(1)

MECH 364: Assignment 3

Requires course text book: Mechanical Vibrations by S.S. Rao (4th Edition).
Solutions will appear approximately ten days after the assignment is posted on VISTA.

Q1. Undamped Free Vibration.

(A) (**T 2.1**) An industrial press is mounted on a rubber pad to isolate it from its foundation. If the rubber pad is compressed by 5 mm by the self-weight of the press, find the natural frequency of the system.

(B) (**T 2.3**) A spring-mass system has a natural frequency of 10 Hz. When the spring constant is reduced by 800 N/m, the natural frequency is altered by 45 %. Find the mass and spring constants of the original system.

(C) (**T 2.33**) The crate of mass, 250 kg, hanging from a helicopter can be modelled as shown below. The rotor blades of the helicopter rotate at 300 rpm. Find the diameter of the steel cables such that the natural frequency of vibration of the crate is at least twice the frequency of the rotor blade. Steel has Young's modulus of $E = 210$ GPa. Ignore the mass of the cables. You may find the formula $k = \frac{AE}{L}$ for axial stiffness of a cable useful here.

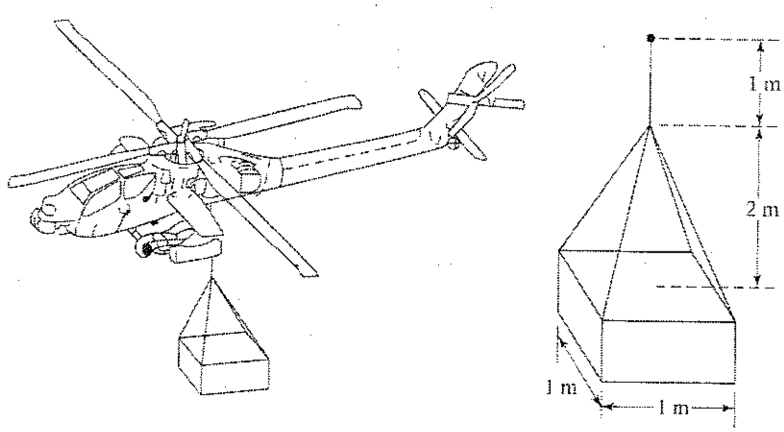


Figure A3.1: Figure for Question 1(C).

ASSIGNMENT # 3: SOLUTIONS

Q1 (A) NATURAL FREQUENCY $\omega_n = \sqrt{\frac{K_{eff}}{M_{eff}}} = \sqrt{\frac{g}{\delta_{st}}}$

$\delta_{st} = 5\text{mm} = 5\times10^{-3}\text{ m}$

$g = 9.81 \text{ m/s}^2$

$\therefore \omega_n = \sqrt{\frac{9.81}{5\times10^{-3}}} = \underline{\underline{44.29 \text{ rad/s}}} = \underline{\underline{7.05 \text{ Hz}}}$

(B) $\omega_n = \sqrt{\frac{K}{M}} = 10 \text{ Hz} = 2\pi\times10 \text{ rad/s}$

K IS REDUCED BY 800 N/m $\quad K_{new} = K - 800$

$\omega_{new} = \omega_n - 45\%.\omega_n = 0.55\omega_n$

$\omega_{new} = \sqrt{\frac{K_{new}}{M}} = \sqrt{\frac{K-800}{M}} = 0.55\times2\pi\times10 \text{ rad/s}$

$$\frac{\omega_{new}^2}{\omega_n^2} = \frac{K_{new}/M}{K/M} = \frac{K_{new}}{K} = \frac{K-800}{K} = \frac{(0.55\times2\pi\times10)^2}{(2\pi\times10)^2}$$

$\Rightarrow \frac{K-800}{K} = (0.55)^2 \Rightarrow \underline{\underline{K = 1146.95 \text{ N/m}}}$

$M = K/\omega_n^2 = \frac{1146.95}{(2\pi\times10)^2} = \underline{\underline{0.2905 \text{ Kg}}}$

Q1)

(c) $K_{vertical} = \frac{AE}{L}$ $\quad L = 1m$

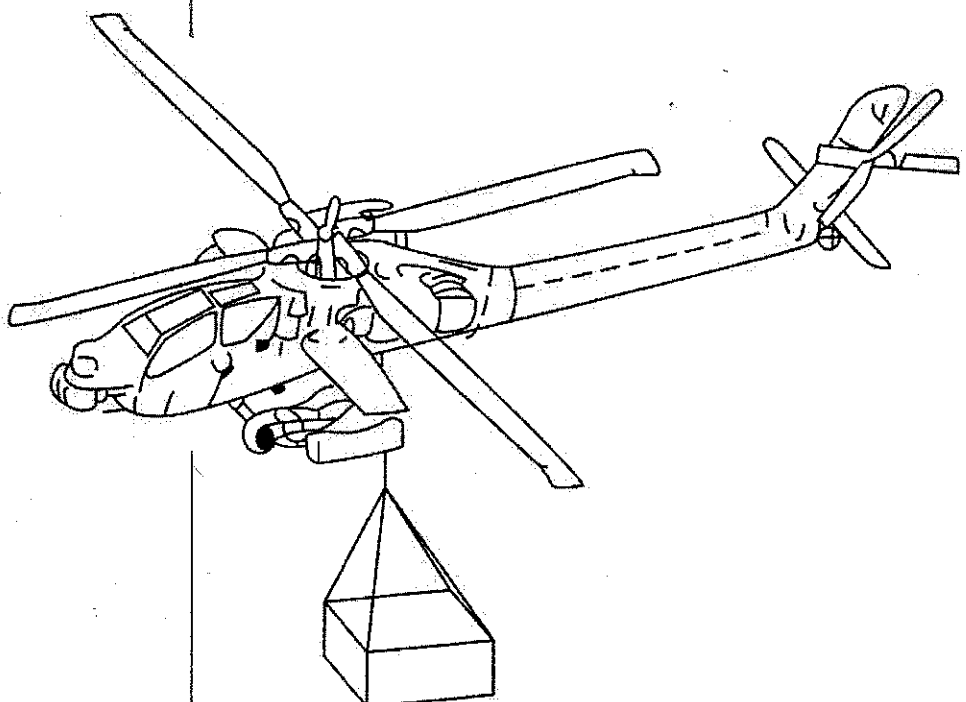
$= A \times 210 \times 10^9$ N/m

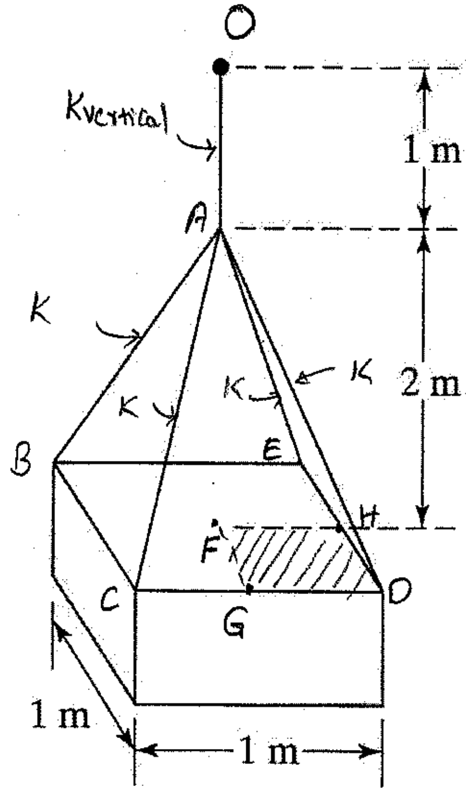
* NEED TO FIND NATURAL FREQUENCY USING EQUIVALENT SYSTEMS CONCEPT.

$K_{AB} = K_{AC} = K_{AE_1} = K_{AD} = K = \frac{AE}{L} = \frac{A \times 210 \times 10^9}{L}$

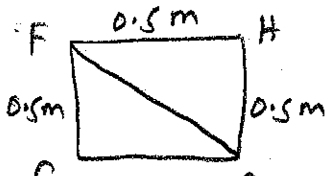
L = LENGTH OF EACH INCLINED CABLE = AD

WE USE THE GEOMETRY INFORMATION





FROM FGHD

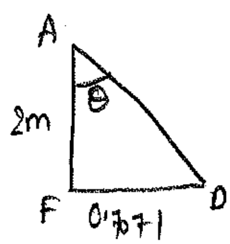


$DF = \sqrt{GF^2 + GD^2}$

$= \sqrt{0.5^2 + 0.5^2}$

$= 0.7071$ m

NOW FROM TRIANGLE AFD



$AD = \sqrt{2^2 + (0.7071)^2} = 2.1213$ m

$\tan\theta = \frac{0.7071}{2} \Rightarrow \theta = 19.471°$

$\therefore$ SPRING CONSTANT OF EACH INCLINED SPRING $= K = \dfrac{AE}{L = 2.1213}$

$$= \frac{A \times 210 \times 10^9}{2.1213}$$

TOTAL VERTICAL STIFFNESS DUE TO 4 SPRINGS / CABLES

$$= 4K\cos^2\theta = \frac{4 \times A \times 210 \times 10^9}{2.1213} \times \cos^2\left(\frac{19.471 \times \pi}{180}\right)$$

$$= 351.98 \times 10^9 \; A \;\; N/m$$

$K_{equivalent} = ?$ DUE TO $K_{vertical}$ & 4K IN SERIES.

$$\frac{1}{K_{equivalent}} = \frac{1}{K_{vertical}} + \frac{1}{4K} = \frac{1}{210 \times 10^9 \times A} + \frac{1}{351.98 \times 10^9 \times A}$$

$$\Rightarrow \quad K_{equivalent} = 131.52 \times 10^9 \times A \quad N/m$$

REQUIREMENT: $\omega_n = \sqrt{\dfrac{K_{equivalent}}{M}} = 2 \times 300 \times \dfrac{2\pi}{60}$ rad/s

$M = 250$ kg $\Rightarrow K_{equivalent} = M \times \left(2 \times 300 \times \dfrac{2\pi}{60}\right)^2$

$$\Rightarrow \quad K_{equivalent} = 131.52 \times 10^9 \times A = 250 \times \left(2 \times 300 \times \frac{2\pi}{60}\right)^2$$

$$\Rightarrow \quad A = \frac{250 \times \left(2 \times 300 \times \frac{2\pi}{60}\right)^2}{131.52 \times 10^9} = 7.504 \times 10^{-6} \; m^2$$

IF d IS DIAMETER OF CABLE $A = \dfrac{\pi d^2}{4} = 7.504 \times 10^{-6} \; m^2$

$\Rightarrow$ DIAMETER $= d = 0.055$ m $= 5.5$ cm

Q2. Damped Free Vibration.

(**T 2.89**) The free vibration response of an electric motor of weight 500 N mounted on different types of foundations are shown below. Identify the following in each case: (i) the nature of damping provided by the foundation, and (ii) the undamped and damped natural frequencies of the electric motor.

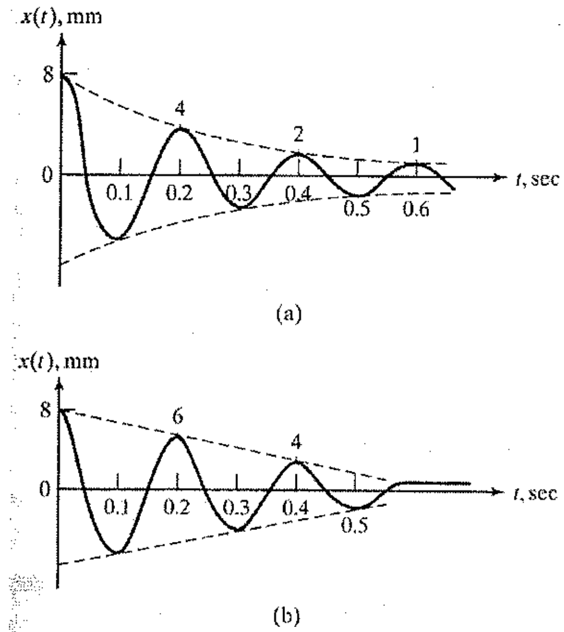


Figure A3.2: Figure for Question 2.

Q3. Undamped Forced Vibration.

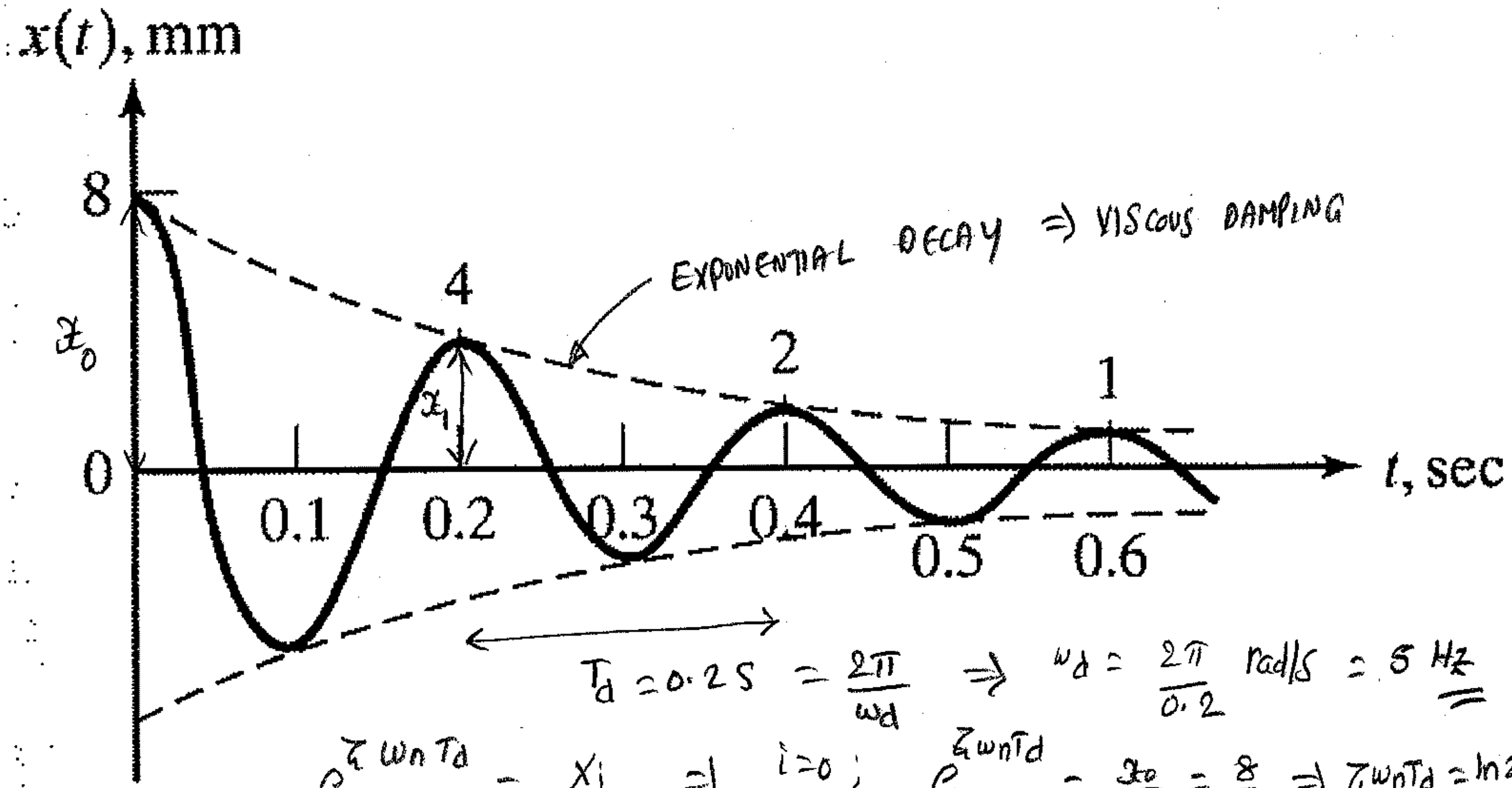
(**T 3.6**) Consider a spring-mass system, with $k = 4000N/m$ and $m = 10kg$, subject to a harmonic force $f(t) = 400\cos 30t$ N. Find (and plot, if possible) the total response of the system under the following initial conditions:

(a) x_0=0.1 m; $\dot{x}_0$=0

(b) x_0=0 m; $\dot{x}_0$=10 m/s

(c) x_0=0.1 m; $\dot{x}_0$=10 m/s.

Q2)

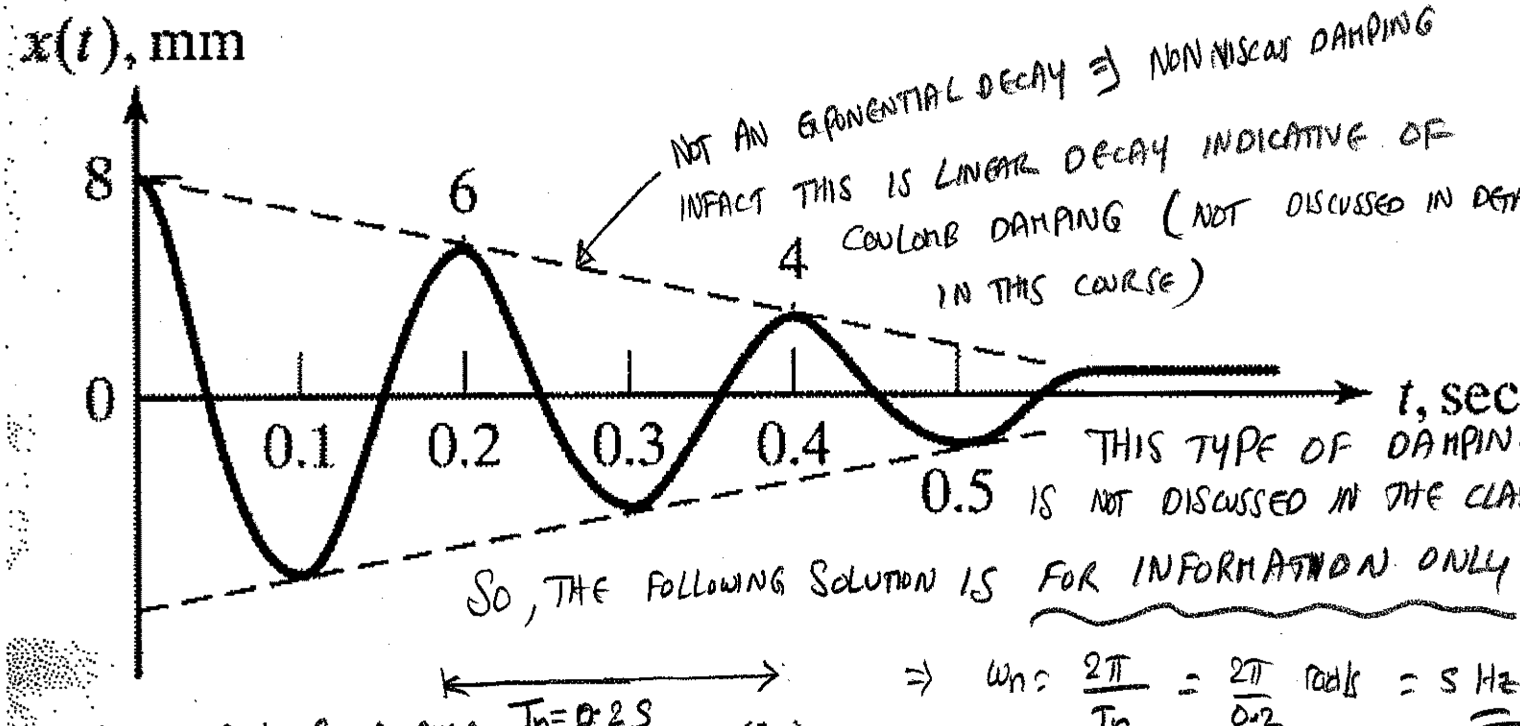


$T_d = 0.2\,s = \frac{2\pi}{\omega_d} \Rightarrow \omega_d = \frac{2\pi}{0.2}$ rad/s $= 5$ Hz

$e^{\zeta\omega_n T_d} = \frac{X_i}{X_{i+1}} \Rightarrow i=0;\quad e^{\zeta\omega_n T_d} = \frac{x_0}{x_1} = \frac{8}{4} \Rightarrow \zeta\omega_n T_d = \ln 2$

$\boxed{\omega_n = \frac{\omega_d}{\sqrt{1-\zeta^2}} = \frac{2\pi\times 5}{\sqrt{1-0.1096^2}} = 31.6065 \text{ rad/s}}$

(a) $\Rightarrow \frac{2\pi\zeta}{\sqrt{1-\zeta^2}} = \ln 2 \Rightarrow \zeta = 0.1096$



NOT AN EXPONENTIAL DECAY ⇒ NON VISCOUS DAMPING

INFACT THIS IS LINEAR DECAY INDICATIVE OF COULOMB DAMPING (NOT DISCUSSED IN DETAIL IN THIS COURSE)

THIS TYPE OF DAMPING IS NOT DISCUSSED IN THE CLASS!

SO, THE FOLLOWING SOLUTION IS FOR INFORMATION ONLY

$T_n = 0.2\,S \Rightarrow \omega_n = \frac{2\pi}{T_n} = \frac{2\pi}{0.2}$ rad/s $= 5$ Hz

(b)

Note: IN COULOMB DAMPING DECAY IS AT ω_n NOT ω_d!

LOGARITHMIC DECREMENT DOES NOT APPLY TO CASE (b)

Note: BOTH CASE (a) & CASE (b) HAVE SAME ω_n!

Q3) $m = 10\ kg$; $K = 4000\ N/m$

$$m\ddot{x} + Kx = 400 \cos 30t$$

$$x(t) = C_1 \cos\omega_n t + C_2 \sin\omega_n t + \frac{400 \cos 30t}{K - M(30)^2}$$

$$\omega_n = \sqrt{\frac{K}{M}} = \sqrt{\frac{4000}{10}} = \sqrt{400} = 20\ rad/s$$

C_1 & C_2 DEPEND ON INITIAL CONDITIONS.

(a) $x_0 = 0.1\,m$, $\dot{x}_0 = 0$

$$x_0 = 0.1 \Rightarrow C_1 + \frac{400}{K - M(30)^2} = 0.1$$

$$\Rightarrow C_1 = 0.1 - \frac{400}{4000 - 10(30)^2} = 0.18$$

$$\dot{x}_0 = 0 \Rightarrow C_2 \omega_n = 0 \Rightarrow C_2 = 0$$

$$\therefore x(t) = 0.18 \cos 20t + 0.08 \cos 30t$$

Note: $\frac{400}{4000 - 10(30)^2} = -0.08$

(b) $x_0 = 0$; $\dot{x}_0 = 10\,m/s$

$$x_0 = 0 \Rightarrow C_1 + \frac{400}{4000 - 10(30)^2} = 0 \Rightarrow C_1 = 0.08$$

$$\dot{x}_0 = 10 \Rightarrow C_2 \omega_n = 10 \Rightarrow C_2 = \frac{10}{20} = 0.5$$

$$\therefore x(t) = 0.08 \cos 20t + 0.5 \sin 20t - 0.08 \cos 30t$$

Q3) c) $x_0 = 0.1$, $\dot{x}_0 = 10$

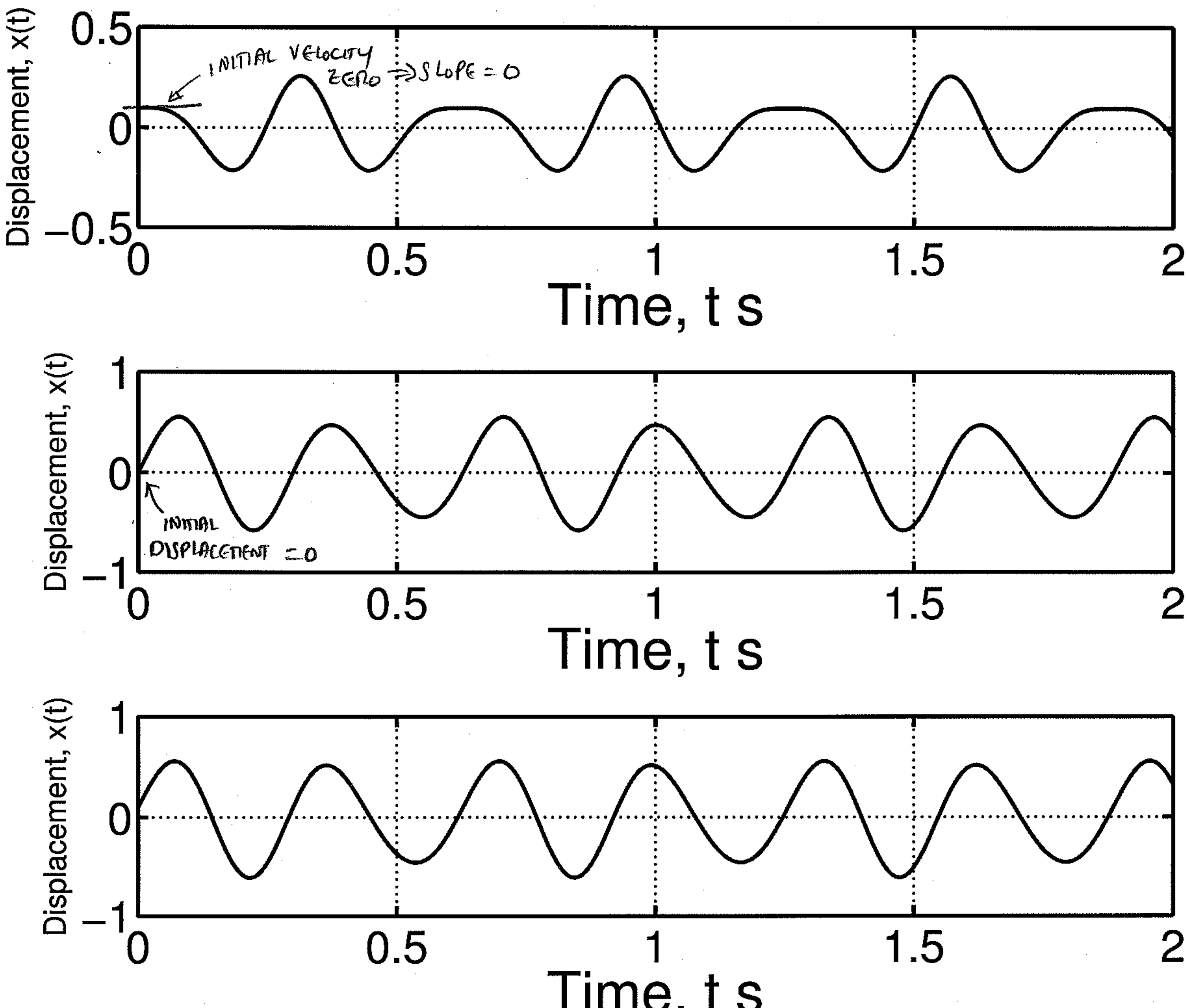
$$x_0 = 0.1 \Rightarrow C_1 + \frac{400}{4000 - 10(30)^2} = 0.1$$

$$\Rightarrow C_1 = 0.18$$

$$\dot{x}_0 = 10 \text{ m/s} \Rightarrow C_2 \omega_n = 10 \Rightarrow C_2 = \frac{10}{20} = 0.5$$

$$\therefore x(t) = 0.18 \cos 20t + 0.5 \sin 20t - 0.08 \cos 30t$$

PLOTS ON NEXT PAGE →


Displacement, x(t)
0.5
0
−0.5
INITIAL VELOCITY ZERO ⇒ SLOPE = 0
0
0.5
1
1.5
2
Time, t s
Displacement, x(t)
1
0
−1
INITIAL DISPLACEMENT = 0
0
0.5
1
1.5
2
Time, t s
Displacement, x(t)
1
0
−1
0
0.5
1
1.5
2
Time, t s

Q4. Damped Forced Vibration (Base Excitation).

(**T 3.26**) The propeller of a ship, of weight $10^5 N$ and polar mass moment of inertia 10,000 kg-m^2, is connected to the engine through a hollow stepped steel propeller shaft, as shown below. Assuming that water provides viscous damping ratio of 0.1, determine the torsional vibratory response of the propeller when the engine induces a harmonic angular displacement of 0.05 $\sin 314.16t$ rad at the base (point A) of the propeller shaft. You may find the formula $k_\theta = \frac{GI_p}{L}$ for torsional stiffness of a shaft useful here. I_p is polar area moment of inertia which depends on the cross sectional geometry.

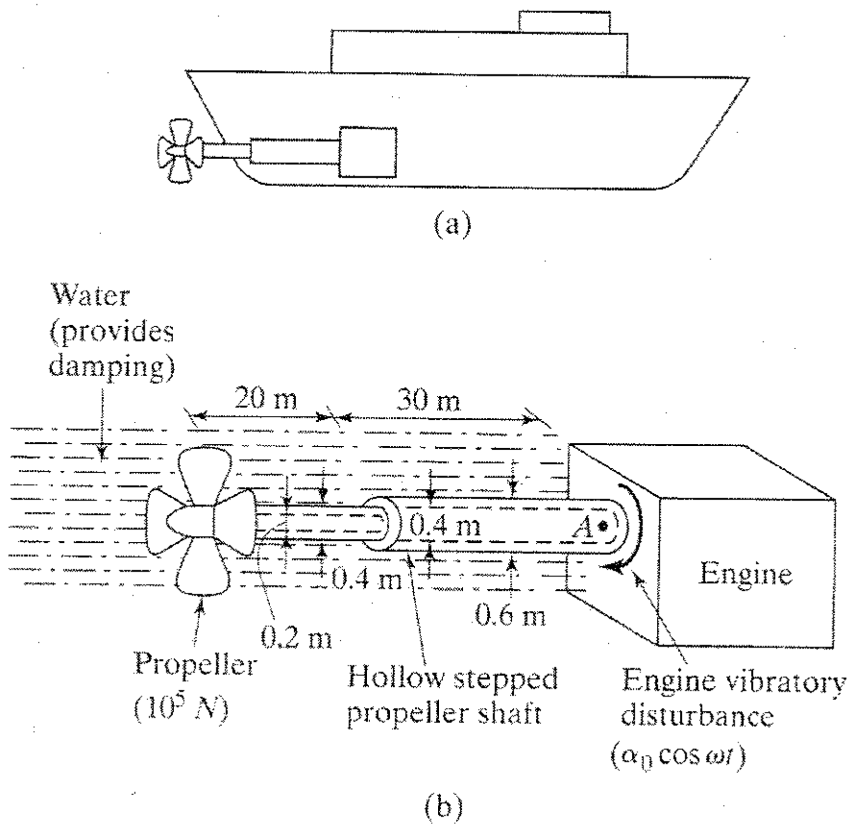
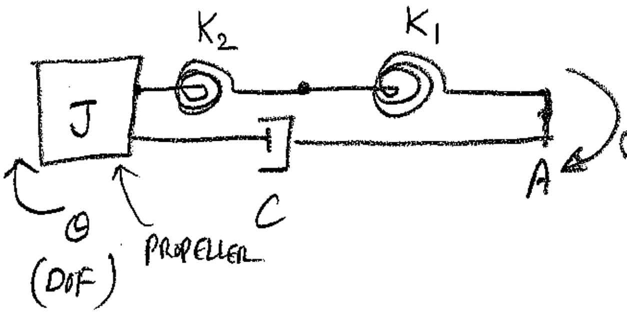


Figure A3.3: Figure for Question 4.

Q4) THE SHAFTS ACTS AS TORSIONAL SPRINGS, APPLYING A RESTORING TORQUE TO THE PROPELLER. WITH THE UNDERSTANDING THAT THE TORSIONAL SPRINGS REPRESENTED AS  (OR)  WE HAVE THE MODEL BELOW APPLY A RESTORING MOMENT



$\alpha = \alpha_0 \cos\omega t$

$= 0.05 \sin 314.16t$ GIVEN

$\alpha_0 = 0.05$ rad ; $\omega = 314.16$

EQUATION OF MOTION: $J\ddot{\theta} + C(\dot{\theta} - \dot{\alpha}) + K_{eq}(\theta - \alpha) = 0$ —(1)

$$\left(1/K_{eq}\right) = \frac{1}{K_1} + \frac{1}{K_2}$$ SHAFTS IN SERIES

$$(1) \Rightarrow J\ddot{\theta} + C\dot{\theta} + K_{eq}\theta = \underbrace{C\dot{\alpha} + K_{eq}\alpha}_{\text{FORCING FUNCTION}} = C\alpha_0\omega\cos\omega t + K_{eq}\alpha_0 \sin\omega t$$

(Note: WE CAN ALSO USE RELATIVE ANGULAR DISPLACEMENT $\phi_r = \theta - \alpha$ IN THAT CASE WE WILL HAVE THE EQUATION OF MOTION

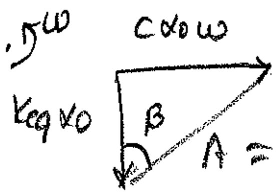
$$J\ddot{\phi}_r + C\dot{\phi}_r + K_{eq}\phi_r = J\ddot{\alpha}$$)

$$\theta = \theta_{particular} = ?$$

WE CAN SIMPLIFY THE FORCE FURTHER AS

$$C\alpha_0\omega\cos\omega t + K_{eq}\alpha_0\sin\omega t = A\sin(\omega t - \beta) \Rightarrow$$ USING ROTATING VECTOR DIAGRAM

ROTATING VECTOR DIAGRAM $\Rightarrow$



$A = \alpha_0\sqrt{K_{eq}^2 + c^2\omega^2}$; $\tan\beta = \frac{C\omega}{K_{eq}}$

Q4) EQUATION OF MOTION:

$$J\ddot{\theta} + C\dot{\theta} + K_{eq}\theta = \alpha_0 \sqrt{K_{eq}^2 + C^2\omega^2}\ \sin(\omega t - \beta)$$

SIMILAR TO

$m\ddot{x} + c\dot{x} + kx = F\sin(\omega t - \beta)$ (SIMILAR TO EXAMPLE 19 SOLVED IN CLASS)

$F = \alpha_0 \sqrt{K_{eq}^2 + C^2\omega^2}$

WE WANT TO FIND THE AMPLITUDE OF PARTICULAR SOLUTION

$$\theta = \theta_p(t) = \Theta_p \sin(\omega t - \beta - \phi)$$

$$\theta_p = \frac{F}{\sqrt{(K_{eq} - J\omega^2)^2 + (C\omega)^2}}$$

$$F = \alpha_0 \sqrt{K_{eq}^2 + c^2\omega^2}$$

NEED TO FIND: (1) K_{eq} FROM SPRINGS IN SERIES FORMULA

(2) C FROM $C = 2\zeta\sqrt{J K_{eq}}$

(3) F FROM $F = \alpha_0 \sqrt{K_{eq}^2 + c^2\omega^2}$

(1) K_{eq}: $G = 80$ GPa for STEEL ; $I_p = \frac{\pi}{32}(d_o^4 - d_i^4)$

$$K_1 = \frac{G I_{p1}}{L_1} = \frac{80\times10^9 \times \frac{\pi}{32}\times(0.6^4 - 0.4^4)}{30}$$ HOLLOW SHAFT

$= 27.2279 \times 10^6$ N-m/rad

$$K_2 = \frac{G I_{p2}}{L_2} = \frac{80\times10^9 \times \frac{\pi}{32} \times (0.4^4 - 0.2^4)}{20} = 9.4248\times10^6 \text{ N-m/rad}$$

$$K_{eq} = \frac{K_1 K_2}{K_1 + K_2} = 7.0013\times10^6 \text{ N-m/rad}$$

(2) $C = 2\zeta\sqrt{J K_{eq}} = 2 \times 0.1 \times \sqrt{10000 \times 7.0013 \times 10^6}$

$= 52919.9 \ \frac{N-m-s}{rad}$

$$\theta_p = \frac{\alpha_0 \sqrt{K_{eq}^2 + c^2\omega^2}}{\sqrt{(K_{eq} - J\omega^2)^2 + (c\omega)^2}} \quad ; \ \omega = 314.16 \ rad/s$$

$$= 0.05 \times \sqrt{\frac{(7.0013 \times 10^6)^2 + (52919.9 \times 314.16)^2}{[7.0013 \times 10^6 - 10000 \times (314.16)^2]^2 + (52919.9 \times 314.16)^2}}$$

$= 9.21 \times 10^{-4}$ rad

MAIN STEPS

1. GET THE PROBLEM INTO FAMILIAR $M\ddot{x} + c\dot{x} + Kx = F \sin\omega t$ FORM
   - (a) USE EQUIVALENT SYSTEMS
   - (b) USE ROTATING VECTOR DIAGRAMS

     CAN BE AVOIDED USING 1 & 2 UNDER NOTE
2. THEN SOLVE THE PROBLEM.

NOTE: 1) YOU CAN CONSIDER $C\dot{x}$ & $K_{eq}x$ AS TWO SEPARATE FORCES AND FIND RESPONSE TO EACH FORCE & ADD BY PROPERLY ACCOUNTING FOR PHASE. THIS IS PRINCIPLE OF SUPER POSITION.

2) WE COULD HAVE USED RELATIVE ANGULAR DISPLACEMENT $\alpha_r = \theta - \alpha$. THIS AVOIDS ROTATING VECTORS.

ALL GIVE SAME ANSWER!