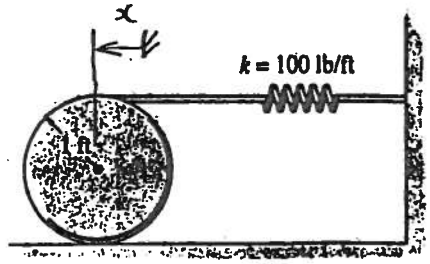
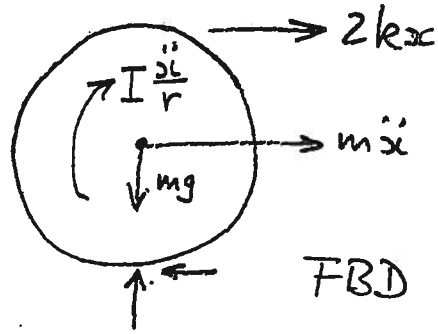
MECH 463 -- Homework 1

1. A 10 lb disk, radius 1 ft, rolls without slipping on a horizontal surface. A spring of stiffness k = 100 lb/ft is attached to the surface of the disk at a point which is highest when the spring is unstretched. Derive the equation of motion and the natural frequency of vibration.



Let x = displacement of centre of disk.
The contact point with the horizontal surface is the instantaneous centre of rotation. Therefore, spring extension $= 2x$.
Angular acceleration, $\alpha = \frac{\ddot{x}}{r}$.



To avoid the unknown reactions at the contact point, take moments about the contact point:

$$I\frac{\ddot{x}}{r} + mr\ddot{x} + 4krx = 0 \qquad \text{where } I = \tfrac{1}{2}mr^2$$

$$\div r \rightarrow \tfrac{3}{2}m\ddot{x} + 4kx = 0 \rightarrow \omega_n = \sqrt{\frac{4k}{\frac{3}{2}m}} = \sqrt{\frac{8k}{3m}} = \sqrt{\frac{8\times 100}{3\times\frac{10}{32.2}}} = 29.3 \text{ rad/s}$$

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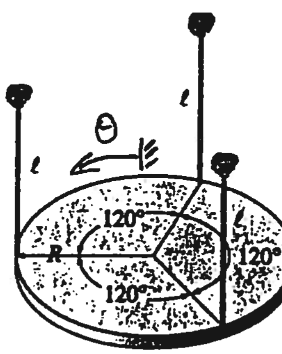
2. A circular plate, of radius R and mass m, is supported by three symmetrically placed strings of length L. Derive the equation of motion and the natural frequency of vibration.



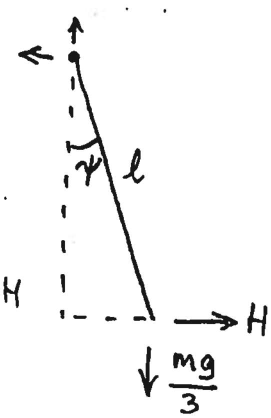
Let θ = rotation angle of plate
ψ = rotation angle of a string

Arc length $= R\theta = l\sin\psi \simeq l\psi$ for small ψ

Horizontal force component on each string $= H$

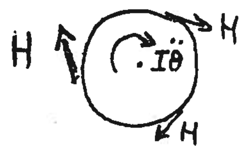
Taking moments about top of a string:

$$Hl\cos\psi - \frac{mg}{3}l\sin\psi = 0 \rightarrow H \simeq \frac{mg}{3l}l\sin\psi = \frac{mg}{3l}R\theta$$



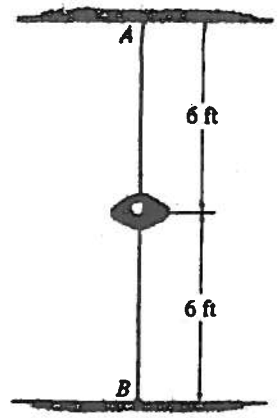
Take moments about centre of disk:

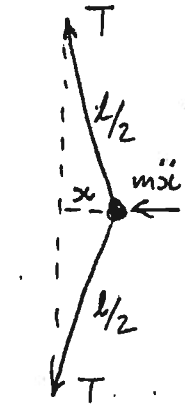
$$I\ddot{\theta} + 3RH = 0 \rightarrow \tfrac{1}{2}mR^2\ddot{\theta} + \frac{mg}{l}R^2\theta = 0$$



$$\rightarrow \ddot{\theta} + \frac{2g}{l}\theta = 0 \qquad \omega = \sqrt{\frac{2g}{l}}$$

3. **A 5 lb weight is supported in the center of a horizontal wire AB that is 12 ft long and under a tension of 20 lb. Derive the equation of motion and the natural frequency of vibration.**





FBD

Let l = length of wire = 12 ft

T = tension in wire = 20 lb

mg = weight at centre = 5 lb

x = displacement of mass

From FBD, horizontal force balance:

$$m\ddot{x} + 2T\frac{x}{l/2} = 0 \quad \rightarrow \quad \underline{m\ddot{x} + \frac{4T}{l}x = 0}$$

Natural frequency, $\omega_n = \sqrt{\frac{4T}{ml}} = \sqrt{\frac{4\times 20}{5/32.2 \times 12}} = 6.55 \text{ rad/s} = \underline{1.043 \text{ Hz}}$

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5. **An oscillating force f(t) of amplitude 0.02 lb and frequency 1 Hz acts on the mass in Q4. Determine the amplitude of vibration.**

With the applied force, the equation of motion becomes

$$m\ddot{x} + \frac{4T}{l}x = F\cos\omega_f t$$

The particular solution gives the steady state response.

Try $x = X\cos\omega_f t \quad \rightarrow \quad \left(-\omega_f^2 m + \frac{4T}{l}\right)X\cos\omega_f t = F\cos\omega_f t$

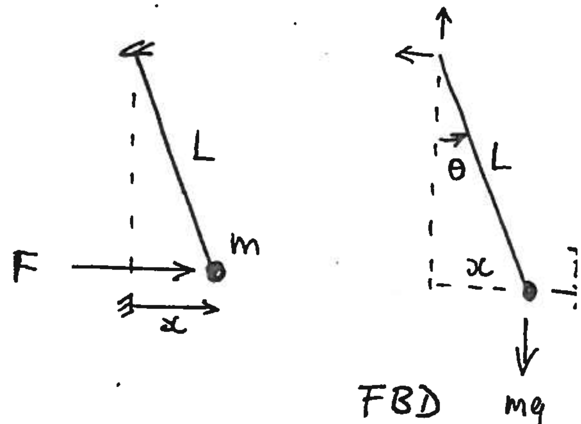
$$\rightarrow X = \frac{F}{\left(\frac{4T}{l} - \omega_f^2 m\right)} = \frac{F/\frac{4T}{l}}{\left(1 - \omega_f^2\frac{ml}{4T}\right)} = \frac{X_0}{\left(1 - \left(\frac{\omega_f}{\omega_n}\right)^2\right)}$$

where $\omega_n^2 = \frac{4T}{ml}$ and $X_0 = F/\frac{4T}{l}$ = static displacement

$$= \frac{0.02}{4\times 20/12} = 0.003$$

$$X = \frac{0.003}{1 - \left(\frac{1}{1.043}\right)^2} = 0.037 \text{ ft} = \underline{0.45 \text{ inches}}$$

4. A constant horizontal force F acts on a simple pendulum of mass m and length L. Determine the effective stiffness $k = F/x$, where x = displacement from the equilibrium position. (This is a statics question.) By analogy to a mass-spring system, determine the natural frequency of vibration.



Let x = horizontal displacement

θ = angular displacement

Take moments about top end of pendulum:

$$F L \cos\theta - mgx = 0 \quad \rightarrow \quad k = \frac{F}{x} = \frac{mg}{L\cos\theta} \simeq \frac{mg}{L} \quad \text{for small } \theta$$

For a mass-spring system $\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{\frac{mg}{L}}{m}} = \sqrt{\frac{g}{L}}$

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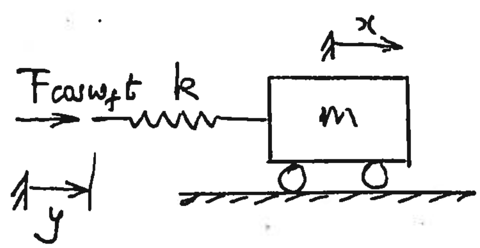
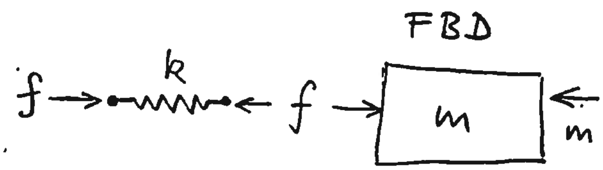
6. An oscillating force f(t) = F cos wft acts on the end of a spring of stiffness k that is attached to a mass m, as shown in the diagram. Derive a formula for the amplitude of vibration of the mass over a range of frequencies wf. Draw a graph illustrating your results and give physical interpretations of significant features.



Let x = displacement of the mass

y = displacement of the force



From FBD, force balances give:

$$m\ddot{x} = f \quad \text{and} \quad f = k(y - x)$$

Let $f = F\cos\omega_f t$, $\quad x = X\cos\omega_f t$.

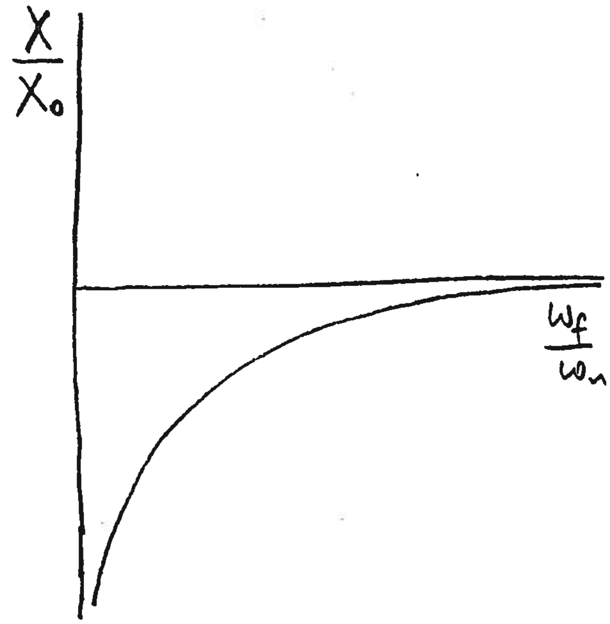
Substitute: $-\omega_f^2 m X \cos\omega_f t = F\cos\omega_f t$

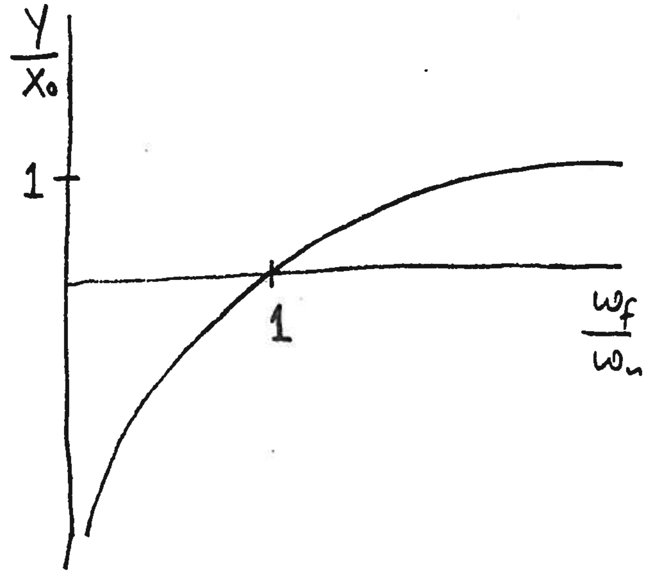
$$\rightarrow X = \frac{F}{-m\omega_f^2} = \frac{F/k}{-\frac{m}{k}\omega_f^2} = \frac{X_o}{-\left(\frac{\omega_f}{\omega_n}\right)^2} \quad \text{where} \quad X_o = F/k, \quad \omega_n^2 = k/m$$

Also $f = k(y - x) \quad \rightarrow \quad y = x + \frac{f}{k} = \left(X + \frac{F}{k}\right)\cos\omega_f t$

$$= Y\cos\omega_f t \quad \text{where} \quad Y = X + X_o$$

$$\rightarrow Y = X + X_o = \frac{\left(\frac{\omega_f}{\omega_n}\right)^2 - 1}{\left(\frac{\omega_f}{\omega_n}\right)^2}$$





The system is degenerate. For a constant force, the mass has a rigid-body motion, and can move sideways without limit. This is like pushing a train along an endless track. In vibration terms, this is equivalent to having a resonance at zero frequency. At high frequencies, the mass cannot respond fast enough to the force, and the vibration amplitude diminishes to zero.

If we look at the displacement amplitude of the force, Y, a similar infinite response occurs at zero frequency. Interestingly, the displacement amplitude of the force is zero at the undamped natural frequency $\omega_n = \sqrt{k/m}$. In this case, all the vibration is in the mass and spring.