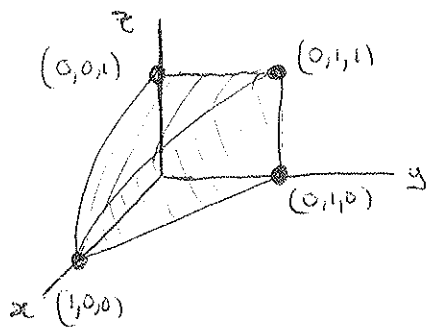
Mech 222 Week2 Test   Math Solutions

1. (10 marks) Consider the iterated integral

$$I = \int_0^1 \int_0^{1-x^2} \int_0^{1-x} f(x, y, z)\, dy\, dz\, dx = \iiint_E f\, dV$$

(a) (5 marks) Sketch (roughly) the region E. Then write this integral as an iterated integral in the five other possible orders.
$E = \{(x, y, z) \mid 0 \le x \le 1,\ 0 \le z \le 1 - x^2,\ 0 \le y \le 1 - x\}$:



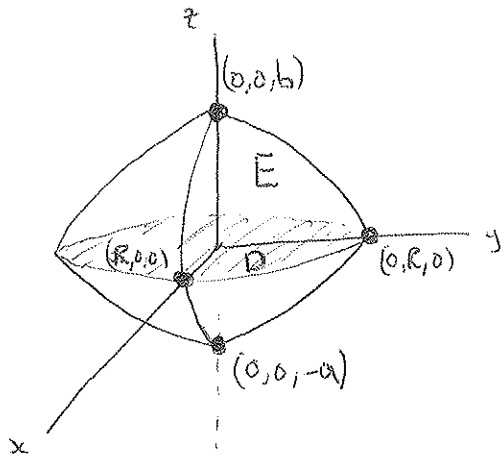
- *over region with parabolic boundary in xz-plane:* $I = \int_0^1 \int_0^{\sqrt{1-z}} \int_0^{1-x} f\, dy\, dx\, dz$
- *over triangle in the xy-plane:* $I = \int_0^1 \int_0^{1-x} \int_0^{1-x^2} f\, dz\, dy\, dx = \int_0^1 \int_0^{1-y} \int_0^{1-x^2} f\, dz\, dx\, dy$
- *over square in yz-plane:* $I = \int_0^1 \int_0^1 \int_0^{\min(\sqrt{1-z}, 1-y)} f\, dx\, dy\, dz = \int_0^1 \int_0^1 \int_0^{\min(\sqrt{1-z}, 1-y)} f\, dx\, dz$

(b) (5 marks) Compute the volume of the region E.
Take $f = 1$ (and any of the possible orders) to compute

$$V = \int_0^1 \int_0^{1-x^2} \int_0^{1-x} f(x, y, z)\, dy\, dz\, dx = \int_0^1 \int_0^{1-x^2} (1 - x) dz dx = \int_0^1 (1 - x)(1 - x^2) dx$$
$$= \int_0^1 (1 - x - x^2 + x^3) dx = 1 - \frac{1}{2} - \frac{1}{3} + \frac{1}{4} = \frac{5}{12}.$$

2. (25 marks) Let E be a uniform solid occupying the region $a\left(\frac{x^2+y^2}{R^2}-1\right) \le z \le b\left(1-\frac{x^2+y^2}{R^2}\right)$ in 3-space. Here a, b, and R are positive constants.

(a) (5 marks) Sketch E.



(b) (10 marks) Find the centroid coordinates of E.

By symmetry,

$$x_c = y_c = 0.$$

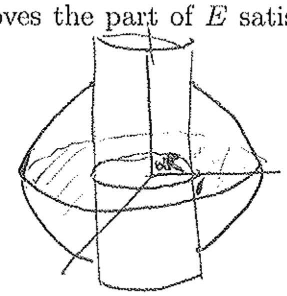
To find z_c, we can set up the triple integrals using the projection of the region in the xy-plane, which is the disk $D = \{(x,y)\ x^2+y^2 \le R^2\}$ of radius R where the two paraboloids intersect. The integrals over D can be done in polar coordinates (a.k.a. cylindrical coordinates in 3-space):

$$\begin{aligned} V &= \iiint_E dV = \iint_D \int_{a(\frac{x^2+y^2}{R^2}-1)}^{b(1-\frac{x^2+y^2}{R^2})} dz\, dA = \iint_D (a+b)(1-\frac{x^2+y^2}{R^2})dA \\ &= (a+b)\int_0^{2\pi}\int_0^R (1-r^2/R^2)r dr\, d\theta = 2\pi(a+b)\int_0^R (r-r^3/R^2)dr \\ &= 2\pi(a+b)(R^2/2-R^2/4) = \frac{1}{2}\pi(a+b)R^2 \end{aligned}$$

$$\begin{aligned} z_c &= \frac{1}{V}\iiint_E z dV = \frac{2}{\pi(a+b)R^2}\iint_D \int_{a(\frac{x^2+y^2}{R^2}-1)}^{b(1-\frac{x^2+y^2}{R^2})} z dz\, dA = \frac{b^2-a^2}{\pi(a+b)R^2}\iint_D (1-\frac{x^2+y^2}{R^2})^2 dA \\ &= \frac{b-a}{\pi R^2}\int_0^{2\pi}\int_0^R (1-\frac{r^2}{R^2})^2 r dr d\theta = \frac{2(b-a)}{R^2}\int_0^R (r-2\frac{r^3}{R^2}+\frac{r^5}{R^4})dr = \frac{2(b-a)}{R^2}\frac{R^2}{6} = \frac{b-a}{3}. \end{aligned}$$

(c) Set up (but do not compute!) iterated integrals to compute the centroid coordinates of each of the solids that remain after

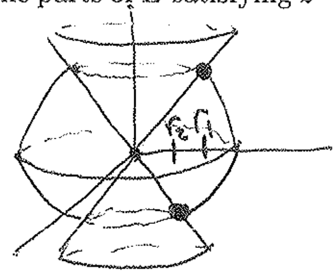
i. (5 marks) a (cylindrical) drill removes the part of E satisfying $x^2 + y^2 \leq \alpha^2 R^2$ ($0 < \alpha < 1$ a constant);



As above, $x_c = y_c = 0$, and the integrals over D above now begin at $r = \alpha R$:

$$z_c = \frac{\int_0^{2\pi} \int_{\alpha R}^{R} \int_{a(\frac{r^2}{R^2}-1)}^{b(1-\frac{r^2}{R^2})} z dz}{\int_0^{2\pi} \int_{\alpha R}^{R} \int_{a(\frac{r^2}{R^2}-1)}^{b(1-\frac{r^2}{R^2})} dz}.$$

ii. (5 marks) a (conical) drill removes the parts of E satisfying $z^2 \geq \beta^2(x^2+y^2)$ ($\beta > 0$ a constant)



Yet again, $x_c = y_c = 0$. The cone intersects the upper paraboloid where

$$z = \beta r = b(1-\frac{r^2}{R^2}) \implies \frac{1}{R^2}r^2+\frac{\beta}{b}r-1 = 0 \implies r = \left(-1+\sqrt{1+\frac{4b^2}{\beta^2 R^2}}\right)\frac{\beta R^2}{2b} =: r_1$$

and, similarly, the lower one at $z = -\beta r_2$, $r_2 := \left(-1+\sqrt{1+\frac{4a^2}{\beta^2 R^2}}\right)\frac{\beta R^2}{2a}$, and we should integrate over the lower and upper halves separately:

$$z_c = \frac{\int_0^{2\pi}\left[\int_0^{r_1}\int_0^{\beta r} z dz + \int_{r_1}^{R}\int_0^{b(1-\frac{r^2}{R^2})} z dz + \int_0^{r_2}\int_{-\beta r}^{0} z dz + \int_{r_2}^{R}\int_{a(\frac{r^2}{R^2}-1)}^{0} z dz\right] d\theta}{\int_0^{2\pi}\left[\int_0^{r_1}\int_0^{\beta r} dz + \int_{r_1}^{R}\int_0^{b(1-\frac{r^2}{R^2})} dz + \int_0^{r_2}\int_{-\beta r}^{0} dz + \int_{r_2}^{R}\int_{a(\frac{r^2}{R^2}-1)}^{0} dz\right] d\theta}$$