Worksheet 9

Felix Funk, MATH Tutorial - Mech 221

1 2D Phase Portraits

Introduction: Drawing Phase Portraits
A linear system of differential equations defined by

$$\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = A \cdot \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \tag{1}$$

can be illustrated by a 2D phase portrait. The eigenvalues and eigenvectors give us essential information about the stability and shape of phase portraits.

Real eigenvectors provide **lines** along which the trajectories enter or leave an equilibrium.

- If the corresponding eigenvalue is positive, then the corresponding solutions leave the equilibrium along the line.
- If the corresponding eigenvalue is negative, then the equilibrium attracts solutions on the line.
- If the eigenvalue is zero, then the whole line consists of equilibria.

When ~~eigenvectors and~~ eigenvalues are **complex**, then the **real component of the eigenvalues** tells, whether a solution

- spirals inwards; $Re(\lambda) < 0$,
- spirals outwards; $Re(\lambda) > 0$,
- or remains on an periodic orbit; $Re(\lambda) = 0$.

The orientation of the spirals and periodic orbits can be obtained by sampling directions using (1).

Problem: Phase Portraits.
Draw the phase portrait to

$$y' = \overbrace{\begin{bmatrix} 1 & 1 \\ 0 & -2 \end{bmatrix}}^{=A} y \tag{2}$$

Example: Obtaining the important information for real eigenvalues.

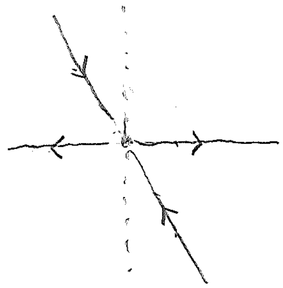
1. Determine the eigenvectors and eigenvalues.

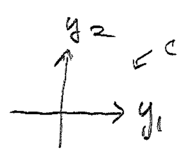
$$\lambda_1 = -2, v_1 = \begin{pmatrix} 1 \\ -3 \end{pmatrix}, \lambda_2 = 1 \quad , v_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

2. Draw the lines corresponding to the eigenvectors. Use the sign to determine the stability along the lines.

3. If you are missing information to complete the phase portrait, sample the directions of individual points, e.g. find the direction in $y = [0, 1]^T$ by calculating the derivative $y' = \begin{bmatrix} 1 & 1 \\ 0 & -2 \end{bmatrix} y$.

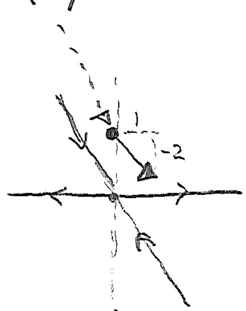
1. $(A + 2I) \cdot v = \begin{pmatrix} 3 & 1 \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ if $3v_1 + v_2 = 0 \qquad v_2 = -3v_1$

2. 

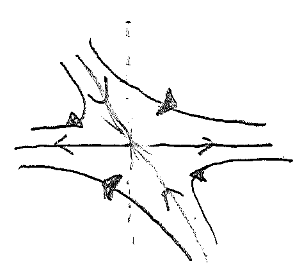


3. To obtain the directions in between, calculate for instance

$y = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$; $\dot{y} = \begin{bmatrix} 1 & 1 \\ 0 & -2 \end{bmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$ (that's the direction)



Fill that up



Problem: Phase Portraits 2.
Draw the phase portrait to

$$y' = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} y \tag{3}$$

Example: Obtaining the important information for complex eigenvalues.

1. Determine the eigenvectors and eigenvalues.

$$\lambda_1 = 1+2i, \cancel{v_1 = \begin{pmatrix} 1 \\ -i \end{pmatrix}}, \lambda_2 = 1-2i, \cancel{v_2 = \begin{pmatrix} 1 \\ i \end{pmatrix}}$$

2. If the eigenvalues are complex: Determine the real part of the eigenvalues and categorize whether the movement is inwards or outwards/ on periodic orbits.

3. To determine the orientation of the spirals or the movement on the periodic orbits, complete the phase portrait, sample the directions of individual points, e.g. find the direction in $y = [0, 1]^T$ by calculating the derivative $y' = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} y$.

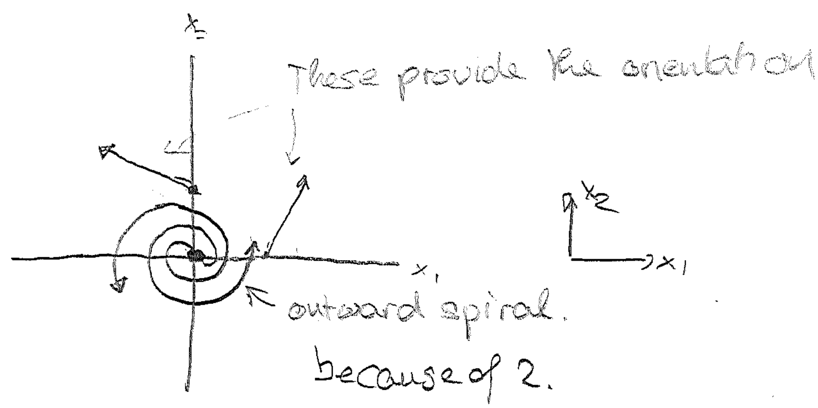
1. $(1-\lambda)^2+4 = 5-2\lambda+\lambda^2$ $\quad \lambda_{1,2} = \frac{2 \pm \sqrt{4-20}}{2} = \frac{2 \pm 4i}{2} = 1 \pm 2i$

2. Because $Re(\lambda_{1,2}) > 0$, we expect outward spirals.

3. Take a point to determine the orientation:

$$y_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \rightarrow \dot{y} = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

$$y_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \dot{y} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$



1.1 Mixed Problems

Problemset: Mixed problems.

1. Draw the phase portrait and use it to find the stability of the origin.

Two distinct real roots

$$x' = x + y \qquad \rightarrow \begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$
$$y' = x - y$$

2. Transform the equation into a linear system of first order differential equations $y''+4y = 0$. Then, draw a phase portrait.

Two imaginary roots

$y_1 = y$, $y_2 = y'$ $\Rightarrow \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ -4 & 0 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$

3. Draw the phase portrait and use it to find the stability of the origin.

Two distinct real

$$x' = x + y \qquad \begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$
$$y' = 2y$$

4. Draw the phase portrait and use it to find the stability of the origin.

Two complex roots

$$y' = \begin{bmatrix} -1 & -1 \\ 1 & -1 \end{bmatrix} y \qquad (4)$$

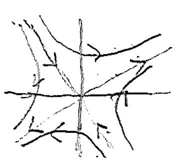
5. Challenging: Draw the phase portrait and use it to find the stability of the origin.

Defective eigenvalues.

$$y' = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} y \qquad (5)$$

1. $(1-\lambda)(-1-\lambda) - 1 = -1 + \lambda^2 - 1 = \lambda^2 - 2 = (\lambda-\sqrt{2})(\lambda+\sqrt{2})$, $1 + v_2 = \sqrt{2}$ $\Rightarrow v_2 = \sqrt{2} - 1 > 0$

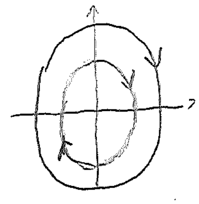
$1 + v_2 = -\sqrt{2}$ $\Rightarrow v_2 = -\sqrt{2} - 1$



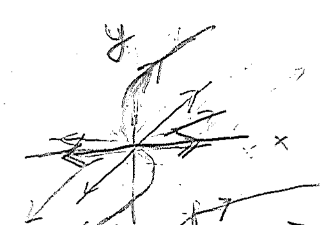
$\rightarrow$ unstable

2. $y'' + 4y = 0 \iff y_1' = y_2,\ y_2' = -4y_1 \Rightarrow \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ -4 & 0 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \rightarrow \lambda_{1,2} = \pm 2i$

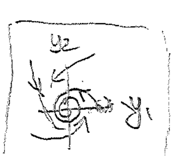
$\begin{pmatrix} 0 & 1 \\ -4 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -4 \end{pmatrix}$ $\qquad Re(\lambda_{1,2}) = 0$ (moves periodically)



~~stable~~



3. $\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \Rightarrow \lambda_1 = 1,\ v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix},\ \lambda_2 = 2\ \ v_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$



4. $(-1-\lambda)^2 + 1 = \lambda^2 + 2\lambda + 2 \rightarrow \frac{-2 \pm \sqrt{4-8}}{2}$

$\rightarrow Re(\lambda) < 0$ (spirals inwards) $\Rightarrow$ stable $\begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

5. $\lambda_{1,2} = 1;\ v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ (Calculating solutions or plugging in