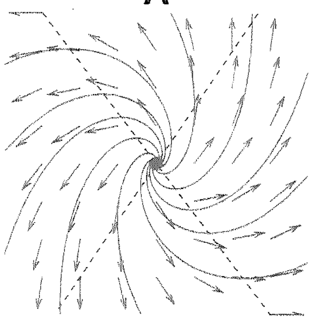
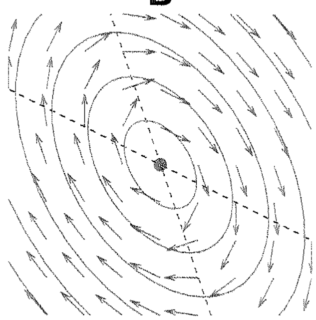
SA 1. [5 marks]

Each figure is the phase portrait of a linear system $\dot{\boldsymbol{x}} = A\boldsymbol{x}$ for some matrix A. Match each figure with the corresponding matrix in the table below.

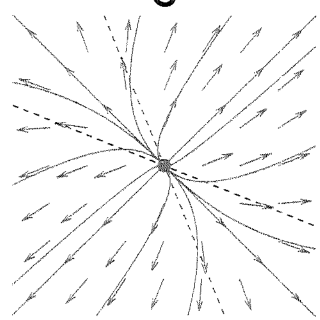
A



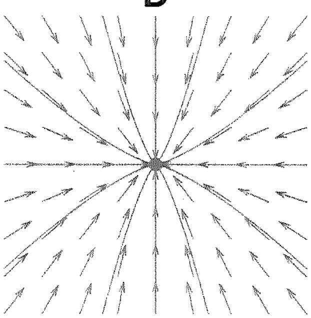
B



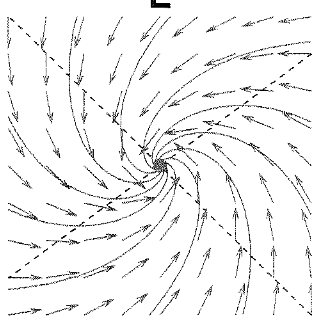
C



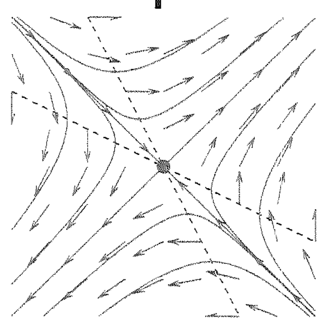
D



E



F



C	$\begin{bmatrix} 5 & 2 \\ 2 & 5 \end{bmatrix}$	F	$\begin{bmatrix} 2 & 4 \\ 4 & 2 \end{bmatrix}$
D	$\begin{bmatrix} -3 & 0 \\ 0 & -4 \end{bmatrix}$	A	$\begin{bmatrix} 3 & -2 \\ 4 & 3 \end{bmatrix}$
B	$\begin{bmatrix} 1 & 2 \\ -3 & -1 \end{bmatrix}$	E	$\begin{bmatrix} -3 & -3 \\ 3 & -4 \end{bmatrix}$

SA 2. [5 marks]

Find values a, b, c and d such that $x_1(t)$ is a solution of the linear system of equations $\dot{x} = Ax$ where

$$x_1(t) = \begin{bmatrix} 1 \\ 0 \\ 2 \\ -1 \end{bmatrix} e^{-3t} \qquad A = \begin{bmatrix} a & 1 & b & 2 \\ -1 & 1 & c & -1 \\ b & 2 & 1 & a \\ c & -1 & d & 1 \end{bmatrix}$$

We must have $A\vec{x}_1 = \dot{\vec{x}}_1$

$$A\vec{x}_1 = \begin{bmatrix} a & 1 & b & 2 \\ -1 & 1 & c & -1 \\ b & 2 & 1 & a \\ c & -1 & d & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 2 \\ -1 \end{bmatrix} e^{-3t} = \begin{bmatrix} a+2b-2 \\ -1+2c+1 \\ b+2-a \\ c+2d-1 \end{bmatrix} e^{-3t}$$

$$\dot{\vec{x}}_1 = \begin{bmatrix} -3 \\ 0 \\ -6 \\ 3 \end{bmatrix} e^{-3t} \Rightarrow \begin{aligned} a+2b-2 &= -3 \quad (1) \\ 2c &= 0 \quad (2) \\ -a+b+2 &= -6 \quad (3) \\ c+2d-1 &= 3 \quad (4) \end{aligned}$$

(1) and (3) $\Rightarrow$

$$\begin{aligned} a+2b &= -1 \\ -a+b &= -8 \\ \hline 3b &= -9 \end{aligned} \qquad \boxed{b=-3} \quad \boxed{a=5}$$

(2) and (4) $\Rightarrow$ $\boxed{c=0}$ $\boxed{d=2}$

Equivalently, we must have $\begin{bmatrix} 1 \\ 0 \\ 2 \\ -1 \end{bmatrix}$ is an eigenvector of A with eigenvalue $\lambda = -3$

$\Rightarrow A\begin{bmatrix} 1 \\ 0 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} -3 \\ 0 \\ -6 \\ 3 \end{bmatrix}$

(c) [2 marks] Write equations (1) and (2) in matrix form $\dot{\boldsymbol{v}} = A\boldsymbol{v}$ as a two-dimensional, first-order, linear, homogeneous system of equations.

$$\frac{dv_a(t)}{dt} + \frac{1}{C}\left(\frac{1}{R_1} + \frac{1}{R_2}\right)v_a(t) - \frac{1}{R_2 C}v_b(t) = 0 \qquad (1)$$

$$-\frac{dv_a(t)}{dt} + 2\frac{dv_b(t)}{dt} + \frac{R_2}{L}v_b(t) = 0 \qquad (2)$$

(1) $\Rightarrow$ $\boxed{\dot{V}_a = \frac{-1}{C}\left(\frac{1}{R_1} + \frac{1}{R_2}\right)V_a + \frac{1}{R_2 C}V_b}$

(2) $\Rightarrow$ $\dot{V}_b = \frac{1}{2}\dot{V}_a - \frac{R_2}{2L}V_b$

Plug (1) into (2)

$$\dot{V}_b = \frac{1}{2}\left(\frac{-1}{C}\left(\frac{1}{R_1} + \frac{1}{R_2}\right)V_a + \frac{1}{R_2 C}V_b\right) - \frac{R_2}{2L}V_b$$

$$\boxed{\dot{V}_b = \frac{-1}{2C}\left(\frac{1}{R_1} + \frac{1}{R_2}\right)V_a + \left(\frac{1}{2R_2 C} - \frac{R_2}{2L}\right)V_b}$$

$$\boxed{\Rightarrow \begin{bmatrix} \dot{V}_a \\ \dot{V}_b \end{bmatrix} = \begin{bmatrix} \frac{-1}{C}\left(\frac{1}{R_1} + \frac{1}{R_2}\right) & \frac{1}{R_2 C} \\ \frac{-1}{2C}\left(\frac{1}{R_1} + \frac{1}{R_2}\right) & \frac{1}{2R_2 C} - \frac{R_2}{2L} \end{bmatrix}}$$

(d) [8 marks] When $R_2 = 100\ \Omega$, $C = 0.01\ F$ and $L = 1\ H$, the system of equations is

$$\begin{bmatrix}\dot{v}_a\\ \dot{v}_b\end{bmatrix} = \begin{bmatrix} -\frac{100}{R_1} - 1 & 1 \\ -\frac{50}{R_1} - \frac{1}{2} & -49.5 \end{bmatrix} \begin{bmatrix} v_a \\ v_b \end{bmatrix}.$$

In this case, find a range of values for R_1 such that $v_a(t)$ and $v_b(t)$ oscillate as they decay to their steady-state values (*i.e.*, they exhibit an underdamped response).

Let $A = \begin{bmatrix} -\frac{100}{R_1} - 1 & 1 \\ -\frac{50}{R_1} - \frac{1}{2} & -49.5 \end{bmatrix}$ We need eigenvalues of A to be complex

Characteristic polynomial: $\det(A - \lambda I)$

$$= \lambda^2 - \left(-\frac{100}{R_1} - 1 - 49.5\right)\lambda + \left(49.5\left(\frac{100}{R_1} + 1\right) + \left(\frac{50}{R_1} + \frac{1}{2}\right)\right)$$

$$= \lambda^2 + \left(\frac{100}{R_1} + 50.5\right)\lambda + \left(\frac{5000}{R_1} + 50\right)$$

We must have $\left(\frac{100}{R_1} + 50.5\right)^2 - 4\left(\frac{5000}{R_1} + 50\right) < 0$

$$\Rightarrow (100 + 50.5R_1)^2 - 20000R_1 - 200R_1^2 < 0$$

$$\underbrace{2350.25R_1^2 - 9900R_1 + 10000 < 0}_{\text{find roots.}}$$

$$R_1 = \frac{9900 \pm \sqrt{9900^2 - 4(2350.25)(10000)}}{2(2350.25)}$$

$$= \frac{9900 \pm 2000}{4700.5} = 1.681,\ 2.532$$

$$\boxed{1.681 < R_1 < 2.532}$$

(e) [7 marks] When $R_1 = R_2 = 1\ \Omega$, $C = 1\ F$ and $L = 0.1\ H$, the system of equations is

$$\begin{bmatrix} \dot{v}_a \\ \dot{v}_b \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ -1 & -4.5 \end{bmatrix} \begin{bmatrix} v_a \\ v_b \end{bmatrix}.$$

Find the general solution in this case.

Let $A = \begin{bmatrix} -2 & 1 \\ -1 & -4.5 \end{bmatrix}$ Characteristic polynomial

$$\Rightarrow \det(A - \lambda I) = \lambda^2 - (-6.5)\lambda + (9+1)$$
$$= \lambda^2 + 6.5\lambda + 10$$

$$\Rightarrow \lambda = \frac{-6.5 \pm \sqrt{6.5^2 - 4(10)}}{2} = \frac{-6.5 \pm 1.5}{2}$$
$$= -4, -5/2$$

$\lambda_1 = -4 \quad (A - \lambda_1 I)\vec{v}_1 = \vec{0}$

$$\left[\begin{array}{cc|c} 2 & 1 & 0 \\ -1 & -0.5 & 0 \end{array}\right] \Rightarrow \vec{v}_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

$\lambda_2 = -5/2 \quad (A - \lambda_2 I)\vec{v}_2 = \vec{0}$

$$\left[\begin{array}{cc|c} 0.5 & 1 & 0 \\ -1 & -2 & 0 \end{array}\right] \Rightarrow \vec{v}_2 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

$$\Rightarrow \boxed{\vec{V}(t) = \begin{bmatrix} V_a(t) \\ V_b(t) \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ -2 \end{bmatrix} e^{-4t} + c_2 \begin{bmatrix} 2 \\ -1 \end{bmatrix} e^{-\frac{5}{2}t} \quad c_1, c_2 \in \mathbb{R}}$$