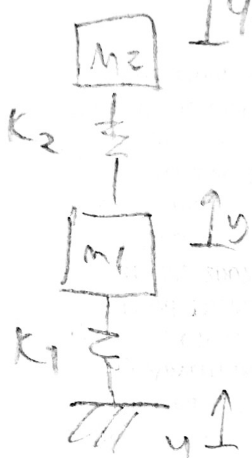
Rathamnoon Prakitpong #53205165

Q1

FBD



$\uparrow y_2$ $\uparrow k_2 y_1$ $\downarrow k_2 y_2$ $\downarrow m_2 \ddot{y}_2$

force balance eq'n

$\Rightarrow 0 = m_2\ddot{y}_2 + k_2 y_2 - k_2 y_1$

$\uparrow y_1$ $\uparrow k_2 y_2$ $\uparrow k_1 u$ $\downarrow k_2 y_1$ $\downarrow k_1 y_1$ $\downarrow m_1 \ddot{y}_1$

$\Rightarrow 0 = m_1\ddot{y}_1 + k_1 y_1 + k_2 y_1 - k_2 y_2 - k_1 u$

State space

$$[x] = \begin{bmatrix} y_1 \\ y_2 \\ \dot{y}_1 \\ \dot{y}_2 \end{bmatrix} \qquad [\dot{x}] = \begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \\ \ddot{y}_1 \\ \ddot{y}_2 \end{bmatrix} \qquad [u] = [u] \qquad [y] = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$[\dot{x}] = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ k_2 & -k_2 & 0 & 0 \\ -(k_2+k_1) & k_2 & 0 & 0 \end{bmatrix}[x] + \begin{bmatrix} 0 \\ 0 \\ 0 \\ k_1 \end{bmatrix}[u]$$

$$[y] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}[x]$$

Q2 $f = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix}$ $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ $u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ $x_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $u_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$$\frac{\partial f}{\partial x} = \begin{bmatrix} -2x_2^2 & -2x_1(2x_2) + u_1 \\ (-u_1)(-e^{-x_1u_1})x_2 + u_2 & -e^{-x_1u_1} \end{bmatrix}$$ $y = [y]$

$$\left.\frac{\partial f}{\partial x}\right|_{\substack{x=x_0 \\ u=u_0}} = \begin{bmatrix} -2(0)^2 & -2(1)2(0) + 1 \\ -1(e^{-1})0 + 0 & -e^{-1(1)} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 \\ 0 & \frac{-1}{e} \end{bmatrix}$$

$$\frac{\partial f}{\partial u} = \begin{bmatrix} x_2 & 0 \\ -x_1(-e^{-x_1u_1})x_2 & x_1 \end{bmatrix}$$

$$\left.\frac{\partial f}{\partial u}\right|_{\substack{x=x_0 \\ u=u_0}} = \begin{bmatrix} 0 & 0 \\ -(1)(-e^{1(1)})0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$[\dot{x}] = \begin{bmatrix} 0 & 1 \\ 0 & -1/e \end{bmatrix}[x] + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}[u]$$

$$\frac{\partial y}{\partial x} = \begin{bmatrix} u_1u_2 & 0 \end{bmatrix}$$

$$\left.\frac{\partial y}{\partial x}\right|_{\substack{x=x_0 \\ u=u_0}} = \begin{bmatrix} 1(0) & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \end{bmatrix}$$

$$\frac{\partial y}{\partial u} = [x_1 u_2 \quad x_1 u_1)$$

$$\left.\frac{\partial y}{\partial u}\right|_{\substack{x=x_0 \\ u=u_0}} = \left[1(0) \quad 1(1)\right] = [0 \quad 1]$$

$$[y] = [0 \quad 1][u]$$

Q3. $A = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$

$A^2 = \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix}$

$A^3 = \begin{bmatrix} -1 & 3 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -8 \end{bmatrix}$

$A^k = \begin{bmatrix} (-1)^k & k(-1)^{k-1} & 0 \\ 0 & (-1)^k & 0 \\ 0 & 0 & (-2)^k \end{bmatrix}$

$e^{At} = I + At + \frac{(At)^2}{2!} + \dots = \begin{bmatrix} e^{-t} & te^{-t} & 0 \\ 0 & e^{-t} & 0 \\ 0 & 0 & e^{-2t} \end{bmatrix}$

$e^{-t} = 1 + -t + \frac{t^2}{2!} - \frac{t^3}{3!} \dots$

$e^{-2t} = 1 + -2t + \frac{4t^2}{2!} \dots$

$te^{-t} = 0 + t + \frac{-2t^2}{2!} + \frac{3t^3}{3!} - \frac{4t^4}{4!} \dots = t^1 - \frac{t^2}{1!} + \frac{t^3}{2!} - \frac{t^4}{3!} \dots$

$= \sum_{n=0}^{\infty} \frac{t^{n+1}}{n!} = te^{-t}$

Q4

$$\dot{x}(t) = \underbrace{\begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix}}_{A} x(t) + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{B} u(t) \rightarrow x[k+1] = A_d x[k] + B_d u[k]$$

$A_d = e^{AT}$ $\quad$ $B_d = \left(\int_0^T e^{A\tau} d\tau\right) B$

$A_d = e^{AT} = e^{At}$ where $t = T$ (sampling time)

find eig of $A \rightarrow 0 = \det(\lambda I - A) = \det\begin{pmatrix} \lambda & -1 \\ 6 & \lambda+5 \end{pmatrix}$

$$0 = \lambda(\lambda+5)+6$$
$$= \lambda^2 + 5\lambda + 6$$
$$= (\lambda+3)(\lambda+2)$$
$$\lambda = -2, -3 \leftarrow \text{distinct} \checkmark$$

eigvals $\rightarrow$ $\lambda_1 = -2 \rightarrow \begin{pmatrix} -2 & -1 \\ 6 & 3 \end{pmatrix} x_1 = 0 \rightarrow x_1 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$

$\lambda_2 = -3 \rightarrow \begin{pmatrix} -3 & -1 \\ 6 & 2 \end{pmatrix} x_2 = 0 \rightarrow x_2 = \begin{pmatrix} 1 \\ -3 \end{pmatrix}$

$T = \begin{bmatrix} 1 & 1 \\ -2 & -3 \end{bmatrix}$

$$A_d = e^{At} = Te^{Dt}T^{-1} = \begin{bmatrix} 1 & 1 \\ -2 & -3 \end{bmatrix}\begin{bmatrix} e^{-2t} & 0 \\ 0 & e^{-3t} \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -2 & -3 \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} e^{-2t} & e^{-3t} \\ -2e^{-2t} & -3e^{-3t} \end{bmatrix}\begin{bmatrix} -3 & -1 \\ 2 & 1 \end{bmatrix}\frac{1}{-3+2}$$

$$= \begin{bmatrix} -3e^{-2t} + 2e^{-3t} & -e^{-2t} + e^{-3t} \\ 6e^{-2t} - 6e^{-3t} & 2e^{-2t} - 3e^{-3t} \end{bmatrix}(-1)$$ $\rightarrow$ substitute t for T (sampling time)

$$= e^{-2T}\begin{bmatrix} 3 & 1 \\ -6 & -2 \end{bmatrix} + e^{-3T}\begin{bmatrix} -2 & -1 \\ 6 & 3 \end{bmatrix}$$

$$B_d = \left(\int_0^T e^{A\tau} \, d\tau \right) \cdot B$$

$$= \left(\int_0^T e^{-2\tau} \begin{pmatrix} 3 & 1 \\ -6 & -2 \end{pmatrix} + e^{-3\tau} \begin{pmatrix} -2 & -1 \\ 6 & 3 \end{pmatrix} d\tau \right) \cdot B$$

$$= \left(\frac{e^{-2\tau}}{-2} \begin{bmatrix} 3 & 1 \\ -6 & -2 \end{bmatrix} + \frac{e^{-3\tau}}{-3} \begin{bmatrix} -2 & -1 \\ 6 & 3 \end{bmatrix} \right) \Bigg|_0^T \cdot B$$

$$= \left(\frac{e^{-2T} - 1}{-2} \begin{bmatrix} 3 & 1 \\ -6 & -2 \end{bmatrix} + \frac{e^{-3T} - 1}{-3} \begin{bmatrix} -2 & -1 \\ 6 & 3 \end{bmatrix} \right) \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= \frac{e^{-2T} - 1}{-2} \begin{bmatrix} 1 \\ -2 \end{bmatrix} + \frac{e^{-3T} - 1}{-3} \begin{bmatrix} -1 \\ 3 \end{bmatrix}$$