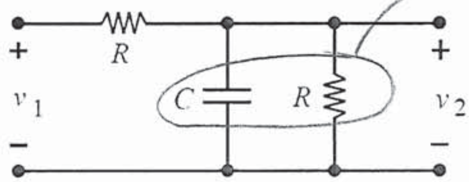
SA 3. [5 marks]

Consider the low-pass filter circuit shown below. Determine the frequency (in terms of R and C) at which the magnitude of the voltage transfer function is equal to $\frac{1}{\sqrt{2}}$ times its maximum value.



$$R \| (1/sC) = \frac{R}{1+sRC}$$

$$\frac{V_2(s)}{V_1(s)} = \frac{(R\| 1/sC)}{(R \| 1/sC)+R} = \frac{1}{2+sRC}$$

$$\frac{V_2(\omega)}{V_1(\omega)} = \frac{1}{(2+j\omega RC)} \cdot \frac{(2-j\omega RC)}{(2-j\omega RC)} = \frac{2}{4+(\omega RC)^2} - j\,\frac{\omega RC}{4+(\omega RC)^2}$$

$$\left|\frac{V_2(\omega)}{V_1(\omega)}\right| = \frac{1}{4+(\omega RC)^2} \cdot \sqrt{4+(\omega RC)^2} = \frac{1}{\sqrt{4+(\omega RC)^2}}$$

$$\left|\frac{V_2(\omega)}{V_1(\omega)}\right|_{max} = \frac{1}{2} \text{ (at } \omega=0) \Rightarrow \frac{1}{\sqrt{2}}\left|\frac{V_2}{V_1}\right|_{max} = \frac{1}{2\sqrt{2}}$$

$$\frac{1}{2\sqrt{2}} = \frac{1}{\sqrt{4+(\omega RC)^2}} \rightarrow \omega = \frac{2}{RC} = \omega_{co}$$

Shortcut: ω at which $\left|\frac{V_2}{V_1}\right| = \frac{1}{\sqrt{2}}\left|\frac{V_2}{V_1}\right|_{max} = \frac{1}{\tau}$

$$\tau = (R \| R)C = \frac{RC}{2} \Rightarrow \omega_{co} = \frac{2}{RC}$$

SA 4. [5 marks]

$= \omega_0$

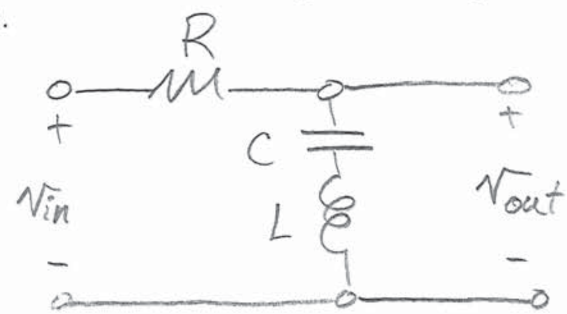
You are asked to design a ~~*band-stop filter*~~ based on a series *RLC* circuit with a center frequency of 4000 *rad/s* and a bandwidth of 400 *rad/s*. Only one inductor of 100 *mH* is available, but resistors and capacitors of any value are available.

(a) [2 marks] Select appropriate values for the *R* and *C* components that result in the specifications given above.

$$\omega_0 = \frac{1}{\sqrt{LC}} \Rightarrow C = \frac{1}{(0.1)(4000)^2} = 0.625\,\mu F$$

$$BW = \frac{R}{L} \Rightarrow R = (0.1)(400) = 40\,\Omega$$

(b) [1 mark] Sketch the circuit diagram clearly defining the input and output voltages.



(c) [2 marks] If a voltage waveform defined by $v_i(t) = 10\sin(4000t)\ V$ is applied to the input, determine the phasor voltage across the <u>inductor</u>.

$v_i(t) = 10\cos(4000t - 90°)\,V$ (i.e. at resonance)

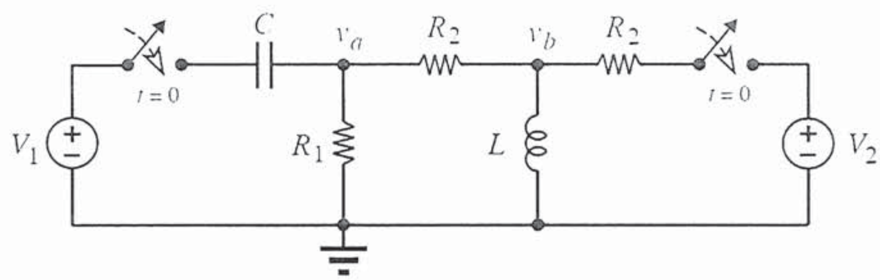
at resonance, $Z_L = j(4000)(0.1) = 400\angle 90°$

$$I_L = \frac{10\angle -90°}{40\angle 0°} = 0.25\angle -90°$$

$$V_L = I_L Z_L = (0.25\angle -90°)(400\angle 90°) = 100\angle 0°\ V.$$

LA 2. [25 marks]

Consider the circuit shown below where V_1 and V_2 are both constant (but not equal) voltages. Prior to $t = 0$, the two switches were open for a sufficiently long time that the circuit is in a steady-state at $t = 0$; both switches are closed simultaneously at $t = 0$.



(a) [4 marks] Show that, for $t > 0$, the node voltages $v_a(t)$ and $v_b(t)$ are described by the following first-order simultaneous differential equations:

$$\frac{dv_a(t)}{dt} + \frac{1}{C}\left(\frac{1}{R_1} + \frac{1}{R_2}\right)v_a(t) - \frac{1}{R_2 C}v_b(t) = 0 \qquad (1)$$

$$-\frac{dv_a(t)}{dt} + 2\frac{dv_b(t)}{dt} + \frac{R_2}{L}v_b(t) = 0 \qquad (2)$$

KCL @ node 'a': $C\frac{d(v_a - V_1)}{dt} + \frac{v_a}{R_1} + \frac{(v_a - v_b)}{R_2} = 0$

or $\frac{dv_a}{dt} + \frac{1}{C}\left(\frac{1}{R_1} + \frac{1}{R_2}\right)v_a - \frac{1}{R_2 C}v_b = 0$ — (1)

KCL @ node 'b': $\frac{(v_b - v_a)}{R_2} + \frac{1}{L}\int v_b\,dt + \frac{(v_b - V_2)}{R_2} = 0$

or $-\frac{1}{R_2}\frac{dv_a}{dt} + \frac{2}{R_2}\frac{dv_b}{dt} + \frac{1}{L}v_b = 0$

or $-\frac{dv_a}{dt} + 2\frac{dv_b}{dt} + \frac{R_2}{L}v_b = 0$ — (2)

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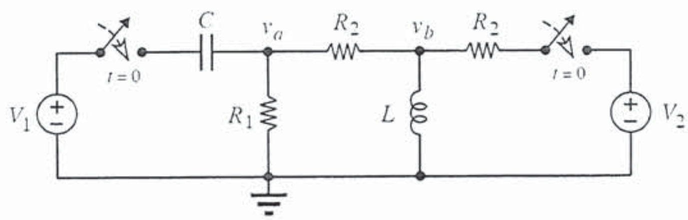
$$\frac{dv_a(t)}{dt} + \frac{1}{C}\left(\frac{1}{R_1} + \frac{1}{R_2}\right)v_a(t) - \frac{1}{R_2 C}v_b(t) = 0 \qquad (1)$$

$$-\frac{dv_a(t)}{dt} + 2\frac{dv_b(t)}{dt} + \frac{R_2}{L}v_b(t) = 0 \qquad (2)$$

(b) [4 marks] Determine the initial value (*i.e.*, at $t = 0+$) of the capacitor current, and the steady-state value (*i.e.*, at $t = \infty$) of the inductor current.

I_C I_L



$$v_C(0-) = v_C(0+) = 0 \Rightarrow v_a(0+) = V_1$$

KCL @ node 'a': $-I_C + \frac{V_1}{R_1} + \frac{V_1 - V_2}{2R_2} = 0$

$$\therefore I_C = \frac{V_1}{R_1} + \frac{V_1 - V_2}{2R_2}$$

$$\text{or } I_C = V_1\left(\frac{R_1 + 2R_2}{2R_1R_2}\right) - \frac{V_2}{2R_2}$$

$v_L(\infty) = 0$ (short cct.)

$$\therefore I_L = \frac{V_2}{R_2}$$

(c) [2 marks] Write equations (1) and (2) in matrix form $\dot{\boldsymbol{v}} = A\boldsymbol{v}$ as a two-dimensional, first-order, linear, homogeneous system of equations.

$$\frac{dv_a(t)}{dt} + \frac{1}{C}\left(\frac{1}{R_1} + \frac{1}{R_2}\right)v_a(t) - \frac{1}{R_2 C}v_b(t) = 0 \qquad (1)$$

$$-\frac{dv_a(t)}{dt} + 2\frac{dv_b(t)}{dt} + \frac{R_2}{L}v_b(t) = 0 \qquad (2)$$

Substitute (1) into (2)

$$\frac{-1}{R_2}\left(\frac{-1}{C}\left(\frac{1}{R_1}+\frac{1}{R_2}\right)V_a + \frac{1}{R_2C}V_b\right) + \frac{2}{R_2}\frac{dV_b}{dt} + \frac{1}{L}V_b = 0$$

$$\frac{2}{R_2}\frac{dV_b}{dt} + \frac{1}{R_2C}\left(\frac{1}{R_1}+\frac{1}{R_2}\right)V_a + \left(\frac{-1}{R_2^2C} + \frac{1}{L}\right)V_b = 0 \qquad (2')$$

$$\Rightarrow \quad \dot{V}_a = \frac{-1}{C}\left(\frac{1}{R_1} + \frac{1}{R_2}\right)V_a + \frac{1}{R_2C}V_b$$

$$\dot{V}_b = \frac{-1}{2C}\left(\frac{1}{R_1}+\frac{1}{R_2}\right)V_a + \frac{R}{2}\left(\frac{1}{R_2^2C} - \frac{1}{L}\right)V_b.$$

$$\Rightarrow \quad \begin{bmatrix}\dot{V}_a \\ \dot{V}_b\end{bmatrix} = \begin{bmatrix} -\frac{1}{C}\left(\frac{1}{R_1}+\frac{1}{R_2}\right) & \frac{1}{R_2C} \\ \frac{-1}{2C}\left(\frac{1}{R_1}+\frac{1}{R_2}\right) & \frac{R}{2}\left(\frac{1}{R_2^2C} - \frac{1}{L}\right)\end{bmatrix}\begin{bmatrix}V_a \\ V_b\end{bmatrix}$$

(d) [8 marks] When R_2 = 100 Ω, C = 0.01 F and L = 1 H, the system of equations is

$$\begin{bmatrix} \dot{v}_a \\ \dot{v}_b \end{bmatrix} = \begin{bmatrix} -\frac{100}{R_1} - 1 & 1 \\ -\frac{50}{R_1} - \frac{1}{2} & -49.5 \end{bmatrix} \begin{bmatrix} v_a \\ v_b \end{bmatrix}.$$

In this case, find a range of values for R_1 such that $v_a(t)$ and $v_b(t)$ oscillate as they decay to their steady-state values (*i.e.*, they exhibit an underdamped response).

We need eigenvalues of $A = \begin{bmatrix} \frac{-100}{R_1} - 1 & 1 \\ \frac{-50}{R_1} - \frac{1}{2} & -49.5 \end{bmatrix}$ to be complex. Compute the characteristic polynomial:

$$\det(A - \lambda I) = \lambda^2 - \left(\frac{-100}{R_1} - 1 - 49.5\right)\lambda + \left(\frac{-100}{R_1} - 1\right)(-49.5) - (1)\left(\frac{-50}{R_1} - \frac{1}{2}\right)$$

$$= \lambda^2 + \left(\frac{100}{R_1} + 50.5\right)\lambda + \left(50 + \frac{5000}{R_1}\right)$$

We need $\quad \left(\frac{100}{R_1} + 50.5\right)^2 - 4\left(50 + \frac{5000}{R_1}\right) < 0$

$$\Rightarrow \quad (100 + 50.5R_1)^2 < 4(50R_1^2 + 5000R_1)$$

$$100^2 + 200(50.5)R_1 + 50.5^2R_1^2 < 200R_1^2 + 4(5000)R_1$$

$$\underbrace{2350.25R_1^2 - 9900R_1 + 10000}_{\text{Find roots}} < 0$$

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$$R_1 = \frac{9900 \pm \sqrt{9900^2 - 4(2350.25)(10000)}}{2(2350.25)}$$

$$= \frac{9900 \pm 2000}{2(2350.25)} = 1.681,\ 2.532$$

$\Rightarrow$ Eigenvalues are complex (solutions oscillate) when $\boxed{1.681 < R_1 < 2.532}$

(e) [7 marks] When $R_1 = R_2 = 1\ \Omega$, $C = 1\ F$ and $L = 0.1\ H$, the system of equations is

$$\begin{bmatrix} \dot{v}_a \\ \dot{v}_b \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ -1 & -4.5 \end{bmatrix} \begin{bmatrix} v_a \\ v_b \end{bmatrix}.$$

Find the general solution in this case.

Let $A = \begin{bmatrix} -2 & 1 \\ -1 & -4.5 \end{bmatrix}$ and compute the characteristic polynomial:

$$\det(A - \lambda I) = \lambda^2 + 6.5\lambda + 10$$

$$\Rightarrow \lambda = \frac{-6.5 \pm \sqrt{6.5^2 - 4(10)}}{2} = \frac{-6.5 \pm 1.5}{2} = \underline{-4, -\tfrac{5}{2}}$$

$\lambda_1 = -4 \quad (A - \lambda_1 I)\vec{v}_1 = \vec{0}$

$$\left[\begin{array}{cc|c} -2+4 & 1 & 0 \\ -1 & -4.5+4 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 2 & 1 & 0 \\ -1 & -0.5 & 0 \end{array}\right] \quad \underline{\vec{v}_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}}$$

$\lambda_2 = -2.5 \quad (A - \lambda_2 I)\vec{v}_2 = \vec{0}$

$$\left[\begin{array}{cc|c} -2+2.5 & 1 & 0 \\ -1 & -4.5+2.5 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 0.5 & 1 & 0 \\ -1 & -2 & 0 \end{array}\right] \quad \underline{\vec{v}_2 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}}$$

$$\Rightarrow \boxed{\begin{bmatrix} v_a(t) \\ v_b(t) \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ -2 \end{bmatrix} e^{-4t} + c_2 \begin{bmatrix} 2 \\ -1 \end{bmatrix} e^{-\frac{5}{2}t} \qquad c_1, c_2 \in \mathbb{R}}$$