Example 3.1
Consider the cascaded lag network shown in Figure 3.1. Determine the input impedance and the output impedance of the circuit.

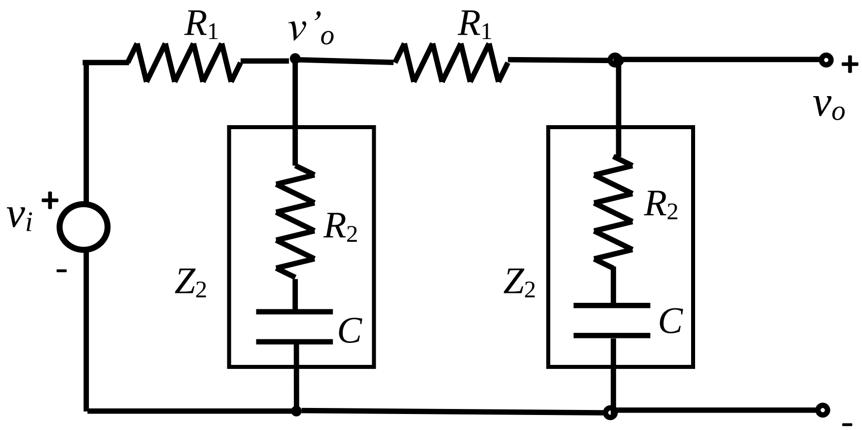

Figure 3.1: Cascaded lag network.

Solution
An equivalent circuit of the given cascaded circuit is shown in Figure S3.1(a).

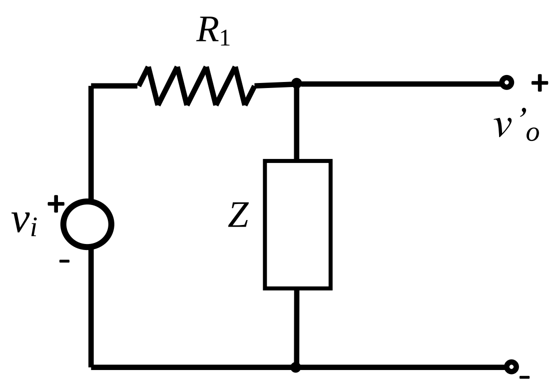

Figure S3.1(a): An equivalent circuit of the cascaded circuit.

Input Impedance:
Since Z is formed by connecting Z_2 and (R_1 + Z_2) in parallel (see Figure 3.1), we have

$$\frac{1}{Z} = \frac{1}{Z_2} + \frac{1}{R_1 + Z_2} \qquad \text{(i)}$$

The input current from the source is

$$i = \frac{v_i}{R_1 + Z}$$

Hence, the input impedance is

$$Z_i = \frac{v_i}{i} = R_1 + Z$$

where, Z is given by (ii).

Output Impedance:

Voltage drop across Z is (from Figure S3.1(a))

$$v_o' = \frac{Z}{R_1 + Z} v_i \qquad \text{(ii)}$$

The output, open-circuit voltage is (see Figure 3.1)

$$v_o = \frac{Z_2}{R_1 + Z_2} v_o'$$

Substitute (ii) into this equation. We get

$$v_o = \frac{Z_2}{(R_1 + Z_2)} \frac{Z}{(R_1 + Z)} v_i \qquad \text{(iii)}$$

To get the short-circuit current, short the output of Figure 3.1. We get the circuit in Figure S3.1(b).

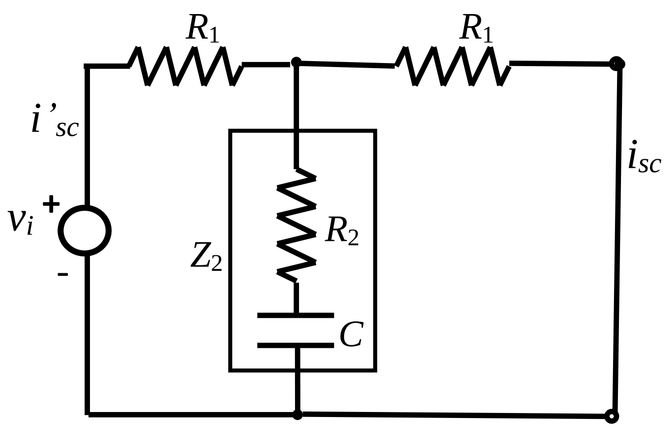

Figure S3.1(b): Circuit with output short-circuited.

The equivalent impedance of the second stage (consisting of Z_2 and R_1 in parallel) of this circuit Z' is given by

$$\frac{1}{Z'} = \frac{1}{Z_2} + \frac{1}{R_1} \text{ or } Z' = \frac{R_1 Z_2}{R_1 + Z_2} \qquad \text{(iv)}$$

Hence, the current at the source under short-circuit conditions is

$$i_{sc}' = \frac{v_i}{R_1 + Z'} \qquad \text{(v)}$$

Since the current is divided proportional to inverse of impedance among parallel branches, the short-circuit current (at the output) is (see Figure S3.1(b))

$$i_{sc} = \frac{Z_2}{R_1 + Z_2} i_{sc}' = \frac{Z_2}{R_1 + Z_2} \frac{v_i}{R_1 + Z'} \qquad \text{(vi)}$$

The output impedance (by definition) is (from (iii) and (vi))

$$Z_o = \frac{v_o}{i_{sc}} = \frac{\dfrac{Z_2}{(R_1+Z_2)}\dfrac{Z}{(R_1+Z)}v_i}{\dfrac{Z_2}{(R_1+Z_2)}\dfrac{v_i}{(R_1+Z')}} = \frac{Z(R_1+Z')}{R_1+Z} = \frac{(R_1+Z')}{R_1/Z+1}$$

Substitute (i) and (iv):

$$Z_o = \frac{\left(R_1 + \dfrac{R_1 Z_2}{R_1+Z_2}\right)}{\left(\dfrac{R_1}{Z_2} + \dfrac{R_1}{R_1+Z_2} + 1\right)} = \frac{R_1 Z_2 (R_1+Z_2+Z_2)}{R_1(R_1+Z_2)+R_1 Z_2 + Z_2(R_1+Z_2)}$$

Or,

$$Z_o = \frac{R_1 Z_2 (R_1+2Z_2)}{R_1^2+3R_1 Z_2+Z_2^2} = \frac{(R_1+2Z_2)}{R_1\ /Z_2+3+Z_2\ /R_1}$$

Think of ways to make this small (to reduce loading).