Worksheet 6

Felix Funk, MATH Tutorial - Mech 221

1 Laplace - Transformation

Reminder: Laplace - Transform. The Laplace transform of a function $f(t)$ is defined by

$$F(s) = L\{f(t)\}(s) = \int_0^\infty e^{-st} f(t)dt. \tag{1}$$

Furthermore, there is an inverse transform $L^{-1}\{F(s)\}(t)$ that satisfies

$$L^{-1}\{F(s)\}(t) = f(t),$$

i.e. the Laplace transform and its inverse cancel. It satisfies four basic properties:

1. **Linearity:**

$$L\{af(t) + bg(t)\}(s) =$$

2. **Differentiation is Transformed to Multiplication:**

$$L\{x'(t)\}(s) =$$

3. **First Shifting Theorem:**

$$L\{e^{-at} f(t)\}(s) =$$

In the subsequent section you will also derive/revise the second shifting theorem: Let $u(t)$ be the Heaviside function as defined (2).

4. **Second Shifting Theorem:**

$$L\{u(t-a)f(t-a)\}(s) =$$

2 The Heaviside Function

Introduction: Heaviside - Function u(t). The Heaviside- function is a step-function that is commonly used to construct discontinuous/piecewise - continuous signals or forces and defined by

$$u(t) = \begin{cases} 0, & \text{if } t < 0, \\ 1, & \text{if } t \geq 0. \end{cases} \tag{2}$$

1. Sketch $u(t), u(t-\pi), sin(t)u(t-\pi)$

2. Determine $L\{u(t-a)\}(s)$. Write $L\{u(t-a)f(t-a)\}(s)$ in terms of $F(s) = L\{f(t)\}(s)$.

3. Model $f(t) = \begin{cases} 1 & \text{if } 0 \leq t < 1 \\ 0 & \text{otherwise} \end{cases}$ using Heaviside-functions. Calculate $L\{f(t)\}(s)$.

4. Write
$$g(t) = \begin{cases} (t-1)^2 & \text{if } 1 \leq t < 2, \\ (3-t) & \text{if } 2 \leq t < 3, \\ 0 & \text{otherwise.} \end{cases}$$
using Heaviside functions. Calculate $L\{g(t)\}(s)$.

3 Solving Differential Equations: Mixed Problems

Problem: 1: Using the Shifting Theorems.
Apply the Laplace transform and use the four basic properties to simplify

$$x'' + x = h(t), \quad h(t) = \begin{cases} 2t^2 e^{5t} & \text{if } 0 \leq t < 1, \\ 0 & \text{otherwise.} \end{cases} \tag{3}$$

for the initial value problem x(0) = 1, x'(0) = 1.

Problem: 2: Solving Homogeneous Systems, optional.
Solve

$$my'' + cy' + ky = 0, \quad y(0) = a, y'(0) = b,$$

where m, c, k are positive constants and the constraint $c^2 - 4km > 0$ is satisfied.

Example: 3: Solving Non-homogeneous Systems. Source: Cole Zmurchok

1. Show $L\{cos(2t)\}(s) = \frac{s}{s^2+4}$.

2. Use partial fractions to show

$$\frac{s}{(s^2+1)(s^2+4)} = \frac{1}{3}\frac{s}{s^2+1} - \frac{1}{3}\frac{s}{s^2+4}.$$

3. Using Laplace-transforms, solve $x'' + x = cos(2t)$ with $x(0) = 0$ and $x'(0) = 1$.

4 Transfer functions

Transfer functions give an algebraic dependence of the output based on the input.

Introduction: Using Transfer functions (Source: Cole Zmurchok)

Consider $Lx = f(t)$ with L a constant coefficient differential operator, with all initial conditions 0. Taking the Laplace Transform gives $A(s)X(s) = F(s)$, so that $X(s) = H(s)F(s)$ for any input $f(t)$. This suggests that $x(t)$ can be found by multiplying $F(s)$ by $H(s)$ in the frequency-domain and subsequently taking the inverse Laplace Transform.

1. Find the transfer function for the ODE $x'' + \omega_0^2 x = f(t)$, assuming all initial conditions are 0.

2. Suppose $f(t) = 1$. Use the transfer function from above to find $x(t)$.

5 Additional Problems

Problem: Problemset.
Solve

1. $y'' + 4y' + 5y = e^{-t}(cos(t) + 3sin(t))$ with $y(0) = 0$ and $y'(0) = 4$.
2. $y'' + y = \begin{cases} 3 & \text{if } 0 \le t < \pi \\ 0 & \text{otherwise} y(0) = 0, y'(0) = 0 \end{cases}$ with $y(0) = 0$ and $y'(0) = 0$.
3. $9y'' + 6y' + y = 3e^{3t}$ with $y(0) = 0$ and $y'(0) = -3$.
4. $y'' - 5y' + 6y = 10e^{t}cos(t)$ with $y(0) = 2$ and $y'(0) = 1$.