1 Rotations in 2D and complex numbers

We are all familiar with complex numbers $\mathbb{C}$, where $z = a + ib$ with $i^2 = -1$, with the familiar multiplication by a scalar, addition, and product

$$z_1 z_2 = (a_1 + ib_1)(a_2 + ib_2) = (a_1 a_2 - b_1 b_2) + i(a_1 b_2 + a_2 b_1)$$

and the properties of commutativity, associativity, distributivity, zero element for addition, unit element for multiplication. The *complex conjugate* of $z = a + ib$ is $\bar{z} = a - ib$, with alternate notation $z^* = a - ib$. We have that $zz^* = a^2 + b^2$. The magnitude or absolute value of z is $|z| = \sqrt{(a^2 + b^2)} = \sqrt{zz^*}$.

Now, we can think of the complex number product $y = sx = (a_s + ib_s)x = s \circ x = f_s(x)$ for fixed s as a *linear function* $f : I\!R^2 \to I\!R^2$, where $x = \begin{bmatrix} a_x \\ b_x \end{bmatrix}$, and the basis vectors are $1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $i = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

Because it is a *linear function*, f_s has a matrix representation, *the columns of which are the images of the basis vectors in* $I\!R^2$. But $s \circ 1 = \begin{bmatrix} a_s \\ b_s \end{bmatrix}$ and $s \circ i = \begin{bmatrix} -b_s \\ a_s \end{bmatrix}$ and therefore

$$(s\circ) = \begin{bmatrix} a_s & -b_s \\ b_s & a_s \end{bmatrix} \quad ; \quad \begin{bmatrix} a_y \\ b_y \end{bmatrix} = \begin{bmatrix} a_s & -b_s \\ b_s & a_s \end{bmatrix} \begin{bmatrix} a_x \\ b_x \end{bmatrix} = \begin{bmatrix} a_s a_x - b_s b_x \\ b_s a_x + a_s b_x \end{bmatrix} \tag{1}$$

We can re-write this equation as

$$\begin{bmatrix} a_y \\ b_y \end{bmatrix} = (a_s^2 + b_s^2) \begin{bmatrix} \frac{a_s}{a_s^2+b_s^2} & -\frac{b_s}{a_s^2+b_s^2} \\ \frac{b_s}{a_s^2+b_s^2} & \frac{a_s}{a_s^2+b_s^2} \end{bmatrix} \begin{bmatrix} a_x \\ b_x \end{bmatrix} = r \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} a_x \\ b_x \end{bmatrix} \tag{2}$$

which we recognize as a rotation plus scaling, with $s = a_s + ib$ written in *polar* form as $s = r(\cos\theta + i\sin\theta)$. This is the familiar polar form of complex number multiplication $z_1 z_2 = r_1 r_2(\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2))$. The product has magnitude equal to the product of the magnitudes $|z_1||z_2|$ and angle equal to the sum of the angles $\theta_1 + \theta_2$.

The polar form of complex numbers is usually written using the complex exponential $\exp i\theta = e^{i\theta}$, which we define here as the series

$$e^{\theta i} = \sum_{k=0} \frac{1}{k!}(\theta i)^k \tag{3}$$

Since $i^2 = -1$, every even term in the series becomes real, and every odd term becomes imaginary:

$$
\begin{aligned}
e^{\theta i} &= 1 + i\theta + \frac{1}{2!}i^2\theta^2 + \frac{1}{3!}i^3\theta^3 + \frac{1}{4!}\theta^4 - ... \\
&= 1 + i\theta - \frac{1}{2!}\theta^2 - i\frac{1}{3!}\theta^3 + \frac{1}{4!}\theta^4 - ... \\
&= (1 - \frac{1}{2!}\theta^2 + \frac{1}{4!}\theta^4 - ...) + i(\theta - \frac{1}{3!}\theta^3 + ...) \\
&= \cos\theta + i\sin\theta \qquad (4)
\end{aligned}
$$

allowing us to write z in polar form as $z = a + ib = r(\cos\theta + i\sin\theta) = re^{i\theta}$. Therefore, with z viewed as a vector in the plane, $e^{i\theta}z$ *rotates* z by an angle θ.

We can easily check that, as expected,

$$
(i\circ)^2 = -I_{2\times 2} \qquad (5)
$$

and therefore we can separate the even and odd powers of $(s\circ)$ to obtain the matrix exponential

$$
\begin{aligned}
e^{\theta(s\circ)} &= I + (s\circ)\theta + \frac{1}{2!}(s\circ)^2\theta^2 + \frac{1}{3!}(s\circ)^3\theta^3 + \frac{1}{4!}(s\circ)^4\theta^4 - ... \\
&= (1 - \frac{1}{2!}\theta^2 + \frac{1}{4!}\theta^4 - ...)I + (\theta - \frac{1}{3!}\theta^3 + ...)(s\circ) \\
&= \cos\theta I + \sin\theta(s\circ) \quad . \qquad (6)
\end{aligned}
$$

2 Quaternions

Quaternions were defined by the Irish mathematician Hamilton in 1843, while researching an extension of the description of rotations in the plane by complex multiplication.

Quaternions were defined as

$$
q = d + ai + bj + ck \quad a, b, c, d \in I\!R \qquad (7)
$$

where the fundamental quaternion units i, j, k are *imaginary numbers* such that

$$
i^2 = j^2 = k^2 = ijk = -1 \quad . \qquad (8)
$$

with addition, multiplication by a scalar, zero and unit elements defined as expected, and with associativity and distributivity properties that apply. Addition of quaternions is commutative but the product is not.

From the definition, we can see that

$$
ij = -ji = k \ ; \ jk = -kj = i \ ; \ ki = -ik = j \ . \qquad (9)
$$

We note that if $b = c \equiv 0$, $c = a \equiv 0$, $a = b \equiv 0$, the corresponding quaternions $q = d + ai$, $q = d + bj$ and $q = d + ck$ are conventional complex numbers. However, unlike complex numbers, the product of quaternions is not commutative in general.

The quaternion product is

$$\begin{aligned} pq &= (d_p + a_p i + b_p j + c_p k)(d_q + a_q i + b_q j + c_q k) \\ &= (d_p 1 + a_p i1 + b_p j1 + c_p k1)d_q + \\ &= (d_p i + a_p ii + b_p ji + c_p ki)a_q + \\ &= (d_p j + a_p ij + b_p jj + c_p kj)b_q + \\ &= (d_p k + a_p ik + b_p jk + c_p kk)c_q \end{aligned} \tag{10}$$

or in matrix form, filling columns as the images of $1, i, j, k$, we obtain the following:

$$\begin{bmatrix} d_{pq} \\ a_{pq} \\ b_{pq} \\ c_{pq} \end{bmatrix} = \begin{bmatrix} d_p & -a_p & -b_p & -c_p \\ a_p & d_p & -c_p & b_p \\ b_p & c_p & d_p & -a_p \\ c_p & -b_p & a_p & d_p \end{bmatrix} \begin{bmatrix} d_q \\ a_q \\ b_q \\ c_q \end{bmatrix} = \left(\begin{bmatrix} d_p \\ a_p \\ b_p \\ c_p \end{bmatrix} \circ \right) \begin{bmatrix} d_q \\ a_q \\ b_q \\ c_q \end{bmatrix} \tag{11}$$

where $(p\circ)$ is the matrix representation of the quaternion product.

With the notation

$$v_p = \begin{bmatrix} a_p \\ b_p \\ c_p \end{bmatrix} \quad v_q = \begin{bmatrix} a_q \\ b_q \\ c_q \end{bmatrix} \quad v_{pq} = \begin{bmatrix} a_{pq} \\ b_{pq} \\ c_{pq} \end{bmatrix} \tag{12}$$

and

$$p = \begin{bmatrix} d_p \\ v_p \end{bmatrix} \quad q = \begin{bmatrix} d_q \\ v_q \end{bmatrix} \quad p \circ q = \begin{bmatrix} d_{pq} \\ v_{pq} \end{bmatrix} \tag{13}$$

we can write

$$\begin{bmatrix} d_p & -a_p & -b_p & -c_p \\ a_p & d_p & -c_p & b_p \\ b_p & c_p & d_p & -a_p \\ c_p & -b_p & a_p & d_p \end{bmatrix} = \begin{bmatrix} d_p & -v_p^T \\ v_p & d_p I + v_p \times \end{bmatrix} \tag{14}$$

and

$$\begin{bmatrix} d_{pq} \\ v_{pq} \end{bmatrix} = \begin{bmatrix} d_p & -v_p^T \\ v_p & d_p I + v_p \times \end{bmatrix} \begin{bmatrix} d_q \\ v_q \end{bmatrix} = \begin{bmatrix} d_p d_q - v_p^T v_q \\ d_q v_p + d_p v_q + v_p \times v_q \end{bmatrix} \tag{15}$$

We note the following:

1. There is some abuse of notation in our study of quaternions, and to keep algebraic manipulation of quaternions p, q etc. simple, p and q can be viewed as elements of the "algebra" of quaternions $\mathbb{H}$, or, alternatively, can be viewed as the coordinate representations of quaternions in $I\!R^4$. In keeping with this convention, we define the scalar product of quaternions p and $q \in I\!R^4$ as the scalar product of their coordinate representations,

$$\begin{aligned} p \cdot q &= p^T q = d_p d_q + a_p a_q + b_p b_q + c_p c_q \\ \|p\| &= \sqrt{p^T p} \end{aligned} \tag{16}$$

2. The matrix $(1\circ)$ is indeed the identity matrix, and $(i\circ)^2 = (j\circ)^2 = (k\circ)^2 = (i\circ)(j\circ)(k\circ) = -I_{4\times4}$.

3. The matrix product

$$(p\circ)(p\circ)^T = (p\circ)^T(p\circ) = (a^2 + b^2 + c^2 + d^2)I_{4\times4} \ . \tag{17}$$

Therefore

$$(p\circ)^{-1} = \frac{1}{a^2 + b^2 + c^2 + d^2}(p\circ)^T \text{ and } |\det(p\circ)| = \pm\sqrt{a^2 + b^2 + c^2 + d^2} \ . \tag{18}$$

So $\frac{1}{\sqrt{a^2+b^2+c^2+d^2}}(p\circ)$ is an orthogonal matrix (all columns and rows are orthogonal to each other and of unit length).
Thus the matrix $\frac{1}{\sqrt{a^2+b^2+c^2+d^2}}(p\circ)$ is a *rotation* $(p\circ) : I\!R^4 \Rightarrow I\!R^4$; hence it does not change the dot products between vectors or the length of vectors. Therefore

$$\|p \circ x\| = \|p\|(\frac{1}{\|p\|}p) \circ x\| = \|p\|\|x\| \tag{19}$$

4. From (15), we recognize a geometric interpretation of the quaternions: with $\mathbf{p}, \mathbf{q}$ as geometric entities, we can write a quaternion as having a *scalar* part $d \in I\!R$ and a *3D vector part* $\underline{v} \in V$ as follows:

$$\mathbf{p} = d_p + \underline{v}_p \ \ ; \ \ \mathbf{q} = d_q + \underline{v}_q \ \ ; \ \ \mathbf{p} + \mathbf{q} = d_p + d_q + \underline{v}_p + \underline{v}_q \tag{20}$$

$$\mathbf{p} \circ \mathbf{q} = (d_p d_q - \underline{v}_p^T \underline{v}_q) + (d_q \underline{v}_p + d_p \underline{v}_q + \underline{v}_p \times \underline{v}_q) \tag{21}$$

Thus p and q can be viewed as coordinates in $I\!R^4$ or in $\mathbb{H}$ of geometric quaternions in $(I\!R, V)$.

5. The sum and the products of quaternions are independent of the coordinate systems used.

6. The *complex conjugate* of a quaternion $p \in \mathbb{H}$, $p = d + ai + bj + ck$ is $p^* = d - ai - bj - ck$. Viewed as a vector the complex conjugate of $\mathbf{p} = d_p + \underline{v}_p$ is $\mathbf{p}^* = d_p - \underline{v}_p$. Note that $(p\circ)p^* = d^2 + a^2 + b^2 + c^2 + \underline{0}$ is the product of two quaternion and it is therefore a quaternion but it has zero vector part and scalar part equal to $p \cdot p = p^T p$.
   The conjugate of the product requires a change in order:
$$(\mathbf{p} \circ \mathbf{q})^* = \mathbf{q}^* \circ \mathbf{p}^* \tag{22}$$

7. If the quaternions $\mathbf{p}$ and $\mathbf{q}$ have zero scalar parts, then $\mathbf{p}\circ\mathbf{q} = 0 - \underline{v}_p^T \underline{v}_q + \underline{v}_p \times \underline{v}_q$. Furthermore, if the vector parts are orthogonal, then the quaternion product reduces to the vector product
   $\mathbf{p} \circ \mathbf{q} = 0 - \underline{v}_p \times \underline{v}_q$ if $\underline{v}_p^T \underline{v}_q = 0$.

8. In general, $\mathbf{p} \circ \mathbf{q} \neq \mathbf{q} \circ \mathbf{q}$. However, when $\underline{v}_p \parallel \underline{v}_q$ then $\mathbf{p} \circ \mathbf{q} = \mathbf{q} \circ \mathbf{p}$.

9. The scalar product and vector product can be derived from the quaternion products:
$$\begin{aligned} \mathbf{p} \circ \mathbf{q} + \mathbf{q} \circ \mathbf{p} &= 2\, \underline{v}_p^T \underline{v}_q + \underline{0} && (23) \\ \mathbf{p} \circ \mathbf{q} - \mathbf{q} \circ \mathbf{p} &= 0 + 2\, \underline{v}_p \times \underline{v}_q && (24) \end{aligned}$$

10. Since $\mathbf{p} \circ \mathbf{p}^* = \mathbf{p}^* \circ \mathbf{p} = \|\mathbf{p}\|^2 + \underline{0}$, the inverse $\mathbf{p}^{-1}$of $\mathbf{p}$ is given by $\mathbf{p}^{-1} = \frac{1}{\|\mathbf{p}\|^2}\mathbf{p}^*$, and for a unit quaternion $\hat{\mathbf{p}}$, its inverse is given by $\hat{\mathbf{p}}^{-1} = \hat{\mathbf{p}}^*$.

11. For any *unit vector* quaternion we have that
$$\begin{aligned} (ai + bj + ck)(ai + bj + ck) &= a^2 i^2 + b^2 j^2 + c^2 k^2 + \\ &\quad\ abij + baji + bcjk + cbkj + acik + caki \\ &= -(a^2 + b^2 + c^2) = -1 && (25) \end{aligned}$$
   In particular, if we let
$$\begin{bmatrix} i' \\ j' \\ k' \end{bmatrix} = Q \begin{bmatrix} i \\ j \\ k \end{bmatrix} \tag{26}$$
   where the matrix Q is a rotation $Q^T Q = QQ^T = I$ and $\det(Q) = 1$, so Q represents a coordinate change, then
$${i'}^2 = {j'}^2 = {k'}^2 = i'j'k' = -1 \ . \tag{27}$$
   and
$$i'j' = -j'i' = k' \ ; \ j'k' = -k'j' = i' \ ; \ k'i' = -i'k' = j' \ . \tag{28}$$

As well, with $\hat{\mathbf{s}} = 0 + \underline{\hat{s}}$, for any unit vector $\underline{\hat{s}}$:

$$\hat{\mathbf{s}} \circ \hat{\mathbf{s}} = (-\underline{\hat{s}}^T \underline{\hat{s}} + \underline{\hat{s}} \times \underline{\hat{s}}) = (-1 + \underline{0}) \ . \tag{29}$$

Finally, we note that for the *unit vector* quaternion with $\hat{\mathbf{s}} = 0 + \underline{\hat{s}}$ with coordinates $\hat{s}$, the matrix $(\hat{s}\circ)$ satisfies the relationship

$$(\begin{bmatrix} 0 \\ \hat{s} \end{bmatrix} \circ)^2 = \begin{bmatrix} 0 & -\hat{s}^T \\ \hat{s} & \hat{s}\times \end{bmatrix}^2 = -I_{4\times 4} \tag{30}$$

12. **Quaternions and exponentials**

With $\mathbf{1} = 1 + \underline{0}$, and $\hat{\mathbf{s}}^k = \hat{\mathbf{s}} \circ ...\hat{\mathbf{s}}$, k-times, we have

$$\begin{aligned} e^{\theta\hat{\mathbf{s}}} &= \mathbf{1} + \theta\hat{\mathbf{s}} + \frac{1}{2!}\theta^2(\hat{\mathbf{s}} \circ \hat{\mathbf{s}}) + \frac{1}{3!}\theta^3\hat{\mathbf{s}}^3 + \frac{1}{4!}\theta^4\hat{\mathbf{s}}^4... \\ &= \mathbf{1} + \theta\hat{\mathbf{s}} - \frac{1}{2!}\theta^2(1 + \underline{0}) - \frac{1}{3!}\theta^3\hat{\mathbf{s}} + \frac{1}{4!}\theta^4(1 + \underline{0})... \\ &= (1 - \frac{1}{2!}\theta^2 + \frac{1}{4!}\theta^4 - ...)\mathbf{1} + (\theta - \frac{1}{3!}\theta^3 + ...)\hat{\mathbf{s}} \\ &= \cos\theta\mathbf{1} + \sin\theta\hat{\mathbf{s}} \end{aligned} \tag{31}$$

Thus, if $\hat{\mathbf{s}}$ is a *unit vector* quaternion, for any $\mathbf{p}$,

$$e^{\theta\hat{\mathbf{s}}} \circ \mathbf{p} \quad = \quad \cos\theta\mathbf{p} + \sin\theta(\hat{\mathbf{s}} \circ \mathbf{p}) \tag{32}$$

We can also use the matrix representation of the quaternion product to compute the exponential mapping $e^{\theta(\hat{s}\circ)} : I\!R^4 \to I\!R^4$:

$$\begin{aligned} e^{\theta(\hat{s}\circ)} &= I + \theta(\hat{s}\circ) + \frac{1}{2!}\theta^2(\hat{s}\circ)^2 + \frac{1}{3!}(\hat{s}\circ)^3 + \frac{1}{4!}(\hat{s}\circ)^4... \\ &= I + \theta(\hat{s}\circ) - \frac{1}{2!}\theta^2 I - \frac{1}{3!}\theta^3(\hat{s}\circ) + \frac{1}{4!}\theta^4 I... \\ &= (1 - \frac{1}{2!}\theta^2 + \frac{1}{4!}\theta^4 - ...)I + (\theta - \frac{1}{3!}\theta^3 + ...)(\hat{s}\circ) \\ &= \cos\theta I + \sin\theta(\hat{s}\circ) \end{aligned} \tag{33}$$

or

$$e^{\theta(\hat{s}\circ)} = \exp\{\theta \begin{bmatrix} 0 & -\hat{s}^T \\ \hat{s} & \hat{s}\times \end{bmatrix}\} = \cos\theta I + \sin\theta \begin{bmatrix} 0 & -\hat{s}^T \\ \hat{s} & \hat{s}\times \end{bmatrix} \tag{34}$$

This is simply a coordinate representation of equation (32), leading to

$$e^{\theta(\hat{s}\circ)}p \quad = \quad \cos\theta p + \sin\theta(\hat{s}\circ)p \tag{35}$$

This coordinate representation is easier to use to perform computations; however, unlike equation (32), it cannot be easily used with right multiplication.

2.1 Quaternion exponentials and the vector rotation formula

Consider again the geometry of 3D rotation Figure 1, repeated here for convenience:

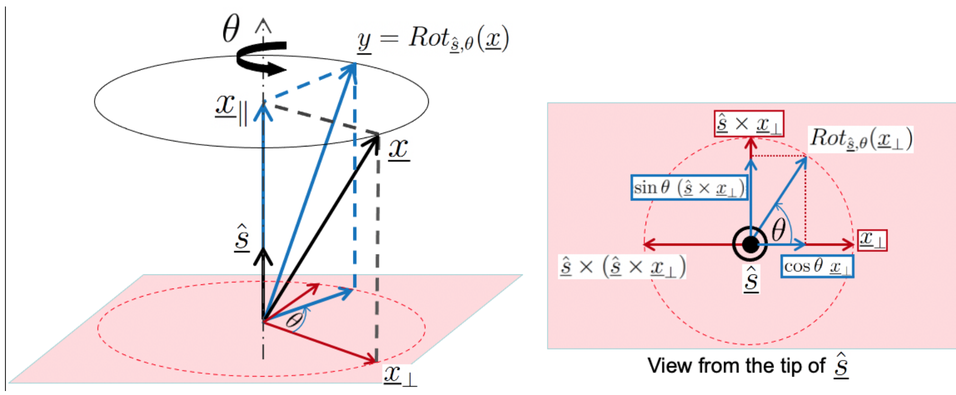

Figure 1: Geometry of 3D rotation: $\underline{x}$ is rotated about the unit axis $\underline{\hat{s}}$ into vector $\underline{y}$. The vector $\underline{x}$ is projected onto the rotation axis ($\underline{x}_{\|}$) and onto the plane perpendicular to the axis ($\underline{x}_{\perp}$). Only $\underline{x}_{\perp}$ is rotated as shown on the right.

Let $\underline{\hat{s}}$ be a unit axis of rotation. Write $\underline{x}$ in terms of its projection $\underline{x}_{\|}$ on $\underline{s}$ and its projection $\underline{x}_{\perp}$ on the plane orthogonal to $\underline{\hat{s}}$:

$$\underline{x} = \underline{x}_{\|} + \underline{x}_{\perp} \ ; \quad \underline{x}_{\|} = (\underline{\hat{s}}^T \underline{x})\underline{s} \ ; \quad \underline{x}_{\perp} = \underline{x} - (\underline{\hat{s}}^T \underline{x})\underline{s} \tag{36}$$

Then,

$$\begin{aligned} e^{\theta \hat{\mathbf{s}}} \circ (0 + \underline{x}_{\perp}) &= \cos\theta(1 + \underline{0}) \circ (0 + \underline{x}_{\perp}) + \sin\theta(0 + \underline{\hat{s}}) \circ (0 + \underline{x}_{\perp}) \\ &= -\sin\theta \underline{\hat{s}}^T \underline{x}_{\perp} + \cos\theta \underline{x}_{\perp} + \sin\theta(\underline{\hat{s}} \times \underline{x}_{\perp}) \\ &= \cos\theta \underline{x}_{\perp} + \sin\theta(\underline{\hat{s}} \times \underline{x}_{\perp}) \end{aligned} \tag{37}$$

which we recognize to be the *vector rotation formula.*

We can also write this in terms of the original "complex numbers" form of the quaternions. If $\underline{\hat{s}} = a_s i + b_s j + c_s k$, with $\sqrt{a_s^2 + b_s^2 + c_s^2} = 1$, $\underline{x}_{\perp} = a_{x\perp} i + b_{x\perp} j + c_{x\perp} k$, $\underline{x}_{\|} = a_{x\|} i + b_{x\|} j + c_{x\|} k$, we can write the rotation $\underline{y} = Rot_{\underline{\hat{s}},\theta}(\underline{x})$ in terms of imaginary numbers

$$\begin{aligned} \underline{y} &= Rot_{\underline{\hat{s}},\theta}(\underline{x}) = Rot_{\underline{\hat{s}},\theta}(\underline{x}_{\|} + \underline{x}_{\perp}) = \underline{x}_{\|} + Rot_{\underline{\hat{s}},\theta}\underline{x}_{\perp} && (38) \\ &= x_{\|} + e^{\theta(a_s i + b_s j + c_s k)} x_{\perp} && (39) \\ &= (a_{x\|} i + b_{x\|} j + c_{x\|}) + e^{\theta(a_s i + b_s j + c_s k)}(a_{x\perp} i + b_{x\perp} j + c_{x\perp} k) && (40) \\ & && (41) \end{aligned}$$

which is similar to complex multiplication $e^{i\theta}z$ of a complex number z by $e^{i\theta}$ in order to rotate it by an angle θ.

Note that in the case of complex number product, $e^{i\theta}z = ze^{i\theta}$, so z is rotated by an angle θ regardless of whether it is multiplied from the left or from the right. For quaternion exponential multiplication, if we change the order of multiplication, then

$$\begin{aligned}(0+\underline{x}_\perp)\circ e^{\theta\hat{\mathbf{s}}} &= \cos\theta(1+\underline{0})\circ(0+\underline{x}_\perp)+\sin\theta\circ(0+\underline{x}_\perp)\circ(0+\underline{\hat{s}}) \\ &= \cos\theta\underline{x}_\perp-\sin\theta(\underline{\hat{s}}\times\underline{x}_\perp)=(0+\underline{x}_\perp)\circ e^{-\theta\hat{\mathbf{s}}}\end{aligned} \tag{42}$$

3 Quaternion Rotation Theorem

The quaternion rotation discussed in the previous section requires that a vector be decomposed along the axis onto the plane perpendicular to it. Fixing that requires for the transformation to first decompose the vector into parallel (to axis) and perpendicular directions, and re-assembling it after rotation. This leads exactly to the rotation matrix described in the mathematical preliminaries.

3.1 Rotation theorem from algebraic definition

Consider a non-zero quaternion $\mathbf{q} = d + \underline{s}$ with angle θ. Then, the function

$$f(\underline{x}) = \mathbf{q} \circ (0 + \underline{x}) \circ \mathbf{q}^{-1} = 0 + Rot_{\underline{s},2\theta}(\underline{x}) \tag{43}$$

is a rotation about $\underline{s}$ of angle 2θ.

Proof:
Without loss of generality, we will consider the case where the quaternion vector par $\underline{s}$ is equal to the third basis vector $\underline{k}$ of the vector space V. We can always change coordinates in V to perform this alignment. We will use the algebraic short hand as introduced by Hamilton, and examine the product

$$(d + ck)x(d + ck)^{-1} = f(x) \tag{44}$$

With $x = a_x i + b_x j + c_x k$, we note that since $f : I\!R^3 \to I\!R^3$ is a linear function, it is fully characterized by its matrix representation. So we only need to find $f(i)$, $f(j)$ and $f(k)$, then form $f(x) = a_x f(i) + b_x f(j) + c_x f(k)$. Starting with k, we obtain:

$$\begin{aligned} f(k) = (d + ck)\, k\, (d - ck) &= (d + ck)k\frac{(d + ck)^*}{d^2 + c^2} \\ &= \frac{(dk - c)(d - ck)}{d^2 + c^2} \\ &= \frac{d^2k - cd - dck^2 + c^2k}{d^2 + c^2} = k \end{aligned} \tag{45}$$

We next find $f(i)$, given by

$$\begin{aligned} f(i) &= (d + ck)i\frac{(d + ck)^*}{d^2 + c^2} \\ &= \frac{(di + cj)(d - ck)}{d^2 + c^2} \\ &= \frac{d^2i + cdj + dcj - c^2i}{d^2 + c^2} \\ &= \frac{d^2 - c^2}{d^2 + c^2}i + \frac{2dc}{d^2 + c^2}j \end{aligned} \tag{46}$$

Finally, we find $f(j)$, given by

$$\begin{aligned}
f(j) &= (d+ck)j\frac{(d+ck)^*}{d^2+c^2} \\
&= \frac{(dj-ci)(d-ck)}{d^2+c^2} \\
&= \frac{d^2j-dci-cdi-c^2j}{d^2+c^2} \\
&= -\frac{2dc}{d^2+c^2}i+\frac{d^2-c^2}{d^2+c^2}j
\end{aligned} \tag{47}$$

Therefore the matrix representation of f is written column-wise as the images of $f(i)$, $f(j)$ and $f(k)$ as follows:

$$A_f = \begin{bmatrix} \frac{d^2-c^2}{d^2+c^2} & -\frac{2dc}{d^2+c^2} & 0 \\ \frac{2dc}{d^2+c^2} & \frac{d^2-c^2}{d^2+c^2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{48}$$

With $\cos\theta = \frac{d}{\sqrt{d^2+c^2}} = \frac{d}{\sqrt{d^2+\|\underline{s}\|^2}}$ and $\sin\theta = \frac{c}{\sqrt{d^2+c^2}} = \frac{\|s\|}{\sqrt{d^2+\|\underline{s}\|^2}}$ we recognize this as a rotation matrix about the k-axis of angle 2θ:

$$A_f = \begin{bmatrix} \cos 2\theta & -\sin 2\theta & 0 \\ \sin 2\theta & \cos 2\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad . \tag{49}$$

Note:

We chose $\cos\theta$ and $\sin\theta$ to have the signs of d and c, respectively. We could have just as well chosen both to be negative. So if $\mathbf{q}$ is a *unit quaternion*, $+\mathbf{q}$ and $-\mathbf{q}$ *represent the same rotation.*

An immediate consequence of this theorem is that sequence of rotations is described by a product of quaternions. Indeed,

$$\begin{aligned}
\mathbf{p}\circ\underbrace{(\mathbf{q}\circ(0+\mathbf{v})\circ\mathbf{q}^{-1})}_{\text{rotation by }\mathbf{q}}\circ\mathbf{p}^{-1} &= \underbrace{(\mathbf{p}\circ\mathbf{q})\circ(0+\mathbf{v})\circ(\mathbf{q}^{-1}\circ\mathbf{p}^{-1})}_{\text{by associativity}} \\
&= \underbrace{(\mathbf{p}\circ\mathbf{q})\circ(0+\mathbf{v})\circ(\mathbf{p}\circ\mathbf{q})^{-1}}_{\text{rotation by }\mathbf{q}\text{, then by }\mathbf{p}}
\end{aligned} \tag{50}$$

Therefore an effective way to keep track of orientation, while explicitly keeping track of the rotation axis and angle is to parametrize rotation by quaternions.

3.2 Quaternion Rotation Theorem - vector geometric version

Let $\mathbf{q} = \cos\theta \mathbf{1} + \sin\theta \hat{\mathbf{s}}$ be the a unit quaternion with axis $\underline{\hat{s}}$ and angle θ. Then, for any vector $\underline{x}$,

$$\mathbf{q} \circ (0 + \underline{x}) \circ \mathbf{q}^{-1} = 0 + Rot_{\underline{\hat{s}},2\theta}(\underline{x}) = \tag{51}$$

meaning that the product $\mathbf{q} \circ (0 + \underline{x}) \circ \mathbf{q}^{-1}$ rotates $\underline{x}$ by an angle 2θ.

Proof:

$$\begin{aligned} Rot_{\hat{\mathbf{s}},2\theta}(\underline{x}) &= Rot_{\hat{\mathbf{s}},2\theta}(\underline{x}_{\|} + \underline{x}_{\perp}) & (52)\\ &= \underline{x}_{\|} + Rot_{\hat{\mathbf{s}},2\theta}(\underline{x}_{\perp}) = \underline{x}_{\|} + \cos 2\theta \underline{x}_{\perp} + \sin 2\theta(\underline{\hat{s}} \times \underline{x}_{\perp}) & (53) \end{aligned}$$

or, in quaternion form, as

$$\begin{aligned} 0 + Rot_{\hat{\mathbf{s}},2\theta}(\underline{x}) &= (0 + \underline{x}_{\|}) + \cos 2\theta(0 + \underline{x}_{\perp}) + \sin 2\theta(0 + \underline{\hat{s}} \times \underline{x}_{\perp}) & (54)\\ &= (0 + \underline{x}_{\|}) + e^{2\theta\hat{s}} \circ (0 + \underline{x}_{\perp}) & (55)\\ &= e^{\theta\hat{s}} e^{-\theta\hat{s}} \circ (0 + \underline{x}_{\|}) + e^{\theta\hat{s}} \circ e^{\theta\hat{s}} \circ (0 + \underline{x}_{\perp}) & (56)\\ &= e^{\theta\hat{s}} \circ (0 + \underline{x}_{\|}) \circ e^{-\theta\hat{s}} + e^{\theta\hat{s}} \circ (0 + \underline{x}_{\perp}) \circ e^{-\theta\hat{s}} & (57)\\ &= e^{\theta\hat{s}} \circ (0 + \underline{x}_{\|} + \underline{x}_{\perp}) \circ e^{-\theta\hat{s}} & (58)\\ &= e^{\theta\hat{s}} \circ (0 + \underline{x}) \circ e^{-\theta\hat{s}} & (59) \end{aligned}$$

where we could change the order of $e^{-\theta\hat{s}}$ and $\underline{x}_{\|}$ because $\underline{x}_{\|}$ is aligned with $\underline{\hat{s}}$, while $(0 + \underline{x}_{\perp}) \circ e^{\theta\hat{s}} = (0 + \underline{x}_{\perp}) \circ e^{-\theta\hat{s}}$ uses the complex conjugate when changing the order according to (42).

3.3 Quaternion Rotation Theorem - matrix representation version

Let $q = \begin{bmatrix} \cos\theta \\ \sin\theta\hat{s} \end{bmatrix}$ be the a unit quaternion with axis $\hat{s}$ and angle θ. Then, for any vector x,

$$q \circ \begin{bmatrix} 0 \\ x \end{bmatrix} \circ q^{-1} = \begin{bmatrix} 0 \\ e^{2\theta\hat{s}\times} x \end{bmatrix} \tag{60}$$

meaning that the product $q \circ \begin{bmatrix} 0 \\ x \end{bmatrix} \circ q^{-1} =$ rotates x by an angle 2θ.

Proof:

We write the matrix representation of the product in (65) as follows:

$$
\begin{aligned}
q \circ \begin{bmatrix} 0 \\ x \end{bmatrix} \circ q^{-1} &= \left(\begin{bmatrix} \cos\theta \\ \sin\theta\hat{s} \end{bmatrix} \circ\right)\left(\begin{bmatrix} 0 \\ x \end{bmatrix} \circ\right) \begin{bmatrix} \cos\theta \\ -\sin\theta\hat{s} \end{bmatrix} & (61)\\
&= \left(\begin{bmatrix} \cos\theta \\ \sin\theta\hat{s} \end{bmatrix} \circ\right) \begin{bmatrix} \sin\theta x^T\hat{s} \\ x\cos\theta + \sin\theta(\hat{s}\times x) \end{bmatrix} & (62)\\
&= \begin{bmatrix} \cos\theta & -\sin\theta\hat{s}^T \\ \sin\theta\hat{s} & \cos\theta I + \sin\theta\hat{s}\times \end{bmatrix} \begin{bmatrix} \sin\theta x^T\hat{s} \\ x\cos\theta + \sin\theta(\hat{s}\times x) \end{bmatrix} & (63)\\
&= \begin{bmatrix} \cos\theta\sin\theta x^T\hat{s} - \sin\theta\hat{s}^T x\cos\theta - \sin^2\theta\hat{s}^T(\hat{s}\times x) \\ \sin^2\theta\hat{s}x^T\hat{s} + \cos^2\theta x + 2\sin\theta\cos\theta\hat{s}\times x + \sin^2\theta\hat{s}\times(\hat{s}\times x) \end{bmatrix}
\end{aligned}
$$

and therefore,

$$
q \circ \begin{bmatrix} 0 \\ x \end{bmatrix} \circ q^{-1} = \begin{bmatrix} 0 \\ \sin^2\theta\hat{s}x^T\hat{s} + \cos^2\theta x + 2\sin\theta\cos\theta\hat{s}\times x + \sin^2\theta\hat{s}\times(\hat{s}\times x) \end{bmatrix} \tag{64}
$$

But $\hat{s}x^T\hat{s} = \hat{s}\hat{s}^T x = ((\hat{s}\times)^2 + \hat{s}^T\hat{s})x = (\hat{s}\times)^2 x + x$, so

$$
\begin{aligned}
q \circ \begin{bmatrix} 0 \\ x \end{bmatrix} \circ q^{-1} &= \begin{bmatrix} 0 \\ \sin^2\theta x + \cos^2\theta x + 2\sin\theta\cos\theta\hat{s}\times x + 2\sin^2\theta\hat{s}\times(\hat{s}\times x) \end{bmatrix} \\
q \circ \begin{bmatrix} 0 \\ x \end{bmatrix} \circ q^{-1} &= \begin{bmatrix} 0 \\ x + \sin 2\theta\,\hat{s}\times x + (1-\cos 2\theta)\,(\hat{s}\times(\hat{s}\times x)) \end{bmatrix} = \begin{bmatrix} 0 \\ e^{2\theta\hat{s}\times}x \end{bmatrix} & (65)
\end{aligned}
$$

We recognize the last entry of (65) to be the rotation about axis $\hat{s}$ of angle 2θ. In terms of matrix transformations, we note that

$$
\begin{bmatrix} \sin\theta x^T\hat{s} \\ x\cos\theta + \sin\theta(\hat{s}\times x) \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta\hat{s}^T \\ -\sin\theta\hat{s} & \cos\theta I + \sin\theta\hat{s}\times \end{bmatrix} \begin{bmatrix} 0 \\ x \end{bmatrix} \tag{66}
$$

and therefore, combining this with (64), we have

$$
\underbrace{\begin{bmatrix} \cos\theta & -\sin\theta\hat{s}^T \\ \sin\theta\hat{s} & \cos\theta I + \sin\theta\hat{s}\times \end{bmatrix}}_{\text{Left rotation } Q_l} \underbrace{\begin{bmatrix} \cos\theta & \sin\theta\hat{s}^T \\ -\sin\theta\hat{s} & \cos\theta I + \sin\theta\hat{s}\times \end{bmatrix}}_{\text{Right rotation } Q_r} = \begin{bmatrix} 1 & 0^T \\ 0 & e^{2\theta\hat{s}\times} \end{bmatrix} \tag{67}
$$

We can re-write these in terms of the quaternion parameters $q = [d_q\, v_q^T]^T$, as:

$$
\underbrace{\begin{bmatrix} d_q & -v_q^T \\ v_q & dI + v_q\times \end{bmatrix}}_{\text{Left rotation } Q_l} \underbrace{\begin{bmatrix} d_q & v_q^T \\ -v_q & dI + v_q\times \end{bmatrix}}_{\text{Right rotation } Q_r} = \begin{bmatrix} 1 & 0^T \\ 0 & e^{2\theta\hat{s}\times} \end{bmatrix} \tag{68}
$$

Both matrices Q_l and Q_r in the above are *rotations* in $I\!R^4$. Q_l is the same as the matrix representation (14).

For a geometric interpretation of these two rotations, consider the case were $\hat{s}$ is aligned with the k-axis of the vector space V, and examine what happens to 1, i, j and k:

$$Q_r \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta k^T \\ -\sin\theta k & \cos\theta I + \sin\theta k\times \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \cos\theta \\ -\sin\theta k \end{bmatrix} \tag{69}$$

$$Q_r \begin{bmatrix} 0 \\ k \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta k^T \\ -\sin\theta k & \cos\theta I + \sin\theta k\times \end{bmatrix} \begin{bmatrix} 0 \\ k \end{bmatrix} = \begin{bmatrix} \sin\theta \\ \cos\theta k \end{bmatrix} \tag{70}$$

$$Q_r \begin{bmatrix} 0 \\ i \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta k^T \\ -\sin\theta k & \cos\theta I + \sin\theta k\times \end{bmatrix} \begin{bmatrix} 0 \\ i \end{bmatrix} = \begin{bmatrix} 0 \\ \cos\theta i + \sin\theta j \end{bmatrix} \tag{71}$$

$$Q_r \begin{bmatrix} 0 \\ j \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta k^T \\ -\sin\theta k & \cos\theta I + \sin\theta k\times \end{bmatrix} \begin{bmatrix} 0 \\ j \end{bmatrix} = \begin{bmatrix} 0 \\ \cos\theta j - \sin\theta i \end{bmatrix} \tag{72}$$

$$Q_l \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta k^T \\ \sin\theta k & \cos\theta I + \sin\theta k\times \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \cos\theta \\ \sin\theta k \end{bmatrix} \tag{73}$$

$$Q_l \begin{bmatrix} 0 \\ k \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta k^T \\ \sin\theta k & \cos\theta I + \sin\theta k\times \end{bmatrix} \begin{bmatrix} 0 \\ k \end{bmatrix} = \begin{bmatrix} -\sin\theta \\ \cos\theta k \end{bmatrix} \tag{74}$$

$$Q_l \begin{bmatrix} 0 \\ i \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta k^T \\ \sin\theta k & \cos\theta I + \sin\theta k\times \end{bmatrix} \begin{bmatrix} 0 \\ i \end{bmatrix} = \begin{bmatrix} 0 \\ \cos\theta i + \sin\theta j \end{bmatrix} \tag{75}$$

$$Q_l \begin{bmatrix} 0 \\ j \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta k^T \\ \sin\theta k & \cos\theta I + \sin\theta k\times \end{bmatrix} \begin{bmatrix} 0 \\ j \end{bmatrix} = \begin{bmatrix} 0 \\ \cos\theta j - \sin\theta i \end{bmatrix} \tag{76}$$

Thus, as seen in the Figure 2, Q_r's action is to rotate vectors in the plane spanned by i and j by an angle θ, and to rotate vectors in the plane spanned by 1 and k by an angle $-\theta$, while Q_l's action is to rotate vectors in the plane spanned by i and j by an angle θ, and to rotate vectors in the plane spanned by 1 and k by an angle θ. In $I\!R^4$, these two planes are invariant under the rotations Q_r and Q_l. When Q_l follows Q_r, vectors in the i -j plane are rotated twice for a rotation angle about k of 2θ. However, vectors in the 1-k plane are rotated by an angle theta but in opposite directions by Q_r and Q_l. Another view is shown in Figure 3. In order to simplify the figures, note that by x, $x_\perp$, $x_\parallel$ we do mean the 4-dimensional vector $0 + x$, $0 + x_\perp$ and $0 + x_\parallel$ in $I\!R^4$.

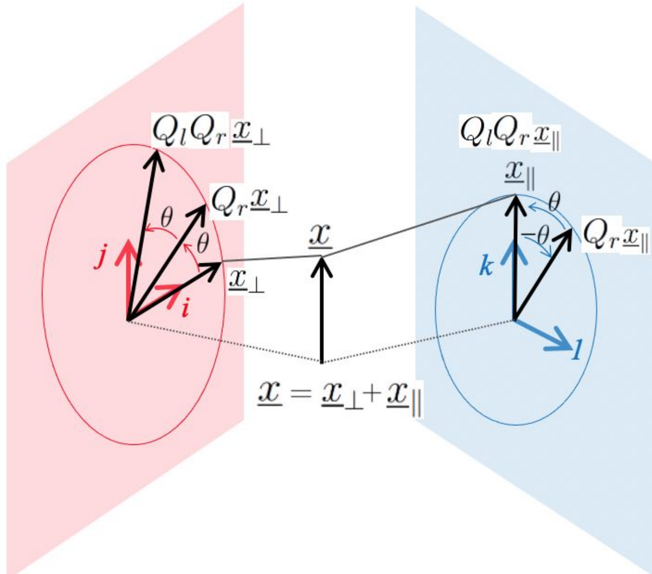

Figure 2: Geometry of quaternion 3D rotation. The first rotation Q_r moves $\underline{x}_\parallel$ off of the rotation axis in the $1-k$-plane by $-\theta$, while Q_l rotates it by $+\theta$. Both Q_l and Q_r rotate the vector by an angle θ in the $i-j$-plane.

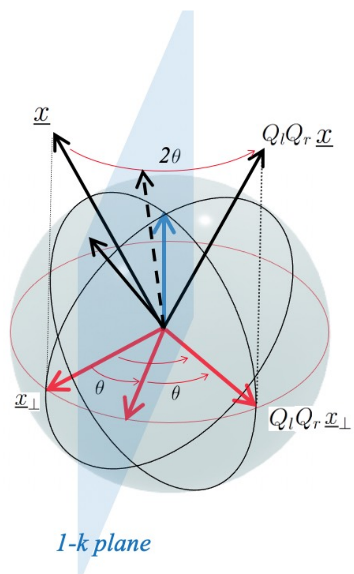

Figure 3: Geometry of quaternion 3D rotation. $Q_l Q_r$ rotates $\underline{x}$ by an angle 2θ about the axis of rotation. Q_r and Q_l also rotate it in the $1-k$-plane, first by $-\theta$, then by $+\theta$.

3.4 Quaternions and angular velocity

Let $\underline{x}_0 \in I\!R^3$ be any fixed vector, and let $\underline{x}$ be its rotated version according to (65). Then the time derivative:

$$\begin{aligned} 0+\underline{\dot{x}} &= \frac{d}{dt}(\mathbf{q} \circ (0+\underline{x}_0) \circ \mathbf{q}^{-1}) \\ &= \dot{\mathbf{q}} \circ (0+\underline{x}_0) \circ \mathbf{q}^{-1} + \mathbf{q} \circ (0+\underline{x}_0) \circ \frac{d}{dt}(\mathbf{q}^{-1}) \end{aligned} \tag{77}$$

But $\mathbf{q} \circ \mathbf{q}^{-1} = 1 + \underline{0}$ implies that $\dot{\mathbf{q}} \circ \mathbf{q}^{-1} + \mathbf{q} \circ \frac{d}{dt}(\mathbf{q}^{-1}) = 0 + \underline{0}$, from which we have that $\frac{d}{dt}(\mathbf{q}^{-1}) = -\mathbf{q}^{-1} \circ \dot{\mathbf{q}} \circ \mathbf{q}^{-1}$, so:

$$\begin{aligned} 0+\underline{\dot{x}} &= \dot{\mathbf{q}} \circ (0+\underline{x}_0) \circ \mathbf{q}^{-1} - \mathbf{q} \circ (0+\underline{x}_0) \circ \mathbf{q}^{-1} \circ \dot{\mathbf{q}} \circ \mathbf{q}^{-1} \\ &= (\dot{\mathbf{q}} \circ \mathbf{q}^{-1}) \circ (\mathbf{q}(0+\underline{x}_0) \circ \mathbf{q}^{-1}) - (\mathbf{q} \circ (0+\underline{x}_0) \circ \mathbf{q}^{-1}) \circ (\dot{\mathbf{q}} \circ \mathbf{q}^{-1}) \\ &= 0 + 2\underline{v}_{\dot{\mathbf{q}} \circ \mathbf{q}^{-1}} \times \underline{x}_0 \end{aligned} \tag{78}$$

where we used $\underline{v}_{\dot{\mathbf{q}} \circ \mathbf{q}^{-1}}$ to denote the vector part of $\dot{\mathbf{q}} \circ \mathbf{q}^{-1}$, and the last equality follows from equation (24). Thus

$$\underline{\dot{x}} = 2\underline{v}_{\dot{\mathbf{q}} \circ \mathbf{q}^{-1}} \times \underline{x}_0 \tag{79}$$

and therefore, by definition of angular velocity, we have that

$$\underline{\omega} = 2\, \underline{v}_{\dot{\mathbf{q}} \circ \mathbf{q}^{-1}} \tag{80}$$

or

$$0 + \underline{\omega} = 2\, \dot{\mathbf{q}} \circ \mathbf{q}^{-1} \tag{81}$$

We can write the last equation in terms of the coordinate representation of the quaternions as follows:

$$\begin{aligned} \begin{bmatrix} 0 \\ \omega \end{bmatrix} = 2\,(\dot{q}\,\circ)\, q^{-1} &= 2 \begin{bmatrix} \dot{d}_q \\ \dot{v}_q \end{bmatrix} \circ \begin{bmatrix} d \\ -v \end{bmatrix} \end{aligned} \tag{82}$$

$$\begin{aligned} &= 2 \begin{bmatrix} \dot{d}_q d_q + \dot{v}_q^T v_q \\ d_q \dot{v}_q - \dot{d}_q v_q - \dot{v}_q \times v_q \end{bmatrix} \\ &= 2 \begin{bmatrix} d_q & v_q^T \\ -v_q & d_q I + v_q \times \end{bmatrix} \begin{bmatrix} \dot{d}_q \\ \dot{v}_q \end{bmatrix} = 2 Q_r \dot{q} \end{aligned} \tag{83}$$

Because Q_r is orthogonal, its inverse is just the transpose, so

$$\dot{q} = \frac{1}{2} \begin{bmatrix} 0 \\ \omega \end{bmatrix} \circ q = \frac{1}{2} \begin{bmatrix} d_q & -v_q^T \\ v_q & d_q I - v_q \times \end{bmatrix} \begin{bmatrix} 0 \\ \omega \end{bmatrix}. \tag{84}$$

Thus to compute angular velocity from the quaternion or the quaternion from the angular velocity we have

$$\omega = [-v_q \quad d_q I - v_q \times]\, \dot{q} \qquad \text{and} \qquad \dot{q} = \frac{1}{2}\begin{bmatrix} -v_q^T \\ d_q I - v_q \times \end{bmatrix} \omega \tag{85}$$

An alternative approach to the above derivation starts with the definition of angular velocity and the explicit expression for the rotation matrix in terms of its quaternion $[d\ v^T]^T = [\cos\frac{\theta}{2}\ \ \hat{s}^T \sin\frac{\theta}{2}]^T$.

$$\begin{aligned}
e^{\theta \hat{s}\times} &= I + \sin\theta(\hat{s}\times) + (1-\cos\theta)(\hat{s}\times)^2 && (86)\\
&= I + 2\cos\frac{\theta}{2}\sin\frac{\theta}{2}(\hat{s}\times) + 2\sin^2\frac{\theta}{2}(\hat{s}\times)^2 && (87)\\
&= I + 2d(v\times) + 2(v\times)^2 && (88)\\
&= \begin{bmatrix} v & dI + v\times \end{bmatrix}\begin{bmatrix} v^T \\ dI + v\times \end{bmatrix} \qquad \text{from equation (67)} && (89)\\
&= Q(d, v) && (90)
\end{aligned}$$

where $q = [d\ a\,b\,c]^T$ with $d = \cos\frac{\theta}{2}$ and $v = [a\,b\,c]^T = \sin\frac{\theta}{2}\hat{s}^T$. We obtain:

$$Q(d,v) = \begin{bmatrix} d^2+a^2-b^2-c^2 & -2dc+2ab & 2db+2ac \\ 2dc+2ab & d^2-a^2+b^2-c^2 & -2da+2bc \\ -2db+2ac & 2da+2bc & d^2-a^2-b^2+c^2 \end{bmatrix} \tag{91}$$

The relationship between angular velocity and quaternion derivatives can be derived from the definition of angular velocity:

$$\omega\times = \begin{bmatrix} 0 & -c_\omega & -b_\omega \\ c_\omega & 0 & a_\omega \\ -b_\omega & a_\omega & 0 \end{bmatrix} = \dot{Q}Q^T = \begin{bmatrix} \dot{q}_{11} & \dot{q}_{12} & \dot{q}_{13} \\ \dot{q}_{21} & \dot{q}_{22} & \dot{q}_{23} \\ \dot{q}_{31} & \dot{q}_{32} & \dot{q}_{33} \end{bmatrix}\begin{bmatrix} q_{11} & q_{21} & q_{31} \\ q_{12} & q_{22} & q_{32} \\ q_{13} & q_{23} & q_{33} \end{bmatrix} \tag{92}$$

So

$$a_\omega = \left(\frac{d}{dt}[-2db+2ac \mid 2da+2bc \mid d^2-a^2-b^2+c^2]\right)\begin{bmatrix} -2dc+2ab \\ d^2-a^2+b^2-c^2 \\ 2da+2bc \end{bmatrix} \tag{93}$$

With a bit of patience and using the fact that $d^2+a^2+b^2+c^2 = 1$ and therefore $d\dot{d} + a\dot{a} + b\dot{b} + c\dot{c} = 0$ we obtain that

$$a_\omega = 2[-a \mid d \mid -c \mid b]\begin{bmatrix} \dot{d} \\ \dot{a} \\ \dot{b} \\ \dot{c} \end{bmatrix} \tag{94}$$

and similarly for b_ω and c_ω for the final relationship

$$\begin{bmatrix} 0 \\ a_\omega \\ b_\omega \\ c_\omega \end{bmatrix} = 2 \begin{bmatrix} d & a & b & c \\ -a & d & -c & b \\ -b & c & d & -a \\ -c & -b & a & d \end{bmatrix} \begin{bmatrix} \dot{d} \\ \dot{a} \\ \dot{b} \\ \dot{c} \end{bmatrix} \tag{95}$$

or

$$\begin{bmatrix} 0 \\ \omega \end{bmatrix} = 2 \left[\begin{array}{cc} d & a \;\; b \;\; c \\ \begin{array}{c} -a \\ -b \\ -c \end{array} & dI + \begin{bmatrix} a \\ b \\ c \end{bmatrix} \times \end{array} \right] \begin{bmatrix} \dot{d} \\ \dot{a} \\ \dot{b} \\ \dot{c} \end{bmatrix} = \begin{bmatrix} d & v^T \\ -v & dI + v\times \end{bmatrix} \dot{q} \tag{96}$$

Quaternion Summary

Quaternion	$\mathbf{q} = d_q + \underline{v}_q$	$\begin{bmatrix} d \\ v \end{bmatrix}$ in frame $\underline{C}$
Quaternion Product	$\mathbf{p} \circ \mathbf{q} = d_p d_q - \underline{v}_p^T \underline{v}_q$ $+ d_q \underline{v}_p + d_p \underline{v}_q + \underline{v}_p \times \underline{v}_q$	$\begin{bmatrix} 0 \\ y \end{bmatrix} = \begin{bmatrix} d_q & -v_q^T \\ v_q & d_q I + v_q \times \end{bmatrix} \begin{bmatrix} 0 \\ x \end{bmatrix}$
Unit quaternion (θ) to rotation (2θ)	$0 + \underline{y} = \mathbf{q} \circ (0 + \underline{x}) \circ \mathbf{q}^{-1}$	$\begin{bmatrix} 0 \\ y \end{bmatrix} = \begin{bmatrix} d_q & -v_q^T \\ v_q & d_q I + v_q \times \end{bmatrix} \begin{bmatrix} d_q & v_q^T \\ -v_q & d_q I + v_q \times \end{bmatrix} \begin{bmatrix} 0 \\ x \end{bmatrix}$
Rotation (θ) to unit quaternion ($\frac{\theta}{2}$)	$\underline{\underline{Q}} = e^{\theta \underline{\hat{s}} \times}$ $\mathbf{q} = \cos\theta + \underline{\hat{s}} \sin\theta$	$q = \begin{bmatrix} d_q \\ v_q \end{bmatrix} = \begin{bmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2} \hat{s} \end{bmatrix}$
Quaternion (θ) to angular velocity	$0 + \underline{\omega} = 2\, \dot{\mathbf{q}} \circ \mathbf{q}^{-1}$	$\omega = [-v_q \quad d_q I - v_q \times]\, \dot{q}$ and $\dot{q} = \frac{1}{2} \begin{bmatrix} -v_q^T \\ d_q I - v_q \times \end{bmatrix} \omega$
Quaternion product $\iff$	$\mathbf{p} \circ \mathbf{q}$ $\underline{\underline{P}}\, \underline{\underline{Q}}$	$\begin{bmatrix} d_{pq} \\ v_{pq} \end{bmatrix} = \begin{bmatrix} d_p & -v_p^T \\ v_p & d_p I + v_p \times \end{bmatrix} \begin{bmatrix} d_q \\ v_q \end{bmatrix}$