Worksheet 1

Felix Funk, MATH Tutorial - Mech 222

1 Review: One-dimensional integration

Problemset: 1. Substitution rule, integration by parts. Solve the following integrals. Take note of the integration variable.

1. $\int_0^{\frac{\pi}{2}} e^{\sin(x)} \cos(x) y \, dx$
2. $\int_1^e e^x log(xy) \, dy, x \geq 0$
3. $\int_0^1 xe^x y^2 \, dx$

2 Iterated Integrals

Introduction: Solving iterated integrals For iterated 2- dimensional integrals of the form

$$\int_a^b \int_c^d f(x, y) \, dxdy$$

we can simply solve the inner integral $\int_c^d f(x, y) \, dx$ first while treating the outer integration variable y as constant. When the inner integral is evaluated, then you can proceed to the outer variable.

Example: 1. Iterated Integral. Solve

$$\int_1^3 \int_1^2 \frac{x^2}{y} + \frac{y^2}{x} \, dxdy$$

1. It is useful to visualize the domain of integration. For that, indicate on an $x - y$ plot the area that is integrated over with respect to x with respect to y.

2. First, solve $\int_1^2 \frac{x^2}{y} + \frac{y^2}{x} dx$. It might help to visualize the integral as $\int_1^2 \frac{x^2}{C} + \frac{C}{x} dx$.

3. Then, solve the outer integral with the correct y values.

Problemset: 2. Iterated Integrals. Solve the following iterated integrals.

1. $\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \int_{-1}^{6} cos(y)\, dxdy$
2. $\int_0^1 \int_0^1 \sqrt{s+t}\, dsdt$
3. Challenging: Find the volume of the solid that lies under the plane $4x+6y-2z+15=0$ and above the rectangle $R = \{(x,y)| -1 \leq x \leq 2, -1 \leq y \leq 1\}$ by formulating the problem as an iterated integral.
4. Challenging: Use symmetry to evaluate $\int \int_R \frac{xy}{1+x^4} dA$ with $R = \{(x,y)|-1 \leq x \leq 1, 0 \leq y \leq 1\}$

Problemset: 3. Iterated Integrals over non-rectangular domains. Solve the following iterated integrals.

1. $\int_0^1 \int_0^{s^2} cos(s^3)\, dtds$
2. $\int_0^1 \int_0^{1-x} 1 - x - y\, dydx$
3. Challenging: $\int_0^1 \int_x^1 sin(y^2)\, dydx$. Hint: Draw the area of integration. Then change the order of integration. Make sure that you are still integrating over the correct domain.

3 Polar coordinates

Introduction: Polar coordinates If the underlying domain is circular, it is usually easier to change into polar coordinates. For this, we substitute $x = r\cos(\theta)$ and $y = r\sin(\theta)$ but need to rescale the function with a factor of r as the inner parts of the circle are stretched apart. Then, r represents the radius and θ the angle. As a reminder. The angle starts at the right part of the x-axis, rotates counter-clockwise and returns back at 2π.

Example: Introducing polar coordinates.. Evaluate the given integral by changing to polar coordinates.

$$\int\int_D x^2 y \, dA$$

where D is the top half of the disk with center the origin and radius 5.

1. First, sketch the domain. Determine the angles and radii that describe the domain.

2. Second, write as an iterated integrate over angle and radius. Don't forget to multiply the inner function with an additional factor of r due to the transformation.

3. Lastly, evaluate the integral.

Problem: 4. Polar coordinates. Evaluate

$$\int\int_D e^{-x^2-y^2} dA,$$

where D is the region bounded by the semicricle $x = \sqrt{4 - y^2}$ and the y-axis.