University of British Columbia
Department of Mechanical Engineering
MECH 463. Midterm 2, November 2, 2020

Allowed Time: 50 min

Materials admitted: Pen, pencil, eraser, straightedge, simple scientific calculator without programming or communication capabilities, Matlab for Q2, personal hand-written notes within one letter-size sheet of paper (one side), timer and document copier apps on your phone (all other phone functionalities are **not** allowed).

There are 2 questions in this exam. You are asked to answer all questions.

The purpose of this test is to evaluate your knowledge of the course material. Orderly presentation demonstrates your knowledge most clearly. Marks are assigned accordingly. A bonus of up to 2 marks will be given for exemplary presentation.

Honour Code: You are asked to behave honourably during this exam and to obey all instructions carefully. Please write and sign the following promise in the space below: "I promise to work honestly on this exam, to obey all instructions carefully, and not to have any unfair advantage over any other students."

Promise:

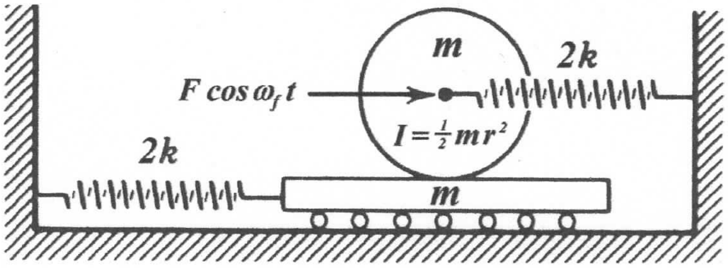
Signed: Name:

	Mark Received	**Maximum Mark**
1		13
2		12
Presentation		2 bonus
Total		25+2

Start Time running

Finish Time running

1. A mechanism within a machine consists of a plate of mass m that moves horizontally on rollers. It is attached to the machine housing by a spring of stiffness 2k. A cylindrical roller of mass m, radius r and moment of inertia I = ½ m r² rolls without slipping on the plate, and is also attached to the machine housing by a spring of stiffness 2k. A harmonic force F cos $\omega_f t$ acts on the roller, as shown in the diagram. Work this question by hand, do not use Matlab.



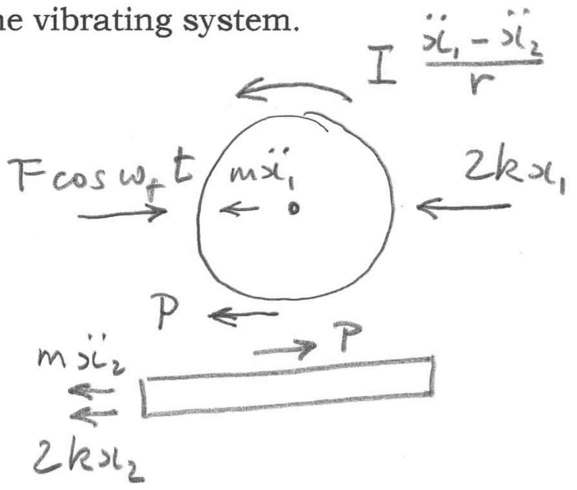
(a) Draw labeled free-body diagrams of the parts of the vibrating system. *(Hint: remember to include the horizontal force between the two masses.)*

(b) Use your free-body diagrams to formulate the equations of motion and express them in matrix form.

(c) Derive a formula for the steady state response of the cylinder. Show the needed steps in detail. At what excitation frequency will there be zero response?

(d) Draw a magnification factor vs. excitation frequency plot for the cylinder and label the key points.

(e) *(Bonus)* Find the natural frequencies of the vibrating system.



(a) Moment about contact point:

$$m\ddot{x}_1 \cdot r + I\,\frac{\ddot{x}_1 - \ddot{x}_2}{r} + 2kx_1 \cdot r = F\cos\omega_f t \cdot r$$

÷r and sub. $I = \frac{1}{2}mr^2$

$$\frac{3}{2}m\ddot{x}_1 - \frac{m}{2}\ddot{x}_2 + 2kx_1 = F\cos\omega_f t$$

Force equilibrium: $m\ddot{x}_1 + 2kx_1 + P = F\cos\omega_f t$

$m\ddot{x}_2 + 2kx_2 - P = 0$

Add to eliminate P: $m\ddot{x}_1 + m\ddot{x}_2 + 2kx_1 + 2kx_2 = F\cos\omega_f t$

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In matrix form:

$$\begin{bmatrix} \frac{3}{2}m & -\frac{m}{2} \\ m & m \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} 2k & 0 \\ 2k & 2k \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} F \\ F \end{bmatrix} \cos\omega_f t$$

Subtract first line from second line to make matrices symmetrical (not an essential step here)

$$\begin{bmatrix} \frac{3}{2}m & -\frac{m}{2} \\ -\frac{m}{2} & \frac{3}{2}m \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} 2k & 0 \\ 0 & 2k \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} F \\ 0 \end{bmatrix} \cos\omega_f t$$

(c)

Try particular solution $\underline{x} = \underline{X}\cos\omega_f t$ and sub. in

$$\underset{\sim}{M}\,\ddot{\underline{x}} + \underset{\sim}{K}\,\underline{x} = \underline{F}\cos\omega_f t \rightarrow \left(-\omega_f^2 \underset{\sim}{M} + \underset{\sim}{K}\right)\underline{X}\cos\omega_f t = \underline{F}\cos\omega_f t$$

For a solution valid for all $t \rightarrow \left(-\omega_f^2 \underset{\sim}{M} + \underset{\sim}{K}\right)\underline{X} = \underline{F}$

$$\rightarrow \begin{bmatrix} 2k - \frac{3}{2}m\omega_f^2 & \frac{m}{2}\omega_f^2 \\ \frac{m}{2}\omega_f^2 & 2k - \frac{3}{2}m\omega_f^2 \end{bmatrix}\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} F \\ 0 \end{bmatrix}$$

Solving by Cramer's Rule:

$$X_1 = \frac{\begin{vmatrix} F & \frac{m}{2}\omega_f^2 \\ 0 & 2k - \frac{3}{2}m\omega_f^2 \end{vmatrix}}{\begin{vmatrix} 2k - \frac{3}{2}m\omega_f^2 & \frac{m}{2}\omega_f^2 \\ \frac{m}{2}\omega_f^2 & 2k - \frac{3}{2}m\omega_f^2 \end{vmatrix}} = \frac{F\left(2k - \frac{3}{2}m\omega_f^2\right)}{\left(2k - \frac{3}{2}m\omega_f^2\right)^2 - \left(\frac{m}{2}\omega_f^2\right)^2}$$

Zero response when $\left(2k - \frac{3}{2}m\omega_f^2\right) = 0 \longrightarrow \underline{\omega_f^2 = \frac{4}{3}\frac{k}{m}}$

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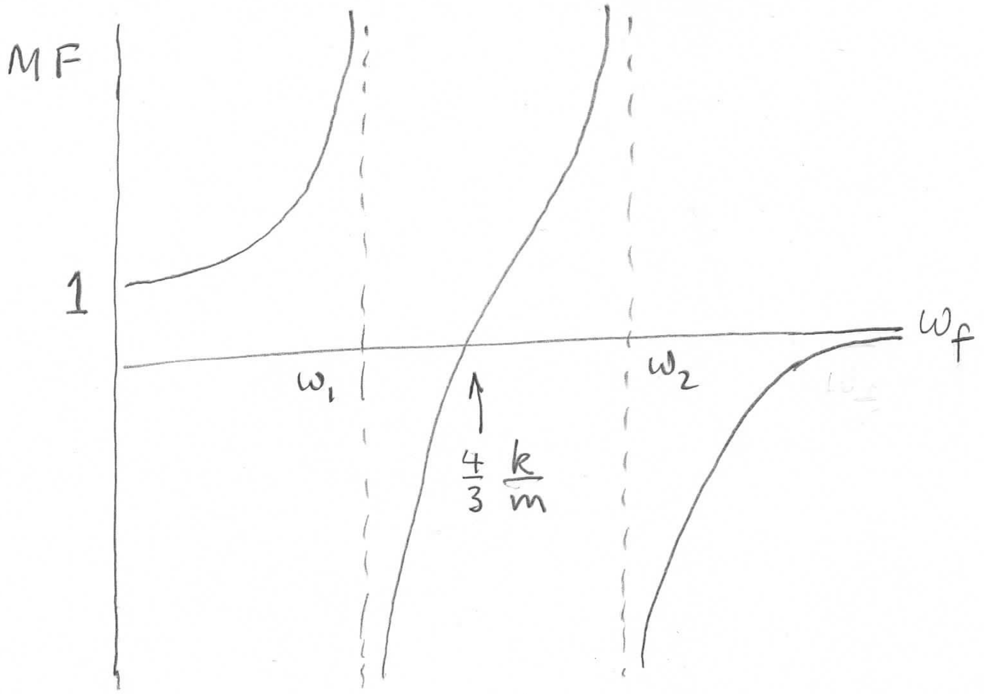
(d)

Static displacement occurs when $\omega_f = 0 \rightarrow F = \frac{2Fk}{(2k)^2} = \frac{F}{2k}$

$$\rightarrow MF = \frac{X_1}{X_{01}} = \frac{1 - \frac{3}{4}\frac{m}{k}\omega_f^2}{(2k - \frac{3}{2}m\omega_f^2)^2 - (\frac{m}{2}\omega_f^2)}$$



(e) The natural frequencies are the roots of the denominator determinant

$$\left(2k - \frac{3}{2}m\omega_f^2\right)^2 - \left(\frac{m}{2}\omega_f^2\right) = 0$$

$$= a^2 - b^2$$

$$= (a+b)(a-b)$$

$$= \left(2k - m\omega_f^2\right)\left(2k - 2m\omega_f^2\right) = 0$$

$$\rightarrow \underline{\omega_2^2 = 2\frac{k}{m}} \qquad \underline{\omega_1^2 = \frac{k}{m}}$$

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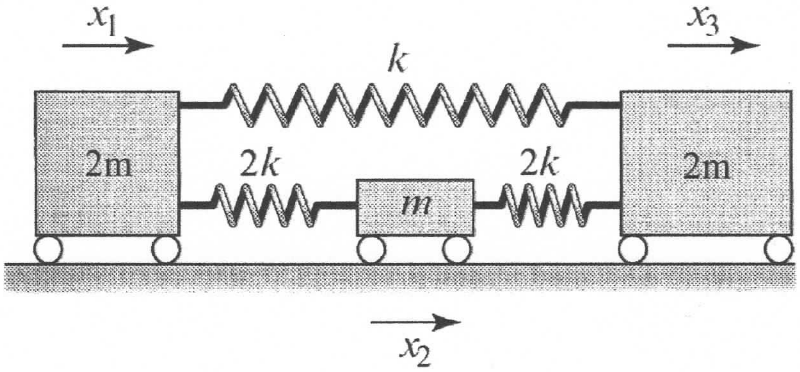
2. A mechanism within a machine consists of three masses connected together by springs as shown in the diagram.



(a) Draw fully labeled free-body diagrams for the various parts of the vibrating systems and use them to derive a matrix equation of motion.

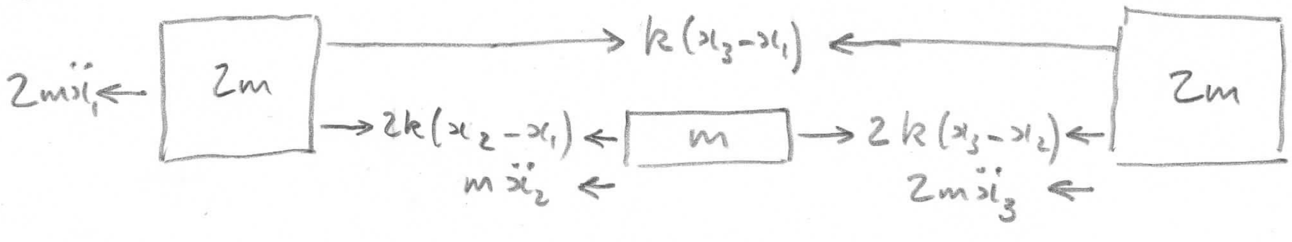
(b) Adapt the following fragment of Matlab code used for the Capstone Laboratory exercise to solve for the natural frequencies and mode shapes of the vibrating system. Take a screenshot of your completed coding and computed results and include it in your exam paper submission.

```
m = 1;
k = 1;
M = [[m 0]' [0 m*R^2/D^2]'];
K = m*g*a1*a2/L1/L2 * [[L1/a2+L2/a1 L1-L2]' ...
 [L1-L2 a2*L1+a1*L2]'];
 [V,w2] = eig(K,M,'vector');
 V(:,:) = V(:,:) ./ V(1,:);
 V
 w2
```

(c) Explain in detail why the Matlab function "eig" is useful for finding the natural frequencies and mode shapes.

(d) Give a physical explanation for each mode shape and corresponding natural frequency. Use the approach that we discussed in class.

(a)



$$2m\ddot{x}_1 - 2k(x_2 - x_1) - k(x_3 - x_1) = 0$$
$$m\ddot{x}_2 + 2k(x_2 - x_1) - 2k(x_3 - x_2) = 0$$
$$2m\ddot{x}_3 + 2k(x_3 - x_2) - k(x_3 - x_1) = 0$$

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In matrix form:

$$\begin{bmatrix} 2m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & 2m \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{bmatrix} + \begin{bmatrix} 3k & -2k & -k \\ -2k & 4k & -2k \\ -k & -2k & 3k \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

(b)

```
m = 1;
k = 1;
M = m*[[2 0 0]' [0 1 0]' [0 0 2]'];
K = k*[[3 -2 -1]' [-2 4 -2]' [-1 -2 3]'];
[V,w2] = eig(K,M,'vector');
V(:,:) = V(:,:) ./ V(1,:);
V
w2

V =

    1.0000    1.0000    1.0000
    1.0000    0.0000   -4.0000
    1.0000   -1.0000    1.0000

w2 =

    0.0000
    2.0000
    5.0000
```

(c) Substitution of $\underline{x} = \underline{X} \cos \omega t$ into $\underset{\sim}{M}\ddot{\underline{x}} + \underset{\sim}{K}\underline{x} = \underline{0}$ gives $(-\omega^2 \underset{\sim}{M} + \underset{\sim}{K})\underline{X} \cos \omega t = \underline{0}$. For a solution valid for all time, this reduces to $(-\omega^2 \underset{\sim}{M} + \underset{\sim}{K})\underline{X} = 0$ and hence $\underset{\sim}{K}\underline{X} = \omega^2 \underset{\sim}{M}\underline{X}$. This is a generalized eigenvalue problem that is solved by "eig". The eigenvalues are the squares of the natural frequencies and the eigenvectors are the mode shapes.

(d) All three masses can roll freely without overall lateral constraint. Therefore, the three masses can move laterally with rigid-body motion with $\omega^2 = 0$.

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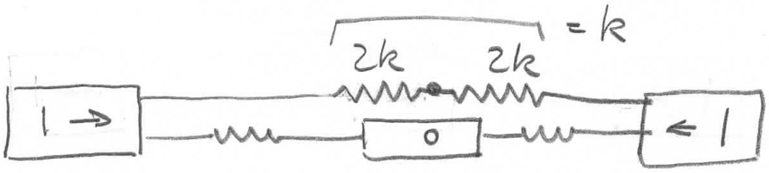
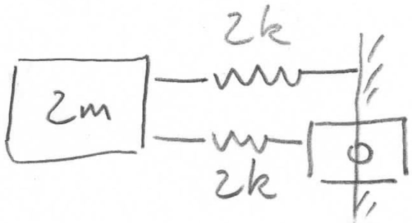
rigid-body motion

The second mode is a symmetrical opposing motion of the two end masses with the centre mass stationary

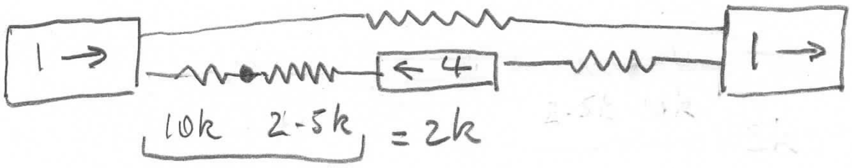


The symmetrical vibration gives a nodal point in the centre of the long spring, dividing it into two 2k sections. Total stiffness = 4k

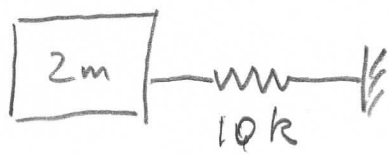


$\rightarrow \omega^2 = \frac{4k}{2m} = 2\frac{k}{m}$

The third mode consists of the two end masses moving together, opposed by the brave little centre mass. To retain the overall centre of mass, the centre mass m must move 4 times that of the outer masses, total 4m.



The long spring is unstretched, so contributes no stiffness. Each short spring is divided into a 4:1 ratio, so 2k gives 10k and 2·5k.



$\rightarrow \omega^2 = \frac{10k}{2m} = 5\frac{k}{m}$