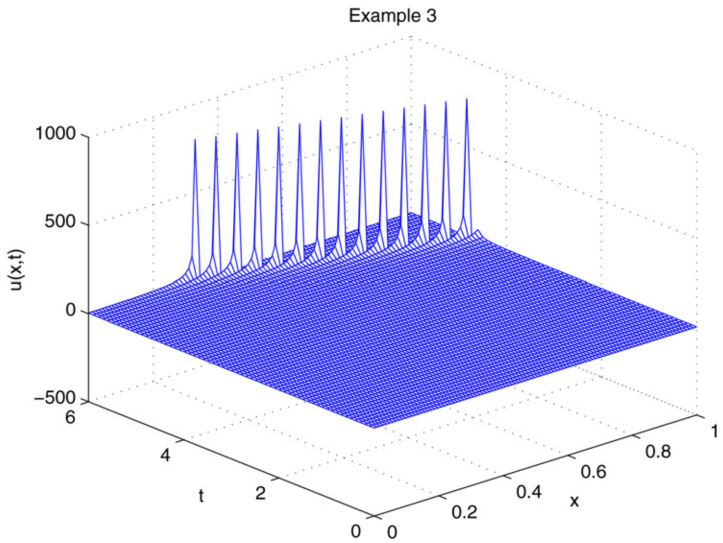

Fig. 3. The graph of the solution with $0 \leq x \leq 1$, up to $t = 6$ for Example 3.

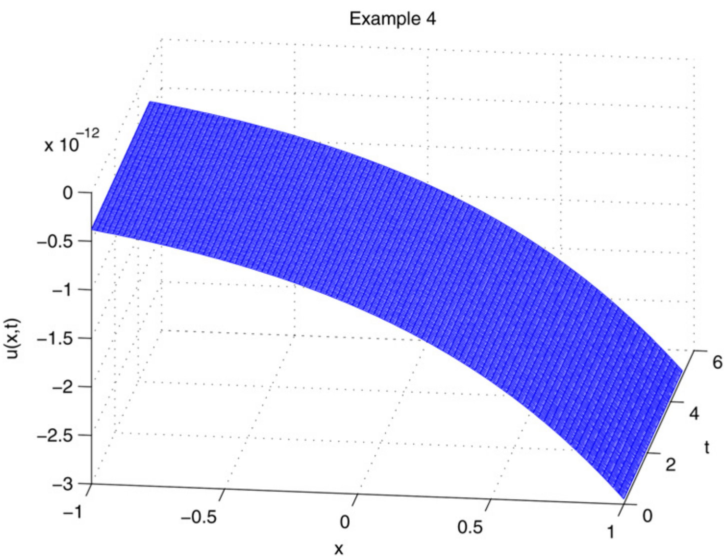

Fig. 4. The graph of the solution with $-1 \leq x \leq 1$, up to $t = 5$ for Example 4.

Example 2. Consider the nonlinear Klein–Gordon equation (1) with $\mu = 1$, $\chi(x, t, u) = u^2$ and $\psi(x, t) = 6tx^3 + (tx)^6 - 6t^3x$ in the interval $0 \leq x \leq 1$. The initial conditions are given by

$$\begin{cases} u(x, 0) = 0, & 0 \leq x \leq 1, \\ u_t(x, 0) = 0, & 0 \leq x \leq 1, \end{cases}$$

with the exact solution $u(x, t) = x^3t^3$. We extract the boundary functions from the exact solution. This example has been solved by our presented method with $h = 0.04$ and $h = 0.02$, $\gamma = \frac{1}{12}$ and $k = 0.0001$. The L_2, L_∞ and Root-Mean-Square (RMS) errors in the solutions are tabulated in Table 2 and compared with results in [4].

Example 3. In this example, we consider the nonlinear Klein–Gordon equation

$$u_{tt} - \frac{5}{2}u_{xx} + u + \frac{3}{2}u^3 = \psi(x, t),$$

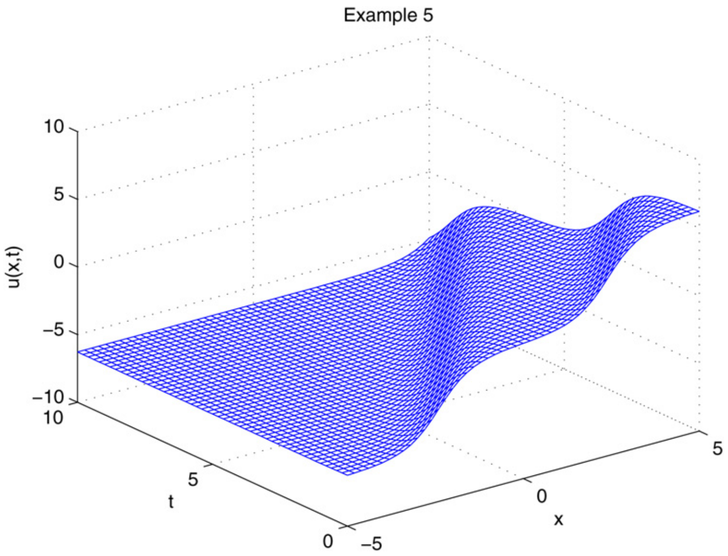

Fig. 5. The graph of the solution with $-5 \le x \le 5$, up to $t = 10$ for Example 5.

Table 2
The computational results for Example 2 with $k = 0.0001$ by: Our method $h = 0.04$, Our method $h = 0.02$ and Method in [4], $h = 0.02$.

T	1	2	3	4	5	10
Our method $h = 0.04$						
L_∞	6.4148×10^{-13}	7.4513×10^{-12}	1.6244×10^{-11}	2.0072×10^{-11}	2.5409×10^{-11}	9.5037×10^{-11}
L_2	1.9129×10^{-12}	2.6181×10^{-11}	5.5256×10^{-11}	7.2621×10^{-11}	8.5371×10^{-11}	1.8202×10^{-10}
RMS	3.8259×10^{-13}	5.2361×10^{-12}	1.1051×10^{-11}	1.4524×10^{-11}	1.7074×10^{-11}	3.6405×10^{-11}
Our method $h = 0.02$						
L_∞	5.5733×10^{-14}	3.0198×10^{-13}	3.5829×10^{-12}	5.1088×10^{-12}	1.4267×10^{-11}	7.2456×10^{-11}
L_2	1.4257×10^{-13}	8.7463×10^{-13}	1.0177×10^{-11}	1.7568×10^{-11}	3.0183×10^{-11}	2.6546×10^{-10}
RMS	2.0162×10^{-14}	1.2369×10^{-13}	1.4392×10^{-12}	2.4846×10^{-12}	4.2685×10^{-12}	2.3400×10^{-11}
T	1	2	3	4	5	
Method in [4], $h = 0.02$						
L_∞	1.1012×10^{-5}	1.6496×10^{-4}	5.9728×10^{-4}	1.8264×10^{-3}	3.6915×10^{-3}	
L_2	5.4998×10^{-5}	1.1522×10^{-3}	3.2588×10^{-3}	9.8191×10^{-3}	1.9139×10^{-2}	
RMS	5.4725×10^{-6}	1.1465×10^{-4}	3.2426×10^{-4}	9.7704×10^{-4}	1.9044×10^{-3}	

in the interval $0 \le x \le 1$. The initial conditions are given by

$$\begin{cases} u(x,0) = B\tan(Kx), & 0 \le x \le 1, \\ u_t(x,0) = BcK\sec^2(Kx), & 0 \le x \le 1, \\ B = \sqrt{\dfrac{2}{3}}, \qquad K = \sqrt{\dfrac{-1}{2(-2.5+c^2)}}, & \end{cases}$$

where the exact solution is $u(x,t) = B\tan(K(x+ct))$. We extract the boundary conditions from the exact solution. This example has been solved by our presented method with $h = 0.02$ and $h = 0.01$, $\gamma = \frac{1}{12}$, $c = 0.5$ and $c = 0.05$ and $k = 0.001$. The L_2, L_∞ and Root-Mean-Square (RMS) errors in the solutions are tabulated in Table 3. We compare our results with the results in [4].

Example 4. Consider the nonlinear Klein–Gordon equation (1) with $\mu = 1$, $\chi(x,t,u) = u + u^3$ and

$$\psi(x,t) = (-2+x^2)\cosh(t+x) + x^6\cosh^3(t+x) - 4x\cosh(t+x),$$

in the interval $-1 \le x \le 1$. The initial conditions are given by

$$\begin{cases} u(x,0) = x^2\cosh(x), & -1 \le x \le 1, \\ u_t(x,0) = x^2\cosh(x), & -1 \le x \le 1, \end{cases}$$

Table 3
The computational results for Example 3 with $k = 0.001$ by: Our method ($c = 0.5$, $h = .02$), ($c = 0.05$, $h = .02$), ($c = 0.5$, $h = .01$), ($c = 0.05$, $h = .01$), and Method in [4] ($c = 0.5$, $h = .01$) and ($c = 0.05$, $h = .01$).

T	1	2	3	4	5	6
Our method	($c = 0.5$, $h = .02$)					
L_∞	1.0801×10^{-7}	3.5077×10^{-7}	1.2429×10^{-6}	7.8735×10^{-6}	5.5050×10^{-4}	4.3849×10^{-4}
L_2	5.3892×10^{-7}	1.8101×10^{-6}	6.3528×10^{-6}	3.8546×10^{-5}	1.7006×10^{-3}	1.1253×10^{-3}
RMS	7.6215×10^{-8}	2.5599×10^{-7}	8.9843×10^{-7}	5.4512×10^{-6}	2.4051×10^{-4}	1.0092×10^{-4}
	($c = 0.05$, $h = .02$)					
L_∞	4.8016×10^{-8}	9.9075×10^{-8}	1.1602×10^{-7}	7.9819×10^{-8}	4.1458×10^{-8}	7.6338×10^{-8}
L_2	2.1426×10^{-7}	5.0993×10^{-7}	5.8932×10^{-7}	3.9845×10^{-7}	2.1622×10^{-7}	3.5869×10^{-7}
RMS	3.0301×10^{-8}	7.2115×10^{-8}	8.3343×10^{-8}	5.6350×10^{-8}	3.0578×10^{-8}	5.0727×10^{-8}
Our method	($c = 0.5$, $h = .01$)					
L_∞	2.6949×10^{-8}	8.7462×10^{-8}	3.0903×10^{-7}	1.9394×10^{-6}	1.1424×10^{-4}	3.0293×10^{-4}
L_2	1.9015×10^{-7}	6.3813×10^{-7}	2.2341×10^{-6}	1.3439×10^{-5}	5.1439×10^{-4}	7.3547×10^{-4}
RMS	1.9015×10^{-8}	6.3813×10^{-8}	2.2344×10^{-7}	1.3439×10^{-6}	5.1439×10^{-5}	9.1233×10^{-5}
	($c = 0.05$, $h = .01$)					
L_∞	1.1986×10^{-8}	2.4733×10^{-8}	2.8958×10^{-8}	1.9916×10^{-8}	1.0340×10^{-8}	3.4954×10^{-8}
L_2	7.5619×10^{-8}	1.7997×10^{-7}	2.0797×10^{-7}	1.4058×10^{-7}	7.6270×10^{-8}	2.5709×10^{-7}
RMS	7.5619×10^{-9}	1.7997×10^{-8}	2.0797×10^{-8}	1.4058×10^{-8}	7.6270×10^{-9}	2.5709×10^{-8}

T	1	2	3	4
Method in [4]	($c = 0.5$, $h = .01$)			
L_∞	5.9964×10^{-6}	2.1973×10^{-5}	9.0893×10^{-5}	8.2945×10^{-4}
L_2	4.0761×10^{-5}	1.5769×10^{-4}	6.4792×10^{-4}	5.3572×10^{-3}
RMS	4.0559×10^{-6}	1.5691×10^{-5}	6.4470×10^{-5}	5.3306×10^{-4}
	($c = 0.05$, $h = .01$)			
L_∞	3.6497×10^{-7}	3.8952×10^{-7}	4.2123×10^{-7}	4.5928×10^{-7}
L_2	1.7861×10^{-6}	1.5383×10^{-6}	1.7275×10^{-6}	2.0097×10^{-6}
RMS	1.7772×10^{-7}	1.5306×10^{-7}	1.7190×10^{-7}	1.9997×10^{-7}

Table 4
The computational results for Example 4 with $k = 0.0001$ by: Our method ($h = .02$), Our method ($h = .01$) and Method in [4] ($h = .01$).

T	1	2	3	4	5
Our method ($h = .02$)					
L_∞	1.1427×10^{-5}	1.2792×10^{-5}	1.5862×10^{-6}	2.2779×10^{-5}	3.9340×10^{-5}
L_2	7.3413×10^{-5}	6.2324×10^{-5}	6.7692×10^{-6}	8.3643×10^{-5}	9.3038×10^{-5}
RMS	1.0382×10^{-6}	8.8140×10^{-6}	9.5731×10^{-6}	1.1828×10^{-5}	2.7330×10^{-5}
Our method ($h = .01$)					
L_∞	3.5666×10^{-6}	3.1949×10^{-6}	3.9619×10^{-6}	5.6889×10^{-6}	6.3356×10^{-6}
L_2	2.5993×10^{-5}	2.2013×10^{-5}	2.3990×10^{-5}	2.9542×10^{-5}	3.2638×10^{-5}
RMS	2.5930×10^{-6}	2.2013×10^{-6}	2.3909×10^{-6}	2.9542×10^{-6}	2.9092×10^{-6}
T	1	2	3	4	5
Method in [4] ($h = .01$)					
L_∞	5.0705×10^{-5}	5.0260×10^{-4}	2.0612×10^{-3}	6.5720×10^{-3}	1.9067×10^{-2}
L_2	2.9474×10^{-4}	2.7082×10^{-3}	9.7246×10^{-3}	2.7881×10^{-2}	7.7337×10^{-2}
RMS	2.0789×10^{-5}	1.9102×10^{-4}	6.8592×10^{-4}	1.9666×10^{-3}	5.4549×10^{-3}

with the exact solution $u(x, t) = x^2 \cosh(x + t)$. We extract the boundary conditions from the exact solution. This example has been solved by our presented method with 0.02 and $h = 0.01$, $\gamma = \frac{1}{12}$ and $k = 0.0001$. In Table 4, the L_2, L_∞ and Root-Mean-Square (RMS) errors in the solutions are tabulated. The results are compared with the results in [4].

Example 5 (*Double Sine-Gordon (DSG) Equation*). The double Sine-Gordon equation (DSG) has very important rule in many fields of science and engineering, for example in condensed matter, quantum optics, and particle physics. This problem has the potential function as, $G(u) = \frac{-4}{1+4|\eta|}(\eta \cos(u) - \cos(\frac{u}{2}))$, where η is an arbitrary constant. The double Sine-Gordon equation is in the following form,

$$u_{tt} - u_{xx} + \frac{-2}{1+4|\eta|}\left(2\eta \sin(\mathrm{u}) - \sin\left(\frac{u}{2}\right)\right) = 0,$$

Table 5
The computational results for Example 5 with $k = 0.01$ by: Our method ($h = .04$) and ($h = .02$).

T	1	2	3	4	5	10
Our method	($h = .04$)					
L_∞	4.1036×10^{-5}	3.8392×10^{-5}	3.7826×10^{-5}	3.7814×10^{-5}	4.3196×10^{-5}	1.0686×10^{-4}
L_2	1.2749×10^{-4}	1.2428×10^{-4}	1.2196×10^{-4}	1.2196×10^{-4}	1.3932×10^{-4}	3.6783×10^{-4}
RMS	2.5498×10^{-5}	2.4857×10^{-5}	2.4393×10^{-5}	2.4393×10^{-5}	2.7865×10^{-5}	7.3566×10^{-5}
	($h = .02$)					
L_∞	1.0585×10^{-5}	9.9785×10^{-6}	9.8403×10^{-6}	9.8306×10^{-6}	1.1177×10^{-5}	2.7511×10^{-5}
L_2	4.6346×10^{-5}	4.5406×10^{-5}	4.4597×10^{-5}	4.4560×10^{-5}	5.0824×10^{-5}	1.3360×10^{-4}
RMS	6.5544×10^{-6}	6.4214×10^{-6}	6.3069×10^{-6}	6.3018×10^{-6}	7.1876×10^{-6}	1.8894×10^{-5}

with the exact solution

$$u(x,t) = 4\pi m \pm 4 \arctan\left(\frac{\sinh(\frac{x-ct}{\sqrt{1-c^2}})}{\cosh(R)}\right)$$

where m is an integer and the signs "+" and "−" correspond to kink and anti-kink solutions respectively. Suppose that $m = 0$, $R = 4$ and $c = 0.8$. We solve this problem for various values of h and for $k = 0.01$. The numerical results are tabulated in Table 5.

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