Worksheet 2

Felix Funk, MATH Tutorial - Mech 222

1 Center of Mass

Introduction: Calculating the center of mass.
The center of mass is the spot, where we can balance a weight perfectly. We can calculate the center of mass $(\bar{x}, \bar{y})$ of a lamina in shape D and density $p(x, y)$ by calculating

$$\bar{x} = \frac{1}{m} \iint_D xp(x,y)dA, \qquad \bar{y} = \frac{1}{m} \iint_D yp(x,y)dA, \qquad m = \iint_D p(x,y)dA.$$

This can easily be extended to three dimensional objects. Formally, the first two integrals resemble a continuous version of the mean/average.

Problemset: 1. Center of Mass.

1. D is a triangular region enclosed by the lines $x = 0, y = x$ and $2x + y = 6$ and a density of $p(x, y) = x^2$.
2. Challenging: A lamina occupies the region inside the circle $x^2 + y^2 = 2y$ but outside the circle $x^2 + y^2 = 1$. The density is inversely proportional to the distance from the origin. Hint: One can show: $\sin^2(x) = 1/2 - 1/2\cos(2x)$.
3. *Express as an iterated integral the solid bounded by the surfaces $x = 2, y = 2, z = 0, x + y - 2z = 2$. Find the center of mass.

2 Cylindrical Coordinates

Introduction: Transforming to cylindrical coordinates A cylinder is 3d shape that has a circular basis with a vertical extension. It is very convenient to transform the circular component into polar coordinates using the transformation.

$$x = r\cos(\theta), y = r\sin(\theta), z = z.$$

This nonlinear transformation requires an additional factor of r that we multiple to the density. With that an integral of the form

$$\iiint_D f(x,y,z)d(x,y,z) \tag{1}$$

turns into

$$\iiint rf(r\cos(\theta), r\sin(\theta), z)d(r,\theta,z).$$

Problemset: 2. Cylindrical Coordinates.

1. Sketch the solid volume that is expressed by the integral below and evaluate:

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{0}^{2} \int_{0}^{r^3} r dz dr d\theta$$

2. Find the mass of the solid S bounded by the paraboloid $z = 4x^2 + 4y^2$ and the plane $z = a$, $a > 0$.

3. *Evaluate by changing into cylindrical coordinates.

$$\int_{-2}^{2} \int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} \int_{\sqrt{x^2+y^2}}^{2} xz dz dx dy.$$

3 Spherical Coordinates

Introduction: Transforming to spherical coordinates Whenever our geometrical object resembles the structure of a ball, the integration process can be simplified by changing the integral (1) into spherical coordinates using the transformation

$$x = \rho\cos(\theta)\sin(\phi), y = \rho\sin(\theta)\sin(\phi), z = \rho\cos(\phi).$$

Again, the xy plane is parameterized by polar coordinates (hence $0 \leq \theta \leq 2\pi$ is a horizontal angle) but also now extended by a vertical angle $0 \leq \phi \leq \pi$. This nonlinear transformation requires an additional factor of $\rho^2 \sin(\phi)$ that we multiply to the density. With that we integrate over

$$\iiint \rho^2 \sin(\phi) f(\rho\cos(\theta)\sin(\phi), y = \rho\sin(\theta)\sin(\phi), z = \rho\cos(\phi)).$$

Problemset: 3. Cylindrical Coordinates.

1. Sketch the solid given by $\frac{3\pi}{4} \leq \phi \leq \pi$.

2. Find the average distance from a ball of radius a to its center.

3. *Challenging: Find the volume of the solid that lies within the sphere $x^2 + y^2 + z^2 = 4$ above the xy plane and below the cone $z = \sqrt{x^2 + y^2}$.