Worksheet 7

Felix Funk, MATH Tutorial - Mech 221

1 Transfer Functions and Steady State Response

Introduction: Application of Transfer Functions.
Transfer functions help us to understand the steady state response of many time invariant systems such as LRC circuits. The following theorem connects transfer functions and steady state analysis.

Theorem. Suppose an output signal y(t) is related to an input signal x(t) in a linear, time-invariant system and let H(s) denote the transfer function $H(s) = \frac{Y(s)}{X(s)}$ relating the Laplace transforms. If $x(t) = x_0 \sin(\omega t)$ then the steady state response of $y(t)$ is

$$y_{ss}(t) = \lim_{t\to\infty} y(t) = x_0|H(j\omega)|\sin(\omega t+\phi), \quad \phi = \arg(H(jw)) = \arctan\left(\frac{Im(H(jw))}{Re(H(jw))}\right)$$

1.1 From ODE to Steady State Response.

Example: Steady State Response.
Calculate the steady state response of the system $y'' + 2y' + 3y = x' + x$ with $x(t) = 2\sin(3t)$.

1. Apply the Laplace Transform and find the transfer function $H(s)$. Assume that $y(0) = y'(0) = 0$.

$$s^2Y + 2sY + 3Y = sX + X \quad \text{because } y(0) = y'(0) = x(0) = 0$$
$$(s^2+2s+3)Y = (s+1)X$$
$$H = \frac{Y}{X} = \frac{s+1}{s^2+2s+3}$$

2. Identify ω and x_0. Calculate $H(j\omega)$.

$$\omega = 3, \quad x_0 = 2 \quad \text{because } x(t) = 2\sin(3t)$$

$$H(j\omega) = H(j3) = \frac{1+3j}{-9+6j+3} = \frac{1+3j}{-6+6j}\cdot\frac{-6-6j}{-6-6j}$$

$$= \frac{-6+18-18j-6j}{36+36} = \frac{12-24j}{72} = +\frac{12}{72} - \frac{24}{72}j$$

$$= +\frac{1}{6} - \frac{1}{3}j$$

3. Calculate the amplitude and the phase shift.

~~Amplitude: $x_0 \cdot |H(j\omega)| = 2 \cdot \sqrt{\frac{1}{36} + \frac{1}{9}} = 2 \cdot \sqrt{\frac{5}{36}}$~~

$= \frac{\sqrt{5}}{3}$

Argument: $\phi = \arctan\left(\frac{-1/3}{1/6}\right) = \arctan(-2)$

4. Collect:

$$y_{ss} = \frac{\sqrt{5}}{3} \sin(3t + \arctan(-2))$$

Problem: Model problem.

A complex system is modelled by the following set of linear ODEs

$$\begin{aligned} v_1' &= u - 3x & (1)\\ v_2' &= u - 2x & (2)\\ x' &= v_1 + v_2 - x. & (3) \end{aligned}$$

$\frac{-x \pm \sqrt{|1 \mp 2|}}{2}$

$u(t)$ expresses an input signal that is split into two intermediate signals $v_1(t)$ and $v_2(t)$ and results in an output signal $x(t)$.

1. Calculate the transfer function $H(s) = \frac{X(s)}{U(s)}$.

$x'' = v_1 + v_2 - x \quad \Rightarrow \quad x'' = v_1' + v_2' - x' = u - 3x + u - 2x - x'$

Assume: Initial conditions 0 (homogeneous solution decays as $x'' + x' + 5x = 0$ is an underdamped system)

~~$H(s) =$~~ $X(s) \cdot (s^2 + 1s + 5) = 2U(s) \Rightarrow H(s) = \frac{X(s)}{U(s)} = \frac{2}{s^2+s+5}$

2. If $u(t) = a\sin(2t) + \sin(3t)$ for a positive constant a, find the steady state response y_{ss}

Due to the linear nature of the system the forcing terms $y_1(t) = a \cdot \sin(2t) + \sin(3t)$ does contribute to two particular solutions $y_1^p(s), y_2^p(s)$ which constitute the steady state response.

$\sqrt{4+16} = \sqrt{20}$ $\sqrt{64+16} = \sqrt{80} = 4\sqrt{5}$

$u_1(t) = a\sin(2t) \Rightarrow x_0' = a, \; \omega_1' = 2 \Rightarrow H(2j) = \frac{2}{-4+2j+5} = \frac{2}{1+2j} = \frac{2-4j}{\sqrt{5}}$

$u_2(t) = \sin(3t) \Rightarrow x_0^2 = 1, \; \omega_2' = 3 \Rightarrow H(3j) = \frac{2}{-9+3j+5} = \frac{2}{-4+3j} = \frac{-8-4j}{20}$

$y_{ss}(t) = a \cdot 2 \cdot \sin(2t + \arctan(-2)) + \frac{\sqrt{5}}{5}\sin(3t + \arctan(+\tfrac{1}{2}))$

3. Find bounds for a such that the steady state response does not surpass the maximum allowed ~~tolerance 0.1.~~ tolerated signal amplitude 0.1

$\arctan(-2) \approx -1.10$
$\arctan(1/2) \approx 0.463$

For any two real numbers holds $|a+b| \le |a| + |b|$

$|y_{ss}(t)| \le \underbrace{|2a\sin(2t + \arctan(-2))|}_{=: y_{ss}^{(1)}(t)} + \underbrace{\left|\frac{\sqrt{5}}{5} \cdot \sin(3t + \arctan(1/2))\right|}_{=: y_{ss}^{(2)}(t)}$

$\le 2|a| + \frac{\sqrt{5}}{5}$ because $|\sin(x)| \le 1$ for any x.

$|a| = a$ (a positive)

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$|y_{ss}(t)| \le 0.1$ is $2a + \frac{\sqrt{5}}{5} \le 0.1 \Rightarrow a \le \frac{1}{10} - \frac{2\sqrt{5}}{10} = \frac{1-2\sqrt{5}}{10\sqrt{5}}$ *

$\Rightarrow$ 4: $1 - 2\sqrt{5} > 0$ if $1 > 2\sqrt{5} \Leftrightarrow 1 > 20$

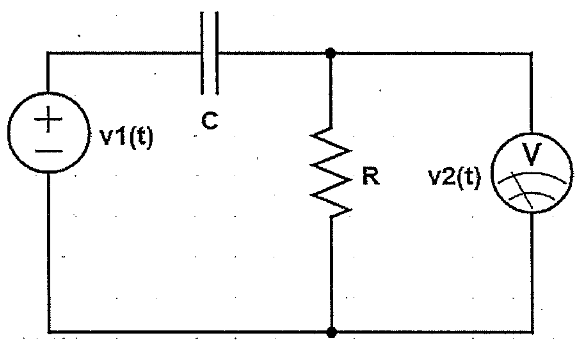
* To show that the signal always surpasses the tolerance 0.1 for any a we can show: For $t^* = -\arctan(-2)\frac{1}{2}$, $y_{ss}^{(2)}(t^*) = 0.85\ldots$, $y_{ss}^{(1)}(t^*) = 0$, $y_{ss}^{(2)}(t^*) = \frac{1}{\sqrt{5}} \cdot 0.85\ldots = 0.380\ldots > 0.1$ independent of a. $\max_{t \ge 0} |y_{ss}(t)| \ge |y_{ss}^{(2)}(t^*)| > 0.1$ for all a.

2 Electrical circuits

Introduction: Resistors, capacitors, inductors, and conservation laws Observe that we can transform many of the basic laws revolving around components found in RLC - circuits using Laplace transforms.

1. Resistor: $v_R = Ri \Rightarrow V_R(s) = RI(s)$
2. Capacitor: $\frac{dv_C}{dt} = \frac{i}{C} \Rightarrow sV_C(s) = \frac{1}{C}I(s)$
3. Inductor: $\frac{di}{dt} = \frac{v_L}{L} \Rightarrow sI(s) = \frac{1}{L}V_L(s)$
4. KVL (Voltage law, sum of voltages around a loop): $\sum_k v_k(t) = 0 \Rightarrow \sum_k V_k(s) = 0$
5. KCL (Current law, sum of currents into a node): $\sum_k i_k(t) = 0 \Rightarrow \sum_k I_k(s) = 0$.

Example: A simple circuit. We consider the following circuit.



1. Use KVL to relate the voltages along the left loop.

$v_1(t) + v_C(t) + v_R(t) = 0$

2. Use the Laplace transform and your knowledge of resistors, capacitors and inductors to link $V_1(s)$ and $V_2(s)$.

Because $v_R(t) + v_2(t) = 0 \Rightarrow V_2(s) = -V_R(s) = -R \cdot I(s)$ (*)

Further, $s \cdot V_C(s) = \frac{1}{C} \cdot I(s)$ (capacitor) and $V_1(s) + V_C(s) + V_R(s) = 0$

$V_1(s) = -V_C(s) - V_R(s) = -\frac{1}{Cs} I(s) + V_2(s) \overset{(*)}{=} \left(\frac{1}{RCs} + 1\right) \cdot V_2(s)$

$\Rightarrow$

3. Obtain the transfer function. For $v_1(t) = \sin(50t)$ what is the steady state voltage response $v_2(t)$ we should observe?

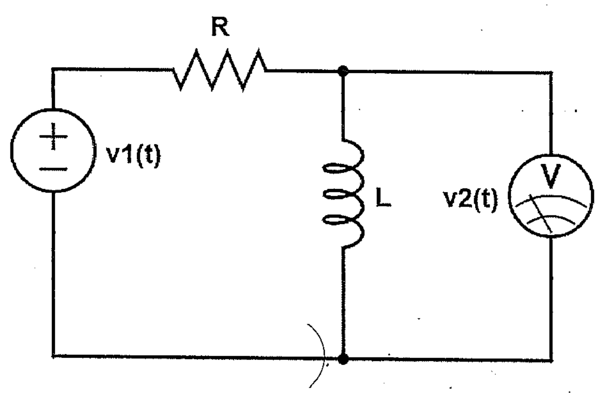
$$H(s) = \frac{V_2(s)}{V_1(s)} = \frac{1+RCs}{1+RCs} \Rightarrow H(50j) = \frac{RC50j}{1+RC50j} = \frac{RC50j+(RC50)^2}{1+(RC50)^2}$$

$$\text{Amplitude: } |H(50j)| = \frac{RC50}{\sqrt{1+(RC50)^2}} \; ; \; \phi = \tan^{-1}\left(\frac{1}{RC50}\right)$$

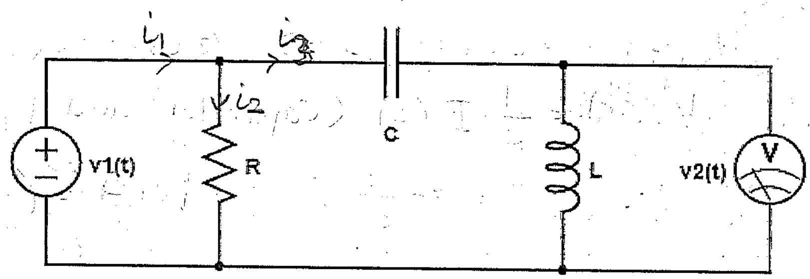
$$v_2(t) = \frac{50RC}{\sqrt{1+(RC50)^2}} \sin(50t+\phi)$$

Problem: Problemset.

- We consider the following circuit. Find the transfer function $H(s) = \frac{V_2(s)}{V_1(s)}$ and use it to obtain information about the amplitude of the steady state response of $v_2(t)$ to the input signal $v_1(t) = \sin(\omega t)$, when ω is very small or very large.



- We consider the following circuit. Find the transfer function $H(s) = \frac{V_2(s)}{V_1(s)}$. What do you observe when the frequency of the input signal $v_1(t) = \sin(\omega t)$ is varied?



Problem set

(1). First loop, second loop

$$V_1(t) + V_R(t) + V_L(t) = 0 \quad \& \quad V_L(t) + V_2(t) = 0$$

Additional knowledge:

$V_R(t) = R \cdot i(t)$

$\frac{di}{dt}(t) = \frac{V_L}{L}$ (In Lapl. : $sI(s) = \frac{1}{L} V_L(s)$)

Laplace: Assuming initial conditions all 0.

$$V_1(s) = -V_R(s) - V_L(s) = -R \cdot I(s) + V_2(s)$$
$$= -\frac{R}{sL} V_L(s) + V_2(s) = \left(1 + \frac{R}{sL}\right) V_2(s)$$

$$H(s) = \frac{V_2(s)}{V_1(s)} = \frac{1}{1+\frac{R}{sL}} = \frac{sL}{sL+R} = \frac{-(s^2L^2) + sLR}{-(sL)^2 + R^2} \quad // \cdot \frac{(-sL+R)}{(-sL+R)}$$

$$H(j\omega) = \frac{(\omega L)^2 + i\omega LR}{(\omega L)^2 + R^2} = \omega L \cdot \frac{(\omega L + iR)}{(\omega L)^2 + R^2}$$

$$|H(j\omega)| = \omega L \cdot \frac{\sqrt{(\omega L)^2 + R^2}}{(\omega L)^2 + R^2} = \frac{\omega L}{\sqrt{(\omega L)^2 + R^2}}$$

For $\omega \to \infty$: $|H(j\omega)| = \frac{\cancel{\omega} \cdot L}{\cancel{\omega} \cdot \sqrt{L^2 + (\frac{R}{\omega})^2}} \xrightarrow[\omega\to\infty]{} \frac{L}{\sqrt{L^2}} = 1,$

hence the output signal experiences no dampening / amplification.

For $\omega \to 0$ $|H(j\omega)| = \frac{\omega L}{\sqrt{(\omega L)^2 + R^2}} \xrightarrow[\omega\to 0]{} \frac{0}{\sqrt{R^2}} = \frac{0}{R} = 0$, i.e small frequencies vanish / are blocked.

→ Highpass filter.

(2)

$$v_1(t) + v_R(t) = 0$$
$$v_R(t) + v_C(t) + v_L(t) = 0$$
$$v_L(t) + v_2(t) = 0$$

$$v_R(t) = R \cdot i_2(t)$$
$$\frac{d}{dt} v_C(t) = \frac{1}{C} \cdot i_3(t)$$
$$\frac{d}{dt} i_3(t) = \frac{1}{L} v_L(t)$$

$$V_1(s) = -V_R(s)$$
$$V_R(s) + V_C(s) + V_L(s) = 0$$

Assuming 0 initial cond. ->

$$V_L(s) = -V_2(s)$$
$$i_1(t) = i_2(t) + i_3(t)$$

$$V_R(s) = R \cdot I_2(s)$$
$$s \cdot V_C(s) = \frac{1}{C} I_3(s)$$
$$s \cdot I_3(s) = \frac{1}{L} V_L(s)$$

$$0 = \underbrace{V_R(s)}_{=-V_1(s)} + V_C(s) + \underbrace{V_L(s)}_{=-V_2(s)}$$

$$V_C(s) = \frac{1}{CS} I_3(s) = \frac{1}{LCS^2} V_L(s) = \frac{-1}{LCS^2} V_2(s)$$

$$\Rightarrow V_1(s) = \left(-\frac{1}{LCS^2} - 1\right) V_2(s)$$

$$H(s) = \frac{V_2(s)}{V_1(s)} = -\frac{LCS^2}{1 + LCS^2}$$

$$H(i\omega) = \frac{LC\omega^2}{1 - LC\omega^2}$$

$$\Rightarrow |H(i\omega)| = \frac{|LC\omega^2|}{|1 - LC\omega^2|} = \frac{LC\omega^2}{|1 - LC\omega^2|}$$

If $1 > LC - \omega^2 \Rightarrow |H(i\omega)| = \dfrac{LC\omega^2}{1 - LC\omega^2}$

$1 < LC - \omega^2 \Rightarrow |H(i\omega)| = \dfrac{LC\omega^2}{|LC\omega^2 - 1|} = \dfrac{LC\omega^2}{LC\omega^2 - 1}$

← this is now positive with same absolute

$1 = LC - \omega^2 \Rightarrow$ Resonance catastrophe.

If $\omega \neq \sqrt{LC - 1}$ → Beats in the swing system.

Please note that the absence of damping in the swing system makes the assumption of 0 initial conditions restrictive