Worksheet 4

Felix Funk, MATH Tutorial - Mech 222

1 Directional Derivatives

Introduction: Directional Derivatives.
To determine the local change of a multivariable function f into a certain direction, we calculate:

$$D_u f(x, y, z) = \nabla f(x, y, z) \cdot u,$$

where u expresses a unit vector, i.e. $|u| = 1$ and $\nabla f = \left[\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right]$.

Problemset: 1. Directional Derivatives.

1. Show that
$$D_u f = |\nabla f| \cos(\theta),$$
where θ denotes the angle between u and ∇f

2. Find the direction(s), in which
   - f has its largest increase,
   - f has its greatest descent,
   - f does not change.

3. Find the directions in which $f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$ in $(1, 2, 2)$ changes most rapidly.

4. After a sandstorm, a mars-rover does not transmit any visual information of its surrounding landscape $f(x, y)$ as it is supposedly stuck in a valley. You only observe three slips of information: $D_{u_1} f = -2$, $D_{u_2} f = 2$ and $D_{u_3} f = 1$ for $u_1 = 1/\sqrt{2} \cdot [1, 1]^T$, $u_2 = 1/\sqrt{2} \cdot [-1, -1]^T$ and $u_3 = 1\sqrt{5} \cdot [2, 1]^T$. Find the direction, in which the rover is most likely to escape the valley in the shortest amount of time.

5. Challenging: Show that every normal line to the sphere $x^2 + y^2 + z^2 = r^2$ passes through the center of the sphere.

2 Maximum and Minimum Values

Introduction: Maximum and Minimum Values

A function $f(x, y)$ attains a minimum value in a point (a, b) if all closeby points (x, y) satisfy $f(x, y) \geq f(a, b)$. Analogously, it attains its maximum value if $f(x, y) \leq f(a, b)$ for all adjacent (x, y) In both scenarios, $D_u f(a, b) = 0$ in any direction u. If we want to determine whether a critical point, in which all partial derivatives are 0, we try the second derivative test. If

$$D(a, b) = f_{xx}(a, b) f_{yy}(a, b) - f_{xy}(a, b)^2$$

- is positive and $f_{xx}(a, b) > 0$, then $f(a, b)$ is a local minimum.
- is positive and $f_{xx}(a, b) < 0$, then $f(a, b)$ is a local maximum.
- is negative, then $f(a, b)$ is neither.

Problemset: 2: Maxima and Minima.

1. Find some values for a, b such that $f(x, y) = ax^2 + by^2$ such that the origin
   - is a local minimum.
   - is a local maximum.
   - is neither.
   - the second derivative test fails but there is a local maximum.

   Test the values using the second derivative test.
2. Find the point on the plane $2x - 3y + 3z = 6$ that is closest to the point $(0, 1, 1)$.
3. Determine all absolute min/max of $f(x, y) = x^2 + y^2 + x^2 y + 3$ within the domain $D = \{(x, y) | |x| \leq 1, |y| \leq 1\}$.

3 Constrained Minima/Maxima

Introduction: Lagrange Multipliers. You can find a max/min of a function $f(x, y, z)$ given the constraint $g(x, y, z) = k$ under the assumption that $\nabla g \neq 0$ on the surface $g(x, y, z) = k$ by using the method of Lagrange Multipliers:

1. Find all (x, y, z, λ) such that $\nabla f(x, y, z) = \lambda \nabla g(x, y, z)$ and $g(x, y, z) = k$.
2. Evaluate f at all points that result from step (a). The largest is the maximum value and the smallest the minimum value if they exist.

It is important to note that usually the resulting system is non-linear. This means that solutions do not necessarily exist.

A useful strategy is usually to solve the equations for λ and with that derive constraints on x, y, z and λ. The most common source of errors is **division by zero** in the process of solving without taking all cases into account.

Problemset: 3. Lagrange Multipliers. Use Lagrange Multipliers to solve the following problems:

1. Find the absolute maximum/minimum of $f(x, y) = x^2 + y^2$, subject to $xy = 1$.
2. Find all min/max of $f(x, y, z) = xyz$, subject to $x^2 + 2y^2 + 3z^2 = 6$.