MECH 421: Mechatronics System Instrumentation
2019/20 Winter Session – Term 2

Homework 5 – Solution

Assigned: Mar 5, 2021
Due: Mar 12, 2021

Problem 1

(a) The equations of motions are

$$\begin{aligned} m_1\ddot{x}_1 &= f_1 - k(x_1 - x_2) \quad \rightarrow \quad m_1\ddot{x}_1 + kx_1 - kx_2 = f_1 \\ m_2\ddot{x}_2 &= f_2 - k(x_2 - x_1) \quad \rightarrow \quad m_2\ddot{x}_2 - kx_1 + kx_2 = f_2, \end{aligned}$$

which can be organized as the following matrix equation.

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} k & -k \\ -k & k \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$$

(b) Taking the Laplace transform leads to

$$\begin{aligned} s^2 \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} + \begin{bmatrix} k & -k \\ -k & k \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} &= \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} \\ \begin{bmatrix} m_1 s^2 + k & -k \\ -k & m_2 s^2 + k \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} &= \begin{bmatrix} F_1 \\ F_2 \end{bmatrix}. \end{aligned}$$

Taking matrix inversion leads to

$$\begin{aligned} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} &= \begin{bmatrix} m_1 s^2 + k & -k \\ -k & m_2 s^2 + k \end{bmatrix}^{-1} \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} \\ &= \frac{1}{(m_1 s^2 + k)(m_2 s^2 + k) - k^2} \begin{bmatrix} m_2 s^2 + k & k \\ k & m_1 s^2 + k \end{bmatrix} \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} \\ &= \frac{1}{m_1 m_2 s^4 + (m_1 + m_2) k s^2} \begin{bmatrix} m_2 s^2 + k & k \\ k & m_1 s^2 + k \end{bmatrix} \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} \end{aligned}$$

(c)

$$H_{11}(s) = \frac{X_1(s)}{F_1(s)} = \frac{m_2 s^2 + k}{m_1 m_2 s^4 + (m_1 + m_2) k s^2}$$

This system is called *collocated system*, because the locations of actuation and sensing are the same. Figure 1 shows the pole-zero map and Bode plot of $H_{11}(s)$.

The system has two poles at the origin, a pair of zeros (called *anti-resonance*), and a pair of poles on the imaginary axis. At low frequencies, the system behaves like a free

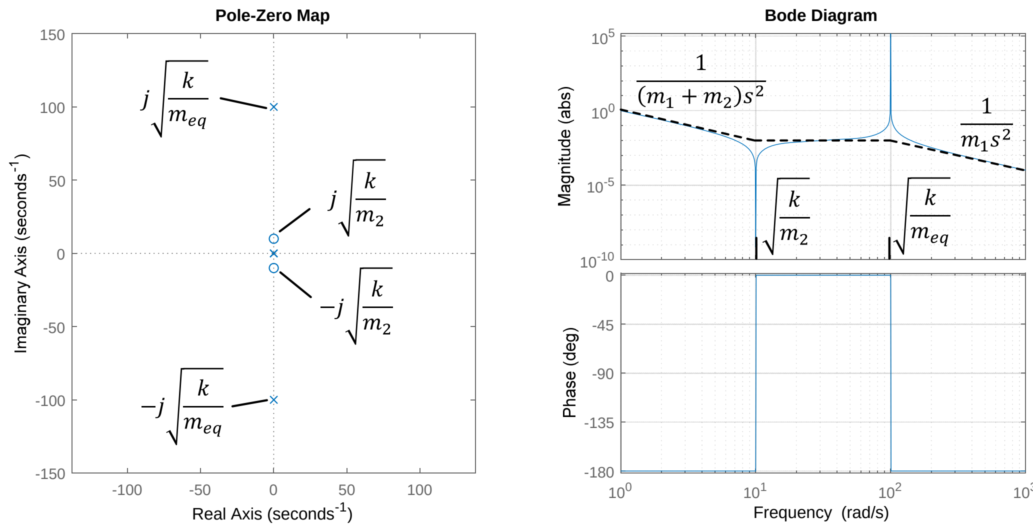


Figure 1: The pole-zero map and Bode plot of $H_{11}(s)$.

mass whose total mass is $m_1 + m_2$. At the anti-resonance, the first mass m_1 does not move and only the second mass m_2 oscillates at the frequency $\sqrt{k/m_2}$. It is as if the first mass m_1 was replaced with a rigid wall and only the second mass m_2 resonates with the spring k. At the resonance, the two masses oscillate against each other at the frequency $\sqrt{k/m_{eq}}$, where $m_{eq} = m_1 \parallel m_2 = \frac{m_1 m_2}{m_1+m_2}$ is the equivalent mass. Above the resonance, the system behaves like a free mass whose total mass is m_1. It is as if the second mass is disconnected from the first mass.

The phase of the collocated system is always bounded within $(-180°, 0°)$.

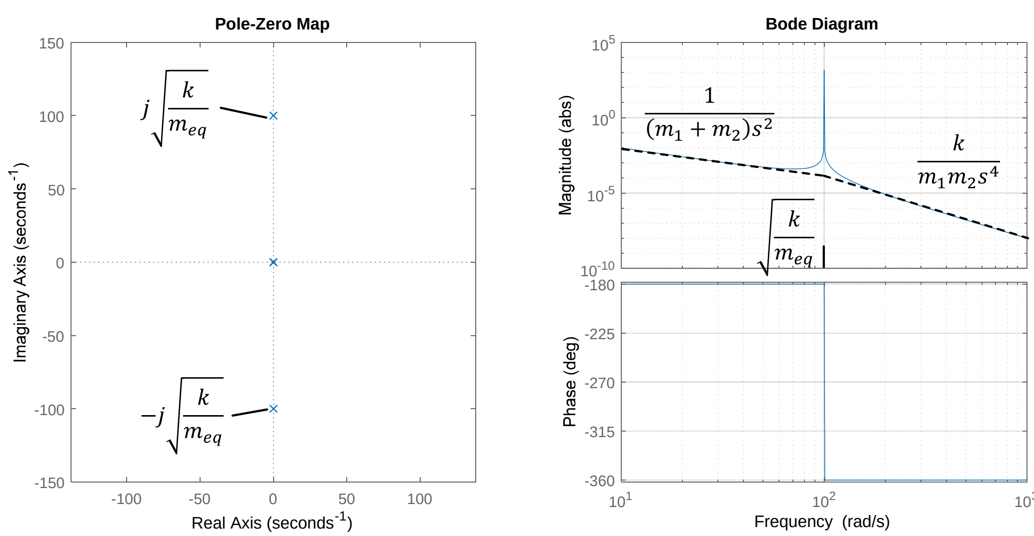


Figure 2: The pole-zero map and Bode plot of $H_{12}(s)$.

$$H_{21}(s) = \frac{X_2(s)}{F_1(s)} = \frac{k}{m_1 m_2 s^4 + (m_1 + m_2) k s^2}$$

This system is called *non-collocated system*, because the locations of actuation and sensing are different. Figure 2 shows the pole-zero map and Bode plot of $H_{21}(s)$.

The system has two poles at the origin and a pair of poles on the imaginary axis. At low frequencies, the system behavior is the same as the collocated system, i.e., a free mass whose total mass is $m_1 + m_2$. At resonance, the two masses oscillate against each other at the frequency $\sqrt{k/m_{eq}}$. Above the resonance, the Bode plot magnitude drops with -4 slope ($-80\,\mathrm{dB/dec}$) and the phase drops to $-360°$.

In general, non-collocated systems are more difficult to control due to the extra phase lag at high frequencies.

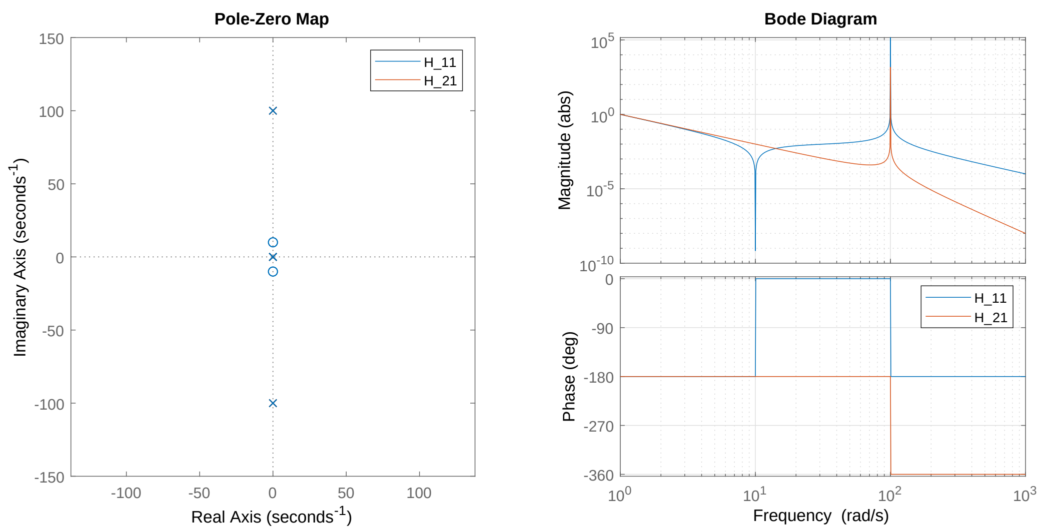


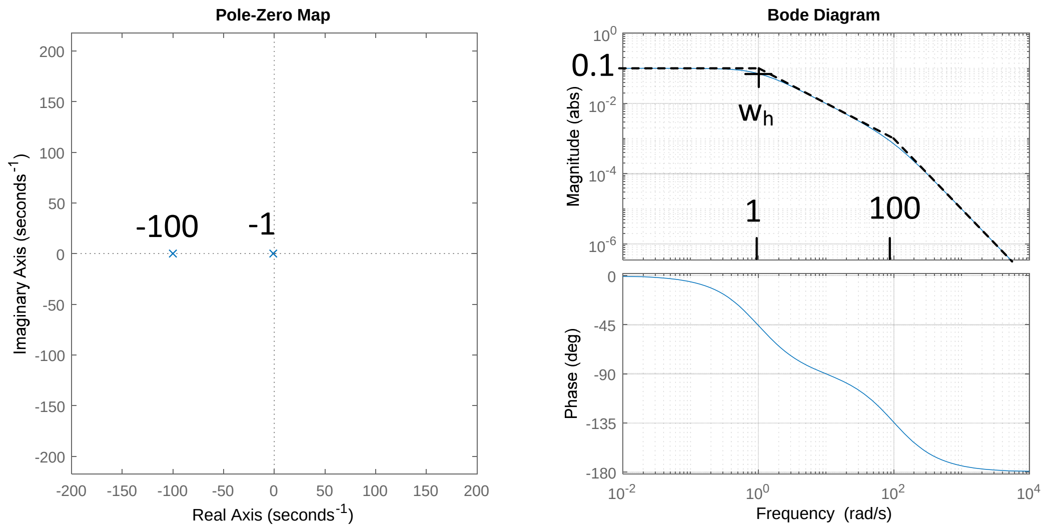
Figure 3: Collocated vs. non-collocated systems.

Figure 3 compares the collocated system and non-collocated system on the same graph ($m_1 = 1/99\,\mathrm{kg}, m_2 = 1\,\mathrm{kg}, k = 100\,\mathrm{N/m}$).

Problem 2

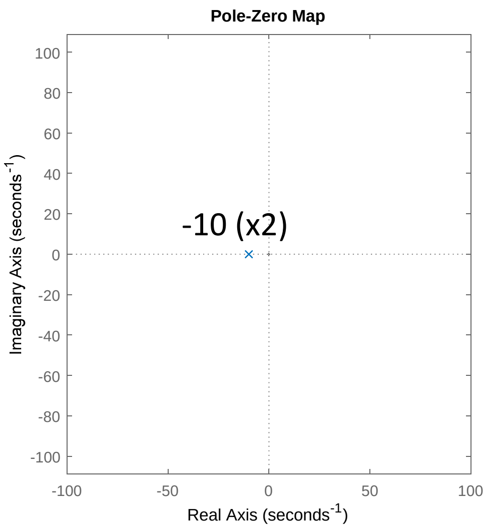
(a)

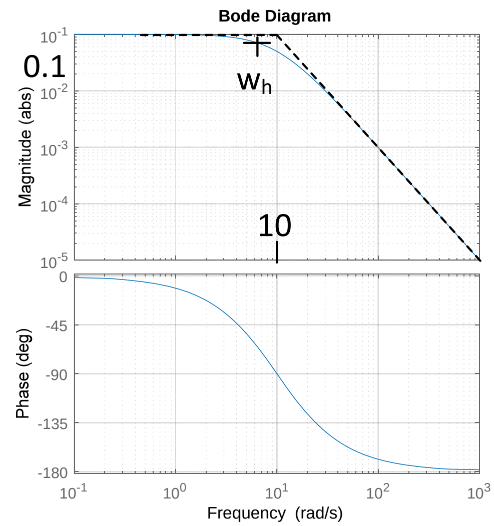
$$H_a(s) = \frac{10}{s^2 + 101s + 100} = \frac{10}{(s+1)(s+100)}$$



(b)

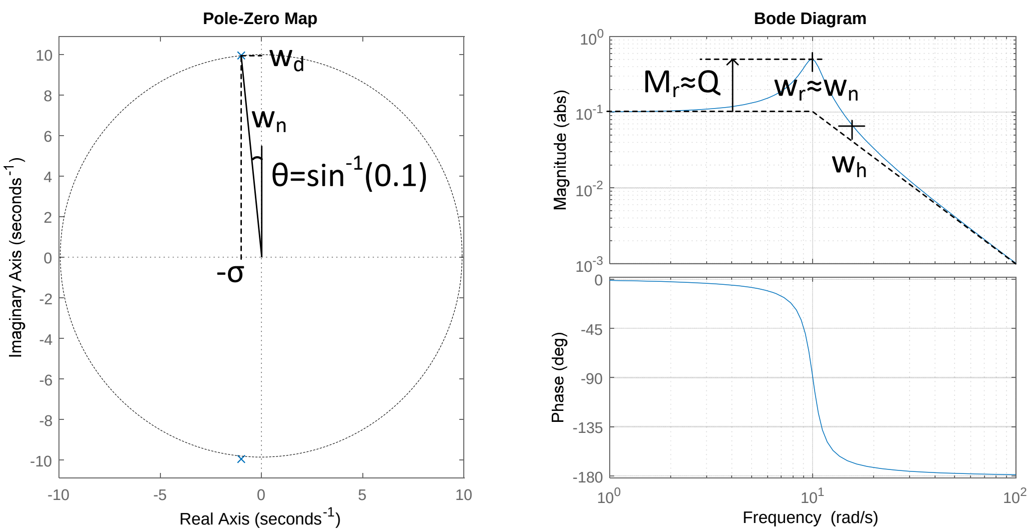
$$H_b(s) = \frac{10}{s^2 + 20s + 100} = \frac{10}{(s+10)^2}$$





(c)

$$H_c(s) = \frac{10}{s^2 + 2s + 100} = \frac{10}{s^2 + 2 \times 0.1 \times 10s + 10^2}$$



Here, the parameter values for the pole-zero map are

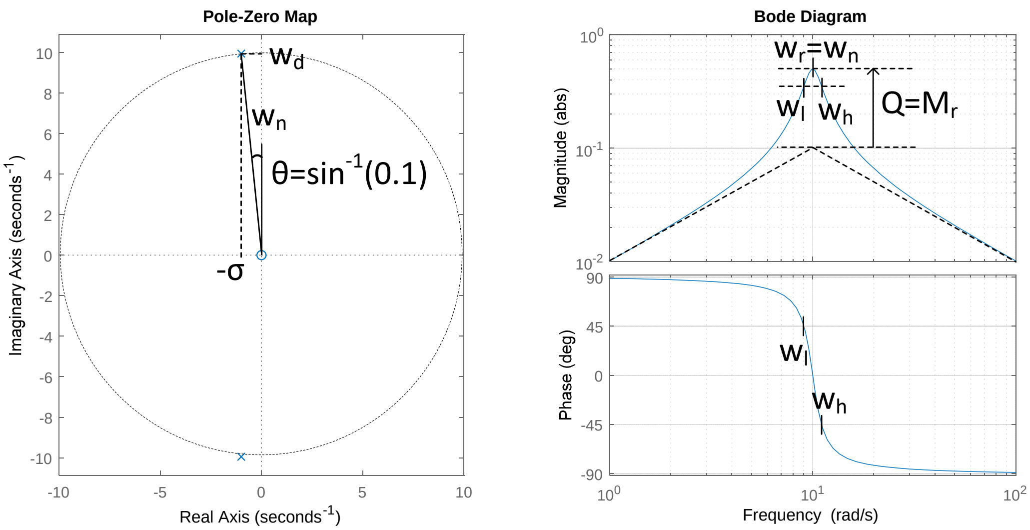
$$\begin{aligned} \omega_n &= 10\,\mathrm{rad/s} & \qquad \omega_d &= \omega_n\sqrt{1-\zeta^2} = 9.9499\,\mathrm{rad/s} \\ \zeta &= 0.1 & \sigma &= \omega_n\zeta = 1\,\mathrm{rad/s}, \end{aligned}$$

and the parameter values for the Bode plot are

$$\begin{aligned} \omega_n &= 10\,\mathrm{rad/s} & \qquad \omega_r &= \omega_n\sqrt{(1-2\zeta^2)} = 9.8995\,\mathrm{rad/s} \\ Q &= \frac{1}{2\zeta} = 5 & M_r &= \frac{1}{2\zeta}\frac{1}{\sqrt{1-\zeta^2}} = 5.0252 \end{aligned}$$

(d)

$$H_d(s) = \frac{s}{s^2 + 2s + 100} = \frac{s}{s^2 + 2 \times 0.1 \times 10s + 10^2}$$



Here, the parameter values for the pole-zero map are

$$\omega_n = 10\,\text{rad/s} \qquad \omega_d = \omega_n\sqrt{1-\zeta^2} = 9.9499\,\text{rad/s}$$
$$\zeta = 0.1 \qquad \sigma = \omega_n\zeta = 1\,\text{rad/s},$$

and the parameter values for the Bode plot are

$$\omega_n = 10\,\text{rad/s} \qquad \omega_r = \omega_n = 10\,\text{rad/s}$$
$$Q = \frac{1}{2\zeta} = 5 \qquad M_r = Q = 5$$