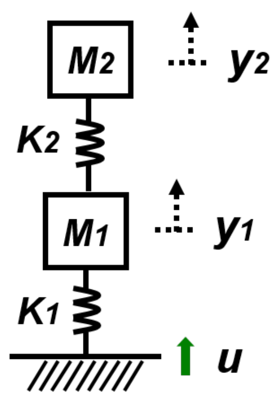
MECH468 Modern Control Engineering
MECH509 Controls

Homework 1. Due: January 29 (Friday), 11:59 pm, 2021.
Solutions

1 Theoretical questions

Q1. Derive the state-space model for the mass-spring system, where the input u is the displacement of the ground (assume that the ground goes up and down, like a bumpy road), and the outputs are the displacement y_1 and y_2 indicated in the figure.



Solution: The equations of motion are

$$\begin{aligned} m_1\ddot{y}_1 &= -k_1(y_1-u)-k_2(y_1-y_2), \\ m_2\ddot{y}_2 &= -k_2(y_2-y_1). \end{aligned}$$

Introduce the state variables as

$$x := \begin{bmatrix} y_1 \\ \dot{y}_1 \\ y_2 \\ \dot{y}_2 \end{bmatrix}.$$

Then, the state-space model becomes

$$
\begin{aligned}
\dot{x} &= \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{k_1+k_2}{m_1} & 0 & \frac{k_2}{m_1} & 0 \\ 0 & 0 & 0 & 1 \\ \frac{k_2}{m_2} & 0 & -\frac{k_2}{m_2} & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ \frac{k_1}{m_1} \\ 0 \\ 0 \end{bmatrix} u \\
y &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} x
\end{aligned}
$$

Q2. Linearize the following nonlinear state-space model around the equilibrium point $x_{1o} = 1$, $x_{2o} = 0$, $u_{1o} = 1$, $u_{2o} = 0$.

$$
\begin{aligned}
\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} &= \begin{bmatrix} -2x_1x_2^2 + x_2u_1 \\ -e^{-x_1u_1}x_2 + x_1u_2 \end{bmatrix} \\
y &= x_1u_1u_2.
\end{aligned}
$$

Solution: The Jacobian calculations are as follows, where f and h denote the right-hand sides of the state equation and of the output equation, respectively.

$$
\frac{\partial f}{\partial x} = \begin{bmatrix} -2x_2^2 & -4x_1x_2 + u_1 \\ u_1e^{-x_1u_1}x_2 + u_2 & -e^{-x_1u_1} \end{bmatrix}, \quad \frac{\partial f}{\partial u} = \begin{bmatrix} x_2 & 0 \\ x_1e^{-x_1u_1}x_2 & x_1 \end{bmatrix}
$$

$$
\frac{\partial h}{\partial x} = \begin{bmatrix} u_1u_2 & 0 \end{bmatrix}, \quad \frac{\partial h}{\partial u} = \begin{bmatrix} x_1u_2 & x_1u_1 \end{bmatrix}
$$

By substituting the operating point into the Jacobian, the linearized model becomes

$$
\begin{aligned}
\delta\dot{x} &= \begin{bmatrix} 0 & 1 \\ 0 & -e^{-1} \end{bmatrix} \delta x + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \delta u \\
\delta y &= \begin{bmatrix} 0 & 0 \end{bmatrix} \delta x + \begin{bmatrix} 0 & 1 \end{bmatrix} \delta u,
\end{aligned}
$$

where

$$
\delta x := x - \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \delta u := u - \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \delta y := y(-0).
$$

Q3. Calculate the matrix exponential e^{At} for the following A matrix.

$$
A = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{bmatrix}
$$

Solution: Since the matrix A has a block-diagonal structure, we can compute the matrix exponential for each diagonal block. For a matrix

$$\tilde{A} := \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix} = -I + \underbrace{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}_{=:S},$$

the matrix exponential is computed as (you can use other methods if you want, like Laplace transform method)

$$\begin{aligned} e^{\tilde{A}t} &= e^{(-I+S)t} \\ &= e^{-It}e^{St}, \ (e^{A+B} = e^A e^B \text{ if } AB = BA) \\ &= e^{-t}I \times \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}, \ (S \text{ is the shift matrix}) \\ &= e^{-t} \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \end{aligned}$$

Therefore,

$$A_d := e^{At} = \begin{bmatrix} e^{-t} & te^{-t} & 0 \\ 0 & e^{-t} & 0 \\ 0 & 0 & e^{-2t} \end{bmatrix}$$

Q4. Discretize the following continuous-time state equation with zero-order-hold, with a sampling time $T > 0$.

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u.$$

Solution: The A-matrix of the discretized system is

$$e^{AT} = \mathcal{L}^{-1} \left\{ \begin{bmatrix} s & -1 \\ 6 & s+5 \end{bmatrix}^{-1} \right\} = e^{-2T} \underbrace{\begin{bmatrix} 3 & 1 \\ -6 & -2 \end{bmatrix}}_{=:M_1} + e^{-3T} \underbrace{\begin{bmatrix} -2 & -1 \\ 6 & 3 \end{bmatrix}}_{=:M_2}$$

because

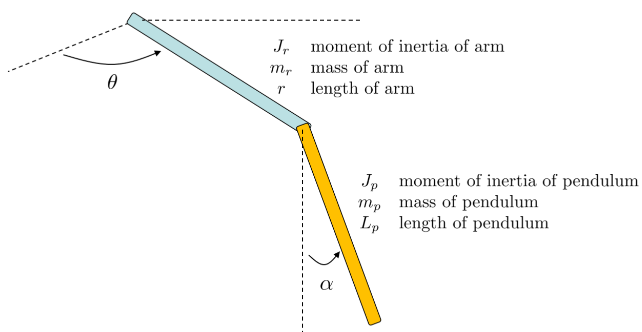
$$\begin{bmatrix} s & -1 \\ 6 & s+5 \end{bmatrix}^{-1} = \frac{1}{(s+2)(s+3)} \begin{bmatrix} s+5 & 1 \\ -6 & s \end{bmatrix} = \frac{1}{s+2} \begin{bmatrix} 3 & 1 \\ -6 & -2 \end{bmatrix} + \frac{1}{s+3} \begin{bmatrix} -2 & -1 \\ 6 & 3 \end{bmatrix}$$

The B-matrix of the discretized system is

$$
\begin{aligned}
B_d &= \int_0^T e^{A\tau} d\tau \cdot B \\
&= \int_0^T (e^{-2\tau} M_1 + e^{-3\tau} M_2) d\tau \cdot B \\
&= \frac{1}{2}(1 - e^{-2T}) M_1 B + \frac{1}{3}(1 - e^{-3T}) M_2 B \\
&= \frac{1}{2}(1 - e^{-2T}) \begin{bmatrix} 1 \\ -2 \end{bmatrix} + \frac{1}{3}(1 - e^{-3T}) \begin{bmatrix} -1 \\ 3 \end{bmatrix}
\end{aligned}
$$

2 Matlab question

Consider a rotary pendulum shown below. This system has been taken from https://www.quanser.com/products/qube-servo-2/.



The equations of motion can be written (no derivation is required here) as

$$(J_r + J_p \sin^2 \alpha)\ddot{\theta} + m_p r \ell \cos \alpha \ddot{\alpha} + 2 J_p \sin \alpha \cos \alpha \dot{\theta} \dot{\alpha} - m_p r \ell \sin \alpha \dot{\alpha}^2 = \tau - b_r \dot{\theta},$$

$$J_p \ddot{\alpha} + m_p r \ell \cos \alpha \ddot{\theta} - J_p \sin \alpha \cos \alpha \dot{\theta}^2 + m_p g \ell \sin \alpha = -b_p \dot{\alpha},$$

where the notations are indicated in the figure, and $\ell := L_p/2$.

If we approximate the system around $\theta = 0$ and $\alpha = 0$, using $\sin \alpha \approx \alpha$, $\cos \alpha \approx 1$ and $\sin^2 \alpha \approx 0$, we can simplify these equations as

$$J_r \ddot{\theta} + m_p r \ell \ddot{\alpha} = \tau - b_r \dot{\theta},$$

$$m_p r\ell\ddot{\theta} + J_p\ddot{\alpha} = -b_p\dot{\alpha} - m_p g\ell\alpha.$$

From these two equations, we can derive

$$\ddot{\theta} = \frac{1}{J_t}\left\{J_p(\tau - b_r\dot{\theta}) + m_p r\ell(b_p\dot{\alpha} + m_p g\ell\alpha)\right\}$$

$$\ddot{\alpha} = \frac{1}{J_t}\left\{-J_r(b_p\dot{\alpha} + m_p g\ell\alpha) - m_p r\ell(\tau - b_r\dot{\theta})\right\}$$

where

$$J_t := J_r J_p - (m_p r\ell)^2.$$

By introducing the state variables as

$$x_1 := \theta,\ x_2 := \dot{\theta},\ x_3 := \alpha,\ x_4 := \dot{\alpha},$$

and the input and outputs as

$$u := \tau,\ y_1 := \theta,\ y_2 = \alpha,$$

we can get the state-space model as

$$\begin{array}{rcl} \dot{x} & = & Ax + Bu, \\ y & = & Cx, \end{array}$$

where

$$A := \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -J_p b_r/J_t & (m_p\ell)^2 rg/J_t & m_p r\ell b_p/J_t \\ 0 & 0 & 0 & 1 \\ 0 & m_p r\ell b_r/J_t & -J_r m_p g\ell/J_t & -J_r b_p/J_t \end{bmatrix}, \quad B := \frac{1}{J_t}\begin{bmatrix} 0 \\ J_p \\ 0 \\ -m_p r\ell \end{bmatrix}$$

$$C := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

The parameter values are given in the table below.

$$J_r := \frac{1}{3}m_r r^2, \quad J_p = \frac{1}{3}m_p L_p^2, \quad \ell = \frac{L_p}{2}.$$

Notation	Meaning	Value and unit
m_r	rotary arm mass	0.095 kg
r	rotary arm length	0.085 m
b_r	viscous friction coefficient	0.001 Nms/rad
m_p	pendulum mass	0.024 kg
L_p	pendulum length	0.129 m
b_p	viscous friction coefficient	5×10^{-5} Nms/rad
g	gravitational acceleration	9.81 m/s^2

Task: Using Simulink, simulate for the case when all the initial states are zero except $\alpha(0) = 0.1$ [rad], and with no input. Plot the outputs $\theta(t)$ and $\alpha(t)$. Add your Matlab code(s) (m-file and Simulink block) in your report.