Worksheet 3

Felix Funk, MATH Tutorial - Mech 221

1 Second Order Linear ODEs

Introduction: Second Order Homogeneous Linear ODEs.
Second order linear ODEs are a powerful tool to model oscillatory systems such as mass-spring systems, electrical circuits and vibrations. We focus more specifically on homogeneous ODEs with constant coefficients, i.e.

$$ay'' + by' + cy = 0 \text{ with } a \neq 0, b, c \text{ in } \mathbb{R}. \tag{1}$$

To solve these equations, one derives the so-called characteristic equation and analyzes its properties in the following steps:

1. Set $y(t) = e^{rt}$ with constant r and substitute into equation (1). One obtains the following equation:

   $$= 0 \tag{2}$$

   This equation is called characteristic equation.

2. The roots of this equation determines essentially the solutions of the system. We are going to differentiate the following three cases.
   (a) There are two distinct real roots r_1, r_2 such that $r_1 \neq r_2$.
   (b) There are two imaginary roots $r_1 = \mu + i\omega, r_2 = \mu - i\omega$ such that $\omega > 0$.
   (c) The two real roots coincide $r_1 = r_2$.

3. In the given cases there are two solutions of the following form:
   (a) Exponential growth or decay: $y_1(t) = e^{r_1 t}$ and $y_2(t) = e^{r_2 t}$,
   (b) Oscillatory motion: $y_1(t) = e^{\mu t} cos(\omega t)$ and $y_2(t) = e^{\mu t} sin(\omega t)$,
   (c) Amplified exponential growth/decay: $y_1(t) = e^{rt}$ and $y_2(t) = xe^{rt}$.

4. The general solution is then a superposition of the two solution, i.e.

   $$y(t) = \alpha y_1(t) + \beta y_2(t) \tag{3}$$

5. If applicable, one can solve for α and β through the corresponding initial value problem $y(t_0) = y_0$ and $y'(t_0) = y_1$.

In the following subsections we have a closer look at the three cases.

1.1 Two Distinct Real Roots

Problem: Model Problem.
Solve the IVP

$$y'' + 5y' - 6y = 0 \tag{4}$$

with the constraints $y(0) = 1, y'(0) = 1$.

Example: Two Distinct Real Roots.

1. Identify the characteristic equation:

2. The two distinct roots are

$$r_1 = \qquad , r_2 =$$

3. Consider

$$y_1(t) = e^{r_1 t}, y_2(t) = e^{r_2 t}.$$

   Show that the provided solutions indeed solve the ODE (4) and sketch the two functions. Sketch $y_1(t)$ and $y_2(t)$

4. The general solution is then

$$y(t) =$$

5. If applicable, use the IVP to solve for α and β

Problem: Problemset 1.
Find the general solution and, if provided, solve the IVP

1. $y'' - 9y = 0$,

2. $y'' + 5y' = 0$ under the constraint $y(0) = 1, y'(1) = 0$.

1.2 Two Imaginary Roots

Problem: Model Problem.
Solve the IVP

$$4y'' + 4y' + \frac{5}{4}y = 0 \tag{5}$$

with the constraints $y(0) = 0, y'(0) = k$.

Example: Two imaginary roots.

1. Identify the characteristic equation:

2. The two imaginary roots are

$$r_1 = \qquad\qquad , r_2 =$$

3. Consider μ and ω as on first page to obtain

$$y_1(t) = e^{\mu t}cos(\omega t), y_2(t) = e^{\mu t}sin(\omega t).$$

   Show that the provided solutions indeed solves the ODE (5). Sketch $y_1(t)$ and $y_2(t)$.

4. The general solution is then

$$y(t) =$$

5. If applicable, use the IVP to solve for α and β.

Problem: Problemset 2.
Find the general solution and, if provided, solve the IVP

1. $2y'' - 4y' + 4y = 0$ under the constraint $y(0) = 0, y'(0) = 0$,
2. $y'' + \omega^2 y = 0$ under the constraint $y(0) = -1, y'(0) = 1$.

1.3 A Single Real Root

Problem: Model Problem.
Solve the IVP

$$y'' + 2y' + y = 0, \tag{6}$$

with the constraints $y(0) = 1, y'(1) = 1$.

Example: A Single Real Root.

1. Identify the characteristic equation:

2. The real root is

$$r =$$

3. Consider

$$y_1(t) = e^{rt}, y_2(t) = te^{rt}.$$

   Show that the provided solutions indeed solve the ODE (6). Sketch $y_1(t)$ and $y_2(t)$.

4. The general solution is then

$$y(t) =$$

5. If applicable, use the IVP to solve for α and β

Problem: Problemset 3.
Find the general solution and, if provided, solve the IVP

1. $2y'' - 4y' + 2y = 0$,
2. $y'' + 6y' + 9y = 0$ with $y(0) = 1, y'(0) = 1$.

2 Mixed Problems

Problem: Challenging Problems. Let a, b, c, C_1, C_2 be constants.

1. Provided is the ODE
$$y'' + ay' + y = 0.$$
For what range of values in a does the system allow oscillations?

2. Construct a second order ODE which has a general solution of the form
$$y(x) = C_1 e^{-2x} cos(3x) + C_2 e^{-2x} sin(3x).$$

3. Construct a second order ODE which has a general solution of the form
$$y(t) = C_1 e^{-at} + C_2 e^{-at} t.$$

4. The system $ay'' + by' + cy = 0$ shall describe a system that is capable of harmonic oscillations (i.e. non-trivial undamped/non-increasing periodic solutions.) What are the restrictions on a, b, c such that y(t) oscillates, harmonically. How many measurements are necessary to determine $y(t)$, uniquely?