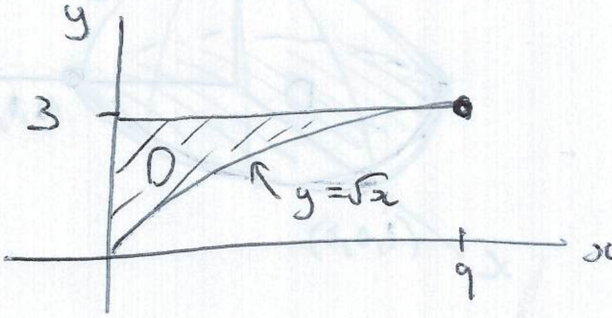
3. For the iterated integral

$$\int_0^9 \int_{\sqrt{x}}^3 \sin(\pi y^3)dydx,$$

sketch the region in the xy-plane that is being integrated over, and then evaluate the integral.

The region D in question is bounded by $x = 0$, $x = 9$, $y = \sqrt{x}$, and $y = 3$:



To evaluate, we should reverse the order of integration (because we don't know an antiderivative for $\sin(\pi y^3)$ with respect to y):

$$\int_0^9 \int_{\sqrt{x}}^3 \sin(\pi y^3)dydx = \iint_D \sin(\pi y^3)\, dA = \int_0^3 \int_0^{y^2} \sin(\pi y^3)dxdy$$

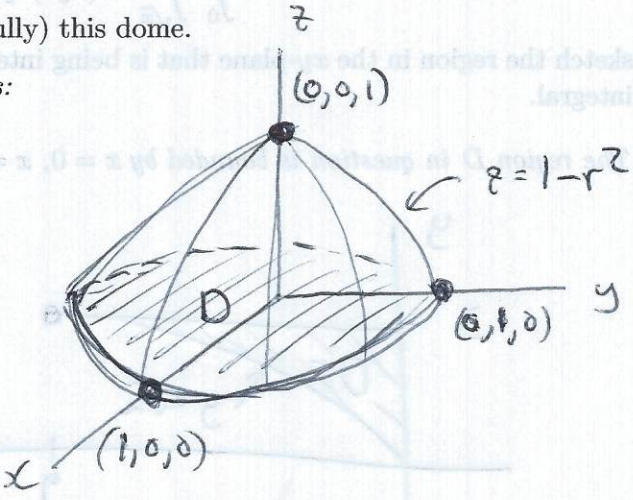
$$= \int_0^3 \left[x \sin(\pi y^3)|_{x=0}^{x=y^2} \right] dy = \int_0^3 y^2 \sin(\pi y^3)dy = -\frac{1}{3\pi} \cos(\pi y^3)|_0^3$$

$$= \frac{1}{3\pi} \left(-\cos(27\pi) + \cos(0)\right) = \frac{2}{3\pi}$$

6. A solid "dome" occupies the region of 3-space bounded by the xy-plane and a paraboloid: $0 \leq z \leq 1 - x^2 - y^2$.

(a) Draw (reasonably carefully) this dome.

The dome looks like this:



(b) Set up and evaluate a double integral to compute the volume of this dome.

Since the dome occupies the region above the unit disk

$$D = \{(x, y) \mid x^2 + y^2 \leq 1\} = \{(r, \theta) \mid 0 \leq \theta < 2\pi, \ 0 \leq r \leq 1\}$$

(in polar coordinates) in the xy-plane, and below the graph $z = f(x, y) = 1 - x^2 - y^2 = 1 - r^2$ (again expressed in polar coordinates), we have (doing the integral in polar coordinates, which is not necessary, but convenient):

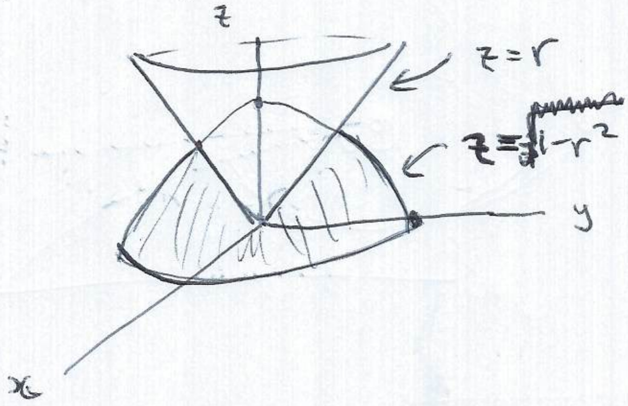
$$V = \int\int_D f dA = \int_0^{2\pi} \int_0^1 (1-r^2) r dr \, d\theta = \int_0^{2\pi} \left(\frac{1}{2} r^2 - \frac{1}{4} r^4 \right) |_0^1 \, d\theta = \int_0^{2\pi} \frac{1}{4} d\theta = \frac{\pi}{2}.$$

(c) Without computing any more antiderivatives: how does the volume change if the dome shape is changed to $0 \leq z \leq 1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}$ for some $a > 0$, $b > 0$.

This shape results from the original one by doing the scaling $x \mapsto ax$, $y \mapsto bx$ (and $z \mapsto z$) (resulting in elliptical horizontal cross-sections), and so the volume gets scaled by ab: $V_{new} = abV$. Alternate solution: set up an iterated integral to compute V_{new} (in rectangular coordinates), and make the changes of variables $x = a\tilde{x}$, $y = b\tilde{y}$ to see that $V_{new} = abV$.

(d) A conical drill bores into the (original $a = b = 1$) dome from above, removing the portion $z \geq \sqrt{x^2+y^2}$. Sketch the remaining part of the dome, set up a double integral to compute its volume, and evaluate.

The paraboloid meets the cone $z = \sqrt{x^2+y^2}$ *where* $r = \sqrt{x^2+y^2} = 1-x^2-y^2 = 1-r^2$, *so* $r^2+r-1=0$, *and* $r = -\frac{1}{2}+\sqrt{\frac{5}{4}} = (\sqrt{5}-1)/2 =: \bar{r}$ *(we ignore the negative root). The new region looks like:*



This volume can be computed in a similar way to the original one, but now the 'roof' is given by the cone $z = r$ *for* $0 \leq r \leq \bar{r}$ *and by the paraboloid* $z = 1 - r^2$ *for* $\bar{r} \leq r \leq 1$ *and we can simply add these two contributions:*

$$V = \int_0^{2\pi}\int_0^{\bar{r}} r\, r dr\, d\theta + \int_0^{2\pi}\int_{\bar{r}}^{1} (1-r^2) r dr\, d\theta = \int_0^{2\pi}\left[\frac{1}{3}\bar{r}^3 + \frac{1}{2} - \frac{1}{4} - \frac{1}{2}\bar{r}^2 + \frac{1}{4}\bar{r}^4\right] d\theta$$

$$= 2\pi\left[\frac{1}{4} + \frac{1}{4}\bar{r}^4 + \frac{1}{3}\bar{r}^3 - \frac{1}{2}\bar{r}^2\right]$$

(which could be further simplified if necessary).