Worksheet 5

Felix Funk, MATH Tutorial - Mech 221

1 Resonance and Forced Oscillation

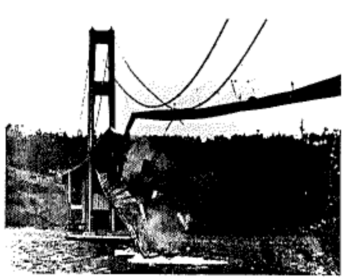
The goal of this section is the analysis and discussion of resonance and forced oscillations in the case of the Tacoma - Narrows bridge.

Introduction: The collapse of a bridge and the resonance hypothesis.
During a heavy storm in 1940, the Tacoma-Narrows bridge collapsed in a rather spectacular manner. One of the earliest and most prevalent hypotheses suggests that resonance and forced oscillations were the primary reasons for the destructive force that broke the bridge apart (see figure 1a.)

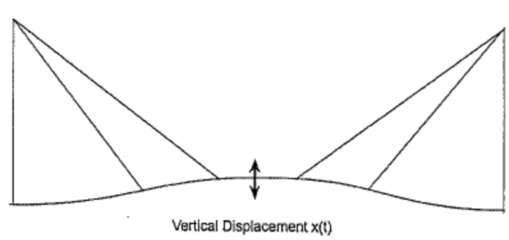
For the analysis we denote $x(t)$ as the vertical displacement of the center of the bridge. In our toy-model the suspension bridge shall be viewed as a mass-spring-damper system of the form

$$x'' + bx' + 100x = k \cdot \cos(\omega_0 t) \tag{1}$$

with some constants $b \geq 0$, $k > 0$ and $\omega_0 > 0$. The right side of the equation models the vertical excitation of the bridge due to the so-called "von Karman vortex" - which are periodically appearing wind-turbulences underneath and above the bridge. For more information, I refer to the more-detailed Wikipedia articles. The central question of our investigation is: Can the forced mass-spring-damper system provide an explanation for the destruction of the bridge?



(a) Collapse of the Tacoma-Narrows bridge. (Source: The Seattle Times)



(b) 1D schematic: Vertical displacement.

Figure 1: Tacoma- Narrows bridge

1.1 Forced Oscillations

Problem: The Impact of Dampening. The vertical displacement of the bridge center is provided by

$$x'' + bx' + 100x = k \cdot \cos(\omega_0 t). \tag{2}$$

Depending on the parameter b: When is the mass-spring-damper system overdamped, critically-damped or underdamped?

$r^2 + br + 100 = 0$

$\Rightarrow r_{1,2} = \frac{-b \pm \sqrt{b^2 - 400}}{2}$

- The system is overdamped for b in $b > 20$
- The system is critically-damped for $b = 20$
- The system is underdamped for b in $0 < b < 20$

Problem: Non-homogeneous solution. Let $b > 0$.

1. Find the general solution to equation (2).
2. The bridge is perfectly calm at the start of the observation. What do you observe during the initial period. What do you observe long-term? Use the MATLAB code to simulate the specific solution, numerically.
3. What is the impact of ω_0 and k on the long-term behaviour? (Hint: How does the homogeneous solution behave long-term? How does the particular solution behave long-term)
4. What happens when b tends to 0, $\omega_0 \neq 10$?

Example: Working economically.

1. The general solution consists of two components, the complimentary/homogeneous solution $x_h(t)$ and the particular solution $x_p(t)$. Hint: Show that the particular solution has the form

$$x_p(t) = A\cos(\omega_0 t) + B\sin(\omega_0 t); \quad A = k \cdot \frac{100 - \omega_0^2}{(b\omega_0)^2 + (100 - \omega_0^2)^2} \quad B = k \cdot \frac{b\omega_0}{(b\omega_0)^2 + (100 - \omega_0^2)^2}.$$

$$x_p'' + bx_p' + 100x_p = (-\omega_0^2 A + b\omega_0 B + 100A)\cos(\omega_0 t) + (-B\omega_0^2 - bA\omega_0 + 100B)\sin(\omega_0 t) = k\cos(\omega_0 t)$$

$\rightarrow$ either solve for A, B or test that A, B work.

$x_h = \alpha x_h^{[1]} + \beta x_h^{[2]}$

For $b > 20$: $x_h^{[1]} = e^{r_1 t}$ $x_h^{[2]} = e^{r_2 t}$, $r_1, r_2 < 0$

$b = 20$: $x_h^{[1]} = e^{-10t}$ $x_h^{[2]} = te^{-10t}$

$b > 20$: $x_h^{[1]} = e^{-\frac{b}{2}t}\cos(\omega_0 t)$

$x_h^{[2]} = e^{-\frac{b}{2}t}\sin(\omega_0 t)$

2. Hint: How can you translate the calm bridge into initial conditions

$$x(0) = x_0 = 0 \quad, x'(0) = x_1 = 0 \quad ?$$

The system is initially disturbed but over time, it stabilizes. This holds for any $b>0$. If the dampening b is smaller then the initial period is larger. In all cases, the system does not equilibriate due to the external forcing.

3. Compute the limits $\lim_{t\to\infty} x_h(t)$ for all of the three possible cases (dependent on b) and $\lim_{t\to\infty} x_p(t)$. What do A and B express? What can you tell about $x(t)$ after a long time? You can identify the impact of k, analytically. The impact of ω_0 is best observed experimentally and depends on the dampening. (Try $b = 0.1, 1, 10$)

$x_h^{[1]}, x_h^{[2]} \xrightarrow[t\to\infty]{} 0$ (due to dampening $b>0$) $\Rightarrow x_h(t) \to 0$

$\lim_{t\to\infty} x_p(t)$ does not exist as the system remains periodic for all times

$|A|, |B|$ are amplitudes of the long-term oscillations (A, B are

After a long time $x(t) \approx x_p(t)$.

The parameter k amplifies the amplitudes in a linear fashion

If k is doubled, then so are A and B.

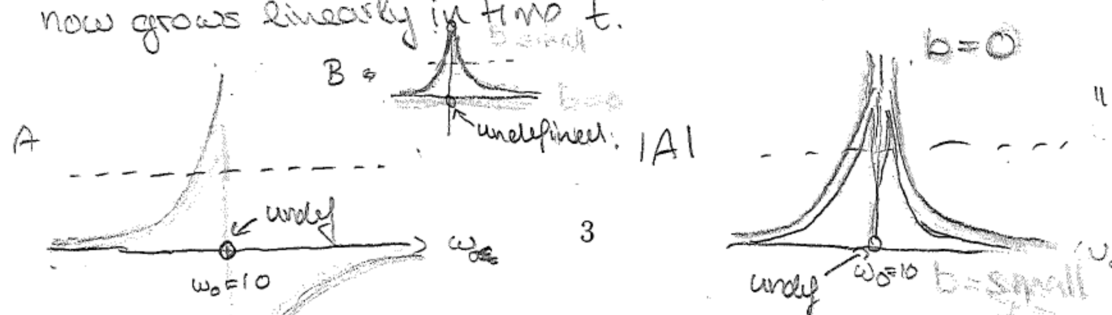
Only for small b, changes in ω_0 seem to be impactful.

4. Find an approximation for A and B for $b \approx 0$. Why does the approximation become invalid for $\omega_0 = 10$? Sketch A ~~and B~~ dependent on the forcing frequency ω_0..

$$A \overset{b\approx 0}{\approx} k \cdot \frac{(100-\omega_0^2)}{\cancel{0+}(100-\omega_0^2)^2} = k \cdot \frac{1}{100-\omega_0^2} \qquad B \overset{b\approx 0}{\approx} k \frac{0}{0+(100-\omega_0^2)^2} = 0$$

When $\omega_0 = 10$ and $b=0$, then $\cos(10t)$, $\sin(10t)$ are also the homogeneous solutions. In this case, the resonance catastrophe occurs, i.e. $y_p(t) = (A\,t)\cos(10t) + (B\,t)\sin(10t)$, ~~and~~ the amplitude now grows linearly in time t.

Sketch of A, |A|



"maximum amplitude that the bridge can withstand" (arbitrary large bound)

Similar idea for B but vanishes with $b=0$

Problem: 1 - Resonance. Let $b = 0$.
Find the general solution to equation (2) depending on w_0. The bridge is again perfectly calm. Use the MATLAB code to explore what happens for different ω_0, numerically. How stable are these results under small perturbations in ω_0? Also, take note how the amplitude change in proximity of $b = 0$.

Problem: A challenging problem: The Resonance Hypothesis.
A prevalent hypothesis for the destruction of the bridge is wind-induced resonance. Find arguments for and against the hypothesis. Compare your ideas with online-resources (a good article is provided by "What to say about Tacoma Narrows Bridge to your introductory physics class", Bernard J. Feldmann.) It's worth noting that vertical oscillations are nowadays not considered the primary reason for the destruction of the bridge (why?)

When $b=0$, $\omega_0 \neq 10$, then $x_h(t) = \alpha\cos(\omega_0 t) + \beta\sin(\omega_0 t)$

$\omega_0 \neq 10$ || $x_p(t) = A\cos(\omega_0 t) + B\sin(\omega_0 t)$ with A, B from previous section. $x(t) = x_h(t) + x_p(t)$. Note, that $x_h(t)$ no longer decays.

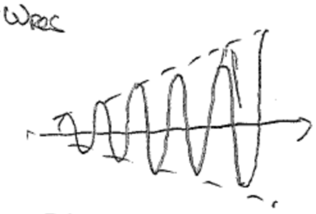
When $\omega_0 = 10$, then $x_p(t) = At\cos(\omega_0 t) + Bt\sin(\omega_0 t)$

With method of undetermined coefficients: $x_p(t) = \frac{k}{20} t\sin(10t)$

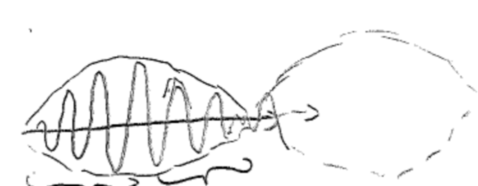
$\omega_0 = 10$ || $x(t) = \alpha\cos(10t) + \beta\sin(10t) + \frac{k}{20} t\sin(10t)$

In proximity of the resonance frequency ω_{RES} we observe drastic changes in amplitudes as the complimentary and particular solutions interfere

$\omega_0 = \omega_{RES}$



At resonance frequency; resonance catastrophe



positive interference, negative interference of harmonic oscillator + forcing

These effects both "vanish" when damping sets in and are only of transient nature. However, the amplitudes can still be destructive to real world objects.

Resonance Hypothesis: My opinion aligns very much with the article (but is certainly not relevant to begin with.)4 One should note that the effects of resonance are prelevant in singular amplitudes even when the system is damped and indicate that the model is qualitatively interesting (to model the vertical displacement.)

* Quantitive studies have shown that resonance is likely not the sole reason for the collapse.

2 Laplace - Transformation

Introduction: Laplace - Transformation. The Laplace - Transformation is an essential tool for solving ODEs as it converts derivatives to multiplicative factors. For this, the solution is transformed from the temporal domain into the domain of frequencies.

The Laplace transform of a function $f(t)$ is defined by

$$L\{f(t)\}(s) = \int_0^\infty e^{-st} f(t) dt. \tag{3}$$

→ Also introduce the inverse transform.

Problem: Using a Laplace Transform to solve an ODE.
Solve the differential equation

$$x'(t) + x(t) = e^{-t} \tag{4}$$

using Laplace-Transformations.

- Show that the Laplace Transform of $f(t) = te^{-t}$ is given by

$$L\{f(t)\}(s) = \frac{1}{(s+1)^2},\ s > -1$$

$$\int_0^\infty e^{-st} te^{-t} dt = \int_0^\infty e^{-(s+1)t} t\, dt = \left[-\frac{1}{s+1} e^{-(s+1)t} t\right]_0^\infty + \int_0^\infty e^{-(s+1)t} \frac{1}{s+1} dt$$

$$= -\frac{1}{s+1}(0-0) + \frac{1}{s+1}\int_0^\infty e^{-(s+1)t} dt$$

$$= \frac{1}{s+1}\left(-\frac{1}{s+1}\right)\left[e^{-(s+1)t}\right]_0^\infty = \frac{1}{(s+1)^2}$$

- Show or recall that the Laplace Transform of $g(t) = e^{at}, a \in \mathbb{R}$ is given by

$$L\{g(t)\}(s) = \frac{1}{s-a}, s > a$$

$$\int_0^\infty e^{-st} e^{at} dt = \int_0^\infty e^{(a-s)t} dt = \frac{1}{a-s}\left[e^{(a-s)t}\right]_0^\infty = \frac{-1}{a-s} = \frac{1}{s-a}$$

- Verify the fundamental properties of the Laplace-Transform. Let f, g, x be arbitrary functions and a, b some constants. Let x grow of at most exponential order.

s //

$$L\{x'(t)\} = sL\{x(t)\} - x(0) \tag{5}$$

Linearity $\rightarrow$

$$L\{af(t) + bg(t)\} = aL\{f(t)\} + bL\{g(t)\} \tag{6}$$

$$\mathcal{L}\{x'(t)\}(s) = \int_0^\infty e^{-st} x'(t)\,dt = \left[- e^{-st} x(t)\right]_0^\infty + s\int_0^\infty e^{-st} x(t)\,dt$$

$$= 0 - x(0) + s \cdot \underbrace{\int_0^\infty e^{-st} x(t)\,dt}_{\mathcal{L}\{x(t)\}(s)}$$

$$\mathcal{L}\{af(t) + bg(t)\} = \int_0^\infty (a f(t) + bg(t))e^{-st}\,dt$$

$$= a \cdot \int_0^\infty f(t) e^{-st}\,dt + b\int_0^\infty g(t)e^{-st} = a\,\mathcal{L}\{f(t)\}(s) + b\,\mathcal{L}\{g(t)\}(s)$$

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- Use the information above to solve the differential equation by applying the Laplace Transform to both sides of equation (4). ~~and using the information above~~.

$$\mathcal{L}\{x'(t) + x(t)\}(s) = \mathcal{L}\{e^{-t}\}(s)$$

$$\overset{\text{Linearity}}{\Rightarrow} \mathcal{L}\{x'(t)\}(s) + \mathcal{L}\{x(t)\}(s) = \mathcal{L}\{e^{-t}\}(s)$$

$$\Rightarrow s\,\mathcal{L}\{x(t)\}(s) - x(0) + \mathcal{L}\{x(t)\}(s) = \frac{1}{s+1}$$

$$\Rightarrow (s+1)\,\mathcal{L}\{x(t)\}(s) - x(0) = \frac{1}{s+1}$$

$$\Rightarrow \mathcal{L}\{x(t)\}(s) = \frac{1}{(s+1)^2} + \frac{x(0)}{s+1} = \mathcal{L}\{te^{-t}\} + x(0)\cdot\mathcal{L}\{e^{-t}\}$$

Now, we apply the inverse transform: $\mathcal{L}^{-1}$ on both sides.

$$x(t) = te^{-t} + x(0)e^{-t}.$$