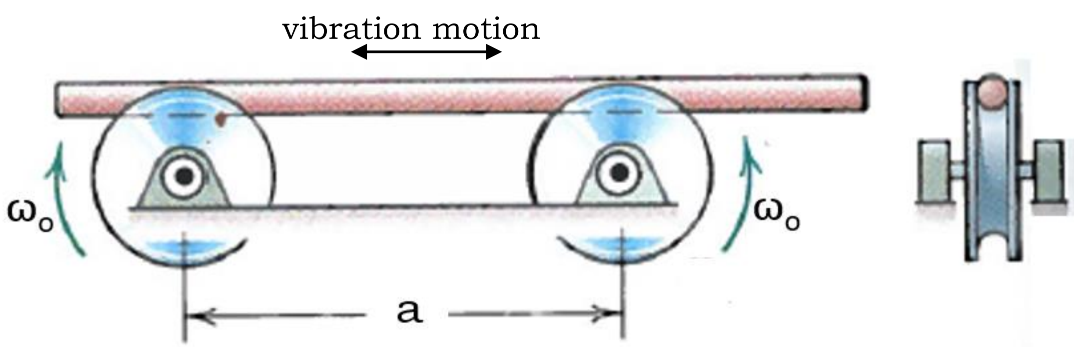
Name:____________________________ Section:_________________________

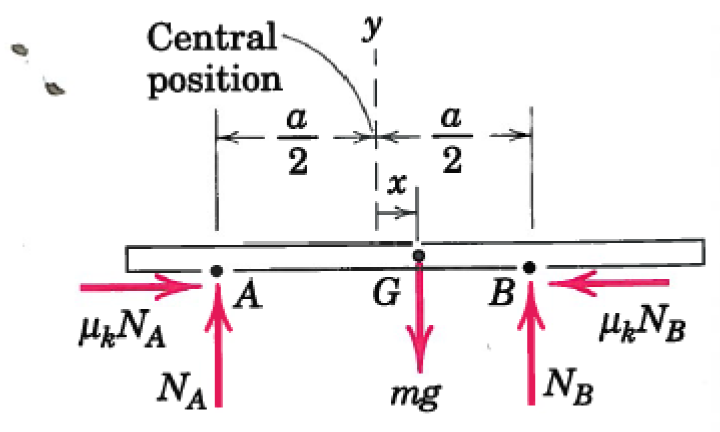
SA 1. [5 marks] In the system below the two fixed counter rotating pulleys rotate with the same speed, ω_o. If the slender bar that is resting on the pulleys is centered on the pulleys it will be stationary as the pulley's rotate beneath it. However, if the slender bar is shifted to the right (as shown below) or left, and then released, then the bar will vibrate back and forth in the horizontal plane.

a) [4 marks] Draw the free body diagram of the displaced slender rod.

b) [1 mark] Explain in one or two sentences why the slender rod vibrates.



a)



b) The bar will vibrate sideways because there will be a larger normal force at B resulting in a larger friction force at B and this will cause the bar to accelerate to the left. Once the bar is displaced to the left the opposite effect will occur and the bar will accelerate to the right.

SA2 [5 Marks]. The transient current (following a switching event) in a series *RLC* circuit is described by the following differential equation (DE):

$$\frac{d^2 i(t)}{dt^2} + 10\frac{di(t)}{dt} + 25i(t) = 0 .$$

(a) [2 Marks]. Determine the damping factor, ζ, and the undamped natural frequency, ω_0.

Characteristic Eqn: $s^2 + 10s + 25 = 0 = s^2 + 2\zeta\omega_0 s + \omega_0^2$

$\Rightarrow \omega_0^2 = 25 \quad \therefore \omega_0 = 5 \text{ r/s}$

$2\zeta\omega_0 = 10 \quad \therefore \zeta = \frac{10\,s^{-1}}{(2)(5s^{-1})} = 1$

(b) [1 Mark]. Determine the general form of the response, *i*(*t*), for *t* > 0. Do not try to solve for the coefficients.

$\zeta = 1 \Rightarrow$ critically damped

$\therefore i(t) = e^{-\omega_0 t}[A + Bt] = e^{-5t}[A + Bt]$

(c) [2 Marks]. Suppose the resistance in the circuit is decreased, but the capacitance and inductance remain unchanged, yielding a slightly different DE. How would such a change affect the general form of the response *i*(*t*). Briefly state your reasoning.

For a series RLC cct, $2\zeta\omega_0 = \frac{R}{L}$, $\omega_0^2 = \frac{1}{LC}$

$\Rightarrow$ If R decreased, L & C unchanged

then ζ decreases ($\therefore \zeta < 1$ here) $\Rightarrow$ underdamped

$\therefore i(t) = e^{-\sigma t}[A\cos\omega_d t + B\sin\omega_d t]$

where $\sigma = \zeta\omega_0$, $\omega_d = \omega_0\sqrt{1-\zeta^2}$

Name:________________________________ Section:______________________________

Prob 1 [25 marks] ***Parts (a), (b), (c) & (d) are separate questions.***

(a) [8 marks] Use the Laplace transform to solve the differential equation

$$y'' + 3y = 1 + \sin(2t)\ , \qquad y(0) = 1, \qquad y'(0) = 1$$

The following partial fraction formulas will be helpful:

$$\frac{1}{s(s^2+a)} = \frac{1}{a}\left(\frac{1}{s} - \frac{s}{s^2+a}\right) \qquad \frac{1}{(s^2+a)(s^2+b)} = \frac{1}{b-a}\left(\frac{1}{s^2+a} - \frac{1}{s^2+b}\right)$$

$$y'' + 3y = 1 + \sin(2t)$$

$$s^2Y(s) - sy(0) - y'(0) + 3Y(s) = \frac{1}{s} + \frac{2}{s^2+4}$$

$$Y(s)(s^2+3) = \frac{1}{s} + \frac{2}{s^2+4} + s + 1$$

$$Y(s) = \frac{1}{s(s^2+3)} + \frac{2}{(s^2+4)(s^2+3)} + \frac{s+1}{s^2+3}$$

$$Y(s) = \frac{1}{3}\left(\frac{1}{s} - \frac{s}{s^2+3}\right) + \frac{2}{4-3}\left(\frac{1}{s^2+3} - \frac{1}{s^2+4}\right) + \frac{s}{s^2+3} + \frac{1}{s^2+3}$$

$$Y(s) = \frac{1}{3}\frac{1}{s} + \frac{2}{3}\frac{s}{s^2+3} + 3\frac{1}{s^2+3} - \frac{2}{s^2+4}$$

$$\boxed{y(t) = \frac{1}{3} + \frac{2}{3}\cos(\sqrt{3}t) + \sqrt{3}\sin(\sqrt{3}t) - \sin(2t)}$$

Name:______________________________ Section:______________________________

(b) [7 marks] Use the **definition** of the Laplace transform to **prove**

$$\mathcal{L}\{te^{2t}\} = \frac{1}{(s-2)^2}\ , \quad s > 2$$

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt$$

$$\mathcal{L}\{te^{2t}\} = \int_0^\infty e^{-st}\, t e^{2t}\, dt$$

$$= \lim_{b\to\infty} \int_0^b t e^{-(s-2)t}\, dt \qquad u = t \quad du = dt$$

$$= \lim_{b\to\infty} \left(t \frac{e^{-(s-2)t}}{-(s-2)} \Big|_0^b - \int_0^b \frac{e^{-(s-2)t}}{-(s-2)} dt \right) \qquad dV = e^{-(s-2)t} dt \quad V = \frac{e^{-(s-2)t}}{-(s-2)}$$

$$= \lim_{b\to\infty} \left(\frac{b e^{-(s-2)b}}{-(s-2)} - \left(\frac{e^{-(s-2)t}}{(s-2)^2} \Big|_0^b \right) \right)$$

$$= \lim_{b\to\infty} \left(\cancel{\frac{b e^{-(s-2)b}}{-(s-2)}}^{\,0} - \left(\cancel{\frac{e^{-(s-2)b}}{(s-2)^2}}^{\,0} - \frac{1}{(s-2)^2} \right) \right)$$

$$= \frac{1}{(s-2)^2}\ , \quad s > 2$$

because we know $\lim_{b\to\infty} e^{-(s-2)b} = 0$ when $s > 2$

and $\lim_{b\to\infty} b e^{-(s-2)b} = 0$ when $s > 2$ (by L'Hopital's Rule).

Name:________________________ Section:______________________

(c) [5 marks] Find the inverse Laplace transform of

$$Y(s) = \frac{s-1}{s^2+2s+5}$$

The polynomial s^2+2s+5 has complex roots and so we can complete the square

$$Y(s) = \frac{s-1}{s^2+2s+5} = \frac{s-1}{(s+1)^2+4}$$

$$Y(s) = \frac{s+1}{(s+1)^2+4} - \frac{2}{(s+1)^2+4}$$

$$\boxed{y(t) = e^{-t}\cos(2t) - e^{-t}\sin(2t)}$$

Name:______________________________ Section:______________________________

(d) [5 marks] Suppose the Laplace transform of $y(t)$ is given by

$$Y(s) = \frac{1}{s^2 + cs + 1}$$

Find the values for the constant c such that $y(t)$ oscillates and

$$\lim_{t\to\infty} y(t) = 0$$

The roots of s^2+cs+1 must be complex for $y(t)$ to oscillate, therefore $c^2-4<0$ ad so $-2<c<2$. The roots are

$$\frac{-c \pm \sqrt{c^2-4}}{2}.$$

Completing the square, we see

$$Y(s) = \frac{1}{s^2+cs+1} = \frac{1}{(s+\frac{c}{2})^2 + (1-\frac{c^2}{4})}$$

$$= \frac{1}{(s+\frac{c}{2})^2 + (\frac{4-c^2}{4})}$$

$$\Rightarrow \quad y(t) = \frac{2e^{-\frac{c}{2}}}{\sqrt{4-c^2}} \sin\left(\frac{\sqrt{4-c^2}}{2} t\right)$$

therefore $y(t) \to 0$ as $t \to \infty$ only if $c > 0$.

$$\Rightarrow \quad \boxed{0 < c < 2}$$