Felix/Solutions

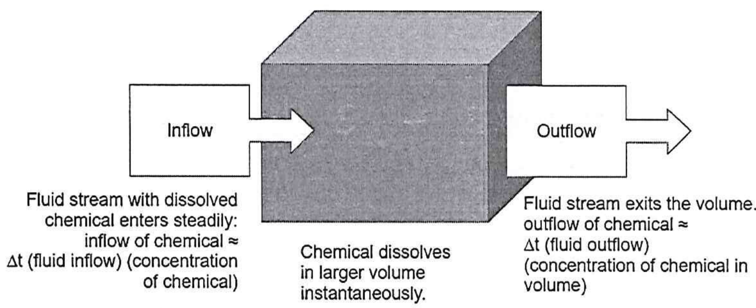
Worksheet 2

Felix Funk, MATH Tutorial - Mech 221

1 Application: Mixing and Conservation of Mass

Many common applications of dynamical systems originate from conservation laws. This is one of them.

Introduction: Conservation of Material.
Under the assumption that material is conserved in a system, one can relate the change of chemical substance within a volume to the inflow and outflow stream.



If a chemical dissolves within a fluid without reacting with it and without dissipation, then the amount of chemical in any given volume changes through the stream of fluid entering and leaving the volume. The inflow into the stream within a time Δt is given by

$$\text{Inflow} \approx \text{rate}_{\text{in}}\text{concentration}_{\text{in}}\Delta t, \tag{1}$$

and the outflow by

$$\text{Outflow} \approx \text{rate}_{\text{out}}\text{concentration}_{\text{out}}\Delta t. \tag{2}$$

The amount of chemical x in the volume is ~~given by~~ changes according to

$$\Delta x \approx \text{Inflow} - \text{Outflow}, \tag{3}$$

$$\frac{\Delta x}{\Delta t} \approx \text{rate}_{\text{in}}\text{concentration}_{\text{in}} - \text{rate}_{\text{out}}\text{concentration}_{\text{out}}. \tag{4}$$

In the limit $\Delta t \to 0$ one obtains an ODE.

Problem: A typical mixing problem. Initially 5 grams of salt are dissolved in 20 liters of water. Brine with concentration of salt 2 grams of salt per liter is added at a rate of 3 liters a minute. The tank is mixed well and is drained at 3 liters a minute. How long does the process have to continue until there are 2 grams of salt in the tank? 1

Exercise: Solving mixing problems.

x(t) denotes the amount of salt in the tank at t^{th} minute.

1. Identify the different rates and concentrations.
   - $r_{in} = \text{rate}_{\text{in}} =$ 3 [l/min]
   - $c_{in} = \text{concentration}_{\text{in}} =$ 2 [g/l]
   - $V(t) =$ Volume at time t = 20 [l]
   - $r_{out} = \text{rate}_{\text{out}} =$ 3 [l/min]
   - $c_{out} = \text{concentration}_{\text{out}} =$ $\frac{x}{V(t)} = \frac{1}{20}x$

2. Derive the ODE:

$$x' = \left[2\cdot 3 - 3\cdot\tfrac{1}{20}x\right]$$

3. What is the IVP? Obtain an explicit formula for $x(t)$. At what time has the initial concentration doubled? 2

$x(0) = 5$ IVP

$$x' + \underbrace{\frac{3}{20}}_{=p(t)}x = \underbrace{6}_{=g(t)}$$

We use integrating factors $r(t) = e^{\int p(t)dt} = e^{3/20\, t}$

Then, $\frac{d}{dt}[\underbrace{e^{3/20t}}_{r(t)}\cdot x] = e^{3/20t}\cdot\underbrace{6}_{g(t)}$ $\quad // \int dt$

$e^{3/20t}\cdot x = \frac{20}{3}\cdot e^{3/20t}\cdot 6 + c$

$$\boxed{x(t) = 40 + c\cdot e^{-3/20t}}$$

$5 = x(0) = 40 + c \Rightarrow \boxed{c = -35}$

$$\boxed{x(t) = 40 - 35\cdot e^{-3/20t}}$$

Check: $x' = +\frac{35\cdot 3}{20}e^{-3/20t}$

$\frac{3}{20}x = -\frac{35\cdot 3}{20}e^{-3/20t} + 40\cdot\frac{3}{20}$

$x' + \frac{3}{20}x = \frac{40\cdot 3}{20} = 6$ ✓

1 As $e^{-3/20t} \underset{t\to\infty}{\to} 0$ and $-35e^{-3/20t}$ increasing in t

⇒ Concentration cannot reduce.

⇒ Will never occure

2 $x(t) = 10$

⇔ $\frac{30}{35} = e^{3/20t}$

⇔ $\underbrace{-\frac{20}{3}}_{<0}\underbrace{\ln\left(\frac{30}{35}\right)}_{<0} = t$ $\quad >0$

As the volume stays the same ⇒ concentr. has doubled

Problem: Additional Mixing Problems.
Solve one of the following two mixing problems.

1. Initially a tank contains 10 liters of pure water. Brine of unknown (but constant) concentration of salt is flowing in at 1 liter per minute. The water is mixed well and drained at 1 liter per minute. In 20 minutes there are 15 grams of salt in the tank. What is the concentration of salt in the incoming brine? ↳ extra condition

2. (Challenging, do it with a partner:) Suppose there are two lakes located on a stream. Clean water flows into the first lake, then the water from the first lake flows into the second lake, and then water from the second lake flows further downstream. The in and out flow from each lake is 500 liters per hour. The first lake contains 100 thousand liters of water and the second lake contains 200 thousand liters of water. A truck with 500 kg of toxic substance crashes into the first lake. Assume that the water is being continually mixed perfectly by the stream. a) Find the concentration of toxic substance as a function of time in both lakes. b) When will the concentration in the first lake be below 0.001 kg per liter? c) When will the concentration in the second lake be maximal?

1. $r_{in} = 1\,[\ell/min]$ c_{in} unknown $\quad r_{out} = 1\,[\ell/min]$ $\quad V(t) = 10\,[\ell]$

$c_{out} = \frac{x}{10}$

$\dot{x} = c_{in} - \frac{x}{10} \Rightarrow \dot{x} + \underbrace{\frac{1}{10}}_{=p(t)} x = \underbrace{c_{in}}_{=f(t)} \qquad r(t) = e^{1/10\, t}$

$x(t) = e^{-1/10\, t} \cdot \int e^{1/10\, t} c_{in}\, dt = e^{-1/10\, t}\left[10\, c_{in}\, e^{1/10\, t} + k\right]$

$= 10 \cdot c_{in} + k e^{-1/10\, t}$

$0 = x(0) = 10 \cdot c_{in} + k$ (IVP)

$15 = x(20) = 10 \cdot c_{in} + k \cdot e^{-2}$ (additional constraint (C))

$-15 = k \cdot (1 - e^{-2})$ (IVP − C)

$\boxed{\frac{-15}{(1-e^{-2})} = k}$ $\qquad c_{in} = \frac{-k}{10} = \frac{3}{2}\frac{1}{(1-e^{-2})}$

2. At the end / Appendix

2 Application: Population Dynamics

Another important application of first order ODEs lies within the field of population dynamics.

Introduction: Replicator Equation.
We aim to predict how two species A and B evolve in a competitive environment. $x = (x_A, x_B)$ denotes the frequency of A and B, respectively, such that $x_A + x_B = 1$, $x_A, x_B > 0$. The main hypothesis of darwinian evolution claims that the two populations reproduce based on how well each is off f_A,f_B compared to the population average $f_{pop}(x) = x_A f_A(x) + x_B f_B(x)$, i.e.

$$\begin{aligned} x'_A &= x_A(f_A(x) - f_{pop}(x)), \\ x'_B &= x_B(f_B(x) - f_{pop}(x)). \end{aligned}$$

The assumption that the population size remains constant $x_A + x_B = 1$ results in the replicator equation.

$$x'_A = x_A(1 - x_A)(f_A(x) - f_B(x)) \tag{5}$$

Example: Frequency Independent Growth.
We assume that the fitness of A and B is constant, i.e $f_A(x) = a$ and $f_B(x) = b$ with positive constants a and b.

1. Use the replicator equation (5) to set up an ODE for A.

$$x'_A = x_A(1-x_A)(a-b)$$

Let $c = a-b$ $\begin{cases} >0 \leadsto A \text{ has advantage} \\ <0 \leadsto B \text{ has disadvantage} \\ =0 \leadsto \text{same} \end{cases}$

put in 1 line

2. Solve for x_A. Find α and β such that that

$$\boxed{\frac{1}{x(1-x)} = \frac{\alpha}{x} + \frac{\beta}{(1-x)}} \tag{6}$$

$1 = \alpha(1-x) + \beta x$

$\Rightarrow$ For $x=0$: $\boxed{1=\alpha}$
$x=1$: $\boxed{1=\beta}$

Separation

$$\left(\frac{1}{x_A} + \frac{1}{1-x_A}\right)dx_A = \frac{1}{x_A(1-x_A)}\,dx_A = c\cdot dt \Rightarrow \ln|x_A| - \ln|1-x_A| = ct + k$$

As x_A in $(0,1)$ $\Rightarrow$ $\ln(x_A) - \ln(1-x_A) = ct+k \Rightarrow \ln\left(\frac{x_A}{1-x_A}\right) = ct+k \Rightarrow \frac{x_A}{1-x_A} = e^{ct}\cdot\tilde{k}$

more space

3. How do population A and B evolve over time for different values of a and b?

If $c>0$: $x_A(t) \xrightarrow[t\to\infty]{} 1 - \lim_{t\to\infty}\frac{1}{1+e^{ct}\tilde{k}} = 1$ as $e^{ct}\to\infty$ $\Rightarrow$ A will outcompete B

$c<0$: $\xrightarrow[t\to\infty]{} 0$ as $e^{ct}\to 0$ $\Rightarrow$ B will outcompete A

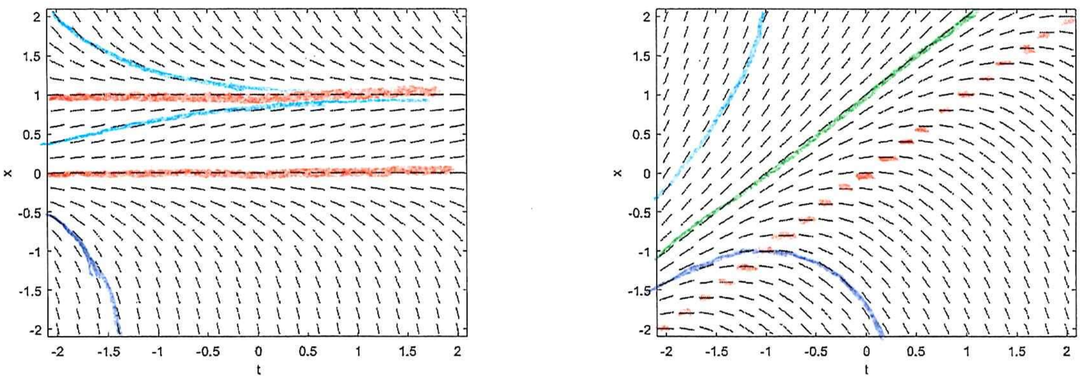
$c=0$: $x_A(t) = 1 - \frac{1}{1+\tilde{k}}$ for all time. $\Rightarrow$ A and B coexist

$\Rightarrow x_A = (1-x_A)e^{ct}\tilde{k} \Rightarrow x_A\cdot(1+e^{ct}\tilde{k}) = e^{ct}\tilde{k}$

$\Rightarrow x_A(t) = \left[1 - \frac{1}{1+e^{ct}\tilde{k}}\right]$

3 Slope Fields

Introduction: Slope fields.
Slope fields indicate the slope x' of the solution x at a given time t. The slope field for a simple case of the replicator equation $x' = x(1-x)$ and the linear ODE $x' = x - t$.



(a) $x' = x(1-x)$ (b) $x' = x - t$

Figure 1: Slope fields

At any point (t, x) the slope x' is calculated.

Problem: Slope fields.

1. Find in the figures above all points with a horizontal slope.

(a) $0 = x' = x(1-x) \Rightarrow x = 0$ or $x = 1$ $(t,0), (t,1)$ for all t in $\mathbb{R}$

(b) $0 = x' = x - t \Rightarrow x = t$ (t,t) for all t in $\mathbb{R}$ have horizontal slope

2. Hypothesize what happens in the limit $t \to \infty$ for different initial values.

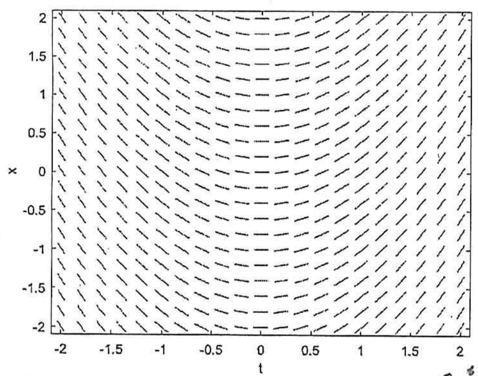
(a) When $x_0 < 0$: $x(t) \xrightarrow[t\to\infty]{} -\infty$, $x_0 = 0$: $x(t) = 0$ for all times, $x_0 > 0$: $x(t) \xrightarrow[t\to\infty]{} 1$

(b) When $x_0 = t_0 + 1$, then $x(t)$ remains on $x(t) = t + 1$. This solution splits the area into two subdomains: When $x_0 < t_0 + 1$, then $x(t) \to -\infty$ and $x_0 > t_0 + 1$, then $x(t) \to \infty$.

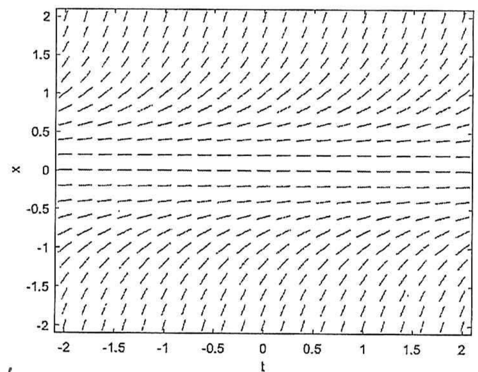
3. The following plots are slope fields to $x' = f(t, x)$ with

$$f_1(t,x) = x^2; \quad f_2(t,x) = t; \quad f_3(t,x) = cos(xt); \quad f_4(t,x) = \frac{x}{(t+3)}.$$

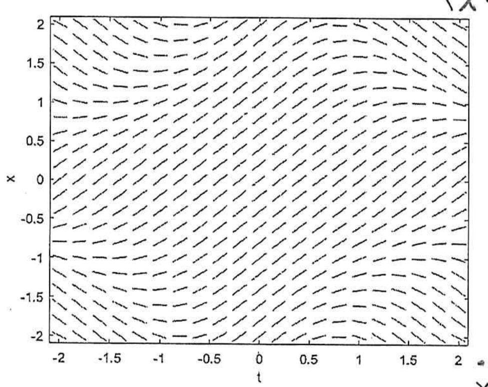
Identify which of the following plots correspond to f_1, f_2, f_3, f_4:



(a) $x' = t$ as $\begin{cases} \dot{x} > 0 & \text{for } t > 0 \\ \dot{x} = 0 & \text{for } t = 0 \\ \dot{x} < 0 & \text{for } t < 0 \end{cases}$

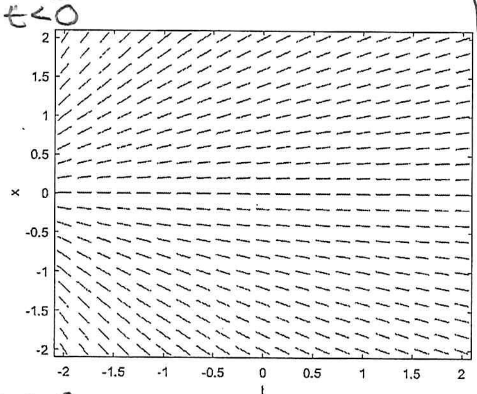


(b) $x' = x^2$ $\left\{ \begin{array}{l} \dot{x} > 0 \\ \dot{x} = 0 \\ \dot{x} > 0 \end{array} \right\}$ for $\left\{ \begin{array}{l} x > 0 \\ x = 0 \\ x < 0 \end{array} \right\}$



(c) $x' = \cos(xt)$

$x = 1$ for $x = 0$ or $t = 0$

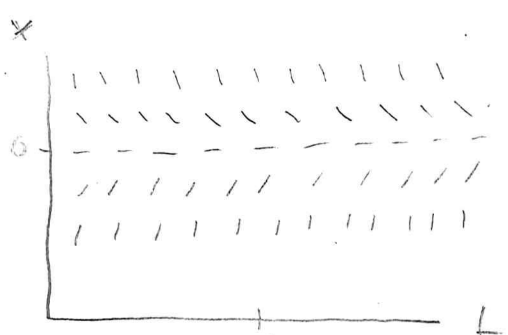


(d) $x' = \dfrac{x}{t+3}$

$\dot{x} = 0$ for $x = 0$
$\dot{x} < 0$ for $x > 0$
$\dot{x} > 0$ for $x < 0$

Figure 2: Slope fields

4. Sketch the slope field to $x' = -x$.



does not depend on t => looks the same along the t-axis

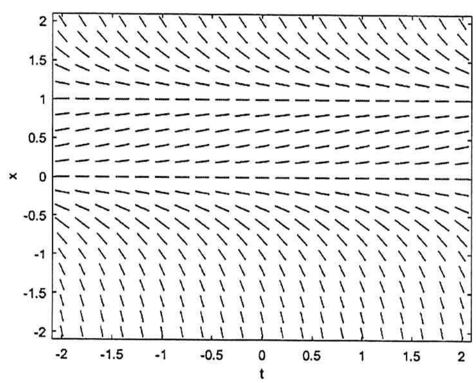
4 Phase Portraits for Autonomous First Order ODEs

For some ODEs it is possible to understand the dynamics of a system without explicitly calculating the solution.

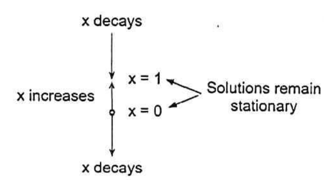
Introduction: Phase Portrait for autonomous first order ODEs.
Those ODEs, whose differential equation does not have an explicit dependence on the independent variable (time, space) such that

$$x'(t) = f(x), \tag{7}$$

can be understood with phase portraits. If one analyzes the slope field of $x' = x(1-x)$ one observes that it is independent of time. This mean that the plot looks identical when shifted to the left or right.



(a) $x' = x(1-x)$



(b) Phase Portraits

The phase portrait depicts through arrows whether a solution decreases or increases. Phase portraits are usually drawn in two steps:

1. Identify those x that satisfy $x' = f(x) = 0$ as the solution does not change there (equilibria).

2. Calculate whether x' is increasing or decreasing in between the equilibria. Use this information to indicate the direction of the arrows.

1 more space

Example: Phase Portrait. Sketch the phase portrait to the ODE

$$x' = x^2 - 1 = (x+1)(x-1) \tag{8}$$

1. Identify equilibria, i.e. these x such that $x' = f(x) = 0$

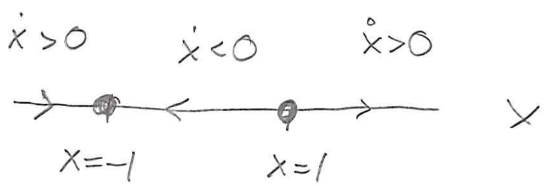
   $x = 1$, $x = -1$ are equilibria

2. Calculate whether x' in increasing or decreasing in between the equilibria.

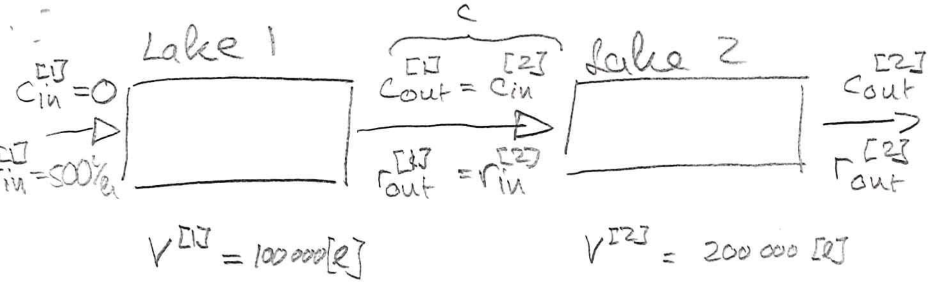
   $x < -1$: $x' = \underbrace{x^2}_{>1} - 1 > 0$ | $x \in (-1, 1)$: $x' = \underbrace{x^2}_{<1} - 1 < 0$

   $x > 1$: $x' = \overbrace{x^2}^{>1} - 1 > 0$

3. Sketch the phase portrait.



Two lakes problem



$r_{in}^{[k]} = r_{out}^{[k]}$

We denote $c^{[1]}_{out} = c^{[2]}_{in} = c$ the outflow of the first & inflow of the second lake.

Notation
$\left[\begin{array}{l} r_{in}^{[k]} \text{ rate; into lake } k \text{ ; } c_{in}^{[k]} \text{ concentration into lake } k. \\ \text{same notation for } r_{out}^{[k]},\ c_{out}^{[k]} \end{array} \right.$

$x(t)$ denotes the toxic substance in lake 1
$y(t)$ — " — 2

ODE
$$\left[\begin{array}{l} x(0) = 500 \ ; \ y(0) = 0 \qquad c = \dfrac{x}{100.000} \\ \dot{x} = \left[0 - \dfrac{x}{200} \right] = -\dfrac{500}{100\,000} x = -\dfrac{1}{200} x \\ \dot{y} = \left[\dfrac{x}{200} - \dfrac{y}{400} \right] \end{array} \right.$$

The first DE is separable
$$\left[\dot{x} = -\frac{1}{200} x \xrightarrow{\text{separation of vars}} \boxed{x(t) = k e^{-\frac{1}{200} t} = 500 \cdot e^{-\frac{1}{200} t}} \qquad 500 = k \Rightarrow \boxed{k = 500} \right.$$

$$\Rightarrow \dot{y} = \frac{500}{200} e^{-\frac{1}{200} t} - \frac{y}{400}$$

$$\dot{y} + \frac{1}{400} y = \frac{5}{2} e^{-\frac{1}{200} t} \qquad \Rightarrow r(t) = e^{\frac{1}{400} t}$$

$$\boxed{y(t)} = e^{-\frac{1}{400} t} \int \underbrace{\frac{5}{2} e^{-\frac{1}{200} t} \cdot e^{\frac{1}{400} t}}_{\frac{5}{2} e^{\left(-\frac{2}{400} + \frac{1}{400}\right) t}} dt$$

$$\frac{5}{2} e^{\overbrace{\left(-\frac{2}{400} + \frac{1}{400}\right)}^{-\frac{1}{400}} t} dt = \frac{5}{2} \cdot (-400) e^{-\frac{1}{400} t} + k$$

$$\boxed{= -1000 e^{-\frac{1}{200} t} + k \cdot e^{-\frac{1}{400} t}}$$

$$0 = y(0) = -1000 + k \Rightarrow k = 1000$$

$$\boxed{y(t) = 1000 \left(-e^{-\frac{1}{200} t} + e^{-\frac{1}{400} t} \right)}$$

Check: $\dot{y} = \left(+5 e^{-\frac{1}{200} t} - 2.5 e^{-\frac{1}{400} t} \right)$; $\left[\dfrac{x}{200} - \dfrac{y}{400} \right] = \dfrac{500}{200} e^{-\frac{1}{200} t} - \dfrac{1000}{400} \left[-e^{-\frac{1}{200} t} + e^{-\frac{1}{400} t} \right]$

$$\Rightarrow \quad \dot{y} = \frac{x}{200} - \frac{y}{400}$$

$$5e^{-\frac{1}{200}t} - 2.5e^{-\frac{1}{400}t} = \frac{5}{2}e^{-\frac{1}{200}t} - \frac{5}{2}\left[-e^{-\frac{1}{200}t} + e^{-\frac{1}{400}t}\right]$$

$$5e^{-\frac{1}{200}t} - 2.5e^{-\frac{1}{400}t} \checkmark$$