Worksheet 9

Felix Funk, MATH Tutorial - Mech 221

1 2D Phase Portraits

Introduction: Drawing Phase Portraits
A linear system of differential equations defined by

$$\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = A \cdot \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \tag{1}$$

can be illustrated by a 2D phase portrait. The eigenvalues and eigenvectors give us essential information about the stability and shape of phase portraits.

Real eigenvectors provide **lines** along which the trajectories enter or leave an equilibrium.

- If the corresponding eigenvalue is positive, then the corresponding solutions leave the equilibrium along the line.
- If the corresponding eigenvalue is negative, then the equilibrium attracts solutions on the line.
- If the eigenvalue is zero, then the whole line consists of equilibria.

When the eigenvalues are **complex**, then the **real component of the eigenvalues** tells, whether a solution

- spirals inwards; $Re(\lambda) < 0$,
- spirals outwards; $Re(\lambda) > 0$,
- or remains on an periodic orbit; $Re(\lambda) = 0$.

The orientation of the spirals and periodic orbits can be obtained by sampling directions using (1).

Problem: Phase Portraits.
Draw the phase portrait to

$$y' = \begin{bmatrix} 1 & 1 \\ 0 & -2 \end{bmatrix} y \tag{2}$$

Example: Obtaining the important information for real eigenvalues.

1. Determine the eigenvectors and eigenvalues.

$$\lambda_1 = \qquad , v_1 = \begin{pmatrix} \quad \\ \quad \end{pmatrix}, \lambda_2 = \qquad , v_2 = \begin{pmatrix} \quad \\ \quad \end{pmatrix}$$

2. Draw the lines corresponding to the eigenvectors. Use the sign to determine the stability along the lines.

3. If you are missing information to complete the phase portrait, sample the directions of individual points, e.g. find the direction in $y = [0, 1]^T$ by calculating the derivative $y' = \begin{bmatrix} 1 & 1 \\ 0 & -2 \end{bmatrix} y$.

Problem: Phase Portraits 2.
Draw the phase portrait to

$$y' = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} y \tag{3}$$

Example: Obtaining the important information for complex eigenvalues.

1. Determine the eigenvalues.

$$\lambda_1 = \qquad , \lambda_2 =$$

2. If the eigenvalues are complex: Determine the real part of the eigenvalues and categorize whether the movement is inwards or outwards/ on periodic orbits.

3. To determine the orientation of the spirals or the movement on the periodic orbits, complete the phase portrait, sample the directions of individual points, e.g. find the direction in $y = [0, 1]^T$ by calculating the derivative $y' = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} y$.

1.1 Mixed Problems

Problemset: Mixed problems.

1. Draw the phase portrait and use it to find the stability of the origin.

$$\begin{aligned} x' &= x + y \\ y' &= x - y \end{aligned}$$

2. Transform the equation into a linear system of first order differential equations $y'' + 4y = 0$. Then, draw a phase portrait.

3. Draw the phase portrait and use it to find the stability of the origin.

$$\begin{aligned} x' &= x + y \\ y' &= 2y \end{aligned}$$

4. Draw the phase portrait and use it to find the stability of the origin.

$$y' = \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix} y \tag{4}$$

5. Challenging: Draw the phase portrait and use it to find the stability of the origin.

$$y' = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} y \tag{5}$$

2 First-order autonomous ODEs

This section is kindly provided by Cole Zmurchok.

Consider the following ODE:

$$x' = f(x) = -x(x-2)(x+2).$$

Finding steady-states The steady states x^* satisfy $f(x^*) = 0$.

1. Find the steady-states of the ODE above.

The **linearized equation** is

$$\frac{d\eta}{dt} = f'(x^*)\eta$$

The linearized system describes the behaviour of the system *near* each of the steady-states.

1. Find the linearized equation for the ODE above at each steady-state, and draw a conclusion about the stability of each steady-state.

Graphical analysis

1. Sketch the phase-portrait (x' versus x) indicating the steady-states, their stability, and the direction of flow on the x-axis with arrows.

   From the phase-plane, sketch several sample solutions in the tx-plane from different initial conditions.

1. Complete a phase-plane analysis for the ODE $x' = x - \tan x$ and sketch a few solutions. If it is too difficult, impossible, or annoying to find the steady-states analytically, use graphical methods alone. Hint: plotting $y_1 = x$ and $y_2 = \tan x$ on the same axis might be fruitful.

2. Complete a phase-plane analysis for the ODE $x' = -x^2(1 - x)$ and sketch several sample solutions.

3 Nonlinear systems

This section is kindly provided by Cole Zmurchok.

Consider the system of ODEs:

$$\dot{x} = f(x, y) = x(3 - x - 2y)$$
$$\dot{y} = g(x, y) = y(2 - x - y)$$

Finding steady-states The steady states (x^*, y^*) satisfy $f(x^*, y^*) = g(x^*, y^*) = 0$.

1. Find the four steady-states of the system ODE above.

The **linearized system** is

$$\frac{d}{dt}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \frac{\partial f}{\partial x}(x^*, y^*) & \frac{\partial f}{\partial y}(x^*, y^*) \\ \frac{\partial g}{\partial x}(x^*, y^*) & \frac{\partial g}{\partial y}(x^*, y^*) \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = A \begin{pmatrix} u \\ v \end{pmatrix}.$$

A is called the **Jacobian matrix** evaluated at the steady-state (x^*, y^*) of the system. **The linearized system describes the behaviour of the system *near* each of the steady-states**.

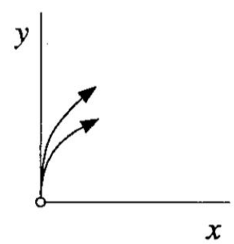
1. Find the Jacobian matrix for the system above.

2. Evaluate the Jacobian matrix at each of the steady-states found above.

Sketching the phase portrait The linearized system describes the behaviour of the system *near* each of the steady-states. For example, for the steady-state $(0, 0)$,

$$A = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix},$$

so the eigenvalues of A ($\lambda = 2, 3$) are positive. From this, we say that $(0, 0)$ is an **unstable node**. Thus, *near* $(0, 0)$ we find:



1. Find the eigenvalues of the Jacobian matrix evaluated at each steady-state. Using what you know about classifying linear systems, classify each steady-state as a saddle, source, sink, spiral source, or spiral sink. *(Note that if the eigenvalues are purely imaginary, or if one or more eigenvalues are zero, then linear stability analysis is* can *be wrong. More work is needed in these cases).*

2. Plot the location of each steady-state in the xy-plane. Using the classification obtained above, sketch a mini phase-portrait around each steady-state. Finally, connect the trajectories as expected to sketch the complete phase-plane diagram.