Worksheet 3

Felix Funk, MATH Tutorial - Mech 222

1 Contour Lines and Partial Derivatives

Introduction: Finding Contour Lines.
Contour lines provide us with a tool to visualize functions over a two-dimensional domain $f(x, y)$. Exemplary, we plot in the xy plane the curves that solve $f(x, y) = K$ for a couple of sensible values of K.

Problemset: 1. Contour Lines. Draw a contour map showing several level curves of

1. $f(x, y) = \frac{x}{y}, y > 0$
2. $f(x, y) = x^y, x > 1$.

If a mountain climber moved from $(2, 1)$ in the positive y-direction, would she descend or ascend based on the contour map?

Introduction: Partial Derivatives.
In high-dimensional spaces, the notion of local change is tightly intertwined with a sense of direction. For instance, in the previous problemset, the climber could also go towards the origin or into the positive/negative x-direction, and the answer to the question whether the climber ascends/descends vary, accordingly. Partial derivatives provide an indicator of the local discrepancies when movement is exerted in parallel to the coordinate axis into the positive axis-direction. For the specific calculation, we can simply fix all variables but one, and take the derivative with respect to the remaining variable. For a function $f(x, y)$ the partial derivative $\frac{\partial f(x,y)}{\partial x}$ expresses the change in the x direction with a constant y.

Problemset: 2. Partial Derivatives. Calculate all second order derivatives of

1. $f(x, y) = \frac{x}{y}, y > 0$

and all first order partial derivatives of

2. $f(x, y) = x^y, x > 1$. Hint: $z = \exp(\ln z)$
3. $f(x, y, z) = xy \sin^{-1}(y, z)$

Use the first two examples to determine the local change in $(2, 1)$ of the climber in the positive y-direction.

Problemset: 3. Partial Derivatives: Implicit Differentiation.
Use implicit differentiation to obtain $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$. Here, x, y are variables and z a function of x and y.

1. $e^z = xyz$. You can assume that $e^z \neq xy$.
2. $yz + x\ln(y) = z^2$. You can assume that $y \neq 2z$.

2 Tangent Plane

Introduction: Tangent Planes.
In a single dimension, we could obtain a linearization of a function z $= f(x)$ in x_0 by calculating the Taylor-Expansion:

$$z \approx f(x_0) + (\frac{\partial f}{\partial x}(x_0))(x - x_0)$$

In multiple dimensions, we take into consideration that the function changes along every coordinate axis. Therefore, a function $z = f(x, y)$ is linearized through

$$z \approx f(x_0, y_0) + (\frac{\partial f}{\partial x})(x_0, y_0)(x - x_0) + (\frac{\partial f}{\partial y})(x_0, y_0)(y - y_0)$$

If the function is linear then the approximation above becomes an equality.

Problemset: 4. Tangent Plane.

1. Find an equation for the tangent plane to $z = \frac{x}{y}$ at $(2, 1, 2)$.
2. Find a linear approximation of $f(x, y) = x^y$ at $(2, 1)$ and use it to approximate $f(2.02, 0.97)$.

3 Chain Rule

Introduction: Chain Rule for Real-Valued Functions
One of the first significant differences in higher dimensions can be observed for the chain rule. Because of the additional dimension, we can chain multiple functions. When the real-valued differentiable function $f(x, y)$ is chained with two differentiable functions g and h, then the differential of $f(g(t), h(t))$ is affected by change in both arguments through its input.

$$\frac{\partial f}{\partial z}(g(t), h(t)) = (\frac{\partial f}{\partial x})(g(t)) \cdot g'(t) + (\frac{\partial f}{\partial y})(h(t)) \cdot h'(t)$$

If one ignores either of the two summands, then we have just a one-dimensional chain rule. Because f has, however, changes in both coordinates, we need to aggregate the change in both directions.

Example: Calculating a Chain Rule.
The climber follows the path $(x, y) = (g(t), h(t))$ with $g(t) = 2 + 1/2 \sin(t)$ and $h(t) = 1 + t$ (t in min) up the mountain of the landscape provided by

$$\hat{f}(x, y) = x^y.$$

Use the contour map to illustrate the path that the climber takes. Find a function that keeps track of the height change during the hike.

1. First, we substitute the path coordinates in $\hat{f}$ and obtain $f(t) = \hat{f}(g(t), h(t))$.

2. Then, calculate the partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$, and the derivatives $g'(t)$ and $h'(t)$.

3. Finally, combine and obtain the derivative with respect to time.

$$\frac{\partial}{\partial t} f(t) =$$

Problemset: 5. Chain Rule.

1. Find $\frac{\partial z}{\partial t}$ for $z = x^2 + y^2 + xy$, and $x = \sin(t), y = e^t$.
2. Find $\frac{\partial z}{\partial t}$ for $z = tan^{-1}(x/y)$ and $x = 1 - e^{-t}$, $y = e^t$