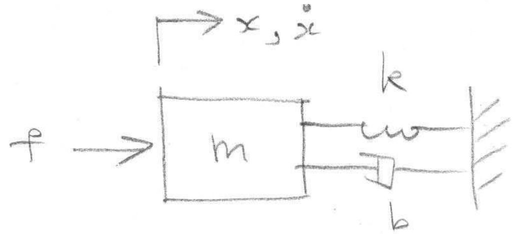
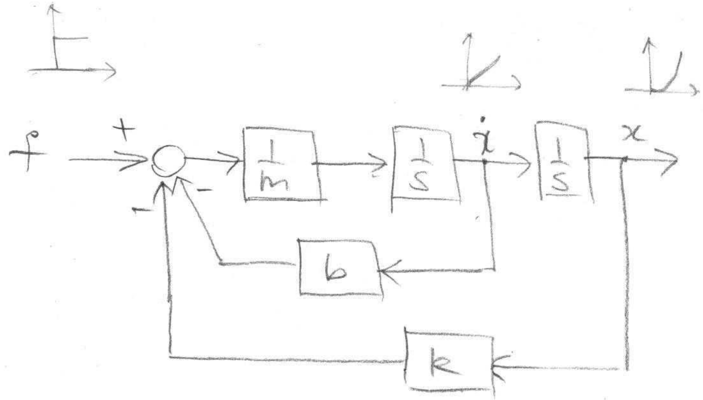
< 2nd-order Systems Review >

- 1 DoF Resonator





- Momentum principle : $m\ddot{x} = \Sigma f$

$$m\ddot{x} = f - b\dot{x} - kx$$

$$\therefore m\ddot{x} + b\dot{x} + kx = f$$

- Take the Laplace transform :

$$\underbrace{(ms^2 + bs + k)}_{\text{Stiffness}} X = F \rightarrow \frac{X}{F} = \underbrace{\frac{1}{ms^2+bs+k}}_{\text{Compliance}}$$

$$\underbrace{\left(ms + b + \frac{k}{s}\right)}_{\text{Impedance}} \dot{X} = F \rightarrow \frac{\dot{X}}{F} = \underbrace{\frac{1}{ms + b + \frac{k}{s}}}_{\text{Admittance}}$$

- Let $G_x(s) = \frac{1}{ms^2+bs+k} = \frac{1}{k} \frac{1}{\left(\frac{s}{\omega_n}\right)^2 + 2\zeta\frac{s}{\omega_n} + 1}$

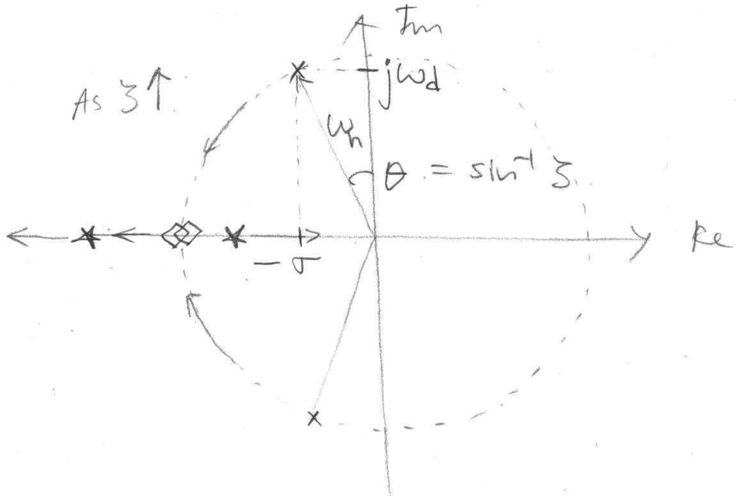
$$= \frac{1}{m} \frac{1}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

- Let $G_v(s) = \frac{s}{ms^2+bs+k} = \frac{1}{\sqrt{mk}} \frac{s/\omega_n}{\left(\frac{s}{\omega_n}\right)^2 + 2\zeta\frac{s}{\omega_n} + 1}$

$$= \frac{1}{m} \frac{s}{s^2 + 2\zeta\omega_n s + 1}$$

- pole – zero map of $G_X(s)$

$$p_{1,2} = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$$
$$= -\sigma \pm j\omega_d$$



× : $\zeta < 1$ under damped

◇ : $\zeta = 1$ critically "

✱ : $\zeta > 1$ over "

- ω_n : natural frequency ($\omega_n = \sqrt{\frac{k}{m}}$)
- ζ : damping ratio. ($\zeta = \frac{b}{2\sqrt{mk}}$)
- σ : decay rate ($\sigma = \zeta\omega_n$)
- ω_d : damped natural freq ($\omega_d = \omega_n\sqrt{1-\zeta^2}$)

– Root Locus with respect to ζ.

, Find the roots of $\sigma(s) = s^2 + 2\zeta\omega_n s + \omega_n^2$ when ζ varies from 0 to ∞.

, Re-write it as $\sigma(s) = \underbrace{(s^2 + \omega_n^2)}_{a(s)} + \zeta\underbrace{(2\omega_n s)}_{b(s)}$

As $\zeta \to 0$, roots of $\sigma(s)$ $\to$ roots of $a(s)$

As $\zeta \to \infty$, roots of $\sigma(s)$ $\to$ roots of $b(s)$

- pole-zero map of $G_V(s)$

: the same as that of $G_X(s)$, but one zero at the origin.

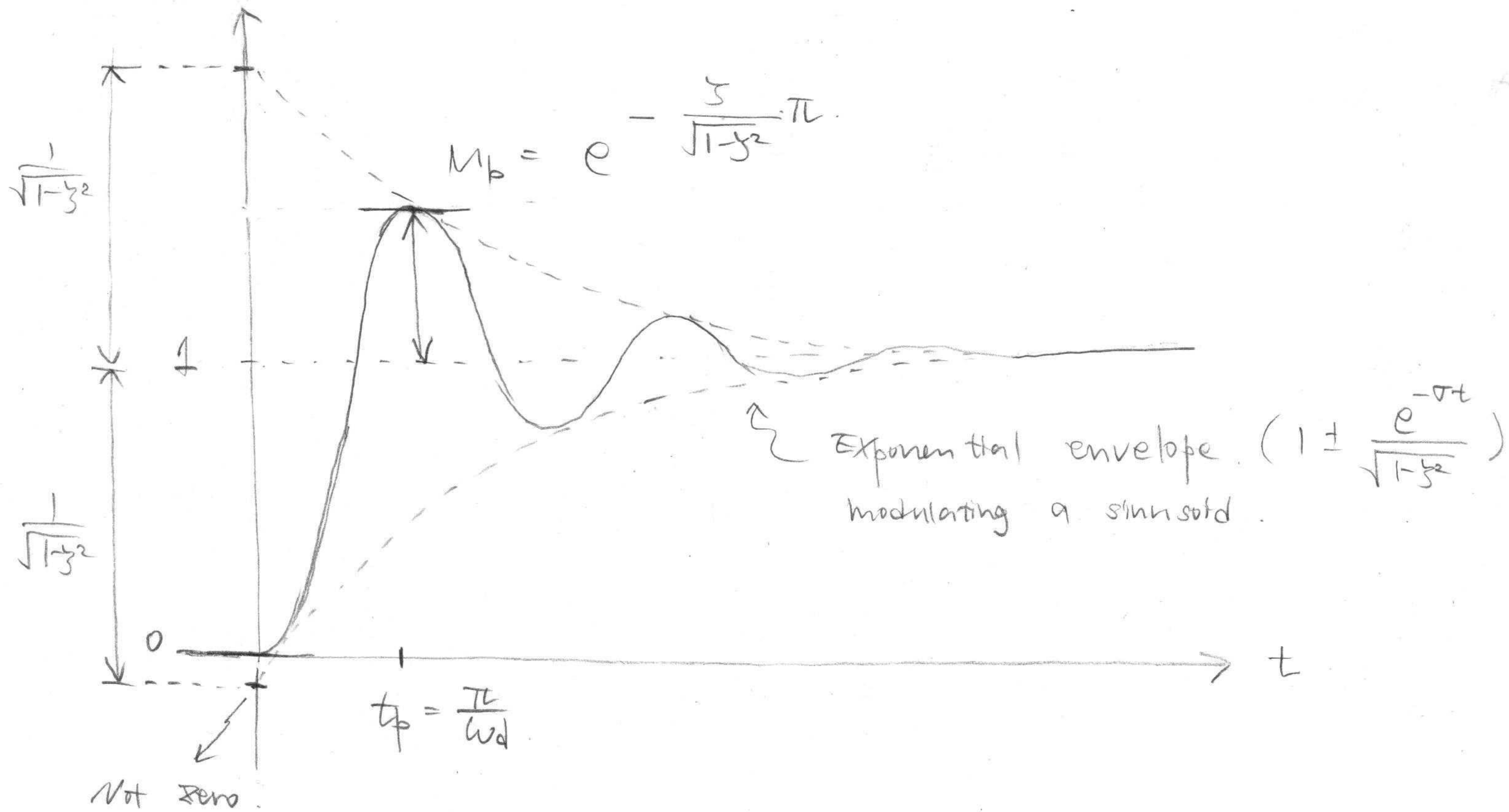
- Step Response of $G_X(s)$

$$x(t) = \left[1 - e^{-\sigma t} \left(\cos \omega_d t + \frac{\sigma}{\omega_d} \sin \omega_d t \right) \right] \frac{u(t)}{k}$$

$$= \left[1 - e^{-\sigma t} M \cdot \cos(\omega_d t + \phi) \right] \frac{u(t)}{k}$$

$$\begin{cases} M = \sqrt{1 + \frac{\sigma^2}{\omega_d^2}} = \frac{1}{\sqrt{1-\zeta^2}} \\ \phi = \tan^{-1}\left(\frac{\sigma}{\omega_d}\right) = \tan^{-1}\left(\frac{\zeta}{\sqrt{1-\zeta^2}}\right) \end{cases}$$

$\hat{x}(t) = k\,x(t)$ "Normalized" step response.



$$\dot{x}(t)\Big|_{t=t_p} = 0 \rightarrow \underset{\sim\sim\sim}{t_p = \frac{\pi}{\omega_d}} \rightarrow x(t_p) = 1 + \underbrace{e^{-\frac{\zeta}{\sqrt{1-\zeta^2}}\pi}}_{M_p}$$

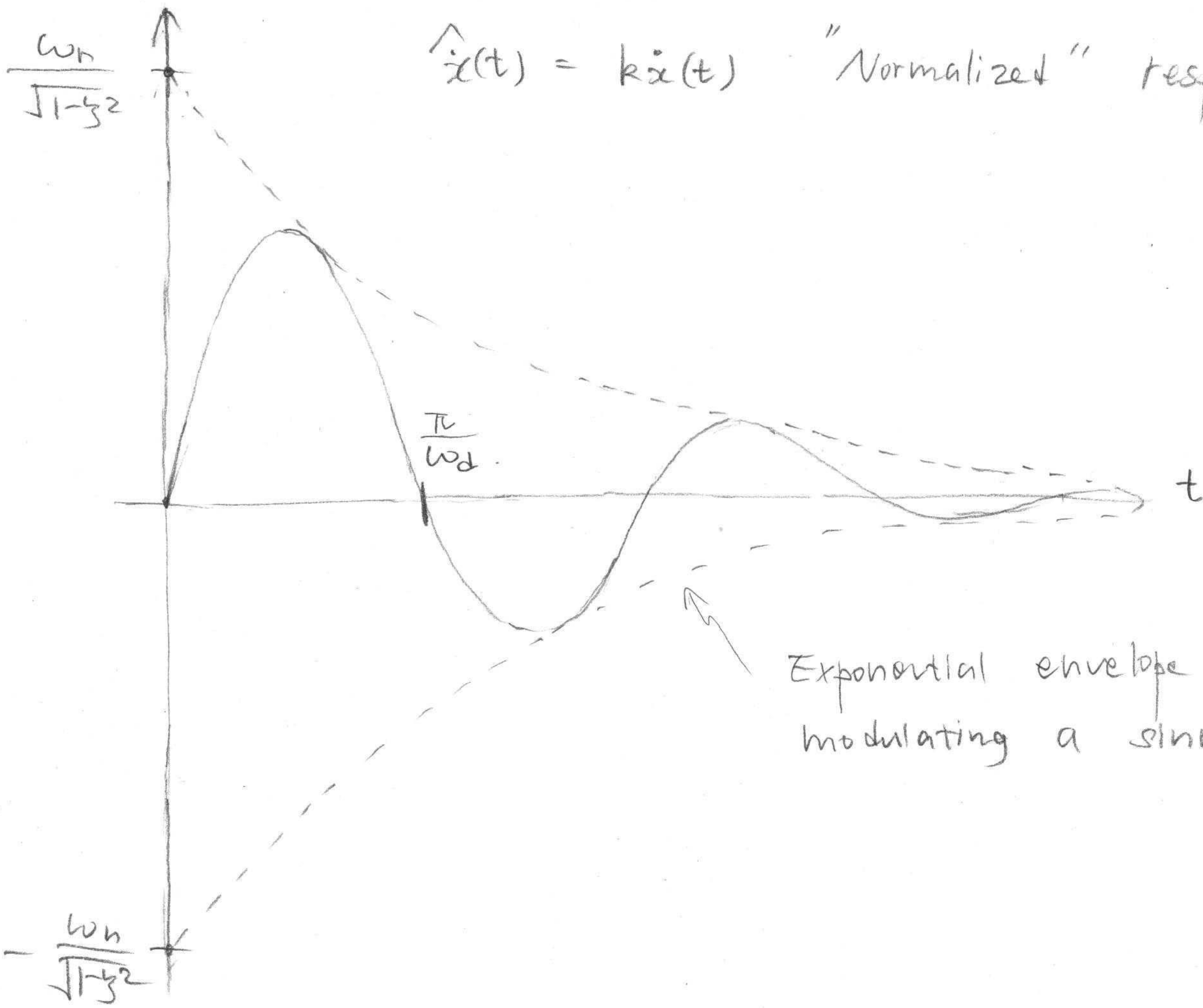
- Step Response of $G_V(s)$

  (= Impulse Response of $G_X(s)$)

$$\dot{x}(t) = \left[e^{-\sigma t} \cdot \left(\omega_d + \frac{\sigma^2}{\omega_d} \right) \cdot \sin \omega_d t \right] \cdot \frac{u(t)}{k}$$

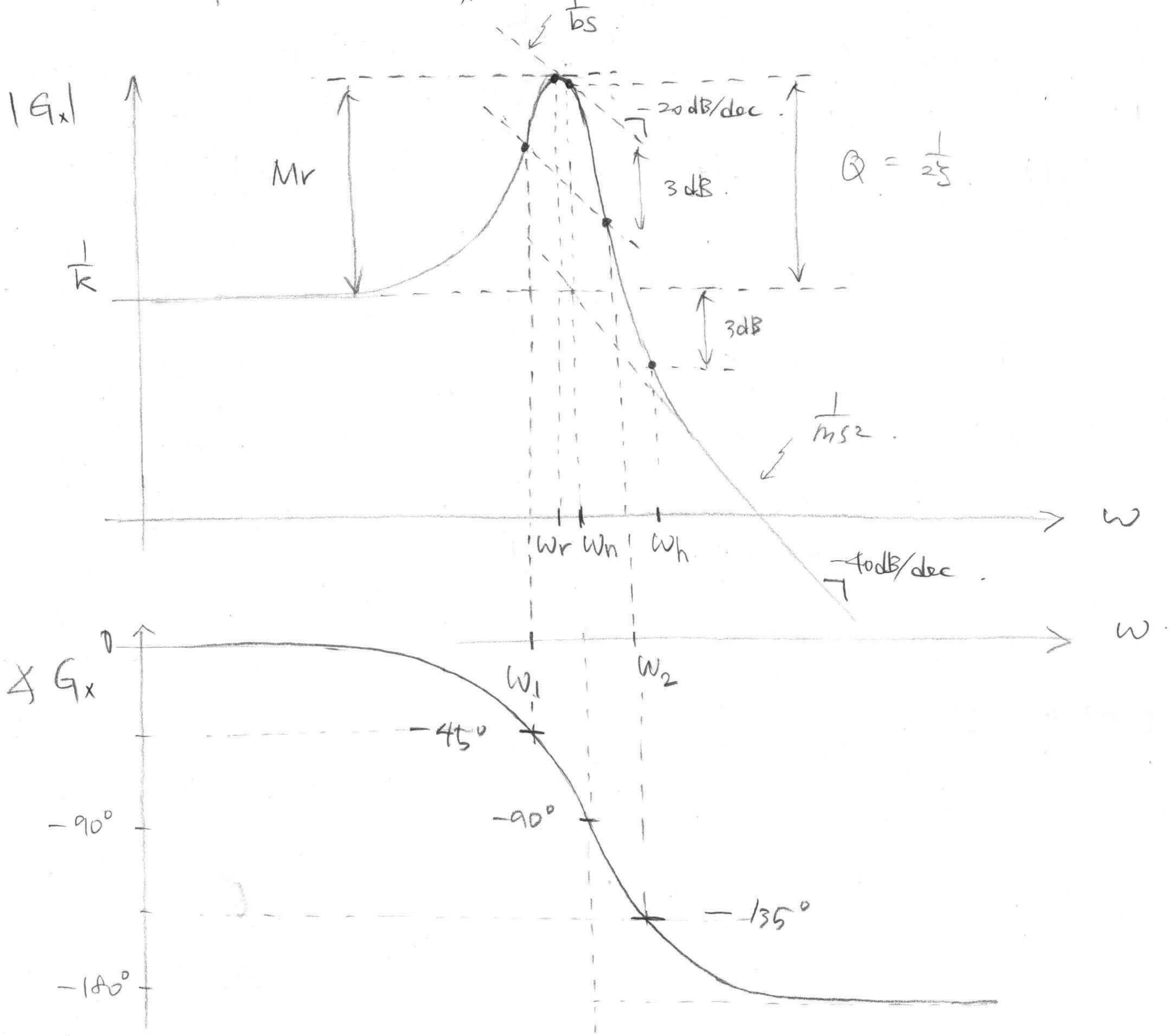
$$= \left[e^{-\sigma t} \left(\frac{\omega_n}{\sqrt{1-\zeta^2}} \right) \cdot \sin \omega_d t \right] \cdot \frac{u(t)}{k}.$$

$\hat{\dot{x}}(t) = k\dot{x}(t)$ "Normalized" response.



Exponential envelope $\left(\pm \frac{\omega_n}{\sqrt{1-\zeta^2}} e^{-\sigma t} \right)$.
modulating a sinusoid.

∘ Bode plot of $G_x(s)$ (for $\zeta < 1$)



- ω_n : natural frequency $\quad \omega_n = \sqrt{\frac{k}{m}}$ for $0 < \zeta < 1$
- ω_r : resonance frequency.

$$\omega_r = \arg\max_{\omega} |G(j\omega)|$$

$$= \omega_n \cdot \sqrt{1 - 2\zeta^2} \quad \text{for } 0 < \zeta < \frac{1}{\sqrt{2}}$$

- M_r : resonance peak

$$M_r = \frac{1}{2\zeta} \frac{1}{\sqrt{1-\zeta^2}}$$

- Q : quality factor

$$Q \triangleq \frac{\omega_n}{\omega_2 - \omega_1} = \frac{1}{2\zeta}$$

- Proof for Resonance Frequency & Resonance Peak

Let $\Omega = \frac{\omega}{\omega_n}$ (normalized freq).

$$G_x(j\omega) = \frac{1}{1-\Omega^2 + 2\zeta\Omega j} \cdot \frac{1}{k}$$

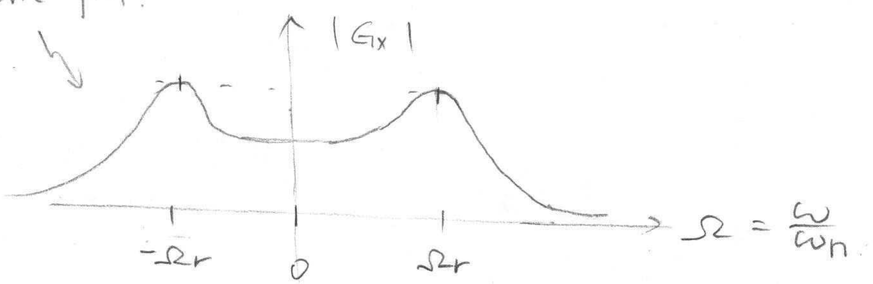
$$|G_x(j\omega)|^2 = \frac{1/k^2}{(1-\Omega^2)^2 + 4\zeta^2\Omega^2} = \frac{1/k^2}{\Omega^4 + (4\zeta^2-2)\Omega^2 + 1}$$

① Find Ω maximizing $|G_x|^2 \longleftrightarrow$ minimizing $\Omega^4 + (4\zeta^2-2)\Omega^2+1$

$$\therefore \frac{d(\cdot)}{d\Omega} \rightarrow 4\Omega^3 + 2(4\zeta^2-2)\Omega = 0$$

$$\Omega(\Omega^2 + 2\zeta^2 - 1) = 0 \qquad \therefore \Omega = \pm\sqrt{1-2\zeta^2} \text{ or } 0$$

linear-scale plot



$$\Omega_r = \sqrt{1-2\zeta^2}$$

$$\therefore \omega_r = \omega_n\sqrt{1-2\zeta^2}.$$

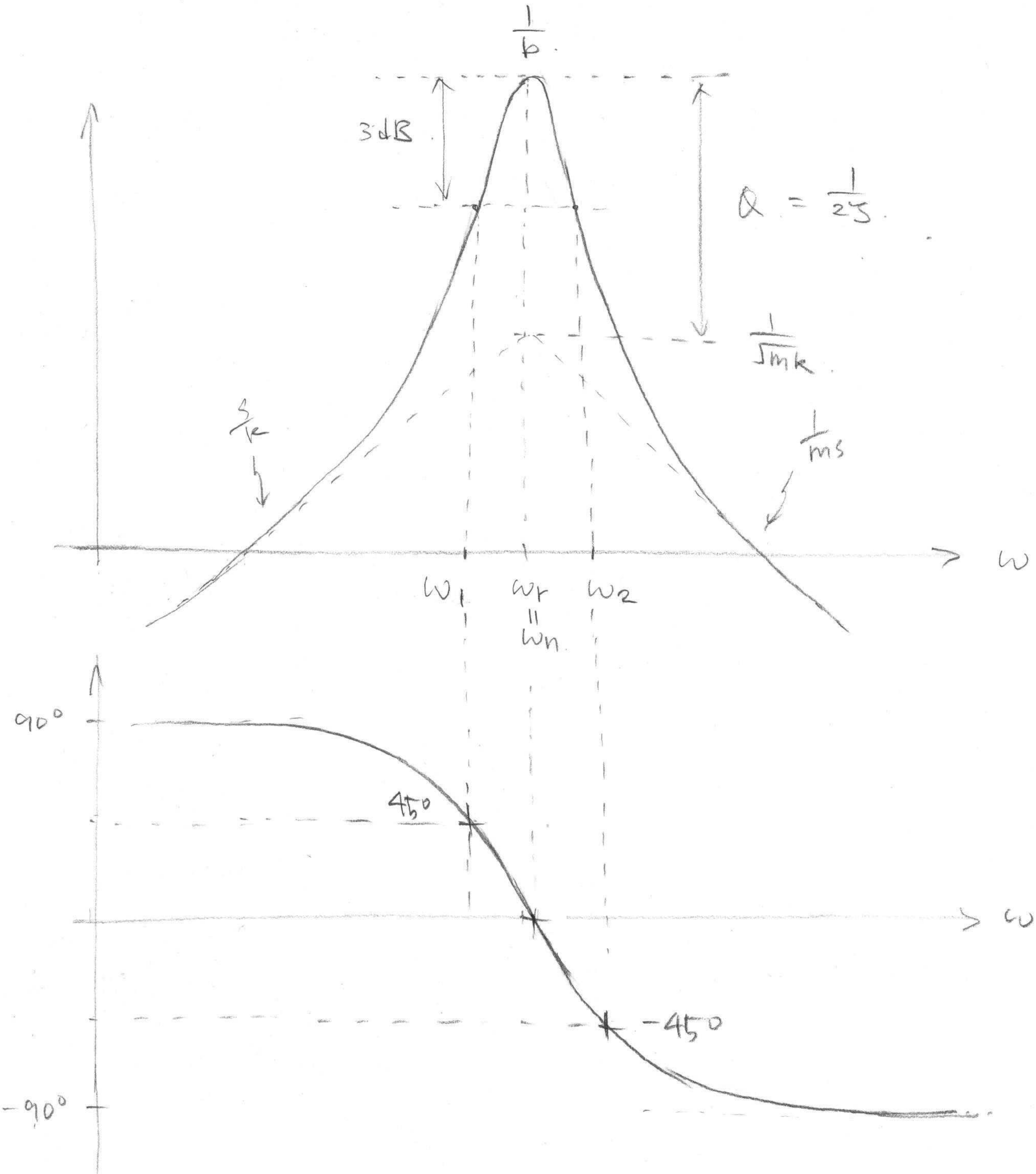
② Find $\left|G_x(j\omega)\right|_{\Omega=\sqrt{1-2\zeta^2}}$.

$$\left|\frac{1/k}{\cancel{1}-\cancel{1}+2\zeta^2 + 2\zeta\sqrt{1-2\zeta^2}\, j}\right| = \frac{1/k}{\sqrt{4\zeta^4 + 4\zeta^2(1-2\zeta^2)}}$$

$$= \frac{1/k}{\sqrt{4\zeta^2 - 4\zeta^4}} = \frac{1/k}{2\zeta\sqrt{1-\zeta^2}}.$$

$$\therefore M_r = \frac{1}{2\zeta\sqrt{1-\zeta^2}}.$$

∘ Bode plot of $G_v(s)$ (for $\zeta < 1$)



- $\omega_r = \omega_n$
- $M_r = Q$.
- Half-power Bandwidth : $BW = \omega_2 - \omega_1$
- Quality factor : $Q \triangleq \dfrac{\omega_n}{\omega_2 - \omega_1} = \dfrac{1}{2\zeta}$, $\omega_n = \sqrt{\omega_1^2 \omega_2^2}$.

- Proof for $Q = \frac{1}{2\zeta}$ and $\omega_n = \sqrt{\omega_1 \omega_2}$.

- Consider a resonator $G(s) = \dfrac{s/\omega_n}{\left(\frac{s}{\omega_n}\right)^2 + 2\zeta\left(\frac{s}{\omega_n}\right) + 1}$

- Define a normalized frequency $\Omega = \dfrac{\omega}{\omega_n}$ $\quad (\Omega = 1 \leftrightarrow \omega = \omega_n)$

$$G(j\omega) = \frac{j\Omega}{(1-\Omega^2) + 2\zeta\Omega j}$$

- max of $|G|$ occurs at $\Omega = 1$ (resonance)

$$\therefore \max |G| = \frac{|j|}{|2\zeta j|} = \frac{1}{2\zeta}.$$

- -3dB below the max $|G|$ is $\frac{1}{2\zeta} \cdot \frac{1}{\sqrt{2}}$.

- The frequencies ω_1 & ω_2 at which $|G| = \frac{1}{2\zeta}\frac{1}{\sqrt{2}}$ form the half-power bandwidth $BW = \omega_2 - \omega_1$.

- Find Ω such that $|G| = \frac{1}{2\zeta}\cdot\frac{1}{\sqrt{2}} \rightarrow |G|^2 = \frac{1}{8\zeta^2}$.

$$|G|^2 = \frac{\Omega^2}{(1-\Omega^2)^2 + 4\zeta^2\Omega^2} = \frac{1}{8\zeta^2}$$

$$\Omega^4 - 2\Omega^2 + 1 + 4\zeta^2\Omega^2 = 8\zeta^2\Omega^2$$

$$\boxed{\Omega^4 - (2 + 4\zeta^2)\Omega^2 + 1 = 0}$$

$$\begin{cases} \Omega_1^2\Omega_2^2 = 1 \\ \Omega_1^2 + \Omega_2^2 = 2 + 4\zeta^2 \end{cases} \Rightarrow \quad \frac{\omega_1^2}{\omega_n^2}\frac{\omega_2^2}{\omega_n^2} = 1 \rightarrow \omega_n = \sqrt{\omega_1\omega_2}$$

$$(\Omega_2 - \Omega_1)^2 = 4\zeta^2$$

$$\Omega_2 - \Omega_1 = 2\zeta$$

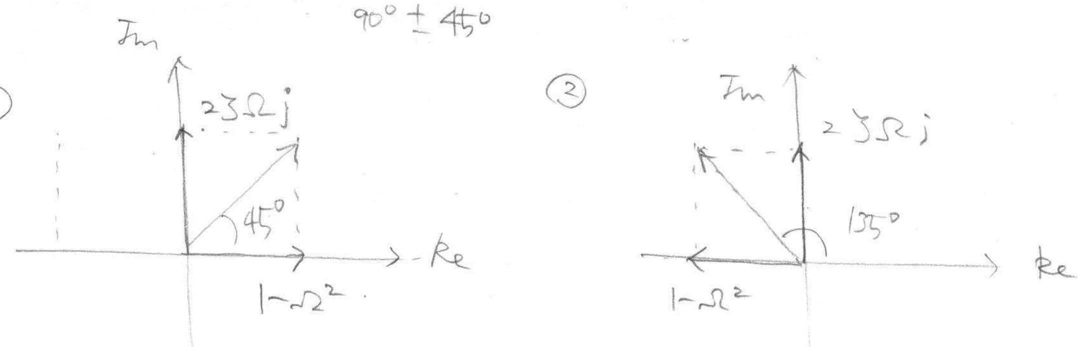
$$\omega_2 - \omega_1 = 2\zeta\omega_n.$$

- $Q \triangleq \dfrac{\omega_n}{\omega_2 - \omega_1} = \dfrac{\omega_n}{2\zeta\omega_n} = \dfrac{1}{2\zeta}$, $\quad \omega_n = \sqrt{\omega_1\omega_2}$

- Proof for $\measuredangle G(j\omega)\Big|_{\omega=\omega_1,\omega_2} = \pm 45^\circ$.

$$G(j\omega) = \frac{j\Omega}{(1-\Omega^2)+2\zeta\Omega j}$$

$$\measuredangle G = 90^\circ - \underbrace{\measuredangle\left[(1-\Omega^2)+2\zeta\Omega j\right]}_{90^\circ \pm 45^\circ}$$



In either cases, $|1-\Omega^2|^2 = |2\zeta\Omega|^2$

$$\Omega^4 - 2\Omega^2 + 1 = 4\zeta^2\Omega^2$$

$$\boxed{\Omega^4 - (2+4\zeta^2)\Omega^2 + 1 = 0}$$

The same equation. → The roots are Ω_1 and Ω_2