a place of mind

MECH468 : Modern Control Engineering
MECH522 : Foundations in Control Engineering

L6 : Discretzation &
Solution to DT LTI SS model

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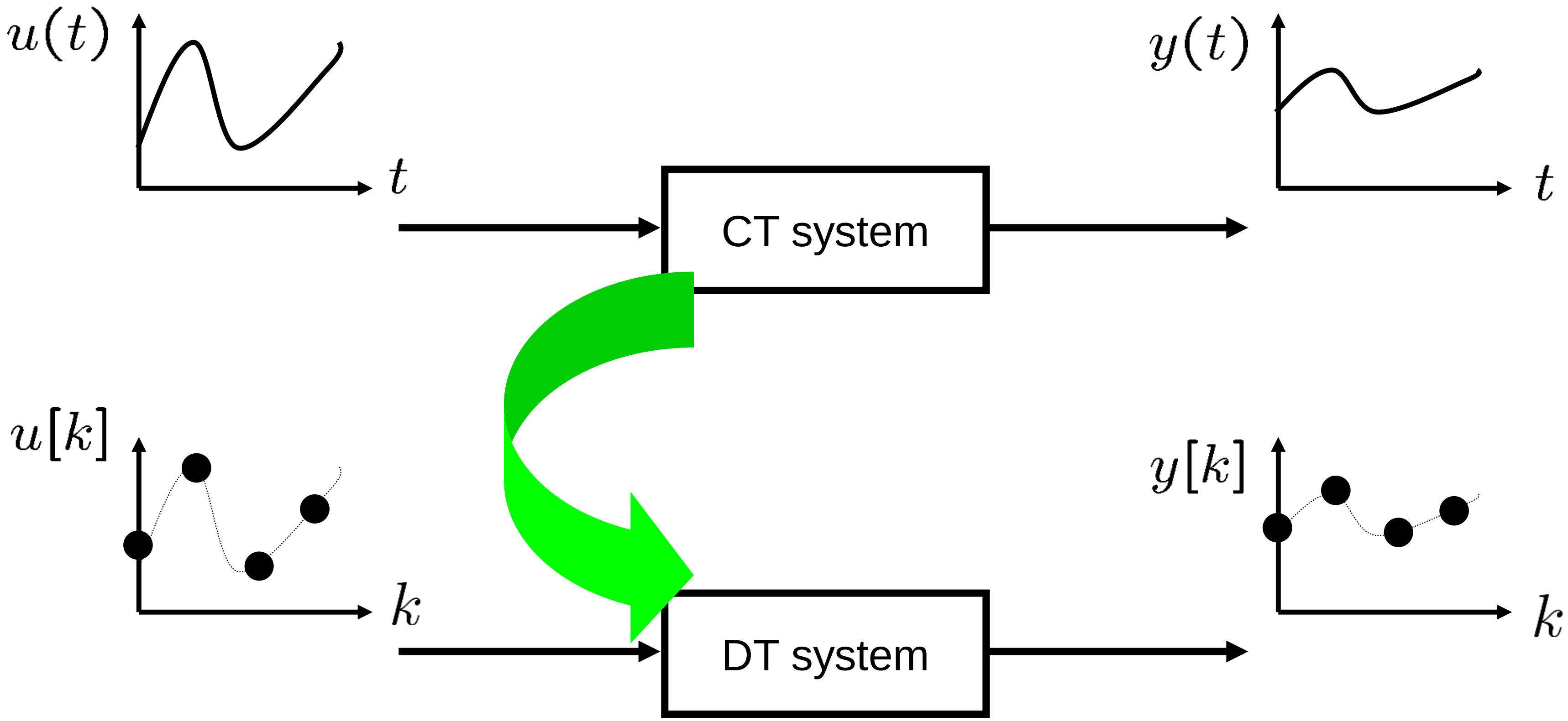
Course plan

Topics	CT	DT
Modeling	✓	→
Stability		
Controllability/observability		
Realization		
State feedback/observer		
LQR/Kalman filter		

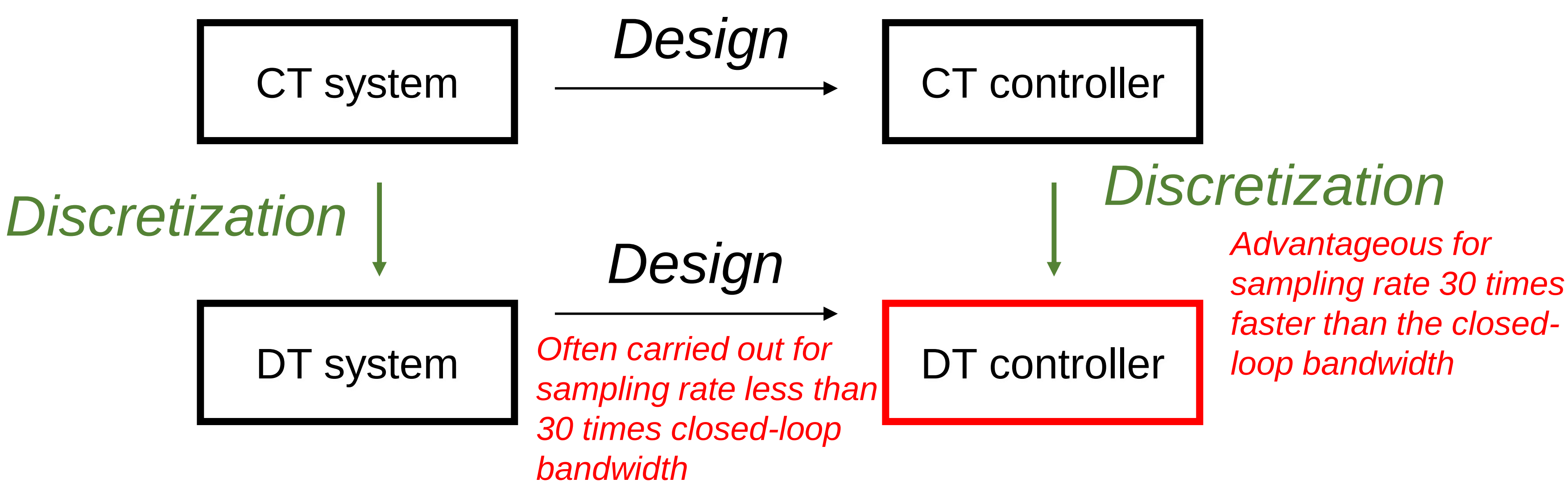
What is "discretization"?

- Approximation of a CT system by a DT system



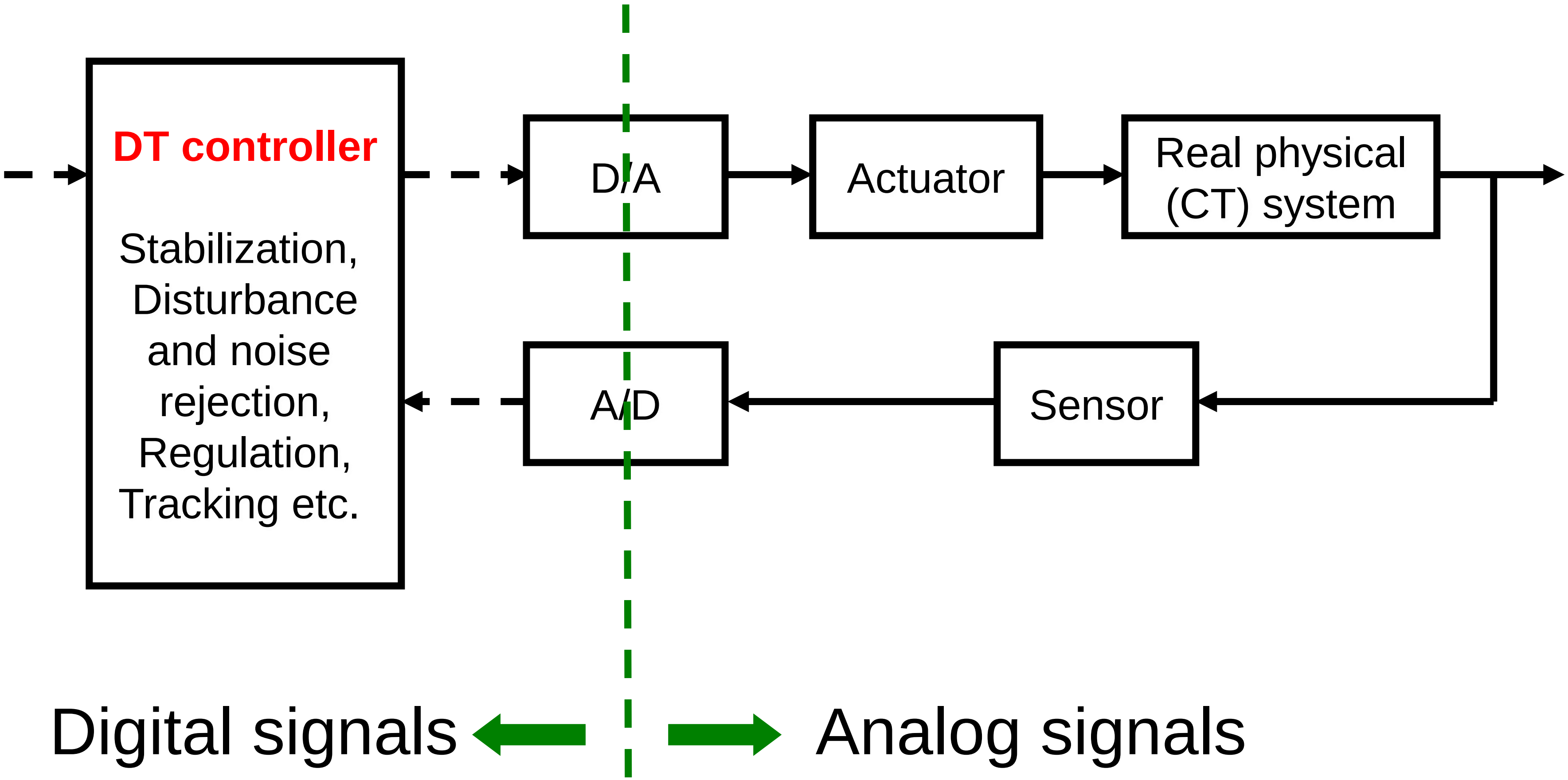
Why “discretization”?

- Digital control (next slide): To realize of a controller in a digital computer, we need a DT controller.



- Digital simulation : Simulation of a CT system is done in discrete-time.

Digital control system

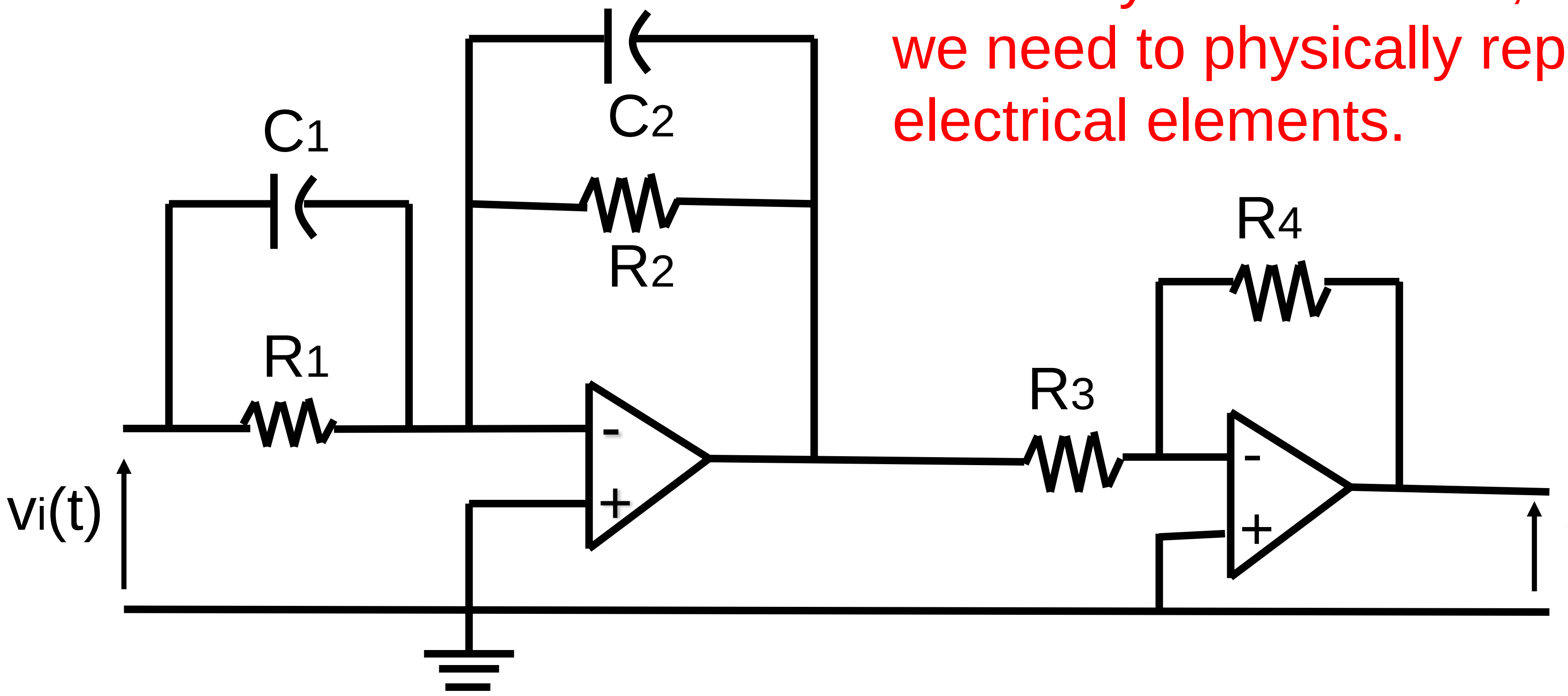


Advantages of digital control

- Reduced cost
  - A single digital computer can replace numerous analog controllers. (→ Reduction in cost!)
- Flexibility in response to design changes (next slide)
  - Any modifications that are required in the future can be implemented with simple software changes rather than expensive hardware modifications.
  - Complex control algorithms can be realized easily.
- Microcontroller examples: Arduino, Raspberry Pi, LabVIEW, dSPACE.

Analog controller inflexibility

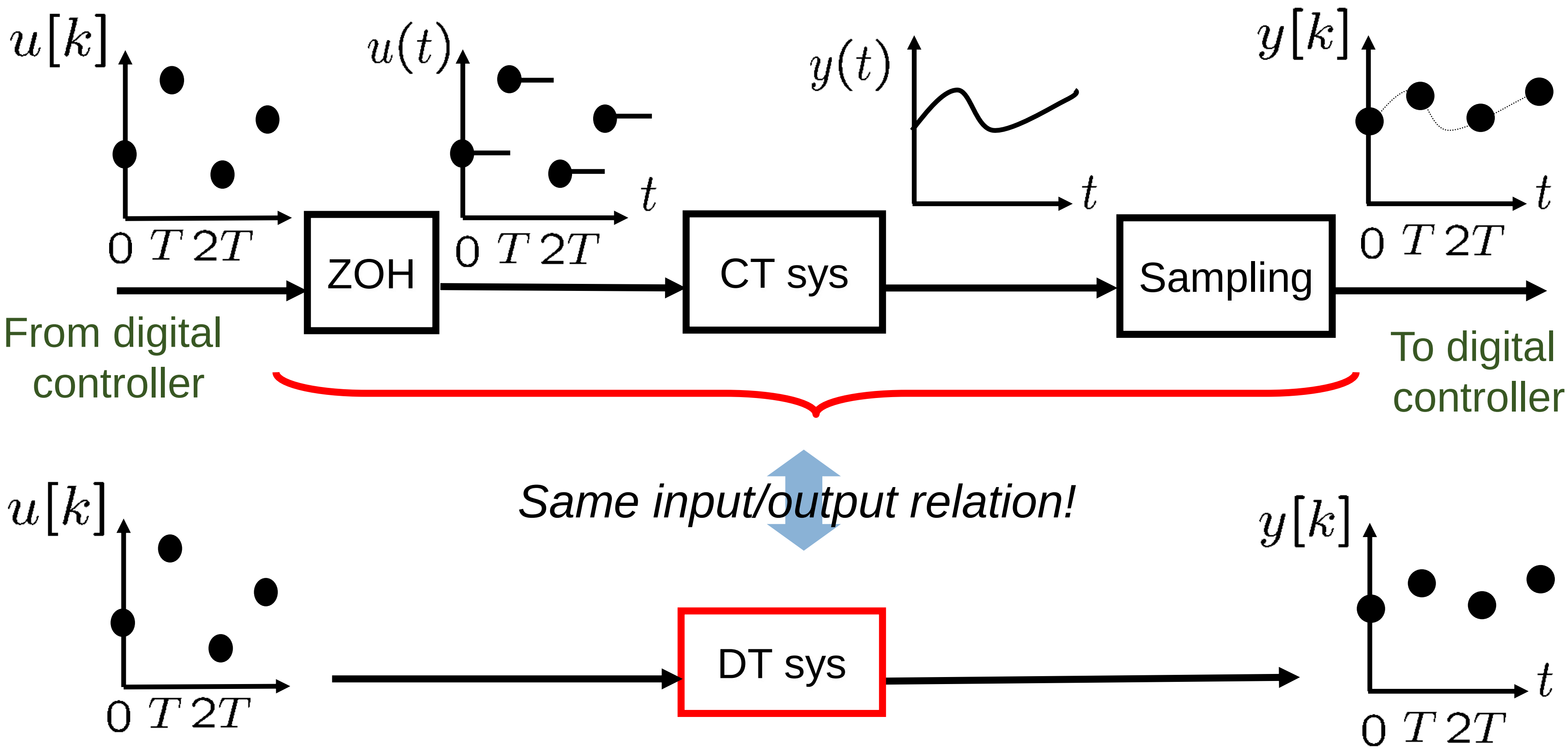
- Lead compensator using operational amplifiers



To modify the controller, we need to physically replace electrical elements.

Discretization by Zero-Order-Hold (ZOH)

- Given a CT system & sample *T*, find a DT system:



ZOH by state-space model

- Continuous-time system $\begin{cases} \dfrac{dx(t)}{dt} = Ax(t) + Bu(t) \\ y(t) = Cx(t) + Du(t) \end{cases}$

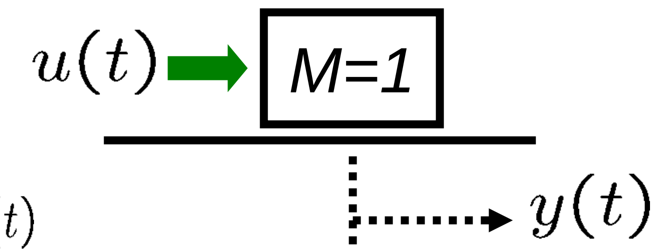
- Discrete-time system obtained by ZOH with sampling time *T* ("c2d.m" in Matlab)

$$\begin{cases} x[k+1] = A_d x[k] + B_d u[k] \\ y[k] = Cx[k] + Du[k] \end{cases} \quad \text{where} \quad \begin{cases} A_d := e^{AT} \\ B_d := \left(\int_0^T e^{A\tau} d\tau \right) \cdot B \end{cases}$$

(C&D: unchanged!)

An example

- Mass with a driving force



$$\begin{cases} \frac{d}{dt}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1/M \end{bmatrix} u(t) \\ y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \end{cases}$$

- Discretization by ZOH with sampling time *T*

$$A_d := e^{AT} = I + AT + \cdots = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix}$$

$$B_d := \left(\int_0^T e^{A\tau} d\tau\right) B = \left(\int_0^T \begin{bmatrix} 1 & \tau \\ 0 & 1 \end{bmatrix} d\tau\right)\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{T^2}{2} \\ T \end{bmatrix}$$

Discretization in Matlab

- Code for discretization

```
>> A = [0 1;0 0];
>> B = [0; 1];
>> C = [1 0];
>> D = 0;
>> sys = ss(A,B,C,D);
>> T = 0.1;
>> sysd = c2d(sys,T);
```

CT system

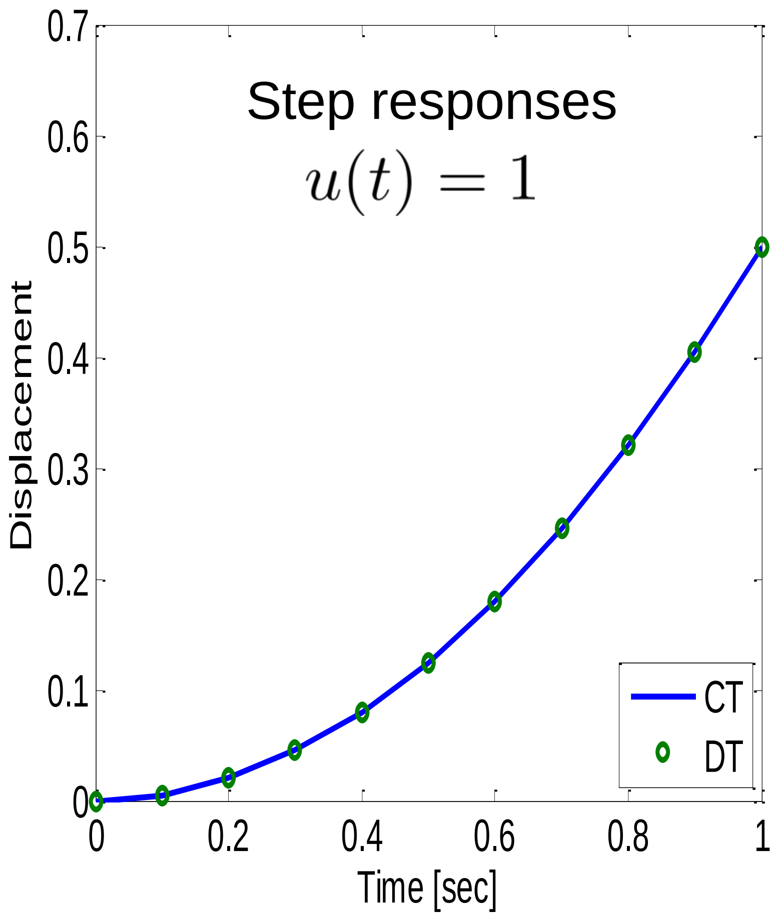
Sampling time

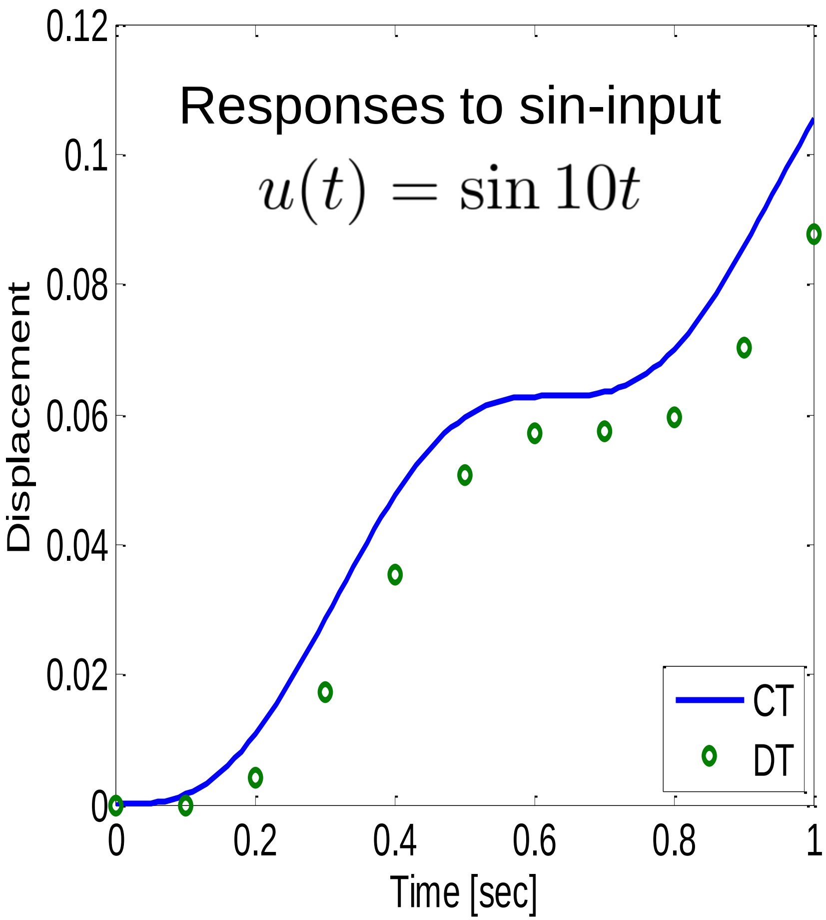
Discretization

Discretization result

```
>> sysd.a
ans =
     1.0000    0.1000
          0    1.0000
>> sysd.b
ans =
     0.0050
     0.1000
```

Simulations in Matlab





Derivation

- Solution to state equation $x(t) = e^{At}x_0 + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau$

- For piecewise constant input $u[k] := u(t), t \in [kT, (k+1)T)$

$$
\begin{aligned}
\underbrace{x((k+1)T)}_{x[k+1]} &= e^{A(k+1)T}x_0 + \int_0^{(k+1)T} e^{A((k+1)T-\tau)}Bu(\tau)d\tau \\
&= e^{AT}\left\{e^{AkT}x_0 + \int_0^{(k+1)T} e^{A(kT-\tau)}Bu(\tau)d\tau\right\} \\
&= e^{AT}x(kT) + \int_{kT}^{(k+1)T} e^{A((k+1)T-\tau)}d\tau \cdot Bu[k] \\
&= \underbrace{e^{AT}}_{Ad}\underbrace{x(kT)}_{x[k]} + \underbrace{\int_0^{T} e^{A\tau}d\tau \cdot B}_{Bd}u[k] \quad \text{(by a variable change)}
\end{aligned}
$$

Solution to DT LTI SS model

- DT LTI state-space model

$$\begin{cases} x[k+1] &= Ax[k] + Bu[k], \ x[0] = x_0 \\ y[k] &= Cx[k] + Du[k] \end{cases}$$

- Solution

$$x[k] = A^k x[0] + \underbrace{\left[B, AB, \cdots, A^{k-1}B\right]} \begin{bmatrix} u[k-1] \\ u[k-2] \\ \vdots \\ u[0] \end{bmatrix}$$

$$y[k] = Cx[k] + Du[k]$$

This matrix will appear later (for *controllability*).

Derivation

- Solve recursively *x[k+1]=Ax[k]+Bu[k]*
  - *k=0*: $x[1] = Ax[0] + Bu[0]$
  - *k=1*: $x[2] = Ax[1] + Bu[1] = A^2x[0] + \begin{bmatrix} B & AB \end{bmatrix} \begin{bmatrix} u[1] \\ u[0] \end{bmatrix}$
  - *k=2*: $x[3] = Ax[2] + Bu[2] = A^3x[0] + \begin{bmatrix} B & AB & A^2B \end{bmatrix} \begin{bmatrix} u[2] \\ u[1] \\ u[0] \end{bmatrix}$
- By induction

$$x[k] = A^k x[0] + [B, AB, \cdots, A^{k-1}B] \begin{bmatrix} u[k-1] \\ u[k-2] \\ \vdots \\ u[0] \end{bmatrix}$$

Summary

- Discretization (Zero-Order Hold)
  - Digital control
  - Formula of discretized state-space model
  - Example with Matlab simulations
- Solution to DT LTI SS models
- Next lecture, stability (VERY IMPORTANT!)

- Now you can solve all the problems in HW1, which is due on January 24 (Friday), 6pm.