Solution to Worksheet 6:

Problemset 1.1:

$$r(t) = (1-t)\cdot P + t\cdot Q = (1-t)\begin{pmatrix}1\\ 1/2\\ 1/3\end{pmatrix} + t\cdot\begin{pmatrix}3\\ 2\\ 1\end{pmatrix} = \begin{pmatrix}1\\ 1/2\\ 1/3\end{pmatrix} + t\cdot\begin{pmatrix}2\\ 3/2\\ 2/3\end{pmatrix} \quad 0\le t\le 1$$

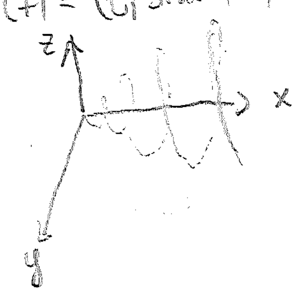
Observe $r(0) = (1-0)P + 0\cdot Q = P$

$r(1) = (1-1)P + 1\cdot Q = Q$

and $r(t)$ is linear.

1.2: $r(t) = (t;\ \sin(t)t;\ \cos(t)t)$

A spiral that grows with time



1.3: $r(t) = (t^2, \sin(t) - t\cos(t), \cos(t) + t\sin(t))$ $t \ge 0$

$r'(t) = (2t, \cos(t) - \cos(t) + t\sin(t), -\sin(t) + \sin(t) + t\cos(t))$

$= (2t, t\sin(t), t\cos(t))$

$r''(t) = (2, \sin(t) + t\cos(t), \cos(t) - t\sin(t))$

$\|r'(t)\| = \sqrt{4t^2 + t^2\sin(t)^2 + t^2\cos(t)^2} = \sqrt{5}\,t$

1.4: [$\|r'(t)\| = K$ then $r'(t) \perp r''(t)$] to show (K is a const.)

We want to show $r'(t) \circ r''(t) = 0$

If $\|r'(t)\| = K$ then $\|r'(t)\|^2 = K^2$.

This is the same as $r'(t)\circ r'(t) = K^2$.

Take the derivative on both sides:

$$\frac{\partial}{\partial t}(\underbrace{r'(t)\cdot r'(t)}_{(r_1'(t)^2 + r_2'(t)^2 + r_3'(t)^2)}) = \frac{\partial}{\partial t}(2\,r'(t)\circ r''(t)) \quad \text{(left term)}$$

$$\frac{\partial}{\partial t}(K^2) = 0$$

Therefore, $2r'(t)\cdot r''(t) = 0 \iff r'(t)\cdot r''(t) = 0$

Problemset 2:

2.1: Evaluate $\int_C x+2y\,dx + x^2 dy$ C: Line from $(0,0)$ to $(2,1)$ and from $(2,1)$ to $(3,0)$

Line param: $(0,0)$ to $(2,1)$: $r_1(t) = t\cdot\begin{pmatrix}2\\1\end{pmatrix} \}\ C_1$ $C = C_1 \oplus C_2$

$(2,1)$ to $(3,0)$: $r_2(t) = \begin{pmatrix}2\\1\end{pmatrix} + t\begin{pmatrix}1\\-1\end{pmatrix} \}\ C_2$ "go C_1 first then C_2."

First line segment || $\int_{C_1} x+2y\,dx + x^2 dy = \underbrace{\int_0^1 [(\underbrace{2t}_{x}) + 2\cdot(\underbrace{t}_{y})]\cdot \underbrace{2}_{x'(t)}\,dt}_{dx\ \text{integral}} + \underbrace{\int_0^1 (\underbrace{2t}_{x^2})^2 \cdot \underbrace{1}_{y'(t)}\,dt}_{dy\ \text{integral}}$

$= \int_0^1 8t + 4t^2 dt = 4 + \frac{4}{3}.$

Second line segment || $\int_{C_2} x+2y\,dx + x^2 dy = \int_0^1 ((\underbrace{2+t}_{x}) + 2\cdot(\underbrace{1-t}_{+2\ y}))\cdot \underbrace{1}_{x'}\,dt + \int_0^1 \underbrace{(2+t)^2}_{x^2}\cdot \underbrace{(-1)}_{y'}dt$

$= \int_0^1 4-t - (4+2t+t^2)\,dt = \int_0^1 \cancel{4}-t-\cancel{4}-2t-t^2 dt$

$= \int_0^1 -3t - t^2 dt = -\frac{3}{2} - \frac{1}{3}$

$\int_C x+2y\,dx + x^2 dy = 4 + \frac{4}{3} - \frac{3}{2} - \frac{1}{3} = \frac{5}{2} + 1 = \frac{7}{2}$

2.2 $\int F\cdot dr = \int_0^1 (\sin(\underbrace{t^3}_{x=t^3}), \cos(\underbrace{-t^2}_{y=-t^2}), \underbrace{t^3\cdot t}_{x\cdot z,\ t^3\ t}) \overset{\text{dot-product}}{\cdot} \underbrace{(3t^2, -2t, 1)}_{r'(t)}\,dt$

$= \int_0^1 3t^2\sin(t^3) + (-2t)\cos(-t^2) + t^4\,dt = [-\cos(t^3), -\sin(-t^2), \frac{t^5}{5}]_0^1$

$= -\cos(1) + \underbrace{\sin(1)}_{\text{or } -\sin(-1)} + \frac{1}{5} + 1 = \frac{6}{5} - \cos(1) + \sin(1)$

2.3 Mass $\int_C K\,ds$ with $C: x^2+y^2=4 \quad x\geq 0$

use param: $r(t) = \langle 2\cdot\cos(t), 2\sin(t)\rangle$ with t in $[-\frac{\pi}{2}, \frac{\pi}{2}]$

$M = \int_C K\,ds = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{4\sin^2(t) + 4\cos^2(t)}\,K\,dt = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 2K\,dt = 2\cdot K\pi$

2.3 continued:

By symmetry: $\bar{y}=0$

For $\bar{x}$ centroid: $\bar{x} = \frac{1}{M}\int_C x \cdot k\, ds = \frac{1}{M}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} k \cdot \underbrace{2\cos(t)}_{x(t)} \cdot \underbrace{\|r'(t)\|}_{\text{arc length}}\, dt$

$= \frac{1}{M}\left[2k\,2\sin(t)\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \frac{1}{M}4k\,[1+1] = \frac{8k}{2k\pi} = 4\frac{1}{\pi}$

Problemset 3:

3.1: $P(x,y) = 3x^2 - 2y^2 \quad Q(x,y) = 4xy + 3$

Check $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \Leftrightarrow -4y = 4y \Rightarrow$ F is not conservative.

3.2: $F(x,y) = (1+xy)e^{xy}\, i + x^2 e^{xy}\, j$

If F is conservative, then $\nabla f = F$: $\frac{\partial}{\partial x} f = (1+xy)e^{xy}$

$\frac{\partial}{\partial y} f = x^2 e^{xy}$

Integrate first $\int x^2 e^{xy}\, dy$: Because in this term we do not have some product, which is usually harder to integrate:

$\int x^2 e^{xy} dy = x^2 \int e^{xy} dy = x^2\left[\frac{1}{x}e^{xy} + C(x)\right] = xe^{xy} + x^2 C(x).$

When we differentiate this w.r.t. x:

$\frac{\partial}{\partial x}(xe^{xy} + x^2 C) = (e^{xy} + xy e^{xy} + 2x C) \overset{!}{=} (1+xy)e^{xy} \Rightarrow C=0.$

$\Rightarrow f(x,y) = xe^{xy}.$

Fundamental Thm of line int: $r(\frac{\pi}{2}) = \begin{pmatrix} 0 \\ 2 \end{pmatrix} \quad r(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$\int F\, dr = \int \nabla f\, dr = f(r(\frac{\pi}{2})) - f(r(0)) = 0 \cdot e^{0\cdot 2} - 1e^{1\cdot 0} = -1$

$$\int_C \sin(y)\,dx + (x\cos(y) - \sin(y))\,dy$$

$$= \int_C \sin(y)\,x' + (x\cos(y) - \sin(y))\,y'\,dt$$

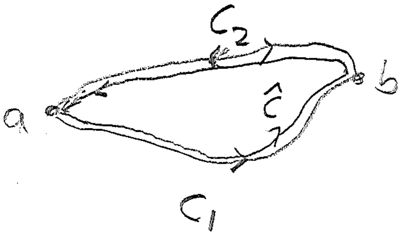
Define $r(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$; then $F(x,y) = \sin(y)\cdot i + (x\cos(y) - \sin(y))\, j$

F is continuous with continuous partial derivatives and $\frac{\partial}{\partial y}\sin(y) = \cos(y) = \frac{\partial}{\partial x}(\quad)$ and hence conservative.

The above integral equals $\int_C F\cdot dr$

Let C_1 & C_2 be two curves that ~~are~~ start in a and end in b.

If we define a curve $\hat{C}$ by moving first along C_1 and then back along C_2, then $\hat{C}$ is a closed curve and



$$\int_{\hat{C}} F\cdot dr = 0$$

and because of the construction of $\hat{C}$

$$(0=)\int_{\hat{C}} F\cdot dr = \int_{C_1} F\cdot dr - \int_{C_2} F\cdot dr \iff \int_{C_1} F\cdot dr = \int_{C_2} F\cdot dr$$

It therefore does not matter whether we take path C_1 or C_2.

To determine the value of the integral: Find f:

Because $\nabla f = F$: $\frac{\partial}{\partial x} f = \sin(y) \Rightarrow f(x,y) = x\sin(y) + g(y)$ (some function of y)

$$\frac{\partial}{\partial y} f = x\cos(y) - \sin(y) \Rightarrow f(x,y) = x\sin(y) + \cos(y)$$

$$\int_C F\cdot dr = \int_C \nabla f\cdot dr = -f((2,0)) + f((1,\pi)) = -\underbrace{2\sin(0)}_{=0}^{y=0} \underbrace{-\cos(0)}_{=1} + \underbrace{1\cdot\sin(\pi)}_{=0} + \underbrace{\cos(\pi)}_{-1}$$

$$= -2$$