2.4. Damped SDOF Response–1

MECH 463: Mechanical Vibrations

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Suggested Readings:

1. Topic 2.4 from notes package for detailed derivations.
2. Sections 2.6 and 3.4 from the course textbook.

Learning Objectives

1. Determine free vibration response of a viscously damped SDOF system.
2. Recognize the trade-offs in transient response design problems.
3. Deduce design guidelines to mitigate vibration response.

2.13 Equations of Motion (NP 2.13)

Thus for the spring-mass system with a viscous damper, we can obtain the following equations of motion:

Fill in the class

$$m\ddot{x} + c\dot{x} + kx = f \tag{1}$$

2.14 Damped Vibration Response (NP 2.14) — # 1

$$m\ddot{x}_h + c\dot{x}_h + kx_h = 0 \quad \text{Homogeneous response/Free vibration} \quad (2a)$$
$$m\ddot{x}_p + c\dot{x}_p + kx_p = f \quad \text{Particular solution/Forced vibration.} \quad (2b)$$

Adding the above two equations we have the TOTAL response, from the principle of superposition

$$m\ddot{x} + c\dot{x} + kx = f, \quad x = x_h + x_p \quad \text{TOTAL response} \quad (3)$$

It is required to specify the initial conditions on the TOTAL response. They can be initial velocity, or initial displacement:

$$x(0) = x_0; \quad \dot{x}(0) = \dot{x}_0 \qquad \text{INITIAL conditions } \underline{\textbf{apply on the TOTAL solution}}. \quad (4)$$

2.15 Free Vibration Response (NP 2.15, T 2.6) — # 1

$$m\ddot{x}_h + c\dot{x}_h + kx_h = 0 \quad \text{Homogeneous response/Free vibration} \tag{5}$$

To solve the above differential equation we assume a solution of the form $x_h = Xe^{st}$ where X and s are to be determined. Let us insert this *trial* solution into the equation of motion Eq.(5)

$$\begin{aligned} & mXs^2e^{st} + csXe^{st} + KXe^{st} = 0 \\ & \Rightarrow \left[ms^2 + cs + k\right] e^{st} = 0 \\ & \Rightarrow \left[ms^2 + cs + k\right] = 0, \quad \because e^{st} \neq 0 \end{aligned}$$

to form the *auxiliary* or *characteristic* equation:

$$ms^2 + cs + k = 0 \tag{6}$$

The two roots of of the above *quadratic* equation are

$$s_{1,2} = \frac{-c \pm \sqrt{c^2 - 4km}}{2m} \tag{7}$$

$$x_h(t) = C_1e^{s_1t} + C_2e^{s_2t} \tag{8}$$

2.15 Free Vibration Response (NP 2.15, T 2.6) — # 2

2.15.1 Underdamping $\zeta < 1$ or $c < c_c$ (See the notes for detailed derivations)

$$x_h(t) = e^{-\zeta\omega_n t}\left[A_1 \cos\omega_d t + A_2 \sin\omega_d t\right] = e^{-\zeta\omega_n t} A\cos(\omega_d t - \phi_0) \quad (9)$$

Q: What can you say about the influence of different parameters on the underdamped free vibration response based on the above?

Fill in the class

2.15 Free Vibration Response (NP 2.15, T 2.6) — # 3

2.15 Free Vibration Response (NP 2.15, T 2.6) — # 4

See notes for derivation.

$$
\begin{aligned}
x = & x_h = e^{-\zeta\omega_n t} A \cos(\omega_d t - \phi_0) \\
\tan\phi_0 = & \frac{\zeta\omega_n x_0 + \dot{x}_0}{\omega_d x_0}; A = \sqrt{x_0^2 + \left[\frac{\zeta\omega_n x_0 + \dot{x}_0}{\omega_d}\right]^2} \\
x = & x_h = e^{-\zeta\omega_n t}\left[x_0 \cos\omega_d t + \frac{\zeta\omega_n x_0 + \dot{x}_0}{\omega_d}\sin\omega_d t\right] \\
\omega_d = & \omega_n\sqrt{1-\zeta^2} \text{ (Undamped natural frequency)}
\end{aligned}
\tag{10}
$$

The following observations are worth making about the free vibration:

1. **Free vibration takes place at the system's damped natural frequency, slightly below the undamped natural frequency**, irrespective of the initial conditions.

2.15 Free Vibration Response (NP 2.15, T 2.6) — # 5

2. Undamped natural frequency depends only on the properties of the system: mass, stiffness, and damping. It increases with an increase in the stiffness or a *decrease* in the mass or damping.

2.15 Free Vibration Response (NP 2.15, T 2.6) — # 6

2.15.2 Critical Damping $\zeta = 1$ or $c = c_c = 2\sqrt{km}$

(See the notes for detailed derivations)

We can let the damping approach the value of 1 in Eq.(10). In this case, $\omega_d = \omega_n\sqrt{1-\zeta^2} \to 0$ and $\cos\omega_d t \to 1$, $\sin\omega_d t \to \omega_d t$. Using these in Eq.(10) gives the free vibration response of a critically damped system as follows.

$$\boxed{x = x_h = e^{-\omega_n t}\left[x_0 + \left(\omega_n x_0 + \dot{x}_0\right) t\right] \qquad (11)}$$

2.15 Free Vibration Response (NP 2.15, T 2.6) — # 7

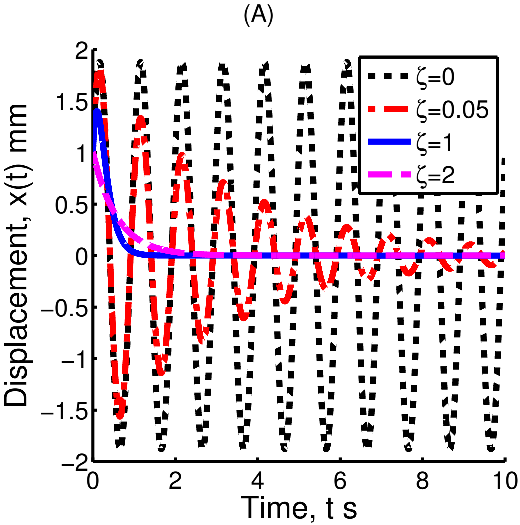
2.15.3 Overdamping $\zeta > 1$ or $c > c_c$

(See the notes for detailed derivations)

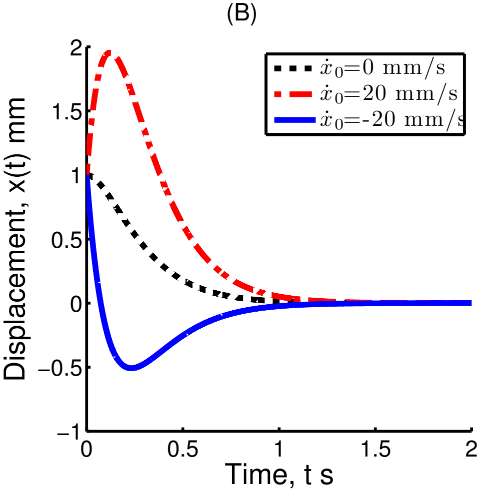
$$\boxed{x = x_h = e^{-\zeta\omega_n t}\left[\frac{\omega_d x_o + \zeta\omega_n x_0 + \dot{x}_0}{2\omega_d}e^{\omega_d t} + \frac{\omega_d x_o - \zeta\omega_n x_0 - \dot{x}_0}{2\omega_d}e^{-\omega_d t}\right]. \quad (12)}$$

2.15 Free Vibration Response (NP 2.15, T 2.6) — # 8

2.15.4 A Comparison of the Three Cases



2.15 Free Vibration Response (NP 2.15, T 2.6) — # 9



2.15 Free Vibration Response (NP 2.15, T 2.6) — # 10

Q: What features do you observe in the responses of the damped systems sketched above?

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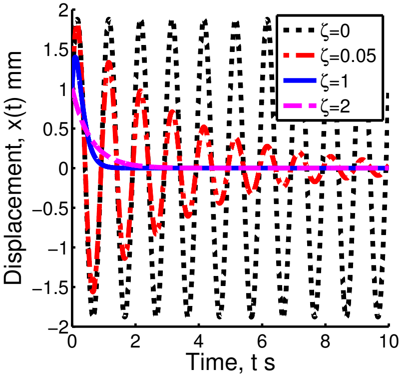
2.15 Free Vibration Response (NP 2.15, T 2.6) — # 11

2.15.5 Logarithmic Decrement



$$\delta = \frac{1}{N} \ln\left(\frac{x_1}{x_{1+N}}\right) = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}} \approx 2\pi\zeta = \frac{2\pi c}{2m\omega_n}. \tag{13}$$

Example 15 — # 1

p. 117 in NP

Example 15 : From the consideration of work performed in harmonic motion show that viscous damper dissipates energy over one cycle while an elastic spring does not. Sketch the rotating vector representation of the underdamped free vibrations.

Fill in the class

Example 15 — # 2

Example 15 — # 3

Example 15 — # 4

Example 16 — # 1

p. 119 in NP

Example 16 : A shock absorber is to be designed to limit its overshoot to 15% of its initial displacement when released. Find the damping ratio ζ_0 required. What will be the overshoot if ζ is made equal to (a) $\frac{3}{4}\zeta_0$, and (b) $\frac{5}{4}\zeta_0$.

Fill in the class

Example 16 — # 2

Example 16 — # 3

Example 16 — # 4

Example 17 — # 1

p. 121 in NP

Example 17 : A railroad car of mass 2000 kg travelling at a velocity of $v = 10$ m/s is stopped at the end of the tracks by a spring-damper system, as shown below. If the stiffness of the spring is $\frac{k}{2} = 40$ N/mm and the damping constant is $c = 20$ N-s/mm, determine (a) the maximum displacement of the car after engaging the springs and damper and (b) the time taken to reach the maximum displacement.

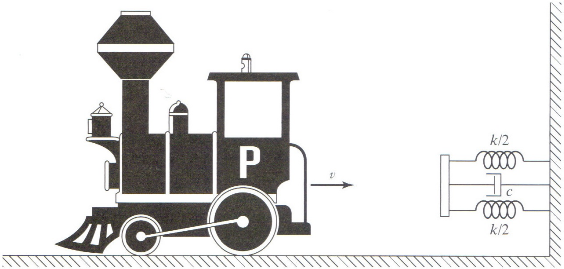


Figure : Figure for example 17.

Example 17 — # 2

Fill in the class

Example 17 — # 3

Example 17 — # 4

Example 17 — # 5

Summary of Free Damped Response

1. There are three categories of damped systems: underdamped, overdamped, and critically damped. Among these three, most of the mechanical systems belong to the underdamped category.
2. Critical damping is engineered when a quick return to the initial configuration in the shortest possible time is desired, such as in recoil mechanisms and shock absorbers.
3. With the passage of time over and critically damped systems gradually turn into underdamped systems.
4. Overdamping does not allow any oscillation but the return to equilibrium is a slow and creeping process.
5. Underdamped response is given by: $x = x_h = e^{-\zeta\omega_n t} A\cos(\omega_d t - \phi_0)$
   $\tan\phi_0 = \frac{\zeta\omega_n x_0 + \dot{x}_0}{\omega_d x_0}$; $A = \sqrt{x_0^2 + \left[\frac{\zeta\omega_n x_0 + \dot{x}_0}{\omega_d}\right]^2}$
   $x = x_h = e^{-\zeta\omega_n t}\left[x_0 \cos\omega_d t + \frac{\zeta\omega_n x_0 + \dot{x}_0}{\omega_d} \sin\omega_d t\right]$
   $\omega_d = \omega_n\sqrt{1-\zeta^2}$ (Undamped natural frequency)
6. Logarithmic decrement is a time-domain measure of damping. It is given by:$\delta = \frac{1}{N}\ln\left(\frac{x_1}{x_{1+N}}\right) = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}} \approx 2\pi\zeta = \frac{2\pi c}{2m\omega_n}$.

Design Guidelines