## Heat Equation

We are supposed to solve the Initial-Boundary Volume Problem.

$$\frac{\partial u}{\partial t} = \nabla^2 u + f \quad \text{in } \Omega \times (0, T], \tag{1}$$

$$u = u_{\mathrm{D}} \text{on } \partial\Omega \times (0, T], \tag{2}$$

$$u = u_0 \text{at } t = 0\,. \tag{3}$$

So we are supposed to find $u(x, y, t)$ in the domain $\Omega$ through $(0, T]$.
We first discretize in time $t$. So say

$$\frac{\partial u}{\partial t} = \frac{u^{n+1}(x, y) - u^n(x, y)}{\Delta t} \tag{4}$$

The $n + 1$ means at the time $t = t_{n+1}$. Thus

$$\frac{u^{n+1}(x, y) - u^n(x, y)}{\Delta t} = \nabla^2 u^{n+1}(x, y) + f^{n+1}(x, y) \tag{5}$$

and

$$u^{n+1}(x, y) - u^n(x, y) = \nabla^2 u^{n+1}(x, y)\Delta t + f^{n+1}(x, y)\Delta t \tag{6}$$

or

$$u^{n+1}(x, y) - \nabla^2 u^{n+1}(x, y)\Delta t = u^n(x, y) + f^{n+1}(x, y)\Delta t \tag{7}$$

Now take a function $v(x, y, t)$ which vanishes on the boundaries and multiply the PDE and the initial conditions by it so

$$\big(u^{n+1}(x, y) - \nabla^2 u^{n+1}(x, y)\Delta t\big)v^{n+1}(x, y) = \big(u^n(x, y) + f^{n+1}(x, y)\Delta t\big)v^{n+1}(x, y) \tag{8}$$

$$u(x, y)v(x, y) = u_0(x, y)v(x, y) \tag{9}$$

We do the initial condition for descritization purposes.
Take the integral over the area $\Omega$.

$$\int_\Omega \big(u^{n+1}(x,y) - \nabla^2 u^{n+1}(x,y)\Delta t\big) v^{n+1}(x,y)\, dA = \int_\Omega \big(u^n(x,y) + f^{n+1}(x,y)\Delta t\big) v^{n+1}(x,y)\, dA \tag{10}$$

$$\int_\Omega u(x,y)v(x,y)\, dA = \int_\Omega u_0(x,y)v(x,y)\, dA \tag{11}$$

Let's handle the main PDE, and let's call $u = u^{n+1}(x,y)$ which is supposed to be sought at every increment and all state variable of $n+1$.

$$\int_\Omega uv\, dA - \int_\Omega v\nabla^2 u\Delta t\, dA = \int_\Omega \big(u^n + f\Delta t\big) v\, dA \tag{12}$$

The second term

$$\Delta t \int_\Omega v\nabla^2 u\, dA = \Delta t \int_\Omega \nabla.(v\nabla u)\, dA - \Delta t \int_\Omega \nabla v.\nabla u\, dA \tag{13}$$

The second integral can be taken over the boundary.

$$\int_\Omega \nabla.(v\nabla u)\, dA = \int_{\partial\Omega} v(\nabla u).\mathbf{n}\, dc \tag{14}$$

so because $v(x,y)$ vanishes on the boundary

$$\int_\Omega \nabla.(v\nabla u)\, dA = \int_{\partial\Omega} v(\nabla u).\mathbf{n}\, dc = 0 \tag{15}$$

and thus the weak form of the PDE becomes

$$\Delta t \int_\Omega v\nabla^2 u\, dA = -\Delta t \int_\Omega \nabla v.\nabla u\, dA \tag{16}$$

$$\int_\Omega uv\, dA + \Delta t \int_\Omega \nabla v.\nabla u\, dA = \int_\Omega \big(u^n + f\Delta t\big) v\, dA \tag{17}$$

$$
\begin{aligned}
a_{n+1}(u,v) &= \int_\Omega uv\,dA + \Delta t\int_\Omega \nabla v.\nabla u\,dA \\
L_{n+1}(u,v) &= \int_\Omega \big(u^n + f\Delta t\big)v\,dA
\end{aligned}
\tag{18}
$$

and

$$
a_{n+1}(u,v) = L_{n+1}(u,v) \tag{19}
$$

Second the initial condition.

$$
\int_\Omega u(x,y)v(x,y)\,dA = \int_\Omega u_0(x,y)v(x,y)\,dA \tag{20}
$$

and

$$
\int_\Omega uv\,dA = \int_\Omega u_0 v\,dA \tag{21}
$$

$$
a_0(u(x,y),v(x,y)) = L_0(u_0(x,y),v(x,y)) \tag{22}
$$

We project the values of the initial conditions to the nodal points.

## Solution

So we are supposed to find the solution

$$
u(x,y;t) \in V \tag{23}
$$

and the test function is

$$
v(x,y;t) \in \hat{V} \tag{24}
$$

We do the descritization so at each time $t$

$$u = \sum_{i=1}^{N} U_i \phi_i \tag{25}$$

$$v = \sum_{i=1}^{N} V_i \phi_i \tag{26}$$

at the time $t = 0$

$$u^0 = \sum_{i=1}^{N} U_i^0 \phi_i \tag{27}$$

**Exact Solution**

Let's say the solution is

$$u = 1 + x^2 + \alpha y^2 + \beta t \tag{28}$$

Pluging it into the PDE gives

$$\beta = 2 + 2\alpha + f(x, y, t) \tag{29}$$

thus

$$f(x, y, t) = \beta - 2 - 2\alpha \tag{30}$$

Let's assume the BC is just what we have for PDE, so

$$u_{\mathrm{D}} = 1 + x^2 + \alpha y^2 + \beta t \quad \text{on} \quad (x, y) \in \Omega \tag{31}$$

so the initial condition is

$$u(x, y, t = 0) = u^0 = 1 + x^2 + \alpha y^2 \tag{32}$$