Worksheet 1

Felix Funk, MATH Tutorial - Mech 221

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Reminder: Separable ODEs.
An ordinary differential equations (ODE) is separable if one can fit the differential equation into the form

$$\frac{dy}{dx} = f(x) \cdot g(y). \tag{1}$$

Frequently, we are interested how a solution $y(x)$ to (1) varies with changes in the independent variable x in general. But sometimes, we require the differential equation to satisfy additional constraints - we want the solution to attain an initial value y_0

$$y(x_0) = y_0. \tag{2}$$

In the latter case, we speak of solving an initial value problem.

Problem: Separable ODE.
Solve

$$y'(x) = xy + x + y + 1 \tag{3}$$

such that $y(0) = 1$.

Exercise: Terminology.
We reduce the problem to its basic components.

1. Identify: What is the ODE? What is the desired function? What is the independent variable?

   The ODE is $y' = xy + x + y + 1$; y is the desired function and x the independent variable

2. Cross out everything that does not apply: The equation (3) is a linear, ~~non-linear~~, first-order, ~~second order~~, ~~higher-order~~, ~~constant coefficient~~, ~~homogeneous~~, non-homogeneous equation.

3. How can you change this equation for it to become a second-order non-linear non-homogeneous equation?

There are many possible solutions. One could be

second-order $\leftarrow$ $y^{(2)} = (x+1)(y+1)^2$ $\rightsquigarrow$ results in terms with y^2 $\rightsquigarrow$ non-linear
and with x (without y) $\rightsquigarrow$ non-homoge.

4. Is this an initial value problem? If so identify x_0, y_0:

$x_0 = 0$
$y_0 = 1$ as $y(0) = 1$

5. Identify the functions f and g

We observe that the right side of (3) factorizes to $(x+1)(y+1)$.
Hence, $f(x) = x+1$ $\quad g(y) = y+1$

Example: Solving separable ODE's: A recipe.
We solve the problem corresponding to (3)

1. Isolate x-terms to the right; y-terms to the left: $\frac{dy}{g(y)} = f(x)dx$.

$$\frac{1}{y+1}\,dy = (x+1)\,dx$$

2. Integrate both sides. $\int \frac{1}{g(y)} \cdot dy = \int f(x)dx$. Don't forget the constant c.

$$\ln|y+1| = \frac{1}{2}x^2 + x + c$$

3. Solve for y, if possible.

$$|y+1| = \exp^{1/2x^2+x+c} \overset{e^c=k}{=} k\,e^{1/2\,x^2+x}$$
$$\Rightarrow\; y+1 = \pm k\,e^{1/2x^2+x} \quad \overset{\tilde{k}=\pm k}{\Rightarrow}\; y = \tilde{k}\,e^{1/2x^2+x} - 1$$

$c, k, \tilde{k}$ some real constants

4. Determine the constant c. Use the initial value (if provided).

We know that $y(0)=1$, i.e $\quad 1 = \tilde{k}\,\underbrace{e^{\frac{1}{2}\cdot 0^2+0}}_{=1} - 1 \iff 2 = \tilde{k}$

Hence, $\boxed{y(x) = 2\cdot e^{1/2x^2+x} - 1}$ solves the IVP
$\hookrightarrow$ initial value problem.

5. Check your solution!

We verify that $\quad y' = (x+1)(y+1)$

Left side:

$$y' = 2\cdot(x+1)\,e^{1/2x^2+x}$$

Right side:

$$(x+1)(y+1) = (x+1)\Big(\underbrace{2e^{1/2x^2+x} - 1}_{\text{plug in } y(x)} + 1\Big) = 2(x+1)e^{1/2x^2+x}$$

Therefore $y(x)$ satisfies

$$y' = (x+1)(y+1).$$

Exercise: Separable ODEs.
Obtain the general solution for one of the following ODEs. Solve the IVP if asked for.

1. $$y' = 2xy,$$

2. $$x' = 3xt^2 - 3t^2, x(0) = 2,$$

3. $$y' = \frac{x^2+1}{y^2+1}, y(0) = 1.$$

Hint: You might not always find an explicit solution.

1. $y' = 2xy$

Isolate: $\frac{1}{y}\,dy = 2x\,dx$

Integrate: $\ln|y| = x^2 + c$

Solve for y: $|y| = k e^{x^2}$

$y \overset{\tilde{k}=\pm k}{=} \tilde{k} e^{x^2}$

$\boxed{y(x) = \tilde{k} e^{x^2}}$ is the general solution.

Check: $y' = 2xy$

$\left.\begin{array}{l} y' = \tilde{k}(2x)e^{x^2} \\ 2xy = 2x \cdot \underbrace{(\tilde{k} e^{x^2})}_{y} \end{array}\right\} \text{“}=\text{”}$

2. $x' = 3xt^2 - 3t^2$, $x(0) = 2$

Observe $3xt^2 - 3t^2 = \underbrace{3t^2}_{g(t)} \underbrace{(x-1)}_{g(x)}$

Isolate: $\frac{1}{x-1}\,dx = 3t^2\,dt$

Integrate: $\ln|x-1| = t^3 + c$

Solve for x: $|x-1| = e^{t^3+c}$

$x = \tilde{k} e^{t^3} + 1$

$\boxed{x(t) = \tilde{k} e^{t^3} + 1}$ is the general solution.

IVP: $x(0) = 2$ (Plug in $t_0 = 0$ and $x_0 = 2$)

$2 = x(0) = \tilde{k} + 1 \Rightarrow \boxed{\tilde{k} = 1}$

Hence, $x(t) = e^{t^3} + 1$ solves IVP.

Check: $x' = 3t^2(x-1)$

Left: $x' = (3t^2)e^{t^3}$

Right: $(3t^2(x-1)) = 3t^2(\underbrace{e^{t^3} + 1}_{x} - 1)$

$= 3t^2 e^{t^3}$

3. $y' = \frac{x^2+1}{y^2+1}$, $y(0) = 1$

Isolate: $(y^2+1)\,dy = (x^2+1)\,dx$

Integrate: $\frac{1}{3}y^3 + y = \frac{1}{3}x^3 + x + c$

There is no explicit form for y (that I know of). We can still identify c, though.

$y(0) = 1$: $\underbrace{\frac{1}{3}1^3 + 1}_{=4/3} = \frac{1}{3}0^3 + 0 + c = c$

($0 \to x_0$, $1 \to y_0$)

The implicit solution is $\boxed{\frac{1}{3}y^3 + y = \frac{1}{3}x^3 + x + \frac{4}{3}}$

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Reminder: Linear ODEs.
A first order linear ODE is an equation of the form

$$y' + p(x)y = f(x). \tag{4}$$

A function r that satisfies $r'(x) = p(x)r(x)$ constitutes the relationship

$$\frac{d}{dx}\left[r(x)y\right] = r(x)y' + r(x)p(x)y = r(x)f(x). \tag{5}$$

The function r is called the integrating factor and can be calculated in the following way:

$$r(x) = e^{\int p(x)dx} \tag{6}$$

Problem: Linear First Order ODE.
Solve the equation

$$y' + 6y = e^x. \tag{7}$$

Example: Use integrating factors.

1. What is $p(x)$? What is $f(x)$?

$$\begin{aligned} p(x) &= 6 \\ f(x) &= e^x \end{aligned}$$

Compare: $y' + \boxed{p(x)}\,y = \boxed{f(x)}$
$y' + \boxed{6}\,y = \boxed{e^x}$

2. Determine the integrating factor. (You don't need to keep track of the integration constant.)

$$r(x) \overset{(6)}{=} e^{\int p(x)dx} = e^{\int 6\,dx} = e^{6x+k} = \tilde{k}e^{6x}$$

this constant becomes irrelevant later but let's keep it for now

3. Multiply equation (7) with the integrating factor and utilize (5): $\frac{d}{dx}\left[\underbrace{r(x)}_{\tilde{k}e^{6x}}y\right] = \underbrace{r(x)}_{\tilde{k}e^{6x}}\underbrace{f(x)}_{e^x}$

$$\frac{d}{dx}\left[\tilde{k}e^{6x}y(x)\right] = \underbrace{\tilde{k}e^{6x}}_{r(x)} \cdot \underbrace{e^x}_{f(x)} = \tilde{k}e^{7x}$$

4. Integrate both sides with respect to x. Don't forget the integration constant.

$$\int \frac{d}{dx}\left[\tilde{k}e^{6x}y(x)\right]dx = \int \tilde{k}e^{7x}dx \overset{\text{Fundamental theorem of calculus}}{\Longrightarrow} \tilde{k}e^{6x}y(x) = \tilde{k}\frac{1}{7}e^{7x} + c$$

5. Isolate y, if possible.

$$y(x) = \frac{1}{7}e^x + \frac{c}{\tilde{k}}e^{-6x}, \quad \frac{c}{\tilde{k}} = \tilde{c}$$

Why can we not neglect e^{-6x} and pull it into $\tilde{c}$?
Both, c and $\tilde{k}$ arise as constants from integration, i.e are real numbers, e.g 1, 15, e, −42
e^{-6x} still varies with x and so does $y(x)$. We would alter $y(x)$ if we dropped it.

6. Determine the integration constant.

We don't have an initial value problem in (7). We can't determine $\tilde{c}$ in this case.

7. Check your solution!

Left: $y' + 6y = \underbrace{\left(\frac{1}{7}e^x + \tilde{c}(-6)e^{-6x}\right)}_{y'} + 6\cdot\underbrace{\left(\frac{1}{7}e^x + \tilde{c}e^{-6x}\right)}_{y}$

Right: e^x

$$= \left(\frac{1}{7} + \frac{6}{7}\right)e^x - 6\tilde{c}e^{-6x} + 6\tilde{c}e^{-6x} = e^x$$

Exercise: Linear ODEs.
Solve the following linear ODEs/ IVPs

1. $$y' + xy = x, y(0) = 0,$$

2. $$y' + cos(x)y = cos(x).$$

1. $\underbrace{y'}+\underbrace{x}_{p(x)}y = \underbrace{x}_{g(x)} \qquad y(0) = 0$

this constant can be dropped

Integrating factor: $r(x) = e^{\int x\,dx} = e^{\frac{1}{2}x^2 + k} = \tilde{k}\, e^{\frac{1}{2}x^2}$

$\frac{d}{dx}[r(x)\,y] = r(x)\,g(x)$

$\downarrow$

$\frac{d}{dx}[\tilde{k}\, e^{\frac{1}{2}x^2} y] = \tilde{k}\, e^{\frac{1}{2}x^2}\, x$

Integrate: $\tilde{k}\, e^{\frac{1}{2}x^2} y = \tilde{k} e^{\frac{1}{2}x^2} + c$
with respect to x.

Solve for y: $y(x) = 1 + \tilde{c}\, e^{-\frac{1}{2}x^2}$

IVP: $y(\underbrace{0}_{x_0}) = \underbrace{0}_{y_0}$ $\qquad 0 = y(0) = 1 + \tilde{c}\, e^{-\frac{1}{2}0^2} = 1 + \tilde{c} \; \to \tilde{c} = -1$

$\boxed{y(x) = 1 - e^{-\frac{1}{2}x^2}}$

Check: $\quad y' + xy = x \qquad y(0)=0$

Left: $y' + xy = \underbrace{\frac{1}{2}(2x)\, e^{-\frac{1}{2}x^2}}_{y'} + x\cdot\underbrace{(1 - e^{-\frac{1}{2}x^2})}_{y}$

$= x e^{-\frac{1}{2}x^2} + x - x e^{-\frac{1}{2}x^2} = x$

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Right: x

2. $y' + \underbrace{\cos(x)}_{p(x)} y = \underbrace{\cos(x)}_{g(x)}$

$r(x) = e^{\int p(x)\,dx} = e^{\sin(x)}$ Integrating factor

$\frac{d}{dx}\left[e^{\sin(x)} y(x)\right] = e^{\sin(x)} \cos(x)$

Integrate w.r.t. x.

$\int \cdot\, dx$
$\Rightarrow e^{\sin(x)} y(x) = e^{\sin(x)} + c$

solve:

$$\boxed{y(x) = 1 + c e^{-\sin(x)}}$$

IVP not available, so keep the constant

check: $y' + \cos(x) y = \cos(x)$

Left $y' + \cos(x) y = \underbrace{c \cdot (-\cos(x)) e^{-\sin(x)}}_{y'} + \cos(x) \underbrace{(1 + c e^{-\sin(x)})}_{y}$

$= -c \cos(x) e^{-\sin(x)} + \cos(x) + c \cos(x) e^{-\sin(x)}$

$= \cos(x)$

Right: $\cos(x)$

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