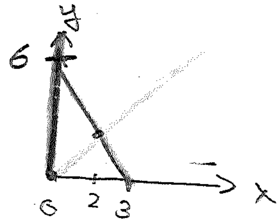
Worksheet 2 - Solution

Problemset 1: 1.



(1) ~~x=0~~
(2) $x=y$
(3) $2x+y=6 \iff y=6-2x$

(2) and (3) intersect when $x \overset{(2)}{=} y \overset{(3)}{=} 6-2x \iff 6=3x \iff 2=x$

First, calculate m:

$$m = \int_0^2 \int_x^{6-2x} \underbrace{x^2}_{p(x,y)} \, dy\, dx = \int_0^2 x^2 \int_x^{6-2x} 1 \, dy \, dx = \int_0^2 \underbrace{x^2 \underbrace{(6-2x-x)}_{6-3x}}_{=6x^2-3x^3} dx$$

$$= \left[2x^3 - \frac{3}{4}x^4\right]_0^2 = 16 - 3\cdot 4 = 4$$

Now, calculate the mean in x direction

$$\bar{x} = \frac{1}{m} \int_0^2 \int_x^{6-2x} \underbrace{x^2}_{p(x,y)} \cdot \underbrace{x}_{\substack{\text{averaging} \\ \text{in x-dir.}}} \, dy\, dx = \frac{1}{m} \int_0^2 \underbrace{x^3(6-3x)}_{6x^3-3x^4} dx = \frac{1}{m}\left[\frac{3}{2}x^4 - \frac{3}{5}x^5\right]_0^2$$

$$= \frac{1}{4}\left[24 - \frac{3\cdot 32}{5}\right] = 6 - \frac{3}{5}\cdot 8 = \frac{30}{5} - \frac{24}{5} = 1.2$$

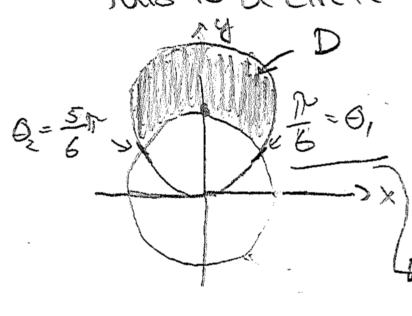
$$\bar{y} = \frac{1}{m} \int_0^2 \int_x^{6-2x} x^2 \cdot y \, dy\, dx = \frac{1}{m}\int_0^2 x^2 \left[\frac{1}{2}y^2\right]_x^{6-2x} dx = \frac{1}{m}\int_0^2 \underbrace{\frac{x^2}{2}(36-24x+\underbrace{4x^2-x^2}_{3x^2})}_{=18x^2-12x^3+\frac{3}{2}x^4} dx$$

$$= \frac{1}{4}\left[6x^3 - 3x^4 + \frac{3}{10}x^5\right]_0^2 = \frac{1}{4}\left[\cancel{48} - \cancel{48} + \frac{3}{10}\cdot 32\right] = 2.4.$$

The center of mass is $(1.2, 2.4)$.

2. $x^2+y^2=2y \overset{\substack{\text{Completing} \\ \text{square}}}{\iff} x^2+(y-1)^2-1=0 \iff x^2+(y-1)^2=1$

This is a circle with radius 1 around $(0,1)$.



Idea: Use polar coordinates:

Lower bound on radius: 1

Upper bound: $x^2+y^2-2y = r^2(\cos(\theta)^2+\sin(\theta)^2) - 2r\sin(\theta) = r^2 - 2r\sin(\theta)$

These bounds come frome $\underset{\substack{\uparrow \\ \text{lower circle}}}{1} = x^2+y^2 = \underset{\substack{\uparrow \\ \text{upper}}}{2y} \Rightarrow y=\frac{1}{2} \Rightarrow \sin(\theta)=\frac{1}{2}$

We are interested in the curve that describes the upper radial bound. This is

$r^2 - 2r\sin(\theta) = 0$ with $r \geq 1$ and $\theta \in (\frac{\pi}{6}, \frac{5\pi}{6})$

We can solve for r:

$r = 2\sin(\theta)$ (≥ 1 as $\sin(\theta) \geq 1/2$ in bounds)

We have to transform the corresponding integrals into polar coordinates. Let D be the domain of the laminar. Then

$$m = \iint_D \underbrace{\frac{1}{\sqrt{x^2+y^2}}}_{p(x,y)} d(x,y) = \int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \int_1^{2\sin(\theta)} \frac{1}{r} \cdot \underbrace{r}_{\text{transformation to polar coordinates}} \, dr \, d\theta$$

$\sqrt{x^2+y^2} = r$ (distance to center)

$$= \int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} (2\sin(\theta) - 1) \, d\theta = \left[-2\cos(\theta) - \theta\right]_{\frac{\pi}{6}}^{\frac{5\pi}{6}} = \left[-2\left(-\frac{\sqrt{3}}{2}\right) - \frac{5\pi}{6} + 2\left(\frac{\sqrt{3}}{2}\right) + \frac{\pi}{6}\right]$$

$$= 2\sqrt{3} - \frac{2\pi}{3} \approx 1.3697$$

Similarly, we set up

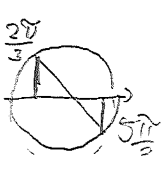
$$\bar{x} = \frac{1}{m} \iint_D \frac{1}{\sqrt{x^2+y^2}} \cdot x \, d(x,y).$$

But here we can observe immediately that $\frac{x}{\sqrt{x^2+y^2}}$ is an odd function in x. Therefore, the weight is centered in 0 (symmetry argument)

$\bar{x} = 0$.

$$\bar{y} = \frac{1}{m} \iint_D \frac{1}{\sqrt{x^2+y^2}} y \, d(x,y) \overset{\text{Polar Coord.}}{=} \frac{1}{m} \int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \int_1^{2\sin(\theta)} \underbrace{r}_{\text{transform}} \cdot \frac{1}{r} \cdot \underbrace{r\sin(\theta)}_{y = r\sin(\theta)} \, dr \, d\theta = \frac{1}{m} \int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \underbrace{\left[\frac{1}{2} r^2\right]_1^{2\sin(\theta)}}_{= \frac{1}{2}(4\sin(\theta)^2 - 1)} \sin(\theta) \cdot 1 \, d\theta$$

$$= \frac{1}{m} \int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} 2\sin(\theta)^3 - \frac{1}{2}\sin(\theta) \, d\theta = \frac{1}{m}\sqrt{3} \approx 1.2645$$



(extra page: there was a simpler way to solve the integral: see last page.)

Problemset 2: Cylindrical Coordinates

1. $\displaystyle\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^2 \int_0^{r^3} r \, dz\, dr\, d\theta$

From outside to inside:

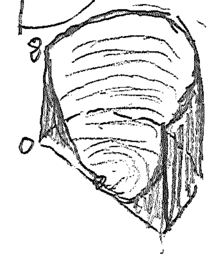
$-\frac{\pi}{2} \le \theta \le \frac{\pi}{2} \Rightarrow$

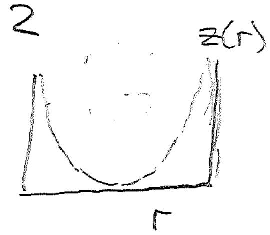
right half circle

$0 \le r \le 2 \Rightarrow$

radius 2

$0 \le z \le r^3 \Rightarrow$





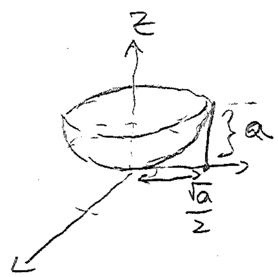
Evaluation:

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^2 \int_0^{r^3} r\, dz\, dr\, d\theta = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 1 \cdot d\theta \cdot \int_0^2 r \cdot \underbrace{[z]_0^{r^3}}_{(r^3-0)} dr\, d\theta = \pi \cdot \left[\frac{1}{5} r^5\right]_0^2$$

$$= \frac{32}{5}\pi$$

2. $\overset{(1)}{z = 4x^2 + 4y^2}$; $\overset{(2)}{z = a}$ $\quad a > 0$

$\Leftrightarrow z = 4 \cdot (x^2 + y^2)$



Transform into cylindrical coordinates:

Conditions:

(1) $z = 4 \cdot (r^2\cos^2(\theta) + r^2 \sin^2(\theta)) = 4r^2$

(2) $z = a$

$4r^2 = a$ if $r = \frac{\sqrt{a}}{2}$ (here the paraboloid and the plane intersect)

We integrate over: $0 \le \theta \le 2\pi$ (full circle)

Over radii in $0 \le r \le \frac{\sqrt{a}}{2}$ Domain D

and height: $4r^2 \le z \le a$

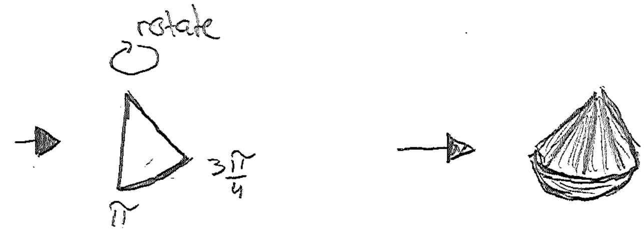
Transformation factor

$$\iiint_D 1 \, d(x,y,z) \overset{\text{cylinder coord}}{=} \int_0^{2\pi} 1 \, d\theta \cdot \int_0^{\frac{\sqrt{a}}{2}} r \int_{4r^2}^{a} 1 \, dz\, dr = 2\pi \cdot \int_0^{\frac{\sqrt{a}}{2}} [ar - 4r^3]\, dr$$

$$= 2\pi \left[\frac{ar^2}{2} - r^4\right]_0^{\frac{\sqrt{a}}{2}} = 2\pi \underbrace{\left[\frac{a^2}{8} - \frac{a^2}{16}\right]}_{\frac{a^2}{16}} = \frac{\pi}{8} a^2$$

Problemset 3:

1. $\frac{3\pi}{4} \leq \phi \leq \pi$



2. $$\iiint_{\text{Ball(radius a)} = B_a} \sqrt{x^2+y^2+z^2}\, d(x,y,z) \overset{\substack{\text{Spherical}\\ \text{coordinates}}}{=} \int_0^{2\pi}\int_0^{\pi}\int_0^{a} \rho \cdot \rho^2 \sin(\phi)\, d\rho\, d\phi\, d\theta$$

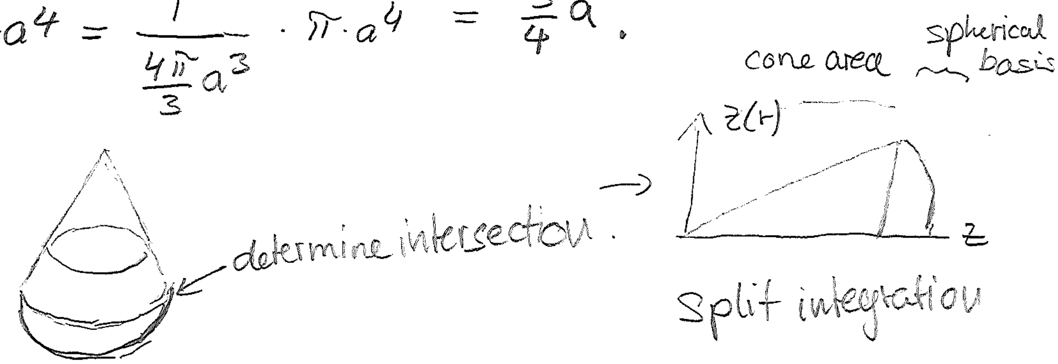
$\rho = \sqrt{x^2+y^2+z^2}$ (radius of ball

$$= \underbrace{\int_0^{2\pi} 1\, d\theta}_{=2\pi} \cdot \underbrace{\int_0^{\pi} \sin(\phi)\, d\phi}_{\substack{[-\cos(\phi)]_0^{\pi} \\ = 1+1=2}} \cdot \underbrace{\int_0^{a} \rho^3\, d\rho}_{\frac{1}{4}a^4} = \pi \cdot a^4$$

Then, the average distance is given by:

$$\frac{1}{V(B_a)} \cdot \pi \cdot a^4 = \frac{1}{\frac{4\pi}{3}a^3} \cdot \pi \cdot a^4 = \frac{3}{4}a.$$

3. Idea: Visualize:



Appendix: Calculation 1.2.

$$\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \sin(\theta)^3 \, d\theta = \int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \sin(\theta)^2 \sin(\theta) \, d\theta \overset{\text{integr.}}{=} \left[\sin(\theta)^2 (-\cos(\theta)) \right]_{\frac{\pi}{6}}^{\frac{5\pi}{6}} + \int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} 2\sin(\theta)\cos(\theta)\cos(\theta) \, d\theta$$

$$= \left[\frac{1}{4} \cdot \frac{\sqrt{3}}{2} + \frac{1}{4}\frac{\sqrt{3}}{2} \right] + 2 \cdot \int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \sin(\theta)(1 - \sin(\theta)^2) \, d\theta$$

$$= \frac{\sqrt{3}}{4} + 2\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \sin(\theta) \, d\theta - 2 \cdot \int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \sin(\theta)^3 \, d\theta$$

$$\Rightarrow 3\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \sin(\theta)^3 \, d\theta = \frac{\sqrt{3}}{4} + 2 \cdot \int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \sin(\theta) \, d\theta$$

$$\Rightarrow \int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \sin(\theta)^3 \, d\theta = \frac{\sqrt{3}}{12} + \frac{2}{3} \int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \sin(\theta) \, d\theta$$

$$2\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \sin(\theta)^3 \, d\theta - \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \sin(\theta) \, d\theta = \frac{\sqrt{3}}{6} + \left(\underbrace{\frac{4}{3}}_{\frac{8}{6}} - \underbrace{\frac{1}{2}}_{\frac{3}{6}} \right) \underbrace{\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \sin(\theta) \, d\theta}_{\left[-\cos(\theta)\right]_{\frac{\pi}{6}}^{\frac{5\pi}{6}} = \frac{\sqrt{3}}{2} \cdot 2} = \frac{\sqrt{3}}{6} + \frac{5}{6}\sqrt{3} = \sqrt{3}$$