University of British Columbia
Department of Mechanical Engineering

MECH468 Modern Control Engineering
MECH522 Foundations in Control Engineering
Final exam

Examiner: Dr. Ryozo Nagamune
April 11 (Wednesday), 2018, noon-2:30pm

Last name, First name

Name: Student #:

Signature:

Exam policies

- Allowed: Lecture note print-outs, hand-written notes, homework, books.
- Not-allowed: PC, calculators, mobile phones.
- Write all your answers on the provided exam booklet.
- Motivate your answers properly. (No chance to defend your answers orally.)
- 100 points in total. Mark will be scaled later.

Before you start ...

- Use washroom before the exam.
- Turn off your mobile phone.
- No eating.
- Questions are NOT allowed.

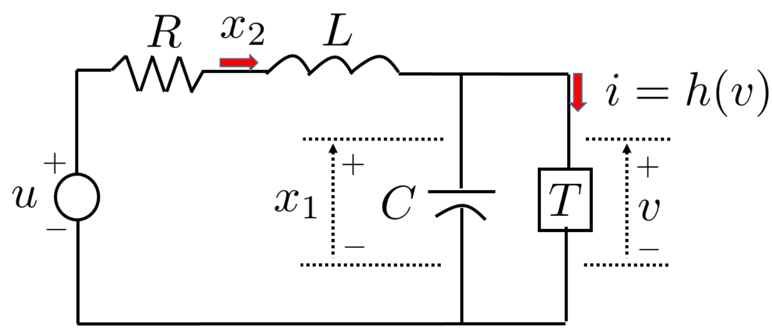
If you finish early ...

- If you would like to leave the room **before 2:20pm**, **raise your hand with this booklet**, and **wait at your seat** until the invigilator comes to you and collects your exam booklet.

To be filled in by the instructor/marker

Problem #	Mark	Full mark
1		20
2		20
3		20
4		20
5		20
Total		100

1. Consider the electric circuit depicted below. Here, the notations R, L and C respectively denote the resistance, inductance and capacitance, and u is the voltage source. An electrical element T has the characteristic $i = h(v)$, where i is the current through T and v is the voltage across T, and h is a nonlinear function which is differentiable with respect to v (i.e., $h'(v)$ exists).



   (a) Let x_1 be the voltage across the capacitance, and x_2 be the current through the inductance. Prove that the state equation for this system is described as follows. (10pt)

$$\begin{aligned} \dot{x}_1(t) &= -\frac{1}{C}h(x_1(t)) + \frac{1}{C}x_2(t) \\ \dot{x}_2(t) &= -\frac{1}{L}x_1(t) - \frac{R}{L}x_2(t) + \frac{1}{L}u(t) \end{aligned}$$

   (b) Linearize the state equation above around the operating point $(x_1, x_2, u) = (x_{10}, x_{20}, u_0)$. (6pt)

   (c) Express x_{20} and u_0 as functions of x_{10}. (4pt)

2. Obtain minimal realizations of the following transfer functions. After obtaining minimal realizations, check if the realization is indeed minimal. (10pt-each)

   (a) $G_1(s) = \begin{bmatrix} \dfrac{s^2+3s+2}{s^2+2s+1} \\ \dfrac{1}{s+2} \end{bmatrix}$

   (b) $G_2(s) = \begin{bmatrix} \frac{1}{s} & \frac{4}{s} \\ \frac{2}{s} & \frac{8}{s} \end{bmatrix}$

3. For the following state-space model, design an observer-based state-feedback controller. For the controller design, select the pole locations so that (20pt)

   - state estimation error converges to zero in about 0.4 second, and
   - (2%) settling time for initial condition excitation becomes about 1 second.

$$\begin{aligned} \dot{x}(t) &= \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \\ y(t) &= \begin{bmatrix} 0 & 1 \end{bmatrix} x(t) \end{aligned}$$

4. Determine whether each statement is True or False.

   - If your answer is 'True', provide an explanation to support your answer.
   - If your answer is 'False', provide a counter-example with two states, with an explanation, to support your answer. In counter-examples, use non-zero B-matrix and non-zero C-matrix.

   One example is given below. (10pt-each)

   Example If a linear time-invariant system is stable, then it is controllable.
   Answer False
   Counter-example

$$\dot{x} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u$$

   Explanation This system is stable because the eigenvalues of A-matrix are -1 and -1, both of which are in the open left-half plane. However, it is not controllable because the rank of the controllability matrix $\mathcal{C} = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$ is one, i.e., $\mathcal{C}$ is not full rank.

   (a) If a linear time-invariant system is unstable, it is not controllable.

   (b) If a linear time-invariant system is detectable, it is observable.

5. Consider the following continuous-time infinite-horizon linear quadratic regulator (LQR) problem, where α is a positive constant.

$$\min_{u(\cdot)} \int_0^\infty \left\{ \alpha x_2(t)^2 + u_1(t)^2 + u_2(t)^2 \right\} dt$$

$$\text{subject to} \quad \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}$$

   (a) Design the LQR control law. (10pt)

   (b) Prove that the designed LQR control law stabilizes closed-loop system for any $\alpha > 0$. (5pt)

   (c) For the closed-loop system with the designed LQR control law and $\alpha = 3$, draw the state trajectory in (x_1, x_2)-plane when the initial state is $x(0) = (1, 1)$. (**Hint:** The state trajectory must converge to $(0, 0)$ (i.e., origin of the (x_1, x_2)-plane because the closed-loop system is stable.) (5pt)

— — — — [END OF FINAL EXAM] — — — —