Name:______________________________ Section:___________________________

Question 1 [8 marks]

Compute the surface integral

$$\iint_S xy\,dS$$

over the surface S given by $z = x^2 + y^2$ with $0 \le x \le 1$ and $0 \le y \le 1$.

$f(x,y) = x^2 + y^2 \qquad f_x = 2x \qquad f_y = 2y$

$$\iint_S xy\,dS = \int_0^1 \int_0^1 xy\sqrt{1+(2x)^2+(2y)^2}\,dx\,dy$$

$$= \int_0^1 \int_0^1 xy\sqrt{1+4x^2+4y^2}\,dx\,dy$$

$$= \int_0^1 \left(y\,(1+4x^2+4y^2)^{3/2}\,\frac{2}{3}\,\frac{1}{8}\,\Big|_0^1 \right) dy$$

$$= \frac{1}{12}\int_0^1 \left(y(5+4y^2)^{3/2} - y(1+4y^2)^{3/2} \right) dy$$

$$= \frac{1}{12}\left((5+4y^2)^{5/2}\,\frac{2}{5}\cdot\frac{1}{8} - (1+4y^2)^{5/2}\,\frac{2}{5}\cdot\frac{1}{8} \right)\Big|_0^1$$

$$= \frac{1}{240}\left(9^{5/2} - 5^{5/2} - (5^{5/2} - 1) \right)$$

$$\boxed{= \frac{244 - 50\sqrt{5}}{240}} \approx 0.5508$$

$$= \frac{122 - 25\sqrt{5}}{120}$$

Name:________________________________ Section:____________________________

Question 2 [9 marks]

Find the surface area of the surface S defined by the parameterization

$$r(u,v) = (u^2 \sin v \cos v\,, u \sin v\,, u \cos v)$$

for $0 \le u \le 1$ and $0 \le v \le 2\pi$.

$$\vec{r}(u,v) = (u^2 \sin(v)\cos(v), u\sin(v), u\cos(v))$$

$$\vec{r}_u = (2u\sin(v)\cos(v), \sin(v), \cos(v))$$

$$\vec{r}_v = (u^2(\cos^2(v) - \sin^2(v)), u\cos(v), -u\sin(v))$$

$$\vec{r}_u \times \vec{r}_v = \Big(\sin(v)(-u\sin(v)) - u\cos^2(v),$$
$$-(2u\sin(v)\cos(v)(-u\sin(v)) - u^2(\cos^2(v) - \sin^2(v))\cos(v)),$$
$$2u\sin(v)\cos(v)\, u\cos(v) - u^2(\cos^2(v) - \sin^2(v))\sin(v) \Big)$$

$$= \Big(-u, -(-u^2\sin^2(v)\cos(v) - u^2\cos^2(v)\cos(v)),$$
$$u^2 \sin(v)\cos^2(v) + u^2\sin^2(v)\sin(v)\Big)$$

$$= (-u, u^2\cos(v), u^2\sin(v))$$

$$\|\vec{r}_u \times \vec{r}_v\| = \sqrt{(-u)^2 + (u^2\cos(v))^2 + (u^2\sin(v))^2}$$

$$= u\sqrt{1+u^2}\,.$$

$$\iint_S dS = \int_0^{2\pi}\int_0^1 u\sqrt{1+u^2}\,du\,dv$$

$$= 2\pi \cdot (1+u^2)^{3/2} \cdot \frac{2}{3}\frac{1}{2} \Big|_0^1 = \boxed{\frac{2\pi}{3}(2\sqrt{2}-1)}$$

Name:______________________________ Section:___________________________

Question 3 [9 marks]

Find the surface integral $\iint \mathbf{F} \cdot d\mathbf{S}$ of the vector field

$$\mathbf{F}(x, y, z) = (z - y^2)\mathbf{i} + (xy + z)\mathbf{j} + (x + y)\mathbf{k}$$

over the surface S defined by $z = 2 - 3x^2 - y^2$ for $0 \le x \le 1$, $0 \le y \le 1$, and with unit normal vector $\mathbf{n}$ pointing in the positive z direction.

$f(x,y) = 2 - 3x^2 - y^2 \quad f_x = -6x \quad f_y = -2y$

$\vec{r}(x,y) = (x, y, f(x,y))$

$\vec{r}_x = (1, 0, -6x) \qquad \vec{r}_x \times \vec{r}_y = (6x, 2y, 1)$

$\vec{r}_y = (0, 1, -2y)$

$$\iint_S \vec{F} \cdot d\vec{S} = \int_0^1 \int_0^1 (z - y^2, xy + z, x + y) \cdot (6x, 2y, 1)\, dx\, dy$$

$$= \int_0^1 \int_0^1 \left(6x(z - y^2) + 2y(xy + z) + (x + y)\right) dx\, dy$$

$$= \int_0^1 \int_0^1 \left(6x(2 - 3x^2 - 2y^2) + 2y(xy + 2 - 3x^2 - y^2) + x + y\right) dx\, dy$$

$$= \int_0^1 \int_0^1 \left(13x - 18x^3 - 10xy^2 + 5y - 6x^2y - 2y^3\right) dx\, dy$$

$$= \int_0^1 \frac{13}{2}x^2 + \frac{18}{4}x^4 - 5x^2y^2 + 5yx - 2x^3y - 2y^3x \Big|_0^1 dy$$

$$= \int_0^1 \left(\frac{13}{2} + \frac{9}{2} - 5y^2 + 5y - 2y - 2y^3 - (0)\right) dy$$

$$= \int_0^1 (2 + 3y - 5y^2 - 2y^3)\, dy$$

$$= 2y + \frac{3}{2}y^2 - \frac{5}{3}y^3 - \frac{1}{2}y^4 \Big|_0^1 = 2 + \frac{3}{2} - \frac{5}{3} - \frac{1}{2} = 3 - \frac{5}{3}$$

$$\boxed{= \frac{4}{3}}$$

Name:______________________________ Section:___________________________

Question 4 [9 marks]

Find the surface integral $\iint \mathbf{F} \cdot d\mathbf{S}$ of the vector field

$$\mathbf{F}(x, y, z) = x^2 y\mathbf{i} + xz\mathbf{j} + yz\mathbf{k}$$

over the surface S of the sphere of radius 1.

Use the Divergence Theorem

$$\iint_S \vec{F} \cdot d\vec{S} = \iiint_E \operatorname{div} \vec{F}\, dV$$

$$= \iiint_E (2xy + 0 + y)\, dV$$

$$= \int_0^{2\pi}\int_0^{\pi}\int_0^1 \left(2(\rho\sin\phi\cos\theta)(\rho\sin\phi\sin\theta) + \rho\sin\phi\sin\theta\right) \cdot \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta.$$

$$= \int_0^{2\pi}\int_0^{\pi}\int_0^1 \left(2\rho^4\sin^3\phi\cos\theta\sin\theta + \rho^3\sin^2\phi\sin\theta\right) d\rho\, d\phi\, d\theta.$$

$$= \int_0^{2\pi}\int_0^{\pi} 2\left(\tfrac{1}{5}\rho^5\right)\sin^3\phi\cos\theta\sin\theta \Big|_0^1 d\phi\, d\theta$$

$$+ \int_0^{2\pi}\int_0^{\pi} \tfrac{1}{4}\rho^4 \sin^2\phi\sin\theta \Big|_0^1 d\phi\, d\theta.$$

$$= \frac{2}{5}\underbrace{\int_0^{2\pi}\cos\theta\sin\theta\, d\theta}_{0} \cdot \int_0^{\pi}\sin^3\phi\, d\phi + \frac{1}{4}\underbrace{\int_0^{2\pi}\sin\theta\, d\theta}_{0} \cdot \int_0^{\pi}\sin^2\phi\, d\phi$$

$$\frac{2}{5}\frac{1}{2}\sin^2\theta\Big|_0^{2\pi} = 0$$

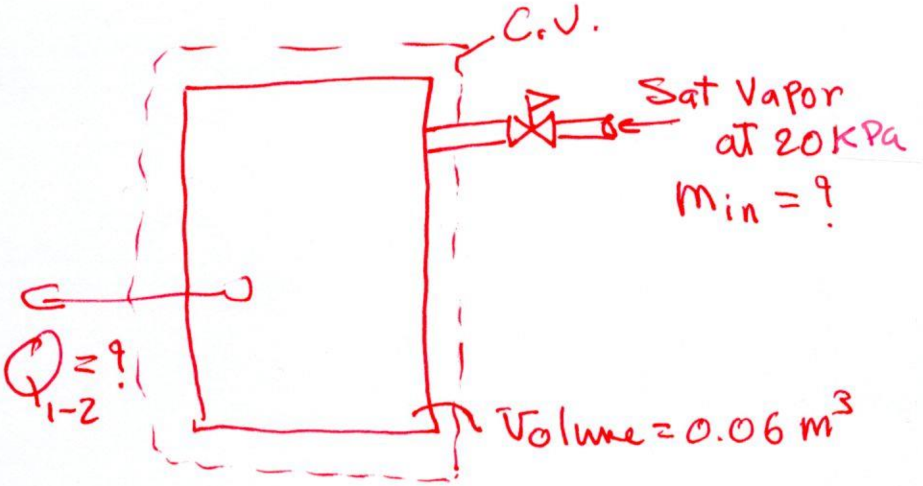
$$\Rightarrow \boxed{\iint_S \vec{F}\cdot d\vec{S} = 0}$$

Solutions

MECH 222-Thermo, Test 5, March 16, 2017

Question 1 (10 marks) A rigid tank has a volume of 0.06 m^3 and initially contains saturated vapor water at a pressure of 20 KPa. As heat is removed from the tank contents, a pressure regulating valve keeps the pressure constant in the tank by allowing saturated vapor to enter. Cooling continues until the quality reaches 15%. Determine:

a. (5 marks) The amount of heat transfer, in kJ
b. (5 marks) The mass of vapor that enters, in kg



* USUF Process.
* ΔKE and ΔPE negligible
* Work = 0
* Constant Pressure Process.

initial conditions

Sat. Vapor at 20 bars

final Conditions

two-phase Mixture at $x = 15\%$

Continuity

$\sum m_{in} - \sum m_{out} = (m_{final} - m_{init})_{C.V.}$

$m_{in} - 0 = m_{final} - m_{initial}$

$m_{initial} = \frac{V}{v_{initial}}$

$P_{sat} = 20$ bars

Sat. Table $\longrightarrow$ P = 20 KPa

Table B.1.2

$u_g = 2456.71$ $\quad h_f = 271.90$

$u_f = 251.35$ $\quad v_f = 0.001017$

$h_g = 2609.7$

$v_g = 7.64937$

$v_{initial} = v_g|_{20 bars}$

$$m_{initial} = \frac{0.06}{7.64937} = 7.844 \times 10^{-3} \text{ kg}$$

$$m_{final} = m_{in} + m_{initial} = m_{in} + 7.844 \times 10^{-3}$$

Energy balance

$$m_{final} u_{final} - m_{initial} u_{initial} = Q_{1-2} - \cancel{W}^{\,0} + m_{in} h_{in}$$

$$u_{final} = (1-x) u_f + x u_g = 0.85 \times 251.35 + 0.15 \times 2456.71$$

$$u_{final} = 582.15 \text{ kJ/kg}$$

$$u_{initial} = u_g = 2456.71 \text{ kJ/kg.}$$

$$m_{final} = \frac{V}{v_{final}} \; ; \quad v_{final} = (1-x) v_f + x v_g$$

$$v_{final} = (1-0.15) \times 0.001017 + 0.15 \times 7.64937$$

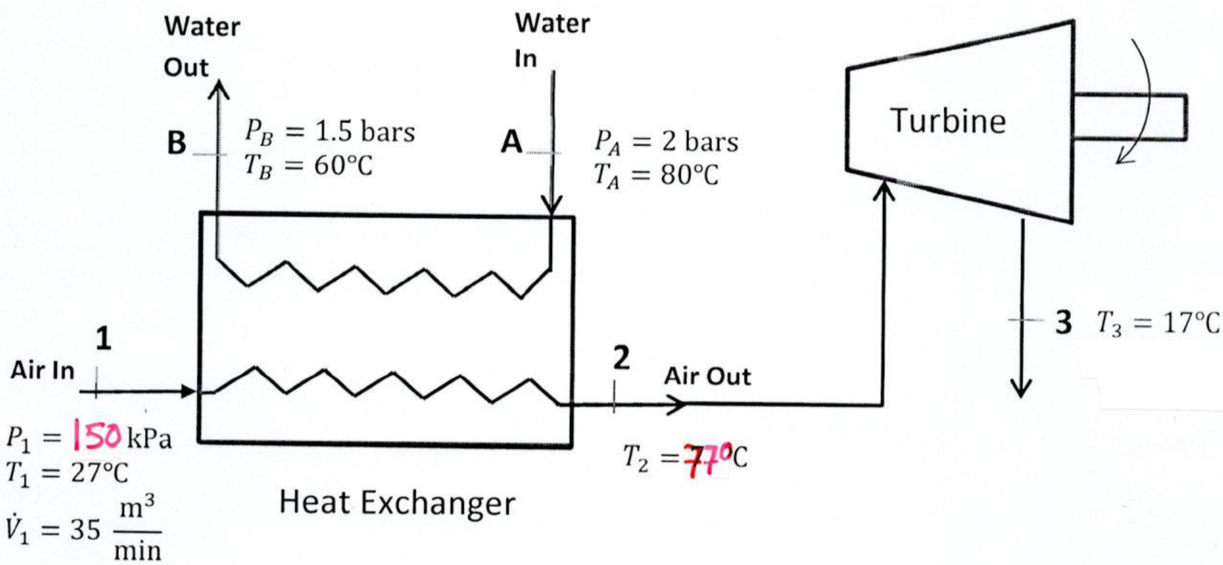
$$m_{final} = \frac{0.06}{1.148} = 0.052 \text{ kg} \qquad v_{final} = 1.148$$

$$0.052 \times 582.15 - 7.844 \times 10^{-3} \times 2456.71 = Q_{1-2} + [0.052 - 7.844 \times 10^{-3}) \times 2609.7$$

✓ $\underline{Q_{1-2}} = 11 - 115.23 = -104.23 \text{ kJ.}$

✓ $\underline{m_{in}} = m_{final} - m_{initial} = (0.052 - 7.844 \times 10^{-3}) = 0.044 \text{ kg.}$

Question 2 (20 marks) Air as an ideal gas is preheated at steady-state through a heat exchanger before entering a turbine generating power. The processes are shown schematically below. Data for various flow streams are shown on the figure. Heat transfer to surroundings can be neglected.



a. (10 marks) Calculate the Turbine power in kW

b. (10 marks) Assuming water specific volume is constant($v = 0.001 \frac{m^3}{kg}$); Calculate the mass flowrate of the heating water in kg/s

Assumptions

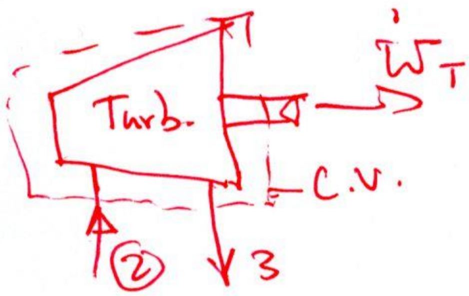
* steady state
* Heat transfer to surroundings negligible
* ΔPE and ΔKE negligible
* air behaves as an ideal gas.
* Water is incompressible with $v = 0.001\ m^3/kg$ constant.
* no pressure drop on the Heat exchanger air side

$\dot{m}_{air} = \rho_1 \dot{V}_1 = \dot{V}_1 / v_1$

$P_1 v_1 = RT_1 \rightarrow v_1 = \frac{0.287 \times 300}{150} = 0.574\ m^3/kg$

$\dot{m}_{air} = \frac{35}{60} \times \frac{1}{0.574} = 1.016\ kg/s$

Table A7.1 air ideal gas $\longrightarrow$ $\begin{cases} T_1 = 300K & h_1 = 300.47 \text{ kJ/kg} \\ T_2 = 350K & h_2 = 350.78 \\ T_3 = 290K & h_3 = 290.43 \end{cases}$



$\dot{W}_T = \dot{m}_{air}(h_2 - h_3)$

$\dot{W}_T = 1.016 \times (350.78 - 290.43)$

$\dot{W}_T = 60.35 \text{ kW}$

b) $\dot{Q}_{1-2} = -\dot{Q}_{AB}$

$\dot{Q}_{1-2} = \dot{m}_{air}(h_2 - h_1) = 1.016 \times (350.78 - 300.47)$

$\dot{Q}_{1-2} = 51.1 \text{ kW} = -\dot{Q}_{AB} \Rightarrow \dot{Q}_{AB} = -51.1 \text{ kW}$

$\dot{Q}_{AB} = \dot{m}_w (h_B - h_A)$

incompressible $h_B \simeq u_f|_{T_B} + v P_B$

$h_A \simeq u_f|_{T_A} + v P_A$

Table B.1.1. $\begin{cases} T_A = 80°C \Rightarrow u_f = 334.84 \text{ kJ/kg} \\ T_B = 60°C \Rightarrow u_f = 251.09 \end{cases}$

$h_B = 251.09 + 0.001 \times 150 = 251.24 \text{ kJ/kg}$

$h_A = 334.84 + 0.001 \times 200 = 335.04 \text{ kJ/kg}$

$$-51.1 = \dot{m}_w (251.24 - 335.04)$$

$$\dot{m}_w = \frac{51.1}{83.8} = 0.61 \ \text{kg/s}$$