Worksheet 6

Felix Funk, MATH Tutorial - Mech 221

1 Laplace - Transformation

Reminder: Laplace - Transform. The Laplace transform of a function $f(t)$ is defined by

$$F(s) = L\{f(t)\}(s) = \int_0^\infty e^{-st} f(t)dt. \tag{1}$$

Furthermore, there is an inverse transform $L^{-1}\{F(s)\}(t)$ that satisfies

$$L^{-1}\{F(s)\}(t) = f(t),$$

i.e. the Laplace transform and its inverse cancel. It satisfies four basic properties:

1. **Linearity:**

$$L\{af(t) + bg(t)\}(s) = a F(s) + b G(s)$$

2. **Differentiation is Transformed to Multiplication:**

$$L\{x'(t)\}(s) = s \cdot \mathcal{L}\{x(t)\}(s) - x(0)$$

3. **First Shifting Theorem:**

$$L\{e^{-at} f(t)\}(s) = F(s+a)$$

In the subsequent section you will also derive/revise the second shifting theorem: Let $u(t)$ be the Heaviside function as defined (2).

4. **Second Shifting Theorem:**

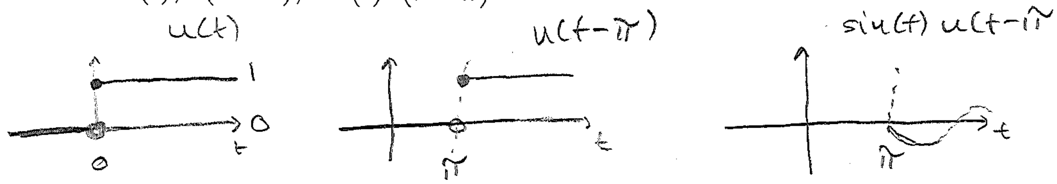
$$L\{u(t-a)f(t-a)\}(s) = e^{-as}\mathcal{L}\{f(t)\}(s) = e^{-as}F(s)$$

2 The Heaviside Function

Introduction: Heaviside - Function u(t). The Heaviside- function is a step-function that is commonly used to construct discontinuous/piecewise - continuous signals or forces and defined by

$$u(t) = \begin{cases} 0, & \text{if } t < 0, \\ 1, & \text{if } t \geq 0. \end{cases} \tag{2}$$

1. Sketch $u(t), u(t-\pi), sin(t)u(t-\pi)$



Proof of Second Shifting Theorem

2. Determine $L\{u(t-a)\}(s)$. Write $L\{u(t-a)f(t-a)\}(s)$ in terms of $F(s) = L\{f(t)\}(s)$.

$$\mathcal{L}\{u(t-a)\}(s) = \int_0^\infty u(t-a)e^{-st}\,dt = \int_a^\infty e^{-st}\,dt = \left[-\frac{1}{s}e^{-st}\right]_a^\infty = \frac{1}{s}e^{-sa}$$

$$\mathcal{L}\{u(t-a)f(t-a)\}(s) = \int_0^\infty u(t-a)f(t-a)e^{-st}\,dt = \int_a^\infty f(t-a)e^{-st}\,dt \overset{\tau = t-a}{=} \int f(\tau)e^{-s(\tau+a)}\,d\tau$$

$$= e^{-sa}\int_0^\infty f(\tau)e^{-s\tau}\,d\tau = e^{-sa}\mathcal{L}\{f(t)\}(s)$$

3. Model $f(t) = \begin{cases} 1 & \text{if } 0 \leq t < 1 \\ 0 & \text{otherwise} \end{cases}$ using Heaviside-functions. Calculate $L\{f(t)\}(s)$.

$$f(t) = u(t) - u(t-1); \quad \mathcal{L}\{f(t)\}(s) = \mathcal{L}\{u(t)\}(s) - \mathcal{L}\{u(t-1)\}(s)$$
$$= \frac{1}{s}e^{-0\cdot s} - \frac{1}{s}e^{-1\cdot s}$$
$$= \frac{1}{s}\cdot(1 - e^{-s})$$

4. Write

$$g(t) = \begin{cases} (t-1)^2 & \text{if } 1 \leq t < 2, \\ (3-t) & \text{if } 2 \leq t < 3, \\ 0 & \text{otherwise.} \end{cases}$$

using Heaviside functions. Calculate $L\{g(t)\}(s)$. Very long and extended answer: next page

$$g(t) = (t-1)^2[u(t-1) - u(t-2)] + (3-t)[u(t-2) - u(t-3)]$$
$$= u(t-1)(t-1)^2 + u(t-2)\underbrace{(-t^2+2t-1+3-t)}_{-t^2+t+2 = (t-2)(-1-t)} + u(t-3)(t-3)$$

$$\mathcal{L}\{g(t)\}(s) = \mathcal{L}\{u(t-1)(t-1)^2\} + \mathcal{L}\{u(t-2)(t-2)(-1-t)\} + \mathcal{L}\{u(t-3)(t-3)\}$$

$\tau = t-2$; $(-1-t) = -(t+1) = -(t-2+2+1)$

$$= e^{-s}\mathcal{L}\{t^2\} + e^{-2s}\mathcal{L}\{-t\cdot(t+2)\} + e^{-3s}\mathcal{L}\{t\}$$
$$= e^{-s}\frac{2}{s^3} + e^{-2s}\left[-\frac{2}{s^3} - 2\cdot\frac{1}{s^2}\right] + e^{-3s}\frac{1}{s^2}$$
$$= \frac{1}{s^2}\cdot\left[2\frac{e^{-s}}{s} + 2e^{-2s}\left(-\frac{1}{s} - 1\right) + e^{-3s}\right]$$

Very long answer to 2.4

$$g(t) = (t-1)^2[u(t-1) - u(t-2)] + (3-t)[u(t-2) - u(t-3)]$$

$$= (t-1)^2 u(t-1) - (t-1)^2 u(t-2) + (3-t)u(t-2) - (3-t)u(t-3)$$

$$= \underbrace{(t-1)^2 u(t-1)}_{(1)} + \underbrace{(\underbrace{-t^2+2t-1+3-t}_{-t^2+t+2})u(t-2)}_{(2)} + \underbrace{(t-3)u(t-3)}_{(3)}$$

Laplace of (1)

$\mathcal{L}\{(1)\} = \mathcal{L}\{(t-1)^2 u(t-1)\}$

Set $\tau_1 = t-1 \xrightarrow{\text{subst.}} \mathcal{L}\{\underbrace{\tau_1^2}_{=f_1(\tau_1)} u(\tau_1)\} \xrightarrow{\text{Resubst.}} \mathcal{L}\{f_1(t-1)u(t-1)\}$

(i.e $f_1(z) = z^2$)

$\Rightarrow \mathcal{L}\{(1)\} = \mathcal{L}\{(t-1)^2 u(t-1)\} = \mathcal{L}\{f_1(t-1)u(t-1)\}$

With $a=1$ use the second shifting theorem $\mathcal{L}\{f_1(t-1)u(t-1)\} = e^{-s}\mathcal{L}\{f_1(t)\}$

$\therefore \mathcal{L}\{(1)\} = e^{-s}\mathcal{L}\{t^2\} = \boxed{e^{-s}\frac{2}{s^3}}$

Laplace of (2)

$\mathcal{L}\{(2)\} = \mathcal{L}\{(-t^2+t+2)u(t-2)\}$

Substitute: $\tau_2 = t-2 \Leftrightarrow t = \tau_2 + 2$

$$(-t^2+t+2) = -(\tau_2+2)^2 + \tau_2 + 2 + 2 = -\tau_2^2 - 2\tau_2 - 4 + \tau_2 + 4$$

$$= -\tau_2^2 - \tau_2 = f_2(\tau_2), \text{ i.e } f_2(z) = -z^2 - z$$

Resubstitute:

$\mathcal{L}\{(2)\} = \mathcal{L}\{f_2(\tau_2)u(\tau_2)\} = \mathcal{L}\{f_2(t-2)u(t-2)\}$

$\underset{a=2}{\overset{\text{2nd shift thm}}{=}} e^{-2s}\mathcal{L}\{f_2(t)\} = \boxed{e^{-2s}\left(-\frac{2}{s^3} - \frac{1}{s^2}\right)}$

Laplace of (3)

$\mathcal{L}\{(3)\} = \mathcal{L}\{(t-3)u(t-3)\}$

Set $\tau_3 = t-3$, substitute $\rightsquigarrow \underbrace{\tau_3}_{=f_3(\tau_3)} u(\tau_3) \xrightarrow{\text{Resubst}} f_3(t-3)u(t-3)$, i.e $f_3(z) = z$

2nd shift thm, $a=3$

$\mathcal{L}\{(3)\} = \boxed{e^{-3s}\left(\frac{1}{s^2}\right)}$

$$\boxed{\Rightarrow \mathcal{L}\{g(t)\} = e^{-s}\frac{2}{s} + e^{-2s}\left(-\frac{2}{s^3} - \frac{1}{s^2}\right) + e^{-3s}\left(\frac{1}{s^2}\right)}$$

3 Solving Differential Equations: Mixed Problems

Problem: 1: Using the Shifting Theorems.
Apply the Laplace transform and use the four basic properties to simplify

$$x'' + x = h(t), \quad h(t) = \begin{cases} 2t^2e^{5t} & \text{if } 0 \le t < 1, \\ 0 & \text{otherwise.} \end{cases} \tag{3}$$

for the initial value problem x(0) = 1, x'(0) = 1.

$$h(t) = 2t^2 e^{5t}\left[u(t) - u(t-1)\right]$$

$$\mathcal{L}\{x'' + x\}(s) = \mathcal{L}\{h(t)\}(s)$$

Linearity (1)

$$\Leftrightarrow \mathcal{L}\{x''\}(s) + \mathcal{L}\{x\}(s) = \mathcal{L}\{e^{5t} \cdot 2t^2 u(t)\}(s) - \mathcal{L}\{e^{5t} \, 2t^2 u(t-1)\}$$

Left Prop. (2) Diff → Mult

Right Prop. (3) 1st shift

$$\Leftrightarrow \mathcal{L}\{x'\} \cdot s - x'(0) + \mathcal{L}\{x\} = \mathcal{L}\{2t^2 u(t)\}(s-5) - \mathcal{L}\{2t^2 u(t-1)\}(s-5)$$

Left Diff → Mult

Right 2nd shift

$$\Leftrightarrow \mathcal{L}\{x\}s^2 - \underbrace{x(0)}_{=1}s - \underbrace{x'(0)}_{=1} + \mathcal{L}\{x\} = \underbrace{e^{-(s-5)0}}_{=1} \mathcal{L}\{2t^2\}(s-5) - \underbrace{e^{-(s-5)1}}_{e^{-s+5}} \mathcal{L}\{2(t+1)^2\}(s-5)$$

Linearity

$$\Leftrightarrow \mathcal{L}\{x\}(s^2+1) - s - 1 = 2 \cdot \frac{2}{(s-5)^3} - 2e^{-s+5} \underbrace{\mathcal{L}\{t^2 + 2t + 1\}(s-5)}_{\frac{2}{(s-5)^3} + 2 \cdot \frac{1}{(s-5)^2} + \frac{1}{(s-5)}}$$

⇝ solvable for $\mathcal{L}\{x\}$

↳ care that the argument of the function has changed

Problem: 2: Solving Homogeneous Systems.
Solve

$$my'' + cy' + ky = 0, \quad y(0) = a, y'(0) = b,$$

where m, c, k are positive constants and the constraint $c^2 - 4km > 0$ is satisfied.

⇝ Next page: Not long but messy calculation.

Example: 3: Solving Non-homogeneous Systems. Source: Cole Zmurchok

1. Show $L\{cos(2t)\}(s) = \frac{s}{s^2+4}$.

$$\mathcal{L}\{\cos(at)\}(s) = \int_0^\infty \cos(at)\, e^{-st}\, dt = \frac{1}{a}\cdot \underbrace{\left[\sin(at)\, e^{-st}\right]_0^\infty}_{=0} - \int \frac{1}{a}\sin(at)\cdot(-s)\, e^{-st} dt$$

$$= + \frac{s}{a}\int \sin(at) e^{-st}\, dt = \frac{s}{a}\cdot\left(\left[-\cos(at)\frac{1}{a}\, e^{-st}\right]_0^\infty - \int_0^\infty \frac{1}{a}\cos(at)\, s e^{-st} dt\right)$$

$$= \frac{s}{a}\left[+1\ \frac{1}{a}\right] - \frac{s^2}{a^2}\mathcal{L}\{\cos(at)\}(s)$$

$$\Rightarrow \left(1+\frac{s^2}{a^2}\right)\cdot\mathcal{L}\{\cos(at)\}(s) = \frac{s}{a^2} \iff \mathcal{L}\{\cos(at)\}(s) = \frac{s}{a^2}\cdot\left(\frac{a^2}{s^2+a^2}\right) = \frac{s}{s^2+a^2}$$

For $a=2$ get result.

2. Use partial fractions to show

$$\frac{s}{(s^2+1)(s^2+4)} = \frac{1}{3}\frac{s}{s^2+1} - \frac{1}{3}\frac{s}{s^2+4}.$$

Multiply by $\cdot (s^2+1)(s^2+4)$

$$\frac{s}{(s^2+1)(s^2+4)} = \frac{As+B}{s^2+1} + \frac{Cs+D}{s^2+4} \Rightarrow \underbrace{s}_{\text{left side}} = \underbrace{\begin{bmatrix} As^3 + Bs^2 + 4As + 4Bs \\ +Cs^3 + Cs \\ + Ds^2 + Ds \end{bmatrix}}_{(A+C)s^3+(B+D)s^2+(4A+C)s+(4B+D)}$$

Terms on the right side

$$\Rightarrow \begin{Bmatrix} A+C=0 & (i) \\ B+D=0 & (ii) \\ 4A+C=1 & (iii) \\ 4B+D=0 & (iv) \end{Bmatrix} \Rightarrow \begin{cases} (ii),(iv) \text{ imply } B=D=0 \\ C=-A \\ 4A+C = 4A-A = 3A \overset{!}{=} 1 \Rightarrow A=\frac{1}{3},\ C=-\frac{1}{3} \end{cases}$$

3. Using Laplace-transforms, solve $x'' + x = cos(2t)$ with $x(0) = 0$ and $x'(0) = 1$.

Use linearity and (Diff to Mult) to obtain

$$\underbrace{\mathcal{L}\{x''\}} + \mathcal{L}\{x\} = \mathcal{L}\{\cos(2t)\}$$

$$s^2\mathcal{L}\{x\} - s\underbrace{x(0)}_{=0} - \underbrace{x'(0)}_{=1} + \mathcal{L}\{x\} = \frac{s}{s^2+4} \Rightarrow \mathcal{L}\{x\} = \left(\frac{s}{s^2+4}+1\right)\frac{1}{s^2+1}$$

$$\mathcal{L}\{x\} = \underbrace{\frac{s}{(s^2+4)(s^2+1)}}_{=\frac{1}{3}\frac{s}{s^2+1} - \frac{1}{3}\frac{s}{s^2+4}} + \frac{1}{s^2+1} \Rightarrow x = \frac{1}{3}(\cos(t)-\cos(2t)) + \sin(t)$$

Problem 3.2 Solving homogeneous systems

$my'' + cy' + ky = 0 \qquad y(0) = a,\ y'(0) = b$

$\Leftrightarrow\ y'' + \frac{c}{m}y' + \frac{k}{m}y = 0 \qquad$ Define $\hat{c} = \frac{c}{m}$ ~~and~~ and $\hat{k} = \frac{k}{m}$

Laplace

Laplace transform:

$$\mathcal{L}\{y''\} + \hat{c}\,\mathcal{L}\{y'\} + \hat{k}\,\mathcal{L}\{y\} = s^2\mathcal{L}\{y\} - s\cdot a - b$$
$$+ \hat{c}s\cdot\mathcal{L}\{y\} - \hat{c}b$$
$$+ \hat{k}\,\mathcal{L}\{y\}$$
$$= (s^2 + \hat{c}s + \hat{k})\mathcal{L}\{y\} - sa - b - \hat{c}b = 0$$

$$\Rightarrow \mathcal{L}\{y\} = \frac{sa + b\cdot(1+\hat{c})}{(s^2 + \hat{c}s + \hat{k})}$$

The roots of the denominator are

$$s_{1,2} = \frac{-\hat{c} \pm \sqrt{\hat{c}^2 - 4\hat{k}}}{2} \quad \underline{\text{real.}} \quad \hat{c}^2 - 4\hat{k} = \frac{c^2}{m^2} - \frac{4k}{m} = \frac{c^2 - 4km}{m^2} > 0$$

distinct

Partial fractions

Partial fractions

form known because two distinct real roots.

$$\frac{sa + b\cdot(1+\hat{c})}{s^2 + \hat{c}s + \hat{k}} = \frac{A}{s - s_1} + \frac{B}{s - s_2}$$

Multiply with $s^2 + \hat{c}s + \hat{k} = (s-s_1)(s-s_2)$:

$$sa + b(1+\hat{c}) = A\cdot(s - s_2) + B\cdot(s - s_1)$$

$s = s_1$: $\quad s_1 a + b(1+\hat{c}) = A\cdot(s_1 - s_2) + 0$

$$\Rightarrow \boxed{A = \frac{s_1 a + b(1+\hat{c})}{s_1 - s_2}},$$

$s = s_2$: $\quad s_2 a + b(1+\hat{c}) = B\cdot(s_2 - s_1)$

$$\Rightarrow \boxed{B = \frac{s_2 a + b(1+\hat{c})}{(s_2 - s_1)}}$$

Note that both A and B are constant (s_1, s_2 don't change with s).

Of the latter the inverse transform is known.

Inverse Laplace

$$\mathcal{L}^{-1}\left\{\frac{A}{s-s_1}\right\} = \mathcal{L}^{-1}\left\{\mathcal{L}\{e^{s_1 t}\cdot A\}\right\} = Ae^{s_1 t}$$

$$\mathcal{L}^{-1}\left\{\frac{B}{s-s_2}\right\} = \mathcal{L}^{-1}\left\{\mathcal{L}\{e^{s_2 t}\,B\}\right\} = Be^{s_2 t}$$

$\Rightarrow y(t) = Ae^{s_1 t} + Be^{s_2 t}$

4 Transfer functions

Transfer functions give an algebraic dependence of the output based on the input.

Introduction: Using Transfer functions (Source: Cole Zmurchok)

Consider $Lx = f(t)$ with L a constant coefficient differential operator, with all initial conditions 0. Taking the Laplace Transform gives $A(s)X(s) = F(s)$, so that $X(s) = H(s)F(s)$ for any input $f(t)$. This suggests that $x(t)$ can be found by multiplying $F(s)$ by $H(s)$ in the frequency-domain and subsequently taking the inverse Laplace Transform.

1. Find the transfer function for the ODE $x'' + \omega_0^2 x = f(t)$, assuming all initial conditions are 0.

$$s^2\mathcal{L}\{x\} \cancel{- s\,x(0) - x'(0)} + \omega_0^2 \mathcal{L}\{x\} = \mathcal{L}\{f(t)\}$$

$$(s^2+\omega_0^2)\mathcal{L}\{x\} = \mathcal{L}\{f(t)\} \Rightarrow \underbrace{\mathcal{L}\{x\}}_{X(s)} = \underbrace{\frac{1}{s^2+\omega_0^2}}_{=H(s)} F(s)$$

2. Suppose $f(t) = 1$. Use the transfer function from above to find $x(t)$.

$$F(s) = \frac{1}{s}$$

$$\Rightarrow X(s) = \frac{1}{s} \cdot \frac{1}{s^2+\omega_0^2}$$

$$X(s) = \frac{A}{s} + \frac{B+Cs}{s^2+\omega_0^2} \quad \text{if} \quad 1 = A(s^2+\omega_0^2) + Bs + Cs^2$$

$$= A\omega_0^2 + Bs + (A+C)s^2$$

$$A = \frac{1}{\omega_0^2}, \quad B = 0, \quad C = -\frac{1}{\omega_0^2}$$

$$\Rightarrow X(s) = \frac{1}{\omega_0^2}\frac{1}{s} - \frac{1}{\omega_0^2}\frac{s}{s^2+\omega_0^2}$$

$$\Rightarrow x(t) = \frac{1}{\omega_0^2} \cdot \underbrace{\mathcal{L}^{-1}\left\{\frac{1}{s}\right\}}_{=1} - \frac{1}{\omega_0^2}\underbrace{\mathcal{L}^{-1}\left\{\frac{s}{s^2+\omega_0^2}\right\}}_{\cos(\omega_0 t)} = \frac{1}{\omega_0^2}(1-\cos(\omega_0 t))$$

5 Additional Problems

Problem: Problemset.
Solve

1. $y'' + 4y' + 5y = e^{-t}(cos(t) + 3sin(t))$ with $y(0) = 0$ and $y'(0) = 4$.
2. $y'' + y = \begin{cases} 3 & \text{if } 0 \leq t < \pi \\ 0 & \text{otherwise} \end{cases}$ with $y(0) = 0$ and $y'(0) = 0$.
3. $9y'' + 6y' + y = 3e^{3t}$ with $y(0) = 0$ and $y'(0) = -3$.
4. $y'' - 5y' + 6y = 10e^{t}cos(t)$ with $y(0) = 2$ and $y'(0) = 1$.