Prob 1. [25 marks] Parts (a), (b) and (c) below are independent.

(a) [12 marks] A force is applied to a mass-spring system (with no damping) which yields the equation of motion

$$y'' + 100y = 1 - \cos(\omega t)$$

Assume $y(0) = y'(0) = 0$, find the range of values for the forcing frequency ω such that the total displacement $y(t)$ satisfies $|y(t)| < 0.04m$. (Hint: Use the inequality $|A\cos(at) + B\cos(bt)| \leq |A| + |B|$).

Find the particular solution: $y_p = A + B\cos(\omega t)$

$$y_p'' = -\omega^2 B\cos(\omega t)$$

$$\Rightarrow y_p'' + 100y_p = -\omega^2 B\cos(\omega t) + 100(A + B\cos(\omega t))$$
$$= 1 - \cos(\omega t)$$

$$\Rightarrow \underline{A = \frac{1}{100}} \qquad (-\omega^2 + 100)B = -1 \qquad \underline{B = \frac{1}{\omega^2 - 100}}$$

Find complimentary solution: $y_c = C_1\cos(10t) + C_2\sin(10t)$

$$\Rightarrow y(t) = C_1\cos(10t) + C_2\sin(10t) + \frac{1}{100} + \left(\frac{1}{\omega^2 - 100}\right)\cos(\omega t)$$

$$y(0) = 0 \Rightarrow 0 = C_1 + \frac{1}{100} + \frac{1}{\omega^2 - 100}$$

$$y'(0) = 0 \Rightarrow C_2 = 0$$

$$\Rightarrow y(t) = \left(\frac{-1}{100} - \frac{1}{\omega^2 - 100}\right)\cos(10t) + \frac{1}{100} + \left(\frac{1}{\omega^2 - 100}\right)\cos(\omega t)$$

use the inequality $|A\cos(at) + B\cos(bt)| \leq |A| + |B|$

$$\Rightarrow |y(t)| \leq \left|\frac{-1}{100} - \frac{1}{\omega^2 - 100}\right| + \left|\frac{1}{\omega^2 - 100}\right| + \frac{1}{100}$$

$$|y(t)| \leq \left|\frac{1}{100} + \frac{1}{\omega^2 - 100}\right| + \left|\frac{1}{\omega^2 - 100}\right| + \frac{1}{100}$$

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Case 1: $\omega > 10$

$$|y(t)| \leq \left(\frac{1}{100} + \frac{1}{\omega^2-100}\right) + \frac{1}{\omega^2-100} + \frac{1}{100} < 0.04$$

$$\frac{2}{\omega^2-100} < 0.02 \Rightarrow 100 < \omega^2 - 100$$

$$\Rightarrow \boxed{\omega > \sqrt{200} \approx 14.14}$$

Case 2: $\omega < 10$

$$\left|\frac{1}{100} + \frac{1}{\omega^2-100}\right| = -\left(\frac{1}{100} + \frac{1}{\omega^2-100}\right)$$

$$\left|\frac{1}{\omega^2-100}\right| = -\frac{1}{\omega^2-100}$$

$$|y(t)| \leq -\left(\frac{1}{100} + \frac{1}{\omega^2-100}\right) - \frac{1}{\omega^2-100} + \frac{1}{100} < 0.04$$

$$\frac{-2}{\omega^2-100} < 0.04$$

$$-50 > \omega^2 - 100 \qquad (\omega^2 - 100 < 0)$$

$$50 > \omega^2 \Rightarrow \boxed{\omega < \sqrt{50} \approx 7.07}$$

(b) [6 marks] Suppose the Laplace transform of $f(t)$ depends on a parameter a and is given by

$$F(s) = \frac{2}{s^2 + as + a + 3}$$

Find the range of values for the parameter a such that $f(t)$ oscillates **and** $f(t) \to 0$ as $t \to \infty$.

Complete the square in the denominator

$$s^2 + as + (a+3) = \left(s + \frac{a}{2}\right)^2 + \left(a + 3 - \frac{a^2}{4}\right)$$

$\Rightarrow$ If $a + 3 - \frac{a^2}{4} > 0$, then

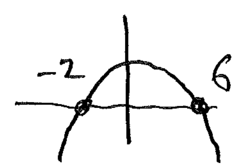
$$f(t) = A e^{-a/2} \cos(\omega t) + B e^{-a/2} \sin(\omega t), \quad A, B \in \mathbb{R}$$

where $\omega = \sqrt{a + 3 - \frac{a^2}{4}}$

$$\Rightarrow a + 3 - \frac{a^2}{4} = \frac{-1}{4}(a^2 - 4a - 12)$$

$$= \frac{-1}{4}(a-6)(a+2)$$

$\Rightarrow -2 < a < 6$ $\hookrightarrow$



$\Rightarrow$ If $a > 0$, then $f(t) \to 0$ as $t \to \infty$

$\Rightarrow$ $\boxed{0 < a < 6}$ guarantees $f(t)$ oscillates and $f(t) \to 0$ as $t \to \infty$.

(c) [7 marks] Use the **definition** of the Laplace transform to **prove**

$$L\{e^{\alpha t}+e^{-\alpha t}\} = \frac{2s}{s^2-\alpha^2}, s > |\alpha|$$

$$\mathcal{L}\{e^{\alpha t}+e^{-\alpha t}\} = \int_0^\infty (e^{\alpha t}+e^{-\alpha t})e^{-st}\,dt$$

$$= \lim_{b\to\infty}\left(\int_0^b e^{(\alpha-s)t}\,dt + \int_0^b e^{-(\alpha+s)t}\,dt\right)$$

$$= \lim_{b\to\infty} \left.\frac{e^{(\alpha-s)t}}{\alpha-s}\right|_0^b + \left.\frac{e^{-(\alpha+s)t}}{-(\alpha+s)}\right|_0^b$$

$$= \lim_{b\to\infty}\left(\frac{e^{(\alpha-s)b}}{\alpha-s} - \frac{1}{\alpha-s} + \frac{e^{-(\alpha+s)b}}{-(\alpha+s)} - \frac{1}{-(\alpha+s)}\right)$$

$$\lim_{b\to\infty} e^{(\alpha-s)b} = \begin{cases} 0, & \alpha-s<0 \\ \infty, & \alpha-s>0 \end{cases}$$ (and integral not defined at $s=\alpha$)

$$\lim_{b\to\infty} e^{-(\alpha+s)b} = \begin{cases} 0, & \alpha+s>0 \\ \infty, & \alpha+s<0 \end{cases}$$ (and integral not defined at $s=-\alpha$)

$$= \frac{1}{s-\alpha} + \frac{1}{s+\alpha}, \quad s>\alpha \text{ and } s>-\alpha$$

$$= \frac{(s+\alpha)+(s-\alpha)}{(s-\alpha)(s+\alpha)}, \quad s>|\alpha|$$

$$= \frac{2s}{s^2-\alpha^2}, \quad s>|\alpha|$$