Worksheet 5

Felix Funk, MATH Tutorial - Mech 221

1 Resonance and Forced Oscillation

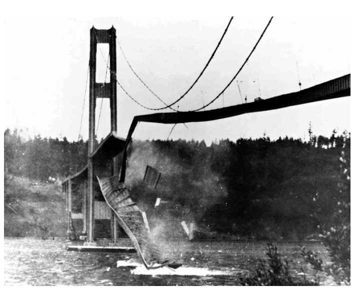
The goal of this section is the analysis and discussion of resonance and forced oscillations in the case of the Tacoma - Narrows bridge.

Introduction: The collapse of a bridge and the resonance hypothesis.
During a heavy storm in 1940, the Tacoma-Narrows bridge collapsed in a rather spectacular manner. One of the earliest and most prevalent hypotheses suggests that resonance and forced oscillations were the primary reasons for the destructive force that broke the bridge apart (see figure 1a.)

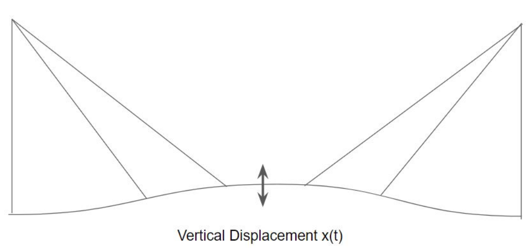
For the analysis we denote $x(t)$ as the vertical displacement of the center of the bridge. In our toy-model the suspension bridge shall be viewed as a mass-spring-damper system of the form

$$x'' + bx' + 100x = k \cdot \cos(\omega_0 t) \tag{1}$$

with some constants $b \geq 0$, $k > 0$ and $\omega_0 > 0$. The right side of the equation models the vertical excitation of the bridge due to the so-called "von Karman vortex" - which are periodically appearing wind-turbulences underneath and above the bridge. For more information, I refer to the more-detailed Wikipedia articles. The central question of our investigation is: Can the forced mass-spring-damper system provide an explanation for the destruction of the bridge?



(a) Collapse of the Tacoma-Narrows bridge. (Source: The Seattle Times)



(b) 1D schematic: Vertical displacement.

Figure 1: Tacoma- Narrows bridge

1.1 Forced Oscillations

Problem: The Impact of Dampening. The vertical displacement of the bridge center is provided by

$$x'' + bx' + 100x = k \cdot \cos(\omega_0 t). \tag{2}$$

Depending on the parameter b: When is the mass-spring-damper system overdamped, critically-damped or underdamped?

- The system is overdamped for b in
- The system is critically-damped for b
- The system is underdamped for b in

Problem: Non-homogeneous solution. Let $b > 0$.

1. Find the general solution to equation (2).
2. The bridge is perfectly calm at the start of the observation. What do you observe during the initial period. What do you observe long-term? Use the MATLAB code to simulate the specific solution, numerically.
3. What is the impact of ω_0 and k on the long-term behaviour? (Hint: How does the homogeneous solution behave long-term? How does the particular solution behave long-term)
4. What happens when b tends to 0, $\omega_0 \neq 10$?

Example: Working economically.

1. The general solution consists of two components, the complimentary/homogeneous solution $x_h(t)$ and the particular solution $x_p(t)$. Hint: Show that the particular solution has the form

   $$x_p(t) = A\cos(\omega_0 t) + B\sin(\omega_0 t); \quad A = k \cdot \frac{100 - \omega_0^2}{(b\omega_0)^2 + (100 - \omega_0^2)^2} \quad B = k \cdot \frac{b\omega_0}{(b\omega_0)^2 + (100 - \omega_0^2)^2}.$$

2. Hint: How can you translate the calm bridge into initial conditions

$$x(0) = x_0 = \qquad , x'(0) = x_1 = \qquad ?$$

3. Compute the limits $\lim_{t\to\infty} x_h(t)$ for all of the three possible cases (dependent on b) and $\lim_{t\to\infty} x_p(t)$. What do A and B express? What can you tell about $x(t)$ after a long time? You can identify the impact of k, analytically. The impact of ω_0 is best observed experimentally and depends on the dampening. (Try $b = 0.1, 1, 10$)

4. Find an approximation for A and B for $b \approx 0$. Why does the approximation become invalid for $\omega_0 = 10$? Sketch A and B dependent on the forcing frequency ω_0.

Problem: 1 - Resonance. Let $b = 0$.
Find the general solution to equation (2) depending on w_0. The bridge is again perfectly calm. Use the MATLAB code to explore what happens for different ω_0, numerically. How stable are these results under small perturbations in ω_0? Also, take note how the amplitude change in proximity of $b = 0$.

Problem: A challenging problem: The Resonance Hypothesis.
A prevalent hypothesis for the destruction of the bridge is wind-induced resonance. Find arguments for and against the hypothesis. Compare your ideas with online-resources (a good article is provided by "What to say about Tacoma Narrows Bridge to your introductory physics class", Bernard J. Feldmann.) It's worth noting that vertical oscillations are nowadays not considered the primary reason for the destruction of the bridge (why?)

2 Laplace - Transformation

Introduction: Laplace - Transformation. The Laplace - transformation is an essential tool for solving ODEs as it converts derivatives to multiplicative factors. For this, the solution is transformed from the temporal domain into the domain of frequencies.

The Laplace transform of a function $f(t)$ is defined by

$$L\{f(t)\}(s) = \int_0^\infty e^{-st} f(t) dt. \tag{3}$$

Furthermore, there is an inverse transform $L^{-1}\{F(s)\}(t)$ that satisfies

$$L^{-1}\{L\{f(t)\}\}(t) = f(t),$$

i.e. the Laplance transform and its inverse cancel.

Problem: Using a Laplace transform to solve an ODE.
Solve the differential equation

$$x'(t) + x(t) = e^{-t} \tag{4}$$

using Laplace-Transformations.

- Show that the Laplace Transform of $f(t) = te^{-t}$ is given by

$$L\{f(t)\}(s) = \frac{1}{(s+1)^2}, s > -1$$

- Show or recall that the Laplace Transform of $g(t) = e^{at}, a \in \mathbb{R}$ is given by

$$L\{g(t)\}(s) = \frac{1}{s-a}, s > a$$

- Verify the fundamental properties of the Laplace-Transform. Let f, g, x be some arbitrary but at most exponentially growing functions, and a, b constants.

$$L\{x'(t)\}(s) = s \cdot L\{x(t)\}(s) - x(0) \tag{5}$$

$$L\{af(t) + bg(t)\}(s) = aL\{f(t)\}(s) + bL\{g(t)\}(s) \tag{6}$$

- Use the information above to solve the differential equation by applying the Laplace transform to both sides of equation (4). Use the inverse transform to obtain $x(t)$.