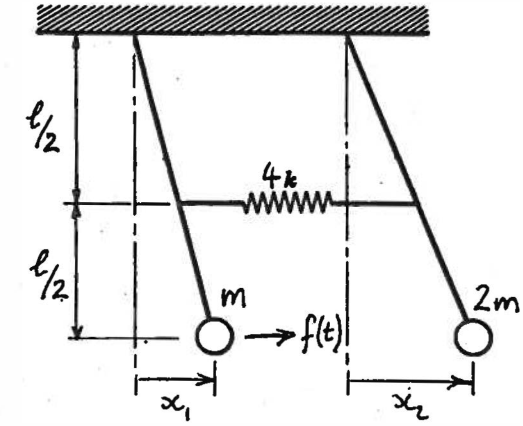
MECH 463 -- Homework 7



1. Two simple pendulums consist of very light, rigid rods of length ℓ, supporting masses m and 2m, as shown in the diagram. The rods are pinned at their upper ends and they have a spring of stiffness 4k pinned at their midpoints. The spring is unstretched when the pendulums are vertical. A horizontal force $f(t) = F \cos\omega_f t$ acts on the mass m. Assume small vibrations.

   (a) Draw a free body diagram of the system and derive the equations of motion in matrix form.

   (b) Determine the natural frequencies and mode shapes of the system. Give physical interpretations of your results.

   (c) Determine the response amplitudes X_1 and X_2 in terms of m, k, etc.

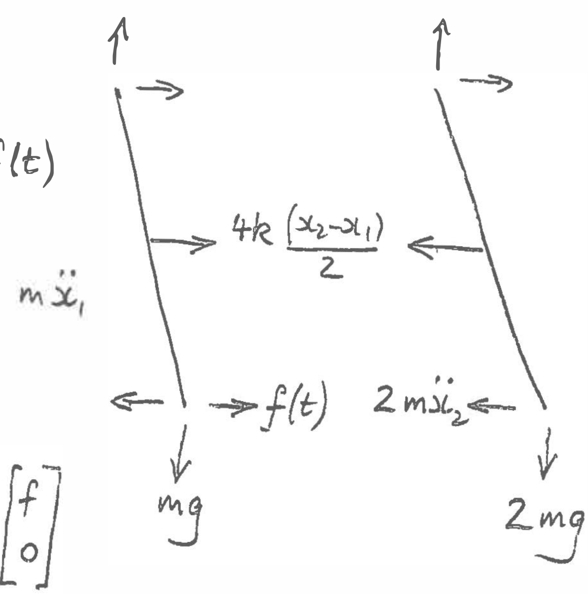
   (d) Determine the excitation frequency ω_f at which the response amplitude X_1 is zero. Determine the corresponding response amplitude X_2.

(a)

Take moments about the pivots:

$$m\ddot{x}_1 \ell + mgx_1 - k\ell(x_2 - x_1) = \ell f(t)$$

$$2m\ddot{x}_2 \ell + 2mgx + k\ell(x_2 - x_1) = 0$$



and put in matrix form

$$\underbrace{\begin{bmatrix} m & 0 \\ 0 & 2m \end{bmatrix}}_{\underset{\sim}{M}} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \underbrace{\begin{bmatrix} \frac{mg}{\ell} + k & -k \\ -k & \frac{2mg}{\ell} + k \end{bmatrix}}_{\underset{\sim}{K}} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} f \\ 0 \end{bmatrix}$$

(b)

For the natural frequencies and mode shapes, it is not necessary to consider the excitation force.

Try solution $\underline{x} = \underline{X}\cos(\omega t + \phi)$ $\rightarrow$ $(\underset{\sim}{K} - \omega^2 \underset{\sim}{M})\underline{X}\cos(\omega t + \phi) = \underline{0}$

For a non-trivial solution valid for all t, $\det(\underset{\sim}{K} - \omega^2 \underset{\sim}{M}) = 0$

$$\rightarrow \left(\frac{mg}{\ell} + k - m\omega^2\right)\left(\frac{2mg}{\ell} + k - 2m\omega^2\right) - k^2 = 0$$

$$= 2m^2\omega^4 - m\left(\frac{4mg}{\ell} + 3k\right)\omega^2 + \frac{2m^2g^2}{\ell^2} + \frac{3mgk}{\ell} + \cancel{k^2} - \cancel{k^2} = 0$$

Divide by $4m^2$

$$\rightarrow \quad \omega^4 - \left(2g/\ell + \tfrac{3}{2}k/m\right)\omega^2 + g^2/\ell^2 + \frac{3gk}{2\ell m} = 0$$

$$\rightarrow \omega^2 = \left[\left(2g/\ell + \tfrac{3}{2}k/m\right) \pm \sqrt{\cancel{\frac{4g^2}{\ell^2}} + \cancel{\frac{6gk}{\ell m}} + \frac{9k^2}{4m^2} - \cancel{\frac{4g^2}{\ell^2}} - \cancel{\frac{6gk}{\ell m}}}\;\right] \Big/ 2$$

$$\omega^2 = g/\ell + \tfrac{3}{4}k/m \pm \tfrac{3}{4}k/m \quad = \underline{g/\ell \text{ and } g/\ell + \tfrac{3}{2}\tfrac{k}{m}}$$

For mode shape, put $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = C\begin{bmatrix} 1 \\ u \end{bmatrix}\cos(\omega t + \phi)$, $\quad \underset{\sim}{x} = C\underset{\sim}{u}\cos(\omega t + \phi)$

$$\rightarrow (\underset{\sim}{K} - \omega^2\underset{\sim}{M})\, C\underline{u}\cos(\omega t + \phi) = \underline{0} \qquad \rightarrow (\underset{\sim}{K} - \omega^2\underset{\sim}{M})\,\underline{u} = \underline{0}$$

for non-trivial solution, valid for all t.

$$\rightarrow \begin{bmatrix} \frac{mg}{\ell} + k - m\omega^2 & -k \\ -k & \frac{2mg}{\ell} + k - 2m\omega^2 \end{bmatrix}\begin{bmatrix} 1 \\ u \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

First equation $\rightarrow \frac{mg}{\ell} + k - m\omega^2 - ku = 0$

$$\rightarrow u = \frac{mg}{k\ell} + 1 - \frac{m}{k}\omega^2 \quad = \underline{1 \quad \text{when } \omega^2 = g/\ell}$$

$$= \underline{-\tfrac{1}{2} \quad \text{when } \omega^2 = g/\ell + \tfrac{3}{2}\tfrac{k}{m}}$$

The first mode is a simple pendulum mode with both pendulums moving in phase and with the same amplitude. Hence, the spring is not active and k is not significant. The second mode involves the spring, and is a vibration about the centre of mass.

For the response to excitation $f(t) = F\cos\omega_f t$, try solution $\underline{x} = \underline{X}\cos\omega_f t \rightarrow (\underset{\sim}{K} - \omega_f^2 \underset{\sim}{M})\underline{X}\cos\omega_f t = \underline{F}\cos\omega_f t.$

For a solution valid for all t, $\rightarrow (\underset{\sim}{K} - \omega_f^2 \underset{\sim}{M})\underline{X} = \underline{F}$

(c)

$$\rightarrow \begin{bmatrix} \frac{mg}{l} + k - m\omega_f^2 & -k \\ -k & \frac{2mg}{l} + k - 2m\omega_f^2 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} F \\ 0 \end{bmatrix}$$

By Cramer's rule:

$$X_1 = \frac{F\left(\frac{2mg}{l} + k - 2m\omega_f^2\right)}{\left(\frac{mg}{l} + k - m\omega_f^2\right)\left(\frac{2mg}{l} + k - 2m\omega_f^2\right) - k^2}$$

$$X_2 = \frac{Fk}{\left(\frac{mg}{l} + k - m\omega_f^2\right)\left(\frac{2mg}{l} + k - 2m\omega_f^2\right) - k^2}$$

(d)

X_1 is zero when $\left(\frac{2mg}{l} + k - 2m\omega_f^2\right) = 0$

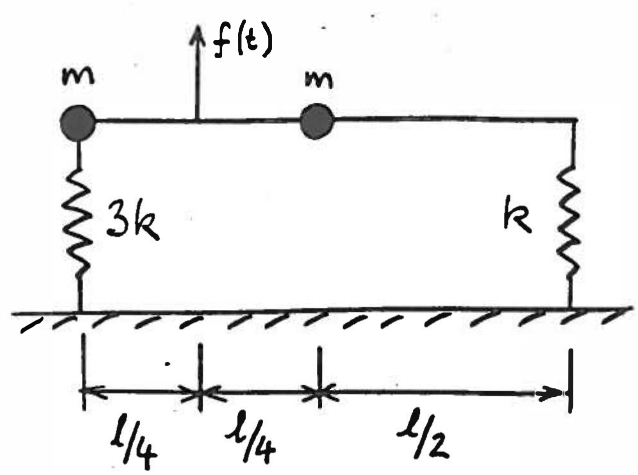
$$\rightarrow \underline{\omega_f^2 = g/l + k/2m}$$

Substitute in X_2 expression

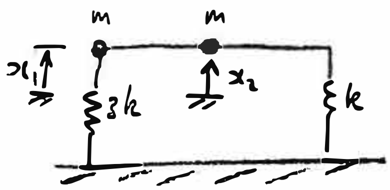
$$X_2 = \frac{Fk}{\left(\cancel{\frac{mg}{l}} + k - \cancel{\frac{mg}{l}} - \frac{k}{2}\right)\left(\cancel{\frac{2mg}{l}} + \cancel{k} - \cancel{\frac{2mg}{l}} - \cancel{k}\right) - k^2}$$

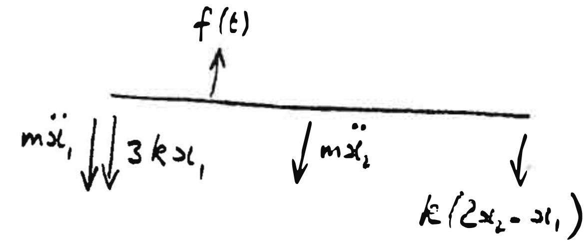
$$\underline{X_2 = -F/k}$$

2. The diagram shows a very idealized 2-DOF model of an automobile. The body of the vehicle is represented as a beam of length ℓ whose mass m is concentrated at its centre. The engine is represented as a concentrated mass m at one end of the beam. The front and rear suspensions are represented as springs of stiffness 3k and k respectively. The gearbox of the vehicle is damaged, and gives rise to a harmonic force $f(t) = F\cos\omega_f t$, acting at a point halfway between the two concentrated masses. Find the vibration amplitude felt by the driver (assumed to be sitting at the centre of the beam). Give a physical explanation of any interesting features that you find.



Choose mass-based coords. x_1 and x_2





$$\Sigma M_2 = 0 \rightarrow \ell/2\left(m\ddot{x}_1 + 3kx_1 - \tfrac{1}{2}f(t) - k(2x_2 - x_1)\right) = 0$$

$$\Sigma M_1 = 0 \rightarrow \ell/2\left(m\ddot{x}_2 - \tfrac{1}{2}f(t) + 2\cdot k(2x_2 - x_1)\right) = 0$$

where $f(t) = F\cos\omega_f t$

In matrix form: $(\div\, \ell/2)$

$$\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} 4k & -2k \\ -2k & 4k \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \tfrac{1}{2}F \\ \tfrac{1}{2}F \end{bmatrix}\cos\omega_f t$$

For steady-state response, try solution $\underline{x} = \underline{X}\cos\omega_f t$

$$\rightarrow \left(-\omega_f^2 \underset{\sim}{M} + \underset{\sim}{K}\right)\underline{X}\cos\omega_f t = \underline{f}\cos\omega_f t$$

This is true for all t

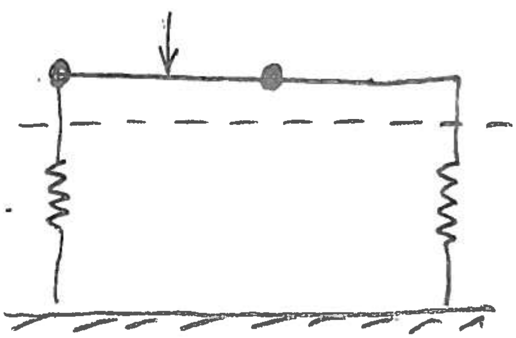
$$\rightarrow \begin{bmatrix} 4k - m\omega_f^2 & -2k \\ -2k & 4k - m\omega_f^2 \end{bmatrix}\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} \tfrac{1}{2}F \\ \tfrac{1}{2}F \end{bmatrix}$$

Solving by Cramer's rule

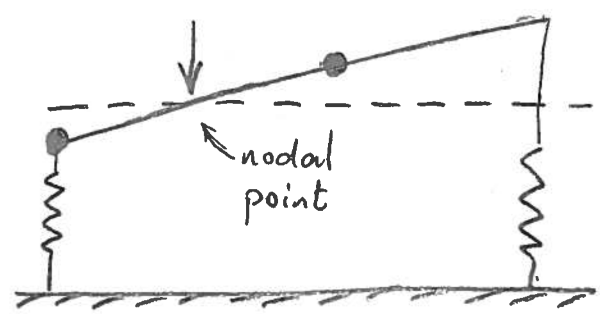
$$\rightarrow X_2 = \frac{\tfrac{1}{2}F\left(4k - m\omega_f^2 + 2k\right)}{\left(4k - m\omega_f^2\right)^2 - (2k)^2} = \frac{\tfrac{1}{2}F\left(6k - m\omega_f^2\right)}{\left(6k - m\omega_f^2\right)\left(2k - m\omega_f^2\right)}$$

Vibration felt by driver $= X_2 = \dfrac{F}{2\left(2k - m\omega_f^2\right)}$

There appears to be only one resonant frequency, $\omega_f^2 = \frac{2k}{m}$, at which $X_2 \to \infty$. (denominator = 0). Since the system has 2-DOF, we would normally expect two resonant frequencies. (The second should be at $\omega_f^2 = \frac{6k}{m}$). The second resonance is not excited because the force $f(t)$ is applied at the nodal point of the second mode.



Mode 1



Mode 2