1.d To check BIBO stability, assume $x(0) = 0$ and find $G(s) = Y(s)/U(s)$

From SS, $G(s) = C(sI - A)^{-1}B + D$

$$A = \begin{pmatrix} 0 & 1 & 0 \\ -1 & -2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$B = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$C = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix}$$

$$D = (0)$$

Therefore,

$$G(s) = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix} \left[\begin{pmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & s \end{pmatrix} - \begin{pmatrix} 0 & 1 & 0 \\ -1 & -2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right]^{-1} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} s & -1 & 0 \\ 1 & s+2 & 0 \\ 0 & 0 & s \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

MATLAB

$$= \begin{pmatrix} 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} \frac{s+2}{s^2+2s+1} & \frac{1}{s^2+2s+1} & 0 \\ \frac{-1}{s^2+2s+1} & \frac{s}{s^2+2s+1} & 0 \\ 0 & 0 & \frac{1}{s} \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{s^2+2s+1} \\ \frac{s}{s^2+2s+1} \\ 0 \end{pmatrix}$$

$$= \frac{s+1}{s^2+2s+1} \longrightarrow \text{poles of } G(s) \text{ are } -1 \text{ \& } -1 \rightarrow \text{BIBO Stable}$$

To check internal stability, assume $u(t) = 0$ (no input)

1.2 $A = \begin{bmatrix} 0 & 1 & 0 \\ -1 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$\det(\lambda I - A) = 0$

$\det \begin{bmatrix} \lambda & -1 & 0 \\ 2 & \lambda+2 & 0 \\ 0 & 0 & \lambda \end{bmatrix} = 0$

$\lambda(\lambda+2)(\lambda) - (-1)2\lambda = 0$

$\lambda^2(\lambda+2) \quad + 2\lambda = 0$

$\lambda^2 + 2\lambda + 2 = 0 \quad \rightarrow \lambda_1 = 0$

$\lambda = \dfrac{-2 \pm \sqrt{4 - 4(2)}}{2}$

$\lambda_{2,3} = -1 \pm i$

$Re[\lambda_2]$ & $Re[\lambda_3]$ are both < 0, so we only check λ_1

$\left.\begin{array}{l} \text{rank}[\lambda_1 I - A] = \text{rank}[-A] = 2 \\ n - m_1 = 3 - 1 = 2 \end{array}\right\} 2 = 2$

So, This system is Marginally stable for $x(t) \neq 0$

1.3 To check controllability, check $\text{rank } C = n$
where $C = [B, AB, A^2B]$

$$A = \begin{bmatrix} 0 & 1 & 0 \\ -1 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$AB = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} \qquad A^2B = \begin{bmatrix} 0 & 1 & 0 \\ -1 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ 3 \\ 0 \end{bmatrix}$$

$$C = [B, AB, A^2B] = \begin{bmatrix} 0 & 1 & -2 \\ 1 & -2 & 3 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \text{rank } C = 2$$

$$n = 3 \quad \rightarrow \quad n = 3 > \text{rank}(C) = 2$$

Therefore, not controllable.

$$\text{Im}(C) = \text{span}\left\{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}\right\} \rightarrow T_C = \begin{bmatrix} 0 & 1 \\ 1 & -2 \\ 0 & 0 \end{bmatrix}$$

$$T_{\bar{C}} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \rightarrow T^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

1.4 To check observability, check rank $O = n$

where $O = \begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix}$

$C = [1 \quad 1 \quad 1]$

$A = \begin{bmatrix} 0 & 1 & 0 \\ -1 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$CA = (-1 \quad -1 \quad 0)$

$CA^2 = (-1 \quad -1 \quad 0)\begin{pmatrix} 0 & 1 & 0 \\ -1 & -2 & 0 \\ 0 & 0 & 0 \end{pmatrix} = [1 \quad 1 \quad 0]$

$O = \begin{bmatrix} 1 & 1 & 1 \\ -1 & -1 & 0 \\ 1 & 1 & 0 \end{bmatrix} \rightarrow \text{rank}(O) = 2 < n = 3$

Therefore, system not obverveable.

Similar to controllability, we can use kernel space to get controllable subspace of this system.

$\text{Ker}(O) = \text{span}\left\{\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}\right\} \Rightarrow T_{\bar{o}} = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$

$\rightarrow T_o = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \rightarrow T^{-1} = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$

1.5 $\mathcal{C} = \begin{bmatrix} 0 & 1 & -2 \\ 1 & -2 & 3 \\ 0 & 0 & 0 \end{bmatrix}$ $\quad \mathcal{O} = \begin{bmatrix} 1 & 1 & 1 \\ -1 & -1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$

$\downarrow \qquad\qquad \downarrow$

$Im(\mathcal{C}) = span\left\{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}\right\}$ $\quad Ker(\mathcal{O}) = span\left\{\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}\right\}$

$T_{c\bar{o}} = Im(c) \cap ker(o)$

$$0 = a\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + b\begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} - c\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

$$0 = \begin{bmatrix} 0 & 1 & +1 \\ 1 & -2 & -1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = t\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, t \in \mathbb{R}$$

$$\left.\begin{array}{l} \begin{bmatrix} 0 & 1 \\ 1 & -2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix} \\ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}\begin{bmatrix} -1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \end{array}\right\} T_{c\bar{o}} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

$T_{co} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ pick simplest $\quad T_{\bar{c}\bar{o}}$ = none, since ker(o) is a line

$T_{\bar{c}o} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad \rightarrow T^{-1} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = T$

$$z = Tx \rightarrow \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} [x]$$

$$A_{new} = TAT^{-1} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ -1 & -2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} -1 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$B_{new} = TB = \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

$$C_{new} = CT^{-1} = (1 \ 1 \ 1) \begin{pmatrix} 1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = [1 \ 0 \ 1]$$

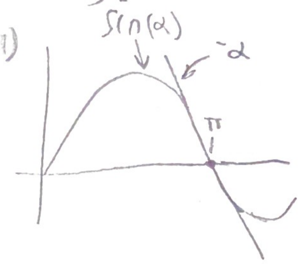
$$\dot{z} = \begin{bmatrix} -1 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} z + \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} u$$

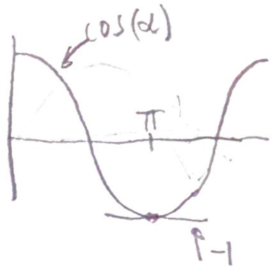
$$y = [1 \ 0 \ 1] z$$

2.1 → For inv pedulum, we flip sign of all terms with g (gravity change direction).

→ since we're now linearizing around $\alpha = \pi$ instead of $\alpha = 0$:

$$\sin\alpha \approx \sin(\pi) + \cos(\pi) \times (\alpha - \pi) \approx \pi - \alpha$$
$$\cos\alpha \approx -1$$
$$\sin^2\alpha \approx 0$$





→ From these changes, we get

$$J_r\ddot{\theta} + -m_p r l \ddot{\alpha} = T - b_r\dot{\theta}$$
$$-m_p r l \ddot{\theta} + J_p \ddot{\alpha} = -b_p\dot{\alpha} - m_p g l(\pi - \alpha)$$

→ we notice that only r terms have flipped signs, so we just flip signs of terms with r in final matrix to get:

→ we notice that $(\pi - \alpha)$ takes place of α, so we can redefine x_3 as $\pi - \alpha$

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & \frac{-J_p b_r}{J_t} & \frac{(m_p l)^2 r g}{J_t} & \frac{-m_p r l b_p}{J_t} \\ 0 & 0 & 0 & -1 \\ 0 & \frac{-m_p r l b_r}{J_t} & \frac{-J_r m_p g l}{J_t} & \frac{-J_r b_p}{J_t} \end{bmatrix} \qquad B = \frac{1}{J_t}\begin{bmatrix} 0 \\ J_p \\ 0 \\ m_p r l \end{bmatrix}$$

added negative because $x_3 = \pi - \alpha$, $\dot{x}_3 = x_4 = -\dot{\alpha}$

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

where $x = \begin{bmatrix} \theta \\ \dot{\theta} \\ \pi - \alpha \\ \dot{\alpha} \end{bmatrix}$ $\quad u = [T] \quad y = \begin{bmatrix} \theta \\ \alpha \end{bmatrix}$

To plot α, take $y_2 = \pi - \alpha \rightarrow \alpha = \pi - y_2$