SA 3. (5 marks)
Match each function $f(t)$ with its Laplace transform $F(s)$ in the table below. (The function $u(t)$ denotes the unit step function.)

	$F(s)$			$f(t)$ $(t \geq 0)$
A	$\dfrac{5(s+1)}{(s^2+2s+2)(s-1)}$		D	$t + e^{2t-4}u(t-2) + (t-1)e^{t-2}u(t-2)$
B	$\dfrac{2}{(s^2+1)^2}$		E	$e^{-t+1}u(t-1)[\cos(t-1) + \sin(t-1)]$
C	$\dfrac{1}{s^2} + \dfrac{2e^{-s}}{s^3} + \dfrac{e^{-2s}}{s+1}$		F	$1 - 4e^{-t}\sin(t)$
D	$\dfrac{1}{s^2} + \dfrac{e^{-2s}}{s-2} + \dfrac{e^{-2s}s}{(s-1)^2}$		B	$\sin(t) - t\cos(t)$
E	$\dfrac{e^{-s}(s+2)}{(s^2+2s+2)}$		C	$t + (t-1)^2u(t-1) + e^{-t+2}u(t-2)$
F	$\dfrac{s^2-2s+2}{s(s^2+2s+2)}$		A	$2e^t - 2e^{-t}\cos(t) + e^{-t}\sin(t)$

SA 4. (5 marks)

Consider a system where an input signal $x(t)$ is related to an output signal $y(t)$ by the differential equation

$$y'' + y' + y = ax' + x$$

Suppose the input is given by $x(t) = \sin(5t)$. Use the transfer function $H(s) = Y(s)/X(s)$ (assuming all initial values for $x(t)$ and $y(t)$ are zero) to find the range of values for the parameter $a > 0$ such that the steady state response satisfies $|y_{ss}(t)| < 1$.

Apply the Laplace transform:

$$s^2 Y(s) + sY(s) + Y(s) = asX(s) + X(s)$$

$$\Rightarrow H(s) = \frac{Y(s)}{X(s)} = \frac{as+1}{s^2+s+1}$$

Then $|y_{ss}(t)| \leq |H(5j)|$

$$|H(5j)| = \left|\frac{a5j+1}{(5j)^2+5j+1}\right| = \left|\frac{1+j5a}{-24+j5}\right| = \frac{\sqrt{1+25a^2}}{\sqrt{24^2+25}}$$

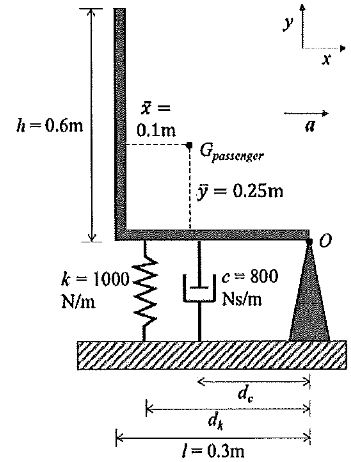
$$\Rightarrow |y_{ss}(t)| < 1 \text{ if } \sqrt{\frac{1+25a^2}{24^2+25}} < 1$$

$$\Rightarrow \frac{1+25a^2}{24^2+25} < 1 \Rightarrow 1+25a^2 < 24^2+25$$

$$a < \sqrt{\frac{24^2+24}{25}} = \sqrt{24} \approx 4.9$$

$$\boxed{a < \sqrt{24} \approx 4.9}$$

(c) [5 marks] The spring is placed at the seatback $d_k = l = 0.3m$. Where should the damper be placed so that the system is critically damped?



$$10.225\ddot{\theta} + 800d_c^2\dot{\theta} + 1000d_k^2\theta = 22.7a$$

System is critically damped when

$$(800d_c^2)^2 - 4(10.225)(1000d_k^2) = 0$$

$$\Rightarrow 800d_c^2 = \sqrt{4(10.225)(1000)(0.3)^2}$$

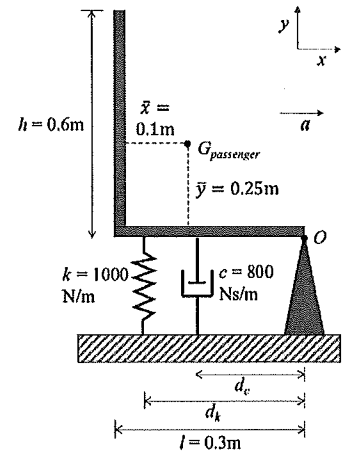
$$d_c = \sqrt{\frac{\sqrt{4(10.225)(1000)(0.3)^2}}{800}}$$

$$\boxed{d_c \approx 0.275\,m}$$

(d) [8 marks] The spring is placed at the seatback $d_k = l = 0.3m$, and the damper is placed at $d_c = 0.26m$. The train accelerates at the rate

$$a(t) = 2.6te^{-0.1t}$$

(in m/s^2) so that train nearly reaches 1G ($9.81m/s^2$) after 10 seconds and then the train reaches a steady speed after 60 seconds. Find the general solution of the equation of motion in part (b).



$$10.225\ddot{\theta} + 800d_c^2\dot{\theta} + 1000d_k^2\theta = 22.7a$$

$$\Rightarrow \quad 10.225\ddot{\theta} + 54.08\dot{\theta} + 90\theta = 59.02\, te^{-0.1t}$$

Complimentary solution: $p(s) = 10.225s^2 + 54.08s + 90$

$$s = \frac{-54.08 \pm \sqrt{54.08^2 - 4(10.225)(90)}}{2(10.225)} = -2.644 \pm 1.345i$$

$$\boxed{\Rightarrow \theta_c(t) = C_1 e^{-2.644t}\cos(1.345t) + C_2 e^{-2.644t}\sin(1.345t)}$$

Particular solution: $\theta_p(t) = (At+B)e^{-0.1t}$

$$\dot{\theta}_p(t) = Ae^{-0.1t} - 0.1(At+B)e^{-0.1t}$$
$$= (A - 0.1B)e^{-0.1t} - 0.1Ate^{-0.1t}$$

$$\ddot{\theta}_p(t) = -0.1(A-0.1B)e^{-0.1t} - 0.1Ae^{-0.1t} + 0.1^2Ate^{-0.1t}$$
$$= (-0.2A + 0.1^2B)e^{-0.1t} + 0.1^2Ate^{-0.1t}$$

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$$10.225\ddot{\theta}_p + 54.08\dot{\theta}_p + 90\theta_p$$

$$= \Big((10.225)(0.1^2 A) + 54.08(-0.1A) + 90A \Big) t e^{-0.1t}$$

$$+ \Big((10.225)(-0.2A + 0.1^2 B) + (54.08)(A - 0.1B) + 90B \Big) e^{-0.1t}$$

$$= 59.02\, t e^{-0.1t}$$

$$\Rightarrow \quad 84.694 A = 59.02 \quad \Rightarrow \quad A = 0.697$$

$$\Rightarrow \quad (10.225)(-0.2)(0.697) + (54.08)(0.697) + 84.694 B = 0$$

$$\Rightarrow B = -0.428$$

$$\boxed{\Rightarrow \quad \theta_p(t) = (0.697t - 0.428)e^{-0.1t}}$$

$$\theta(t) = \theta_c(t) + \theta_p(t)$$