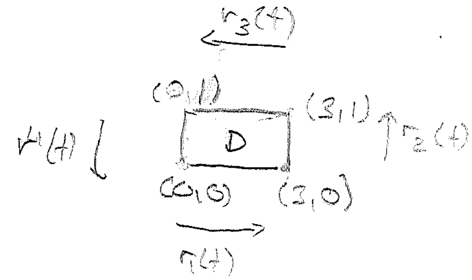
Worksheet 8

Problemset: 1 Green's Theorem.

1.1 → Direct evaluation: $\int_C xy\,dx + x^2\,dy$



$C_1: r_1(t) = 3t\,i \quad t \in [0,1]$

$C_2: r_2(t) = 3i + tj$

$C_3: r_3(t) = (3-3t)\,i + j$

$C_4: r_4(t) = (1-t)\,j$

$\int_C xy\,dx + x^2\,dy$

① Individual calculations:

$$\int_{C_1} xy\,dx + x^2\,dy = \int_0^1 \underbrace{0}_{y=0} + \underbrace{0}_{y'=0}\,dt = 0$$

$$\int_{C_2} xy\,dx + x^2\,dy = \int_0^1 3t\cdot\underbrace{0}_{x'} + \underbrace{9}_{x^2}\cdot\underbrace{1}_{y'}\,dt = 9$$

$$\left[t - \frac{t^2}{2}\right]_0^1 \quad 1 - \frac{1}{2}$$

$$\int_{C_3} xy\,dx + x^2\,dy = \int_0^1 \underbrace{(3-3t)}_{x}\cdot\underbrace{1}_{y}\cdot\underbrace{(-3)}_{x'} + \underbrace{(3-3t)^2}_{x^2}\cdot\underbrace{0}_{y'}\,dt = -9\int_0^1 (1-t)\,dt = -9\cdot\frac{1}{2}$$

$$\int_{C_4} xy\,dx + x^2\,dy = \int_0^1 \underbrace{0}_{x}\underbrace{(1-t)}_{y}\underbrace{0}_{x'}\,dt + \underbrace{0}_{x^2}\cdot\underbrace{(-1)}_{y'}\,dt = 0$$

② $$\int_C xy\,dx + x^2\,dy = \int_{C_1 \oplus C_2 \oplus C_3 \oplus C_4} xy\,dx + x^2\,dy = 0 + 9 - 9\cdot\frac{1}{2} = 9\frac{1}{2}$$

$C_1 \oplus C_2 \oplus C_3 \oplus C_4$ ← concatenating the four curves

→ Green's Theorem: [pos orientation ↺, simple bounded surface, P, Q are cont. and have cont. partial deriv. ...]

$$\int_C \underbrace{xy}_{P}\,dx + \underbrace{x^2}_{Q}\,dy = \iint_D 2x - y\;d(x,y) = \int_0^3\int_0^1 2x - y\;dy\,dx = \int_0^3 2x - \frac{1}{2}\,dx = 9 - \frac{3}{2} = \frac{9}{2}$$

1.2 Recall from earlier tutorials:

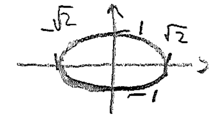
(*) If $f(y)$ is an odd function, i.e $f(-y) = -f(y)$, then for every $a \geq 0$ $\int_{-a}^{a} f(y)\,dy$ as $\int_{-a}^{a} f(y)\,dy = \int_{0}^{a} f(y)\,dy + \int_{-a}^{0} f(y)\,dy = \int_{0}^{a} f(y)\,dy + \int_{0}^{a} f(-y)\,dy$

$$\overset{\text{odd function}}{=} \int_0^a f(y)\,dy + \int_0^a -f(y)\,dy = \int_0^a \underbrace{f(y) - f(y)}_{=0}\,dy = 0$$

The area inside of the ellipse D is enclosed by the simple closed curve. C and $P(x,y) = y^4$, $Q(x,y) = 2xy$ have continuous partial derivatives in $\mathbb{R}^2$. We can apply Green's Theorem:

$$\frac{\partial Q}{\partial x} = 2y \; ; \; \frac{\partial P}{\partial y} = 4y^3 \Rightarrow \int_C y^4 dx + 2xy\,dy = \iint_D 2y - 4y^3\,d(x,y) = J$$

(I) If we remain in carthesian coordinates, then



$$y \in [-1, 1]$$
$$x \in [-\sqrt{2-2y^2}, \sqrt{2-2y^2}]$$

and

$$J = \iint_D 2y - 4y^3 d(x,y) = \int_{-1}^{1}\int_{-\sqrt{2-2y^2}}^{\sqrt{2-2y^2}} 2y - 4y^3\,dx\,dy = \int_{-1}^{1} 2\cdot\sqrt{2-2y^2}\,(2y-4y^3)\,dy.$$

Define, $f(y) = 2\cdot\sqrt{2-2y^2}\,(2y-4y^3)$. We show that $f(y)$ is an odd function

$$f(-y) = 2\cdot\sqrt{2-2(-y)^2}\,(2(-y) - 4(-y)^3) = -[2\sqrt{2-2y^2}\; 2y - 4y^3] = -f(y).$$

with $a=1$ and the result above (*), we find

$$\int_{-1}^{1} 2\sqrt{2-2y^2}\,(2y-4y^3)\,dy = \int_{-a}^{a} f(y)\,dy \overset{(*)}{=} 0.$$

(II) An alternative way to show this result transforms the integral J into polar coordinates.

In this case, parameterize the inside of the ellipse $D: x^2 + 2y^2 \leq 2$ in terms of radius r and angle θ: $x = r\cos(\theta)$ $y = r\sin(\theta)$.

In this case D: $\underbrace{r^2\cos^2(\theta) + 2r^2\sin^2(\theta) \leq 2}_{\Leftrightarrow\; r^2 \leq \frac{2}{2\sin^2(\theta)+\cos^2(\theta)}}$, i.e $\begin{cases} \theta \in [0, 2\pi) \\ r \in (0, \underbrace{\sqrt{\frac{2}{2\sin^2(\theta)+\cos^2(\theta)}}}_{=r_u(\theta)}] \end{cases}$

Then, $J = \iint_D 2y - 4y^3 d(x,y) = \int_0^{2\pi}\int_0^{r_u(\theta)} (2r\sin(\theta) - 4r^3\sin^3(\theta))\,r\,dr\,d\theta$

and you can compute the integral in a long and tedious calculation.

Problemset 2:

2.1: We use essentially that derivation is interchangeably for cont. 2nd order part. derivatives.

$$\mathrm{curl}(F) = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) i + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right) j + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) k$$

Then, $F = \nabla f = \underbrace{\frac{\partial}{\partial x} f}_{P}\, i + \underbrace{\frac{\partial}{\partial y} f}_{Q}\, j + \underbrace{\frac{\partial}{\partial z} f}_{R}\, k$:

Determine the three individual terms: cont. 2nd order part. deriv. } interchange $\frac{\partial}{\partial y}\left(\frac{\partial}{\partial z} f\right) = \frac{\partial}{\partial z}\left(\frac{\partial}{\partial y} f\right)$

$$\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} = \frac{\partial}{\partial y}\left(\frac{\partial}{\partial z} f\right) - \frac{\partial}{\partial z}\left(\frac{\partial}{\partial y} f\right) = 0$$

$$\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} = \frac{\partial}{\partial z}\left(\frac{\partial}{\partial x} f\right) - \frac{\partial}{\partial x}\left(\frac{\partial}{\partial z} f\right) = 0$$

and $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial y} f\right) - \frac{\partial}{\partial y}\left(\frac{\partial}{\partial x} f\right) = 0$

$$\mathrm{curl}(\nabla f) = 0i + 0j + 0k = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

2.2: Define: $f(x,y,z) = \frac{1}{2}x^2 + \frac{1}{2}y^2 + \frac{1}{2}z^2$.

Then, $(\nabla f)(x,y,z) = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = xi + yj + zk = F$ and f has cont. second order partial derivatives, Hence, $\mathrm{curl}(\nabla f) = 0$ by 2.1.

It is tempting to define $g(x,y,z) = \sqrt{x^2+y^2+z^2}$ as $\nabla g = \frac{1}{\sqrt{x^2+y^2+z^2}} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = G$ but here g is not necessarily cont. differentiable in $(0,0,0)$ and 2.1 not applicable.

~~Hence~~ We need to determine the derivatives of G and explicitly calculate.

$$\frac{\partial P}{\partial y} = \frac{\partial}{\partial y}\left(\frac{1}{\sqrt{x^2+y^2+z^2}}\, x\right) = -\frac{1}{2} \cdot \frac{1}{(x^2+y^2+z^2)^{3/2}} \cdot 2y \cdot x = \frac{xy}{\sqrt{(x^2+y^2+z^2)^3}} = \frac{-xy}{h(x,y,z)}$$

with $h(x,y,z) = (x^2+y^2+z^2)^{3/2}$

By symmetry:

$$\frac{\partial P}{\partial z} = \frac{-xz}{h(x,y,z)}, \quad \frac{\partial Q}{\partial x} = \frac{-xy}{h(x,y,z)}, \quad \frac{\partial Q}{\partial z} = \frac{-yz}{h(x,y,z)}, \quad \frac{\partial R}{\partial x} = \frac{-xz}{h(x,y,z)}, \quad \frac{\partial R}{\partial y} = \frac{-yz}{h(x,y,z)}$$

Hence,

$$\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} = -\frac{yz}{h(x,y,z)} + \frac{yz}{h(x,y,z)} = 0 \quad \ldots \Rightarrow \mathrm{curl}(G) = 0.$$

fill in dots

2.3 For F to be continuously differentiable, we need a function f such that $\nabla f = F$:

The first component xyz^3 suggests $\frac{\partial}{\partial x} f(x,y,z) = xyz^3 \Leftrightarrow f(x,y,z) = \frac{x^2}{2} yz^3 + g(y,z)$ where g is a function that does not depend on x.

Further, $\frac{\partial}{\partial z}\left(\frac{x^2}{2} yz^3 + g(y,z)\right) = b\, x^2 y z^2$

$\Leftrightarrow \quad \frac{3}{2} x^2 y z^2 + \frac{\partial}{\partial z} g(y,z) = b x^2 y z^2 \Leftrightarrow b = \frac{3}{2}$ and $\frac{\partial}{\partial z} g(y,z) = 0$,

i.e. $g(y,z) = g(y)$ (no dependence on z)

Lastly $\frac{\partial}{\partial y}\left(\left(\frac{x^2}{2} yz^3\right) + g(y)\right) = \frac{1}{2} x^2 y^a z^3$

$\Leftrightarrow \quad \frac{x^2}{2} z^3 + g'(y) = \frac{1}{2} x^2 y^a z^3 \Leftrightarrow g'(y) = 0$ and $a = 0$.

Hence, $a=0$, $b=3/2$ provides $f(x,y,z) = \frac{1}{2} x^2 y z^3$ such that $\nabla f = F$, i.e F is conservative.

<u>Curl:</u>

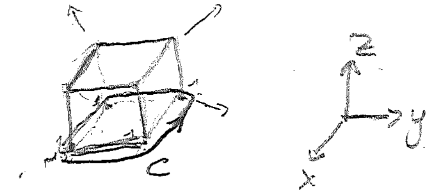
$\frac{\partial P}{\partial y} = xz^3$, $\frac{\partial P}{\partial z} = 3xyz^2$; $\frac{\partial Q}{\partial x} = xy^\alpha z^3$, $\frac{\partial Q}{\partial z} = \frac{3}{2} x^2 y^\alpha z^2$; $\frac{\partial R}{\partial y} = bx^2z^2$ $\quad \frac{\partial R}{\partial x} = 2b\,xyz^2$

$$\mathrm{curl}(F) = \begin{pmatrix} bx^2z^2 - \frac{3}{2} x^2 y^\alpha z^2 \\ 3xyz^2 - 2b\,xyz^2 \\ xy^\alpha z^3 - xz^3 \end{pmatrix} = \begin{pmatrix} x^2z^2\left(b - \frac{3}{2} y^\alpha\right) \\ xyz^2(3-2b) \\ xz^3(y^a - 1) \end{pmatrix}$$

For $a=0$, $b=3/2$ $\mathrm{curl}(F) = \begin{pmatrix}0\\0\\0\end{pmatrix}$ and all the part. derivatives of F are cont.

This is consistent with <u>2.1</u>

Problemset 3



3.1 $F = \begin{pmatrix} xyz \\ xy \\ x^2yz \end{pmatrix}$

The surface of the cube is oriented outward. The achieve a positive orientation you need to imagine walking down the curve C on top of the surface (the top is given by the normal vector). If the surface is always to the left then the orientation is positive.

The curve C is given by the vertices $(-1,-1,-1), (1,-1,-1), (1,1,-1)$ and $(-1,1,-1)$, with param:

$$r_1(t) = \begin{pmatrix} -1+2t \\ -1 \\ -1 \end{pmatrix} \quad r_2(t) = \begin{pmatrix} 1 \\ -1+2t \\ -1 \end{pmatrix} \quad r_3(t) = \begin{pmatrix} 1-2t \\ 1 \\ -1 \end{pmatrix} \quad r_4(t) = \begin{pmatrix} -1 \\ 1-2t \\ -1 \end{pmatrix} \quad t \in [0,1]$$

~~we~~ The vector field has continuous part. derivatives.

Then,

$$\iint_S \text{curl}(F)\, dS \overset{\text{Stoke's:}}{=} \int_C F \cdot dr = 0 \quad \text{because}$$

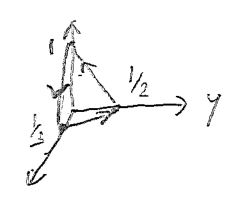
r_1: $$\int_0^1 \begin{pmatrix} (-1+2t)\cdot 1 \\ * \\ * \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} dt = \int_0^1 2\cdot(-1+2t)\, dt = -2+2 = \underline{0}$$

$\uparrow$ irrelevant as mult. with 0.

r_2: $$\int_0^1 \begin{pmatrix} * \\ (-1+2t) \\ * \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} dt = \int_0^1 2\cdot(-1+2t)\, dt = \underline{0}$$

r_3: $$\int_0^1 \begin{pmatrix} -1+2t \\ * \\ * \end{pmatrix} \cdot \begin{pmatrix} -2 \\ 0 \\ 0 \end{pmatrix} dt = \underline{0} \qquad r_4: \int_0^1 \begin{pmatrix} * \\ -1+2t \\ * \end{pmatrix} \begin{pmatrix} 0 \\ -2 \\ 0 \end{pmatrix} dt = \underline{0}$$

3.2



The ~~normal~~ graph of the surface is given by $\left.\begin{array}{l} \overbrace{\boxed{\begin{array}{l} 0 \le y \le \frac{1}{2} \\ 0 \le x \le \frac{1}{3} - \frac{2}{3}y \end{array}}}^{=D} \\ \text{and } z = 1-3x-2y \end{array}\right\} =: S$

$n = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$

Determine the curl of $F(x,y,z) = \begin{pmatrix} 1 \\ x+y \\ xy - \sqrt{z} \end{pmatrix}$ $\quad \text{curl}(F) = \begin{pmatrix} x-y \\ 0-y \\ 1-0 \end{pmatrix} = \begin{pmatrix} x-y \\ -y \\ 1 \end{pmatrix}$

Hence, $$\int_C F \cdot dr = \iint_S \begin{pmatrix} x-y \\ -y \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} dS = \iint_D 3x-5y+1\, d(x,y) = \int_0^{1/2}\int_0^{1/3-\frac{2}{3}y} 3x-5y+1\, dx\, dy$$

$$= \int_0^{1/2} \frac{3}{2}(1/3 - 2/3\, y)^2 - (5y+1)(1/3 - 2/3\, y)\, dy = \int_0^{1/2} (1/3 - 2/3\, y)(1/2 - y - 5y - 1)\, dy = \ldots$$

$$\int_0^{1/2} \left(\tfrac{1}{3} - \tfrac{2}{3}y\right)\cdot\left(-\tfrac{1}{2} - 6y\right) dy = \int_0^{1/2} -\tfrac{1}{6} + \tfrac{1}{3}y - 2y + 4y^2 \, dy$$

$$= \int_0^{1/2} -\tfrac{1}{6} \cancel{*} - \tfrac{5}{3}y + 4y^2 \, dy = -\frac{1}{12} - \frac{5}{3}\cdot\frac{1}{2}\left(\frac{1}{2}\right)^2 + \frac{4}{3}\cdot\left(\frac{1}{2}\right)^3$$

$$= -\frac{1}{12} - \frac{5}{24} + \frac{4}{24} = \frac{-3}{24} = -\frac{1}{8}$$

3.3

$$F(x,y,z) = \begin{pmatrix} -2yz \\ y \\ 3x \end{pmatrix}$$ ← cont. partial deriv.

The curve C is given by

$$z=1 = 5 - x^2 - y^2 \iff 4 = x^2 + y^2, \quad z = 1$$

$\cancel{z=1}$

$$\iff (2\cos(\theta), 2\sin(\theta), 1) = r(\theta) \quad \theta \in [0, 2\pi]$$

Then,

$$\iint_S \mathrm{curl}(F)\, dS \overset{\text{Stoke's}}{=} \int_C F \cdot dr = \int_0^{2\pi} \begin{pmatrix} -2(2\sin(\theta)\cdot 1) \\ 2\sin(\theta) \\ 3\cdot(2\cos(\theta)) \end{pmatrix} \cdot \underbrace{\begin{pmatrix} -2\sin(\theta) \\ 2\cos(\theta) \\ 0 \end{pmatrix}}_{r'} d\theta$$

$$= \int_0^{2\pi} 8\sin(\theta)^2 + 4\sin(\theta)\cos(\theta)\, d\theta$$

$$= 8\cdot\underbrace{\int_0^{2\pi} \sin(\theta)^2 d\theta}_{\pi} + 4\cdot\underbrace{\int_0^{2\pi} \sin(\theta)\cos(\theta) d\theta}_{[\sin(\theta)^2]_0^{2\pi}} = 8\pi + 4\cdot 0 = 8\pi$$