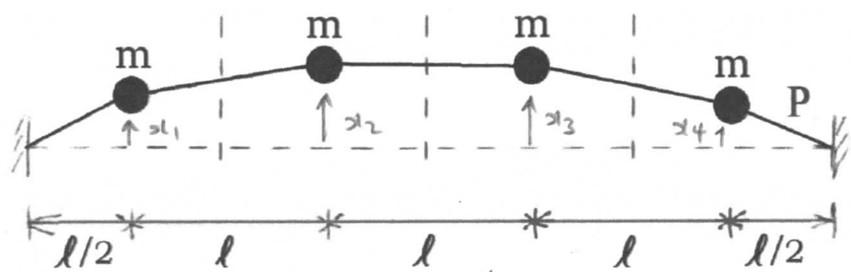
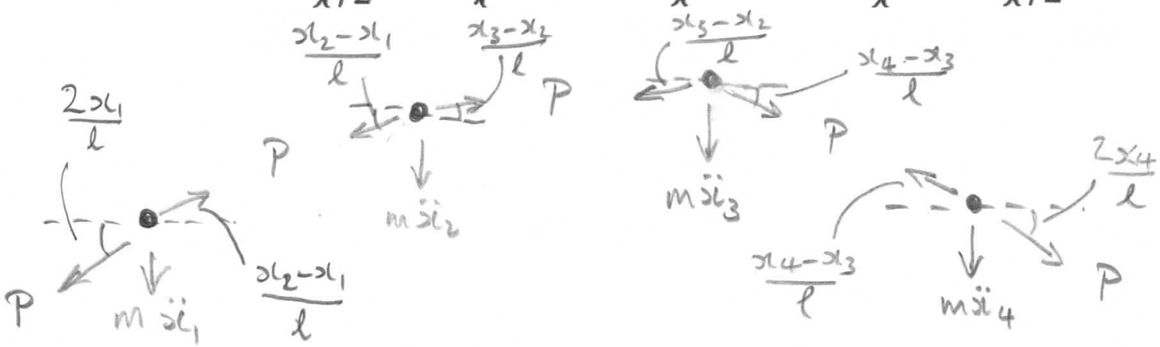
# MECH 463 -- Homework 12

1. A uniform string of length L, mass density ρ and cross-section area A is stretched to a tension P. It is desired to model the string as a sequence of n equal segments, each of length $\ell = L/n$ and mass $m = \rho A \ell$. The mass of each segment is centred within the segment, so the distances of the first and last masses from the fixed ends are $\ell/2$, while the distances between all the interior masses are $\ell$. Consider the case where n = 4, draw a free-body diagram and formulate the matrix equation of motion. Examine the structure of your matrices and then generalize them for larger n. Program your equations into Matlab and compute the first three natural frequencies and plot the corresponding mode shapes for n = 10, 20, 40, 80. Compare your results with the theoretical solution of a vibrating string.




Use small-angle approximations to define the angles.

Vertical equilibrium in the FBDs.

$$-P\frac{2x_1}{\ell} - m\ddot{x}_1 + P\frac{x_2-x_1}{\ell} = 0 \qquad -P\frac{x_2-x_1}{\ell} - m\ddot{x}_2 - P\frac{x_2-x_3}{\ell} = 0$$

$$-P\frac{x_3-x_2}{\ell} - m\ddot{x}_3 - P\frac{x_4-x_3}{\ell} = 0 \qquad -P\frac{x_4-x_3}{\ell} - m\ddot{x}_4 - P\frac{2x_4}{\ell} = 0$$

Rearranging and putting into matrix form

$$\begin{bmatrix} m & 0 & 0 & 0 \\ 0 & m & 0 & 0 \\ 0 & 0 & m & 0 \\ 0 & 0 & 0 & m \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \\ \ddot{x}_4 \end{bmatrix} + \begin{bmatrix} 3k & -k & 0 & 0 \\ -k & 2k & -k & 0 \\ 0 & -k & 2k & -k \\ 0 & 0 & -k & 3k \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

As expected from the mass-based coordinates, the mass matrix is diagonal. It has $m$ at each position, where $m = \rho A l = \rho A L/n$. The stiffness matrix has a tri-diagonal structure because each mass is connected only to its adjacent masses. The main diagonal has $2k$ at each position, except for the first and last, with $3k$ because of the $l/2$ spacings at the ends. The upper and lower adjacent diagonals have $-k$ at all positions, where $k = P/l = \frac{nP}{L}$.

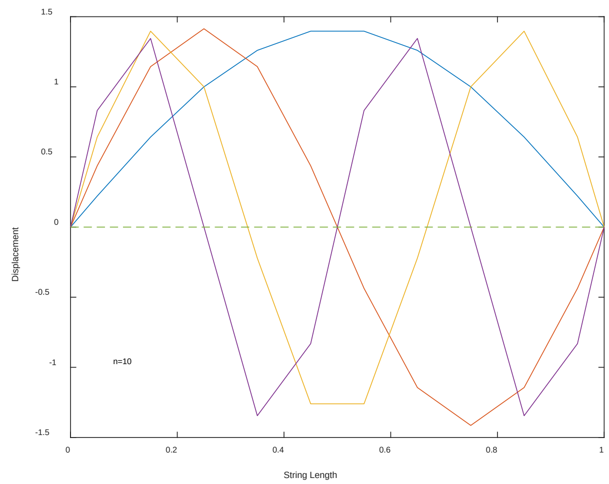
These matrices were coded using Matlab, with the eigensolution done by function eig. The computations were done using unit values of $\rho$, $A$, $P$ and $L$. For given values of these quantities, the computed natural frequencies need be multiplied by $\sqrt{\frac{P}{\rho A L^2}}$.

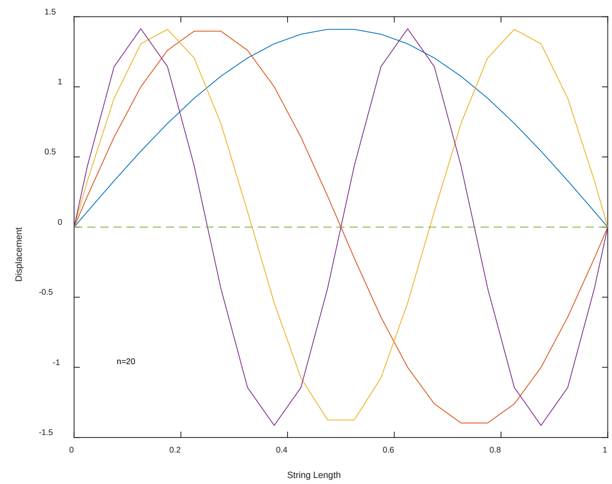
From the analytical solution to the wave equation, the "exact" solution is $\omega_n = \frac{n\pi}{L}\sqrt{\frac{P}{\rho A}}$.

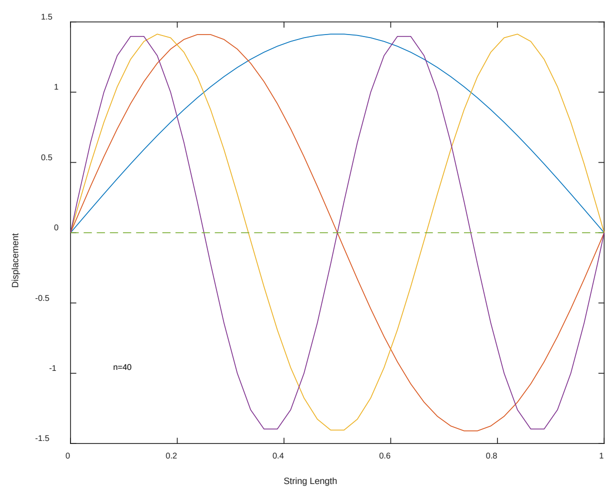
For unit values of $\rho$, $A$, $P$ and $L$, the computed $\omega_n = n\pi$.

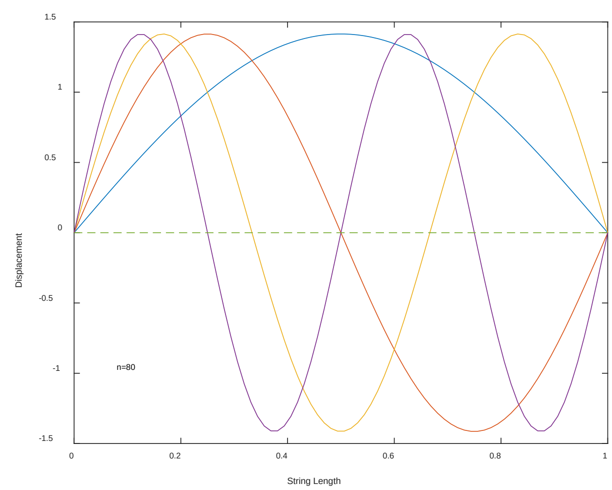
```
% MECH 463 Homework 12 Q1
%
% Global variables:
% -----------------
% D         eigenvalue matrix
% i         figure index
% j         mode shape index
% K         stiffness matrix
% k0        diagonal of K matrix
% k1        super and sub diagonal of K
% M         mass matrix
% m0        diagonal of M matrix
% N         set of n values used
% n         number of segments
% U         mode shape matrix
% V         eigenvector matrix
% wn        natural frequencies
% x         distance along string

clear all;
close all;

% Do calculations for n = 10, 20, 40, 80
N = [10 20 40 80];
for i = 1:1:4
    n = N(i);

%   Assign mass and stiffness matrices
    m0 = linspace(1/n,1/n,n);
    M = diag(m0);
    k0 = linspace(2*n,2*n,n);
    k0(n) = 3*n;
    k0(1) = 3*n;
    k1 = linspace(-n,-n,n-1);
    K = diag(k0) + diag(k1,1) + diag(k1,-1);

%   Solve generalized eigenvalue problem
    [V,D] = eig(K,M);

%   Extract the natural frequencies from V
    wn(i,1:4) = sqrt(diag(D(1:4,1:4)));

%   Plot mode shapes.  Improve their appearance
%   by giving them consistent signs and adding
%   zeroes at their ends
    figure (i)
    x(n+2) = 1;
    x(1)   = 0;
    x(2:n+1) = linspace(0.5/n,(n-0.5)/n,n);
    U(n+2,1:4) = 0;
    U(1,1:4)   = 0;
    for j = 1:1:4
        if V(1,j) > 0
            U(2:n+1,j) =  V(1:n,j);
        else
            U(2:n+1,j) = -V(1:n,j);
        end
    end
    plot(x,U(:,1:4))
    hold on
    plot([0 1],[0 0], '--')
    xlabel('String Length')
    ylabel('Displacement')
    text(0.08,-1,strcat('n= ',num2str(N(i),2)))
end

% Display the natural frequency estimates
disp(' ')
disp('   n =       Natural 
Frequencies/sqrt(P/rho.A.L^2)')
disp([N' wn(:,1:4)])

% Display the theoretical natural frequencies
disp('   Theory    Natural 
Frequencies/sqrt(P/rho.A.L^2)')
disp([0 linspace(pi, 4*pi, 4)])
```

```
>> HW12_Q1

n =       Natural Frequencies/sqrt(P/rho.A.L^2)
10.0000    3.1287    6.1803    9.0798   11.7557
20.0000    3.1384    6.2574    9.3378   12.3607
40.0000    3.1408    6.2767    9.4030   12.5148
80.0000    3.1414    6.2816    9.4193   12.5535

Theory    Natural Frequencies/sqrt(P/rho.A.L^2)
      0    3.1416    6.2832    9.4248   12.5664
```


1.5
1
0.5
0
-0.5
-1
-1.5
Displacement
n=10
0
0.2
0.4
0.6
0.8
1
String Length


1.5
1
0.5
0
-0.5
-1
-1.5
Displacement
n=20
0
0.2
0.4
0.6
0.8
1
String Length

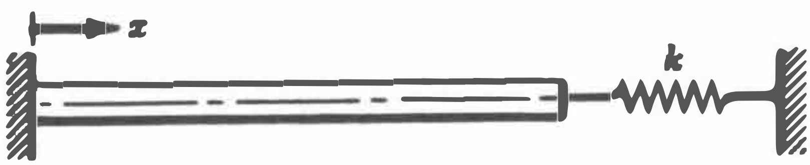

1.5
1
0.5
0
-0.5
-1
-1.5
Displacement
n=40
0
0.2
0.4
0.6
0.8
1
String Length


1.5
1
0.5
0
-0.5
-1
-1.5
Displacement
n=80
0
0.2
0.4
0.6
0.8
1
String Length

2. A uniform rod of length L, cross-section area A, Young's modulus E and mass density ρ is rigidly fixed at its left end and connected to a spring of stiffness k at its right end. Solve for the natural frequencies and mode shapes of the system starting from the wave equation for longitudinal vibrations:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

where u(x,t) is the longitudinal vibrational displacement, and c = √(E/ρA) is the wave speed. Leave your equations in symbolic form, but indicate how the roots could be evaluated if numerical answers were required. *Hint: The boundary condition at the right end is ∂u/∂x(L) = -(k/EA) u(L).*




Try a separable solution $u(x,t) = X(x)\,T(t)$

Sub in wave equation: $X(x)\,\ddot{T}(t) = c^2 X''(x)\,T(t)$

$\rightarrow \dfrac{X''(x)}{X(x)} = \dfrac{1}{c^2}\dfrac{\ddot{T}(t)}{T(t)} = $ a constant $= -\beta^2$ for convenience

$\rightarrow \ddot{T} + (c\beta)^2 T = 0 \rightarrow T(t) = A\cos\omega t - B\sin\omega t$ where $\omega = c\beta$

$X'' + \beta^2 X = 0 \rightarrow X(x) = C\cos\beta x + D\sin\beta x$

Left b.c. $X(0) = 0 \rightarrow C = 0$

Right b.c. Force $= -k\,X(L)$

Strain at right $= X'(L)$

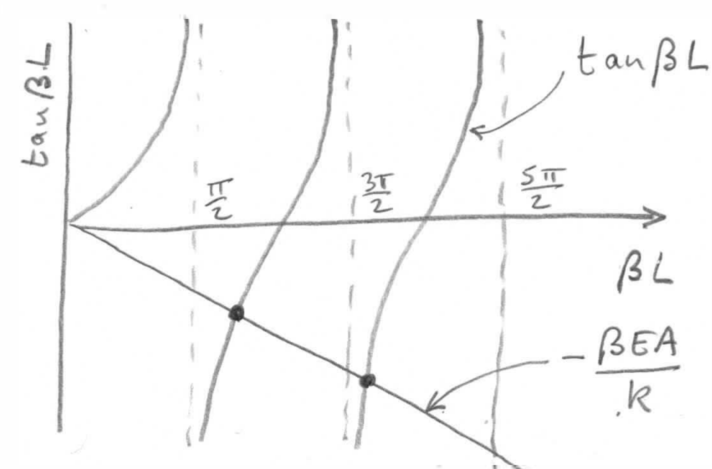
Stress at right $= E X'(L)$

Force at right $= EA X'(L)$

$\rightarrow X'(L) = -\frac{k}{EA} X(L)$

$\rightarrow \beta D\cos\beta L = -\frac{k}{EA} D\sin\beta L$

$\rightarrow \tan\beta L = -\dfrac{\beta EA}{k}$




Roots give $\beta L$ values

$\omega = \beta c = \beta L \cdot \frac{c}{L} = \beta L \cdot \frac{1}{L}\sqrt{\frac{E}{\rho}}$