a place of mind

MECH468 : Modern Control Engineering
MECH509 : Controls

L4 : Linearization

Dr. Ryozo Nagamune

Department of Mechanical Engineering

University of British Columbia

Zoom lecture to be recorded and posted on Canvas

Review and today's topic

Physical system

Mech, Elec, Chem,
Aero, Bio, Econ, etc.

Modeling

(Lecture 3)

Nonlinear SS model

$$\begin{cases} \dot{x} &= f(x,u) \\ y &= h(x,u) \end{cases}$$

Linear SS model

$$\begin{cases} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{cases}$$

Linearization

(Today's topic)

Acronym
SS: State-space

Linear SS control theory

- deals with linear SS models
- deals with various physical systems in a UNIFIED way.

Linear system (review)

- A system having *Principle of Superposition*
  - Assume $\left.\begin{array}{l} x_i(t_0) \\ u_i(t), t \geq t_0 \end{array}\right\} \Rightarrow y_i(t), t \geq t_0, \quad i = 1, 2$
  - Then

$$\boxed{\left.\begin{array}{l} \alpha_1 x_1(t_0) + \alpha_2 x_2(t_0) \\ \alpha_1 u_1(t) + \alpha_2 u_2(t), t \geq t_0 \end{array}\right\} \Rightarrow \alpha_1 y_1(t) + \alpha_2 y_2(t), t \geq t_0 \quad \forall \alpha_1, \alpha_2 \in \mathbb{R}}$$

- A nonlinear system is a system which does not satisfy the principle of superposition.

Linear and nonlinear SS models

- Linear state-space model: Right-hand sides of the state-space model is linear with respect to *x* and *u*.

$$\begin{cases} \dot{x} & = & Ax + Bu \\ y & = & Cx + Du \end{cases}$$

- Nonlinear state-space model: Right-hand sides of the state-space model has nonlinear terms with respect to *x* and *u*.

$$\begin{cases} \dot{x} & = & f(x, u) \\ y & = & h(x, u) \end{cases}$$

Examples of nonlinear terms

$$x_1^2, \ x_1 x_2, \ x_1 u, \ \sin(x_1), \ \sqrt{x_1}$$

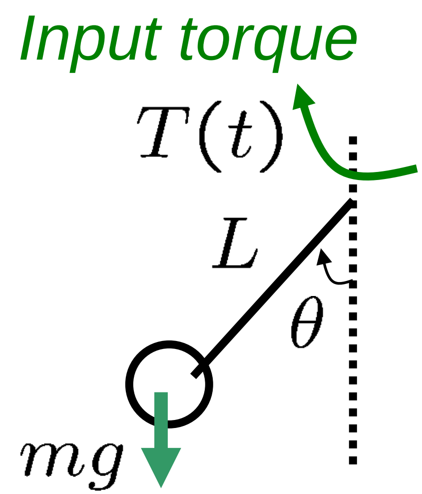
Why linearization?

- Real systems are inherently nonlinear. (Linear systems do not exist!)
  - *Ex.* $f(t)=Kx(t)$ holds only around an operating range.
- Nonlinear systems are difficult to deal with mathematically and theoretically.
- Many control analysis/design techniques are available for linear systems.
- Linear approximation is often good enough for control system analysis and design purposes.
- How to linearize nonlinear systems?

Today's outline

- Examples of nonlinear systems
- Linearization of simple functions (1-dim, 2-dim)
- Linearization of nonlinear systems
- Examples revisited

A pendulum



- Motion of the pendulum

$$mL^2\ddot{\theta}(t) = T(t) - mgL\sin\theta(t)$$

- Define state variables

$$x_1(t) := \theta(t),\ x_2(t) := \dot{\theta}(t)$$

$$\Rightarrow \begin{cases} \frac{d}{dt}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} x_2(t) \\ -\frac{g}{L}\underline{\sin x_1(t)} + \frac{1}{mL^2}T(t) \end{bmatrix} \\ y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \end{cases}$$

Nonlinear!

Water level control

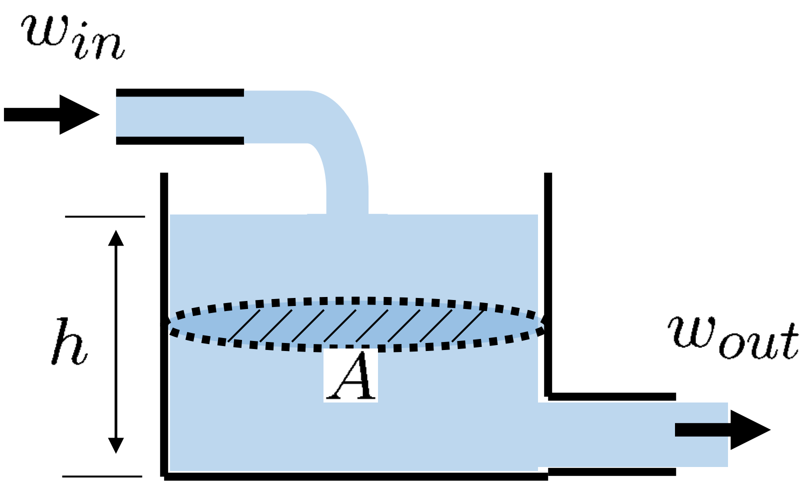
- Mass flow equation

$$\rho A\dot{h}(t) = -w_{out}(t) + w_{in}(t) = -\frac{(\rho g)^{1/\alpha}}{R}h(t)^{1/\alpha} + w_{in}(t)$$

$$\Rightarrow \begin{cases} \dot{h}(t) = -\dfrac{(\rho g)^{1/\alpha}}{\rho A R}h(t)^{1/\alpha} + \dfrac{1}{\rho A}w_{in}(t) \\ y(t) = h(t) \end{cases}$$

$\alpha = 1 \Rightarrow$ linear

$\alpha \neq 1 \Rightarrow$ nonlinear



w_{in}, w_{out} : mass flow rate
h : water height
A : tank area
ρ : liquid density
R, α : constant depending on restriction. $1 \leq \alpha \leq 2$

Today's outline

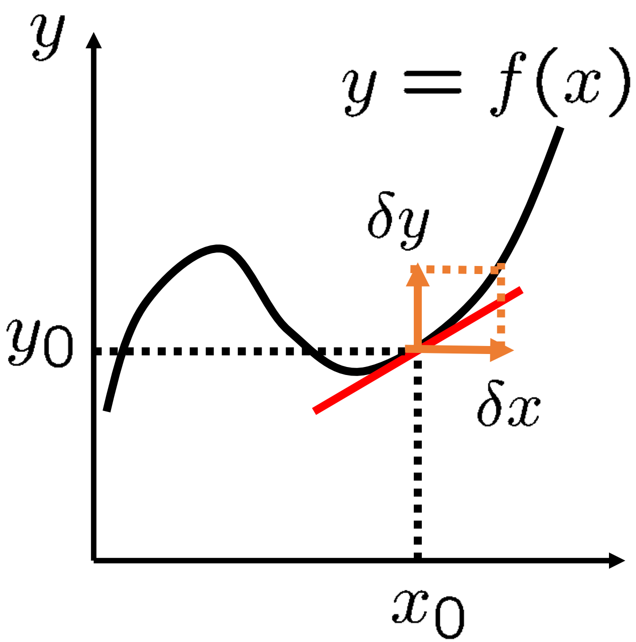
- Examples of nonlinear systems
- Linearization of simple functions (1-dim, 2-dim)
- Linearization of nonlinear systems
- Examples revisited

Linearization: 1-dim case

- Linearize a function *y=f(x)* around *x=x0* (scalar)
  - Consider a solution (*x0*,*y0*)

$$y_0 = f(x_0)$$

  - If *x* perturbs from *x0*, then *y* also perturbs from *y0*.



$$\begin{aligned} \cancel{y_0} + \delta y &= f(x_0 + \delta x) \\ &= \cancel{f(x_0)} + \left.\frac{df}{dx}\right|_{x=x_0} \delta x + \underline{H.O.T.} \end{aligned}$$

(Taylor expansion)

Negligible for small δx

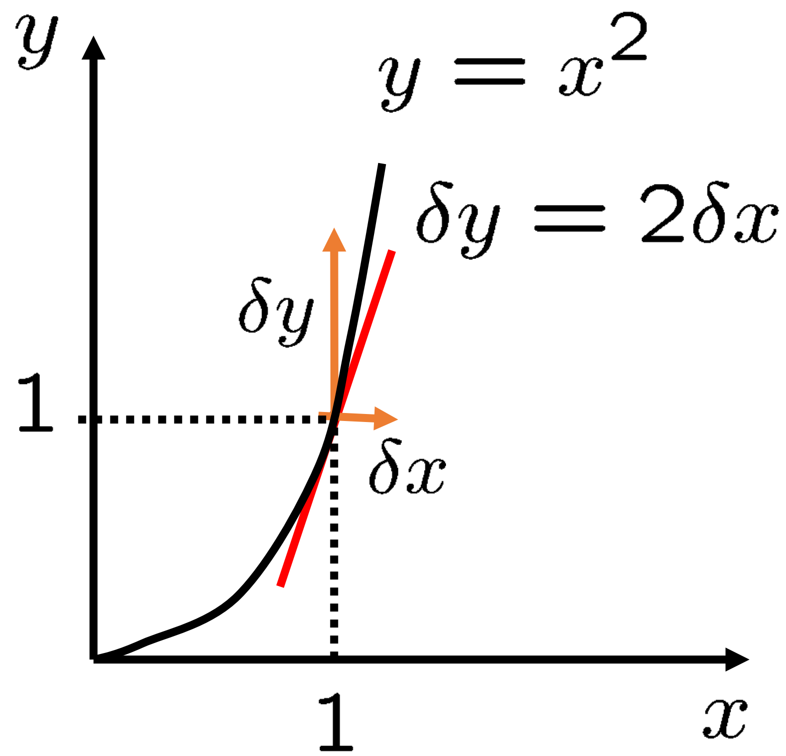
$$\Rightarrow \quad \delta y = \left.\frac{df}{dx}\right|_{x=x_0} \delta x$$

Example: 1-dim case

- Linearization of a function *y=x^2* around *x=1*.

$$\begin{cases} \delta y = \left.\dfrac{df}{dx}\right|_{x=x_0} \delta x \\ \left.\dfrac{df}{dx}\right|_{x=1} = 2x|_{x=1} = 2 \end{cases}$$

$\Rightarrow$ $\delta y = 2\delta x$

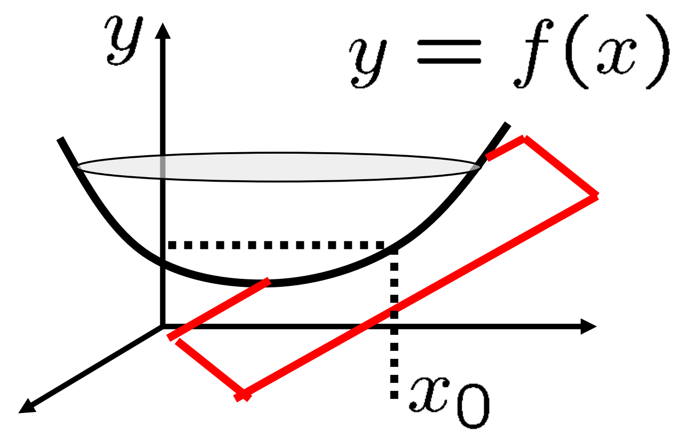


Linearization: 2-dim case

- Linearize a function *y=f(x)* around $x=x_0 \in \mathbb{R}^2$
  - Consider a solution

    $$y_0 = f(x_0)$$

  - If *x* perturbs from x_0, then *y* also perturbs from y_0.



$$
\begin{aligned}
\cancel{y_0} + \delta y &= f(x_0 + \delta x) \\
&= \cancel{f(x_0)} + \left.\frac{\partial f}{\partial x_1}\right|_{x=x_0} \delta x_1 + \left.\frac{\partial f}{\partial x_2}\right|_{x=x_0} \delta x_2 + \underbrace{H.O.T.}_{\text{Negligible for small } \delta x}
\end{aligned}
$$

$$
\Rightarrow \delta y = \begin{bmatrix} \left.\frac{\partial f}{\partial x_1}\right|_{x=x_0} & \left.\frac{\partial f}{\partial x_2}\right|_{x=x_0} \end{bmatrix} \begin{bmatrix} \delta x_1 \\ \delta x_2 \end{bmatrix} = \underbrace{\left.\frac{\partial f}{\partial x}\right|_{x=x_0}}_{\textit{Jacobian}} \delta x
$$

Example: 2-dim case

- Linearize a function below around $x_0 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$$y = x_1^2 + \sin(x_1 x_2^2)$$

- Linearized equation $\delta y = \left.\frac{\partial f}{\partial x}\right|_{x=x_0} \delta x$

    - Jacobian computation

$$\left.\frac{\partial f}{\partial x}\right|_{x=x_0} = \left.\begin{bmatrix} 2x_1 + x_2^2 \cos(x_1 x_2^2) & 2x_1 x_2 \cos(x_1 x_2^2) \end{bmatrix}\right|_{x=x_0}$$

$$= \begin{bmatrix} 4 + \cos 2 & 4 \cos 2 \end{bmatrix}$$

Today's outline

- Examples of nonlinear systems
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- Examples revisited

Linearization of nonlinear systems

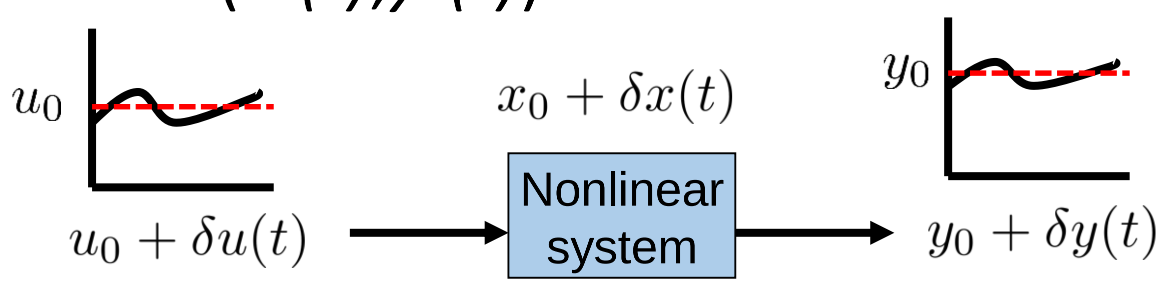
$$\begin{cases} \dot{x} & = & f(x,u) \\ y & = & h(x,u) \end{cases}$$

- Suppose that (*$x_0(t)$,$u_0(t)$,$y_0(t)$*) satisfies

$$\begin{cases} \dot{x}_0(t) & = & f(x_0(t),u_0(t)) \\ y_0(t) & = & h(x_0(t),u_0(t)) \end{cases}$$

Such trajectories (or points if *x_0* and *u_0* are constants) are called equilibrium trajectories (equilibrium points).

- If *$u(t)$* perturbs from *$u_0(t)$*, then *$x(t)$* and *$y(t)$* also perturb from (*$x_0(t)$,$y_0(t)$*).



Linearization of state equation

$$\frac{d}{dt}(\cancel{x_0(t)} + \delta x(t)) = f(x_0(t) + \delta x(t), u_0(t) + \delta u(t))$$

$$= \cancel{f(x_0(t), u_0(t))}$$

$$+ \left.\frac{\partial f}{\partial x}\right|_{(x_0(t),u_0(t))} \delta x(t) + \left.\frac{\partial f}{\partial u}\right|_{(x_0(t),u_0(t))} \delta u(t) + \underline{H.O.T.}$$

Negligible for small δx(t) & δu(t)

$$\Rightarrow \quad \frac{d}{dt}(\delta x(t)) = \underbrace{\left.\frac{\partial f}{\partial x}\right|_{(x_0(t),u_0(t))}}_{A(t)} \delta x(t) + \underbrace{\left.\frac{\partial f}{\partial u}\right|_{(x_0(t),u_0(t))}}_{B(t)} \delta u(t)$$

Often, remove "δ". Then, $\dot{x}(t) = A(t)x(t) + B(t)u(t)$

Linearization of output equation

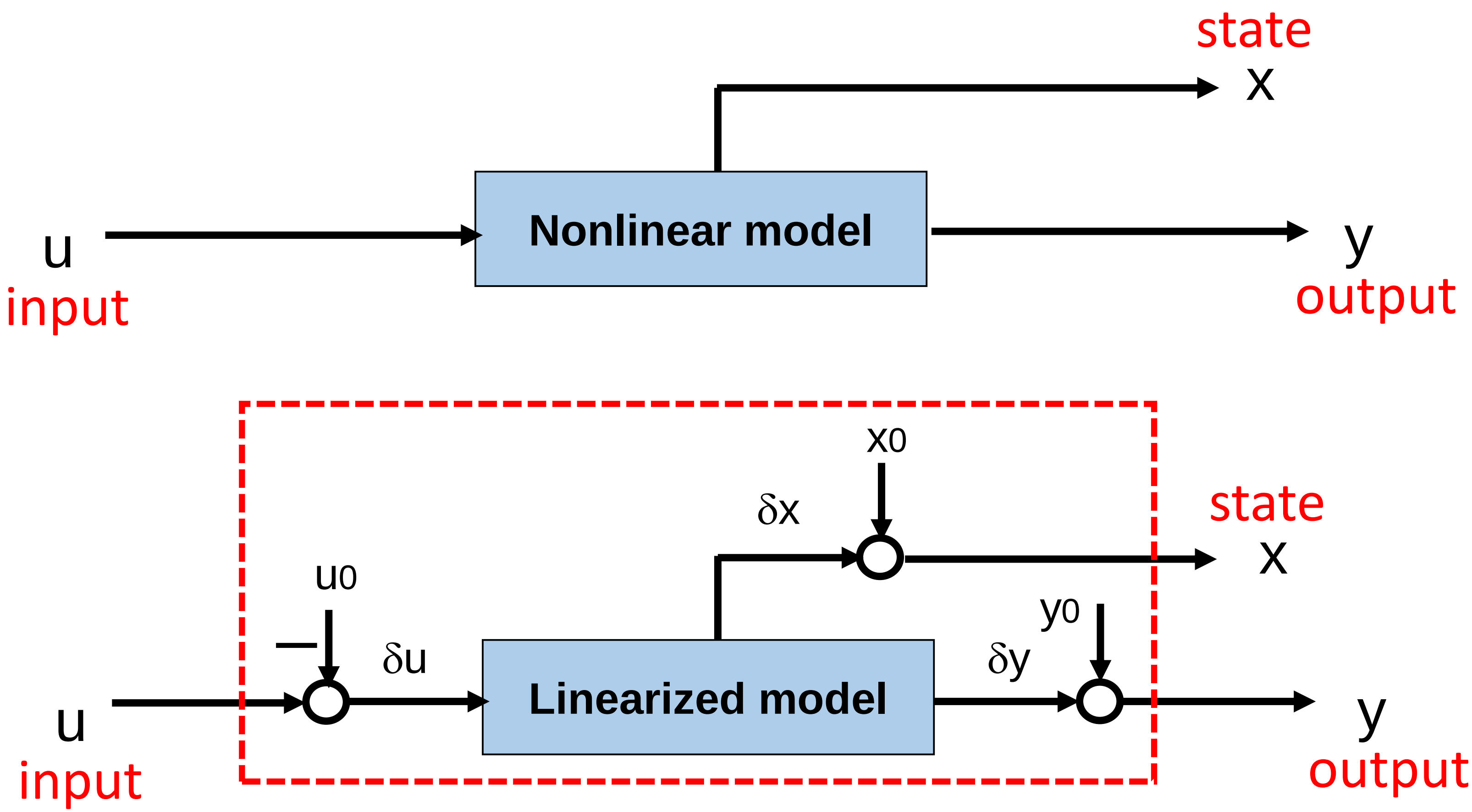
$$
\begin{aligned}
\cancel{y_0(t)} + \delta y(t) &= h(x_0(t) + \delta x(t), u_0(t) + \delta u(t)) \\
&= \cancel{h(x_0(t), u_0(t))} \\
&\quad + \left.\frac{\partial h}{\partial x}\right|_{(x_0(t),u_0(t))} \delta x(t) + \left.\frac{\partial h}{\partial u}\right|_{(x_0(t),u_0(t))} \delta u(t) + \underline{H.O.T.}
\end{aligned}
$$

Negligible for small δx(t) & δu(t)

$$
\Rightarrow \quad \delta y(t) = \underbrace{\left.\frac{\partial h}{\partial x}\right|_{(x_0(t),u_0(t))}}_{C(t)} \delta x(t) + \underbrace{\left.\frac{\partial h}{\partial u}\right|_{(x_0(t),u_0(t))}}_{D(t)} \delta u(t)
$$

Often, remove "δ". Then, $y(t) = C(t)x(t) + D(t)u(t)$

Comparison between nonlinear and its linearized models



Today's outline

- Examples of nonlinear systems
- Linearization of simple functions (1-dim, 2-dim)
- Linearization of nonlinear systems
- Examples revisited

A pendulum revisited

- Nonlinear model $\begin{cases} \frac{d}{dt}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} x_2(t) \\ -\frac{g}{L}\sin x_1(t) + \frac{1}{mL^2}u(t) \end{bmatrix} \\ y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \end{cases}$

- Linearization around $(x_1(t), x_2(t), u(t)) = 0$

Note that these trajectories (points) satisfies state-space model.

$$f(x,u) = \begin{bmatrix} x_2 \\ -\frac{g}{L}\sin x_1 + \frac{1}{mL^2}u \end{bmatrix} \Rightarrow \frac{\partial f}{\partial x} = \begin{bmatrix} 0 & 1 \\ -\frac{g}{L}\cos x_1 & 0 \end{bmatrix} \quad \frac{\partial f}{\partial u} = \begin{bmatrix} 0 \\ \frac{1}{mL^2} \end{bmatrix}$$

$$\Rightarrow \quad \dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -\frac{g}{L} & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ \frac{1}{mL^2} \end{bmatrix} u(t)$$

Water level control revisited

- Nonlinear model $\begin{cases} \dot{x}(t) &= \tilde{A}x(t)^{1/\alpha} + \tilde{B}u(t) \\ y(t) &= x(t) \end{cases}$ $\quad 1 \leq \alpha \leq 2$

- Linearization around $(x(t), u(t)) = \left(x_0, -\frac{\tilde{A}}{\tilde{B}} x_0^{1/\alpha} \right)$

Note that these trajectories (points) satisfies state-space model.

$$f(x,u) = \tilde{A}x^{1/\alpha} + \tilde{B}u \Rightarrow \frac{\partial f}{\partial x} = \frac{\tilde{A}}{\alpha} x^{(1-\alpha)/\alpha}$$

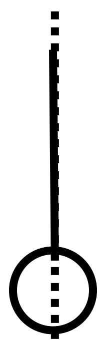
$$\Rightarrow \dot{x}(t) = \frac{\tilde{A}}{\alpha} x_0^{(1-\alpha)/\alpha} x(t) + \tilde{B}u(t)$$

Note that (x,u) are deviations from linearization points.

Equilibrium trajectory/point selection

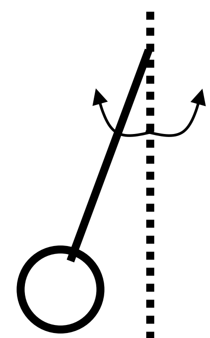
- Select an equilibrium trajectory/point around which:
  - you want to analyze the system, and
  - you want to design a feedback controller.
- To consider a regulation problem at:

Ex. $\theta = 0$



$\theta_0 = 0$

Ex. $\theta(t) = \sin \omega t$



$\theta_0(t) = \sin \omega t$

Summary

- Linearization of nonlinear systems
- Examples
  - Pendulum
  - Water level control
  - Inverted pendulum (Appendix)
- Next, solution to state-space models

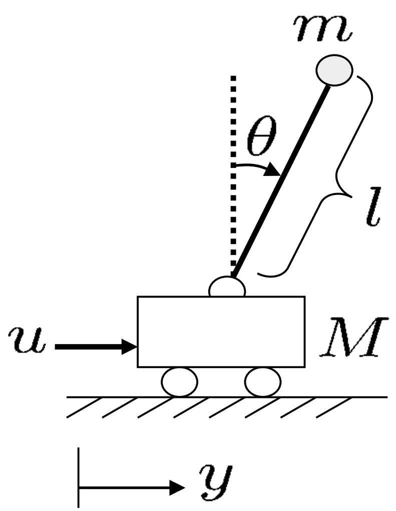
$$\begin{cases} \dfrac{dx(t)}{dt} &= A(t)x(t) + B(t)u(t) \\ y(t) &= C(t)x(t) + D(t)u(t) \end{cases}$$

$\Rightarrow$ How to compute *x(t)* & *y(t)*?

Nonlinear system example
Cart with an inverted pendulum

- Equation of motion

$$\begin{cases} M\ddot{y}(t) &= u(t) - m\frac{d^2}{dt^2}(y(t) + l\sin\theta(t)) \\ ml^2\ddot{\theta}(t) &= mgl\sin\theta(t) - ml\ddot{y}(t)\cos\theta(t) \end{cases}$$



- Define state variables $x(t) := \begin{bmatrix} \theta(t) \\ \dot{\theta}(t) \\ y(t) \\ \dot{y}(t) \end{bmatrix}$

$$\Rightarrow \begin{cases} \dot{x}(t) &= f(x(t), u(t)) \\ \begin{bmatrix} \theta(t) \\ y(t) \end{bmatrix} &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} x(t) \end{cases}$$

Nonlinear!