Worksheet 4

Felix Funk, MATH Tutorial - Mech 221

1 Non-homogeneous second order linear ODEs

The first part of the worksheet discusses how to treat second order linear constant-coefficient ODEs with non-homogeneous right-hand-side. The second part of the worksheet is about mass-damper-spring systems as an illustration of homogeneous second order linear ODEs.

Introduction: Non-homogeneous Second Order Linear ODEs.
In the last worksheet, we have explored the different dynamics that one can capture in systems of the form.

$$ay_h'' + by_h' + cy_h = 0 \tag{1}$$

These solutions y_h will be called homogeneous solutions as they solve the homogeneous problem (i.e. the right side of the equation is 0.)

In applications, we frequently observe ODEs of the form:

$$ay'' + by' + cy = f(t) \tag{2}$$

We will discuss: How can we find a solution? What is the form of a general solution to equation (2)?

Problem: The form of the general solution. Let $y_{p,1}$ and $y_{p,2}$ are two particular solutions to the non-homogeneous differential equation (2). Show: The two particular solution only differ by a homogeneous solution. Hint: Consider $y = y_{p,1} - y_{p,2}$.

Conclusion: The General Solution.
Thus, the general solution to equation (2) is given by $y(t) = y_p(t) + y_h(t)$, where y_p is any (guessed) non-homogeneous solution and y_h is a homogeneous solution.

Introduction: Method of Undetermined Coefficients: Guessing y_p.
To obtain a particular solution, sometimes educated guessing is the most straight-forward approach. We use versions of the inhomogeneity $f(t)$ as a guess for the particular solution y_p. In the following subsections, we explore polynomial, oscillatory, and exponential inhomogeneities $f(t)$.

1.1 Method of Undetermined Coefficients: $f(t)$ is a Polynomial.

Problem: Polynomial $f(t)$.
Find a particular solution to:

$$y'' + 2y' + 2y = 5t + 1, \tag{3}$$

Example: Educated Guess: Polynomial.

1. Identify $f(t) =$

2. $f(t)$ is a polynomial. Mimic $f(t)$ by guessing a polynomial with undetermined coefficients.

$$y_p(t) = At + B$$

.

3. Substitute $y_p(t)$ into equation (3).

4. Determine A, B.

Problem: General solution and IVP.
Find the general solution to equation (3). Solve the IVP $y(0) = 1, y'(0) = 2$

Example: General Solution.

1. Find the general homogeneous solution y_h.

2. Combine:

$$y(t) = y_h(t) + y_p(t) =$$

3. Solve the IVP.

Problem: 1.1. Find the general solution to

$$y'' + y = t. \tag{4}$$

Problem: 1.2. Find the solution to

$$y'' + 2y' + y = t^2 \tag{5}$$

that satisfies the conditions y(0) = -1, y'(0)=0.

1.2 Method of Undetermined Coefficients: $f(t)$ is Periodic.

Problem: Periodic $f(t)$.
Find a particular solution to:

$$y'' - 4y' + 4y = cos(4t), \tag{6}$$

Example: Educated Guess: Cosines and Sines.

1. Identify $f(t) =$
2. $f(t)$ undergoes periodic motion. Mimic $f(t)$ by guessing a periodic function with undetermined coefficients.

$$y_p(t) = Acos(4t) + Bsin(4t)$$

.

3. Substitute $y_p(t)$ into equation (6).
4. Determine A, B.

Problem: General solution.
Find the general solution to equation (6).

Example: General solution.

1. Find the general homogeneous solution y_h.
2. Combine:

$$y(t) = y_h(t) + y_p(t) =$$

Problem: 2.1. Find the particular solution to

$$y'' - y = \sin(t). \tag{7}$$

Problem: 2.2. Find a particular solution to

$$y'' + y = \sin(t). \tag{8}$$

1. Determine the homogeneous solution y_h, first.

2. Observe, that the previous guess fails.

3. Now, try the guess $y_p(t) = At\cos(t) + Bt\sin(t)$. Hypothesize, what one can do, when the educated guess coincides with a homogeneous solution.

1.3 Method of Undetermined Coefficients: $f(t)$ Grows Exponentially.

Problem: Exponential $f(t)$.
Find a solution to

$$y'' - 2y' = e^{3t}, \tag{9}$$

that satisfies y(0) = 0, y'(0) = 1.

Example: Educated Guess: Exponential.

1. Identify $f(t) =$
2. $f(t)$ grows exponentially. Mimic $f(t)$ by guessing a periodic function with undetermined coefficients.
$$y_p(t) = Ae^{3t}$$
.
3. Substitute $y_p(t)$ into equation (9).
4. Determine A.
5. Find the general homogeneous solution y_h.
6. Combine:
$$y(t) = y_h(t) + y_p(t) =$$
7. Solve the IVP.
$$y(t) = \qquad\qquad\qquad\qquad\qquad\qquad .$$

Problem: 2.1. Find the general solutions to

$$
\begin{aligned}
2y'' - 8y' &= e^{-t}. && (10)\\
y'' - 100y &= e^{10t} && (11)
\end{aligned}
$$

2 Application: Mass-Spring-Damper System

Introduction: Spring- Mass- Damper Systems.
To model mass-spring-damper systems, we use a second order system of the form

$$x'' + bx' + cx = 0,$$

where $b \geq 0$ models dampening and c is a positive spring-related constant. Let $x(t)$ denote the vertical displacement at time t. We distinguish four cases according to the diagram below depending on the roots of the system.

Problem: A damped oscillator. For the mass-spring damper system with varying dampening

$$x'' + bx' + 4x = 0, \tag{12}$$

find all b such that $x(t)$ exerts

- free motion,
- underdampened motion,
- critically-damped motion,
- overdamped motion.