THE UNIVERSITY OF BRITISH COLUMBIA
FACULTY OF APPLIED SCIENCE
DEPARTMENT OF MECHANICAL ENGINEERING

MECH 221

TEST #7, November 17th, 2016

Suggested Time: 100 minutes **Allowed Time:** 110 minutes

Materials admitted: Pencil, eraser, straightedge, MECH 2 Approved Calculator (Sharp EL-510), one 3x5 inch index card or sheet of paper for hand-written notes.

There are 4 Short-Answer Questions and 2 Long-Answer Problems on this test. All questions must be answered.

Provide **all** work and solutions **on this test**. Orderly presentation of work is required for solutions to receive full credit. **Illegible work, or answers that do not include supporting calculations and explanations will NOT BE MARKED**.

FILL OUT THE SECTION BELOW AND WRITE YOUR NAME ON THE TOP OF ALL TEST PAGES. Do this during the examination time as additional time will not be allowed for this purpose.

NAME:____________________________________Section__________

SIGNATURE:______________________________

STUDENT NUMBER:________________________

Question	Mark Received	Maximum Mark
SA 1		5
SA 2		5
SA 3		5
SA 4		5
Prob 1		25
Prob 2		25

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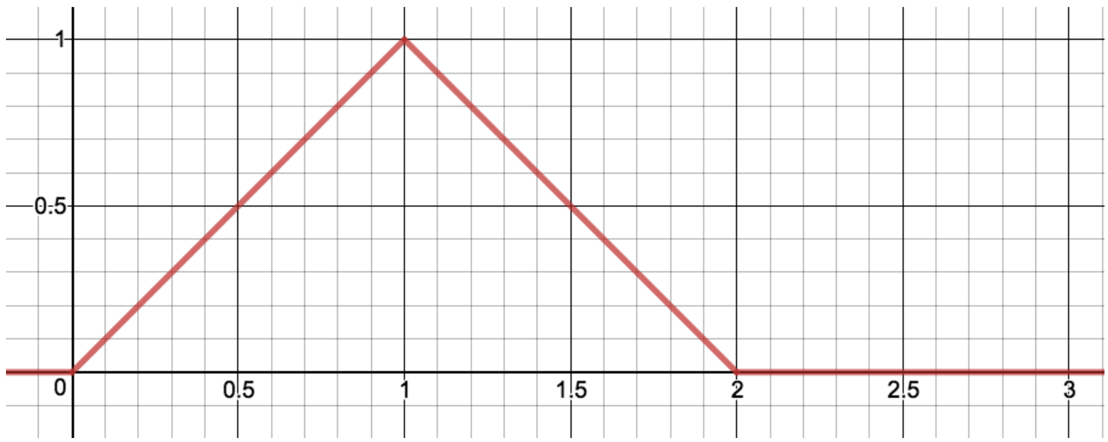
SA1 [5 marks]. Find the Laplace transform $Y(s)$ of the solution $y(t)$ of the differential equation

$$2y'' + 3y' - y = f(t),\ \ y(0) = 0,\ y'(0) = 0$$

where the forcing function $f(t)$ is given by



(Note: Do not solve for $y(t)$.)

First, find a formula for $f(t)$

$$\begin{aligned} f(t) = {} & t\big(u(t) - u(t-1)\big) + (2-t)\big(u(t-1) - u(t-2)\big) \\ & = t \cdot u(t) - 2(t-1) \cdot u(t-1) + (t-2) \cdot u(t-2) \end{aligned}$$

Use the formula

$$\mathcal{L}\{u(t-a)f(t-a)\} = e^{-as}F(s)$$

to compute the Laplace transform of $f(t)$

$$F(s) = \frac{1}{s^2} - \frac{2e^{-s}}{s^2} + \frac{e^{-2s}}{s^2}$$

Applying the Laplace transform to the differential equation yields

$$2s^2Y(s) + 3sY(s) - Y(s) = F(s)$$

$$Y(s) = \frac{F(s)}{2s^2 + 3s - 1} = \frac{1 - 2e^{-s} + e^{-2s}}{s^2(2s^2 + 3s - 1)}$$

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SA2 [5 marks]. Suppose an output signal $y(t)$ is related to an input signal $x(t)$ by the differential equation

$$y'' + 3y' + y = 3x' - x\,.$$

(a) [1 mark]. Find the transfer function $H(s) = \dfrac{Y(s)}{X(s)}$.

Apply the Laplace transform to the equation and solve

$$s^2Y(s) + 3sY(s) + Y(s) = 3sX(s) - X(s)\ ,\quad H(s) = \frac{Y(s)}{X(s)} = \frac{3s-1}{s^2+3s+1}$$

(b) [4 marks]. Suppose $x(t) = \sin(\omega t)$. Find values $\omega > 0$ such that the amplitude of the steady state response is less than 1. In other words, find values $\omega > 0$ such that

$$\lim_{t\to\infty}\left|y(t)\right| = \left|y_{ss}(t)\right| < 1$$

The amplitude of the steady state response is given by

$$|H(j\omega)| = \left|\frac{3j\omega - 1}{-\omega^2 + 3j\omega + 1}\right| = \frac{\sqrt{9\omega^2+1}}{\sqrt{9\omega^2+(1-\omega^2)^2}} = \frac{\sqrt{9\omega^2+1}}{\sqrt{\omega^4+7\omega^2+1}} < 1$$

Therefore the question is

$$\sqrt{9\omega^2+1} < \sqrt{\omega^4+7\omega^2+1}$$

$$9\omega^2 + 1 < \omega^4 + 7\omega^2 + 1$$

$$2\omega^2 < \omega^4$$

$$2 < \omega^2$$

and so $\omega > \sqrt{2}$.

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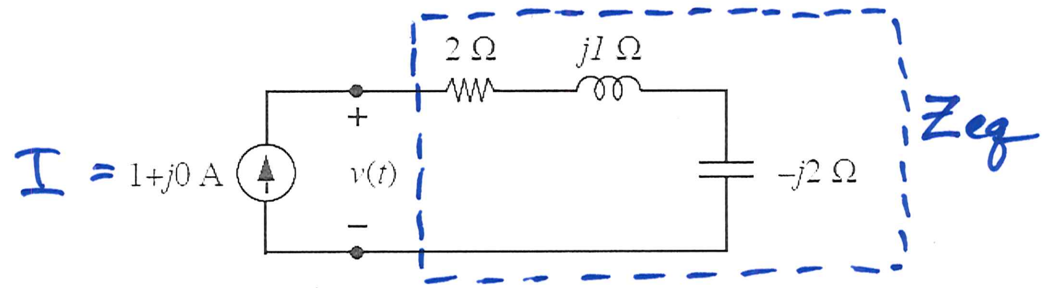
SA 4 [5 Marks]. Consider the following circuit that is operating in the sinusoidal steady-state at angular frequency $\omega = 500$ rad/s.



(a) [3 marks]. Determine the time-domain voltage $v(t)$.

$$Z_{eq} = 2\Omega + j(1\Omega - 2\Omega) = 2 - j1\,\Omega$$
$$= 2.24\angle{-26.57^\circ}\,\Omega$$

$$V = I \cdot Z_{eq}$$
$$= (1\angle 0^\circ)(2.24\angle{-26.57^\circ})$$
$$= 2.24\angle{-26.57^\circ}\ V.$$

$$\therefore\ v(t) = 2.24\cos(500t - 26.57^\circ)\ V.$$

(b) [2 marks]. Determine the values of the *inductance* (in Henries) and *capacitance* (in Farads).

$$\omega = 500\ rs^{-1} \Rightarrow j1\Omega = j(500)L$$
$$\therefore\ L = 2mH = 2\times10^{-3}\ H.$$
$$-j2\Omega = -j\frac{1}{(500)C}$$
$$\therefore\ C = 1mF = 1\times10^{-3}\ F.$$

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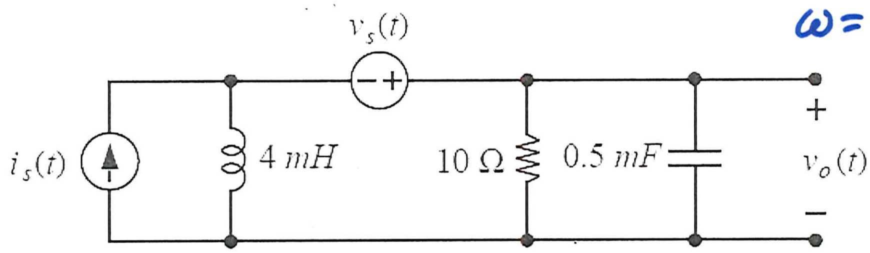
Prob 1 [25 Marks]. Consider the circuit shown below that is operating in the steady-state with $i_s(t) = 4\cos(1000t)\ A$ and $v_s(t) = 12\cos(1000t + 90°)\ V$.

$\omega = 1000\ rs^{-1}$



(a) [3 marks]. Determine the phasor impedances for the inductor and capacitor, and write the time-domain source functions $i_s(t)$ and $v_s(t)$ as phasors $\mathbf{I}_s$ and $\mathbf{V}_s$.

$$\omega = 1000\ rs^{-1} \Rightarrow Z_L = j\omega L = j(1000)(4mH) = j4\ \Omega$$

$$Z_C = \frac{1}{j\omega C} = -j\frac{1}{(1000)(0.5mF)} = -j2\ \Omega$$

$$I_s = 4\angle 0° = 4 + j0\ A.$$

$$V_s = 12\angle 90° = 0 + j12\ V.$$

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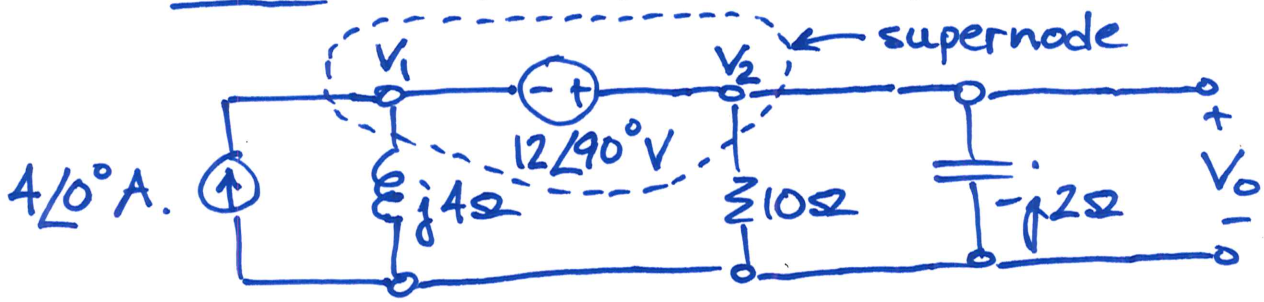
Prob 1, *Cont'd.*

(b) [7 marks]. Use nodal analysis to determine $\mathbf{V}_O$, the phasor representation of $v_o(t)$. (*Suggestion: it may prove useful to redraw the circuit here with all of the components labeled in phasor notation.*)



KCL @ supernode: $-4\angle 0° + \frac{V_1}{4\angle 90°} + \frac{V_2}{10} + \frac{V_2}{2\angle -90°} = 0$

constraint: $V_1 = V_2 - 12\angle 90°$

also, $V_2 = V_o$

$\rightarrow \frac{V_o}{4\angle 90°} - \frac{12\angle 90°}{4\angle 90°} + \frac{V_o}{10} + \frac{V_o}{2\angle -90°} = 4\angle 0°$

$$V_o\left(\underbrace{0.25\angle -90° + 0.1 + 0.5\angle 90°}_{\substack{=0.1+j0.25 \\ =0.269\angle 68.2°}}\right) = \underbrace{4\angle 0° + 3\angle 0°}_{=7\angle 0°}$$

$\therefore V_o = \frac{7\angle 0°}{0.269\angle 68.2°} = 26\angle -68.2° \text{ V.}$

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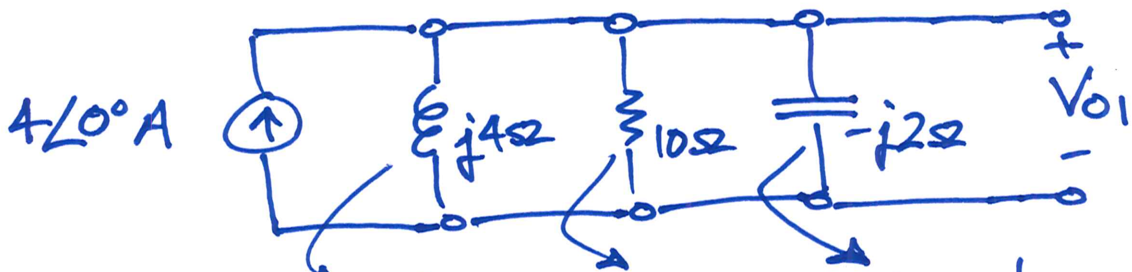
Prob 1, *Cont'd.*

(c) [12 marks]. Use *superposition analysis* to determine the phasor voltage $\mathbf{V}_O$, and write the corresponding time-domain function $v_o(t)$.

Eq. cct. for current source alone:



$Y_L = \frac{1}{4\angle 90^\circ} = 0.25\angle{-90^\circ}$ $\quad Y_R = 0.1$ $\quad Y_C = \frac{1}{2\angle{-90^\circ}} = 0.5\angle 90^\circ$

$$
\begin{aligned}
Y_{eq} &= Y_L + Y_R + Y_C \\
&= 0.25\angle{-90} + 0.1 + 0.5\angle 90^\circ \\
&= -j0.25 + 0.1 + j0.5 \\
&= 0.1 + j0.25 \\
&= 0.269\angle 68.2^\circ
\end{aligned}
$$

$$
V_{01} = \frac{4\angle 0^\circ}{0.269\angle 68.2^\circ} = 14.87\angle{-68.2^\circ}\ \text{V.} = 5.522 - j13.81\ \text{V.}
$$

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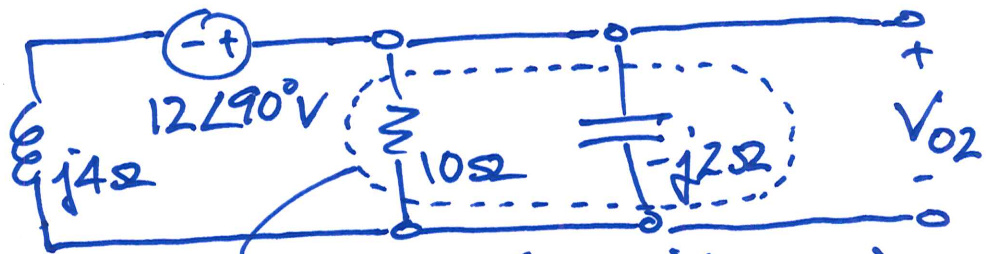
Eq. cct. for voltage source alone:



$$Z_{eq} = \frac{(10\angle 0°)(2\angle -90°)}{(10-j2)} \quad (10-j2) = 10.2\angle -11.31°$$

$$= 1.96\angle -78.69°$$

$$= 0.385 - j1.923\ \Omega$$

Voltage Div:

$$V_{O2} = (12\angle 90°) \cdot \frac{(1.96\angle -78.69°)}{\underbrace{0.385 + j(4-1.923)}_{2.11\angle 79.5°}}$$

$$= 11.15\angle -68.2°\ V.$$

$$= 4.14 - j10.35\ V.$$

$$\therefore V_o = V_{O1} + V_{O2} = (5.522 + 4.14) - j(13.81 + 10.35)$$

$$= 25.9\angle -68.8°$$

$$\therefore v_o(t) = 25.91 \cos(1000t - 68.8°)\ V.$$

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Prob 1, *Cont'd.*

(d) [3 marks]. Suppose the source functions $i_s(t)$ and $v_s(t)$ were instead operating at <u>different</u> frequencies, say ω_1 and ω_2. Briefly, in just a few sentences, outline how you would approach solving for $v_0(t)$.

You must use superposition. The component impedances must be calculated separately for each of the equiv. ccts. The final summation must be done in the time-domain.