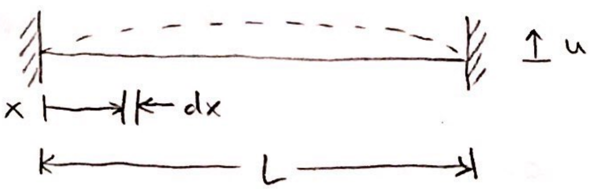
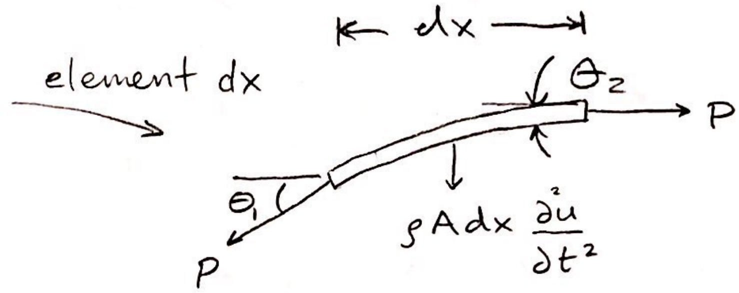
Lecture 19

Continuous Systems

Vibrating String - Tension is constant





Tension = P

Mass density = ρ

X-section area = A

Length - L

Length coord - x

Lateral displacement = $u(x,t)$

$$\begin{cases} \theta_1 = \dfrac{\partial u}{\partial x} \\ \theta_2 = \dfrac{\partial u}{\partial x} + \dfrac{\partial}{\partial x}\left(\dfrac{\partial u}{\partial x}\right) dx \end{cases}$$

Vertical Force Balance:

$$P \frac{\partial u}{\partial x} + \rho A dx \frac{\partial^2 u}{\partial t^2} = P\left(\frac{\partial u}{\partial x} + \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) dx\right)$$

$$\Rightarrow \frac{\partial^2 u}{\partial t^2} = \frac{P}{\rho A} \frac{\partial^2 u}{\partial x^2}$$

$$\Rightarrow \underbrace{\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}}_{\text{Wave equation}} \quad \text{where} \quad \underbrace{c = \sqrt{\frac{P}{\rho A}}}_{\text{Wave speed}}$$

Try separable solution $u(x,t) = X(x) \cdot T(t)$

Here, $X(x)$ is mode shape, $T(t)$ is vibration.

Sub $u(x,t)$ into wave equation:

$$X(x)\,\ddot{T}(t) = c^2 X''(x)\,T(t)$$ dots for time, dash for spac

Rearrange: $$\frac{X''}{X} = \frac{1}{c^2}\frac{\ddot{T}}{T} = \text{a constant}, -\beta^2$$

$$\Rightarrow X'' + \beta^2 X = 0 \quad \text{and} \quad \ddot{T} + (\beta c)^2 T = 0$$

$$\Rightarrow \ddot{T} + \omega^2 T = 0 \quad \text{where } \omega = \beta c$$

Solutions: $T(t) = A\cos(\omega t) - B\sin(\omega t)$

$X(x) = C\cos(\beta x) - D\sin(\beta x)$

Full solution: $u(x,t) = \big(C\cos(\beta x) - D\sin(\beta x)\big)\big(A\cos(\omega t) - B\sin(\omega t)\big)$

Note only 3 of A, B, C, D are independent

Bounday conditions: $u(x=0,t) = 0$, $u(x=L,t) = 0$ $\Rightarrow$ $X(0) = 0$, $X(L) = 0$

$$\Rightarrow X(x) = C\cos(\beta x) - D\sin(\beta x)$$

$$X(0) = C - 0 = 0 \Rightarrow C = 0$$

$$X(L) = -D\sin(\beta L) = 0$$

For non-trivial solution, $D \neq 0 \Rightarrow \sin(\beta L) = 0 \Rightarrow \beta L = n\pi$

So, $\beta = \frac{n\pi}{L} \Rightarrow \omega = \beta c = \frac{n\pi c}{L}$ for $n = 1, 2, 3\ldots$

Full solution: $u(x,t) = \sum_{n=1}^{\infty} \underbrace{\sin\left(\frac{n\pi x}{L}\right)}_{\text{mode shape}} \underbrace{\left(A_n \cos(\omega_n t) - B_n \sin(\omega_n t)\right)}_{\text{vibrations}}$

Note 'n' is not 'natural' but for n^{th} mode shape