0.1. **Steinberg fact.** Consider χ a Dirichlet character of modulus N and conductor N_χ. Write $N = N_\chi \cdot N'$. Let d_χ denote the order of χ.

Fact: if $q \nmid N_\chi$ and $q || N'$, then $S_1^{\mathrm{new}}(N, \chi) = 0$.

The reason is easy. At such a q, the associated Galois representation is Steinberg and thus has infinite image. But this is not possible for a weight 1 form.

Here' something new. If $\mathrm{ord}_\ell(N) = \mathrm{ord}_\ell(N_\chi)$, then the Galois representation at ℓ has an unramified quotient which sends Frob_ℓ to a_ℓ. In particular, for $\sigma \in G_{\mathbb{Q}_\ell}$ lifting Frob_ℓ, we have

$$\rho(\sigma) = \begin{pmatrix} \chi(\sigma) a_\ell^{-1} & 0 \\ 0 & a_\ell \end{pmatrix}.$$

If $\overline{d}_\sigma$ is the projective order of $\rho(\sigma)$, then we have

$$\chi(\sigma)^{\overline{d}_\sigma} a_\ell^{-\overline{d}_\sigma} = a_\ell^{\overline{d}_\sigma}.$$

Write $\chi = \chi_\ell \cdot \chi^\ell$ which χ_ℓ is a character of modulus a power of ℓ and χ^ℓ has modulus prime to ℓ. Then

$$\chi_\ell(\sigma)^{\overline{d}_\sigma} \chi^\ell(\ell)^{\overline{d}_\sigma} a_\ell^{-\overline{d}_\sigma} = a_\ell^{\overline{d}_\sigma}$$

nothing that $\chi^\ell(\sigma) = \chi^\ell(\ell)$ as σ lifts Frob_ℓ.

Choosing σ such that $\sigma_\ell(\sigma) = 1$ then gives

$$a_\ell^{2\overline{d}} = \chi^\ell(\ell)^{2\overline{d}}$$

where $\overline{d}$ is in $\{1, 2, 3, 4, 5\}$.

On the other hand, if D is the gcd of two elements of $\{1, 2, 3, 4, 5\}$, then

$$\chi_\ell(\sigma)^D \chi^\ell(\ell)^D a_\ell^{-D} = a_\ell^D$$

and

$$\chi^\ell(\ell)^D a_\ell^{-D} = a_\ell^D.$$

In particular,

$$\chi_\ell(\sigma)^D = 1$$

for all σ. Thus the order of χ_ℓ divides D. This puts a new condition on when weight 1 forms can exist!

0.2. **Fourier coefficients.** Let $f = \sum_n a_n q^n$ with $a_1 = 1$ be a newform in $S_1(N, \chi)$.

Proposition 0.1.

(1) *if* $\ell \nmid N$, *then*
$$a_\ell^2 = c\chi(\ell)$$
where $c = 0, 1, 2$ *or* 4.

(2) *if* $\ell | N_\chi$, $\ell \nmid N'$, *then*
$$a_\ell^{2e} = 1$$
where $e = d_\chi \overline{d} / \gcd(d_\chi, \overline{d})$ *and* $\overline{d}$ *is a projective order: that is,* $\overline{d} = 1, 2, 3, 4$ *or* 5.

(3) *if* $\ell | N'$, *then*
$$a_\ell = 0.$$

Proof. The first part is Buzzard-Lauder, Lemma 1(b).

For the second part, if $\ell | N_\chi$ but $\ell \nmid N'$, then $\pi_\ell(f)$ is the ramified principal series $\pi(\chi_1, \chi_2)$ where χ_2 is unramified with $\chi_2(\mathrm{Frob}_\ell) = a_\ell$ and $\chi_1\chi_2 = \chi$ (Loeffler-Weinstein, Prop 2.8). In particular, ρ_f at ℓ is simply the direct sum $\chi_1 \oplus \chi_2$ (noting that the representation must be semi-simple as it is finite order). Thus if $\sigma \in \mathrm{Frob}_\ell I_\ell$ with I_ℓ equal to inertia at ℓ, then
$$\rho_f(\sigma) = \begin{pmatrix} \chi(\sigma)/a_\ell & 0 \\ 0 & a_\ell \end{pmatrix}$$
and
$$\rho_f(\sigma)^e = \begin{pmatrix} a_\ell^{-e} & 0 \\ 0 & a_\ell^e \end{pmatrix}$$
is a diagonal matrix as both d_χ and $\overline{d}$ divide e. Thus, $a_\ell^e = a_\ell^{-e}$ and $a_\ell^{2e} = 1$ as desired.

For the third part, $\pi_\ell(f)$ is supercuspidal again by the same Loeffler-Weinstein reference above which implies $a_\ell = 0$. $\square$

Let $\pi_\ell(x)$ denote the minimum polynomial of a_ℓ over $\mathbb{Q}$ and set d_ℓ equal to the degree of this polynomial. Let $d_\chi = [\mathbb{Q}(\chi) : \mathbb{Q}]$ which is the order of χ.

Proposition 0.2. *We have*
$$d_\ell \leq \gcd(d_\ell, d_\chi) \cdot \dim S_1(N, \chi).$$

Proof. If K_f is the field of Fourier coefficients of f, then $[K_f : \mathbb{Q}(\chi)] \leq \dim S_1(N, \chi)$ as all of the $\mathbb{Q}(\chi)$-Galois conjugates of f are in this weight 1 space. Thus
$$\begin{aligned} S_1(N, \chi) &\geq [K_f : \mathbb{Q}(\chi)] \\ &\geq [\mathbb{Q}(\chi, a_\ell) : \mathbb{Q}(\chi)] \\ &= [\mathbb{Q}(a_\ell) : \mathbb{Q}(\chi) \cap \mathbb{Q}(a_\ell)] \\ &= [\mathbb{Q}(a_\ell) : \mathbb{Q}] / [\mathbb{Q}(\chi) \cap \mathbb{Q}(a_\ell) : \mathbb{Q}] \\ &\geq d_\ell / \gcd(d_\ell, d_\chi). \end{aligned}$$
Here the first equalities follows since $K_f/\mathbb{Q}$ is an abelian extension and last follows since $[\mathbb{Q}(\chi) \cap \mathbb{Q}(a_\ell) : \mathbb{Q}]$ divides both d_χ and d_ℓ. $\square$

Let $\rho : G_{\mathbb{Q}} \to \mathrm{GL}_2(\mathbb{C})$ be an Artin representation with conductor N and determinant χ. In this note, we will describe all possible values of the trace of Frobenius at a prime $\ell \nmid N$.

Theorem 0.3. *For $\ell \nmid N$, let $A = \rho(\mathrm{Frob}_\ell) \in \mathrm{GL}_2(\mathbb{C})$ and let $\overline{A}$ denote the image of A in $\mathrm{PGL}_2(\mathbb{C})$. If the order of A is d, the order of $\overline{A}$ is $\overline{d}$, and the order of $\chi(\ell) \in \mathbb{C}^\times$ is r, then*

$$d = \begin{cases} \frac{2r\overline{d}}{\gcd(r,\overline{d})} & \text{if } 2 \mid \frac{r}{\gcd(r,\overline{d})} \\ \\ \frac{r\overline{d}}{\gcd(r,\overline{d})} \text{ or } \frac{2r\overline{d}}{\gcd(r,\overline{d})} & \text{otherwise} \end{cases}$$

Proof. Let $\langle A \rangle$ denote the cyclic subgroup of $\mathrm{GL}_2(\mathbb{C})$ generated by A and similarly for $\overline{A}$. We then have an exact sequence

$$1 \to \langle A \rangle \cap \mathbb{C}^\times \to \langle A \rangle \to \langle \overline{A} \rangle \to 1.$$

Since $\overline{d}$ is the smallest positive integer such that $A^{\overline{d}}$ is diagonal, this sequence is the same as

$$1 \to \langle A^{\overline{d}} \rangle \to \langle A \rangle \to \langle \overline{A} \rangle \to 1.$$

Thus, we have

$$d = \overline{d} \cdot \mathrm{ord}(A^{\overline{d}}).$$

To prove this theorem, we must then compute $\mathrm{ord}(A^{\overline{d}})$. To this end, since $A^{\overline{d}}$ is diagonal with determinant $\chi(\ell)^{\overline{d}}$, we have

$$A^{\overline{d}} = \begin{pmatrix} \sqrt{\chi(\ell)^{\overline{d}}} & 0 \\ 0 & \sqrt{\chi(\ell)^{\overline{d}}} \end{pmatrix} \text{ or } \begin{pmatrix} -\sqrt{\chi(\ell)^{\overline{d}}} & 0 \\ 0 & -\sqrt{\chi(\ell)^{\overline{d}}} \end{pmatrix}$$

where $\sqrt{\chi(\ell)^{\overline{d}}}$ is some fixed square root of $\chi(\ell)^{\overline{d}}$. Thus, the order of $A^{\overline{d}}$ is simply the multiplicative order of $\pm\sqrt{\chi(\ell)^{\overline{d}}}$.

We have $r = \mathrm{ord}(\chi(\ell))$. Thus, $\mathrm{ord}(\sqrt{\chi(\ell)^{\overline{d}}}) = \dfrac{2r}{\gcd(r,\overline{d})}$. Now if ζ is a primitive $4n$-th root of unity, then the same is true of $-\zeta$. However, if ζ is a primitive $2n$-th root of unity with n odd, then $-\zeta$ is either a primitive $2n$-th root of unity or a primitive n-root of unity. The theorem follows from this. $\square$

Remark 0.4. We note that one can easily use this theorem to compute the trace of Frobenius as one can readily compute the trace of matrix given its order and determinant. Indeed, if A has order d and determinant m, then it's eigenvalues are ζ_d and $\zeta_d^{-1}m$ where ζ_d is a primitive d-th root of unity and thus the trace of A is $\zeta_d + m\zeta_d^{-1}$. (NO! THIS IS WRONG BUT EASILY FIXED – NEED TO DEAL WITH DIVISORS OF d.)

Here's a table of traces of matrices with determinant 1:

d	$\zeta_d + \zeta_d^{-1}$	trace
1	1 + 1	2
2	-1 + -1	-2
3	$\zeta_3 + \zeta_3^{-1}$	-1
4	$\zeta_4 + \zeta_4^{-1}$	0
5	$\zeta_5 + \zeta_5^{-1}$	$\frac{-1\pm\sqrt{5}}{2}$
6	$\zeta_6 + \zeta_6^{-1}$	1
8	$\zeta_8 + \zeta_8^{-1}$	$\pm\sqrt{2}$
10	$\zeta_{10} + \zeta_{10}^{-1}$	$\frac{1\pm\sqrt{5}}{2}$

Here's a table of traces of matrices with determinant -1:

d	$\zeta_d - \zeta_d^{-1}$	trace
2	-1 - -1	0
4	$\zeta_4 - \zeta_4^{-1}$	$\pm 2i$
8	$\zeta_8 - \zeta_8^{-1}$	$\pm\sqrt{-2}$
12	$\zeta_{12} - \zeta_{12}^{-1}$	$\pm i$
20	$\zeta_{20} - \zeta_{20}^{-1}$	$\pm\sqrt{\frac{-3\pm\sqrt{5}}{2}}$

Now we use this theorem to compare to Serre's claims on page 263 at the bottom of his weight 1 article. Namely, let $N = q$ a prime and take χ to be a quadratic character. In this case, the projective image of $\rho(G_{\mathbb{Q}})$ is either S_4 or A_5.

Let's first analyze the S_4 case. In this group the possible orders are 1,2,3, and 4. (These are the possible values of $\overline{d}$ from the last theorem.) We proceed in two cases: $\chi(\ell) = 1$ or $\chi(\ell) = -1$; that is, $r = 1$ or $r = 2$. When $r = 1$, we have $\frac{r\overline{d}}{\gcd(r,\overline{d})} = 1$ is always odd and $\frac{r\overline{d}}{\gcd(r,\overline{d})} = \overline{d}$. Thus

- $\overline{d} = 1 \implies d = 1$ or $2 \implies$ trace $= 2$ or -2
- $\overline{d} = 2 \implies d = 2$ or $4 \implies$ trace $= -2$ or 0
- $\overline{d} = 3 \implies d = 3$ or $6 \implies$ trace $= -1$ or 1
- $\overline{d} = 4 \implies d = 4$ or $8 \implies$ trace $= 0$ or $\pm\sqrt{2}$

Thus, the possible traces are $0, \pm 1, \pm 2, \pm\sqrt{2}$. This exactly matches Serre would predicts that the traces have squares equal to 0,1,2 and 4.

When $r = 2$, we have

- $\overline{d} = 1 \implies \frac{r}{\gcd(r,\overline{d})} = 2 \implies d = 4 \implies$ trace $= \pm 2i$
- $\overline{d} = 2 \implies \frac{r}{\gcd(r,\overline{d})} = 1 \implies d = 2$ or $4 \implies$ trace $= 0, \pm 2i$
- $\overline{d} = 3 \implies \frac{r}{\gcd(r,\overline{d})} = 2 \implies d = 12 \implies$ trace $= \pm i$
- $\overline{d} = 4 \implies \frac{r}{\gcd(r,\overline{d})} = 1 \implies d = 4$ or $8 \implies$ trace $= \pm 2i, \pm\sqrt{-2}$

Thus, the possible traces are $0, \pm i, \pm 2i, \pm\sqrt{-2}$. This exactly matches Serre would predicts that the traces have squares equal to 0,-1,-2 and -4. Great!

Now on to A_5. We then have $\overline{d} = 1, 2, 3$, or 5. Again, taking $r = 1$, we have $\frac{r\overline{d}}{\gcd(r,\overline{d})} = 1$ is always odd and $\frac{r\overline{d}}{\gcd(r,\overline{d})} = \overline{d}$. Thus

- $\overline{d} = 1 \implies d = 1$ or $2 \implies$ trace $= 2$ or -2
- $\overline{d} = 2 \implies d = 2$ or $4 \implies$ trace $= -2$ or 0
- $\overline{d} = 3 \implies d = 3$ or $6 \implies$ trace $= -1$ or 1
- $\overline{d} = 5 \implies d = 5$ or $10 \implies$ trace $= \frac{-1\pm\sqrt{5}}{2}$ or $\frac{1\pm\sqrt{5}}{2}$

Thus, the possible traces are $0, \pm 1, \pm 2, \frac{\pm 1 \pm \sqrt{5}}{2}$. This exactly matches Serre would predicts that the traces have squares equal to 0,1,4 and $\frac{3\pm\sqrt{5}}{2}$.

When $r = 2$, we have

- $\overline{d} = 1 \implies \frac{r}{\gcd(r,\overline{d})} = 2 \implies d = 4 \implies$ trace $= \pm 2i$
- $\overline{d} = 2 \implies \frac{r}{\gcd(r,\overline{d})} = 1 \implies d = 2$ or $4 \implies$ trace $= 0, \pm 2i$
- $\overline{d} = 3 \implies \frac{r}{\gcd(r,\overline{d})} = 2 \implies d = 12 \implies$ trace $= \pm i$
- $\overline{d} = 5 \implies \frac{r}{\gcd(r,\overline{d})} = 2 \implies d = 20 \implies$ trace $= \pm\sqrt{\frac{-3\pm\sqrt{5}}{2}}$.

Thus, the possible traces are $0, \pm i, \pm 2i, \pm\sqrt{\frac{-3\pm\sqrt{5}}{2}}$. This exactly matches Serre would predicts that the traces have squares equal to 0,-1,-4, and $\frac{-3\pm\sqrt{5}}{2}$. Great!