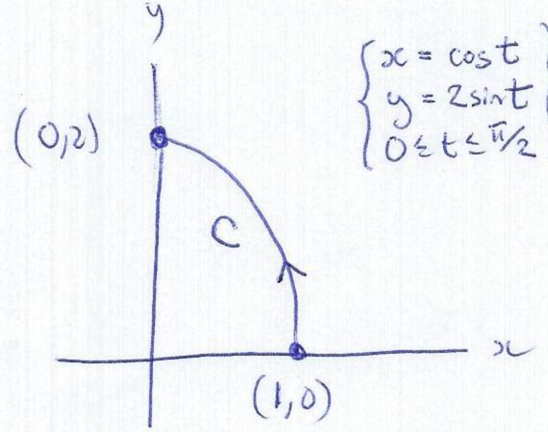
MECH 222: WEEKLY TEST 6 MATH SOLUTIONS

1. (10 marks) Let C be the 1/4-ellipse $x^2 + y^2/4 = 1$, $x \geq 0$, $y \geq 0$ in the first quadrant of the xy-plane.

   (a) (5 marks) Parameterize the curve C, and use your parameterization to write (but **not** evaluate) an integral giving its length.



$$\begin{cases} x = \cos t \\ y = 2\sin t \\ 0 \le t \le \pi/2 \end{cases} \qquad \vec{r} = \langle \cos t, 2\sin t\rangle, \quad \vec{r}\,' = \langle -\sin t, 2\cos t\rangle, \quad \|\vec{r}\,'\| = \sqrt{\sin^2 t + 4\cos^2 t}$$

$$L = \int_C ds = \int_0^{\pi/2} \|\vec{r}\,'\|\,dt = \boxed{\int_0^{\pi/2} \sqrt{\sin^2 t + 4\cos^2 t}\,dt}$$

(alternate: $\begin{cases} x = \sqrt{1 - y^2/4} \\ y = y \\ 0 \le y \le 2 \end{cases}$ $\quad \vec{r}\,' = \left\langle -\frac{y}{4\sqrt{1-y^2/4}}, 1 \right\rangle$ $\quad \|\vec{r}\,'\| = \sqrt{\frac{y^2}{16(1-y^2/4)} + 1} = \sqrt{\frac{16-3y^2}{16-4y^2}}$

$$L = \int_C ds = \int_0^2 \sqrt{\frac{16-3y^2}{16-4y^2}}\,dy$$

alternate: $\begin{cases} x = x \\ y = 2\sqrt{1-x^2} \\ x: 1 \to 0 \end{cases}$ ... etc.)

   (b) (5 marks) Compute the work done by a force $\mathbf{F} = \mathbf{F_1} + \mathbf{F_2}$ on an object that moves along C from $(1,0)$ to $(0,2)$, where $\mathbf{F_1}(x,y) = -y\hat{\mathbf{i}} + x\hat{\mathbf{j}}$, $\qquad \mathbf{F_2} = \nabla(e^{xy}x^2y^2)$.

- $W = \int_C \vec{F}\cdot d\vec{r} = \int_C \vec{F}_1\cdot d\vec{r} + \int_C \vec{F}_2\cdot d\vec{r}$
- $\int_C \vec{F}_1\cdot d\vec{r} = \int_0^{\pi/2} \langle -2\sin t, \cos t\rangle \cdot \langle -\sin t, 2\cos t\rangle\,dt = \int_0^{\pi/2} 2(\sin^2 t + \cos^2 t)\,dt = \pi$
- $\int_C \vec{F}_2\cdot d\vec{r} = \int_C \nabla(e^{xy}x^2y^2)\cdot d\vec{r} \underset{\text{FTC}}{=} e^{xy}x^2y^2 \Big|_{(1,0)}^{(0,2)} = 0$

$\Rightarrow W = \boxed{\pi}$

(alternate: $\int_C \vec{F}_1\cdot d\vec{r} = \int_0^2 \left\langle -\frac{y}{4\sqrt{1-y^2/4}}, 1\right\rangle \cdot \left\langle -y, \sqrt{1-y^2/4}\right\rangle dy$

$$= \int_0^2 \left(\frac{y^2}{4\sqrt{1-y^2/4}} + \sqrt{1-y^2/4} \right) dy = \int_0^2 \frac{1}{\sqrt{1-y^2/4}}\,dy$$

$\begin{cases} y = 2\sin\theta \\ dy = 2\cos\theta\,d\theta \end{cases}$ $= 2\int_0^{\pi/2} \frac{\cos\theta}{\cos\theta}\,d\theta = \pi$)

or: use x parameterization ...