< Digital Control System >

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- Objectives
  - Understand the architecture of digital control systems.
  - Review on $\mathcal{Z}$-transform & Discrete-time Fourier Transform (DTFT)
  - Discrete-time control design via approximate mapping.

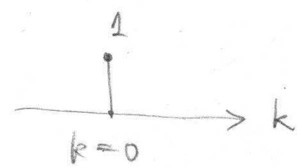
- Discrete-time Systems. (LTI)

$$x[k] \longrightarrow \boxed{H(z)} \longrightarrow y[k]$$

- Note that DT signals & systems can exist for its own sake.
  No requirement for underlying CT signals & systems.

- Impulse response

  when $x[k] = \delta[k]$ "Kronecker delta"



$$y[k] = \underbrace{h[k]}_{\text{Impulse resp.}}$$

- Transfer function.

$$H(z) = \mathcal{Z}\{h[k]\}$$

$$= \sum_{k=-\infty}^{\infty} h[k] z^{-k}$$

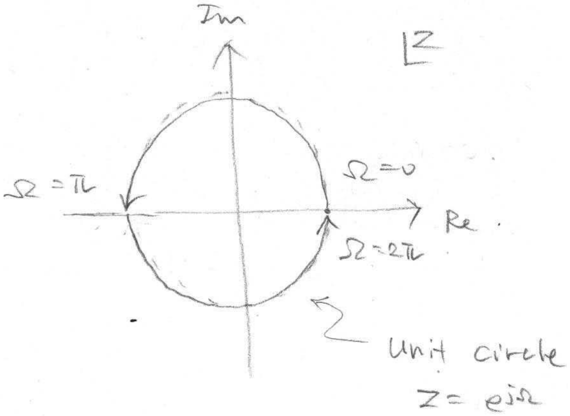
$$= \sum_{k=0}^{\infty} h[k] z^{-k} \quad \text{if } h[k] = 0 \text{ for } k<0 \quad \text{"Causal syst"}$$

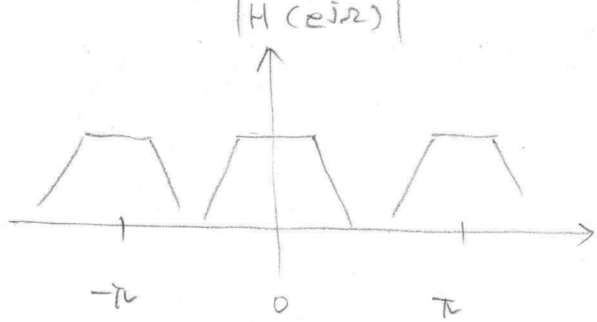
- DT frequency response.

$$H(e^{j\Omega}) = \mathcal{F}\{h[k]\} = \sum_{k=-\infty}^{\infty} h[k] e^{-j\Omega k} = H(z)\Big|_{z=e^{j\Omega}}$$

↳ Discrete-time Fourier transform (DTFT)

- Relation between $H(z)$ and $H(e^{j\Omega})$



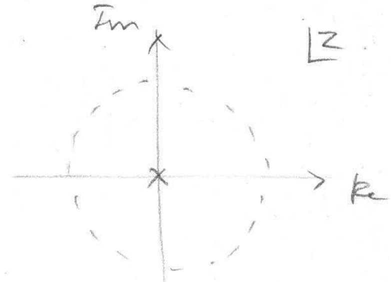


DTFT is periodic !

$$H(e^{j\Omega}) = H(e^{j(\Omega + 2\pi)})$$

- Unit delay

$\delta[k]$ $\qquad$ $h[k] = \delta[k-1]$

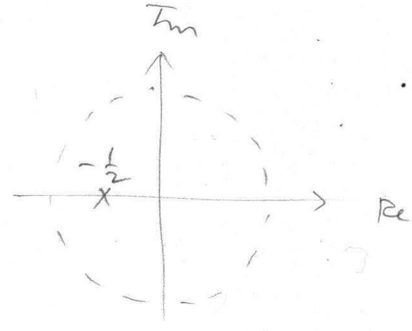


$$H(z) = \sum_{k=0}^{\infty} \delta[k-1] z^{-k} = z^{-1} = \frac{1}{z}$$

$$H(e^{j\Omega}) = e^{-j\Omega} \iff e^{-j\omega T}$$

Similarity

Example $x[k]$ $y[k]$



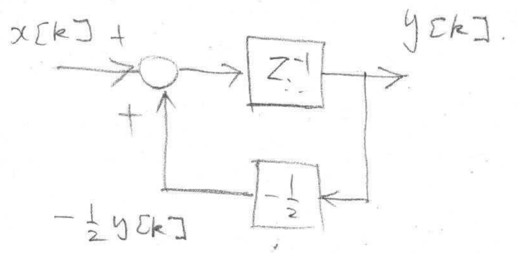
$$y[k] = x[k-1] - \frac{1}{2} y[k-1]$$ "Diffence Eqn"

$\leftrightarrow$ Differential.

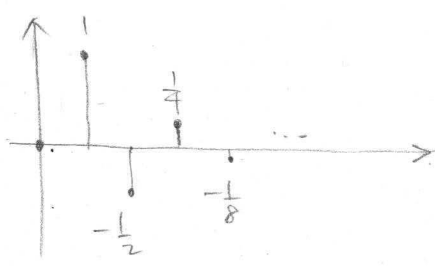
$$Z\{\cdot\} \Rightarrow Y(z) = z^{-1} X(z) - \frac{1}{2} z^{-1} Y(z)$$

$$\left(1 + \frac{1}{2} z^{-1}\right) Y(z) = z^{-1} X(z) \qquad \therefore \frac{Y(z)}{X(z)} = \frac{z^{-1}}{1 + \frac{1}{2} z^{-1}} = \frac{1}{z + \frac{1}{2}}.$$

- Block diagram



- Impulse response

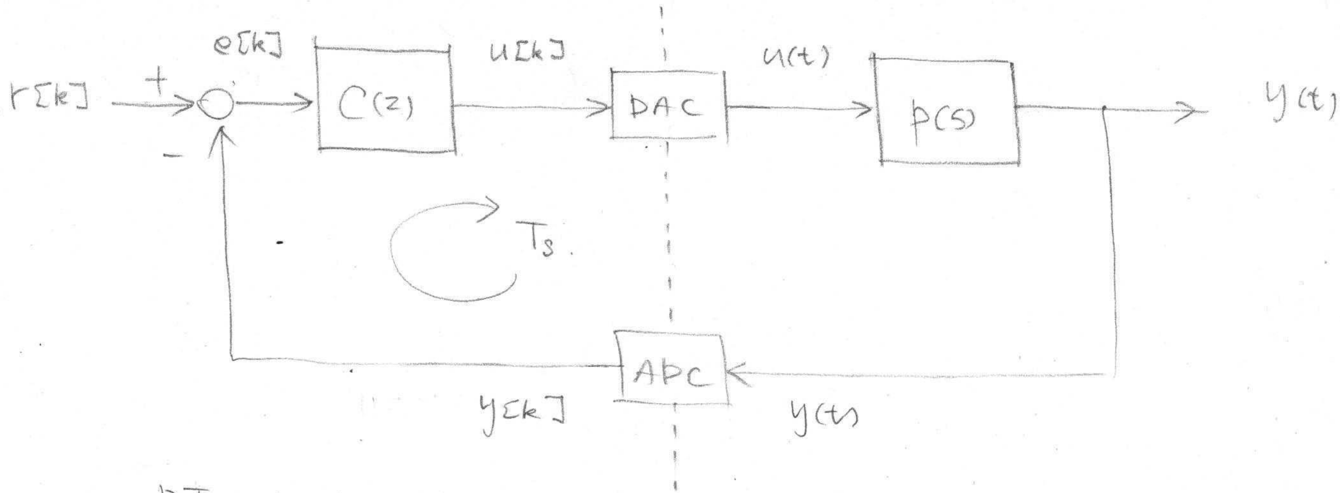


$$H(e^{j\Omega}) = \frac{1}{e^{j\Omega} + \frac{1}{2}}$$

$$|H| = \frac{1}{\sqrt{(e^{j\Omega} + \frac{1}{2})(e^{-j\Omega} + \frac{1}{2})}}$$

$$= \frac{1}{\sqrt{\frac{5}{4} + \cos\Omega}}$$

- Sampled-data system (DT control of CT system)



- DT signals
- DT systems (diffence eqn)

- CT signals
- CT systems (differential eqn.)

- Analog to digital converters. (ADC) Ideal model: $y[k] = y(kT_s)$

  - Sample rate [S/s = Hz] : $\frac{1}{T_s}$

    Simultateous sampling vs. multiplexed

  - Resolution

    Actual ADC has finite resolution.

    DT signals "quantized" in amplitude $\longrightarrow$ "Digital signal"

    Quantization error is typically handeled as an additive random noise.

  - Latency.

    - Actual ADC takes finite time for conversion.

      Latency is not the same as the sampling time T_s.

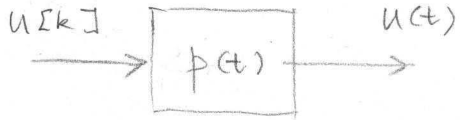
      It can be larger than T_s (e.g. delta sigma ADC).

    - Two popular types. $\begin{cases} \text{Successive - approximation register (SAR)} \\ \text{Delta-sigma modulation} \end{cases}$

      SAR is more popular for control because of its low latency.

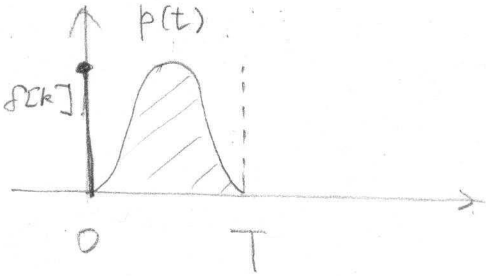
- Digital to analog converters (DAC)

  - Generates a continuous-time pulse $p(t)$ for each DT sample.
  - Two types
    - ① pulse Amplitude Modulation (PAM).
    - ② pulse width modulation (PWM)

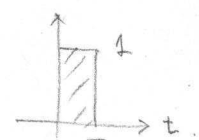


① PAM.

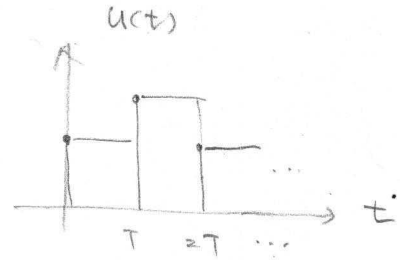
$$u(t) = \sum_{k=-\infty}^{\infty} u[k] \cdot p(t-kT)$$



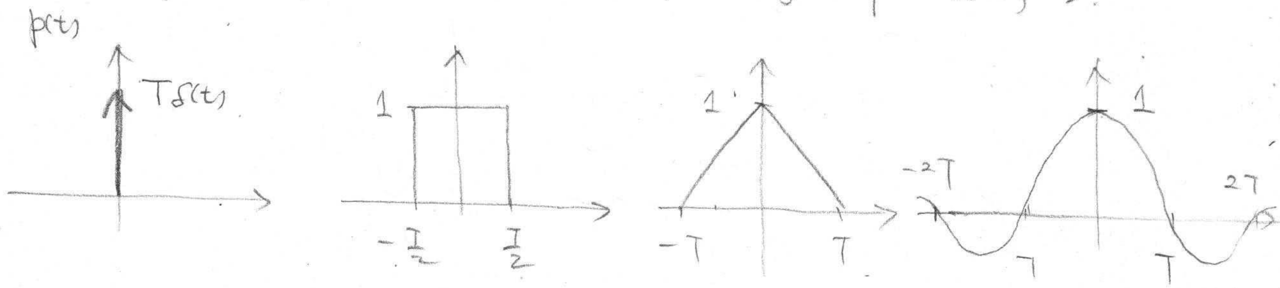
Typically $\int_{-\infty}^{\infty} p(t) = T$, e.g.



- Most popular type is "Zero-order hold"

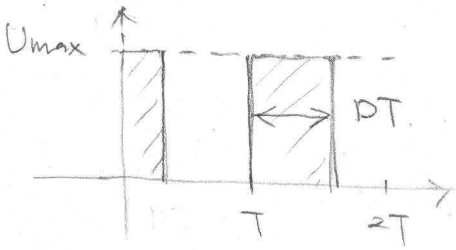


- Other types of PAM include. (signal processing).



Ideal LPF. $\rightarrow$ $p(t) = \dfrac{\sin\left(\frac{\pi t}{T}\right)}{\frac{\pi t}{T}}$

② PWM



- $u(t) = \sum_{k=-\infty}^{\infty} U_{max} \, P_D(t-kT)$
- If $\int_{<T>} u(t)\, dt = u[k] \cdot T$ it will do a similar job.

$\Rightarrow U_{max} \, D[k] \, T = u[k] \, T$

- (Works well if $p(s)$ is low-passive in nature)

$\therefore D[k] = \dfrac{u[k]}{U_{max}}$ (when $D<0$, use $-U_{max}$)

- Sampling frequency : $f_s = \frac{1}{T_s}$ $\left(\omega_s = \frac{2\pi}{T_s}\right)$

  Select f_s such that $f_s > 10 f_c$ at least ($f_c = 2\pi\,\omega_c$)

  $f_s > 20 f_c$: less worries about ΔT effect

  $f_s < 20 f_c$: Need to account for ΔT effect