CRITICAL SLOPE p-ADIC L-FUNCTIONS – DRAFT

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1. Introduction

Let p be a prime number, and let $f = \sum_n a_n q^n$ denote a normalized cuspidal eigenform of weight $k+2$ on $\Gamma_0(N)$ with $p \nmid N$. If f is a p-ordinary form, then by [1, 17] we can attach a p-adic L-function to f which interpolates special values of its L-series. On the other hand, if f is non-ordinary at p, we have *two* p-adic L-functions attached to f, one for each root of $x^2 - a_p x + p^{k+1}$. These two roots correspond to the two p-stabilizations of f to level $\Gamma_0(Np)$, and, more precisely, we are attaching a p-adic L-function to each of these forms.

In the case when f is p-ordinary, one of these p-stabilizations is p-ordinary and the other has slope $k+1$ (critical slope). The methods of [1, 17] only apply to forms of slope strictly less than $k+1$, which is why in this case we only have one p-adic L-function. It is the goal of this paper to give a natural construction of p-adic L-functions of critical slope forms, and thus to construct the "missing" p-adic L-function in the ordinary case.

The basic starting point of our method is the theory of overconvergent modular symbols developed by the second author. Let $\mathcal{D}_k$ denote the space of locally analytic distributions on $\mathbb{Z}_p$ endowed with the weight k action. This distribution space admits a surjective map to $\mathcal{P}_k := \mathrm{Sym}^k(\mathbb{Q}_p^2)$, and thus we get an induced Hecke-equivariant map

$$H^1_c(\Gamma, \mathcal{D}_k) \longrightarrow H^1_c(\Gamma, \mathcal{P}_k)$$

which we refer to as the *specialization* map. Here $\Gamma = \Gamma_1(N) \cap \Gamma_0(p)$. The target of this map is finite-dimensional while the source is certainly infinite-dimensional. Nonetheless, we have the following comparison theorem of the second author (see [16]).

Theorem 1.1. *We have*

$$H^1_c(\Gamma, \mathcal{D}_k)^{(<k+1)} \xrightarrow{\cong} H^1_c(\Gamma, \mathcal{P}_k)^{(<k+1)}.$$

That is, the specialization map is an isomorphism on the subspace where U_p acts with slope strictly less than $k+1$.

This comparison theorem should be viewed as the analogue of Coleman's theorem on small slope overconvergent modular forms being classical.

Let f now be a normalized cuspidal eigenform of level Γ, which we assume for simplicity has its Fourier coefficients in $\mathbb{Z}_p$. (Note that we are certainly allowing the possibility that f is old at p.) Consider the modular symbol $\phi_f \in H^1_c(\Gamma, \mathcal{P}_k)$ attached to f. If f is of non-critical slope, by Theorem 1.1, ϕ_f lifts uniquely to a Hecke-eigensymbol $\Phi_f \in H^1_c(\Gamma, \mathcal{D}_k)$. Moreover, if we "integrate" this symbol from ∞ to 0, the resulting distribution we get is exactly the p-adic L-function of f (see [16] and [13, Prop 6.3]).

Thus, to define critical slope p-adic L-functions, it is natural to examine the specialization map on the slope $k+1$ subspace. The following theorem is proven in this paper.

Theorem 1.2. *Let f be an eigenform in $S_{k+2}(\Gamma, \overline{\mathbb{Q}}_p)$ with slope $k+1$. Then*

$$H^1_c(\Gamma, \mathcal{D}_k)_{(f)} \xrightarrow{\sim} H^1_c(\Gamma, \mathcal{P}_k)_{(f)}$$

is an isomorphism if and only if $f \notin \mathrm{im}(\theta^{k+1})$.

Here, the subscript (f) denotes the generalized eigenspace on which the Hecke-algebra acts via the eigenvalues of f, and $\theta^{k+1} : M^{\dagger}_{-k}(\Gamma, \overline{\mathbb{Q}}_p) \longrightarrow M^{\dagger}_{k+2}(\Gamma, \overline{\mathbb{Q}}_p)$ denotes the p-adic θ-operator which acts on q-expansions by $(q\frac{d}{dq})^{k+1}$.

In particular, if f is not in the image of θ^{k+1}, using the same arguments as above, we can associate a unique Hecke-eigensymbol $\Phi_f \in H^1_c(\Gamma, \mathcal{D}_k)$ which specializes to ϕ_f. We then simply *define* the p-adic L-function of f to be the value of Φ_f when integrated from ∞ to 0.

Many examples of these critical slope p-adic L-functions are computed in [13]. Their zeroes appear to contain interesting patterns which encode the classical μ- and λ-invariants of the corresponding ordinary p-adic L-function.

The case when f is in the image of θ^{k+1} remains an interesting one. In this situation, we know that there is some non-zero Hecke-eigensymbol $\Phi_f \in H^1_c(\Gamma, \mathcal{D}_k)$ in the kernel of specialization with the same system of eigenvales as f. In fact, there are two such symbols, one in each of the eigenspaces of complex conjugation. Numerical experiments from [13] suggest that there is a unique such symbol in each eigenspace; if this is true, we could define a p-adic L-function, at least up to scaling. However, we have been unable to establish this claim even in a particular case.

In the course of the paper, we actually first prove a weaker version of the above theorem (see Theorem 6.7). The conclusion of this weaker theorem is the same, but a hypothesis stronger than $f \notin \mathrm{im}(\theta^{k+1})$ is assumed. We chose to include both proofs of these theorems in this paper, as the proof of the weaker theorem is self-contained, while the proof of the stronger theorem relies on the results of [15]. Moreover, the proof of the weaker theorem may be more easily generalized to a wider class of reductive groups. Lastly, the fact that the weaker theorem follows from the stronger theorem is non-trivial as it relies upon the existence of companion forms.

We now sketch a proof of the non-critical slope comparison theorem, and then sketch the two proofs dealing with the critical slope subspace.

Using the Riesz decomposition[1] (review in section 4), it follows that the specialization map restricted to the slope less than $k+1$ subspace

$$H^1_c(\Gamma, \mathcal{D}_k)^{(<k+1)} \overset{\rho_k^*}{\twoheadrightarrow} H^1_c(\Gamma, \mathcal{P}_k)^{(<k+1)}$$

is surjective. Moreover, the kernel of the specialization map can be identified with

$$H^1_c(\Gamma, \mathcal{D}_{-2-k})(k+1);$$

here, we are twisting by the $(k+1)$-st power of the determinant, and thus the Hecke operator T_n acts by $n^{k+1}T_n$ (see section 3.3). In particular, U_p acts with slope at

[1]For the classical Riesz decomposition, one should be working Banach spaces. However, $\mathcal{D}_k$ is a Frechet space and not a Banach spaces. This is the primary reason we introduce in this paper the larger space $\mathbf{D}_k$ of rigid analytic distributions.

least $k+1$ on this space. It follows that the specialization map restricted to the slope less than $k+1$ subspace is also injective, proving the comparison theorem.

To deal with the critical slope case, assume that f is an eigenform of slope $k+1$. We wish to show (under some hypotheses) that the f-isotypic subspace

$$\left(H^1_c(\Gamma, \mathcal{D}_{-2-k})(k+1)\right)_{(f)}$$

vanishes. Assume the contrary, and let Ψ denote some non-zero Hecke-eigensymbol in this subspace. Let Ψ_0 denote the untwisted symbol in $H^1_c(\Gamma, \mathcal{D}_{-2-k})$. Since Ψ has slope $k+1$, the symbol Ψ_0 has slope 0.

Approach 1: By scaling, we may assume that Ψ_0 takes values in $\mathcal{D}^0_{-2-k}$, the unit ball of $\mathcal{D}_{-2-k}$. In section 3.4, we introduce a descending filtration $\operatorname{Fil}^r \mathcal{D}^0_m$ for any *negative* weight m, such that any normalized eigensymbol taking values in $\operatorname{Fil}^r \mathcal{D}^0_m$ has slope bounded below by r. Specifically, for $r=1$, the subspace $\operatorname{Fil}^1 \mathcal{D}^0_m$ consists of distributions whose total measure is divisible by p. Since Ψ_0 has slope zero, its image $\overline{\Psi}_0$ in

$$H^1_c(\Gamma, \mathcal{D}^0_{-2-k}/\operatorname{Fil}^1 \mathcal{D}^0_{-2-k})$$

is a non-zero eigensymbol. Moreover, the quotient $\mathcal{D}^0_{-2-k}/\operatorname{Fil}^1 \mathcal{D}^0_{-2-k}$ is simply $\mathbb{F}_p$ with a certain non-trivial matrix action. Thus, this non-zero eigensymbol is related to weight 2 modular forms with some nebentype. One computes and finds that the existence of such an eigensymbol implies that system of eigenvalues of $\overline{\theta} f$ occurs in $S_2(\Gamma_1(Np), \omega^{k+2}, \overline{\mathbb{F}}_p)$. Here $\overline{\theta}$ is the classical θ-operator on mod p modular forms. Thus, we deduce that the specialization map restricted to the f-isotypic subspace is an isomorphism as long as the system of eigenvalues of $\overline{\theta} f$ does not occur in the above space of modular forms.

Approach 2: By [15], systems of Hecke-eigenvalues in $H^1_c(\Gamma, \mathcal{D}_k)$ are in one-to-one correspondence with systems of Hecke-eigenvalues in $M^\dagger_{k+2}(\Gamma, \overline{\mathbb{Q}}_p)$ for any p-adic weight k. In particular, the eigensymbol Ψ_0 corresponds to some overconvergent modular form $g \in M^\dagger_{-k}(\Gamma, \overline{\mathbb{Q}}_p)$ such that the system of eigenvalues of $\theta^{k+1} g$ is the same as the system of eigenvalues of f. By looking at q-expansions, we see that $f = \theta^{k+1} g$, and deduce that $f \in \operatorname{im}(\theta^{k+1})$. Thus, the specialization map restricted to the f-isotypic subspace is an isomorphism as long as f is not in the image of θ^{k+1}.

We note that if f is in the image of θ^{k+1}, using companion forms, one can check that the system of eigenvalues attached to $\overline{\theta} f$ does occur in $S_2(\Gamma_1(Np), \omega^{k+2}, \overline{\mathbb{F}}_p)$ (see Remark 7.4). Thus, the second approach yields a stronger theorem than the first approach.

We close this introduction by mentioning two other approaches to constructing critical slope p-adic L-functions. The first combines Perrin-Riou's dual exponential map with the existence of Kato's zeta element (see [11, 3.2.2] and [4, Remark 4.12]). The second uses Emerton's theory of $\widehat{H}^1$ (see [6]). At the end of section 8, we recall these two methods and discuss the role played by the condition $f \notin \operatorname{im}(\theta^{k+1})$.

The format of the paper is as follows: in the following section we review the basic definitions of modular symbols. In the third section, we introduce the relevant spaces of distributions, and the filtration on them described above. In the fourth section, we review the Riesz decomposition. In the fifth section, we prove the non-critical comparison theorem. In the sixth section, we prove a critical slope comparison theorem by the first approach outlined above, and in the seventh section, we follow the second approach. In the final section, we discuss p-adic L-functions.

2. Modular Symbols

2.1. **Basic definitions.** Let p be a prime and N an integer prime to p. Set $\Gamma = \Gamma_1(N) \cap \Gamma_0(p)$ and

$$S_0(p) = \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \in M_2(\mathbb{Z}) \text{ such that } p \nmid a, p \mid c \text{ and } ad - bc \neq 0 \right\}.$$

If M is a right $\mathbb{Z}[S_0(p)]$-module, let $\widetilde{M}$ denote the associated locally constant sheaf on the open modular curve Y_Γ, and let

$$H^1_c(\Gamma, M) := H^1_c(Y_\Gamma, \widetilde{M})$$

denote the space of one-dimensional compactly supported cohomology with coefficients in $\widetilde{M}$.

The space $H^1_c(\Gamma, M)$ admits a description in terms of modular symbols. Indeed, let $\Delta_0 := \mathrm{Div}^0(\mathbb{P}^1(\mathbb{Q}))$ denote the set of degree 0 divisors on $\mathbb{P}^1(\mathbb{Q})$ endowed with a left $\mathrm{GL}_2(\mathbb{Q})$-action via linear fraction transformations. The space $\mathrm{Hom}(\Delta_0, M)$ admits a right action of $S_0(p)$ by

$$(\phi \big| \gamma)(D) = \phi(\gamma D) \big| \gamma$$

where $D \in \Delta_0$ and $\gamma \in S_0(p)$. By [2, Proposition 4.2], there exists a canonical isomorphism

$$H^1_c(\Gamma, M) \cong \mathrm{Hom}_\Gamma(\Delta_0, M) \tag{1}$$

where the target of the map is the set of Γ-invariant homomorphisms.

If $\mathcal{H}$ denotes the free polynomial algebra over $\mathbb{Z}$ generated by the Hecke operators T_ℓ for $\ell \nmid Np$ and U_q for $q | Np$, then both $H^1_c(\Gamma, M)$ and $\mathrm{Hom}_\Gamma(\Delta_0, M)$ are naturally $\mathcal{H}$-module. For instance, the U_p-operator on $\mathrm{Hom}_\Gamma(\Delta_0, M)$ can be explicitly realized by

$$\phi \big| U_p = \sum_{a=0}^{p-1} \phi \big| \left(\begin{smallmatrix} 1 & a \\ 0 & p \end{smallmatrix} \right).$$

The isomorphism (1) is Hecke-equivariant, and henceforth, we will tacitly identify these two spaces. Also, as the congruence subgroup Γ is fixed throughout the paper, we simply write $H^1_c(M)$ for $H^1_c(\Gamma, M)$ and refer to it as the space of M-valued modular symbols (of level Γ).

2.2. **Miscellany.** If M is a Banach space and if Γ acts by unitary operators on M, then $H^1_c(M)$ is also a Banach space under the norm

$$||\Phi|| = \sup_{D \in \Delta_0} ||\Phi(D)||.$$

This supreme exists as $||\Phi(D)||$ is constant on each of the finitely many Γ-orbits of Δ_0.

For $r \in \mathbb{Z}$, let $M(r)$ denote the $S_0(p)$-module whose underlying set is M and whose $S_0(p)$-action is twisted by the r^{th} power of the determinant. For an $\mathcal{H}$-module X, set $X(r)$ to be the $\mathcal{H}$-module whose underlying set is X and whose Hecke action by T_ℓ (resp. U_q) is given by ℓT_ℓ (resp. qU_q). We then have the tautological isomorphism

$$H^1_c(M(r)) \cong H^1_c(M)(r).$$

We also mention that the matrix $\iota = \left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ normalizes Γ, and thus induces an involution on $H^1_c(M)$; if 2 acts invertibly on M, this involution gives a decomposition

$$H^1_c(M) = H^1_c(M)^+ \oplus H^1_c(M)^-$$

into ± 1-eigenspaces for ι.

3. Distributions

3.1. **Definitions.** For each $r \in |\mathbb{C}_p^\times|$, let

$$B[\mathbb{Z}_p, r] = \{z \in \mathbb{C}_p \mid \text{there exists some } a \in \mathbb{Z}_p \text{ with } |z - a| \le r\}.$$

Then $B[\mathbb{Z}_p, r]$ is the $\mathbb{C}_p$-points of a $\mathbb{Q}_p$-affinoid variety. For example, if $r \ge 1$ then $B[\mathbb{Z}_p, r]$ is the closed disc in $\mathbb{C}_p$ of radius r around 0. If $r = \frac{1}{p}$ then $B[\mathbb{Z}_p, r]$ is the disjoint union of the p discs of radius $\frac{1}{p}$ around the points $0, 1, \ldots, p-1$.

Let $\mathbf{A}[r]$ denote the $\mathbb{Q}_p$-Banach algebra of $\mathbb{Q}_p$-affinoid functions on $B[\mathbb{Z}_p, r]$. For example, if $r \ge 1$

$$\mathbf{A}[r] = \left\{ f(z) = \sum_{n=0}^{\infty} a_n z^n \in \mathbb{Q}_p[[z]] \text{ such that } \{|a_n| \cdot r^n\} \to 0 \right\}.$$

The norm on $\mathbf{A}[r]$ is given by the supremum norm. That is, if $f \in \mathbf{A}[r]$ then

$$||f||_r = \sup_{z \in B[\mathbb{Z}_p, r]} |f(z)|_p.$$

For $r_1 > r_2$, there is a natural restriction map $\mathbf{A}[r_2] \to \mathbf{A}[r_1]$ that is injective, completely continuous and has dense image. We define

$$\mathcal{A} = \varinjlim_{s>0} \mathbf{A}[s]$$

which is naturally identified with the space of locally analytic $\mathbb{Q}_p$-valued functions on $\mathbb{Z}_p$. Its topology is given by the inductive limit topology. Note that there are natural continuous inclusions

$$\mathbf{A}[r] \hookrightarrow \mathcal{A}$$

with dense image.

Set $\mathbf{D}[r]$ (resp. $\mathcal{D}$) equal to the space of continuous $\mathbb{Q}_p$-linear functionals on $\mathbf{A}[r]$ (resp. $\mathcal{A}$) endowed with the strong topology. Note that

$$\mathcal{D} = \varprojlim_{s>0} \mathbf{D}[s]$$

with the projective limit topology.

We have that $\mathbf{D}[r]$ is a Banach space under the norm

$$||\mu||_r = \sup_{\substack{f \in \mathbf{A}[r] \\ f \neq 0}} \frac{|\mu(f)|}{||f||_r}$$

while $\mathcal{D}$ has its topology defined by the family of norms $\{||\cdot||_s\}$ for $s \in |\mathbb{C}_p^\times|$ with $s > 0$. By duality, we have continuous linear injective maps

$$\mathcal{D} \hookrightarrow \mathbf{D}[r].$$

When $r = 1$, we simply write $\mathbf{A}$ for $\mathbf{A}[1]$ and $\mathbf{D}$ for $\mathbf{D}[1]$.

3.2. **The action of $\Sigma_0(p)$.** Let

$$\Sigma_0(p) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{Z}_p) \text{ such that } p \nmid a,\ p \mid c \text{ and } ad - bc \neq 0 \right\}$$

be the p-adic version of $S_0(p)$. Fix an integer k, and let $\Sigma_0(p)$ act on $\mathbf{A}[r]$ on the left by

$$(\gamma \cdot_k f)(z) = (a + cz)^k \cdot f\left(\frac{b + dz}{a + cz}\right)$$

where $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Sigma_0(p)$ and $f \in \mathbf{A}[r]$. Then $\Sigma_0(p)$ acts on $\mathbf{D}[r]$ on the right by

$$(\mu\big|_k\gamma)(f) = \mu(\gamma \cdot_k f).$$

where $\mu \in \mathbf{D}[r]$. These two actions then induce actions on $\mathcal{A}$ and $\mathcal{D}$.

To emphasis the role of k in this action, we will include it as a subscript such as $\mathbf{A}_k[r]$. When $r = 1$, we write $\mathbf{A}_k$ and $\mathbf{D}_k$ for $\mathbf{A}_k[1]$ and $\mathbf{D}_k[1]$.

Remark 3.1. We note that the space $\mathcal{A}_k$ can be viewed as a "locally analytic induction". Indeed, let N^{opp} (resp. T) denote the subgroup of lower triangular (resp. diagonal) matrices in $\mathrm{GL}_2(\mathbb{Q}_p)$. Let I denote the Iwahori subgroup of $\mathrm{GL}_2(\mathbb{Z}_p)$, and let X denote its image in $N^{\mathrm{opp}}\backslash \mathrm{GL}_2(\mathbb{Q}_p)$. Then X inherits a natural right action by $\Sigma_0(p)$. Also, $\mathbb{Z}_p$ injects into X by sending z to $\left(\begin{smallmatrix} 1 & z \\ 0 & 1 \end{smallmatrix}\right)$.

Let λ denote the character of T that maps $\left(\begin{smallmatrix} a & 0 \\ 0 & d \end{smallmatrix}\right)$ onto a^k, and set

$$\mathcal{A}_\lambda := \{f : X \to \mathbb{Q}_p : f \text{ is locally analytic and } f(tx) = \lambda(t)f(x) \text{ for } t \in T\}.$$

Here, a function on X is locally analytic if its restriction to the image of $\mathbb{Z}_p$ in X is locally analytic. One then verifies that restriction to $\mathbb{Z}_p$ induces a $\Sigma_0(p)$-isomorphism between $\mathcal{A}_\lambda$ and $\mathcal{A}_k$. Similarly, $\mathbf{A}[r]$ can be viewed as a rigid analytic induction.

3.3. **Finite-dimensional quotients.** Assume for this section that k is a non-negative integer. Let $\mathcal{P}_k$ denote the space of homogenous polynomials of degree k in two variables over $\mathbb{Q}_p$. We view $\mathcal{P}_k$ as a right $\mathrm{GL}_2(\mathbb{Q}_p)$-module by

$$(P\big|\sigma)(X,Y) = P((X,Y)\cdot\sigma^*) = P(dX - cY, -bX + aY)$$

where $\sigma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ and $\sigma^* = \left(\begin{smallmatrix} d & -b \\ -c & a \end{smallmatrix}\right)$. (We note that $\mathcal{P}_k$ is isomorphic as a representation space to $\mathrm{Sym}^k(\mathbb{Q}_p^2)$.)

If D_k equals either $\mathcal{D}_k$ or $\mathbf{D}_k[r]$, we have a $\Sigma_0(p)$-equivariant surjection

$$D_k \xrightarrow{\rho_k} \mathcal{P}_k$$
$$\mu \mapsto \int (zX - Y)^k \, d\mu.$$

Here the integration is done coefficient by coefficient.

It is possible to explicitly describe the kernel of ρ_k. Taking k-derivatives yields a $\Sigma_0(p)$-equivariant map

$$A_k \longrightarrow A_{-2-k}(k+1)$$

where A_k equals either $\mathcal{A}_k$ or $\mathbf{A}_k[r]$. (Here the action on the target of the map is twisted by the $(k+1)^{\mathrm{st}}$ power of the determinant.) A direct computation shows that the dual of this map fits into the short exact sequence

$$0 \longrightarrow D_{-2-k}(k+1) \longrightarrow D_k \xrightarrow{\rho_k} \mathcal{P}_k \longrightarrow 0. \tag{2}$$

Moreover, when $D_k = \mathbf{D}_k[r]$, this is an exact sequence of Banach spaces.

3.4. **A filtration on $\mathbf{D}_{-k}$.** Let k be a positive integer. Set

$$\mathbf{D}^0_{-k} = \left\{\mu \in \mathbf{D}_{-k} \text{ with } \mu(z^j) \in \mathbb{Z}_p \text{ for all } j \geq 0\right\}$$

which is the unit ball of $\mathbf{D}_{-k}$, and consider the decreasing filtration of $\mathbf{D}^0_{-k}$ given by

$$\mathrm{Fil}^r \mathbf{D}^0_{-k} = \left\{\mu \in \mathbf{D}^0_{-k} \text{ such that } \mu(z^j) \in p^{r-j}\mathbb{Z}_p\right\}.$$

Proposition 3.2. *The filtration*

$$\mathbf{D}^0_{-k} \supset \mathrm{Fil}^1 \mathbf{D}^0_{-k} \supset \cdots \supset \mathrm{Fil}^r \mathbf{D}^0_{-k} \supset \cdots$$

is stable under the weight $-k$ action of $\Sigma_0(p)$.

Proof. For $\mu \in \mathrm{Fil}^r \mathbf{D}^0_{-k}$, we have

$$(\mu\big|\gamma)(z^j) = \mu\left((a+cz)^{-k-j}(b+dz)^j\right).$$

Note that

$$(a+cz)^{-k-j}(b+dz)^j = a^{-k-j}(1+a^{-1}cz)^{-k-j}(b+dz)^j =$$

$$a^{-k-j}\left(\sum_{i=0}^{\infty}\binom{-k-j}{i}a^{-i}c^iz^i\right)\left(\sum_{i=0}^{j}\binom{j}{i}b^{j-i}d^iz^i\right) = \sum_{i=0}^{\infty}a_iz^i,$$

since $-k-j<0$. Moreover, since $a \in \mathbb{Z}_p^\times$ and $c \in p\mathbb{Z}_p$, a direct computation shows that $\mathrm{ord}_p(a_i) \geq i-j$. Substituting back in yields

$$(\mu\big|\gamma)(z^j) = \sum_{i=0}^{\infty} a_i\mu(z^i).$$

Thus, to show that $\mu\big|\gamma \in \mathrm{Fil}^r \mathbf{D}^0_{-k}$, we need to show that $\sum_{i=0}^{\infty} a_i\mu(z^i)$ is divisible by p^{r-j} for $j<r$.

To see this, note that $\mu(z^i)$ is divisible by p^{r-i} as $\mu \in \mathrm{Fil}^r \mathbf{D}^0_{-k}$. For i between 0 and j, $\mu(z^i)$ is thus divisible by p^{r-j} . For i between j and r, $a_i\mu(z^i)$ is divisible by $p^{i-j}\cdot p^{r-i} = p^{r-j}$. Thus, $(\mu\big|\gamma)(z^j)$ is divisible by p^{r-j}, and $\mu\big|\gamma$ is in $\mathrm{Fil}^r \mathbf{D}^0_{-k}$. $\square$

Remark 3.3. Note that the subset

$$\left\{\mu \in \mathbf{D}^0_{-k} \text{ such that } \mu(z^j) = 0 \text{ for } j<r\right\}$$

is not preserved by the weight k action of $\Sigma_0(p)$. Indeed, the $\Sigma_0(p)$-closure of this space is $\mathrm{Fil}^r \mathbf{D}^0_{-k}$.

Remark 3.4. In this paper, we only make use of the first step of this filtration. In [13], we make more extensive use of an analogous filtration on $\mathbf{D}_k$ for $k \geq 0$ in order to do explicit computations with overconvergent modular symbols.

The following lemma will be useful in our study of critical slope modular symbols. In what follows, $\mathbb{F}_p(a^j)$ denotes the $\Sigma_0(p)$-module $\mathbb{F}_p$ on which $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ acts by a^j.

Lemma 3.5. *We have*

$$\mathbf{D}^0_{-k}/\mathrm{Fil}^1 \mathbf{D}^0_{-k} \cong \mathbb{F}_p(a^{-k})$$

as $\Sigma_0(p)$-modules.

Proof. As a group, $\mathbf{D}^0_{-k}/\operatorname{Fil}^1\mathbf{D}^0_{-k}$ is isomorphic to $\mathbb{F}_p$ and is generated by any μ such that $\mu(\mathbf{1}) \neq 0$. To see the $\Sigma_0(p)$-action, note that

$$\begin{aligned}(\mu\big|\gamma)(\mathbf{1}) &= \mu\left((a+cz)^{-k}\right) = a^{-k}\mu\left((1+a^{-1}cz)^{-k}\right)\\ &= a^{-k}\mu\left(\sum_{i=0}^{\infty}\binom{-k}{i}a^{-i}c^iz^i\right)\\ &\equiv a^{-k}\mu(\mathbf{1}) \pmod p\end{aligned}$$

which proves the claim. □

4. Slope decompositions

Let X be a vector space over $\mathbb{Q}_p$ equipped with an endomorphism U. If $h \in \mathbb{R}$, we define $X^{(<h)}$ to be the subspace on which U acts with slope less than h. In this section, we observe that the association of X to $X^{(<h)}$ is well-behaved when X is a Banach space and U is completely continuous.

Let $\mathcal{C}$ denote the category of Banach spaces over $\mathbb{Q}_p$ which are equipped with a completely continuous operator U. The maps of the category are U-equivariant linear maps.

Theorem 4.1 (Riesz decomposition)**.** *Let $X \in \mathcal{C}$. For each irreducible polynomial Q in $\mathbb{Q}_p[T]$, the space X decomposes into a direct sum of two closed subspaces preserved by U:*

$$X \cong X(Q) \oplus X'(Q)$$

such that $Q(U)$ is nilpotent on $X(Q)$ and invertible on $X'(Q)$. Moreover, $X(Q)$ is finite-dimensional over $\mathbb{Q}_p$.

Proof. See [14, pg. 82 - Remarques 3]. □

Lemma 4.2. *With X and Q as above, we have*

(1) $X(Q) = \bigcup_n \ker(Q(U)^n)$;
(2) $X'(Q) = \bigcap_n \operatorname{im}(Q(U)^n)$.

In particular, the Riesz decomposition is unique.

Proof. For the first part, $X(Q) \subseteq \bigcup_n \ker(Q(U)^n)$ as $Q(U)$ is nilpotent on $X(Q)$. Moreover, the projection of $\bigcup_n \ker(Q(U)^n)$ to $X'(Q)$ is zero since $Q(U)$ acts invertibly on $X'(Q)$. Thus, the above containment is an equality.

For the second part, since $Q(U)$ acts invertibly on $X'(Q)$, we have $X'(Q) \subseteq \bigcap_n \operatorname{im}(Q(U)^n)$. Since $X(Q)$ is finite-dimensional, $Q(U)^n$ annihilates $X(Q)$ for some n. Applying $Q(U)^n$ to the Riesz decomposition of X then yields $\operatorname{im}(Q(U)^n) = X'(Q)$, proving the lemma. □

Corollary 4.3. *Let X and Y be in $\mathcal{C}$ with $f : X \to Y$ a U-equivariant linear map. Then $f(X(Q)) \subseteq Y(Q)$ and $f(X'(Q)) \subseteq Y'(Q)$.*

Proof. This follows immediately from the previous lemma as both $\bigcup_n \ker(Q(U)^n)$ and $\bigcap_n \operatorname{im}(Q(U)^n)$ have this property. □

Corollary 4.4. *Let*

$$0 \to X_1 \to X_2 \to X_3 \to 0$$

be an exact sequence in $\mathcal{C}$. Then for any irreducible polynomial $Q \in \mathbb{Q}_p[T]$,

$$0 \to X_1(Q) \to X_2(Q) \to X_3(Q) \to 0$$

is exact.

Proof. Exactness is clear except for the surjection $X_2(Q) \to X_3(Q)$. Take $x_3 \in X_3(Q)$, and let $x_2 + x_2'$ be some preimage with $x_2 \in X_2(Q)$ and $x_2' \in X_2'(Q)$. As the image of x_2' lands in $X_3'(Q)$, we have that x_2 maps to x_3. □

Definition 4.5. For $X \in \mathcal{C}$ and $h \in \mathbb{R}$, set

$$X^{(<h)} := \bigoplus_{v(Q)<h} X(Q)$$

where Q runs over all monic irreducible polynomials of $\mathbb{Q}_p[T]$, and $v(Q)$ denotes the valuation of any root of Q. We define $X^{(=h)}$ and $X^{(\leq h)}$ similarly.

Lemma 4.6. *For $X \in \mathcal{C}$ and $h \in \mathbb{R}$, we have $X^{(<h)}$, $X^{(=h)}$ and $X^{(\leq h)}$ are all finite-dimensional vector spaces.*

Proof. Let P_U be the characteristic power series of U acting on X. Then $X(Q) \neq 0$ if and only if the reciprocal of a root of Q is a root of P_U. Since each $X(Q)$ is finite-dimensional and P_U has only finitely roots with slope less than or equal to h, the lemma follows. □

Proposition 4.7. *If*

$$0 \to X_1 \to X_2 \to X_3 \to 0$$

is an exact sequence in $\mathcal{C}$, then

$$0 \to X_1^{(<h)} \to X_2^{(<h)} \to X_3^{(<h)} \to 0$$

is exact. The same assertion is true for $X_i^{(=h)}$ and $X_i^{(\leq h)}$.

Proof. This follows from Corollary 4.4 as $X_i^{(<h)}$ is a direct sum over various $X(Q)$. □

5. Comparison theorem

As the map $\rho_k : D_k \to \mathcal{P}_k$ defined in section 3.3 is $\Sigma_0(p)$-equivariant, it induces a map on cohomology

$$H^1_c(D_k) \xrightarrow{\rho_k^*} H^1_c(\mathcal{P}_k)$$

which we call the *specialization* map. (Recall that D_k equals either $\mathcal{D}_k$ or $\mathbf{D}_k[r]$.)

In this section, we sketch a proof of a theorem of second author which states that ρ_k^* is a isomorphism on the subspace of slope strictly less than $k+1$.

5.1. Some lemmas.

Lemma 5.1. *We have*

$$0 \to H^1_c(\mathbf{D}_{-2-k})(k+1) \to H^1_c(\mathbf{D}_k) \xrightarrow{\rho_k^*} H^1_c(\mathcal{P}_k) \to 0$$

is an exact sequence of Banach spaces.

Proof. The sequence

$$0 \to \mathbf{D}_{-2-k}(k+1) \to \mathbf{D}_k \xrightarrow{\rho_k} \mathcal{P}_k \to 0$$

induces

$$H^1_c(\mathbf{D}_{-2-k}(k+1)) \to H^1_c(\mathbf{D}_k) \to H^1_c(\mathcal{P}_k) \to H^2_c(\mathbf{D}_{-2-k}(k+1)).$$

Using the description of H^1_c in terms of modular symbols, we see that the first map is injective. By Poincare duality, $H^2_c(\mathbf{D}_{-2-k})$ is dual to $H^0(\mathbf{D}_{-2-k})$, and is thus isomorphic to the Γ-coinvariants of $\mathbf{D}_{-2-k}$. As these coinvariants are zero, the lemma follows. $\square$

Lemma 5.2. *For any $h \in \mathbb{R}$, the natural map*

$$H^1_c(\mathcal{D}_k)^{(<h)} \xrightarrow{\sim} H^1_c(\mathbf{D}_k)^{(<h)}$$

is an isomorphism.

Proof. Viewing these cohomology groups in terms of modular symbols, it is immediate that this map is injective. Thus, we only need to show that any $\Phi \in H^1_c(\mathbf{D}_k)^{(<h)}$ actually takes values in $\mathcal{D}_k$.

Since U_p acts invertibly on $H^1_c(\mathbf{D}_k)^{(<h)}$, for each $n \geq 1$, $\Phi = \Psi \big| U_p^n$ for some $\Psi \in H^1_c(\mathbf{D}_k)$. For $D \in \mathrm{Div}^0(\mathbb{P}^1(\mathbb{Q}))$ and $g \in \mathbf{A}_k[1]$, we have

$$\begin{aligned}\Phi(D)(g) = (\Psi|U_p^n)(D)(g) &= \sum_{a=0}^{p^n-1} \left(\Psi\big| \left(\begin{smallmatrix} 1 & a \\ 0 & p^n \end{smallmatrix}\right)\right)(D)(g) \\ &= \sum_{a=0}^{p^n-1} \left(\Psi\left(\left(\begin{smallmatrix} 1 & a \\ 0 & p^n \end{smallmatrix}\right) D\right) \big| \left(\begin{smallmatrix} 1 & a \\ 0 & p^n \end{smallmatrix}\right)\right)(g) = \sum_{a=0}^{p^n-1} \Psi\left(\left(\begin{smallmatrix} 1 & a \\ 0 & p^n \end{smallmatrix}\right) D\right)\left(\left(\begin{smallmatrix} 1 & a \\ 0 & p^n \end{smallmatrix}\right) g\right).\end{aligned}$$

If h is any element of $\mathbf{A}_k[p^{-n}]$, then $\left(\begin{smallmatrix} 1 & a \\ 0 & p^n \end{smallmatrix}\right) h$ extends naturally to an element of $\mathbf{A}_k[1]$. Thus, the above computation shows that $\Phi(D)$ extends to $\mathbf{D}_k[p^{-n}]$ for all n, and thus to $\mathcal{D}_k$. $\square$

5.2. **Proof of comparison theorem.**

Theorem 5.3 (Stevens)**.** *We have*

$$H^1_c(\mathcal{D}_k)^{(<k+1)} \xrightarrow{\sim} H^1_c(\mathcal{P}_k)^{(<k+1)}$$

is an isomorphism.

Proof. From Lemma 5.1, we have that

$$0 \to H^1_c(\mathbf{D}_{-2-k})(k+1) \to H^1_c(\mathbf{D}_k) \xrightarrow{\rho_k^*} H^1_c(\mathcal{P}_k) \to 0$$

is an exact sequences of Banach spaces. By Proposition 4.7, passing to the slope less than $k+1$ subspace gives an exact sequence

$$0 \to H^1_c(\mathbf{D}_{-2-k})(k+1)^{(<k+1)} \to H^1_c(\mathbf{D}_k)^{(<k+1)} \to H^1_c(\mathcal{P}_k)^{(<k+1)} \to 0.$$

As U_p preserves the unit ball of $H^1_c(\mathbf{D}_{-2-k})$, all of its eigenvalues on this space are p-adic integers. Thus, $H^1_c(\mathbf{D}_{-2-k})(k+1)^{(<k+1)} = 0$, and the theorem follows from Lemma 5.2. $\square$

Remark 5.4. This theorem can be viewed as an overconvergent modular symbol version of Coleman's theorem that non-critical overconvergent modular forms are classical.

6. The critical slope subspace I

In this section, we study the restriction of the specialization map to the subspace where U_p acts with slope equal to $k+1$.

6.1. **Some lemmas on filtrations.** Let k be a positive integer.

Lemma 6.1. *If $\mu \in \operatorname{Fil}^r \mathbf{D}^0_{-k}$, then $\mu\big|\left(\begin{smallmatrix} 1 & a \\ 0 & p \end{smallmatrix}\right) \in p^r \mathbf{D}^0_{-k}$.*

Proof. We have that

$$(\mu\big|\left(\begin{smallmatrix} 1 & a \\ 0 & p \end{smallmatrix}\right))(z^j) = \mu\left((a+pz)^j\right) = \sum_{i=0}^{j} \binom{j}{i} a^{j-i} p^i \mu(z^i)$$

which is divisible by p^r as $\mu(z^i) \in p^{r-i}\mathbb{Z}_p$. □

Lemma 6.2. *We have that*

$$H^1_c(\operatorname{Fil}^r \mathbf{D}^0_{-k})\big|U_p \subseteq p^r H^1_c(\mathbf{D}^0_{-k}).$$

Proof. For $\Phi \in H^1_c(\operatorname{Fil}^r \mathbf{D}^0_{-k})$, we have

$$(\Phi\big|U_p)(D) = \sum_{a=0}^{p-1} \Phi(\left(\begin{smallmatrix} 1 & a \\ 0 & p \end{smallmatrix}\right)D)\big|\left(\begin{smallmatrix} 1 & a \\ 0 & p \end{smallmatrix}\right),$$

which, by Lemma 6.1, is divisible by p^r as $\Phi(\left(\begin{smallmatrix} 1 & a \\ 0 & p \end{smallmatrix}\right)D) \in \operatorname{Fil}^r \mathbf{D}^0_{-k}$. □

Lemma 6.3. *If $\Phi \in H^1_c(\mathbf{D}^0_{-k})$ is a U_p-eigensymbol with slope h and $||\Phi|| = 1$, then the image of Φ in $H^1_c(\mathbf{D}^0_{-k}/\operatorname{Fil}^r \mathbf{D}^0_{-k})$ is non-zero for $r > h$.*

Proof. This lemma follows from Lemmas 6.1 and 6.2. □

6.2. **Some linear algebra.** Recall that $\mathcal{H}$ denotes the free polynomial algebra over $\mathbb{Z}$ generated by the Hecke operators T_ℓ for $\ell \nmid Np$ and U_q for $q|Np$. We define a *system of eigenvalues* of $\mathcal{H}$ over a ring R to be a homomorphism $\eta : \mathcal{H} \to R$. If M is a (right) $\mathcal{H}$-module, we say that a system of eigenvalues η occurs in M, if there is some $m \in M$ such that $m\big|T = \eta(T)m$ for all $T \in \mathcal{H}$.

Let V be a vector space over $\mathbb{Q}_p$ with an action of $\mathcal{H}$. For $T \in \mathcal{H}$ and $\alpha \in \overline{\mathbb{Q}}_p$, let $V_{(\alpha,T)}$ denote the pseudoeigenspace of T acting on $V \otimes \overline{\mathbb{Q}}_p$ with eigenvalue α. For η a system of eigenvalues of $\mathcal{H}$ over $\overline{\mathbb{Q}}_p$, we define

$$V_{(\eta)} = \bigcap_{T \in \mathcal{H}} V_{(\eta(T),T)},$$

the η-isotypic subspace of V. Note that a system of eigenvalues η occurs in V if and only if $V_{(\eta)} \neq 0$.

Lemma 6.4. *Let V', V, and V'' be finite-dimensional K-vector spaces equipped with an action of $\mathcal{H}$. If*

$$0 \to V' \to V \to V'' \to 0$$

is an $\mathcal{H}$-equivariant exact sequence, then

$$0 \to V'_{(\eta)} \to V_{(\eta)} \to V''_{(\eta)} \to 0$$

is exact.

Proof. By basic linear algebra (i.e. Jordan canonical form), for a fixed $T \in \mathcal{H}$, passing to the $\eta(T)$-pseudoeigenspace of T preserves exact sequences. Since $\mathcal{H}$ is a commutative algebra, forming the η-isotypic subspace is done by repeatedly restricting to pseudoeigenspaces of elements of $\mathcal{H}$. □

Remark 6.5. Passage to eigenspaces (as opposed to pseudoeigenspaces) does not preserve exact sequences. For instance, let $V = \mathbb{Q}_p e_1 \oplus \mathbb{Q}_p e_2$, and let $V'' = V/\mathbb{Q}_p e_1$. Let T act on V by the matrix $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ and on V'' trivially. The natural map $V \to V''$ is then T-equivariant, and the image of e_2 in V'' is an eigenvector. However, no preimage of this vector in V is an eigenvector.

If $f = \sum_n a_n q^n$ is an eigenform in $S_k(\Gamma, \overline{\mathbb{Q}}_p)$, then $\eta_f : \mathcal{H} \to \overline{\mathbb{Q}}_p$ given by $\eta_f(T_\ell) = a_\ell$ and $\eta_f(U_q) = a_q$ is the system of eigenvalues attached to f. To simplify notation, we write $V_{(f)}$ for $V_{(\eta(f))}$.

6.3. **A lemma relating modular symbols and modular forms.**

Lemma 6.6. *A non-Eisenstein system of eigenvalues for $\mathcal{H}$ occurs in $H^1_c(\Gamma, \overline{\mathbb{F}}_p(d^j))$ if and only if it occurs in $S_2(\Gamma_1(Np), \omega^j, \overline{\mathbb{F}}_p)$.*

Proof. The kernel and cokernel of the natural map

$$H^1_c(\Gamma, \overline{\mathbb{F}}_p(d^j)) \to H^1(\Gamma, \overline{\mathbb{F}}_p(d^j))$$

are both Eisenstein. Furthermore, restriction induces an $\mathcal{H}$-isomorphism

$$H^1(\Gamma, \overline{\mathbb{F}}_p(d^j)) \to H^1(\Gamma_1(Np), \overline{\mathbb{F}}_p)^{(\omega^j)}.$$

By [2, Lemma 2.6 and Proposition 2.5(b)], systems of eigenvalues occur in the target of this map if and only if they occur in $S_2(\Gamma_1(Np), \omega^j, \overline{\mathbb{F}}_p)$. $\square$

6.4. **Main theorem I.** Let $\mathfrak{m}$ denote the maximal ideal of the ring of integers of $\overline{\mathbb{Q}}_p$. For a normalized eigenform $f \in S_k(\Gamma, \overline{\mathbb{Q}}_p)$, let $\overline{f}$ in $S_k(\Gamma, \overline{\mathbb{F}}_p)$ denote the reduction of f modulo $\mathfrak{m}$. Let $\overline{\theta}$ denote the θ-operator defined on mod p modular forms which acts on q-expansions by $q\frac{d}{dq}$, and which increases the weight by $p+1$.

Theorem 6.7. *Let f be a normalized eigenform of $S_k(\Gamma, \overline{\mathbb{Q}}_p)$ with slope $k+1$. Assume:*

1. *the residual representation associated to f is irreducible;*
2. *the system of eigenvalues attached to $\overline{\theta f}$ does not occur in $S_2(\Gamma_1(Np), \omega^{k+2}, \overline{\mathbb{F}}_p)$.*

Then,

$$H^1_c(\Gamma, \mathcal{D}_k)_{(f)} \xrightarrow{\sim} H^1_c(\Gamma, \mathcal{P}_k)_{(f)}$$

is an isomorphism.

Proof. By Proposition 4.7 and Lemma 5.1, we have an exact sequence

$$0 \to H^1_c(\mathbf{D}_{-2-k})(k+1)^{(=k+1)} \to H^1_c(\mathbf{D}_k)^{(=k+1)} \to H^1_c(\mathcal{P}_k)^{(=k+1)} \to 0$$

of finite-dimensional vector spaces. By Lemma 6.4, passing to f-isotypic subspaces gives an exact sequence

$$0 \to H^1_c(\mathbf{D}_{-2-k})(k+1)_{(f)} \to H^1_c(\mathbf{D}_k)_{(f)} \to H^1_c(\mathcal{P}_k)_{(f)} \to 0.$$

Seeking a contradiction, assume that $\Psi \in H^1_c(\mathbf{D}_{-2-k})(k+1)_{(f)}$ is some non-zero $\mathcal{H}$-eigenvector, and let Ψ_0 be the corresponding untwisted $\mathcal{H}$-eigensymbol in $H^1_c(\mathbf{D}_{-2-k})$.

By scaling, we may assume that $||\Psi_0|| = 1$; that is, Ψ_0 takes values in $\mathbf{D}^0_{-2-k}$. Since Ψ has slope $k+1$, we have that Ψ_0 has slope 0. Thus, by Lemma 6.3, the reduction of Ψ_0 modulo $\mathrm{Fil}^1 \mathbf{D}^0_{-2-k}$ is non-zero.

By Lemma 3.5, we have

$$H^1_c(\mathbf{D}^0_{-2-k}/\operatorname{Fil}^1 \mathbf{D}^0_{-2-k}) \cong H^1_c(\mathbb{F}_p(a^{-2-k})).$$

Twisting by $\det^{k+2}$, we see that the system of eigenvalues associated to $\overline{\theta f}$ occurs in

$$H^1_c(\overline{\mathbb{F}}_p(a^{-2-k}))(k+2) \cong H^1_c(\overline{\mathbb{F}}_p(a^{-2-k} \cdot (ad)^{k+2})) \cong H^1_c(\overline{\mathbb{F}}_p(d^{k+2})).$$

Thus, by Lemma 6.6, this system of eigenvalues occurs in $S_2(\Gamma_1(Np), \omega^{k+2}, \overline{\mathbb{F}}_p)$ contradicting the assumptions of the theorem.

Finally, by Lemma 5.2,

$$H^1_c(\mathcal{D}_k)_{(f)} \cong H^1_c(\mathbf{D}_k)_{(f)} \cong H^1_c(\mathcal{P}_k)_{(f)},$$

proving the theorem. □

6.5. **On the condition of Theorem 6.7.** The conditions of Theorem 6.7 are of an explicit nature which allows one to verify whether or not they hold for a particular modular form. In this section, we give examples of forms that satisfy and fail these conditions.

Example 6.8. Let g be the normalized newform on $\Gamma_0(11)$ that corresponds to $X_0(11)$. Let $p = 3$, and let f correspond to the critical 3-stabilization of g to level 33. One immediately sees that the residual representation of f is irreducible at 3, and thus the first condition of Theorem 6.7 holds. To verify the second condition, we need to know the complete list of systems of eigenvalues which occur in

$$S_2(\Gamma_1(33), \omega^2, \overline{\mathbb{F}}_p) = S_2(\Gamma_0(33), \overline{\mathbb{F}}_p).$$

By [2, Corollary 1.2], any system of eigenvalues that occurs in this space lifts to a system of eigenvalues in $S_2(\Gamma_0(33), \overline{\mathbb{Q}}_p)$. This latter space of modular forms is three-dimensional. Two dimensions are accounted for by the two oldforms arising from g. There is a single newform h that corresponds to an elliptic curve of conductor 33. This modular form h is congruent to f mod 3, and thus there is only one system of eigenvalues occurring in $S_2(\Gamma_0(33), \overline{\mathbb{F}}_p)$. In particular, the system of eigenvalues attached to $\overline{\theta f}$ does not occur in this space.

The following proposition gives a criterion that guarantees that a modular form *fails* the second condition of Theorem 6.7. Its proof relies upon the existence of mod p companion forms.

Proposition 6.9. *Let $f \in S_{k+2}(\Gamma_1(N), \mathbb{Q}_p)$ be a p-ordinary eigenform, and assume that the residual representation of f is split locally at p. Then the system of eigenvalues attached to $\overline{\theta f}$ occurs in $S_2(\Gamma_1(Np), \omega^{k+2}, \overline{\mathbb{F}}_p)$.*

Proof. By [9, 3], there exists a mod p companion form g on $\Gamma_1(N)$ with weight $p-1-k$ such that $\overline{\theta f} = \overline{\theta}^{k+2}\overline{g}$. Thus, the system of eigenvalues associated to $\overline{\theta f}$ occurs in

$$S_{p-1-k+(k+2)(p+1)}(\Gamma_1(N), \overline{\mathbb{F}}_p) = S_{kp+3p+1}(\Gamma_1(N), \overline{\mathbb{F}}_p).$$

By [2, Lemma 3.3], this system of eigenvalues occurs in

$$S_2(\Gamma_1(Np), \omega^{kp+3p-1}, \overline{\mathbb{F}}_p) = S_2(\Gamma_1(Np), \omega^{k+2}, \overline{\mathbb{F}}_p)$$

as claimed. □

Example 6.10. Let $E/\mathbb{Q}$ be any curve with CM, and let g be the corresponding normalized newform. Let p be a good ordinary prime for E, and let f be the criticial p-stabilization of g. Since E has CM, the p-adic representation attached to f (or g) is split locally at p (see [8, Prop 3]). Thus certainly the residual representation attached to f is split at p, and the hypothesis of Proposition 6.9 is satisfied. In particular, the form f fails the conditions of Theorem 6.7.

In [13], computations were done for f corresponding to the CM elliptic curve $X_0(32)$ with $p = 5$. In these computations, an approximation to an overconvergent Hecke-eigensymbol was found with the same eigenvalues as f. However, this symbol was in the kernel of specialization, and no symbol was found which specialized to the classical modular symbol attached to f.

Remark 6.11. Although we do expect that there are modular forms which do not satisfy the conclusions of Theorem 6.7, the hypotheses of this theorem are too strict. Namely, let f be a modular form whose residual representation is split locally at p, but whose p-adic representation is not. Then the hypotheses of Theorem 6.7 fail, but, as we will see in the next section, the conclusions of the theorem are still valid.

7. The critical slope subspace II

In this section we strengthen the results of Theorem 6.7 using the following theorem of the second author.

Theorem 7.1 (Stevens). *A system of eigenvalues of $\mathcal{H}$ occurs in $H^1_c(\Gamma, \mathcal{D}_k)$ if and only if it occurs in $M^\dagger_{k+2}(\Gamma, \overline{\mathbb{Q}}_p)$.*

Proof. See [15]. □

7.1. **Main theorem II.** Let θ^{k+1} denote the θ-operator on overconvergent modular forms which takes $M^\dagger_{-k}(\Gamma, \overline{\mathbb{Q}}_p)$ to $M^\dagger_{k+2}(\Gamma, \overline{\mathbb{Q}}_p)$ and which acts on q-expansions by $(qd/dq)^{k+1}$.

Theorem 7.2. *Let f be an eigenform in $S_{k+2}(\Gamma, \overline{\mathbb{Q}}_p)$ with slope $k+1$. Then*

$$H^1_c(\Gamma, \mathcal{D}_k)_{(f)} \xrightarrow{\sim} H^1_c(\Gamma, \mathcal{P}_k)_{(f)}$$

is an isomorphism if and only if $f \notin \mathrm{im}(\theta)$.

Proof. As in the proof of Theorem 6.7, we have an exact sequence

$$0 \to H^1_c(\mathbf{D}_{-2-k})(k+1)_{(f)} \to H^1_c(\mathbf{D}_k)_{(f)} \to H^1_c(\mathcal{P}_k)_{(f)} \to 0.$$

Assume there is a non-zero $\mathcal{H}$-eigensymbol $\Psi \in H^1_c(\mathbf{D}_{-2-k})(k+1)_{(f)}$, and let Ψ_0 be the untwisted eigensymbol in $H^1_c(\mathbf{D}_{-2-k})$. By Theorem 7.1, there is some overconvergent eigenform $g \in M^\dagger_{-k}(\Gamma, \overline{\mathbb{Q}}_p)$ with the same system of eigenvalues as Ψ_0. The system of eigenvalues of $\theta^{k+1}g \in M^\dagger_{k+2}(\Gamma, \overline{\mathbb{Q}}_p)$ then equals the system of eigenvalues of f. Thus, looking at q-expansion, we deduce that $f = \theta^{k+1}g$.

Conversely, assume that $f = \theta^{k+1}g$ for some $g \in M^\dagger_{-k}(\Gamma, \overline{\mathbb{Q}}_p)$. By Theorem 7.1, there is some $\mathcal{H}$-eigensymbol Ψ_0 in $H^1_c(\mathbf{D}_{-2-k})$ with the same system of eigenvalues as g. Twisting this symbol then gives an $\mathcal{H}$-eigensymbol Ψ in $\left(H^1_c(\mathbf{D}_{-2-k})(k+1)\right)_{(f)}$. The image of Ψ in $H^1_c(\mathbf{D}_k)_{(f)} \cong H^1_c(\mathcal{D}_k)_{(f)}$ is then a non-zero symbol in the kernel of specialization. □

Remark 7.3. When f is in the image of θ^{k+1}, the above theorem implies that there is an $\mathcal{H}$-eigensymbol $\Psi \in H^1_c(\mathcal{D}_k)^{\pm}_{(f)}$ which is in the kernel of specialization. However, from Theorem 7.1, one cannot conclude that this symbol (up to scaling) is unique. Indeed, it is theoretically possible that there are multiple $\mathcal{H}$-eigenvectors in $H^1_c(\mathcal{D}_k)^{\pm}_{(f)}$, while in $(M^\dagger_k)_{(f)}$, $\mathcal{H}$ acts non-semisimply and there is only a one-dimensional space of overconvergent eigenforms.

Remark 7.4. We note that Theorem 7.2 does indeed imply Theorem 6.7. If f is in the image of θ^{k+1}, then by [7, Prop 1.2] the p-adic representation associated to f is split at p and, in particular, the same is true for the associated residual representation. The implication then follows from Proposition 6.9.

8. p-adic L-functions

Let $f = \sum a_n q^n$ be a normalized eigenform in $S_{k+2}(\Gamma, \overline{\mathbb{Q}}_p)$ with slope $h < k+1$. In this case, there is a p-adic L-function $\mu_f \in \mathcal{D}$ which interpolates the special values of twists of the complex L-series of f. Specifically, if χ is a finite order character of $\mathbb{Z}_p^\times$ with conductor p^n and j is an integer between 0 and k, then

$$\mu_f(x^j \cdot \chi) = \frac{1}{a_p^n} \cdot \frac{p^{n(j+1)}}{(-2\pi i)^j} \cdot \frac{j!}{\tau(\chi^{-1})} \cdot \frac{L(f, \chi^{-1}, 1)}{\Omega_f^{\pm}} \tag{3}$$

where $\tau(\chi^{-1})$ is a Gauss sum and $\Omega_f^{\pm}$ are certain complex periods. We note that the p-adic L-function μ_f is uniquely determined by this interpolation property and by a bound on its growth (i.e. that it is h-admissible).

We now describe an alternative construction of this p-adic L-function via overconvergent modular symbols to motivate our definition of p-adic L-functions for critical slope forms.

Let K_f denote the finite extension of $\mathbb{Q}_p$ containing the Fourier coefficients of f. By Eichler-Shimura theory and multiplicity one, the f-isotypic subspace of $H^1_c(\Gamma, \mathcal{P}_k))^{\pm} \otimes K_f$ is one-dimensional. Let $\phi_f^{\pm}$ denote a non-zero element of this subspace, normalized to have size 1, and set $\phi_f = \phi_f^+ + \phi_f^-$.

Since we are assuming that f is non-critical, by Theorem 5.3, there is a unique overconvergent modular symbol $\Phi_f \in H^1_c(\mathcal{D}_k) \otimes K_f$ which specializes to ϕ_f. The following theorem relates Φ_f to the p-adic L-function of f.

Proposition 8.1. *With f, ϕ_f and Φ_f as above, we have*

$$\Phi_f(\{\infty\} - \{0\})\big|_{\mathbb{Z}_p^\times} = \mu_f,$$

the p-adic L-function of f.

Proof. See [16] or [13, Prop 6.3]. □

We now consider the case where f has slope equal to $k+1$. In light of Proposition 8.1, we make the following definition of the p-adic L-function of f.

Definition 8.2. Let f be an eigenform in $S_{k+2}(\Gamma, \overline{\mathbb{Q}}_p)$ of slope $k+1$ which is not in the image of θ^{k+1}. Let Φ_f be the unique overconvergent eigensymbol of Theorem 7.2 which specializes to ϕ_f. We define the p-adic L-function of f to be

$$\mu_f := \Phi_f(\{\infty\} - \{0\})\big|_{\mathbb{Z}_p^\times},$$

which is a locally analytic distribution on $\mathbb{Z}_p^\times$.

Proposition 8.3. *Let f be an eigenform in $S_{k+2}(\Gamma, \overline{\mathbb{Q}}_p)$ of slope $k+1$ which is not in the image of θ. Then μ_f is a $(k+1)$-admissible distribution. Further, μ_f satisfies the interpolation property in (3).*

Proof. The admissibility claim follows from [13, Lemma 6.2]. The interpolation property is a formal consequence of Φ_f being a U_p-eigensymbol lifting ϕ_f as in Proposition 8.1. □

Remark 8.4. Since μ_f is a $(k+1)$-admissible distribution, it is *not* uniquely determined by the above interpolation property. To uniquely determine this distribution by interpolation, one would also need to specify its values at the characters of the form $x^{k+1}\chi$. We point out here that our method of producing μ_f from overconvergent modular symbols does not directly give a way of understanding its values at such characters.

Remark 8.5. We now sketch an alternative construction of a critical slope p-adic L-functions given by combining Perrin-Riou's dual exponential map with Kato's zeta-element (see [11] and [4] for more details).

Let f now be an eigenform on $\Gamma_0(N)$ with $p \nmid N$, and let V_f denote the p-adic representation attached to f. Consider Perrin-Riou's dual-exponential map (see [12] and [10, 2.1]),

$$\varprojlim_n H^1(\mathbb{Q}_{n,p}, V_f) \xrightarrow{\exp^*} \mathcal{D}_k \otimes D_{\mathrm{cris}}(V_f)$$

where $\mathbb{Q}_{n,p}$ is the n-th layer of the local cyclotomic $\mathbb{Z}_p$-extension of $\mathbb{Q}_p$. Kato's zeta-element $\mathbf{z}(f) = (z_n(f)) \in \lim_n H^1(\mathbb{Q}_n, V_f)$ is a norm-coherent system of global cohomology classes; here, $\mathbb{Q}_n$ is the n-th layer of the cyclotomic $\mathbb{Z}_p$-extension of $\mathbb{Q}$.

Let $\mathbf{z}_p(f)$ denote the restriction to p of $\mathbf{z}(f)$, and let

$$L_p(f) = \exp^*(\mathbf{z}_p(f)) \in \mathcal{D}_k \otimes D_{\mathrm{cris}}(V_f).$$

We have that $D_{\mathrm{cris}}(V_f)$ decomposes under the φ-action into eigenspaces with eigenvalues α and β, the roots of $x^2 - a_p x + p^{k+1}$. Let $L_{p,\alpha}(f)$ and $L_{p,\beta}(f)$ denote the two projections of $L_p(f)$ onto these eigenspaces, which we can identify as locally analytic p-adic distributions.

If f is non-ordinary at p, then these two p-adic distributions are precisely the p-adic L-functions attached to f. If f is ordinary at p, and α is a p-unit, then $L_{p,\alpha}(f)$ is the ordinary p-adic L-function of f. In this case, one *defines* $L_{p,\beta}(f)$ to be the critical slope p-adic L-function attached to f. This distribution satisfies the interpolation property of equation (3).

We remark that if f is locally split at p (which conjecturally is equivalent to f being in the image of θ^{k+1}), then $L_{p,\beta}(f)$ is identically zero.

Remark 8.6. Emerton's theory of $\widehat{H}^1$ also gives rise to critical slope p-adic L-functions. **To be filled in ...**

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