Corrected version: Thank you for your corrections!

Worksheet 1: Solution

Problemset 1:

1. $\int_0^{\pi/2} e^{\sin(x)}\cos(x)\, y\, dx = y \int_0^{\pi/2} e^{\sin(x)}\cos(x)\, dx = y\left[e^{\sin(x)}\right]_0^{\pi/2} = y(e-1)$

2. $\int_1^e e^x \log(xy)\, dy = e^x \int_1^e \log(x) + \log(y)\, dy = e^x \log(x)(e-1) + e^x\left[y\log(y) - y\right]_1^e$

$= e^x\log(x)(e-1) + e^x\left[e\underbrace{\log(e)}_{=1} - e - (1\underbrace{\log(1)}_{=0} - 1)\right]$

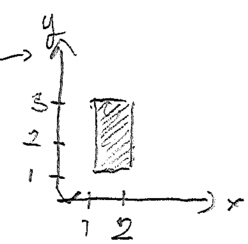
$= e^x\log(x)(e-1) + e^x$

3. $\int_0^1 x e^x y^2\, dx = y^2 \cdot \underbrace{\int_0^1 x e^x\, dx}_{=1 \;\rightarrow\; \text{partial integr./wolfram alpha.}} = y^2$

Example 1:

$$\int_1^3 \int_1^2 \frac{x^2}{y} + \frac{y^2}{x}\, dx\, dy$$

1. Visualize domain of integration $\int_1^3 \int_1^2 dx\, dy$ (3: upper, 1: lower)



2. Solve: $\int_1^2 \frac{x^2}{y} + \frac{y^2}{x}\, dx = \frac{1}{y}\underbrace{\int_1^2 x^2\, dx}_{=\left[\frac{1}{3}x^3\right]_1^2} + y^2 \underbrace{\int_1^2 \frac{1}{x}\, dx}_{\left[\log(x)\right]_1^2} = \frac{1}{3y}[8-1] + y^2\left[\log(2) - \underbrace{\log(1)}_{=0}\right]$

$$= \frac{7}{3y} + \log(2)\, y^2$$

3. Solve $\int_1^3 \frac{7}{3}\frac{1}{y} + \log(2)\, y^2\, dy = \frac{7}{3}\left[\log(y)\right]_1^3 + \log(2)\left[\frac{1}{3}y^3\right]_1^3$

$= \frac{7}{3}\log(3) + \frac{26}{3}\log(2)$

Problemset 2:

1. $\int_{\frac{\pi}{6}}^{\pi/2} \int_{-1}^{6} \cos(y)\, dx\, dy = 7 \cdot \int_{\pi/6}^{\pi/2} \cos(y)\, dy = 7 \cdot \left[\sin(y)\right]_{\pi/6}^{\pi/2} = 7 \cdot \left[1 - \frac{1}{2}\right] = \frac{7}{2}$

2. $\int_0^1 \int_0^1 \sqrt{s+t}\, ds\, dt \overset{s+t=u}{=} \int_0^1 \int_t^{1+t} \sqrt{u}\, du\, dt = \int_0^1 \left[\frac{2}{3}u^{3/2}\right]_t^{1+t} dt = \int_0^1 \frac{2}{3}(1+t)^{3/2} - \frac{2}{3}t^{3/2}\, dt$

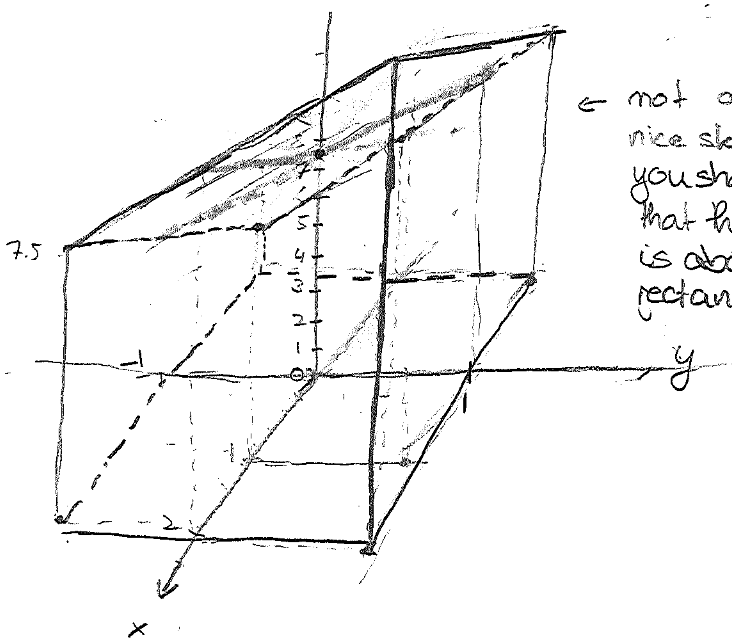
$= \frac{2}{3}\cdot\frac{2}{5}\left[2^{5/2} - 1\right] - \frac{2}{3}\frac{2}{5}[1-0] = \frac{4}{15}\left[2^{5/2} - 1\right] - \frac{4}{15}$

3. Sketch the plane :

Set $x=0$: $6y-2z+15=0 \Rightarrow z = 3y + 7.5$

$y=0$: $4x-2z+15=0 \Rightarrow z = 2x + 7.5$

$z=0$: $4x+6y+15=0 \Rightarrow y = -\frac{2}{3}x - \underbrace{\frac{15}{6}}_{=\frac{5}{2}=2.5}$



← not a nice sketch but you should see that the plane is above the rectangle

As we are interested in the solid between the rectangle an the plane, we need to make sure that the plane does not attain negative values.

Idea: we can use the height as the density of the integral :

$$\int_{-1}^{2}\int_{-1}^{1} 2x+3y+7.5 \quad dy\,dx$$

$4x+6y-2z+15=0$

$\Leftrightarrow \boxed{z = 2x+3y+7.5}$

⇝ solve

4. Advanced hint : You can show that for any odd function (ie $f(-x)=-f(x)$)

$\int_{-1}^{1} f(x)\,dx = 0$ → You can use that to show $\iint_R \frac{xy}{1+x^4}\,dA = 0$

Problemset 3:

1. $\int_0^1 \int_0^{s^2} \cos(s^3)\, dt\, ds = \int_0^1 s^2 \cos(s^3)\, ds = \left[\frac{1}{3} \sin(s^3) \right]_0^1 = \frac{1}{3} [\sin(1) - 0] \approx 0.2805$

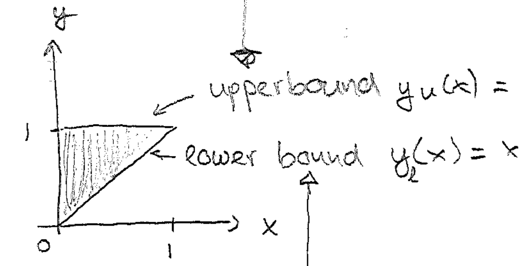
2. $\int_0^1 \int_0^{1-x} 1-x-y \, dy dx = \int_0^1 (1-x) [y]_0^{1-x} + \left[-\frac{1}{2} y^2\right]_0^{1-x} dx = \int_0^1 (1-x)^2 + \left[-\frac{1}{2}(1-x)^2\right] dx$

$= \frac{1}{2} \int_0^1 (1-x)^2 dx = \frac{1}{2} \left[-\frac{1}{3} (1-x)^3 \right]_0^1 = \frac{1}{6}$

3. Change of basis:

$\int_0^1 \int_x^1 \sin(y^2)\, dy\, dx$

(1) Visualize domain:



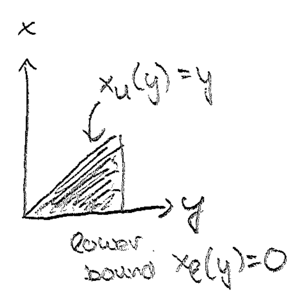
(2) Outer integral first: (You might have to split this if there are different local behaviours.)

y
1
0

$\int_0^1 \cdot \quad dy$

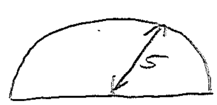
(3) Determine the upper/lower boundaries: write in terms of the outer variable:

$\int_0^1 \int_0^y \sin(y^2)\, dx\, dy = \int_0^1 y \sin(y^2)\, dy = \left[\frac{-1}{2} \cos(y^2)\right]_0^1 = -\frac{1}{2}\cos(1) + \frac{1}{2}$



Example: Introducing polar coordinates:

1.



$\left.\begin{array}{l} 0 \le r \le 5 \\ 0 \le \theta \le \pi \end{array}\right\}$ transformed domain

Rectangular domain

2. $\iint_D x^2 y\, dy = \int_0^5 \int_0^\pi \underbrace{r}_{\text{transformation factor}} \cdot \overbrace{r^2 \cos(\theta)^2}^{x^2 = (r \cos(\theta))^2} \overbrace{r \sin(\theta)}^{y = (r \sin(\theta))} d\theta\, dr = \int_0^5 r^4 dr \cdot \int_0^\pi \cos(\theta)^2 \sin(\theta)\, d\theta$

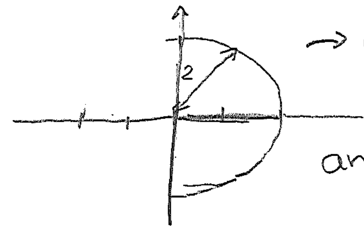
3. $\int_0^\pi \cos(\theta)^2 \sin(\theta)\, d\theta = \left[-\frac{1}{3} \cos(\theta)^3 \right]_0^\pi = \left[-\frac{1}{3}(-1-1) \right] = \frac{2}{3}$

$\int_0^5 r^4 dr = \left[\frac{1}{5} r^5 \right]_0^5 = 5^4 = 625 \quad \Rightarrow \quad \iint_D x^2 y\, dy = 5^4 \cdot \frac{2}{3}$

Problem 4:

$$\iint_D e^{-x^2-y^2}\,dA$$

D



$\rightarrow$ radii: $0 \leq r \leq 2$

angles: $0 \leq \theta \leq \frac{\pi}{2}$ (upperhalf)

$\frac{3\pi}{2} \leq \theta \leq 2\pi$

or: $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$

$$\Rightarrow \iint_D e^{-x^2-y^2}\,dA = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^2 \underset{\uparrow}{r}\; e^{-r^2\cos(\theta)^2-r^2\sin(\theta)^2}\,dr\,d\theta$$

transformation factor

Because: $\cos(\theta)^2 + \sin(\theta)^2 = 1$:

Rectangular domain

$$\ldots = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^2 r\,e^{-r^2}\,dr\,d\theta = \underbrace{\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 1\,d\theta}_{=\pi} \cdot \underbrace{\int_0^2 r e^{-r^2}\,dr}_{=\left[-\frac{1}{2}e^{-r^2}\right]_0^2}$$

$$= \pi \cdot \left[-\frac{1}{2}e^{-4} + \frac{1}{2}\right]$$