Find the ZOH Equivalent of given systems with T = 0.01

1) $G_p(s) = \dfrac{K}{s+a}$ $\quad K = 10 \quad a = 3$

$$G_{P}(z) = (1 - z^{-1})\, Z\left\{\frac{10}{s(s+3)}\right\}$$

$$\frac{10}{s(s+3)} = \frac{A}{s} + \frac{B}{s+3}$$

$$A = \lim_{s \to 0} \frac{s\,10}{s(s+3)} = \frac{10}{3} \qquad B = \lim_{s \to -3} \frac{(s+3)\,10}{s(s+3)} = -\frac{10}{3}$$

$$G_{P}(z) = (1 - z^{-1})\, Z\left\{\frac{10}{3s} - \frac{10}{3(s+3)}\right\}$$

$$= \frac{10}{3}(1 - z^{-1})\, Z\left\{\frac{1}{s} - \frac{1}{s+3}\right\}$$

$$= \frac{10}{3}(1 - z^{-1})\left\{\frac{1}{1-z^{-1}} - \frac{1}{1-e^{-0,03}z^{-1}}\right\}$$

$$= \frac{10}{3}(1 - z^{-1})\left(\frac{1 - e^{-0,03}z^{-1} - 1 + z^{-1}}{(1-z^{-1})(1-e^{-0,03}z^{-1})}\right)$$

$$G_{P}(z) = \frac{10}{3}\left(\frac{z^{-1}(1-e^{-0,03})}{(1-e^{-0,03}z^{-1})}\right) = \frac{0.0985\, z^{-1}}{-0,97\, z^{-1} + 1}$$

$$G_{P(s)} = \frac{w_n^2}{s^2 + 2\zeta w_n + w_n^2} \qquad w_n = 10 \text{ rad/sn} \qquad \zeta = 0.8$$

$$G_{P(s)} = \frac{100}{s^2+16s+100} \qquad \Delta = b^2 - 4ac = 164 - 400 = -144$$

$$\Delta < 0$$

$$G_{P(z)} = (1-z^{-1})\, Z\left\{\frac{100}{s(s^2+16s+100)}\right\}$$

$$\frac{100}{s(s^2+16s+100)} = \frac{A}{s} + \frac{B_1 s + B_2}{(s^2+16s+100)}$$

$$A = \lim_{s\to 0} s \frac{100}{s(s^2+16s+100)} = 1$$

$$100 = A(s^2+16s+100) + s(B_1 s + B_2) \Rightarrow B_1 = -1 \quad B_2 = -16$$

$$\frac{100}{s(s^2+16s+100)} = \frac{1}{s} - \frac{s+16}{(s^2+16s+100)}$$

By completing the square in denominator;

$$\frac{1}{s} - \frac{s+8}{(s+8)^2+6^2} - \frac{8}{(s+8)^2+6^2} = \frac{1}{s} - \frac{s+8}{(s+8)^2+6^2} - \frac{\left(\frac{8}{6}\right)6}{(s+8)^2+6^2}$$

$$G_{p(z)} = (1-z^{-1})\left(\frac{1}{1-z^{-1}} - Z\left\{e^{-8t}\cos 6t\right\} - \frac{8}{6} Z\left\{e^{-8t}\sin 6t\right\}\right)$$

| Time Domain | Laplace Domain | Z domain |
|---|---|---|
| $e^{-\alpha t}\sin(wt)$ | $\frac{w}{(s+\alpha)^2+w^2}$ | $\frac{e^{-\alpha T}z^{-1}\sin(wt)}{1-2e^{-\alpha T}z^{-1}\cos(wT)+e^{-2\alpha T}z^{-2}}$ |
| $e^{-\alpha t}\cos(wt)$ | $\frac{s+\alpha}{(s+\alpha)^2+w^2}$ | $\frac{e^{-\alpha T}z^{-1}\cos(wt)}{1-2e^{-\alpha T}z^{-1}\cos(wT)+e^{-2\alpha T}z^{-2}}$ |

$$G_{p(t)} = \frac{1.85z^{-2} - 2.84z^{-1} + 1}{0.852z^{-2} - 1.84z^{-1} + 1}$$

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5)

$$G_{p(s)} = \frac{w_n^2}{s^2 + 2\zeta w_n + w_n^2} \qquad w_n = 20\,[rad/sn] \quad \zeta = 1.25$$

$$G_{p(s)} = \frac{400}{s^2+50s+400} \qquad \Delta = b^2 - 4ac = 2500 - 1600 = 900$$

$$\Delta > 0$$

Root 1 $-10$ Root 2 $= -40$

$$G_{p(z)} = (1-z^{-1})\, Z\left\{\frac{400}{s(s^2+50s+400)}\right\}$$

Partial Fraction:

$$\frac{400}{s(s^2+50s+400)} = \frac{400}{s(s+10)(s+40)} = \frac{1}{s} + \frac{-\frac{4}{3}}{s+10} + \frac{\frac{1}{3}}{s+40}$$

$$G_{p(z)} = (1-z^{-1})\, Z\left\{\frac{1}{s} + \frac{-4/3}{s+10} + \frac{1/3}{s+40}\right\}$$

$$G_{p(z)} = (1-z^{-1})\left\{\frac{1}{1-z^{-1}} - \frac{4}{3}\frac{1}{1-e^{-0,1}z^{-1}} + \frac{1}{3}\frac{1}{1-e^{-0.4}z^{-1}}\right\}$$

$$G_{p(t)} = \frac{0.01444z^{-2}+0.017z^{-1}}{0.607z^{-2}-1.58z^{-1}+1}$$

4)

$$G_{p(s)} = K \cdot \frac{s+a/10}{s+a} \qquad K=0,81 \qquad a=2$$

$$\frac{G_{p(s)}}{s} = 0,81 \quad \frac{s+0,2}{s(s+2)}$$

$$\frac{s+0,2}{s(s+2)} = \frac{A}{s} + \frac{B}{s+2}$$

$$A = \lim_{s\to 0} \frac{s+0,2}{s+2} = 0,1 \qquad B = \lim_{s\to -2} \frac{s+0,2}{s} = 0,9$$

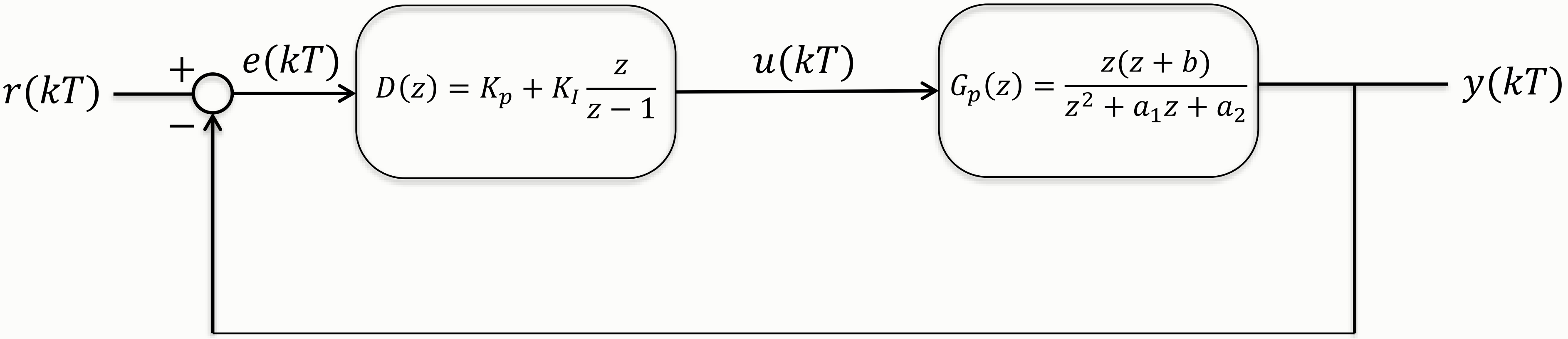
$$G_{p(z)} = (1-z^{-1})\, z\left\{0,81\ \frac{s+0,2}{s(s+2}\right\} = 0,81\,(1-z^{-1})\, z\left\{\frac{0,1}{s}+\frac{0,9}{s+2}\right\}$$

$$G_{p(z)} = (1-z^{-1})\left\{\frac{0,081}{1-z^{-1}} + \frac{0,729}{1-e^{-0,02}z^{-1}}\right\}$$

$$\boxed{G_{p(t)} = \frac{0,808z^{-1}-0,81}{0,98z^{-1}-1}}$$

Lead compensator

# Question 2




**a)** Find the closed-loop transfer function of the system with the unity feedback.

$$D(z) = K_p + K_I \frac{z}{z-1} = \frac{K_p(z-1) + K_I(z)}{z-1}$$

$$G(z)D(z) = \frac{z(z+b)\big(K_p(z-1) + K_I(z)\big)}{(z^2 + a_1 z + a_2)(z-1)}$$

# Question 2 (Cont.)

$$G_c(z) = \frac{G(z)D(z)}{1 + G(z)D(z)} = \frac{z(z+b)\left(K_p(z-1) + K_I(z)\right)}{(z^2 + a_1 z + a_2)(z-1) + z(z+b)\left(K_p(z-1) + K_I(z)\right)}$$

$$\boxed{G_c(z) = \frac{(K_p + K_I)z^3 + \left(b(K_p + K_I) - K_p\right)z^2 - bK_p z}{(K_p + K_I)z^3 + \left(b(K_p + K_I) + a_1 - K_p - 1\right)z^2 + \left(a_2 - a_1 - bK_p\right)z - a_2}}$$

# Question 2 (Additional)

**b)** Express $y(k)$ in terms of past/present inputs and outputs.

$$\frac{y(k)}{r(k)} = G_c(z) = \frac{n_3 z^3 + n_2 z^2 + n_1 z}{d_3 z^3 + d_2 z^2 + d_1 z + d_0} = \frac{n_3 + n_2 z^{-1} + n_1 z^{-2}}{d_3 + d_2 z^{-1} + d_1 z^{-2} + d_0 z^{-3}}$$

$$n_3 = K_p + K_I$$

$$n_2 = b\left(K_p + K_I\right) - K_p$$

$$n_1 = -bK_p$$

$$d_3 = K_p + K_I$$

$$d_2 = b\left(K_p + K_I\right) + a_1 - K_p - 1$$

$$d_1 = a_2 - a_1 - bK_p$$

$$d_0 = -a_2$$

# Question 2 (Additional)

$$\frac{y(k)}{r(k)} = \frac{n_3 + n_2 z^{-1} + n_1 z^{-2}}{d_3 + d_2 z^{-1} + d_1 z^{-2} + d_0 z^{-3}}$$

$$\Rightarrow d_3 y[k] + d_2 y[k-1] + d_1 y[k-2] + d_0 y[k-3] = n_3 r[k] + n_2 r[k-1] + n_1 r[k-2]$$

$$\boxed{y[k] = \frac{1}{d_3}(n_3 r[k] + n_2 r[k-1] + n_1 r[k-2] - d_2\, y[k-1] - d_1 y[k-2] - d_0 y[k-3])}$$

$y[k]$ is directly affected by $r[k]$, so the system has zero delay terms.

# Question 2 (Additional)

**c)** Express the control law. (i.e. $u[k] = ??r[k\} + ??y[k]$)

$$e[k] = r[k] - y[k]$$

$$u[k] = D(z)e(k) = D(z)(r[k] - y[k])$$

$$\Rightarrow \; u[k] = \frac{K_p(z-1) + K_I(z)}{z-1}(r[k] - y[k])$$

$$\Rightarrow \; u[k+1] - u[k] = (K_p + K_I)r[k+1] - K_p r[k] - (K_p + K_I)y[k+1] + K_p y[k]$$

Substituting $k+1$ with $k$

$$\boxed{u[k] = u[k-1] + (K_p + K_I)r[k] - K_p r[k-1] - (K_p + K_I)y[k] + K_p y[k-1]}$$

# Question 3

$$G(z) = \frac{y(k)}{u(k)} = \frac{b_0 z^2 + b_1 z + b_2}{z^3(z^2 + a_1 z + a_2)}$$

**a)** Express $y(k)$ in terms of past/present inputs and outputs.

$$G(z) = \frac{y[k]}{u[k]} = \frac{b_0 z^2 + b_1 z + b_2}{(z^5 + a_1 z^4 + a_2 z^3)} = \frac{b_0 z^{-3} + b_1 z^{-4} + b_2 z^{-5}}{1 + a_1 z^{-1} + a_2 z^{-2}}$$

$$y[k] + a_1 y[k-1] + a_2 y[k-2] = b_0 u[k-3] + b_1 u[k-4] + b_2 u[k-5]$$

$$y[k] = b_0 u[k-3] + b_1 u[k-4] + b_2 u[k-5] - a_1 y[k-1] - a_2 y[k-2]$$

There is a delay equal to three sample times because the effect of the input u[k] is not seen in the output until y[k+3], or in other words, the current output $y[k]$ is affected by $u[k-3]$ and the inputs before but not affected by $u[k-2], u[k-1], u[k]$.

# Question 3 (Cont.)

$$G(z) = \frac{y(k)}{u(k)} = \frac{b_0 z^2 + b_1 z + b_2}{z^3(z^2 + a_1 z + a_2)}$$

**b)** What is the DC gain of the given system?

$$DC\ gain: \lim_{s \to 0} G(s) = \lim_{z \to 1} G(z)$$

$$z = e^{sT}$$

$$\boxed{DC\ gain: \lim_{z \to 1} G(z) = \lim_{z \to 1} \frac{b_0 z^2 + b_1 z + b_2}{z^3(z^2 + a_1 z + a_2)} = \frac{b_0 + b_1 + b_2}{1 + a_1 + a_2}}$$

# Question 3 (Cont.)

$$G(z) = \frac{y(k)}{u(k)} = \frac{b_0 z^2 + b_1 z + b_2}{z^3(z^2 + a_1 z + a_2)}$$

**c)** What is the steady-state error of the system to the following input (step function)?

$$u(t) = 2$$

$$e = u - y = u - Gu = u(1 - G)$$

$$e_{ss}: \lim_{t \to \infty} e(t) = \lim_{s \to 0} se(s) = \lim_{z \to 1} (1 - z^{-1}) e(z)$$

$$\Rightarrow \ e_{ss}: \lim_{z \to 1} (1 - z^{-1}) e(z) = \lim_{z \to 1} (1 - z^{-1}) \frac{2}{1 - z^{-1}} \left(1 - \frac{b_0 z^2 + b_1 z + b_2}{z^3(z^2 + a_1 z + a_2)}\right)$$

$$\boxed{e_{ss} = 2(1 - DC\ gain) = \frac{2(1 + a_1 + a_2 - b_0 - b_1 - b_2)}{1 + a_1 + a_2}}$$