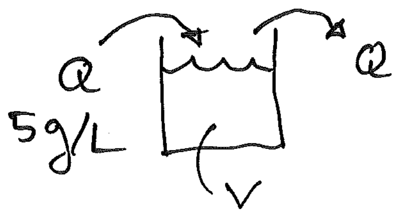
**Prob 1 [25 marks] Parts (a), (b), (c) and (d) below are independent.**

(a) [9 marks] A tank contains a volume $V$ (measured in litres, L) of a solution with 35g/L salt concentration. The engineers controlling the system would like to reduce the salt concentration by pouring a 5g/L salt solution into the tank and pumping the solution out of the tank at the *same* rate so that the volume $V$ of solution in the tank does not change over time. What flow rate should the engineers choose so that the concentration in the tank is 20g/L after 2 hours? (The answer depends on the volume $V$.)




Let M be the mass of salt in the tank.

$$\frac{M(0)}{V} = 35 g/L \qquad \frac{dM}{dt} = 5\frac{g}{L} \cdot Q - \frac{M}{V} Q$$

$$\frac{dM}{dt} = 5Q - \frac{Q}{V} M \Rightarrow M' + \frac{Q}{V} M = 5Q$$

Let $p(t) = \frac{Q}{V}$, $f(t) = 5Q$ and $r(t) = e^{\int p(t) dt} = e^{\frac{Q}{V} t}$

$$\Rightarrow M(t) = \frac{1}{e^{\frac{Q}{V}t}} \left( \int e^{Qt/V} \cdot 5Q \, dt \right)$$

$$= e^{-Qt/V} \left( 5Q \frac{V}{Q} e^{Qt/V} + C \right)$$

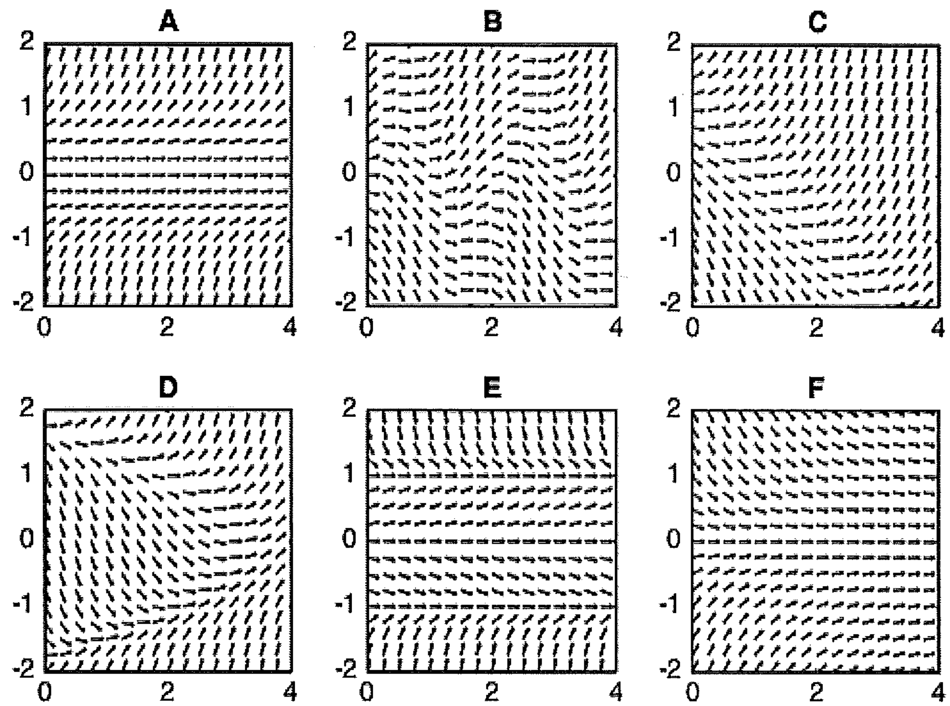
$$M(t) = 5V + C e^{-Qt/V}$$

Plug in $M(0) = 35V$, $\quad 35V = 5V + C \quad \underline{C = 30V}$

Plug in $M(2) = 20V \quad 20V = 5V + 30V e^{-2Q/V}$

$$\Rightarrow \frac{1}{2} = e^{-2Q/V} \Rightarrow \boxed{Q = \frac{\ln(2)}{2} V} \ L/h.$$

(b) Each figure is the slope field of a first order differential equation.




(i) [2 marks] Which figures represent <u>autonomous</u> differential equation? (-1 mark for each incorrect response.)

A E

(ii) [2 marks] The function $y(t) = e^t - t$ is a solution of the differential equation represented by which slope field?

$y(0) = 1 \quad y'(0) = 0 \Rightarrow$ C

(iii) [2 marks] Which figure represents the slope field of the differential equation $y' = t + y^2 - 3$?

D

(c) [5 marks] Let $y(t)$ be the unique solution of the differential equation

$$y' = \sin(y) + \cos(y)$$

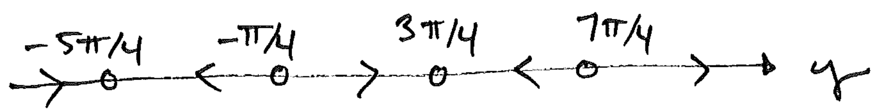
satisfying the initial condition $y(0) = -1$. Compute $\lim_{t\to\infty} y(t)$.

This is an autonomous equation. Find critical points $\Rightarrow$ $0 = \sin(y) + \cos(y)$

$$\sin(y) = -\cos(y)$$

$$\Rightarrow \quad y = \frac{3\pi}{4}, \frac{7\pi}{4}, \frac{11\pi}{4}, \ldots, \frac{-\pi}{4}, \frac{-5\pi}{4}, \frac{-9\pi}{4}, \ldots$$

Sketch phase line




$\Rightarrow$ If $y(t)$ is the solution for $y(0)=-1$ then $\boxed{\lim_{t\to\infty} y(t) = -\frac{5\pi}{4}}$

$\left(\text{since } -\frac{5\pi}{4} < -1 < \frac{-\pi}{4}.\right)$

(d) [5 marks] Give an example of a first order autonomous differential equation with exactly three critical points such that two are stable and one is unstable.

Choose any numbers $\alpha < \beta < \gamma$

$$\Rightarrow \quad y' = -(y-\alpha)(y-\beta)(y-\gamma)$$