

Mathematics of Finance Project 1

Renee Prager and M.K.

September 2025

1 Amortized Loan Formula

Consider a loan of principal P . The loan is to be repaid in T years with m fixed payments per year. The annual interest rate is r and is compounded periodically based on payment frequency (i.e., per-payment rate is r/m). Here we ignore the day-count complications and assume payments are made at the end of each payment period.

1.1

Derive the installment amount C as a function of P , T , m , and r .

In class we found that for an amortized loan, the loan installment equation is

$$C = \frac{Pr}{1 - (1 + r)^{-T}},$$

derived from the present value formula

$$P = C \frac{1 - (1 + r)^{-T}}{r}.$$

Here we are told that the interest per period is $\frac{r}{m}$ and the total number of payments is mT , so if we plug them into the equation we get

$$C = \frac{P \left(\frac{r}{m} \right)}{1 - \left(1 + \frac{r}{m} \right)^{-mT}}.$$

1.2

For each payment, calculate the amount that goes into principal and interest, respectively.

The interest portion (I_k) is the interest rate times the balance before payment (B_{k-1}) for payment k :

$$I_k = \frac{r}{m} B_{k-1}.$$

The principal portion is the fixed payment at time k (C) minus what was given as interest:

$$P_k = C - I_k = C - \frac{r}{m} B_{k-1}.$$

1.3

Write a Python function that takes in arguments P , T , m , and r , and outputs (a) C (b) the amount that goes into principal and interest for each payment. Python code in Appendix

1.4

An adjustable-rate mortgage (ARM) is a home loan with an interest rate that is fixed for an initial period and then fluctuates based on market conditions. With $P = 1,000,000$, $T = 30$ (years), and $m = 12$,

- (a) For the first 5 years the bank asks for an interest rate $r_1 = 6\%$, calculate the amount of each payment C_1 .
- (b) For the next year the bank asks for an interest rate $r_2 = 8\%$, calculate the amount of each payment C_2 .
- (c) For the next year the bank asks for an interest rate $r_3 = 10\%$, calculate the amount of each payment C_3 .
- (d) For the rest of the years the bank asks for an interest rate $r_4 = 12\%$, calculate the amount of each payment C_4 .

Given:

$$P = \$1,000,000, \quad T = 30, \quad m = 12$$

- (a) First 5 years: Interest rate = 6%**

$$\frac{r}{m} = \frac{0.06}{12} = 0.005, \quad mT = 30 \cdot 12 = 360$$

$$C = \frac{1,000,000(0.005)}{1 - (1.005)^{-360}} \approx \$5,995$$

- (b) Year 6: Interest rate = 8%**

There are now 25 years left, so

$$\frac{r}{m} = \frac{0.08}{12}, \quad mT = 25 \cdot 12 = 300$$

The loan balance at this point can be found using the equation

$$B = P \frac{(1+i)^n - (1+i)^p}{(1+i)^n - 1},$$

where

$$n = 360, \quad p = 60, \quad i = 0.005.$$

So the balance is

$$B \approx \$930,600.$$

Now we compute the new payment:

$$C = \frac{930,600 \left(\frac{0.08}{12}\right)}{1 - \left(1 + \frac{0.08}{12}\right)^{-300}} \approx \$7,172.$$

(c) **Year 7: Interest rate = 10%**

$$\frac{r}{m} = \frac{0.10}{12} = 0.008333, \quad mT = 24 \cdot 12 = 288$$

The loan balance is

$$B \approx \$918,529.$$

The new installment is

$$C = \frac{918,529(0.008333)}{1 - (1.008333)^{-288}} \approx \$8,454.$$

(d) **Remaining Time: Interest rate = 12%**

$$\frac{r}{m} = \frac{0.12}{12} = 0.01, \quad mT = 23 \cdot 12 = 276$$

The remaining balance is

$$B \approx \$908,270.$$

Thus the installment for the remaining loan term is

$$C = \frac{908,270(0.01)}{1 - (1.01)^{-276}} \approx \$9,739.$$

1.2

1)

Proposition 1 states that a contingent claim $C(T)$ can be replicated by investing in a portfolio (θ_1, θ_2) of the risk-free asset and the underlying risky asset. That is,

$$C(T) = \theta_1 A(T) + \theta_2 S(T).$$

Then,

$$C(0) = \theta_1 A(0) + \theta_2 S(0),$$

otherwise an arbitrage opportunity would arise.

Under a simple market model, a similar argument can be made to derive the fair price of a put option $P(0)$.

We know that

$$P(T) = \max(K - S(T), 0).$$

If a put option is replicated so that

$$P(T) = \theta_1 A(T) + \theta_2 S(T),$$

then in order to prevent arbitrage, it must be that

$$P(0) = \theta_1 A(0) + \theta_2 S(0).$$

2)

Using the above proposition, we can price a put option with:

$$K = 19.5, \quad A(0) = 100, \quad A(T) = 105, \\ S(0) = 20, \quad S_u(T) = 22, \quad S_d(T) = 19.$$

We replicate the option.

Upper state:

$$\theta_1 A(T) + \theta_2 S_u(T) = \max(K - S_u(T), 0) \\ \theta_1(105) + \theta_2(22) = \max(19.5 - 22, 0) = 0.$$

Lower state:

$$\theta_1 A(T) + \theta_2 S_d(T) = \max(K - S_d(T), 0) \\ \theta_1(105) + \theta_2(19) = \max(19.5 - 19, 0) = 0.5.$$

Solving the System

Solve the up and down equations together:

$$-\theta_2(19) + 0.5 = -\theta_2(22) \\ -3\theta_2 = 0.5 \quad \Rightarrow \quad \theta_2 = -\frac{1}{6}.$$

Substitute into the upper-state equation:

$$\theta_1(105) + \left(-\frac{1}{6}\right)(22) = 0 \\ \theta_1 = \frac{11}{315}.$$

Put Option Price

Plug into the pricing equation:

$$P(0) = \theta_1 A(0) + \theta_2 S(0) \\ P(0) = \left(\frac{11}{315}\right)(100) + \left(-\frac{1}{6}\right)(20) \approx 0.15873.$$

A Python Code

```
# -*- coding: utf-8 -*-
"""project 1 code

Automatically generated by Colab.

Original file is located at
https://colab.research.google.com/drive/16Kzo32EH_lDU18th9kT2j--bPeosGCB4

Project 1, Question 3
"""

# P is the principal of the loan
P=100
# T is the lifespan of the loan
T=30
#r is the annual rate
r=0.1
# m is the number of payments per year
m=12

#formula for fixed payment (C)
P * (1-(1+(r/m))**(-1))/((1+r/m)**(-1)-(1+r/m)**(-T*m-1))

def my_loan_calculator (principal,life_span,payments_per_year,rate):

    #compute the discount factor for one period
    discount_factor= (1+rate/payments_per_year)**(-1)

    #compute payment per period
    amount_per_payment= principal*(1-discount_factor)/(discount_factor-discount_factor**(life_

        return amount_per_payment

#calculate each payment using given values
my_loan_calculator(100,20,12,0.05)

# Loan Schedule
def loan_schedule(principal,life_span,payments_per_year,rate):

    #loan schedule calculated using loan calculator
    payment_amount=my_loan_calculator(principal,life_span,payments_per_year,rate)

    #start with initial loan balance
    balance=principal
```

```

#loop for all payments (here 360)
for i in range(life_span*payments_per_year):

    #calculate interest owed this payment
    accrued_interest=balance*rate/payments_per_year

    #calculate principal portion owed this payment
    principal_paid = payment_amount - accrued_interest

    print(f"At time {i+1}, the balance before payment is {balance}, the principal\\
paid is {principal_paid}, the accrued interest is {accrued_interest}")

    #update balance
    balance=balance-(payment_amount-accrued_interest)

print(f"At time {i+1}, the final balance is {balance}")

return balance

loan_schedule(100,10,2,0.05)

```