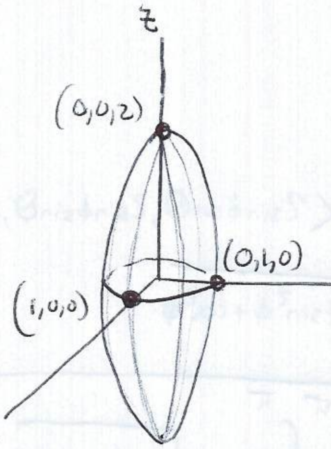
2. (10 marks) Let S be the ellipsoid $x^2 + y^2 + \frac{z^2}{4} = 1$.

(a) (4 marks) Write a parameterization of the surface S.



• parametrize S using "spherical angles"

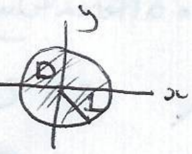
$$\boxed{\begin{gathered}\vec{r}(\phi,\theta) = \langle \sin\phi\cos\theta,\ \sin\phi\sin\theta,\ 2\cos\phi \rangle \\ 0 \le \theta \le 2\pi \\ 0 \le \phi \le \pi\end{gathered}}$$

(• alternately, we could split S into upper and lower (say) halves, and parametrize each as graphs:

$$S = S_+ \cup S_-$$

$$S_+ : \vec{r}(x,y) = \langle x, y, 2\sqrt{1-x^2-y^2} \rangle,\ (x,y) \in D = \{x^2+y^2 \le 1\}$$

$$S_- : \vec{r}(x,y) = \langle x, y, -2\sqrt{1-x^2-y^2} \rangle,\ (x,y) \in D$$)



(b) (6 marks) Write the surface area of S as a double integral (with respect to your parameters from (a), or in some other way), but do *not* try to compute it.

- $\vec{r}_\phi = \langle \cos\phi\cos\theta, \cos\phi\sin\theta, -2\sin\phi \rangle$

$\vec{r}_\theta = \langle -\sin\phi\sin\theta, \sin\phi\cos\theta, 0 \rangle$

$\vec{r}_\phi \times \vec{r}_\theta = \langle 2\sin^2\phi\cos\theta, 2\sin^2\phi\sin\theta, \sin\phi\cos\phi \rangle = \sin\phi \langle 2\sin\phi\cos\theta, 2\sin\phi\sin\theta, \cos\phi \rangle$

$\Rightarrow \|\vec{r}_\phi \times \vec{r}_\theta\| = \sin\phi\sqrt{4\sin^2\phi(\cos^2\theta+\sin^2\theta)+\cos^2\phi} = \sin\phi\sqrt{4\sin^2\phi+\cos^2\phi}$

$$\Rightarrow A(S) = \iint_S dS = \int_0^{2\pi}\int_0^{\pi} \|\vec{r}_\phi \times \vec{r}_\theta\|\, d\phi\, d\theta = \boxed{\int_0^{2\pi}\int_0^{\pi} \sin\phi\sqrt{4\sin^2\phi+\cos^2\phi}\, d\phi\, d\theta}$$

$$\left(= 2\pi\int_0^{\pi} \sin\phi\sqrt{4-3\cos^2\phi}\, d\phi \right)$$

(• alternately, use the graph parameterization: for the upper half,

$z = 2\sqrt{1-x^2-y^2} =: f(x,y)$ $\qquad f_x = \dfrac{-2x}{\sqrt{1-x^2-y^2}}, \quad f_y = \dfrac{-2y}{\sqrt{1-x^2-y^2}}$

$$\Rightarrow \sqrt{1+f_x^2+f_y^2} = \sqrt{1+\frac{4(x^2+y^2)}{1-x^2-y^2}} = \sqrt{\frac{1+3(x^2+y^2)}{1-(x^2+y^2)}}$$

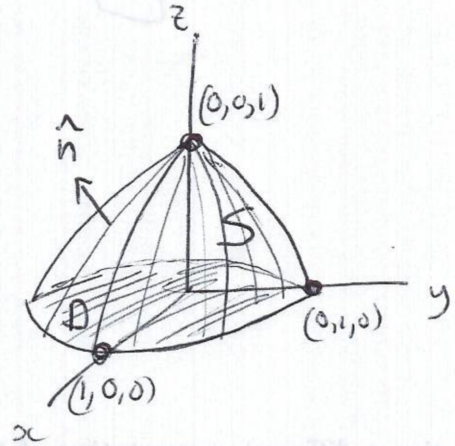
$$\Rightarrow A(S) = 2A(S_+) = 2\iint_{S_+} dS = \boxed{2\iint_{D=\text{unit disk}} \sqrt{\frac{1+3(x^2+y^2)}{1-(x^2+y^2)}}\, dA}$$

$$\left(= 4\pi\int_0^1 \sqrt{\frac{1+3r^2}{1-r^2}}\, r\, dr \right)$$

)

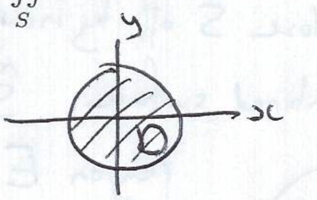
6. (25 marks) Let S be the surface $z = 1 - x^2 - y^2$, $z \geq 0$, oriented so that the unit normal has non-negative z-component, and let $\mathbf{F}$ be the vector field $\mathbf{F} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + (1+z)\hat{\mathbf{k}}$.

(a) (4 marks) Sketch the surface S.



(b) (10 marks) Compute (directly) the surface integral $\iint_S \mathbf{F} \cdot d\mathbf{S}$.

• with $z = f(x,y) = 1 - x^2 - y^2$, $(x,y) \in D$



$f_x = -2x$, $f_y = -2y$

$$\Rightarrow \iint_S \vec{F} \cdot d\vec{S} = \iint_D \langle -f_x, -f_y, 1 \rangle \cdot \langle x, y, 1+z \rangle \, dA$$

$$= \iint_D \langle 2x, 2y, 1 \rangle \cdot \langle x, y, 2 - x^2 - y^2 \rangle \, dA$$

$$= \iint_D (2x^2 + 2y^2 + 2 - x^2 - y^2) \, dA = \iint_D (2 + x^2 + y^2) \, dA$$

$$= \int_0^{2\pi} \int_0^1 (2 + r^2) r \, dr \, d\theta = 2\pi \int_0^1 (2r + r^3) \, dr = 2\pi \left(1 + \frac{1}{4} \right)$$

polar coords.

$$= \boxed{\frac{5\pi}{2}}$$

(c) (2 marks) Compute $\nabla \cdot \mathbf{F}$, the divergence of $\mathbf{F}$,

- $\nabla \cdot \vec{F} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(1+z) = 1+1+1 = \boxed{3}$

(d) (9 marks) Use the Divergence Theorem to give a different computation of the integral $\iint\limits_S \mathbf{F} \cdot d\mathbf{S}$.

- if we close S off by including the base D (with downward normal), the combined surface $\tilde{S} := S \cup D$ is closed, bounding a solid region E, and we may apply the Divergence Theorem:

$$\iint\limits_{\tilde{S}} \vec{F} \cdot d\vec{S} = \iiint\limits_E \nabla \cdot \vec{F} \, dV = 3 \iiint\limits_E dV$$

$$= 3 \iint\limits_D \int_0^{1-x^2-y^2} dz \, dA = 3 \iint\limits_D (1-x^2-y^2) \, dA$$

$$\overset{\text{polar}}{=} 3 \cdot 2\pi \cdot \int_0^1 (1-r^2) r \, dr = 6\pi\left(\frac{1}{2} - \frac{1}{4}\right) = 3\pi/2$$

- so $\frac{3\pi}{2} = \iint\limits_{\tilde{S}} \vec{F} \cdot d\vec{S} = \iint\limits_S \vec{F} \cdot d\vec{S} + \iint\limits_D \vec{F} \cdot d\vec{S} \Rightarrow \iint\limits_S \vec{F} \cdot d\vec{S} = \frac{3\pi}{2} - \iint\limits_D \vec{F} \cdot d\vec{S}$

- on D, $\hat{n} = -\hat{k}$, and $z = 0$ so $\vec{F} \cdot \hat{n} = -(1+z) = -1 \Rightarrow \iint\limits_D \vec{F} \cdot d\vec{S} = -\iint\limits_D dS = -\pi$

$$\Rightarrow \iint\limits_S \vec{F} \cdot d\vec{S} = \frac{3\pi}{2} + \pi = \boxed{\frac{5\pi}{2}}$$