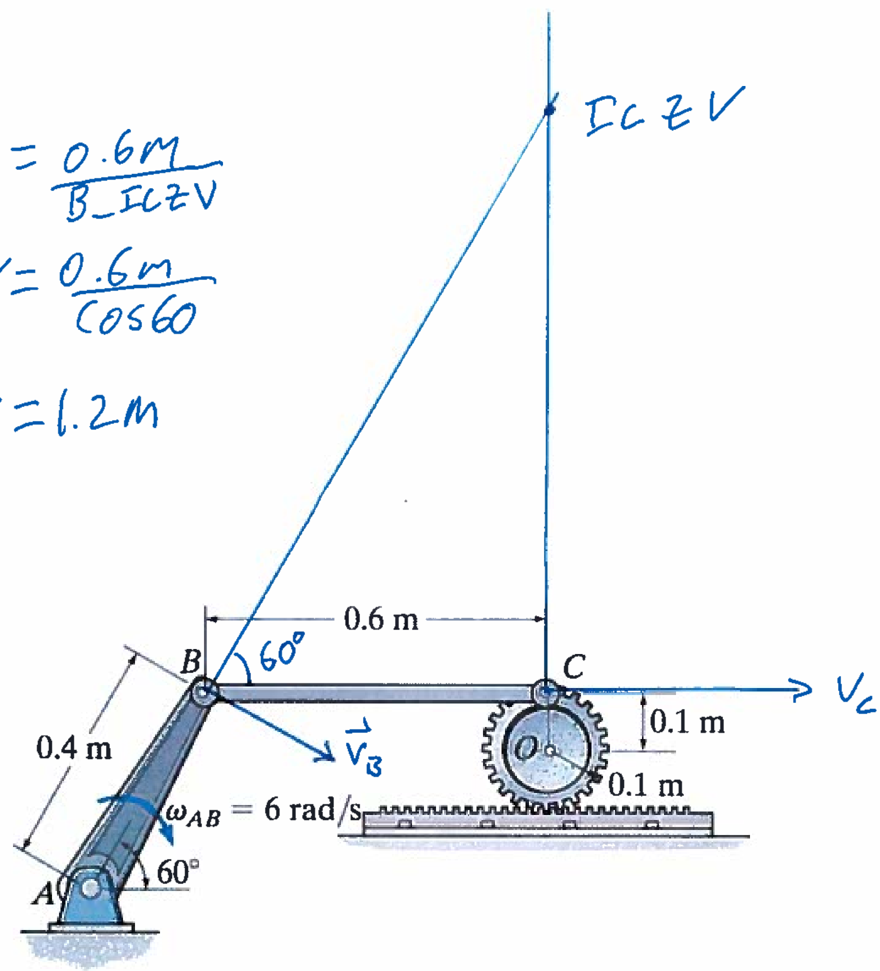
Name:________________________ Section:________________________

SA1 [5 Marks] Consider the mechanism below and visualize how it moves.

$\cos 60 = \frac{0.6m}{B\_ICZV}$

$B\_ICZV = \frac{0.6m}{\cos 60}$

$B\_ICZV = 1.2m$

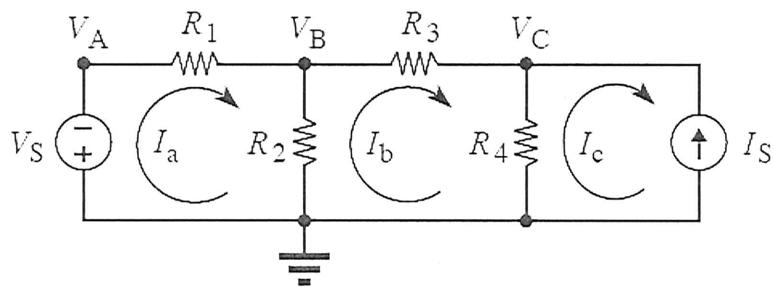


To get credit for the following questions you will need to show your solution method by drawing or by calculations:

a) [2 marks] Bar BC rotates **counterclockwise**. Using a straightedge, on the diagram, draw **and label** a vector showing

1. Direction of velocity of point B, $\mathbf{v}_B$.
2. Direction and velocity of point C, $\mathbf{v}_C$.

SA1 [5 Marks]. Consider the following circuit for which all the resistor and source values are known:



(a) [2.5 Marks] Write the minimal set of equations that could be solved for the labeled node voltages V_A, V_B and V_C.

$$V_A = -V_S$$

$$\frac{V_B - V_A}{R_1} + \frac{V_B}{R_2} + \frac{V_B - V_C}{R_3} = 0$$

$$\frac{V_C - V_B}{R_3} + \frac{V_C}{R_4} - I_S = 0$$

(b) [2.5 Marks] Write the minimal set of equations that could be solved for the labeled mesh currents I_a, I_b and I_c.

$$V_S + I_a R_1 + (I_a - I_b) R_2 = 0$$

$$(I_b - I_a) R_2 + I_b R_3 + (I_b - I_c) R_4 = 0$$

$$I_c = -I_S$$

Prob 1 [25 marks] ***This question has 5 parts: (a), (b), (c), (d) & (e).***

Consider the following mixing problem for parts (a), (b) and (c) below.
A small tank initially contains 100L of pure water. At time $t = 0$, a salt-water solution with concentration 20g/L is poured into the tank at 1L/min and solution from the tank is drained at 2L/min. Assume the solution in the tank is perfectly mixed.

(a) [5 marks] Show that the total mass $M(t)$ of salt in the tank (as a function of time t) satisfies the differential equation

$$\frac{dM}{dt} = 20 - \frac{2M}{100 - t}$$

(Note that M is measured in grams and t is measured in minutes.)

Let $C(t)$ be the concentration (in g/L) of the salt solution and let $V(t)$ be the volume of solution. By the conservation of mass, we have

$$\frac{dM}{dt} = \left(\begin{matrix}\text{mass}\\ \text{In}\end{matrix}\right) - \left(\begin{matrix}\text{mass}\\ \text{out}\end{matrix}\right)$$

$$= 20\frac{g}{L} \cdot 1\frac{L}{\text{min}} - C(t) \cdot 2\frac{L}{\text{min}}$$

V_{out} V_{in}

Since $C(t) = M(t)/V(t)$ and $V(t) = 100 - 2t + t$
$= 100 - t$

$$\boxed{\frac{dM}{dt} = 20 - \frac{M}{100-t} \cdot (2)}$$

(b) [10 marks] Solve the equation in part (a) and find a formula for the **concentration** $C(t)$ of the salt solution in the tank (in g/L) as a function of time t.

$$\frac{dM}{dt} = 20 - \frac{2M}{100-t} \implies M' + \underbrace{\frac{2}{100-t}}_{p(t)} M = \underbrace{20}_{f(t)}$$

This is a linear equation and we compute the integrating factor: $r(t) = e^{\int p(t)\,dt}$

$$\int p(t)\,dt = \int \frac{2}{100-t}\,dt = -2\ln(100-t)$$

$$\implies r(t) = e^{-2\ln(100-t)} = (100-t)^{-2}$$

Therefore, $M(t) = (100-t)^2 \int (100-t)^{-2}(20)\,dt$

$$= (100-t)^2\left(\frac{20}{100-t} + K\right)$$

$$= 20(100-t) + K(100-t)^2 \text{ for some } K.$$

Since $M(0) = 0$, $K = \frac{-20}{100} = -\frac{1}{5}$

$$\implies M(t) = 20(100-t) - \frac{1}{5}(100-t)^2$$

$$\implies \boxed{C(t) = 20 - \frac{1}{5}(100-t)}$$

(c) [3 marks] What is the concentration of the solution in the tank at the instant just before it is empty?

Since $V(t) = 100 - t$, the tank is empty when $t = 100$, therefore

$$\boxed{c(100) = 20 - \frac{1}{5}(0) = 20\ g/L}$$

Consider the following differential equation in parts (d) and (e):

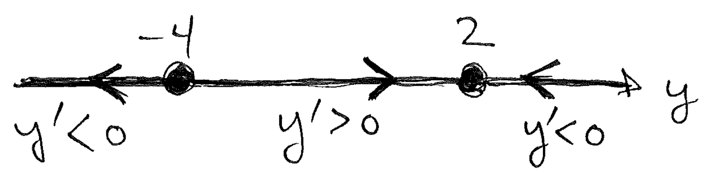
$$y' = -y^2 - 2y + 8 \quad (*)$$

(d) [5 marks] Sketch the phase line for the first order equation (*) above including critical points and flow directions.

$$y' = -y^2 - 2y + 8$$
$$= (2-y)(y+4)$$

Critical points are

$y = 2$, $y = -4$



(e) [2 marks] Compute the steady state

$$\lim_{t\to\infty} y(t)$$

for the solution $y(t)$ of the differential equation (*) above satisfying $y(2) = 1$. Justify your answer.

The initial condition $y(2) = 1$ lies in the region where $y' > 0$ $(-4 < y < 2)$

therefor $\boxed{\lim_{t\to\infty} y(t) = 2}$