

Worksheet 7

Felix Funk, MATH Tutorial - Mech 221

1 Transfer Functions and Steady State Response

Introduction: Application of Transfer Functions.

Transfer functions help us to understand the steady state response of many time invariant systems such as LRC circuits. The following theorem connects transfer functions and steady state analysis.

Theorem. Suppose an output signal $y(t)$ is related to an input signal $x(t)$ in a linear, time-invariant system and let $H(s)$ denote the transfer function $H(s) = \frac{Y(s)}{X(s)}$ relating the Laplace transforms. If $x(t) = x_0 \sin(\omega t)$ then the steady state response of $y(t)$ is

$$y_{ss}(t) = \lim_{t \rightarrow \infty} y(t) = x_0 |H(j\omega)| \sin(\omega t + \phi), \quad \phi = \arg(H(j\omega)).$$

1.1 From ODE to Steady State Response.

Example: Steady State Response.

Calculate the steady state response of the system $y'' + 2y' + 3y = x' + x$ with $x(t) = 2 \sin(3t)$.

1. Apply the Laplace Transform and find the transfer function $H(s)$. Assume that $y(0) = y'(0) = 0$.
2. Identify ω and x_0 . Calculate $H(j\omega)$.

3. Calculate the amplitude and the phase shift.

4. Collect:

$$y_{ss} =$$

Problem: Model problem.

A complex system is modelled by the following set of linear ODEs

$$v_1' = u - 3x \quad (1)$$

$$v_2' = u - 2x \quad (2)$$

$$x' = v_1 + v_2 - x. \quad (3)$$

$u(t)$ expresses an input signal that is split into two intermediate signals $v_1(t)$ and $v_2(t)$ and results in an output signal $x(t)$.

1. Calculate the transfer function $H(s) = \frac{X(s)}{U(s)}$.

2. If $u(t) = a \sin(2t) + \sin(3t)$ for a positive constant a , find the steady state response y_{ss}

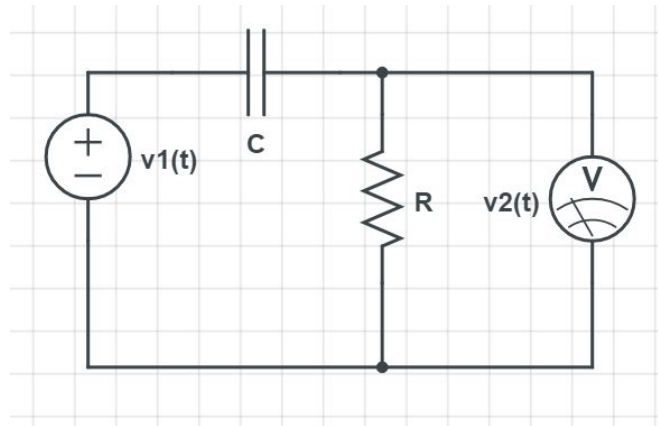
3. Find bounds for a such that the steady state response does not surpass the maximum allowed tolerance 0.1.

2 Electrical circuits

Introduction: Resistors, capacitors, inductors, and conservation laws Observe that we can transform many of the basic laws revolving around components found in RLC - circuits using Laplace transforms.

1. Resistor: $v_R = Ri \Rightarrow V_R(s) = RI(s)$
2. Capacitor: $\frac{dv_C}{dt} = \frac{i}{C} \Rightarrow sV_C(s) = \frac{1}{C}I(s)$
3. Inductor: $\frac{di}{dt} = \frac{v_L}{L} \Rightarrow sI(s) = \frac{1}{L}V_L(s)$
4. KVL (Voltage law, sum of voltages around a loop): $\sum_k v_k(t) = 0 \Rightarrow \sum_k V_k(s) = 0$
5. KCL (Current law, sum of currents into a node): $\sum_k i_k(t) = 0 \Rightarrow \sum_k I_k(s) = 0$.

Example: A simple circuit. We consider the following circuit.

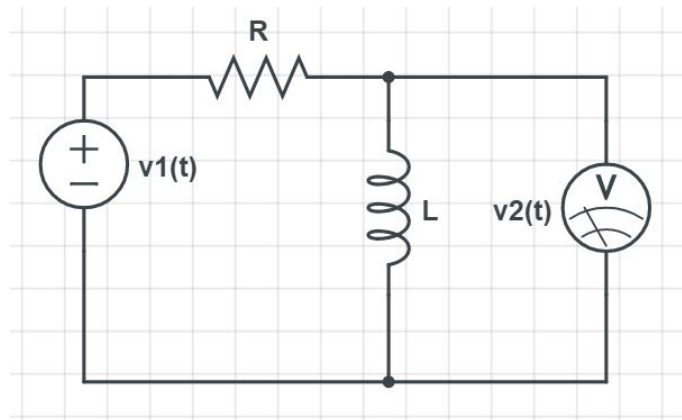


1. Use KVL to relate the voltages along the left loop.
2. Use the Laplace transform and your knowledge of resistors, capacitors and inductors to link $V_1(s)$ and $V_2(s)$.

3. Obtain the transfer function. For $v_1(t) = \sin(50t)$ what is the steady state voltage response $v_2(t)$ we should observe?

Problem: Problemset.

- We consider the following circuit. Find the transfer function $H(s) = \frac{V_2(s)}{V_1(s)}$ and use it to obtain information about the amplitude of the steady state response of $v_2(t)$ to the input signal $v_1(t) = \sin(\omega t)$, when ω is very small or very large.



- We consider the following circuit. Find the transfer function $H(s) = \frac{V_2(s)}{V_1(s)}$. What do you observe when the frequency of the input signal $v_1(t) = \sin(\omega t)$ is varied?

