Worksheet 5

Felix Funk, MATH Tutorial - Mech 222

1 Parametric Surfaces

Introduction: Parametrizations and Equations and Surface Areas For surfaces, we usually work with two different representations: One which describes the surface as a smooth parametric form $r(u,v) = x(u,v)i + y(u,v)j + z(u,v)k$ and one that describes it through an equations such as $x^2 + y^2 = z$ or $x + 4y - 5z = 0$ or a graph $z = g(x,y) = 10x - 2y + 5$. The equation usually helps us to find intersection lines and curves. The parametric form and graph form is essential to determine surface integrals, which are the focus of our analysis in this week's worksheet.

Example: The Parametric Form for Planes. Let P be the plane given by the three non-collinear points $A = (2,3,4), B = (3,2,4)$, and $C = (4,3,4)$. Find the parametric form and the equation that describes the plane uniquely:

1. Find the parametric form: Determine two directional vectors $w_1 = B - A$, $w_2 = C - A$. Then,

$$w_1 = \begin{pmatrix} 3 \\ 2 \\ 4 \end{pmatrix} - \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \qquad w_2 = \begin{pmatrix} 4 \\ 3 \\ 4 \end{pmatrix} - \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}$$

$$P(u,v) = A + u \cdot w_1 + v \cdot w_2 = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} + u \cdot \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + v \cdot \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}.$$

2. We can write this as a planar equation by computing the normal vector n, which is orthogonal on w_1 and w_2. Argue: $((x,y,z) - A) \cdot n = 0$ describes all points in plane P.

$$n = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \text{ as } \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = 0 \text{ and } \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} = 0.$$

Then, $\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix} - \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} \right) \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = 0$ describes the plane. $\Leftrightarrow z - 4 = 0$

$d = \begin{pmatrix} x \\ y \\ z \end{pmatrix} - \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}$ describes all directional vectors from $\begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}$ to point $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$

$d \cdot n = 0$ indicates all those which are orthogonal to n.

With that the solutions $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ are in the plane.

$\frac{18^2}{4} = 81 + 9$

3. Check that A, B and C are solutions of your planar equation. How can you obtain from the equation above a parametric representation again?

   A, B, C : $z = 4 \Rightarrow z - 4 = 0$ ✓

   To obtain the parameterization, find three points that do not form a line e.g. $\tilde{A} = \begin{pmatrix} 0 \\ 0 \\ 4 \end{pmatrix}$, $\tilde{B} = \begin{pmatrix} 1 \\ 0 \\ 4 \end{pmatrix}$ $\tilde{C} = \begin{pmatrix} 0 \\ 1 \\ 4 \end{pmatrix}$ (these solve the eq.) Then repeat 1.

For other nonlinear equations, one can solve for one of the variables to determine the graph:

$$z = g(x, y), (x, y) \in D \tag{1}$$

e.g. for a paraboloid $x^2 + 2y^2 = z$, the graph is $g(x, y) = x^2 + 2y^2$.

Introduction: Surface Area. The parametric representation and the graph are the two basis on how to calculate a surface area.

- For the parametric representation:

  $$r(u, v) = x(u, v)i + y(u, v)j + z(u, v)k, \quad (u, v) \in D$$

  the surface area of S is

  $$A(S) = \iint_D |r_u \times r_v| dA,$$

  where $r_u = \frac{\partial x}{\partial u}i + \frac{\partial y}{\partial u}j + \frac{\partial z}{\partial u}$. Here, $|r_u \times r_v|$ denotes the length of the normal vector on r_u and r_v.

- For the graph representation:

  $$z = g(x, y), \quad (x, y) \in D.$$

  we can also determine the surface area using:

  $$A(S) = \iint_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} dA.$$

Problem: Determining a planar surface area. Find the area of the part of the plane $3x + 2y + z = 6$ that lies in the first octant. Use the parametric representation as well as the graph representation.

Problemset: 1. Surface Areas. Calculate the surface area of

1. the part of the surface $-1 = 3x + 2y^2 - z$ above the triangle given by the vertices $(0, 0)$, $(0, 1)$ and $(2, 1)$.

2. the spiral $r(u, v) = u\cos(v)i + u\sin(v)j + vk$ in the domain $0 \leq u \leq 1, 0 \leq v \leq \pi$. Hint: You might find the substitution $u = \tan(\hat{u})$ and/or a computer algebra program useful.

2 Surface Integrals

Introduction: Surface Integrals: Surface Area Integration with Densities. Usually, the surface integral comes with a density function $f(x,y,z)$. We can generalize the two forms above:

- For the parametric representation:

$$\iint_S f(x,y,z)dS = \iint_D f(r(u,v))|r_u \times r_v|dA.$$

- For the graph form:

$$\iint_S f(x,y,z)dS = \iint_D f(x,y,g(x,y))\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}dA.$$

Problemset: 2. Surface Integrals. Determine the following surface integrals.

1. $\iint_S xzdS$, where S is the part of the plane $2x+2y+z=4$ in the first octant.
2. $\iint_S (x^2z+y^2z)dS$, where S is the hemisphere $x^2+y^2+z^2=1, z\geq 0$.

3 Surface Integrals over Vector Fields

Introduction: Surface Integrals over Vector Fields. Frequently, we are not interested in the area of the surface itself but e.g. how much liquid flows through a section of a surface. In this case, you want to obtain the integral over a vector field (i.e. the fluid stream) **orthogonal** to the surface. As every three-dimensional surface usually comes with two orthogonal directions, you usually have to choose the correct sign for the normal vector (e.g. the one that points in the direction of the stream). For simple closed surfaces, we commonly indicate whether the normal vector points inside (negative orientation) or outside (positive direction). The flux of a vector field F across the surface S is then given by

$$\iint_S F\cdot dS = \iint_S F\cdot ndS,$$

which is a surface integral over the scalar product of the vector field F with the unit normal vector n.

Example: Surface Integral over a Vector Field. Calculate the surface integral over the vector field $F(x,y,z)=zi+yj+xk$, where S is the spiral from problem 1.2 and the normal is pointed upwards.

1. Find the normal vector. What factor do you have to rescale it with to obtain the **unit** normal vector?

The normal vector is given in solution 1.2 $n^0 = \begin{pmatrix} \sin(v) \\ -\cos(v) \\ u \end{pmatrix}$. Hence, $|n^0| = \sqrt{1+u^2} \Rightarrow$ unit normal: $n = \frac{1}{|n^0|} n^0 = \frac{1}{\sqrt{1+u^2}} \begin{pmatrix} \sin(v) \\ -\cos(v) \\ u \end{pmatrix}$

2. Remember that we have to substitute $x = u\cos(v), y = u\sin(v), z = v$ into the vector field as in section 2:

$$F(x,y,z)\cdot n = \underbrace{\begin{pmatrix} z \\ y \\ x \end{pmatrix}}\cdot(\underbrace{u\cos(v)}_{x}, \underbrace{u\sin(v)}_{y}, v) \cdot \frac{1}{1+\sqrt{u^2}}\begin{pmatrix} \sin(v) \\ -\cos(v) \\ u \end{pmatrix}$$

3. Integrate over the scalar product. $F(x,y,z)$

$$= \frac{1}{\sqrt{1+u^2}}(v\sin(v) + u\sin(v)\cos(v) + u^2\cos(v))$$

$$\iint_S \underbrace{F \cdot n} \, dS = \iint \underbrace{\frac{1}{\sqrt{1+u^2}}(v\sin(v) + u\sin(v)\cos(v) + u^2\cos(v))}_{f(r(u,v)) \text{ as in prev. section}} \cdot \underbrace{\sqrt{1+u^2}}_{|r_u \times r_v|} d(u,v)$$

$$= \int_0^\pi\int_0^1 v\sin(v) + u\sin(v)\cos(v) + u^2\cos(v) \, du\,dv$$

$$= \int_0^\pi v\sin(v) + \frac{1}{2}\sin(v)\cos(v) + \underbrace{\frac{1}{3}\cos(v)\,dv}_{\int_0^\pi \cos(v)\,dv = 0}$$

$$= [-v\cos(v)]_0^\pi + \int_0^\pi \cos(v)\,dv + \frac{1}{4}[\sin(v)^2]_0^\pi$$

$$= \pi + 0 + 0 = \boxed{\pi}$$

Problem: 3. Surface Integrals over Vector Fields. Determine the surface integrals within the following vector fields:

1. $F(x,y,z) = xi - zj + yk$. S is the part of the sphere $x^2+y^2+z^2 = 4$ in the first octant, and the normal points in negative direction.

4 The Divergence Theorem

Introduction: Divergence Theorem The Divergence Theorem is one of the most impressive theorems of vector calculus. It connects surface integration with integration over a volume. Let E be a simple bounded region and S the boundary surface of E with an positive orientation (normal points outwards). If F is a continuous vectorfield with continuous partial derivatives on a region containing E, then

$$\iint F\cdot dS = \iiint_E \mathrm{div} F dV.$$

The divergence of a vectorfield $F = Pi+Qj+Rk$ is defined as follows: $\mathrm{div}F = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$.

Example: Divergence Theorem. Determine the flux of the vector field $F(x,y,z) = yi + yj + xk$ over the sphere $x^2+y^2+z^2 = 4$.

1. Determine the divergence of F:

$$\mathrm{div}(F) = \underbrace{\frac{\partial}{\partial x}y}_{=0} + \underbrace{\frac{\partial}{\partial y}y}_{=1} + \underbrace{\frac{\partial}{\partial z}x}_{=0} = 1$$

2. Use the divergence theorem to obtain an integral over the ball contained within the sphere.

$$\iint F \cdot dS \overset{\text{Div Thm}}{=} \iiint_{\text{Ball (radius}=2)} 1 \, dV = V(\text{Ball (Radius } 2)) = \frac{4\pi}{3} 2^3$$

Problemset: 4. Divergence Theorem. Use the divergence theorem to evaluate.

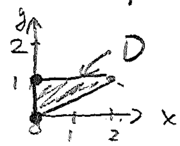
1. S is the surface of the solid bounded by $y^2 + z^2 = 1$ and the planes $x = -1; x = 2$. Determine the surface integral over $F(x, y, z) = 3xy^2 i + x\exp(z) j + z^3 k$

2. Find the surface integral over $F(x, y, z) = (\cos(z) + xy^2)i + x\exp(-z)j + (\sin(y) + x^2 z)k$, where S is the surface of the solid bounded by the paraboloid $z = x^2 + y^2$ and the plane $z = 4$.

Solutions to Worksheet 5 :

Problemset 1:

1. Determine the surface area of $-1 = 3x + 2y^2 - z$ above the triangle $\Delta(0,0), (0,1), (2,1)$.



$z = 3x + 2y^2 + 1 = g(x,y)$

I use the graph form:

$$A(S) = \iint_D \sqrt{1 + 3^2 + (4y)^2}\, dA = \int_0^1 \int_0^{2y} \sqrt{10 + 16y^2}\, dx\, dy = \int_0^1 \sqrt{10 + 16y^2}\, 2y\, dy$$

$$= \left[(10 + 16y^2)^{3/2} \cdot \frac{2}{3} \cdot \frac{1}{16} \right]_0^1 = (26)^{3/2} \frac{1}{24} - (10)^{3/2} \frac{1}{24}$$

2. The normal vector is given by

$$n = \begin{vmatrix} i & j & k \\ \hline & \nabla_u r & \\ \hline & \nabla_v r & \end{vmatrix} = \begin{vmatrix} i & j & k \\ \cos(v) & \sin(v) & 0 \\ -u\sin(v) & u\cos(v) & 1 \end{vmatrix} = \sin(v)\, i \mp \cos(v)\, j + u\, k$$

↳ Because $u \in [0,1]$ the normal faces upward.

$$|n| = \sqrt{\sin^2(v) + \cos^2(v) + u^2} = \sqrt{1 + u^2}$$

$u = \tan(\tilde{u})$

$$A(S) = \iint_D \sqrt{1+u^2}\, dA = \int_0^{\pi} 1\, dv \cdot \int_0^1 \sqrt{1+u^2}\, du = \pi \int_0^{\frac{\pi}{4}} \frac{1}{\cos(\tilde{u})^3}\, d\tilde{u}$$

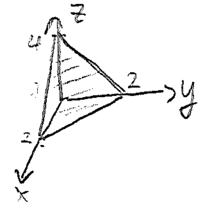
$$\boxed{\begin{aligned} \frac{du}{d\tilde{u}} &= \frac{1}{\cos^2(\tilde{u})} \\ \tan^{-1}(0) &= 0 \\ \tan^{-1}(1) &= \frac{\pi}{4} \end{aligned}} \quad \boxed{\sqrt{1 + \tan(\tilde{u})^2} = \sqrt{1 + \frac{\sin^2(\tilde{u})}{\cos^2(\tilde{u})}} = \sqrt{\frac{1}{\cos^2(\tilde{u})}} = \frac{1}{\cos(\tilde{u})}}$$

$$\overset{\text{Wolfram Alpha}}{=} \pi \left(\frac{1}{2} \left[\sqrt{2} + \sinh^{-1}(1) \right] \right)$$

! **Problem: Determining a pla....** (see last page)

Problemset 2:

1. Solve for z: $z(x,y) = 4 - 2x - 2y$



$0 = z = 4 - 2x - 2y$

$y = 2 - x$

$y = 0$: $z = 4 - 2x$

$x = 0$: $z = 4 - 2y$

We see in sketch: $\boxed{\begin{aligned} 0 &\leq x \leq 2 \\ 0 &\leq y \leq 2 - x \end{aligned}}$

↳ domain

$$\iint_S xz\, dA = \iint_D x \cdot (4 - 2x - 2y) \sqrt{1 + (-2)^2 + (-2)^2}\, dy\, dx = 3 \int_0^2 x \int_0^{2-x} (4 - 2x - 2y)\, dy\, dx$$

$$= 3 \int_0^2 x \left[(4-2x)(2-x) - (2-x)^2 \right] dx = 3 \int_0^2 x \cdot \left[(2-x)(4 - 2x - 2 + x) \right] dx$$

$$= 3 \int_0^2 x \left[\underbrace{(2-x)^2}_{4 - 4x + x^2} \right] dx = 3 \left[2x^2 - \frac{4}{3}x^3 + \frac{1}{4}x^4 \right]_0^2 = 3 \left[8 - \frac{32}{3} + 4 \right] = \frac{16}{3}$$

2.2:

$$\iint_S x^2 z + y^2 z \, dS \qquad S: x^2+y^2+z^2=1 \qquad z \geq 0$$

$$0 \leq \varphi \leq 2\pi \qquad 0 \leq \theta \leq \frac{\pi}{2}$$

Introduce polar coordinates with fixed radius.

$$r(\varphi,\theta) = \langle \cos(\varphi)\sin(\theta),\ \sin(\varphi)\sin(\theta),\ \cos(\theta) \rangle$$

$$n = \begin{vmatrix} i & j & k \\ -\sin(\varphi)\sin(\theta) & \cos(\varphi)\sin(\theta) & 0 \\ \cos(\varphi)\cos(\theta) & \sin(\varphi)\cos(\theta) & -\sin(\theta) \end{vmatrix}$$

$$= -i\cos(\varphi)\sin(\theta)^2 - j\sin(\varphi)\sin(\theta)^2 + k\underbrace{\left[-\sin(\varphi)^2\sin(\theta)\cos(\theta) - \cos(\varphi)^2\sin(\theta)\cos(\theta)\right]}_{= -\sin(\theta)\cos(\theta)}$$

$$|n| = \sqrt{\cos(\varphi)^2\sin(\theta)^4 + \sin(\varphi)^2\sin(\theta)^4 + \sin(\theta)^2\cos(\theta)^2}$$

$$= \sqrt{\sin(\theta)^4 + \sin(\theta)^2\cos(\theta)^2} = |\sin(\theta)| \underset{0 \leq \theta \leq \frac{\pi}{2}}{=} \sin(\theta)$$

$$\int_0^{\frac{\pi}{2}}\int_0^{2\pi} \underbrace{\left[\cos(\varphi)^2\sin(\theta)^2\cdot\cos(\theta) + \sin(\varphi)^2\sin(\theta)^2\cos(\theta)\right]}_{g(r(\varphi,\theta))} \underbrace{\sin(\theta)}_{|r_\varphi \times r_\theta|} \, d\varphi \, d\theta$$

$$= \int_0^{\frac{\pi}{2}} \cos(\theta)\sin(\theta)^3 \, d\theta \int_0^{2\pi} 1 \, d\varphi = \left[\frac{1}{4}(\sin(\theta))^4\right]_0^{\frac{\pi}{2}} \cdot 2\pi = \frac{\pi}{2}$$

Problem 3:

1. $F(x,y,z) = xi - zj + yk$ part of the sphere $x^2+y^2+z^2=4$ $\quad x,y,z \geq 0$

$$\iint_S F \cdot dS = \iint_S F \cdot n \, dS = \iint_S \langle x, -z, y\rangle \cdot \begin{pmatrix} -x \\ -y \\ -z \end{pmatrix} \frac{1}{2} \, dS = \frac{1}{2}\iint_S (-x^2) \, dS$$

$g(x,y,z) = x^2+y^2+z^2$ has the gradient $\nabla g = 2\cdot\begin{pmatrix} x \\ y \\ z \end{pmatrix}$

- The gradient is normal on the contour line $k = g(x,y,z)$
- $-$ sign for negative orientation
- $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \frac{1}{\left\|\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right\|} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}\frac{1}{2}$ is normalized

Then, from previous q. and $r(\varphi,\theta) = \langle 2\cdot\cos(\varphi)\sin(\theta),\ 2\sin(\varphi)\sin(\theta),\ 2\cos(\theta)\rangle$

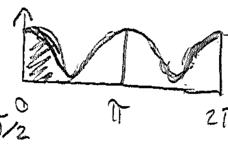
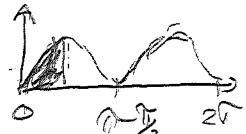
$$|n| = 4\sin(\theta)$$

$$\dots = \int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} \underbrace{-\frac{1}{2}\left(4\cdot\cos(\varphi)^2\sin(\theta)^2\right)}_{-\frac{1}{2}x^2} \cdot \underbrace{4\sin(\theta)}_{|n|} \, d\varphi \, d\theta$$

$$= -8\cdot\int_0^{\pi/2}\cos(\varphi)^2 \, d\varphi \int_0^{\pi/2}\sin(\theta)^3 \, d\theta$$

Problem 3.1 continued:

$$\int_0^{\pi/2} \cos(\varphi)^2 d\varphi =$$

observe that  $\int_0^{\pi/2} \cos(\varphi)^2 d\varphi \overset{(*)}{=} \int_0^{\pi/2} \sin(\varphi)^2 d\varphi$ 

and, hence, $\int_0^{\pi/2} \cos(\varphi)^2 d\varphi = \frac{1}{2}\int_0^{\pi/2} \cos(\varphi)^2 d\varphi + \underbrace{\frac{1}{2}\int_0^{\pi/2} \sin(\varphi)^2 d\varphi}_{(*)} = \frac{1}{2}\cdot\int_0^{\pi/2} \underbrace{\cos(\varphi)^2 + \sin(\varphi)^2}_{=1} d\varphi$

$$= \frac{1}{2}\frac{\pi}{2} = \frac{\pi}{4}$$

$$\int_0^{\pi/2} \sin(\theta)^3 d\theta = \int_0^{\pi/2} \sin(\theta)\overbrace{(1-\cos(\theta)^2)}^{\sin(\theta)^2} d\theta = \left[-\cos(\theta) + \frac{1}{3}\cos(\theta)^3\right]_0^{\pi/2}$$

$\sin(\theta)$ $\quad -\cos(\theta)^2\sin(\theta)$

$$= [-0 + 0 - (\underbrace{-\cos(0)}_{=-1} + \frac{1}{3}\underbrace{\cos(0)^3}_{=1})]$$

$$= 1 - \frac{1}{3} = \frac{2}{3}$$

Hence, $\iint_S F \cdot dS = -8 \cdot \frac{\pi}{4} \cdot \frac{2}{3}$.

Problemset 4: Divergence Theorem.

1. $\mathrm{div}(F) = \frac{\partial}{\partial x}(3xy^2) + \frac{\partial}{\partial y}(x\exp(z)) + \frac{\partial}{\partial z}(z^3) = 3y^2 + 3z^2$

   Use div-Thm and cylinder coordinates $x = x$ $\quad y = r\cos(\theta)$, $z\ r\sin(\theta)$

   $$\iint_S F \cdot dS = \iiint_E 3y^2 + 3z^2\, d(x,y,z) = \int_{-1}^{2}\int_0^{2\pi}\int_0^1 3r^2 \cdot r\, dr\, d\theta\, dx = \frac{3}{4}\cdot 2\pi \cdot 1 = \frac{6}{4}\pi$$

2. $\mathrm{div}(F) = \frac{\partial}{\partial x}(\cos(z) + xy^2) + \frac{\partial}{\partial y}(x\exp(-z)) + \frac{\partial}{\partial z}(\sin(y) + x^2 z)$

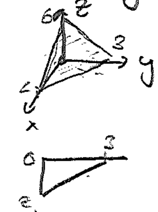
   $$= y^2 + 0 + x^2$$

   $$\iint_S F \cdot dS = \iiint_E \mathrm{div}(F)\, dV = \int_0^{2\pi}\int_0^1\int_r^4 r^2 \cdot r\, dz\, dr\, d\theta = 2\pi \cdot \int_0^1 (4r^3 - r^4)\, dr = 2\pi \cdot (1 - \frac{1}{5})$$

   $$= \frac{10}{5}\pi$$

Problem: Determining a planar surface area: $3x + 2y + z = 6 \Rightarrow z = 6 - 3x - 2y$



Graph: $$\int_0^2 \int_0^{3-\frac{3}{2}x} \underbrace{\sqrt{1+9+4}}_{\sqrt{14}} \, dy \, dx$$

$$\int_0^2 \sqrt{14}\left(3 - \tfrac{3}{2}x\right) dx = \sqrt{14}\left(6 - \frac{3}{4} \cdot 4\right) = \sqrt{14} \cdot 3$$

$z = 0$

$6 = 3x + 2y$

$y = \frac{6}{2} - \frac{3}{2}x$

$= 3 - \frac{3}{2}x$

Planar: Representation

$$r(u,v) = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} + u \cdot \begin{pmatrix} -2 \\ 3 \\ 0 \end{pmatrix} + v \begin{pmatrix} -2 \\ 0 \\ 6 \end{pmatrix}$$

$$\begin{vmatrix} i & j & k \\ -2 & 3 & 0 \\ -2 & 0 & 6 \end{vmatrix} = 18i - 12j + 6k$$

~~Dx~~ $$\int_0^1 \int_0^u \sqrt{18^2 + 12^2 + 6^2} \, dv \, du = \int_0^1 u \sqrt{504} \, du = \frac{1}{2} \cdot 3 \cdot \sqrt{56} = 3\sqrt{14}$$