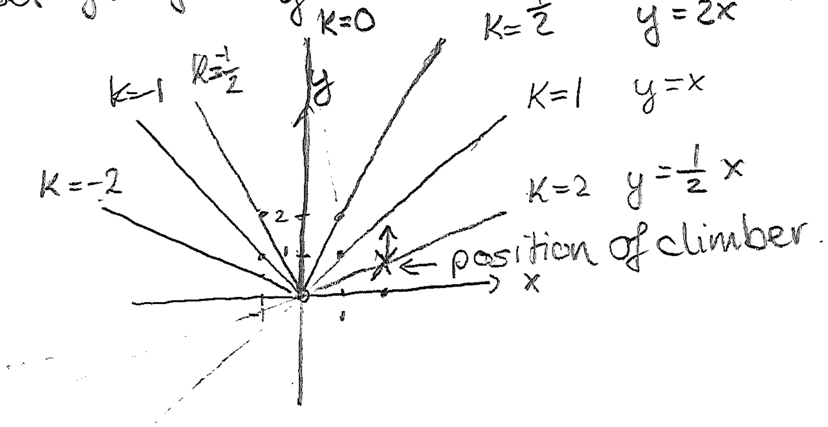
Worksheet 5 - Solution

Contour lines:

Problemset 1

1. $f(x,y) = \frac{x}{y}$, $y > 0$

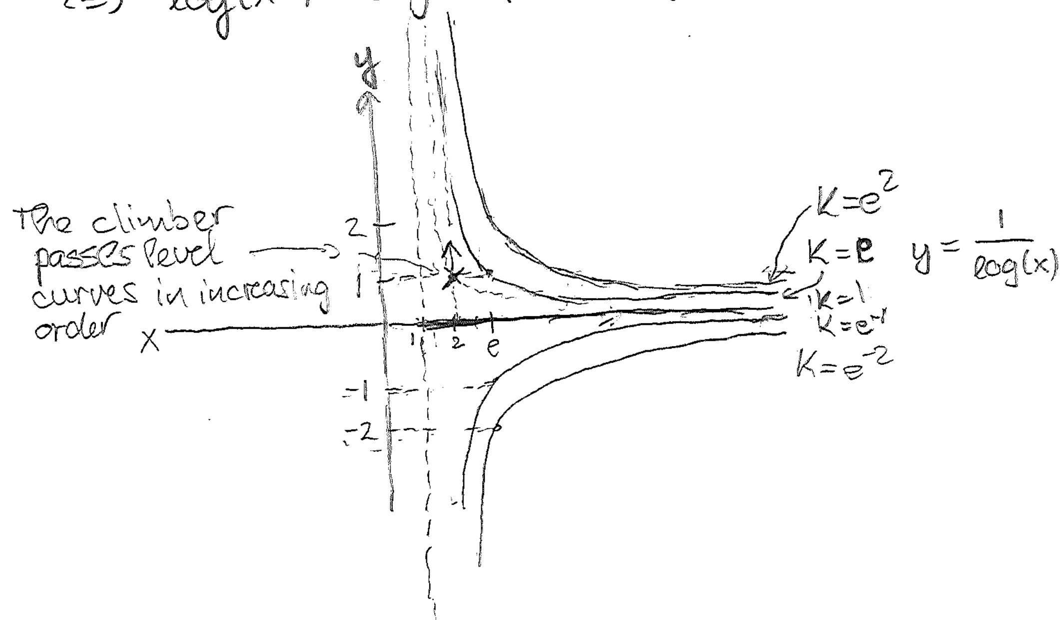
Set $f(x,y) = \frac{x}{y} = K$. Then, $y = \frac{x}{K}$ for $K \neq 0$ and $x = 0$ for $K = 0$



As the climber move into the positive y-direction, she passes level curves of decreasing heights. She descends.

2. $f(x,y) = x^y = K$, $x > 1$

$\Leftrightarrow \log(x^y) = \log(K) \Leftrightarrow y \log(x) = \log(K) \Rightarrow y = \frac{\log(K)}{\log(x)}$



The climber ascends.

Problemset 2:

1. $f(x,y) = \frac{x}{y}$

First order: $\frac{\partial f}{\partial x}(x,y) = \frac{1}{y}$; $\frac{\partial f}{\partial y}(x,y) = -\frac{x}{y^2}$

Second order:

$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)(x,y) = 0$; $\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)(x,y) = -\frac{1}{y^2}$

$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)(x,y) = -\frac{1}{y^2}$, $\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)(x,y) = 2\frac{x}{y^3}$

2. $f(x,y) = x^y = \exp(\ln(x^y)) = \exp(y \ln(x)) = e^{y \ln(x)}$

For a fixed y: x^y is just a polynomial:

$\frac{\partial}{\partial x} f(x,y) = y \cdot x^{y-1}$

For the derivative with respect to y:

$\frac{\partial}{\partial y} f(x,y) = \ln(x)\, e^{\ln(x) y} = \ln(x) \cdot x^y$

3. $f(x,y,z) = xy \sin^{-1}(y \cdot z)$

$\frac{\partial f}{\partial x}(x,y,z) = y \sin^{-1}(y,z)$

$\frac{\partial f}{\partial y}(x,y,z) \overset{\text{Prod. rule}}{=} x \sin^{-1}(y \cdot z) + xy \frac{1}{\sqrt{1-(yz)^2}} \cdot z$ using $\frac{d}{dt} \sin^{-1}(t) = \frac{1}{\sqrt{1-t^2}}$

$\frac{\partial f}{\partial z}(x,y,z) = xy^2 \frac{1}{\sqrt{1-y^2z^2}}$

· We now use partial derivatives to determine the local change of the climber

To obtain the change of the climber into the positive y-direction, we evaluate

1. $\frac{\partial f}{\partial y}(2,1) = -\frac{x}{y^2}\Big|_{(x,y)=(2,1)} = -\frac{2}{1} = -2 \Rightarrow$ descend

2. $\frac{\partial f}{\partial y}(2,1) = \ln(x) \cdot x^y\Big|_{(x,y)=(2,1)} = \ln(2)\, 2^1 = \ln(2^2) = \ln(4) > \ln(e) = 1$
$\Rightarrow$ ascend.

Problemset 3.:

1. $e^z = xyz$

Differentiate on both sides with respect to x

$$\left.\begin{aligned} \frac{\partial}{\partial x}(e^z) &= e^z \frac{\partial z}{\partial x} \\ \frac{\partial}{\partial x}(xyz) &= yz + xy\frac{\partial z}{\partial x} \end{aligned}\right\} \Rightarrow \frac{\partial z}{\partial x}(e^z - xy) = yz \Rightarrow \frac{\partial z}{\partial x} = \frac{yz}{e^z - xy}$$

Differentiate on both sides with respect to y:

$$\left.\begin{aligned} \frac{\partial}{\partial y}(e^z) &= e^z \frac{\partial z}{\partial y} \\ \frac{\partial}{\partial y}(xyz) &= xz + xy\frac{\partial z}{\partial y} \end{aligned}\right\} \Rightarrow \text{Symmetry } \frac{\partial z}{\partial y} = \frac{xz}{e^z - xy}$$

2. $yz + x\ln(y) = z^2$

$$\frac{\partial}{\partial x} // \Rightarrow y\frac{\partial z}{\partial x} + \ln(y) = 2\cdot z\cdot\frac{\partial z}{\partial x} \Leftrightarrow \frac{\partial z}{\partial x}(y-2z) = -\ln(y)$$

$$\Leftrightarrow \frac{\partial z}{\partial x} = \frac{-\ln(y)}{y-2z}$$

$$\frac{\partial}{\partial y} // \Rightarrow z + y\frac{\partial z}{\partial y} + \frac{x}{y} = 2z\frac{\partial z}{\partial y} \Leftrightarrow \frac{\partial z}{\partial y}(y-2z) = -\left(z+\frac{x}{y}\right)$$

$$\Leftrightarrow \frac{\partial z}{\partial y} = -\frac{z+\frac{x}{y}}{y-2z}$$

Problemset 4:

1. We observe $z(2,1) = \frac{2}{1} = 2$. Define $f(x,y) = \frac{x}{y}$.

we have observed that

$$\frac{\partial f}{\partial x}(x,y) = \frac{1}{y}, \quad \frac{\partial f}{\partial y}(x,y) = -\frac{x}{y^2}.$$

Therefore

$$z = 2 + \left(\frac{1}{y}\right)_{(x,y)=(2,1)}(x-2) + \left(-\frac{x}{y}\right)_{(x,y)=(2,1)}(y-1) = 2 + (x-2) - 2(y-1) = x - 2y + 2$$

is the planar equation. ($z = x - 2y + 2$)

2. $$z \approx \underbrace{2^1}_{f(2,1)} + y\cdot x^{y-1}|_{(x,y)=(2,1)}(x-2) + (\log(x)\,x^y)|_{(x,y)=(2,1)}(y-1)$$

$$= 2 + 2\cdot 1\,(x-2) + \ln(4)\,(y-1)$$

In (2.02, 0.97) the linear approximation results in:

$$z \approx 2 + 0.04 + \underbrace{\ln(4)}_{\approx 1.39}(-0.03) \approx 1.998$$

Example : Chainrule

$\hat{f}(x,y) = x^y \quad g(t) = 2+\frac{1}{2}\sin(t) \quad h(t) = 1+t$

1. $f(t) = (1+\frac{1}{2}\sin(t))^{1+t}$

2. Recall $\frac{\partial f}{\partial x} = \frac{\partial \hat{f}}{\partial x} = y \cdot x^{y-1}$ | $g'(t) = \frac{1}{2}\cos(t)$, $h'(t) = 1$

$\frac{\partial f}{\partial y} = \frac{\partial \hat{f}}{\partial y} = \ln(x)\, x^y$

3. Then, $\frac{\partial f}{\partial t}(t) = \left(\frac{\partial f}{\partial x}\right)(g(t), h(t)) \cdot g'(t) + \left(\frac{\partial f}{\partial y}\right)(g(t), h(t)) \cdot h'(t)$

$$= \underbrace{(1+t)}_{h(t)} \underbrace{\left(2+\frac{1}{2}\sin(t)\right)^{t}}_{g(t)^{(h(t)-1)}} \cdot \underbrace{\frac{1}{2}\cos(t)}_{g'(t)} + \underbrace{\ln\left(2+\frac{1}{2}\sin(t)\right)}_{\ln(g(t))} \cdot \underbrace{\left(2+\frac{1}{2}\sin(t)\right)^{1+t}}_{g(t)^{h(t)}} \cdot \underbrace{1}_{h'(t)}$$

Problem set 5:

1. $z = x^2+y^2+xy \qquad x = \sin(t) \qquad y = e^t$

$\frac{\partial z}{\partial x} = 2x+y \qquad \frac{\partial z}{\partial y} = 2y+x \qquad x'(t) = \cos(t) \qquad y'(t) = e^t$

$$\frac{\partial z}{\partial t} = (2\sin(t)+e^t)\cdot\cos(t) + (2e^t + \sin(t))\, e^t$$
$$= 2\sin(t)\cos(t) + e^t(\cos(t)+\sin(t)) + 2e^{2t}$$

2. $z = \tan^{-1}\left(\frac{x}{y}\right)$ and $y = e^t \quad x = 1-e^{-t}$ $\qquad (\tan^{-1})'(t) = \frac{1}{t^2+1}$

$\frac{\partial z}{\partial x} = \frac{1}{(\frac{x}{y})^2+1} \cdot \frac{1}{y} \qquad \frac{\partial z}{\partial y} = \frac{1}{(\frac{x}{y})^2+1}\left(-\frac{x}{y^2}\right)$, $y'(t) = e^t \quad x'(t) = e^{-t}$

$$\frac{\partial z}{\partial t} = \frac{1}{1+\left(\frac{1-e^{-t}}{e^t}\right)^2} \cdot \frac{1}{e^t} \cdot e^{-t} + \frac{1}{1+\left(\frac{1-e^{-t}}{e^t}\right)^2} \cdot \left(-\frac{1-e^{-t}}{e^{2t}}\right) e^t$$

$$= \frac{1}{1+\left(\frac{1-e^{-t}}{e^t}\right)^2} \cdot \left[\frac{1 - e^t + 1}{e^{2t}}\right] = \frac{2-e^t}{e^{2t}} \cdot \frac{1}{1+\left(\frac{1-e^{-t}}{e^t}\right)^2}.$$