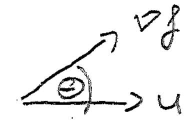
Solutions to worksheet 4:

Problemset 1:

definition of scalar product.
scalar product

1.1. $D_u f(x,y,z) \overset{(1)}{=} \nabla f \bullet u \overset{(2)}{=} |\nabla f| \cdot \underbrace{|u|}_{=1} \cdot \cos(\theta) \overset{(3)}{=} |\nabla f| \cdot \cos(\theta)$

where θ is the angle between ∇f and u.



We used

(1) the definition of D_u

(2) the definition of a scalar product

(3) that $|u|=1$

1.2. This equality above can be used to make observations: $D_u f(x,y,z)$ is largest when $\cos(\theta)=1$, i.e. $u=\nabla f$, is smallest if $\cos(\theta)=-1$, i.e $u=-\nabla f$, and, finally, $D_u f(x,y,z)=0$ if $\cos(\theta)=0$, i.e $u \perp \nabla f$.

1.3. Determine $\nabla f(x,y,z)$. By 2, we know that $u=\pm\nabla f$ maximizes the rate of change.

$$\nabla f(x,y,z) = \frac{1}{2\sqrt{x^2+y^2+z^2}} \begin{pmatrix} 2x \\ 2y \\ 2z \end{pmatrix} \Rightarrow u^* = \frac{\pm 1}{\sqrt{x^2+y^2+z^2}} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

In the point $(1,2,2)$:

$$u^* = \pm \frac{1}{\sqrt{9}} \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} = \pm \frac{1}{3} \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$$

1.4. $u_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ $u_2 = \frac{-1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ $u_3 = \frac{1}{\sqrt{5}} \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$D_{u_1} f = -2$, $D_{u_2} f = 2$ $D_{u_3} f = 1$

Just looking at the directional derivatives u_2 seems to be a good choice as the slope is largest among the three directions u_1, u_2, u_3. But can we do better? From 1.2, we know that we find the steepest increase in direction ∇f, and this includes $\frac{\partial}{\partial x} f, \frac{\partial}{\partial y} f$.

We use the definition of directional derivatives:

(1) $-2 = D_{u_1} f = \nabla f \bullet u_1 = \begin{pmatrix} \frac{\partial}{\partial x} f \\ \frac{\partial}{\partial y} f \end{pmatrix} \bullet \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} = \frac{1}{\sqrt{2}} \frac{\partial}{\partial x} f + \frac{1}{\sqrt{2}} \frac{\partial}{\partial y} f$

(2) $2 = D_{u_2} f = \nabla f \bullet u_2 = \frac{-1}{\sqrt{2}} \frac{\partial}{\partial x} f + \left(\frac{-1}{\sqrt{2}}\right) \frac{\partial}{\partial y} f$ (3) $1 = D_{u_3} f = \frac{2}{\sqrt{5}} \frac{\partial}{\partial x} f + \frac{1}{\sqrt{5}} \frac{\partial}{\partial y} f$

Because (1), (2) are equivalent, solve (1,3) for $\frac{\partial}{\partial x} f, \frac{\partial}{\partial y} f$.

$f_x = \sqrt{5} + 2\sqrt{2}$; $f_y = -4\sqrt{2} - \sqrt{5}$ and hence $\nabla f = \begin{pmatrix} \sqrt{5}+2\sqrt{2} \\ -4\sqrt{2}-\sqrt{5} \end{pmatrix}$ points into the right direction

1.5. Define $f(x,y,z) = x^2 + y^2 + z^2$.

Then f is <u>constant</u> (does not change) along the surface $(*)$ $f(x,y,z) = x^2+y^2+z^2 = r^2$.

From 1.2, we know that ∇f points perpendicular off the level set,

For a given point (x_0, y_0, z_0) on $(*)$ $\nabla f(x_0, y_0, z_0) = 2 \cdot \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix}$.

The normal line is given by

$$\ell(x,y,z) = \underbrace{\begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix}}_{\text{point on surface}} + t \cdot 2 \underbrace{\begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix}}_{\text{direction/normal vector}}$$

$\ell(x,y,z) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ for $t = -\frac{1}{2}$, ie the origin is intersected by the line ℓ.

<u>Problemset 2</u>

2.1. • $a = 1$, $b = 1$ $\quad f(x,y) = x^2 + y^2$

$D(0,0) = 2 \cdot 2 - 0^2 = 4 > 0$ and $f_{xx}(0,0) = 2 > 0$ $\Rightarrow$ local min.

• $a = -1$, $b = -1$ $\quad f(x,y) = -x^2 - y^2$

$D(0,0) = (-2) \cdot (-2) - 0^2 = 4 > 0$ and $f_{xx}(0,0) = -2 < 0$ $\Rightarrow$ local max.

• $a = 1$, $b = -1$ $\quad f(x,y) = x^2 - y^2$

$D(0,0) = (-2) \cdot 2 - 0^2 = -4 < 0$ $\Rightarrow$ neither.

• $a = 0$ $b = -1$ $\quad f(x,y) = -y^2$

$D(0,0) = 0 \cdot (-2) - 0 = 0$. Test fails.

But f is constant in x and $y = 0$ maximum of $f(x,y) = -y^2$.

Hence, $(0,0)$ a local maximum.

2.2. By solving for z we obtain $z(x,y) = -\frac{2}{3}x + y + 2$

We want to minimize the distance to the ~~p~~ point $(0,0,1)$, which is given by

$f(x,y) = \sqrt{(x-0)^2 + (y-0)^2 + (z(x,y) - 1)^2}$. This minimization is equivalent to minimizing

the squared function $g(x,y) = x^2 + y^2 + (-\frac{2}{3}x + y + 1)^2$

We first need to find these x,y ~~that~~ where the gradient vanishes $\nabla g(x,y) = 0$

(1) $\frac{\partial}{\partial x} g(x,y) = 2x + 2\cdot(-\frac{2}{3}x+y+1)(-\frac{2}{3}) = \cancel{2x}\ \frac{26}{9}x - \frac{4}{3}y - \frac{4}{3} = 0$

(2) $\frac{\partial}{\partial y} g(x,y) = +2y + 2(-\frac{2}{3}x+y+1)(1) = 2y - \frac{4}{3}x + 2y + 2 = -\frac{4}{3}x + 4y + 2 = 0$

(2) $\Leftrightarrow y = \frac{1}{4}\cdot(+\frac{4}{3}x - 2) = \frac{1}{3}x - \frac{1}{2}$

(1) $\overset{\text{use (2)}}{\Leftrightarrow} \frac{26}{9}x - \frac{4}{3}(\frac{1}{3}x - \frac{1}{2}) - \frac{4}{3} = 0 \Leftrightarrow \frac{22}{9}x = \frac{4}{6} \Leftrightarrow x = \frac{2}{3}\cdot\frac{9}{22} = \frac{6}{22} = \frac{3}{11}$

$y \overset{(2)}{=} \frac{1}{3}x - \frac{1}{2} = \frac{1}{3}\cdot(\frac{6}{22}) - \frac{1}{2} = \frac{1}{11} - \frac{1}{2} = \frac{2-11}{22} = -\frac{9}{22}$

Hence, $\boxed{x = \frac{3}{11},\ y = -\frac{9}{11}}$ is a candidate for a minimum.

Then, $\frac{\partial}{\partial x}\left(\frac{\partial}{\partial x} g(x,y)\right) = \frac{26}{9}$, $\frac{\partial}{\partial y}\left(\frac{\partial}{\partial x} g(x,y)\right) = -\frac{4}{3}$

$\frac{\partial}{\partial y}\left(\frac{\partial}{\partial y} g(x,y)\right) = 4$

$D_{(\frac{3}{2},\frac{9}{4})} = \frac{26}{9}\cdot 4 - (-\frac{4}{3})^2 = \frac{108}{9} - \frac{16}{9} > 0$ and $\frac{\partial^2}{\partial x^2} g(x,y) = \frac{26}{9} > 0$

$\Rightarrow$ local minimum.

Intuition tells us that it has to be the global minimum.

2.3. First, we look for all loc. min/max in $\mathring{D} = \{(x,y) : |x|<1, |y|<1\}$

$f(x,y) = x^2 + y^2 + x^2 y + 3$

$\frac{\partial}{\partial x} f(x,y) = 2x + 2xy = 2x(1+y)$; $\frac{\partial}{\partial y} f(x,y) = 2y + x^2$

$\nabla f = 0$ iff $2x(1+y) = 0$ and $y = -\frac{x^2}{2}$.

$\Rightarrow [x=0$ and $y=0]$ or $[y=-1$ and $x=\sqrt{2}]$

Because $x = \sqrt{2} > 1 \Rightarrow x=0$ and $y=0$ is the only relevent loc. extremum in

$\frac{\partial^2}{\partial x^2} f(0,0) = 2(1+y)|_{y=0} = 2$; $\frac{\partial^2}{\partial x \partial y} f(0,0) = 2x|_{x=0} = 0$

$\frac{\partial^2}{\partial y^2} f(0,0) = 2 \Rightarrow D_{(0,0)} = 2\cdot 2 - 0^2 = 4 > 0$, $\frac{\partial^2}{\partial x^2} f(0,0) = 2 > 0$

$(0,0)$ is a loc. minimum with $f(0,0) = 3$

Now, we need to check the four boundaries:

For $|x| = 1$: $f(x,y) = 1 + y^2 + y + 3$ with $f'(y) = 2y + 1 = 0$ iff $y = -\frac{1}{2}$, $f''(y) = 2 > 0$ (min)

$f(\pm x, -\frac{1}{2}) = \frac{1}{4} - \frac{1}{2} + 3 = \frac{11}{4}$

$f(x, -1) = 1 + 1 - 1 + 3 = 4$; $f(x, 1) = 1 + 1 + 1 + 3 = 6$

For $y=1$: $f(x,+1) = x^2+1+1+3 = x^2+5$ $\Big\} \Rightarrow f'(x) = 2x = 0 \quad f''(0) = 0 \to$ saddle
$y=-1$: $f(x,-1) = x^2+3$

As a consequence, $(0,0)$ is the $\overset{f(0,0)=3}{\text{minimizer}}$ and the maximum is attained on the boundary in $(1,1)$ and $(-1,1)$ with a value of 6.

Problemset 3:

1. $\nabla f(x,y) = \lambda \nabla g(x,y) \iff \begin{pmatrix} 2x \\ 2y \end{pmatrix} = \begin{pmatrix} \lambda y \\ \lambda x \end{pmatrix} \begin{matrix} (1) \\ (2) \end{matrix}$

$$xy = 1 \quad (3)$$

$\overset{(3)}{\Rightarrow}$ $x,y \neq 0$ as $xy=1$: $\lambda = \frac{2x}{y}$ (1), $\lambda = \frac{2y}{x}$ (2) $\Big\}$ $\frac{2x}{y} = \lambda = \frac{2y}{x} \iff x^2 = y^2 \iff |x| = |y|$

with $xy = 1$: $x = \pm 1$, $y = x$

Hence, $(1,1)$; $(-1,-1)$ are candidates for a max/min.

$f(1,1) = 2$; $f(-1,-1) = 2$

As $x = \frac{1}{y} \Rightarrow x^2+y^2 = \frac{1}{y^2} + y^2 \xrightarrow[y\to\infty]{} \infty$ (and similarly with x),

there is no maximum.

The minimum is 2 (by comparing the two values.

2. Find all min/max of $f(x,y,z) = xyz$ subject to $x^2+2y^2+3z^2=6$

What I usually try.
↓

$$\nabla f = \lambda \nabla g \iff \begin{pmatrix} yz \\ xz \\ xy \end{pmatrix} = \begin{pmatrix} \lambda \cdot 2x \\ \lambda \cdot 4y \\ \lambda \cdot 6z \end{pmatrix} \begin{matrix} (1) \\ (2) \\ (3) \end{matrix}$$

$$x^2+2y^2+3z^2 = 6 \quad (4)$$

Phase 1: Solve for λ

Case $x \neq 0$: $\boxed{\lambda = \frac{yz}{2x}}$ $(*)$, (1) into (2) $\quad xz = \lambda \cdot 4y = \frac{yz}{2x} \cdot 4 \cdot y = \frac{2y^2z}{x}$

Phase 2: Use λ to find constraints on x,y,z

Case $z \neq 0$: $\boxed{x^2 = 2y^2}$ (5)

Into (3): $xy = \frac{yz}{2x} \cdot 6z = 3\frac{yz^2}{x}$

Case $y \neq 0$: $\boxed{x^2 = 3 \cdot z^2}$ (6)

Phase 3: Use constraints in $g(x,y,z) = c$

Into (4): $x^2+2y^2+3z^2 \overset{(5,6)}{=} 3x^2 = 6 \Rightarrow x^2 = 2$

$\Rightarrow \boxed{y^2 = 1}$; $\boxed{z^2 = \frac{2}{3}}$, i.e. $|x| = \sqrt{2}$; $|y| = 1$; $|z| = \sqrt{\frac{2}{3}}$

Phase 4
Clean up the mess.

8 possible cases:

	x	y	z	$f(x,y,z)=xyz$
(i)	$\sqrt{2}$	1	$\sqrt{\frac{2}{3}}$	$\frac{2}{\sqrt{3}}$
(ii)	$-\sqrt{2}$	1	$\sqrt{\frac{2}{3}}$	$-\frac{2}{\sqrt{3}}$
(iii)	$\sqrt{2}$	-1	$\sqrt{\frac{2}{3}}$	$-\frac{2}{\sqrt{3}}$
(iv)	$-\sqrt{2}$	-1	$\sqrt{\frac{2}{3}}$	$\frac{2}{\sqrt{2}}$
(v)	$\sqrt{2}$	1	$-\sqrt{\frac{2}{3}}$	$-\frac{2}{\sqrt{3}}$
(vi)	$-\sqrt{2}$	1	$-\sqrt{\frac{2}{3}}$	$+\frac{2}{\sqrt{2}}$
(vii)	$\sqrt{2}$	-1	$-\sqrt{\frac{2}{3}}$	$\frac{2}{\sqrt{3}}$
(viii)	$-\sqrt{2}$	-1	$-\sqrt{\frac{2}{3}}$	$-\frac{2}{\sqrt{3}}$

At last, we need to deal with the case where $x=0$ or $y=0$ or $z=0$:
In all these cases, $f(x,y,z)=0$ with $-\frac{2}{\sqrt{3}} < 0 < \frac{2}{\sqrt{3}}$ (neither min or max).
The minimum is $-\frac{2}{\sqrt{3}}$ and maximum $\frac{2}{\sqrt{3}}$.