

Heat Equation

We are supposed to solve the Initial-Boundary Value Problem.

$$\frac{\partial u}{\partial t} = \nabla^2 u + f \quad \text{in } \Omega \times (0, T], \quad (1)$$

$$u = u_D \text{ on } \partial\Omega \times (0, T], \quad (2)$$

$$u = u_0 \text{ at } t = 0. \quad (3)$$

So we are supposed to find $u(x, y, t)$ in the domain Ω through $(0, T]$.

We first discretize in time t . So say

$$\frac{\partial u}{\partial t} = \frac{u^{n+1}(x, y) - u^n(x, y)}{\Delta t} \quad (4)$$

The $n + 1$ means at the time $t = t_{n+1}$. Thus

$$\frac{u^{n+1}(x, y) - u^n(x, y)}{\Delta t} = \nabla^2 u^{n+1}(x, y) + f^{n+1}(x, y) \quad (5)$$

and

$$u^{n+1}(x, y) - u^n(x, y) = \nabla^2 u^{n+1}(x, y) \Delta t + f^{n+1}(x, y) \Delta t \quad (6)$$

or

$$u^{n+1}(x, y) - \nabla^2 u^{n+1}(x, y) \Delta t = u^n(x, y) + f^{n+1}(x, y) \Delta t \quad (7)$$

Now take a function $v(x, y, t)$ which vanishes on the boundaries and multiply the PDE and the initial conditions by it so

$$(u^{n+1}(x, y) - \nabla^2 u^{n+1}(x, y) \Delta t) v^{n+1}(x, y) = (u^n(x, y) + f^{n+1}(x, y) \Delta t) v^{n+1}(x, y) \quad (8)$$

$$u(x, y) v(x, y) = u_0(x, y) v(x, y) \quad (9)$$

We do the initial condition for discretization purposes.

Take the integral over the area Ω .

$$\begin{aligned} \int_{\Omega} (u^{n+1}(x, y) - \nabla^2 u^{n+1}(x, y) \Delta t) v^{n+1}(x, y) dA = \\ \int_{\Omega} (u^n(x, y) + f^{n+1}(x, y) \Delta t) v^{n+1}(x, y) dA \end{aligned} \quad (10)$$

$$\int_{\Omega} u(x, y) v(x, y) dA = \int_{\Omega} u_0(x, y) v(x, y) dA \quad (11)$$

Let's handle the main PDE, and let's call $u = u^{n+1}(x, y)$ which is supposed to be sought at every increment and all state variable of $n + 1$.

$$\int_{\Omega} uv dA - \int_{\Omega} v \nabla^2 u \Delta t dA = \int_{\Omega} (u^n + f \Delta t) v dA \quad (12)$$

The second term

$$\Delta t \int_{\Omega} v \nabla^2 u dA = \Delta t \int_{\Omega} \nabla \cdot (v \nabla u) dA - \Delta t \int_{\Omega} \nabla v \cdot \nabla u dA \quad (13)$$

The second integral can be taken over the boundary.

$$\int_{\Omega} \nabla \cdot (v \nabla u) dA = \int_{\partial\Omega} v(\nabla u) \cdot \mathbf{n} dc \quad (14)$$

so because $v(x, y)$ vanishes on the boundary

$$\int_{\Omega} \nabla \cdot (v \nabla u) dA = \int_{\partial\Omega} v(\nabla u) \cdot \mathbf{n} dc = 0 \quad (15)$$

and thus the weak form of the PDE becomes

$$\Delta t \int_{\Omega} v \nabla^2 u dA = -\Delta t \int_{\Omega} \nabla v \cdot \nabla u dA \quad (16)$$

$$\int_{\Omega} uv dA + \Delta t \int_{\Omega} \nabla v \cdot \nabla u dA = \int_{\Omega} (u^n + f \Delta t) v dA \quad (17)$$

$$\begin{aligned}
a_{n+1}(u, v) &= \int_{\Omega} uv \, dA + \Delta t \int_{\Omega} \nabla v \cdot \nabla u \, dA \\
L_{n+1}(u, v) &= \int_{\Omega} (u^n + f \Delta t) v \, dA
\end{aligned} \tag{18}$$

and

$$a_{n+1}(u, v) = L_{n+1}(u, v) \tag{19}$$

Second the initial condition.

$$\int_{\Omega} u(x, y) v(x, y) \, dA = \int_{\Omega} u_0(x, y) v(x, y) \, dA \tag{20}$$

and

$$\int_{\Omega} uv \, dA = \int_{\Omega} u_0 v \, dA \tag{21}$$

$$a_0(u(x, y), v(x, y)) = L_0(u_0(x, y), v(x, y)) \tag{22}$$

We project the values of the initial conditions to the nodal points.

Solution

So we are supposed to find the solution

$$u(x, y; t) \in V \tag{23}$$

and the test function is

$$v(x, y; t) \in \hat{V} \tag{24}$$

We do the descritization so at each time t

$$u = \sum_{i=1}^N U_i \phi_i \quad (25)$$

$$v = \sum_{i=1}^N V_i \phi_i \quad (26)$$

at the time $t = 0$

$$u^0 = \sum_{i=1}^N U_i^0 \phi_i \quad (27)$$

Exact Solution

Let's say the solution is

$$u = 1 + x^2 + \alpha y^2 + \beta t \quad (28)$$

Plugging it into the PDE gives

$$\beta = 2 + 2\alpha + f(x, y, t) \quad (29)$$

thus

$$f(x, y, t) = \beta - 2 - 2\alpha \quad (30)$$

Let's assume the BC is just what we have for PDE, so

$$u_D = 1 + x^2 + \alpha y^2 + \beta t \quad \text{on} \quad (x, y) \in \Omega \quad (31)$$

so the initial condition is

$$u(x, y, t = 0) = u^0 = 1 + x^2 + \alpha y^2 \quad (32)$$

Thus, the exact solution of the heat equation

$$\frac{\partial u}{\partial t} = \nabla^2 u + \beta - 2 - 2\alpha \quad (33)$$

with the bounday conditions everywhere on the boundary:

$$u_D = 1 + x^2 + \alpha y^2 + \beta t \quad (34)$$

and the initial condition everywhere

$$u^0 = 1 + x^2 + \alpha y^2 \quad (35)$$

is

$$u(x, y, t) = 1 + x^2 + \alpha y^2 + \beta t \quad (36)$$

Finite Element Solution