Worksheet 3

Felix Funk, MATH Tutorial - Mech 221

1 Second Order Linear ODEs

Introduction: Second Order Homogeneous Linear ODEs.
Second order linear ODEs are a powerful tool to model oscillatory systems such as mass-spring systems, electrical circuits and vibrations. We focus more specifically on homogeneous ODEs with constant coefficients, i.e.

$$ay'' + by' + cy = 0 \text{ with } a \neq 0, b, c \text{ in } \mathbb{R}. \tag{1}$$

To solve these equations, one derives the so-called characteristic equation and analyzes its properties in the following steps:

1. Set $y(t) = e^{rt}$ with constant r and substitute into equation (1). One obtains the following equation:

   $$ar^2e^{rt} + bre^{rt} + ce^{rt} = 0 \quad //: e^{rt}$$

   $$ar^2 + br + c = 0 \tag{2}$$

   This equation is called characteristic equation.

2. The roots of this equation determines essentially the solutions of the system. We are going to differentiate the following three cases.

   (a) There are two distinct real roots r_1, r_2 such that $r_1 \neq r_2$.

   (b) There are two imaginary roots $r_1 = \mu + i\omega, r_2 = \mu - i\omega$ such that $\omega > 0$.

   (c) The two real roots coincide $r_1 = r_2$.

3. In the given cases there are two solutions of the following form:

   (a) Exponential growth or decay: $y_1(t) = e^{r_1 t}$ and $y_2(t) = e^{r_2 t}$,

   (b) Oscillatory motion: $y_1(t) = e^{\mu t} cos(\omega t)$ and $y_2(t) = e^{\mu t} sin(\omega t)$,

   (c) Amplified exponential growth/decay: $y_1(t) = e^{rt}$ and $y_2(t) = \overset{t}{\cancel{x}} e^{rt}$. ← typo

4. The general solution is then a superposition of the two solution, i.e.

   $$y(t) = \alpha y_1(t) + \beta y_2(t) \tag{3}$$

5. If applicable, one can solve for α and β through the corresponding initial value problem $y(t_0) = y_0$ and $y'(t_0) = y_1$.

In the following subsections we have a closer look at the three cases.

1.1 Two Distinct Real Roots

Problem: Model Problem.
Solve the IVP

$$y'' + 5y' - 6y = 0 \tag{4}$$

with the constraints $y(0) = 1, y'(0) = 1$.

Example: Two Distinct Real Roots.

1. Identify the characteristic equation:

$$r^2 + 5r - 6 = 0$$

2. The two distinct roots are

$$r_{1,2} = \frac{-5 \pm \sqrt{25 + 4 \cdot 6}}{2} = -\frac{5}{2} \pm \frac{7}{2}$$

$$r_1 = 1 \quad , r_2 = -6$$

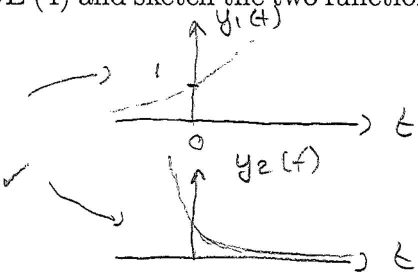
3. Consider

$$y_1(t) = e^{r_1 t}, y_2(t) = e^{r_2 t}.$$

Show that the provided solutions indeed solve the ODE (4) and sketch the two functions.
Sketch $y_1(t)$ and $y_2(t)$

$e^t : e^t + 5e^t - 6e^t = 0 \checkmark$

$e^{-6t} : (36) e^{-6t} - 30e^{-6t} - 6e^{-6t} = 0 \checkmark$



4. The general solution is then

$$y(t) = \alpha e^{t} + \beta e^{-6t}$$

5. If applicable, use the IVP to solve for α and β

$1 = y(0) = \alpha + \beta$

$1 = y'(0) = \alpha - 6\beta$

$\alpha + \beta = \alpha - 6\beta \Rightarrow 5\beta = 0 \Rightarrow \beta = 0$

$1 = \alpha + 0 = \alpha$

$y(t) = e^t$

Problem: Problemset 1.
Find the general solution and, if provided, solve the IVP

(1) • $y'' - 9y = 0$,

(2) • $y'' + 5y' = 0$ under the constraint $y(0) = 1, y'(1) = 0$.

(1) $y'' - 9y = 0$

characteristic equation: $r^2 - 9 = 0 \overset{\text{3rd Binomial law}}{\Longrightarrow} (r-3)(r+3) = 0$

Hence, $y_1(t) = e^{r_1 t} = e^{3t}$, $y_2(t) = e^{r_2 t} = e^{-3t}$

$y(t) = \alpha e^{3t} + \beta e^{-3t}$

(2) $y'' + 5y' = 0$

characteristic equation: $r^2 + 5r = 0 \Rightarrow r \cdot (r+5) = 0$

Hence, $y_1(t) = e^{0t} = 1$, $y_2(t) = e^{-5t}$

$y(t) = \alpha + \beta e^{-5t}$

$1 = y(0) = \alpha + \beta$

$0 = y'(1) = -5\beta e^{-5} \Rightarrow \beta = 0 \Rightarrow \alpha = 1$

$y(t) = 1$

1.2 Two Imaginary Roots

Problem: Model Problem.
Solve the IVP

$$4y'' + 4y' + \underset{4}{5}y = 0 \tag{5}$$

with the constraints $y(0) = 0, y'(0) = k$.

Example: Two imaginary roots.

1. Identify the characteristic equation:

$$4r^2 + 4r + 5 = 0$$

2. The two imaginary roots are

$$r_{1,2} = \frac{-4 \pm \sqrt{16 - 20}}{8} = -\frac{1}{2} \pm \frac{\sqrt{-4}}{8}$$

$$= -\frac{1}{2} \pm \frac{2i}{8} = -\frac{1}{2} \pm \frac{1}{4}i$$

$$r_1 = \underbrace{-\frac{1}{2}}_{=\mu} + \underbrace{\frac{1}{4}}_{=\omega}i, \; r_2 = -\frac{1}{2} - \frac{1}{4}i$$

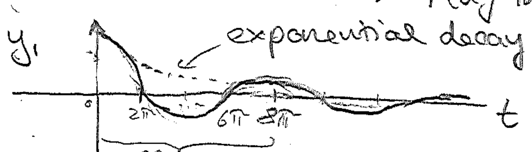
3. Consider

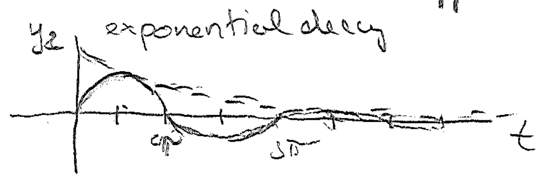
$$y_1(t) = e^{\mu t} cos(\omega t), y_2(t) = e^{\mu t} sin(\omega t). \qquad \mu = -\frac{1}{2} \quad \omega = \frac{1}{4}$$

oscillatory motions

Show that the provided solutions indeed solves the ODE (5). Sketch $y_1(t)$ and $y_2(t)$

+Take derivatives -> Plug into (5) or clever solution in Appendix





4. The general solution is then

$$y(t) = \alpha e^{-\frac{1}{2}t} \cos(\tfrac{1}{4}t) + \beta e^{-\frac{1}{2}t} \sin(\tfrac{1}{4}t)$$

5. If applicable, use the IVP to solve for α and β

$$0 = \alpha \cos(\tfrac{1}{4}0) = \alpha \Rightarrow \alpha = 0$$

$$k = \sin(\tfrac{1}{4}0) \cdot \ldots + \beta \cdot \left(-\tfrac{1}{2} e^{-\frac{1}{2}\cdot 0} \underbrace{\sin(\tfrac{1}{4}\cdot 0)}_{=0} + \underbrace{e^{-\frac{1}{2}\cdot 0}}_{=1} \underbrace{\cos(\tfrac{1}{4}\cdot 0)\cdot \tfrac{1}{4}}_{=1}\right)$$

$$= \beta/4 \Rightarrow \beta = 4k$$

$$y(t) = 4k e^{-\frac{1}{2}t} \sin(\tfrac{1}{4}t)$$

Problem: Problemset 2.

Find the general solution and, if provided, solve the IVP

(1) • $2y'' - 4y' + 4y = 0$ under the constraint $y(0) = 0, y'(0) = 0,$

(2) • $y'' + \omega^2 y = 0$ under the constraint $y(0) = -1, y'(0) = 1.$

(1) $\Leftrightarrow y'' - 2y' + 2y = 0$

characteristic eq.: $r^2 - 2r + 2 = 0$

$$r_{1,2} = \frac{2 \pm \sqrt{4-8}}{2} = \underbrace{1}_{\mu} \pm \underbrace{i}_{\omega \cdot i} \Rightarrow \omega = 1$$

$$y(t) = e^t \cdot (\alpha \cos(t) + \beta \sin(t))$$

$0 = y(0) = \alpha \quad \Rightarrow \alpha = 0$

$0 = y'(0) = \underbrace{e^0}_{=1} \beta \underbrace{\cos(0)}_{=1} + \underbrace{\sin(0)}_{=0} \cdot [\ldots] = \beta \Rightarrow \beta = 0$

$y(t) = 0$

(2) $y'' + \omega^2 y = 0 \quad \Rightarrow \quad r^2 + \omega^2 = 0 \quad \Rightarrow (r + i\omega)(r - i\omega) = 0$

$y(t) = \alpha \cos(\omega t) + \beta \sin(\omega t)$

$-1 = y(0) = \alpha \quad \Rightarrow \alpha = -1$

$1 = y'(0) = \alpha\omega \underbrace{(-\sin(\omega t))}_{=0} + \beta\omega \underbrace{\cos(\omega t)}_{=1} \Big|_{t=0} = \beta\omega \Rightarrow \beta = \frac{1}{\omega}$

$$y(t) = -\cos(\omega t) + \frac{1}{\omega} \cdot \sin(\omega t)$$

1.3 A Single Real Root

Problem: Model Problem.
Solve the IVP

$$y'' + 2y' + y = 0, \tag{6}$$

with the constraints $y(0) = 1, y'(0) = 1$.

Typo

Example: A Single Real Root.

1. Identify the characteristic equation:

$$r^2 + 2r + 1 = 0$$

2. The real root is

$$(r+1)^2 = 0$$

$$r = -1$$

3. Consider

$$y_1(t) = e^{rt}, \quad y_2(t) = te^{rt}$$

Show that the provided solutions indeed solve the ODE (6).

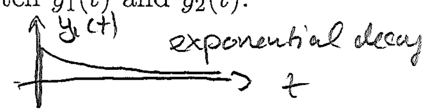
$$(-1)^2 e^{-t} + 2\cdot(-1)\, e^{-t} + e^{-t} = 0;$$

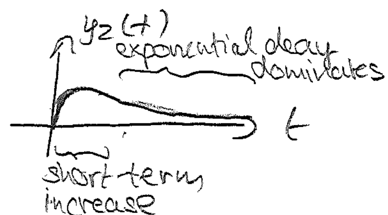
$$y_2'(t) = e^{-t}(1-t)$$

$$y_2''(t) = -e^{-t} - e^{-t}(1-t) = -e^{-t}\cdot(2-t)$$

$$y_2'' + 2y_2' + y_2 = -e^{-t}(2-t) + e^{-t}(1-t)\cdot 2 + te^{-t} = e^{-t}\cdot(-2+t+2-2t+t) = 0$$

Sketch $y_1(t)$ and $y_2(t)$.





4. The general solution is then

$$y(t) = \alpha e^{-t} + \beta\, te^{-t}$$

5. If applicable, use the IVP to solve for α and β

$$\boxed{1 = y(0) = \alpha e^{-0} + 0 = \alpha}$$

$$1 = y'(0) = -\alpha e^{-0} + \beta[e^{-0} - 0\cdot e^{-0}] = -\alpha + \beta = -1 + \beta$$

$$\boxed{\beta = 2}$$

$$y(t) = e^{-t} + 2te^{-t} = (1+2t)e^{-t}$$

Problem: Problemset 3.
Find the general solution and, if provided, solve the IVP

(1)• $2y'' - 4y' + 2y = 0$,

(2)• $y'' + 6y' + 9y = 0$ with $y(0) = 1, y'(0) = 1$.

(1) $y'' - 2y' + y = 0$

char. eq.: $r^2 - 2r + 1 = 0 \Rightarrow (r-1)^2 = 0 \Rightarrow r = 1$

$y(t) = \alpha e^t + \beta t e^t$

(2) $y'' + 6y' + 9y = 0 \Rightarrow r^2 + 6r + 9 = 0 \Rightarrow (r+3)^2 = 0 \Rightarrow r = -3$

$y(t) = \alpha e^{-3t} + \beta t e^{-3t}$

$1 = y(0) = \alpha \Rightarrow \alpha = 1$

$1 = y'(0) = -3\alpha + \beta \cdot (\underbrace{e^{-3\cdot 0}}_{=1} + \underbrace{0\cdot(-3)e^{-3\cdot 0}}_{=0}) = -3\alpha + \beta = -3 + \beta$

$\Rightarrow \beta = 4$

$y(t) = e^{-3t} + 4te^{-3t} = (1+4t)e^{-3t}$

2 Challenging Problems

Problem: Mixed Problems. Let a, b, c, C_1, C_2 be constants.

1. Provided is the ODE
$$y'' + ay' + y = 0.$$
For what range of values in a does the system allow oscillations?

2. Construct a second order ODE which has a general solution of the form
$$y(x) = C_1 e^{-2x} cos(3x) + C_2 e^{-2x} sin(3x).$$

3. Construct a second order ODE which has a general solution of the form
$$y(t) = C_1 e^{-at} + C_2 e^{-at} t.$$

4. A ~~harmonic~~ harmonic oscillator is described through the equation
$$ay'' + by' + cy = 0.$$
How many measurements are necessary to determine $y(t)$, uniquely? Are you sure?

1. Look at char. eq.: $r^2 + ar + 1 = 0 \Rightarrow r_{1,2} = \frac{-a \pm \sqrt{a^2-4}}{2}$

$\sqrt{a^2-4}$ is imaginary if and only if $|a| < 2$.
Only for imaginary roots does the system allow oscillatory solutions, i.e cosine / sine solutions. $\Rightarrow a$ in $(-2, 2)$

2. e^{-2x} ↳ $e^{\mu x} \leadsto \mu = -2$; $\underbrace{\cos(3x)}$ ↳ $\cos(\omega x)$ $\omega = 3$ } by comparison.

Hence, $r_{1,2} = \mu \pm \omega = -2 \pm 3i$ — 3rd binomial

$$(r - r_1)(r - r_2) = (\underbrace{r+2}_{a} - \underbrace{3i}_{b})(\underbrace{r+2}_{a} + \underbrace{3i}_{b}) \overset{\checkmark}{=} (r+2)^2 - (\underbrace{3i}_{=-1})^2$$
$$= r^2 + 4r + 4 + 9 = r^2 + 4r + 13 \Rightarrow y'' + 4y' + y = 0$$

3. Amplified exponential growth: Comparison $r_{1,2} = (-a)$
Hence, $(r + a)^2 = 0$ is the characteristic eq.: $r^2 + 2ar + a^2 = 0$
$\Rightarrow y'' + 2ay' + a^2 = 0$

4. Question:

(*) $ay'' + by' + cy = 0$ describes a system that is capable of harmonic oscillations (i.e undampened/non-increasing periodic solutions.) What are the restrictions on a, b, c such that $y(t)$ exerts nontrivial periodic oscillations? How many measurements (i.e initial cond. + additional info) are sufficient to uniquely determine the solution to (*)?

Answer: 3 measurements. (2 initial + 1 additional)

Let' assume $a = 0$, then $by' + cy = 0$.

If $\begin{cases} b = 0, & \text{then } cy = 0 \text{ (does not allow oscillations)} \\ b \neq 0, & \text{then } y' = -\frac{c}{b} y \quad (y(t) = e^{-\frac{c}{b}t}) \text{ (does not allow oscillations)} \end{cases}$

Therefore, we can divide by a and substitute $\hat{b} = \frac{b}{a}$, $\hat{c} = \frac{c}{a}$

$$y'' + \hat{b} y' + \hat{c} y = 0$$ ⇝ a does not affect model (can be dropped)

The characteristic equation yields: $r^2 + \hat{b} r + \hat{c} = 0$

$$r_{1,2} = \frac{-\hat{b} \pm \sqrt{\hat{b}^2 - 4\hat{c}}}{2} = \underbrace{-\frac{\hat{b}}{2}}_{=\hat{\mu}} \pm \frac{\sqrt{\hat{b}^2 - 4\hat{c}}}{2}$$

For harmonic oscillations, $y(t)$ should not vary exponentially in t, i.e $e^{\hat{\mu} t}$. Therefore $\hat{\mu} = 0$, $\Leftrightarrow \hat{b} = 0$.

The only remaining parameter is $\hat{c}$, which determines

$\frac{\sqrt{-4\hat{c}}}{2} = \sqrt{-\hat{c}} = i\omega$, $\omega = \sqrt{\hat{c}}$. The occurence of oscillations restricts $\hat{c}$ to positive constants.

The solution is then. $y(t) = \alpha \cos(\omega t) + \beta \sin(\omega t)$.

For three free parameters at least three measurements have to be made.

Three measurements ar enough.

With initial conditions $y(0) = y_0$, $y'(0) = 0$

$\boxed{y_0 = y(0) = \alpha}$, $\boxed{0 = y'(0) = \omega\beta \Rightarrow \beta = 0}$

Another measurement could be the time of first return to 0, i.e $y(t_1) = 0$

$0 = y(t_1) = y_0 \cos(\omega t_1) \Rightarrow \cos(\omega t_1) = 0 \Rightarrow \omega t_1 = \frac{\pi}{2} \Rightarrow \boxed{\omega = \frac{\pi}{2} \frac{1}{t_1}}$

Appendix:

// Showing that $y_1(t) = \cos(\omega t)e^{\mu t}$ and $y_2(t) = \sin(\omega t)e^{\mu t}$ solutions.

From characteristic equations we know that

$y_1^*(t) = e^{r_1 t}$ and $y_2^*(t) = e^{r_2 t}$ are solutions to $\boxed{ay'' + by' + cy = 0}$ (*)

with $r_1 = \mu + i\omega$ $\quad r_2 = \mu - i\omega$

Using the Euler-formula: $e^{i\theta} = \cos(\theta) + i\sin(\theta)$, we expand:

$$y_1^*(t) = e^{r_1 t} = \underbrace{e^{(\mu + i\omega)t}}_{\to e^{\mu t}\cdot e^{i\omega t}} = e^{\mu t}[\cos(\omega t) + i\sin(\omega t)]$$

$$y_2^*(t) = e^{r_2 t} = \underbrace{e^{(\mu - i\omega)t}}_{\to e^{\mu t}\cdot e^{-i\omega t}} = e^{\mu t}[\cos(\omega t) - i\sin(\omega t)]$$

(here we use y-axis symm of cos: $\cos(-\theta) = \cos(\theta)$ and symmetry to origin of sin: $\sin(-\theta) = -\sin(\theta)$)

The linear homogeneous ODE (*) satisfies the two properties:

(i) If $x_1(t)$ and $x_2(t)$ solve (*), then so does $y(t) = x_1(t) + x_2(t)$:

Explanation

$y''(t) = x_1''(t) + x_2''(t)$; $\quad y'(t) = x_1'(t) + x_2'(t)$.

Hence, $ay'' + by' + cy = a\cdot(x_1'' + x_2'') + b(x_1' + x_2') + c(x_1 + x_2)$

$$= \underbrace{ax_1'' + bx_1' + cx_1}_{\text{solution to (*)} \ =0} + \underbrace{ax_2'' + bx_2' + cx_2}_{\text{solution to (*)} \ =0}$$

$= 0$, which is why y is a solution.

(ii) If $x(t)$ solves (*), then so does $y(t) = d\cdot x(t)$ (for any const d)

↳ this could be real or complex)

Explanation

$y''(t) = dx''(t)$ $\quad y'(t) = dx'(t)$

Hence, $ay'' + by' + cy = a\,dx'' + b\,dx' + c\,dx$

$$= d\cdot(\underbrace{ax'' + bx' + cx}_{\text{solution to (*)} \ =0}) = 0,$$

which is why y is a solution.

Then $y_1(t) = \frac{1}{2}[y_1^*(t) + y_2^*(t)] = e^{\mu t}\cos(\omega t)$ is a solution

(because: $y^*(t) \overset{\text{Def}}{=} y_1^*(t) + y_2^*(t)$ solves (*) by (i) and $y_1(t) = \underbrace{\tfrac{1}{2}}_{=c} y^*(t)$ solves (*) by (ii)

Similarly, $y_2(t) = \frac{1}{2i}[y_1^*(t) - y_2^*(t)] = e^{\mu t}\sin(\omega t)$ is a solution to (*)