Why $+y_h(t)$ a solution part. problem? Solutions

Worksheet 4

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1 Non-homogeneous second order linear ODEs

The first part of the worksheet discusses how to treat second order linear constant-coefficient ODEs with non-homogeneous right-hand-side. The second part of the worksheet is about mass-damper-spring systems as an illustration of homogeneous second order linear ODEs.

Introduction: Non-homogeneous Second Order Linear ODEs.
In the last worksheet, we have explored the different dynamics that one can capture in systems of the form.

$$ay_h'' + by_h' + cy_h = 0 \tag{1}$$

These solutions y_h will be called homogeneous solutions as they solve the homogeneous problem (i.e. the right side of the equation is 0.)

In applications, we frequently observe ODEs of the form:

$$ay'' + by' + cy = f(t) \tag{2}$$

We will discuss: How can we find a solution? What is the form of a general solution to equation (2)?

Problem: The form of the general solution. Let $y_{p,1}$ and $y_{p,2}$ are two particular solutions to the non-homogeneous differential equation (2). Show: The two particular solution only differ by a homogeneous solution. Hint: Consider $y = y_{p,1} - y_{p,2}$.

$$\begin{aligned} ay'' + by' + cy &= ay_{p,1}'' - ay_{p,2}'' + by_{p,1}' - by_{p,2}' + cy_{p,1} - cy_{p,2} \\ &= ay_{p,1}'' + by_{p,1}' + cy_{p,1} - (ay_{p,2}'' + by_{p,2}' + cy_{p,2}) \\ &= f(t) - f(t) = 0 \quad \Rightarrow y \text{ solves the homogeneous eq.} \end{aligned}$$

Conclusion: The General Solution.
Thus, the general solution to equation (2) is given by $y(t) = y_p(t) + y_h(t)$, where y_p is any (guessed) non-homogeneous solution and y_h is a homogeneous solution.

Introduction: Method of Undetermined Coefficients: Guessing y_p.
To obtain a particular solution, sometimes educated guessing is the most straight-forward approach. We use versions of the inhomogeneity $f(t)$ as a guess for the particular solution y_p. In the following subsections, we explore polynomial, oscillatory, and exponential inhomogeneities $f(t)$.

1.1 Method of Undetermined Coefficients: $f(t)$ is a Polynomial.

Problem: Polynomial $f(t)$.
Find a particular solution to:

$$y'' + 2y' + 2y = 5t + 1, \tag{3}$$

Example: Educated Guess: Polynomial.

1. Identify $f(t) = 5t+1$
2. $f(t)$ is a polynomial. Mimic $f(t)$ by guessing a polynomial with undetermined coefficients.

$$y_p(t) = At + B$$

3. Substitute $y_p(t)$ into equation (3).

$$2A + \boxed{2A}t + 2B = \boxed{5}t + 1$$

4. Determine A, B.

$2A = 5$ (for the linear coeff to agree) $\Rightarrow \boxed{A = \frac{5}{2}}$
$2A + 2B = 1 \Rightarrow 5 + 2B = 1 \Rightarrow \boxed{B = \frac{-4}{2} = -2}$

Problem: General solution and IVP.
Find the general solution to equation (3). Solve the IVP $y(0) = 1, y'(0) = 2$

Example: General Solution.

1. Find the general homogeneous solution y_h. $y'' + 2y' + 2y = 0$ (Homog. eq)

$\Rightarrow r^2 + 2r + 2 \quad r_{1,2} = \frac{-2 \pm \sqrt{4-8}}{2} = -1 \pm \sqrt{-4} = -1 \pm 2i$

$y_h(t) = e^{-t}[\alpha\cos(2t) + \beta\sin(2t)]$

2. Combine:

$$y(t) = y_h(t) + y_p(t) = e^{-t}[\alpha\cos(2t) + \beta\sin(2t)] + [\tfrac{5}{2}t - 2]$$

3. Solve the IVP. $1 = y(0) = \alpha - 2 \iff \boxed{\alpha = 3}$

$y'(t) = -e^{-t}[\alpha\cos(2t) + \beta\sin(2t)] + e^{-t}[-2\alpha\sin(2t) + 2\beta\cos(2t)] + \frac{5}{2}$

$2 = y'(0) = -\underbrace{[\alpha]}_{=3} + [2\beta] + \frac{5}{2} \iff 2\beta = 2 + 3 - \frac{5}{2} = \frac{5}{2}$

$\iff \beta = \frac{5}{4}$

$y(t) = e^{-t}[3\cos(2t) + \frac{5}{4}\cos(2t)] + [\frac{5}{2}t - 2]$

Problem: 1.1. Find ~~particular~~ the general ~~solutions~~ solution to

$$y'' + y = t. \tag{4}$$

$y(t) = At+B \Rightarrow At+B = t \Rightarrow A = 1,\ B = 0 \Rightarrow y_p(t) = t$

$r^2+1 = 0 \overset{\alpha = \pm i}{\Rightarrow} y_h(t) = \alpha\cos(t) + \beta\sin(t)$

$y(t) = \alpha\cos(t) + \beta\sin(t) + t$

Reduce space //

Problem: 1.2. Find the solution to

$$y'' + 2y' + y = t^2 \tag{5}$$

that satisfies the conditions y(0) = -1, y'(0)=0.

$y_p(t) = At^2 + Bt + C \quad y'(t) = 2At + B \quad y'' = 2A$

$2A + 2(2At+B) + (At^2+Bt+C)$

$= At^2 + (4A+B)t + 2A + 2B + C \overset{!}{=} t^2 + 0\cdot t + 0$

Compare coefficients:

t^2: $\boxed{A = 1}$

t: $4A + B = 4 + B = 0 \Rightarrow \boxed{B = -4}$ $\Rightarrow y_p(t) = t^2 - 4t + 6$

t^0: $2A + 2B + C = 2 - 8 + C = 0 \Rightarrow \boxed{C = 6}$

$y_h(t)$: $r^2 + 2r + 1 = (r+1)^2 \overset{r=-1}{\Rightarrow} y_h(t) = \alpha e^{-t} + \beta t e^{-t}$

$y(t) = y_h(t) + y_p(t) = \alpha e^{-t} + \beta t e^{-t} + t^2 - 4t + 6$

$-1 = y(0) = \alpha + 6 \Rightarrow \alpha = -7$

$0 = y'(0) = \underbrace{-\alpha}_{=7} + \beta - 4 \Rightarrow \beta = -3$

$y(t) = -7e^{-t} - 3te^{-t} + t^2 - 4t + 6$

1.2 Method of Undetermined Coefficients: $f(t)$ is Periodic.

Problem: Periodic $f(t)$.
Find a particular solution to:

$$y'' - 4y' + 4y = cos(4t), \tag{6}$$

Example: Educated Guess: Cosines and Sines.

1. Identify $f(t) =$ $\cos(4t)$

2. $f(t)$ undergoes periodic motion. Mimic $f(t)$ by guessing a periodic function with undetermined coefficients.

$$y_p(t) = Acos(4t) + Bsin(4t)$$

3. Substitute $y_p(t)$ into equation (6). $y_p'(t) = -4A\sin(4t) + 4B\cos(4t)$
$y_p''(t) = -16A\cos(4t) - 16B\sin(4t)$

$$-16A\cos(4t) - 16B\sin(4t) - 4\cdot[-4A\sin(4t) + 4B\cos(4t)] + 4[A\cos(4t) + B\sin(4t)] \stackrel{!}{=} \cos(4t)$$

4. Determine A, B. Comparison of cosines & sines:

cos: $(-16A - 16B + 4A) = 1$

sin: $(-16B + 16A + 4B) = 0$

From second eq: $A = \frac{1}{16} \cdot 12B = \frac{3}{4}B$

From first: $-12A - 16B = -9B - 16B = -25B = 1 \Rightarrow B = \frac{-1}{25}$

$A = -\frac{3}{100}$

Problem: General solution.
Find the general solution to equation (6).

Example: General solution.

1. Find the general homogeneous solution y_h.

$r^2 - 4r + 4 = 0 \Rightarrow (r-2)^2 : \ r_{1,2} = 2$

$y_h(t) = e^{2t} \cdot (\alpha + \beta t)$

2. Combine:

$$y(t) = y_h(t) + y_p(t) = e^{2t}(\alpha + \beta t) - \frac{3}{100}\cos(4t) - \frac{1}{25}\sin(4t)$$

Problem: 2.1. Find particular solutions to

$$y'' - y = \sin(t). \qquad y_p(t) = A\sin(t) + B\cos(t) \tag{7}$$

$$-A\sin(t) - B\cos(t) - A\sin(t) - B\cos(t) = \sin(t)$$

$$\Rightarrow A = -\frac{1}{2} \quad B = 0$$

$$y_p(t) = -\frac{1}{2}\sin(t)$$

Problem: 2.2. Find a particular solution to

$$y'' + y = \sin(t). \tag{8}$$

1. Determine the homogeneous solution y_h, first. $r^2 + 1 = 0$

   $y_h = \alpha\cos(t) + \beta\sin(t)$

2. Observe, that the previous guess fails.

   Using $y_p(t) = A\cos(t) + B\sin(t)$ yields $y_p''(t) + y_p = 0$

3. Now, try the guess $y_p(t) = At\cos(t) + Bt\sin(t)$. Hypothesize, what one can do, when the educated guess coincides with a homogeneous solution.

$$y_p(t) = t \cdot [A\cos(t) + B\sin(t)]$$

$$y_p' = [A\cos(t) + B\sin(t)] + t \cdot [-A\sin(t) + B\cos(t)]$$

$$y_p'' = [-A\sin(t) + B\cos(t)] + [-A\sin(t) + B\cos(t)] + t[-A\cos(t) - B\sin(t)]$$

$$y_p'' + y_p = -2A\sin(t) + 2B\cos(t) \stackrel{!}{=} \sin(t)$$

$$\boxed{A = -\frac{1}{2}} \quad \boxed{B = 0}$$

$$y_p(t) = -\frac{1}{2}t\cos(t)$$

Use $y_p(t) = t\, y_{guess}(t)$ ...

1.3 Method of Undetermined Coefficients: $f(t)$ grows Exponentially.

Problem: Exponential $f(t)$.
Find a solution to

$$y'' - 2y' = e^{3t}, \tag{9}$$

that satisfies y(0) = 0, y'(0) = 1.

Example: Educated Guess: Exponential.

1. Identify $f(t) =$ e^{3t}
2. $f(t)$ grows exponentially. Mimic $f(t)$ by guessing a periodic function with undetermined coefficients.
$$y_p(t) = Ae^{3t}$$
.

Ref //

3. Substitute $y_p(t)$ into equation ~~(6)~~. (8)
$$9Ae^{3t} - 6Ae^{3t} = e^{3t} \Rightarrow 3A = 1$$
4. Determine A.
$$A = \frac{1}{3}$$
5. Find the general homogeneous solution y_h.
$$r^2 - 2r = r\cdot(r-2)$$
$$y_h(t) = \alpha + \beta\cdot e^{2t}$$
6. Combine:
$$y(t) = y_h(t) + y_p(t) = \alpha + \beta e^{2t} + \frac{1}{3}e^{3t}$$
7. Solve the IVP.
$$0 = y(0) = \alpha + \beta + \frac{1}{3}$$
$$1 = y'(0) = 2\beta + 1 \Rightarrow \beta = 0 \Rightarrow \alpha = -\frac{1}{3}$$
$$y(t) = -\frac{1}{3} + \frac{1}{3}e^{3t}$$

change // **Problem: 2.1.** Find ~~particular~~ the general solutions to

$$
\begin{aligned}
2y'' - 8y' &= e^{-t}. && (10)\\
y'' - 100y &= e^{10t} && (11)
\end{aligned}
$$

(10) Part: $y_p(t) = Ae^{-t}$

$2Ae^{-t} + 8Ae^{-t} = e^{-t} \Rightarrow 10A = 1 \Rightarrow A = \frac{1}{10}$

Homog: $2(r^2 - 4r) = 0 \iff r(r-4) = 0 \Rightarrow r_1 = 0, r_2 = 4$

$y_h(t) = \alpha + \beta e^{4t}$

$y(t) = \alpha + \beta e^{4t} + \frac{1}{10} e^{-t}$

(11) Homog: $y_h(t) = \alpha e^{10t} + \beta e^{-10t}$ as $r^2 - 100 = 0 \iff r_1 = 10, r_2 = -10$

e^{10t} is a homog. solution.

choose: $y_p(t) = Ate^{10t}$ $\quad y_p' = Ae^{10t} + t \cdot 10 A e^{10t}$

$y_p'' = 10Ae^{10t} + 10Ae^{10t} + t\,100A e^{10t}$

$y_p'' - 100y_p = 20Ae^{10t} + t\,100Ae^{10t} - 100\,tAe^{10t} \overset{!}{=} e^{10t} = f(t)$

$\Rightarrow 20Ae^{10t} = e^{10t} \Rightarrow A = \frac{1}{20}$

$y(t) = y_h(t) + y_p(t)$

$= \alpha e^{10t} + \beta e^{-10t} + \frac{1}{20} t e^{10t}.$

2 Application: Mass-Spring-Damper System

Introduction: Spring- Mass- Damper Systems.
To model mass-spring-damper systems, we use a second order system of the form

$$x'' + bx' + cx = 0,$$

where $b \geq 0$ models dampening and c is a positive spring-related constant. Let $x(t)$ denote the vertical displacement at time t. We distinguish four cases according to the diagram below depending on the roots of the system.

Problem: A damped oscillator. For the mass-spring damper system with varying dampening

$$x'' + bx' + 4x = 0, \tag{12}$$

find all b such that $x(t)$ exerts

- free motion,
- underdampened motion,
- critically-damped motion,
- overdamped motion.

$r_{1,2} = \dfrac{-b \pm \sqrt{b^2-16}}{2} = \underbrace{-\dfrac{b}{2}}_{\text{Real}} \pm \underbrace{\sqrt{b^2-16}}_{\text{Potentially imaginary}}$

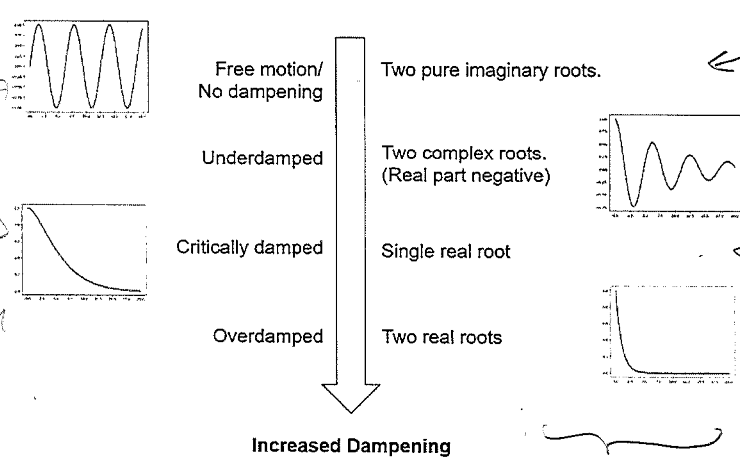
Only for $\boxed{b=0}$: pure imag roots

If $b^2-16<0$, then $\sqrt{b^2-16} = i\underbrace{\sqrt{16-b^2}}_{>0} \rightarrow$ complex

For $\boxed{0<b<4}$ underdamped

For $\boxed{b=4}$: $r_{1,2} = -\dfrac{b}{2} \pm 0 \Rightarrow$ critically-damped

For $\boxed{4<b}$ $\sqrt{b^2-16} > 0$, i.e. $r_{1,2}$ real $\Rightarrow$ overdamped



$\leftarrow b=0$

$\leftarrow b<0<4$

$\leftarrow b=4$

$\leftarrow 4<b$

One can see that free motion and critically-damped motion are solely theoretical classifications. ...

practically relevant cases