Worksheet 8

Felix Funk, MATH Tutorial - Mech 221

1 Revision on Eigenvalues and Eigenvectors

Problem: Eigenvalues and Eigenvectors.
Calculate the eigenvalues and (generalized) eigenvectors of the following matrices:

$$B_1 = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad B_3 = \begin{bmatrix} -3 & 0 \\ 0 & -3 \end{bmatrix}, \quad B_4 = \begin{bmatrix} -3 & 1 \\ 0 & -3 \end{bmatrix}, \quad B_5 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

B_4 and B_5 have defective eigenvalues and therefore generalized eigenvectors. If you need a reminder for these, then read the section on repeated eigenvalues with defects, first, and come back after that.

Reminder: Repeated Eigenvalues With Defects. Sometimes, eigenvalues coincide $\lambda = \lambda_1 = \lambda_2 = \ldots = \lambda_m$ but we can only find k linear independent eigenvectors $v_1, \ldots, v_k$, $k < m$. In this case, we have to find $m - k$ generalized eigenvectors. Not every eigenvector has a defect and some eigenvectors have a more severe defects than others. When an eigenvector is defective we find a w such that $(\lambda I - A)w = v$. If the defect is more severe, we can find x such that $(\lambda I - A)x = w$, etc. Let us illustrate this problem using the following matrix

switch

Let

$$C = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 1 & -1 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

1. Find the eigenvalue λ of C:

2. Observe that C has only two eigenvectors $v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ and $v_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$.

3. Show that v_1 does not have a defect.

4. Calculate the generalized eigenvectors w_2 and x_2 to the eigenvector v_2 such that $(I\lambda - C)w_2 = v_2$ and $(I\lambda - C)x_2 = w_2$

See in 2 pages.

2 Solving Linear Systems of ODEs

Introduction: Eigenvalue method Using eigenvalues and eigenvectors, we can find the general solution to linear systems of n differential equations given in the form

$$y' = Ay, \qquad \text{with } y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} \text{ and an } n \times n \text{ constant coefficient matrix } A. \tag{1}$$

In the following subsections we distinguish between four distinct cases: **distinct real eigenvalues, complex pairs of eigenvalues, repeated eigenvalues**, and **repeated eigenvalues with defects.**

2.1 Distinct real eigenvalues

Introduction: Distinct real eigenvalues If the matrix A has n distinct eigenvalues $\lambda_1, \dots \lambda_n$ with corresponding real eigenvectors $v_1, \dots v_n$, then the general solution has the form

$$y(t) = c_1 e^{\lambda_1 t} v_1 + c_2 e^{\lambda_2 t} v_2 + \dots c_n e^{\lambda_n t} v_n$$

with coefficients $c_1, c_2, \dots c_n$.

Example: Model problem. Solve the IVP

$$y' = B_1 y \quad \text{with } B_1 = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix} \text{ and } y(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

1. The eigenvalues of B_1 are $\lambda_1 = 1$ and $\lambda_2 = 2$. The corresponding eigenvectors are $v_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $v_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

2. The general solution is

$$y(t) = c_1 e^{t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 e^{2t} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

3. Solve the IVP by determining c_1, c_2.

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 0 \end{pmatrix} \Rightarrow c_1 = -1 \Rightarrow 1 = -1 + c_2, \; c_2 = 2$$

$$y(t) = \begin{pmatrix} -e^{t} + 2 \cdot e^{2t} \\ e^{t} \end{pmatrix}$$

1 Revision on Eigenvalues & Eigenvectors

- $B_1 = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix} \rightsquigarrow \boxed{\lambda_1 = 1,\ \lambda_2 = 2}$

$\underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}}_{\lambda_1 I_2 - B_1} = \begin{pmatrix} -1 & -1 \\ 0 & 0 \end{pmatrix}$ 　　 or nullspace, kernel, ...

$\mathrm{Kern}(\lambda_1 I_2 - B_1) = \mathrm{span}\left(\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right)$

$\boxed{\Rightarrow v_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}}$

$\mathrm{Kern}(\lambda_2 I_2 - B_1) = \mathrm{Kern}\left(\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} - \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}\right) = \mathrm{Kern}\left(\begin{pmatrix} 0 & -1 \\ 0 & 1 \end{pmatrix}\right) = \mathrm{span}\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right)$

$\boxed{\Rightarrow v_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}}$

- $B_2 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \Rightarrow \lambda^2 + 1 = 0$ (as $\det(\lambda I - B_2) = 0$)

$\Rightarrow \lambda_{1,2} = \pm i$ $(\lambda_1 = -i,\ \lambda_2 = i)$

$\lambda_1 I_2 - B_2 = \begin{pmatrix} -i & -1 \\ 1 & -i \end{pmatrix}$ $v_1 \in \mathrm{Kern}(\lambda_1 I_2 - B_2)$ if $\begin{pmatrix} -i & -1 \\ 1 & -i \end{pmatrix}\begin{pmatrix} v_1^{(1)} \\ v_1^{(2)} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\Rightarrow -i v_1^{(1)} - v_1^{(2)} = 0 \Rightarrow v_1^{(2)} = -i v_1^{(1)}$

$\Rightarrow v_1 = \begin{pmatrix} 1 \\ -i \end{pmatrix}$

$\lambda_2 I_2 - B_2 = \begin{pmatrix} i & -1 \\ 1 & i \end{pmatrix} \Rightarrow v_2 \in \mathrm{Kern}(\lambda_2 I_2 - B_2)$ if $i v_2^{(1)} - v_2^{(2)} = 0$

$\Rightarrow v_2 = \begin{pmatrix} 1 \\ i \end{pmatrix}$

- B_3 is a diagonal matrix $\Rightarrow \lambda_1 = \lambda_2 = -3$ (eigenvalues on diagonal) and the eigenvectors $v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

- B_4 is an upper diagonal matrix $\Rightarrow \lambda_1 = \lambda_2 = -3$ (eigenvalues on diag)

However, $\mathrm{Kern}((-3) \cdot I_2 - B_4) = \mathrm{Kern}\begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} = \mathrm{span}\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right)$

we only find a single eigenvector $v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

With that, let's look for a generalized eigenvector.

The gen. eigenvector w_1 satisfies $(-(-3)I_2 + B_4)\cdot w_1 = v_1$ ← (I wrote it down first in the wrong order, care)

$$\begin{pmatrix} 0 & +1 \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} w_1^{(1)} \\ w_1^{(2)} \end{pmatrix} = \begin{pmatrix} v_1^{(1)} \\ v_1^{(2)} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{cases} \cancel{w_1^{(1)} \cdot 0} - w_1^{(2)} = 1 \\ 0 = 0 \end{cases} \qquad w_1^{(1)} \text{ arbitrary} \Rightarrow \boxed{w_1 = \begin{pmatrix} 0 \\ (+1) \end{pmatrix}}$$

• $B_5 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \rightsquigarrow \boxed{\lambda_1 = 0,\ \lambda_2 = 0,\ \lambda_3 = -1}$

$\boxed{v_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}}$ (from (upper) diagonal form)

but $\mathrm{Kern}(B_5) = \mathrm{Kern}(+B_5)$ has dimension 1.

Hence, we are miss the one eigenvector.

The generalized eigenvector w_2 satisfies $(\underbrace{\lambda}_{=0} B_5)\cdot \begin{pmatrix} w_1^{(1)} \\ w_1^{(2)} \\ w_1^{(3)} \end{pmatrix} = \begin{pmatrix} v_1^{(1)} \\ v_1^{(2)} \\ v_1^{(3)} \end{pmatrix}$

$$\Rightarrow \begin{bmatrix} 0 & +1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \cdot \begin{pmatrix} w_1^{(1)} \\ w_1^{(2)} \\ w_1^{(3)} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \left.\begin{matrix} +w_1^{(2)} = 1 \\ 0 = 0 \\ w_1^{(3)} = 0 \end{matrix}\right\} \quad \boxed{w_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}}$$

• C 1. Eigenvalues on diagonal (upper diagonal form) $\Rightarrow \lambda = 2$ $\quad \lambda_1 = \lambda_2 = \lambda_3 = \lambda_4$

2. $C\cdot v_1 = 2\cdot v_1,\ C\cdot v_2 = 2\cdot v_2$ ✓

3. Suppose v_1 had a defect.

$\underbrace{(\ )}_{C-2I_4} w_1 = v_1$ would imply $\begin{pmatrix} 0 & & & \\ & 0 & +1 & -1 \\ & & 0 & +1 \\ & & & 0 \end{pmatrix} \cdot \begin{pmatrix} w_1^{(1)} \\ w_1^{(2)} \\ w_1^{(3)} \\ w_1^{(4)} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$

$\Rightarrow 0 = 1$ contradiction. $\Rightarrow w_1$ does not exist.

4. $\underbrace{\begin{pmatrix} 0 & & & \\ & 0 & +1 & -1 \\ & & 0 & +1 \\ & & & 0 \end{pmatrix}}_{X} \cdot \begin{pmatrix} w_2^{(1)} \\ w_2^{(2)} \\ w_2^{(3)} \\ w_2^{(4)} \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \Rightarrow +w_2^{(3)} - w_2^{(4)} = 1$

$-w_2^{(4)} = 0 \Rightarrow w_2^{(3)} = 1$

$\Rightarrow w_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$

Similarly: $x_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ so that $X\cdot x_2 = w_2$

2.2 Complex eigenvalues

Introduction: Pair of complex eigenvalues
We can observe complex conjugate pairs of eigenvalues $\lambda = \mu + i\omega$ and $\lambda^* = \mu - i\omega$ with eigenvectors v and v^*. The contribution of the pair of complex eigenvalues to the general solution is

$$c_1 \cdot Re(e^{(\mu+i\omega)t}v) + c_2 \cdot Im(e^{(\mu+i\omega)t}v). \tag{2}$$

Using the Euler identity, one can obtain two linear independent solutions.

solve for x''

Example: Model problem. Given is the second order equation $\boxed{x'' + x = 0.}$ Transform the system in a first system of differential equations by utilizing $y_1 = x$ and $y_2 = x'$. Solve the IVP $x(0) = 0, x'(0) = 1$.

1. Find B_2 such that $y' = B_2 y$. Set $y_1 = x$, $y_2 = x'$ $\Rightarrow y_1' = y_2$; $y_2' = x'' = -x = -y_1$

   $\Rightarrow \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}'(t) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}(t)$ $\quad B_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

2. The eigenvalues of B_2 are $\lambda_1 = -i$ and $\lambda_2 = +i$. The corresponding eigenvectors are $v_1 = \begin{bmatrix} 1 \\ -i \end{bmatrix}$ and $v_2 = \begin{bmatrix} 1 \\ i \end{bmatrix}$. (Choose either λ_1 and v_1 or λ_2 and v_2)

3. The general solution is

   $e^{-it} = \cos(-t) + i\sin(-t) = \cos(t) - i\sin(t)$

   $Re\left(e^{-it}\begin{pmatrix} 1 \\ -i \end{pmatrix}\right) = Re\begin{pmatrix} \cos(t) - i\sin(t) \\ -i\cos(t) - \sin(t) \end{pmatrix} = \begin{pmatrix} \cos(t) \\ -\sin(t) \end{pmatrix}$

   $Im\left(e^{-it}\begin{pmatrix} 1 \\ -i \end{pmatrix}\right) = \begin{pmatrix} -\sin(t) \\ -\cos(t) \end{pmatrix}$

   $y(t) = c_1 \cdot \begin{pmatrix} \cos(t) \\ -\sin(t) \end{pmatrix} + c_2 \begin{pmatrix} -\sin(t) \\ -\cos(t) \end{pmatrix}$

4. Solve the IVP by determining c_1, c_2.

   $0 = x(0) = y_1(0)$
   $1 = x'(0) = y_2(0)$

   $\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} c_1 \\ -c_2 \end{pmatrix} \Rightarrow c_1 = 0, \; c_2 = -1$

   $y(t) = - \begin{pmatrix} -\sin(t) \\ -\cos(t) \end{pmatrix} = \begin{pmatrix} \sin(t) \\ \cos(t) \end{pmatrix}$

5. Obtain $x(t)$.

   $x(t) = y_1(t) = \sin(t)$

   check: $\underbrace{\sin(t)''}_{x''} + \underbrace{\sin(t)}_{x} = 0$ ✓

   $x(0) = \sin(0) = 0$
   $x'(0) = \sin'(0) = \cos(0) = 1$

2.3 Repeated Eigenvalues Without Defects

Introduction: Repeated Eigenvalues Without Defects. Sometimes, eigenvalues coincide, i.e. we have $\lambda = \lambda_1 = \lambda_2 = \ldots = \lambda_m$. However, we may still find m linear independent eigenvectors $v_1, \ldots, v_m$ then we can still find the contribution of the eigenvalue to the solution analogous to the first subsection

$$c_1 e^{\lambda t} v_1 + c_2 e^{\lambda t} v_2 + \ldots + c_m e^{\lambda t} v_m$$

with arbitrary coefficients $c_1, c_2, \ldots c_m$.

Example: Model problem. Solve the IVP

$$y' = B_3 y \quad \text{with } B_3 = \begin{bmatrix} -3 & 0 \\ 0 & -3 \end{bmatrix} \text{ and } y(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

1. The eigenvalues of B_3 are $\lambda_1 = -3$ and $\lambda_2 = -3$. The corresponding eigenvectors are $v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

2. The general solution is

$$y(t) = \begin{pmatrix} c_1 e^{-3t} \\ c_2 e^{-3t} \end{pmatrix}$$

3. Solve the IVP by determining c_1, c_2.

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

$$y(t) = \begin{pmatrix} 0 \\ e^{-3t} \end{pmatrix}$$

2.4 Repeated Eigenvalues with Defects

Introduction: Solving Systems of Differential Equations with Defective Eigenvalues Defective eigenvectors result in special terms. If an eigenvector has a defect of 1, i.e we find w (as defined on the front page) but no further defect, then the defective eigenvector contributes

$$c_1ve^{\lambda t} + c_2(vt + w)e^{\lambda t}.$$

If the defect is 2, then the contribution is

$$c_1ve^{\lambda t} + c_2(vt + w)e^{\lambda t} + c_3(v\frac{t^2}{2} + wt + x)e^{\lambda t}.$$

Example: Model problem. Find the solution to Compare to front page

$$y' = Cy, y(0) = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}.$$

1. The eigenvector v_1 does not have a defect, hence it contributes to the general solution:

$$c_1 \cdot \underbrace{\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}}_{\leftarrow v_1} e^{2t}$$

2. The eigenvector v_2 has a defect of order 2 and contributes to the general solution

$$e^{2t} \cdot \left(c_2 \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + c_3 \cdot \begin{pmatrix} 0+0 \\ t+0 \\ 0+1 \\ 0+0 \end{pmatrix} + c_4 \left(\begin{pmatrix} 0 \\ t^2/2 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ +t \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ +1 \\ +1 \end{pmatrix} \right) \right)$$

3. The general solution is

$$y(t) = c_1 \begin{pmatrix} e^{2t} \\ 0 \\ 0 \\ 0 \end{pmatrix} + c_2 \cdot \begin{pmatrix} 0 \\ e^{2t} \\ 0 \\ 0 \end{pmatrix} + c_3 \begin{pmatrix} 0 \\ te^{2t} \\ +e^{2t} \\ 0 \end{pmatrix} + c_4 \begin{pmatrix} 0 \\ t^2/2\, e^{2t} \\ (+t+1)e^{2t} \\ +e^{2t} \end{pmatrix}$$

4. Solve the IVP by determining the four coefficients.

$$\begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \\ +c_3+c_4 \\ +c_4 \end{pmatrix} \Rightarrow c_1 = 0,\ c_2 = 1,\ c_3 = (-1),\ c_4 = +1$$

$$y(t) = \cancel{\begin{pmatrix} 0 \\ e^{2t} + te^{2t} + t^2/2\, e^{2t} \\ e^{2t}(-1) \\ e^{2t} \end{pmatrix}} = e^{2t} \begin{pmatrix} 0 \\ 1-t+t^2/2 \\ +t \\ 1 \end{pmatrix}$$

2.5 Mixed Problems

Problemset: 1.

1. A system of equations is given by

$$\begin{aligned} y_1' &= \alpha y_1 + y_2, \\ y_2' &= \beta y_2 + y_3, \\ y_3' &= \gamma y_3, \end{aligned}$$

for some real constants α, β, γ.

   (a) Let α, β, γ be real and pairwise different. For which values of α, β, γ is the ~~system~~ origin unstable ~~/ stable~~?

   (b) For some real constant let $q = \alpha = \beta = \gamma$. Determine $y(t)$ for the initial value problem $y_1(0) = 1$ and $y_2(0) = y_3(0) = 0$. For which values of q is the system origin unstable?

2. Find the general solution to $y' = B_4 y$ and $y' = B_5 y$.

3. Transform the mass-spring-damper system $my'' + cy' + ky = 0$ into a system of first order equations. Show that the characteristic equation has repeated roots if and only if the transformed system has an eigenvalue with defect.

1 (a) $\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}'(t) = \begin{pmatrix} \alpha & 1 & 0 \\ 0 & \beta & 1 \\ 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}(t)$

$\lambda_1 = \alpha,\ \lambda_2 = \beta,\ \lambda_3 = \gamma$

$\Rightarrow y(t) = c_1 e^{\alpha t} v_1 + c_2 e^{\beta t} v_2 + c_3 e^{\gamma t} v_3$. As v_1, v_2, v_3 eigenvectors $v_1, v_2, v_3 \neq 0$

~~There is a~~ If any of the three > 0, then there is for $\begin{cases} y(0) = v_1 \text{ (for } \alpha > 0) \\ y(0) = v_2 \text{ (for } \beta > 0) \\ y(0) = v_3 \text{ (for } \gamma > 0) \end{cases}$

$y(t) = \begin{cases} e^{\alpha t} v_1, \text{ if } \alpha > 0 \\ e^{\beta t} v_2, \text{ if } \beta > 0 \\ e^{\gamma t} v_3, \text{ if } \gamma > 0 \end{cases}$ with $\|y(t)\| \to \infty$ as $\begin{cases} e^{\alpha t} \to \infty \text{ if } \alpha > 0 \\ e^{\beta t} \to \infty \text{ if } \beta > 0 \\ e^{\gamma t} \to \infty \text{ if } \gamma > 0 \end{cases}$

(b)

1. (b) $A = \begin{pmatrix} q & 1 & 0 \\ & q & 1 \\ & & q \end{pmatrix} \Rightarrow \lambda_1 = \lambda_2 = \lambda_3 = q$ and $v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

sole eigenvalue.

As $(A - q I_3) \cdot \underbrace{\begin{pmatrix} 0 \\ +1 \\ 0 \end{pmatrix}}_{w_1} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ $\qquad A - q \cdot I_3 = \begin{pmatrix} 0 & +1 & 0 \\ 0 & 0 & +1 \\ 0 & 0 & 0 \end{pmatrix}$

$(A - q I_3) \underbrace{\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}}_{= x_1} = \begin{pmatrix} 0 \\ +1 \\ 0 \end{pmatrix}$

$\Rightarrow y(t) = c_1 \cdot e^{qt} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_2 e^{qt} \begin{pmatrix} t \\ +1 \\ 0 \end{pmatrix} + c_3 e^{qt} \begin{pmatrix} t^2/2 \\ t \\ 1 \end{pmatrix}$

For $q < 0 \Rightarrow e^{qt}, e^{qt} t, e^{qt} t^2 \xrightarrow[t \to \infty]{} 0$, hence stable

For $q = 0 \Rightarrow t^2/2 \to \infty \Rightarrow$ unstable (whenever $c_3 \neq 0$

For $q > 0 \Rightarrow e^{qt} \to \infty$ unstable.

$\Rightarrow$ If $q < 0 \to$ stable

2. Similar to before $y(t) = \begin{pmatrix} c_1 e^{-3t} + c_2 \cdot t e^{-3t} \\ c_2 e^{-3t} \end{pmatrix}$ $\quad c_1, c_2 \in \mathbb{R}$

for B_4, $\dot{y} = B_4 y$

and $y(t) = \begin{pmatrix} c_1 + c_2 t \\ c_2 \\ c_3 e^{-t} \end{pmatrix}$ for $c_1, c_2, c_3 \in \mathbb{R}$ for B_5, $\dot{y} = B_5 y$.

3. $m \neq 0 \Rightarrow y'' + \frac{c}{m} y' + \frac{k}{m} y = 0$

Transformed sys : $y_1 = y$, $y_2 = y'$

$\Rightarrow y_1' = y_2$

$\underset{\text{"}y''\text{"}}{y_2'} = -\frac{c}{m} y' - \frac{k}{m} y = -\frac{c}{m} y_2 - \frac{k}{m} y_1$. Def. $\hat{c} = \frac{c}{m}$

$-\hat{c} y_2 - \hat{k} y_1$. $\hat{k} = \frac{k}{m}$

Then $y'' + \hat{c} y' + \hat{k} y = 0$

if and only if $\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}'(t) = \begin{pmatrix} 0 & 1 \\ -\hat{k} & -\hat{c} \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}(t)$.

Assume char. eq. has repeated root, show: λ repeated with defect.

The characteristic eq $r^2 + \hat{k} r + \hat{c} = 0$ has repeated roots

if $r^2 + \hat{c} r + \hat{k} = (r \mp r_1)^2 \Rightarrow \hat{c} = -2r_1$, $\hat{k} = r_1^2$

for some $r_1 \in \mathbb{R}$ $= (r^2 \mp 2rr_1 + r_1^2)$ ← repeated root

$A = \begin{pmatrix} 0 & 1 \\ -r_1^2 & +2r_1 \end{pmatrix}$ $\dot{y} = A \cdot y$ ~~Eigenvalues: $\lambda \cdot (\lambda + r_1^2) \mp 2r_1$~~

~~$= \lambda^2 + 2r_1^2 \mp 2r_1 = 0$~~

~~$\Rightarrow \lambda^2 \cdot \lambda + \lambda r_1^2 - 2r_1 = (\lambda + \frac{r_1^2}{2})^2 - \frac{r_1^4}{4}$~~ $+4r_1$

Eigenvalues: $-\lambda \cdot (\lambda \mp 2r_1) + r_1^2 = \lambda^2 - 2r_1 \lambda + r_1^2 = (\lambda - r_1)^2 = 0$

~~because~~ for $\lambda_1 = r_1$ $\lambda_2 = r_1 \Rightarrow \lambda = r_1$

Further, $(\lambda I - A) = \begin{pmatrix} \lambda & -1 \\ r_1^2 & \lambda \mp 2r_1 \end{pmatrix} = \begin{pmatrix} r_1 & -1 \\ r_1^2 & -r_1 \end{pmatrix}$

$\Rightarrow \operatorname{Kern}(\lambda I - A) = \operatorname{span}\begin{pmatrix} 1 \\ -r_1 \end{pmatrix} \neq \mathbb{R}^2$.

Hence, there is only a single eigenvector to λ. $\rightsquigarrow$ λ has defect and is repeated

Now, let λ^* be the repeated with defect. Show that the char. eq has a repeated root

$\lambda \cdot (\lambda + \hat{c}) + \hat{k} = 0$ is the eigenvalue equation

$\Rightarrow \lambda^2 + \hat{c}\lambda + \hat{k} = 0$. λ^* is a repeated root (by assumption), i.e.

$\lambda^2 + \hat{c}\lambda + \hat{k} = 0 = (\lambda - \lambda^*)^2 \Rightarrow$ Be renaming $\lambda = r$

and $\lambda^* = r_1 \Rightarrow$ Characteristic eq has repeated root.

3 Application: Mathematics of Love, Nullclines and Phase Planes

I'm following Cole Zmurchok's example and sharing his code for the following section.

Introduction: Romeo and Juliet
Let $R(t)$ denote Romeo's feelings for Juliet at time t, and $J(t)$ denote Juliet's feelings for Romeo at time t where positive values indicate positive feelings and negative values indicate negative feelings.

Suppose that Romeo and Juliet respond to their own feelings as well as each other's feelings, so that

$$\frac{dR}{dt} = a_R R + b_R J,$$
$$\frac{dJ}{dt} = a_J J + b_J R,$$

where $a_R, b_R, a_J, b_J \in \mathbb{R}$. **Given $a_R, b_R, a_J, b_J \in \mathbb{R}$ can we describe the evolution of Romeo and Juliet's feelings for each other over time?**

1. Write this first-order constant coefficient linear system as a matrix ODE.

$$x'(t) = \begin{pmatrix} R \\ J \end{pmatrix}'(t) = \begin{pmatrix} a_R & b_R \\ b_J & a_J \end{pmatrix} \begin{pmatrix} R \\ J \end{pmatrix}(t) = A \cdot x \qquad x = \begin{pmatrix} R \\ J \end{pmatrix}$$

2. Download the code from the webpage to get a handle on the system behaviour. How does the system behave in the following cases?

   (a) Case A: $a_R = 2$, $b_R = 1$, $b_J = 0$, $a_J = 1$.

   (b) Case B: $a_R = 0$, $b_R = 1$, $b_J = -1$, $a_J = 0$.

   (c) Case C: $a_R = -3$, $b_R = 0$, $b_J = 0$, $a_J = -3$.

   (d) Case D: $a_R = -3$, $b_R = 1$, $b_J = 0$, $a_J = -3$.

3. For each of these cases we can find the explicit solution $x(t) = \begin{bmatrix} R(t) \\ J(t) \end{bmatrix}$, find the nullclines, sketch the phase-plane and classify the stability of the origin. Suppose $R(0) = J(0) = 1$.

   Example: Case A.
   $a_R = 2$, $b_R = 1$, $b_J = 0$, $a_J = 1$. This gives

$$\frac{dx}{dt} = Ax = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix} \cdot x$$

   Find or recall the general solution $\vec{x}(t)$ from the previous sections.

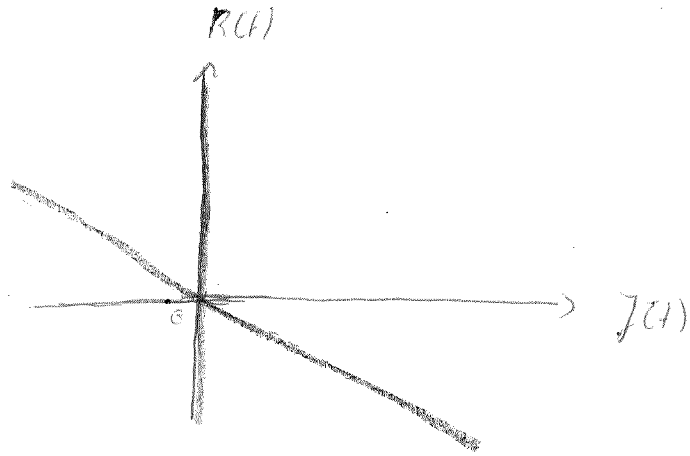
   $x(t)$ as on page 3.

Find the nullclines.

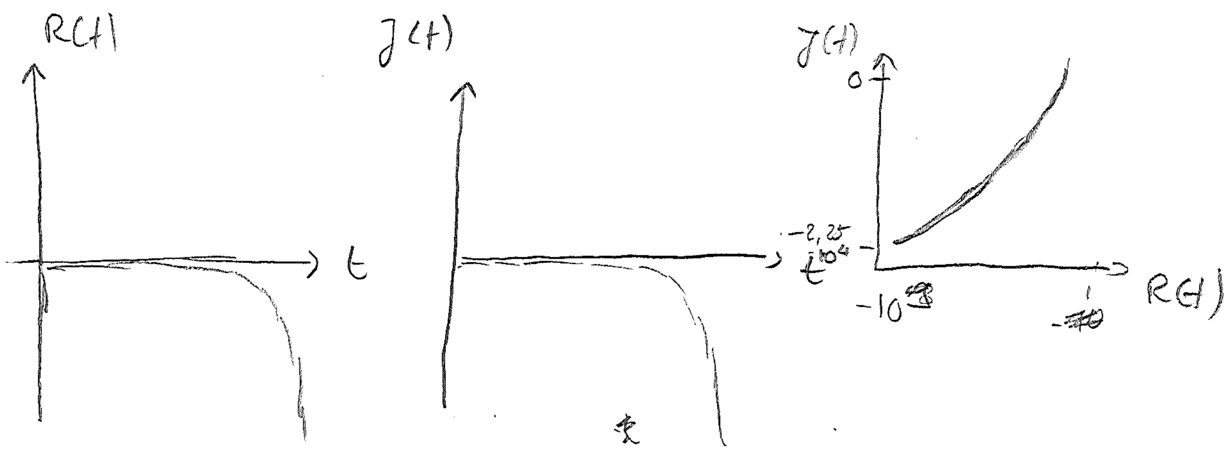
$0 = \dot{R} = 2R + 1J$

$\Leftrightarrow R = -\frac{1}{2}J$

$0 = \dot{J} = J$



Sketch the solutions, $R(t)$, and $J(t)$, and sketch the phase-plane ($J(t)$ versus $R(t)$).



Your turn. Find the explicit solution $x(t) = \begin{bmatrix} R(t) \\ J(t) \end{bmatrix}$ for Case B, Case C and Case D. Find the nullclines, sketch the phase plane, and classify the stability of the origin.

More?

Can you predict the outcome of the love affair given any set of parameters $\{a_R, b_R, b_J, a_J\}$?

Can you improve my code to compare the results of `ode45` with the exact solution of the system? Can you plot nullclines? Can you plot a vector field and several trajectories (such as Figure 3.5-3.7 in WFT.pdf?)