Worksheet 7

Felix Funk, MATH Tutorial - Mech 222

1 Green's Theorem

Introduction: Green's Theorem

Let C be a positive oriented, piecewise smooth, simple closed curve in the plane and let D be the region bounded by C. If P and Q have continuous partial derivatives on an open region that contains D, then

$$\int_C P dx + Q dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA.$$

Problemset: 1. Green's Theorem.

1. Evaluate the line integral directly and by using Green's Theorem:

$$\int_C xy dx + x^2 dy,$$

   C is the rectangle with vertices $(0, 0), (3, 0), (3, 1)$, and $(0, 1)$.

2. Use Green's theorem to evaluate the line integral

$$\int_C y^4 dx + 2xy dy$$

   along the ellipse C given by $x^2 + 2y^2 = 2$.

2 Curl

Introduction: Curl For a vector field $F = Pi + Qj + Rk$ the curl of a vector field is defined by

$$\text{curl}(F) = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) i + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right) j + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) k$$

One of the essential properties of the curl is based on the link to conservative vector fields.

If F is a vector field defined on all $\mathbb{R}^3$ whose component functions have continuous partial derivatives and $\text{curl}(F) = 0$, then F is a conservative vector field.

Problemset: 2. Curl.

1. Show: If f has continuous second derivatives, then $\text{curl}(\nabla f) = 0$.
2. Find the curl of $F(x, y, z) = xi + yj + zk$ and $G(x, y, z) = \frac{1}{\sqrt{x^2+y^2+z^2}}(xi + yj + zk)$.
3. $F(x, y, z) = xyz^3 i + \frac{1}{2}x^2 y^a z^3 j + bx^2 yz^2 k$. Determine a and b such that F is a conservative vector field. Find the curl for arbitrary $a \geq 0$ and arbitrary b. Relate your result to the first part of the problemset.

3 Stoke's Theorem

Introduction: Stoke's Theorem Let S be an oriented piecewise-smooth surface that is bounded by a simple, closed, piecewise-smooth boundary curve C with positive orientation. Let F be a vector field whose components have continuous partial derivatives on an open region in $\mathbb{R}^3$ that contains S. Then,

$$\int_C F \cdot dr = \iint_S \text{curl}(F) \cdot dS$$

To find the positive orientation of the curve, imagine walking on top of the surface (this is where the normal vector points) along the curve. During the entire journey, the surface should be always on the left side.

Problemset: 3. Stoke's Theorem. Use Stoke's Theorem to solve the following problems.

1. A vector field $F(x, y, z) = xyzi + xyj + x^2 yzk$. S consists of the top and the four sides (but not the bottom) of the cube with vertices $(\pm 1, \pm 1, \pm 1)$ with the surface oriented outwards. Evaluate $\iint_S \text{curl}(F) \cdot dS$.
2. Use Stoke's Theorem to evaluate $\int_c F \cdot dr$. C is oriented counterclockwise as viewed from above:
$$F(x, y, z) = i + (x + yz)j + (xy - \sqrt{z})k$$
and C is the boundary of the part of the plane $3x + 2y + z = 1$ in the first octant.
3. $F(x, y, z) = -2yzi + yj + 3xk$. S is the part of the paraboloid $z = 5 - x^2 - y^2$ that lies above the plane $z = 1$ oriented outward. Evaluate $\iint_S \text{curl}(F) \cdot dS$.

Final Notes

It was a pleasure teaching you, friends. Please take care of yourself and don't stress yourself out for the exam. It is only an assessment amongst many. Follow your heart and your dreams and I am confident that all of you can shoot for the moon.

Take it easy, folks, and good luck for whatever is to come. Cheers, Felix.