Worksheet 6

Felix Funk, MATH Tutorial - Mech 222

1 Curves in Space

Introduction: Curves in Space In this worksheet we are interested in parameterizing curves in space. For this, we have to define a path $r(t) = x(t)i + y(t)j + z(t)k$. In this setup $r'(t) = x'(t)i + y'(t)j + z'(t)k$ describes the change in all spatial dimensions as time progresses. If $r(t)$ tracks particle movement, then $r'(t)$ denotes the velocity, its length $|r'(t)|$ the speed, and $r''(t)$ the particle's acceleration.

Problemset: 1. Curves in Space.

1. Find the parametric form for the line segment which starts in $P = (1, 1/2, 1/3)$ and ends in $Q = (3, 2, 1)$.
2. Sketch $r(t) = (t, t\sin(t), t\cos(t))$.
3. Find the velocity, acceleration, and speed of a particle with the position function: $r(t) = (t^2, \sin(t) - t\cos(t), \cos(t) + t\sin(t)), t \geq 0$.
4. Challenging: Show that if a particle moves with constant speed, then the velocity and acceleration vectors are orthogonal.

2 Line Integrals

2.1 Line Integrals in Planes/Space

Introduction: Line Integrals. There are various notations for the integration procedure along lines. When you want to integrate a function over a curve, then you have to integrate with respect to arclength.

- In 2D: $\int_C f(x,y)ds = \int_a^b f(x(t),y(t))\sqrt{\left(\frac{\partial x}{\partial t}\right)^2 + \left(\frac{\partial y}{\partial t}\right)^2}dt$
- In 3D: $\int_C f(x,y,z)ds = \int_a^b f(x(t),y(t),z(t))\sqrt{\left(\frac{\partial x}{\partial t}\right)^2 + \left(\frac{\partial y}{\partial t}\right)^2 + \left(\frac{\partial z}{\partial t}\right)^2}dt$

Choosing the right function, this can provide us with the length or the mass of a wire, for instance. Alternatively, we also see integrals with respect to x or y, which can expressed in terms of the parameterization by the following equalities

- $\int_C f(x,y)dx = \int_a^b f(x(t),y(t))x'(t)dt$
- $\int_C f(x,y)dy = \int_a^b f(x(t),y(t))y'(t)dt$

with simple generalization to three dimensions.

And finally we can also use directional derivatives to evaluate work done by a forcefield while a particle follows a curve. $r(t)$ describes the particle movement as in the previous section and the integration process aggregates contributions along its normalized velocity vector. In this case,

$$\int_C F \cdot dr = \int_a^b F(r(t)) \cdot r'(t)dt.$$

Problemset: 2. Line Integrals.

1. Evaluate $\int_C (x+2y)dx + x^2dy$; where C consists of line segments from $(0,0)$ to $(2,1)$ and $(2,1)$ to $(3,0)$.
2. Determine the line integral along the vector field $F(x,y,z) = \sin(x)i + \cos(x)j + xzk$; and generated by $r(t) = t^3i - t^2j + tk, \quad 0 \le t \le 1$.
3. A thin wire is bent into the shape of a semicircle $x^2 + y^2 = 4; x \ge 0$. We assume constant linear density k. Find the mass and center of mass of the wire.

3 Fundamental Theorem of Line Integrals

Introduction: Fundamental Theorem of Line Integrals. Let $r(t), a \leq t \leq b$ define a smooth curve C. When we find a function f with continuous partial derivatives along the curve C such that $\nabla f = F$ then

$$\int_C \nabla f dr = f(r(b)) - f(r(a)).$$

So one would think that the integral should be independent of the path that we are taking for integration. But only for special - so called conservative - vector fields, one finds that this is the case.

You can sometimes check a simple criterion, whether such a function f exists: If $F(x, y) = P(x, y)i + Q(x, y)j$ with continuous first-order partial derivatives on a simply-connected region D satisfies

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \text{ for all } (x, y) \in D.$$

One important result for conservative vector fields F is that any closed curve $\int_C F \cdot dr = 0$ as the start and end point agree.

Problemset: 3. Fundamental Theorem of Line Integrals. Use the Fundamental Theorem of Line Integrals to solve the following questions.

1. Determine whether or not $F(x, y) = (3x^2 - 2y^2)i + (4xy + 3)j$ is a conservative vector field. If it is, find $F = \nabla f$.

2. Find a function f such that $F = \nabla f$ and use this to evaluate $\int_C F \cdot dr$ with

   $$F(x, y) = (1 + xy)e^{xy}i + x^2e^{xy}j$$

   along the curve $C : r(t) = \cos(t)i + 2\sin(t)j; \quad 0 \leq t \leq \pi/2.$

3. Challenging: Show that the line integral $\int_C \sin(y)dx + (x\cos(y) - \sin(y))dy$ is independent of the specific path taken from $(2, 0)$ to $(1, \pi)$ and evaluate the integral. Hints: First, find a vector field F and a path r that expresses the integral above as $\int_C F \cdot dr$ and show that F is conservative. Second, construct a closed curve using two arbitrary paths that start and end at the same point.