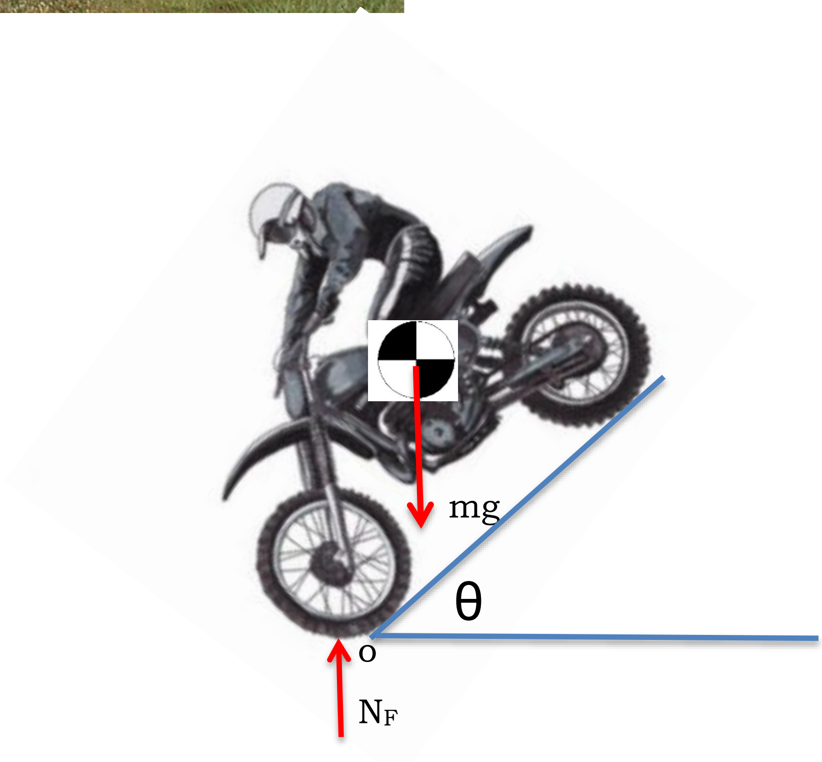
**SA 1**. [5 marks] At the instant shown in the photograph below the motorcycle is stationary.

a) [2 marks] Draw the free body diagram of the motorcycle and rider. What physical measurements would you have to take to define the quantities in the free body diagram?

b) [3 mark] Write the equations of motion of the motorcycle.



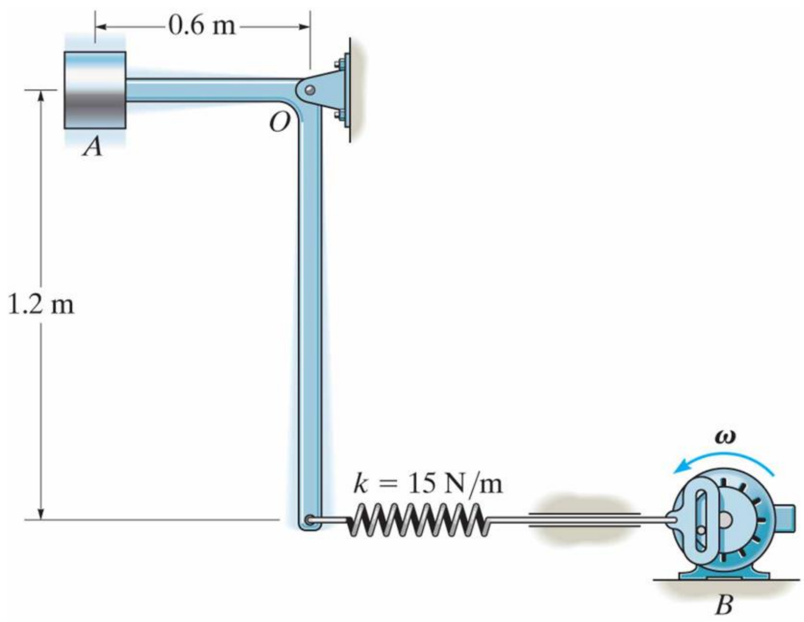

a) We would need to know the location of the centre of gravity of the combined motorcycle and rider and the mass of the combined motorcycle and rider and the vector from the centre of gravity to the front wheel contact point $\mathbf{r}_{o/CG}$.

b)

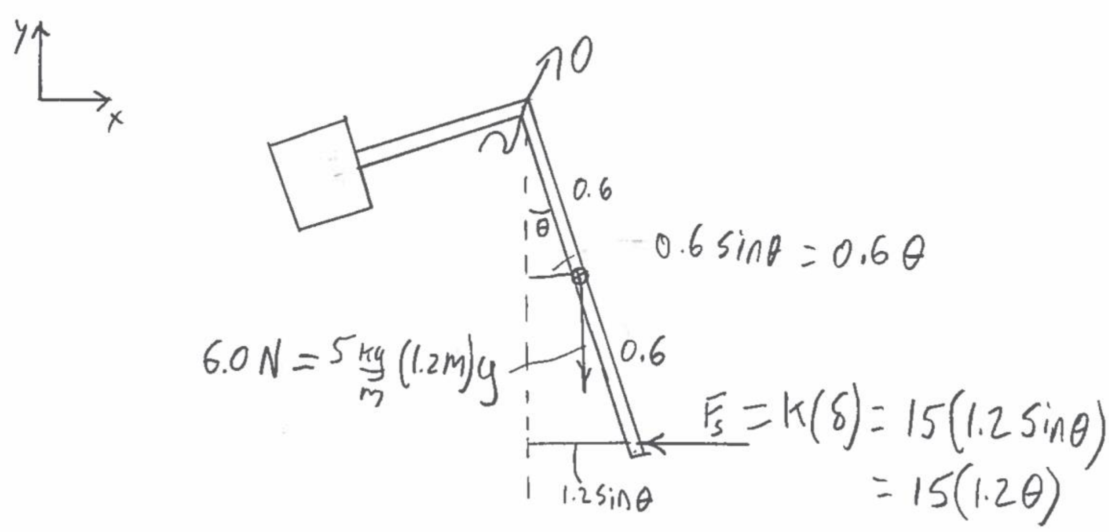
$$\sum M_o = I_o \ddot{\theta}$$

$$r_{CG/o} \times mg = I_o \ddot{\theta}$$

**Prob 1 [25 Marks]** The small block at A has width 0.2m and height 0.3m, and unknown mass $m$. The block is mounted on the slender bent rod with mass 5 kg/m. Before switching on the motor at B, the system is at equilibrium as shown in the figure below. The length of the shorter end of the rod is 0.5m.




A) [2 marks] Draw the free body diagram of the system before the motor at B is switched on.



B) [8 marks] Use the small angle approximation to **prove** that the equation of motion of the system (before the motor at B is switched on) is approximately

$$(0.37083m + 3.0883)\ddot{\theta} + 56.916\theta = 0$$

where $m$ is the unknown mass of the block, and $\theta$ is the angular position (positive in the counter-clockwise direction) relative to position shown in the figure. **Do not solve this equation.**

The rotation of the rigid body about the point $O$ is:

$$\sum M_O = I_O\ddot{\theta}$$

The sum of the moments about $O$ are:

$$\sum M_O = -\,5(1.2)(9.81)(0.6)\theta - 1.2^2(15)\theta = -56.916\theta$$

Here we used the small angle approximation $\sin\theta \approx \theta$. The moment of inertia of the body about $O$ is

$$I_O = \frac{1}{12}m(0.2^2 + 0.3^2) + m(0.6)^2 + \frac{1}{12}5(0.5)(0.5)^2 + 5(0.5)(0.25)^2 + \frac{1}{12}5(1.2)(1.2)^2 + 5(1.2)(0.6)^2 = 0.37083m + 3.0883$$

The equation of motion is

$$(0.37083m + 3.0883)\ddot{\theta} + 56.916\theta = 0$$

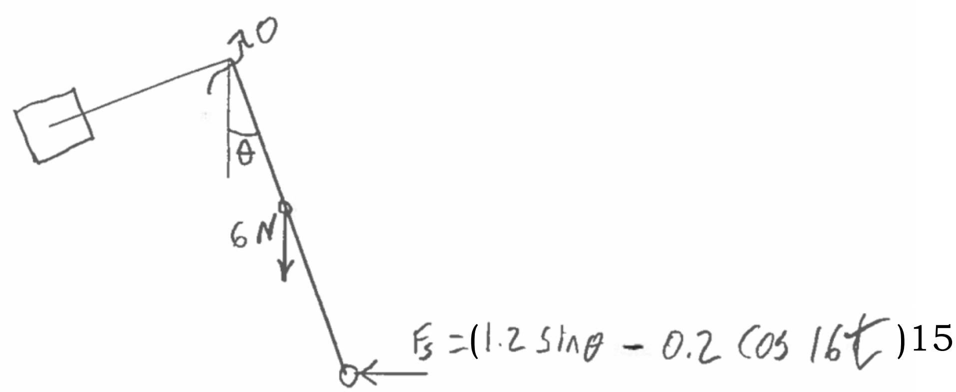
C) [2 marks] Find the value of the mass $m$ that makes the natural frequency of the system equal to 2 radians per second.

From the equation in part B, we see that the natural frequency is

$$\omega_n = \sqrt{\frac{56.916}{0.37083m + 3.0883}}$$

Set $\omega_n = 2$ and solve for $m = 30.043\ kg$.

D) [5 marks] The mass of the block is now known to be 5kg, the motor at B is switched on and the motion at B is given by $\delta_B = 0.2\cos(16t)$. The position $\delta_B$ is measured in meters (the positive direction is to the right) and $t$ is measured in seconds. Draw the free body diagram and determine the equation of motion.




*as before but with the* $0.2\cos 16t$ *m forcing function on the right side of the spring*

$$(0.37083m + 3.0883)\ddot{\theta} + 56.916\theta = 3.6\cos 16t$$

E) [8 marks] Find the general solution of the equation in part D.

The equation is given by

$$4.94225\ddot{\theta} + 53.316\theta = 3.6\cos(16t)$$
$$\ddot{\theta} + 10.787\theta = 0.72838\cos(16t)$$

The natural frequency in this case is $\omega_n = 3.2844$ and the complimentary solution is

$$\theta_c(t) = C_1\sin(3.2844t) + C_2\cos(3.2844t)\ , \quad C_1, C_2 \in \mathbb{R}$$

A particular solution is of the form $\theta_p(t) = A\cos(16t)$ and we compute

$$\ddot{\theta}_p + 10.787\theta_p = A(-(16^2) + 10.787)\cos(16t) = 0.72838\cos(16t)$$

therefore $A(-(16^2) + 10.787) = 0.72838$ and so $A = -0.0029794$. The general solution is

$$\theta(t) = -0.0029794\cos(16t) + C_1\sin(3.2844t) + C_2\cos(3.2844t)$$

for $C_1, C_2 \in \mathbb{R}$.

**THE UNIVERSITY OF BRITISH COLUMBIA**
**FACULTY OF APPLIED SCIENCE**
# DEPARTMENT OF MECHANICAL ENGINEERING

# MECH 221

# TEST #7, November 17th, 2016

**Suggested Time**: 100 minutes **Allowed Time:** 110 minutes

**Materials admitted**: Pencil, eraser, straightedge, MECH 2 Approved Calculator (Sharp EL-510), one 3x5 inch index card or sheet of paper for hand-written notes.

There are 4 Short-Answer Questions and 2 Long-Answer Problems on this test. All questions must be answered.

Provide **all** work and solutions **on this test**. Orderly presentation of work is required for solutions to receive full credit. **Illegible work, or answers that do not include supporting calculations and explanations will NOT BE MARKED**.

**FILL OUT THE SECTION BELOW AND WRITE YOUR NAME ON THE TOP OF ALL TEST PAGES. Do this during the examination time as additional time will not be allowed for this purpose.**

NAME:_______________________________________Section__________

SIGNATURE:_________________________________

STUDENT NUMBER:__________________________

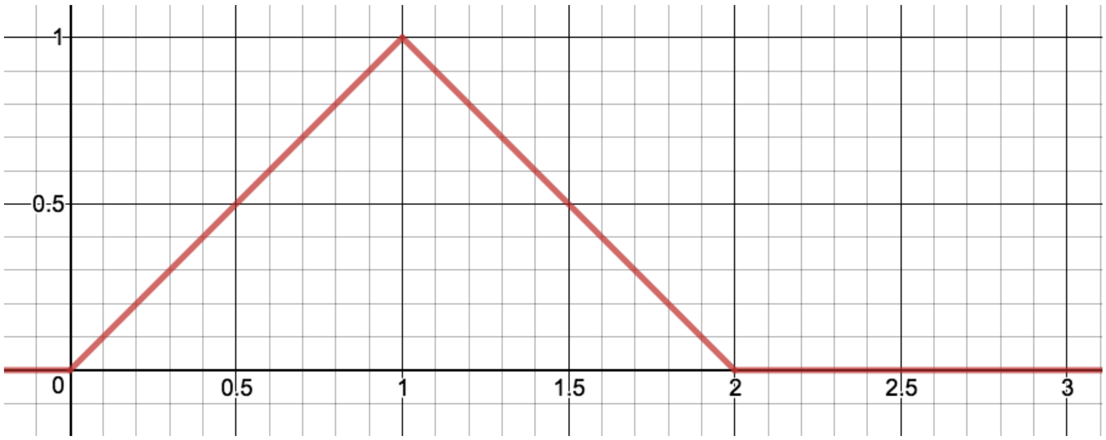
| Question | Mark Received | Maximum Mark |
|---|---|---|
| SA 1 | | 5 |
| SA 2 | | 5 |
| SA 3 | | 5 |
| SA 4 | | 5 |
| Prob 1 | | 25 |
| Prob 2 | | 25 |

Name:______________________________ Section:__________________________

**SA1 [5 marks].** Find the Laplace transform $Y(s)$ of the solution $y(t)$ of the differential equation

$$2y'' + 3y' - y = f(t), \; y(0) = 0, \; y'(0) = 0$$

where the forcing function $f(t)$ is given by




**(Note: Do not solve for $y(t)$.)**

First, find a formula for $f(t)$

$$\begin{aligned} f(t) = &\; t\big(u(t) - u(t-1)\big) + (2-t)\big(u(t-1) - u(t-2)\big) \\ &= t \cdot u(t) - 2(t-1) \cdot u(t-1) + (t-2) \cdot u(t-2) \end{aligned}$$

Use the formula

$$\mathcal{L}\{u(t-a)f(t-a)\} = e^{-as}F(s)$$

to compute the Laplace transform of $f(t)$

$$F(s) = \frac{1}{s^2} - \frac{2e^{-s}}{s^2} + \frac{e^{-2s}}{s^2}$$

Applying the Laplace transform to the differential equation yields

$$2s^2Y(s) + 3sY(s) - Y(s) = F(s)$$

$$Y(s) = \frac{F(s)}{2s^2 + 3s - 1} = \frac{1 - 2e^{-s} + e^{-2s}}{s^2(2s^2 + 3s - 1)}$$

Name:______________________________ Section:__________________________

**SA2 [5 marks].** Suppose an output signal $y(t)$ is related to an input signal $x(t)$ by the differential equation

$$y'' + 3y' + y = 3x' - x\,.$$

(a) [1 mark]. Find the transfer function $H(s) = \dfrac{Y(s)}{X(s)}$.

Apply the Laplace transform to the equation and solve

$$s^2Y(s) + 3sY(s) + Y(s) = 3sX(s) - X(s)\ , \quad H(s) = \frac{Y(s)}{X(s)} = \frac{3s-1}{s^2+3s+1}$$

(b) [4 marks]. Suppose $x(t) = \sin(\omega t)$. Find values $\omega > 0$ such that the amplitude of the steady state response is less than 1. In other words, find values $\omega > 0$ such that

$$\lim_{t\to\infty}|y(t)| = |y_{ss}(t)| < 1$$

The amplitude of the steady state response is given by

$$|H(j\omega)| = \left|\frac{3j\omega - 1}{-\omega^2 + 3j\omega + 1}\right| = \frac{\sqrt{9\omega^2+1}}{\sqrt{9\omega^2+(1-\omega^2)^2}} = \frac{\sqrt{9\omega^2+1}}{\sqrt{\omega^4+7\omega^2+1}} < 1$$

Therefore the question is

$$\sqrt{9\omega^2+1} < \sqrt{\omega^4+7\omega^2+1}$$

$$9\omega^2 + 1 < \omega^4 + 7\omega^2 + 1$$

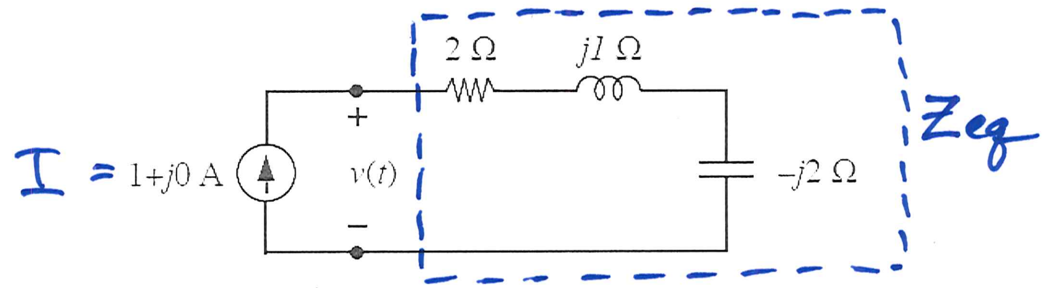
$$2\omega^2 < \omega^4$$

$$2 < \omega^2$$

and so $\omega > \sqrt{2}$.

Name:______________________________ Section:________________________

**SA 4 [5 Marks].** Consider the following circuit that is operating in the sinusoidal steady-state at angular frequency $\omega = 500$ rad/s.




(a) [3 marks]. Determine the time-domain voltage $v(t)$.

$$Z_{eq} = 2\Omega + j(1\Omega - 2\Omega) = 2 - j1\,\Omega$$
$$= 2.24\angle{-26.57^\circ}\,\Omega$$

$$V = I \cdot Z_{eq}$$
$$= (1\angle 0^\circ)(2.24\angle{-26.57^\circ})$$
$$= 2.24\angle{-26.57^\circ}\ V.$$

$$\therefore\ v(t) = 2.24\cos(500t - 26.57^\circ)\ V.$$

(b) [2 marks]. Determine the values of the *inductance* (in Henries) and *capacitance* (in Farads).

$$\omega = 500\ rs^{-1} \Rightarrow j1\Omega = j(500)L$$
$$\therefore\ L = 2mH = 2\times10^{-3}\ H.$$
$$-j2\Omega = -j\frac{1}{(500)C}$$
$$\therefore\ C = 1mF = 1\times10^{-3}\ F.$$

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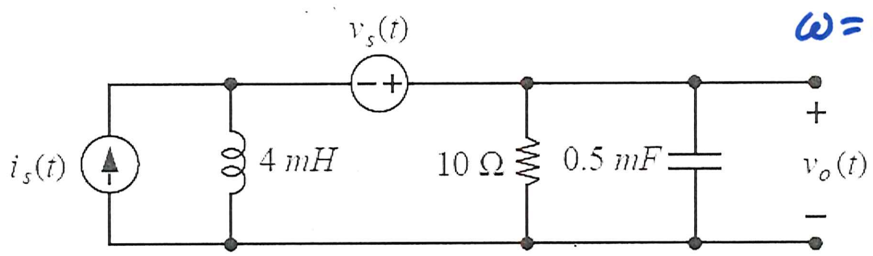
Name:________________________ Section:________________________

**Prob 1 [25 Marks].** Consider the circuit shown below that is operating in the steady-state with $i_s(t) = 4\cos(1000t)\ A$ and $v_s(t) = 12\cos(1000t + 90°)\ V$.

$\omega = 1000\ rs^{-1}$




(a) [3 marks]. Determine the phasor impedances for the inductor and capacitor, and write the time-domain source functions $i_s(t)$ and $v_s(t)$ as phasors $\mathbf{I}_s$ and $\mathbf{V}_s$.

$\omega = 1000\ rs^{-1} \Rightarrow Z_L = j\omega L = j(1000)(4mH) = j4\ \Omega$

$Z_C = \frac{1}{j\omega C} = -j\frac{1}{(1000)(0.5mF)}$

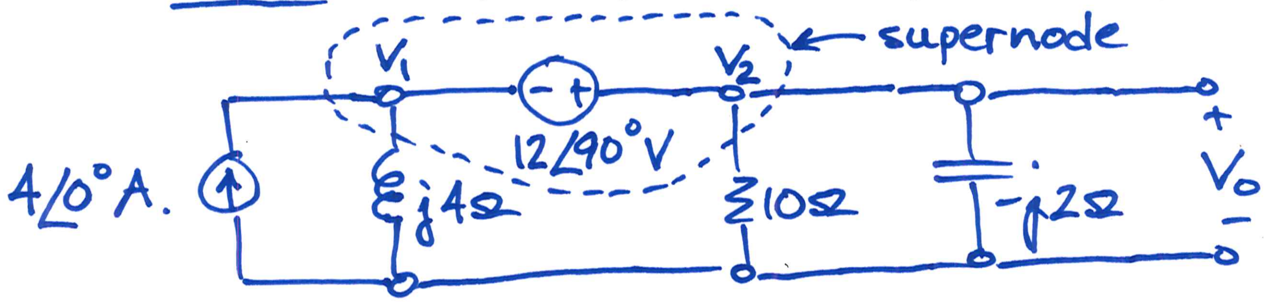
$= -j2\ \Omega$

$I_s = 4\angle 0° = 4 + j0\ A.$

$V_s = 12\angle 90° = 0 + j12\ V.$

**Prob 1**, *Cont'd.*

(b) [7 marks]. Use nodal analysis to determine $\mathbf{V}_O$, the phasor representation of $v_o(t)$. (*Suggestion: it may prove useful to redraw the circuit here with all of the components labeled in phasor notation.*)




KCL @ supernode: $-4\angle 0^\circ + \dfrac{V_1}{4\angle 90^\circ} + \dfrac{V_2}{10} + \dfrac{V_2}{2\angle -90^\circ} = 0$

constraint: $V_1 = V_2 - 12\angle 90^\circ$

also, $V_2 = V_o$

$$\rightarrow \frac{V_o}{4\angle 90^\circ} - \frac{12\angle 90^\circ}{4\angle 90^\circ} + \frac{V_o}{10} + \frac{V_o}{2\angle -90^\circ} = 4\angle 0^\circ$$

$$V_o\Big(\underbrace{0.25\angle -90^\circ + 0.1 + 0.5\angle 90^\circ}_{\substack{=0.1+j0.25 \\ =0.269\angle 68.2^\circ}}\Big) = \underbrace{4\angle 0^\circ + 3\angle 0^\circ}_{=7\angle 0^\circ}$$

$$\therefore V_o = \frac{7\angle 0^\circ}{0.269\angle 68.2^\circ} = 26\angle -68.2^\circ \text{ V.}$$

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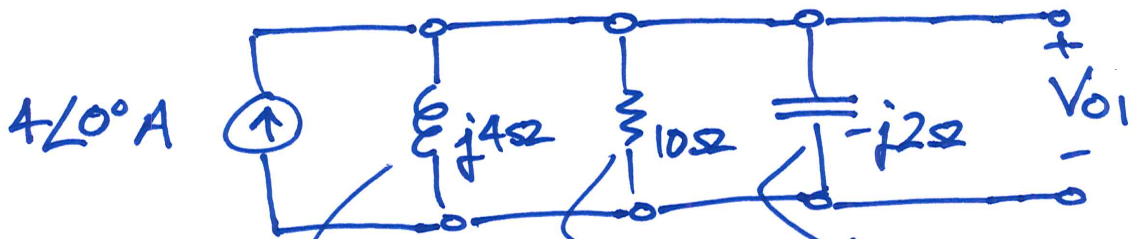
Name:________________________ Section:____________________

**Prob 1**, *Cont'd.*

(c) [12 marks]. Use *superposition analysis* to determine the phasor voltage $\mathbf{V}_O$, and write the corresponding time-domain function $v_o(t)$.

Eq. cct. for current source alone:




$$Y_L = \frac{1}{4\angle 90^\circ} = 0.25\angle -90^\circ \qquad Y_R = 0.1 \qquad Y_C = \frac{1}{2\angle -90^\circ} = 0.5\angle 90^\circ$$

$$\begin{aligned} Y_{eq} &= Y_L + Y_R + Y_C \\ &= 0.25\angle -90 + 0.1 + 0.5\angle 90^\circ \\ &= -j0.25 + 0.1 + j0.5 \\ &= 0.1 + j0.25 \\ &= 0.269\angle 68.2^\circ \end{aligned}$$

$$\begin{aligned} V_{o1} = \frac{4\angle 0^\circ}{0.269\angle 68.2^\circ} &= 14.87\angle -68.2^\circ \text{ V.} \\ &= 5.522 - j13.81 \text{ V.} \end{aligned}$$

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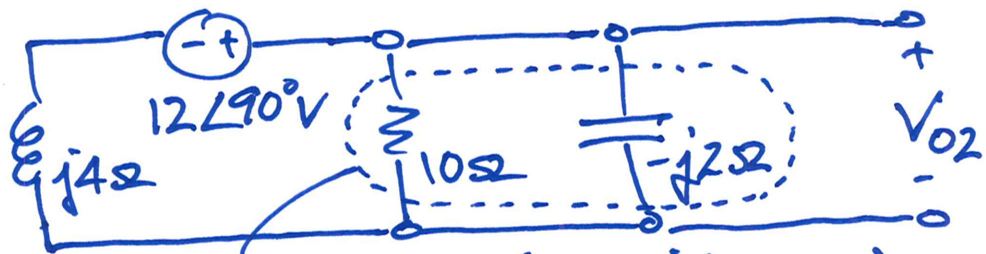
Eq. cct. for voltage source alone:




$$Z_{eq} = \frac{(10\angle 0^\circ)(2\angle -90^\circ)}{\underbrace{(10-j2)}_{=10.2\angle -11.31^\circ}}$$

$$= 1.96\angle -78.69^\circ$$

$$= 0.385 - j1.923\ \Omega$$

Voltage Div:

$$V_{O2} = (12\angle 90^\circ) \cdot \frac{(1.96\angle -78.69^\circ)}{\underbrace{0.385 + j(4-1.923)}_{2.11\angle 79.5^\circ}}$$

$$= 11.15\angle -68.2^\circ\ V.$$

$$= 4.14 - j10.35\ V.$$

$$\therefore V_O = V_{O1} + V_{O2} = (5.522+4.14) - j(13.81+10.35)$$

$$= 25.9\angle -68.8^\circ$$

$$\therefore v_o(t) = 25.91\cos(1000t - 68.8^\circ)\ V.$$

Name:________________________ Section:______________________

**Prob 1**, *Cont'd.*

(d) [3 marks]. Suppose the source functions $i_s(t)$ and $v_s(t)$ were instead operating at <u>different</u> frequencies, say $\omega_1$ and $\omega_2$. Briefly, in just a few sentences, outline how you would approach solving for $v_0(t)$.

You must use superposition. The component impedances must be calculated separately for each of the equiv. ccts. The final summation must be done in the <u>time</u>-domain.