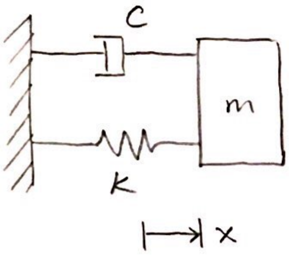
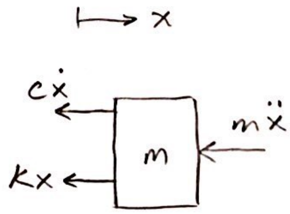
Lecture 4

MECH 463
Sept 17

Damping



FBD:



$$m\ddot{x} + c\dot{x} + kx = 0$$

Damper force is same direction as for spring

Use solution #3:

$$x = Ge^{\lambda t} \Rightarrow \dot{x} = \lambda G e^{\lambda t}, \quad \ddot{x} = \lambda^2 G e^{\lambda t}$$

$$(m\lambda^2 + c\lambda + k) Ge^{\lambda t} = 0$$

$G \neq 0$ generally, so part in brackets $= 0$.

Characteristic equation: $m\lambda^2 + c\lambda + k = 0$

Quadratic equation: $\lambda_1, \lambda_2 = -\frac{c}{2m} \pm \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}}$

Solution: $x = Ge^{\lambda_1 t} + He^{\lambda_2 t}$

Useful notation:

$\omega_N^2 = \frac{k}{m}$ (undamped natural frequency)

$\zeta = \frac{c}{2\sqrt{km}}$ (Damping factor)

$\omega_D = \sqrt{1 - \zeta^2}\, \omega_N$ (Damped natural frequency)

Rewrite quadratic solution:

$$\lambda_1, \lambda_2 = \frac{-c}{2\sqrt{km}} \cdot \sqrt{\frac{k}{m}} \pm \sqrt{\left(\frac{k}{m}\right)\left[\left(\frac{c}{2\sqrt{km}}\right)^2 - 1\right]}$$

$$\lambda_1, \lambda_2 = \left(-\zeta \pm i\sqrt{1-\zeta^2}\right)\omega_N$$

$$\lambda_1, \lambda_2 = -\zeta\omega_N + i\omega_D \Rightarrow 3 \text{ solutions, } \zeta \lesseqgtr 1$$

$\boxed{\text{Underdamping } (\zeta < 1)}$ → Oscillation

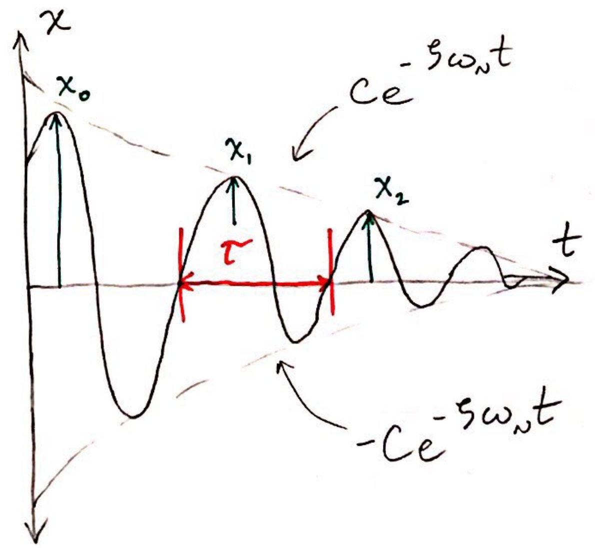
$$x = Ge^{(-\zeta + i\sqrt{1-\zeta^2})\omega_N t} + He^{(-\zeta - i\sqrt{1-\zeta^2})\omega_N t}$$

$$x = e^{-\zeta\omega_N t}\left(Ge^{i\omega_D t} + He^{-i\omega_D t}\right)$$

$$x = e^{-\zeta\omega_N t}\left(A\cos(\omega_D t) - B\sin(\omega_D t)\right) \quad \text{and} \quad \begin{cases} A = H + G \\ B = i(H - G) \end{cases}$$

$$\boxed{x = \underbrace{e^{-\zeta\omega_N t}}_{\text{decay w/ time}} \cdot \left[\underbrace{C}_{\text{amplitude}}\underbrace{\cos}_{\text{oscillation}}(\omega_D t + \underbrace{\phi}_{\text{phase}})\right]}$$

plot:



$$\omega_D = \frac{2\pi}{\tau}$$

τ is period

x_N is an amplitude

Logarithmic decrement

$$x = Ce^{-\zeta\omega_N t}\left(\cos(\omega_D t + \phi)\right)$$

$$\frac{x_0}{x_N} = \frac{Ce^{-\zeta\omega_N t_0}\left(\cos(\omega_D t_0 + \phi)\right)}{Ce^{-\zeta\omega_N t_N}\left(\cos(\omega_D t_N + \phi)\right)}$$

$$\frac{x_0}{x_N} = \frac{\left(Ce^{-\zeta\omega_N t_0}\right)\left(\cos(\omega_D t_0 + \phi)\right)}{\left[Ce^{\zeta\omega_N\left(t + \frac{2\pi n}{\omega_D}\right)}\right]\cos(\omega_D t_0 + 2\pi n + \phi)}$$

$$\frac{x_0}{x_N} = \frac{\left(Ce^{-\zeta\omega_N t_0}\right)\left(\cos(\omega_D t_0 + \phi)\right)}{\left(Ce^{-\zeta\omega_N t_0}\right)\left(e^{-\frac{\zeta\omega_N 2\pi n}{\omega_D}}\right)\left(\cos(\omega_D t_0 + \phi)\right)}$$

$$\Rightarrow \boxed{\frac{x_0}{x_N} = e^{2\pi n\zeta\left(\frac{\omega_N}{\omega_D}\right)}}$$

Define logarithmic decrement, δ

$$\delta = \frac{1}{n}\ln\left(\frac{x_0}{x_n}\right)$$

$$\delta = 2\pi\frac{\omega_N\zeta}{\omega_D} = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}}$$

$$\Rightarrow \boxed{\zeta = \frac{\delta}{\sqrt{4\pi^2 + \delta^2}} \simeq \frac{\delta}{2\pi}}$$

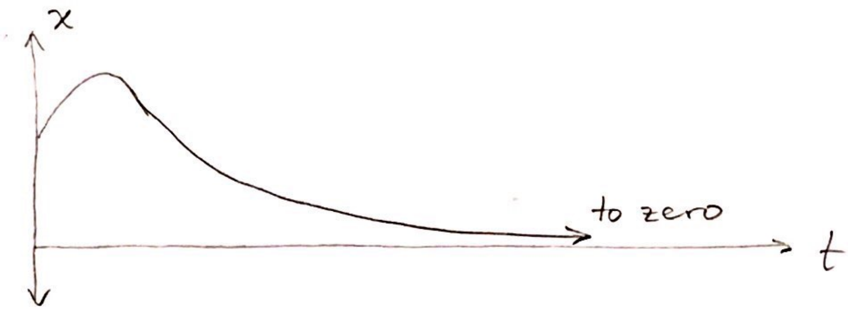
for small δ

$\boxed{\text{Overdamping } (\zeta > 1)}$ → No oscillation

$$x = Ge^{\lambda_1 t} + He^{\lambda_2 t}$$

$$x = Ge^{-(\zeta + \sqrt{\zeta^2 - 1})\omega_n t} + He^{-(\zeta - \sqrt{\zeta^2 - 1})\omega_n t}$$

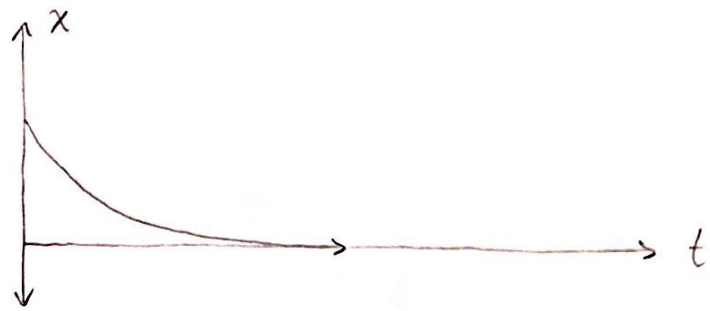
Plot:



$\boxed{\text{Critical Damping } (\zeta = 1)}$ → Quickest to equilibrium

$$x = (G + Ht)e^{-\omega_n t}$$

Plot:



$\boxed{\text{Forced Response (next lecture)}}$