MECH 421: Mechatronics System Instrumentation
2020/2021 Winter Session – Term 2

Homework 3

Assigned: Feb 5, 2021
Due: Feb 12, 2021

Problem 1

Let us consider an op-amp circuit in Figure 1. We assume that the op-amp input impedance is infinite, the output impedance is R_o, and the open-loop gain is $A(s)$. Note that the output impedance R_o is pulled out of the op-amp.

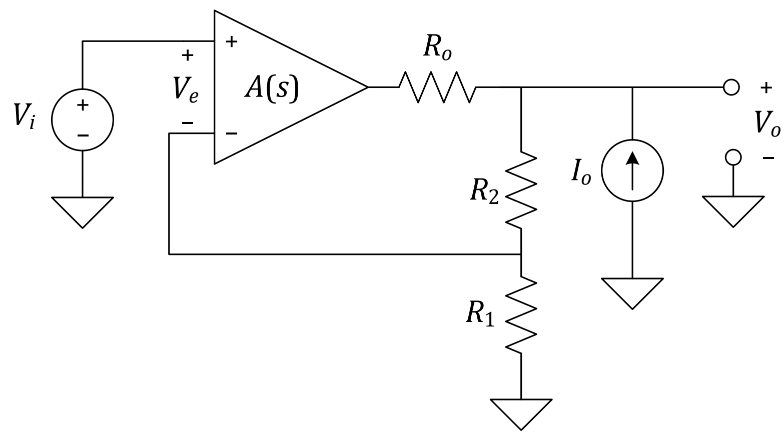
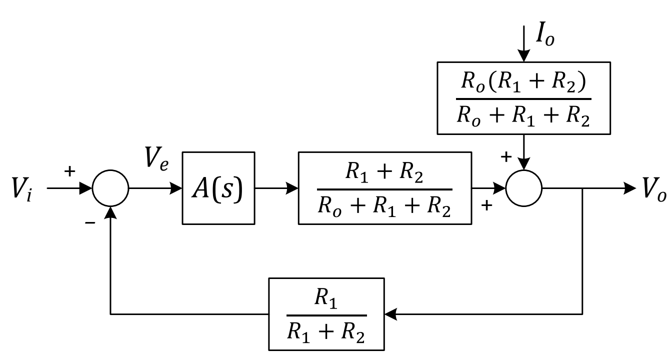


Figure 1: Schematic of a non-inverting amplifier.

(a) Draw a block diagram that shows the relation between the input voltage V_i, disturbance current I_o, and the output voltage V_o. The block diagram should show a feedback loop around $A(s)$.

Answer:



Refer to the **Homework 2 – Problem 1** for the derivation.

(b) Find the expression for the loop transfer function $L(s)$. Also, find the expression for the output impedance $Z_o(s) = V_o/I_o$ in terms of $L(s)$, R_o, R_1, and R_2.

Answer:

The loop transfer function is

$$L(s) = A(s)\frac{R_1}{R_o + R_1 + R_2}.$$

The output impedance is

$$\begin{aligned} Z_o(s) &= R_o \parallel (R_1 + R_2) \left(\frac{1}{1 + L(s)}\right) \\ &= \frac{R_o(R_1 + R_2)}{R_o + R_1 + R_2} \left(\frac{1}{1 + L(s)}\right). \end{aligned}$$

(c) Assuming the op-amp high-frequency open-loop gain is zero, i.e., $A(j\omega)|_{\omega\to\infty} = 0$, find the expression for the high-frequency output impedance $Z_o(j\omega)|_{\omega\to\infty}$ in terms of R_o, R_1, and R_2. Also, find the expression for the output impedance normalized to the high-frequency value, i.e.,

$$\hat{Z}(s) \equiv \frac{Z_o(s)}{Z_o(j\omega)|_{\omega\to\infty}}$$

in terms of $L(s)$.

Answer:

If $A(j\omega)_{\omega\to\infty} = 0$, then $L(j\omega)_{\omega\to\infty} = 0$ as well. Therefore,

$$\begin{aligned} Z_o(j\omega)|_{\omega\to\infty} &= \frac{R_o(R_1 + R_2)}{R_o + R_1 + R_2} \cancelto{1}{\left(\frac{1}{1 + L}\right)} \\ &= \frac{R_o(R_1 + R_2)}{R_o + R_1 + R_2} \end{aligned}$$

Thus, the normalized output impedance is

$$\hat{Z}(s) = \frac{1}{1 + L(s)} \equiv S(s)$$

(d) Using MATLAB, draw the Bode plots of $L(s)$ and $\hat{Z}(s)$ for

$$R_o = 50\,\Omega \qquad R_1 = 1\,\mathrm{k}\Omega \qquad R_2 = 1\,\mathrm{k}\Omega \qquad A(s) = \frac{10^4}{(s + 1)(0.001s + 1)}$$

on the same graph. Find the gain crossover frequency ω_c and phase margin ϕ_m of $L(s)$, and also find the cutoff frequency ω_l where $|\hat{Z}(j\omega_l)| = 0.7071(-3\,\text{dB})$. Discuss how the shapes of $|L(j\omega)|$ and $|\hat{Z}(j\omega)|$ are related at frequencies below ω_c.

Answer:

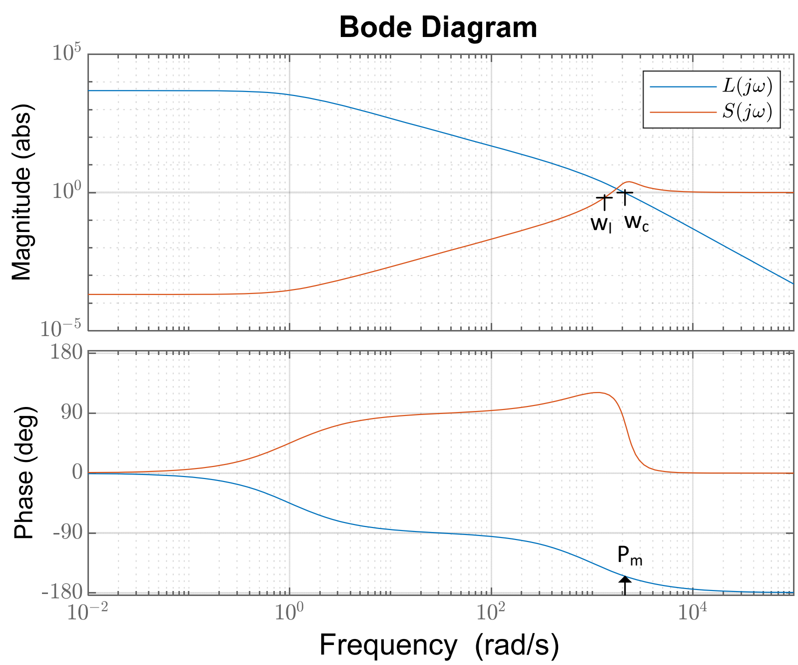
The gain crossover frequency and phase margin of $L(s)$ are

$$\omega_c = 2098\,\text{rad/s}$$
$$\phi_m = 25.5^\circ$$

The cutoff frequency ω_l where $|S(j\omega_l)| = 0.7071(-3\,\text{dB})$ is

$$\omega_l = 1370\,\text{rad/s}$$

The Bode plots are as follows.



Note that below ω_l, the shapes of $|L(j\omega)|$ and $|S(j\omega)|$ are symmetric with respect to the unity line (zero dB line) and the shapes of $\angle L(j\omega)$ and $\angle S(j\omega)$ are symmetric with respect to the zero angle line.

(e) Find the output dc impedance $Z_o(j\omega)|_{\omega=0}$

Answer:

$$
\begin{aligned}
Z_o(j\omega)|_{\omega=0} &= \frac{R_o(R_1+R_2)}{R_o+R_1+R_2}\left(\frac{1}{1+L(j\omega)|_{\omega=0}}\right) \\
&= \frac{R_o(R_1+R_2)}{R_o+R_1+R_2}\left(\frac{1}{1+\frac{R_1}{R_o+R_1+R_2}A(j\omega)|_{\omega=0}}\right) \\
&= 48.78\,\Omega\frac{1}{1+0.4878\times 10^4} \\
&= 0.01\,\Omega
\end{aligned}
$$

Alternatively, one can find the approximate output DC impedance as follows. Since $R_o \ll R_1 + R_2$,

$$
R_o \parallel (R_1+R_2) \approx R_o = 50\,\Omega
$$

$$
f = \frac{R_1}{R_o+R_1+R_2} \approx \frac{R_1}{R_1+R_2} = 0.5
$$

Also, since $fA(j\omega)|_{\omega=0} \gg 1$

$$
\frac{1}{1+fA(j\omega)|_{\omega=0}} \approx \frac{1}{fA(j\omega)|_{\omega=0}} = \frac{1}{5000}
$$

Therefore,

$$
Z_o(j\omega)|_{\omega=0} \approx \frac{50\,\Omega}{5000} = 0.01\,\Omega.
$$

Note that this is quite close to the exact value.

Problem 2

Let us consider a brushed dc motor driven by a voltage amplifier shown in Figure ??. Here, L_m is the winding inductance, R_m is the winding resistance, J_m is the rotor rotational inertia, and K_t is the motor torque constant. There is a wheel mounted on the motor shaft, whose rotational inertia is J_w. On the current return path, there is a shunt resistor R_s to measure the current through the motor winding.

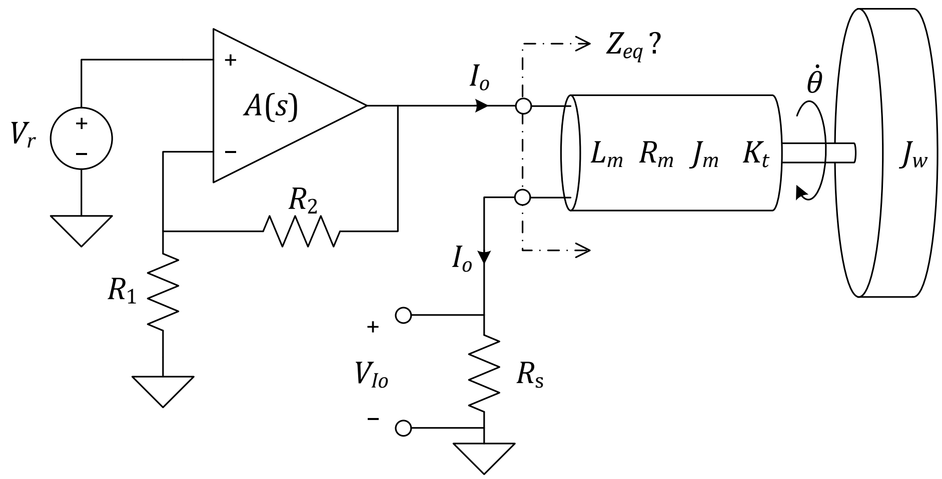
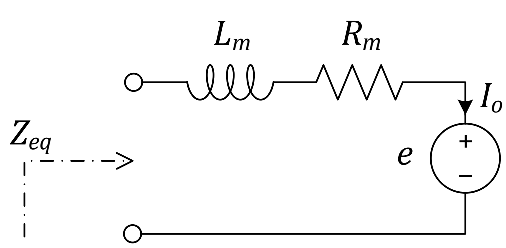


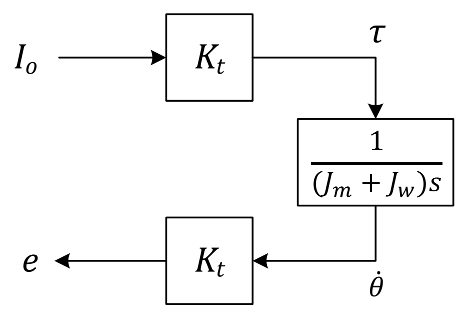
Figure 2: Brushed dc motor driven by an op-amp circuit.

(a) Find the electrical impedance $Z_{eq}(s)$ looking into the electrical port of the motor.

Answer:

The electrical part of the motor can be modeled as follows.





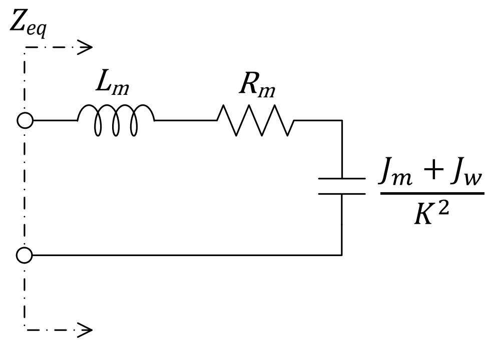
Here, the back EMF is modeled as a speed-dependent voltage source e. The apparent impedance of this element is the ratio between the voltage drop across the element and the current through the element, i.e., $e/I_o = \frac{K_t^2}{(J_m+J_w)s}$, which can be found from the block diagram on the right. Therefore, the total impedance $Z_{eq}(s)$ is

$$Z_{eq}(s) = L_m s + R_m + \frac{K^2}{(J_m + J_w)s}$$

(b) Draw an equivalent circuit diagram where all mechanical elements are referred to the electrical domain as passive electrical elements. Find the parameter(s) of the equivalent circuit element(s) in terms of L_m, R_m, J_m, K_t, and J_w.

Answer:

When the mechanical load is pure inertia, the speed-dependent voltage source appears as a capacitor whose capacitance is $\frac{J_m+J_w}{K_t^2}$. Therefore, an equivalent circuit diagram is as follows.



(c) Find the analytic expression for the transconductance from $V_r(s)$ to $I_o(s)$. Then, draw the Bode plot of $I_o(s)/V_r(s)$ using MATLAB for

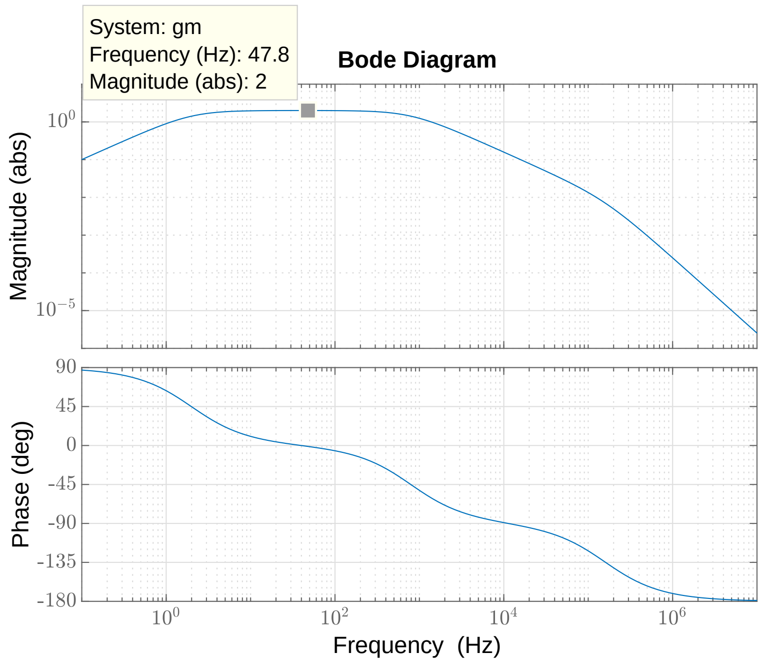
$$A(s) = \frac{10^7}{s} \qquad R_1 = 1\,\mathrm{k\Omega} \qquad R_2 = 9\,\mathrm{k\Omega}$$
$$R_m = 4.8\,\Omega \qquad L_m = 1\,\mathrm{mH} \qquad R_s = 0.2\,\Omega$$
$$K_t = 250\,\mathrm{mNm/A} \qquad J_m = 1\,\mathrm{kg\cdot cm^2} \qquad J_w = 9\,\mathrm{kg\cdot cm^2}$$

Answer:

The analytical expression for the transconductance is

$$\frac{I_o(s)}{V_r(s)} = \underbrace{\frac{A(s)}{1+A(s)\frac{R_1}{R_1+R_2}}}_{V_o/V_r} \underbrace{\frac{1}{L_m s + R_m + R_s + \frac{K^2}{(J_m+J_w)s}}}_{I_o/V_o}$$

Note that the second factor I_o/V_o also depends on the shunt resistor R_s. If $R_s \ll R_m$, we can ignore the effect of R_s. The bode plot of I_o/V_r is as follows.



Note that there are three break frequencies in the magnitude curve, and the bandwidth of the transconductance is limited between the first two break frequencies. The first one around 20 Hz is related with the motor mechanical time constant $\tau_m = J_{\text{net}}/b_{\text{net}}$, where $b_{\text{net}} = \frac{K_t^2}{R}$ is the apparent damping. Note that this time constant not only depends on the rotor inertia but also depends on the load ineartia. The second one around 1 kHz is related with the motor electrical time constant $\tau_e = L/R$. The last one around 200 kHz is from the voltage amplifier bandwidth.

(d) Let us assume that the shunt resistor R_s is rated for 1 W power. That is, the resistor fails when it dissipates more than 1 W. What is the maximum rms current $I_{o,\text{rms}}$ allowed for the shunt resistor?

Answer:

The maximum permissible rms current for the shunt resistor is

$$I_{o,\text{rms}} = \sqrt{\frac{1\,\text{W}}{R_s}} = 2.2361\,\text{A}_{\text{rms}}.$$

(e) Let the input voltage V_r be sinusoidal at 50 Hz. What is the maximum rms voltage $V_{r,\text{rms}}$ that the shut resistor can accommodate for its power rating?

Answer:

The transconductance at 50 Hz is 2 A/V. Therefore, the maximum permissible rms reference voltage at 50 Hz is

$$V_{r,\mathrm{rms}} = \frac{I_{o,\mathrm{rms}}}{2\,\mathrm{A/V}} = 1.118\,\mathrm{V_{rms}}$$