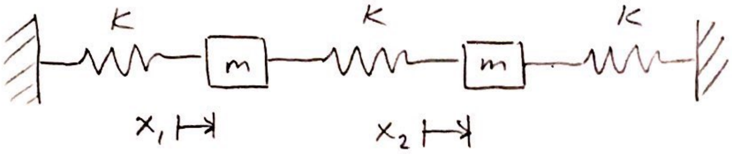
2 - DOF Vibration



Only because of symmetry not generally

$$\begin{cases} x_1 = C_1 \cos(\omega_1 t + \phi_1) + C_2(\omega_2 t + \phi_2) \\ x_2 = u_1 C_1 \cos(\omega_1 t + \phi_1) + u_2 C_2 \cos(\omega_2 t + \phi_2) \end{cases}$$ and $u_1 = 1$, $u_2 = -1$

Choose initial conditions such that $C_2 = 0$:

$$\Rightarrow \begin{cases} x_1 = C_1 \cos(\omega_1 t + \phi_1) \\ x_2 = u_1 C_1 \cos(\omega_1 t + \phi_2) \end{cases}$$

<u>For n degrees of freedom</u>:

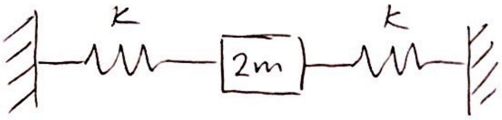
1) There are n natural frequencies (not always distinct.
2) When vibrating at a given ω_N all parts vibrate either in phase or out of phase.
3) At each natural frequency there is a definite ratio of the vibration amplitude of each part
4) 2n initial conditions are required to specify the motion. (typically x and $\dot{x}$)

<u>For system above</u>:

$\omega_1^2 = \frac{k}{m}$ $\quad \omega_2^2 = \frac{3k}{m}$ $\quad u_1 = 1$ $\quad u_2 = -1$

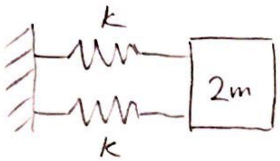
For m_1, m_2 vibrating in phase, spring in middle does not deform $\Rightarrow$ vibration mode 1.

(middle $k=0$)



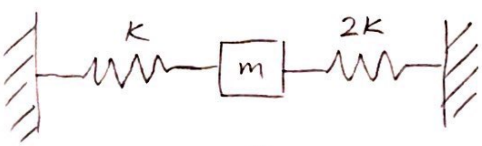
(middle $k=\infty$)

$\Updownarrow$ equivalent



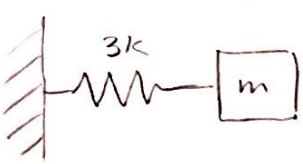
Springs add "opposite" of resistors

For m_1, m_2 totally out of phase, a middle point is stationary $\Rightarrow$ vibration mode 2.



A spring of half length has twice the stiffness

$\Updownarrow$ equivalent



$\boxed{\text{Matrix Solutions}}$ $[X]$-matrix, $\vec{x}$-vector

$$\begin{cases} m\ddot{x}_1 + 2kx_1 - kx_2 = 0 \\ m\ddot{x}_2 - kx_1 + 2kx_2 = 0 \end{cases} \Rightarrow \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} 2k & -k \\ -k & 2k \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow [M]\ddot{\vec{x}} + [K]\vec{x} = \vec{0} \leftarrow \text{zero vector}$$

mass matrix $\uparrow$ $\quad$ $\uparrow$ stiffness matrix

The matrix equation has the same form as the scalar equation:

$$[M]\ddot{\vec{x}} + [k]\vec{x} = \vec{0} \iff m\ddot{x} + kx = 0$$

1 DOF: $x = C\cos(\omega t + \phi)$

2 DOF: $x_1 = X_1 \cos(\omega t + \phi)$
$x_2 = X_2 \cos(\omega t + \phi)$ $\iff \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \cos(\omega t + \phi)$

Differentiate twice:

$$\ddot{x}_1 = -\omega^2 X_1 \cos(\omega t + \phi)$$
$$\ddot{x}_2 = -\omega^2 X_2 \cos(\omega t + \phi) \iff \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} = -\omega^2 \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \cos(\omega t + \phi)$$

Sub into matrix equation:

$$\left(-\omega_2 \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} + \begin{bmatrix} 2k & -k \\ -k & 2k \end{bmatrix}\right) \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \cos(\omega t + \phi) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow -\omega_2\left([M] + [k]\right)\vec{X} \cos(\omega t + \phi) = \vec{0}$$

General matrix equation:

$$[A]\vec{x} = \vec{y} \Rightarrow \vec{x} = [A]^{-1}\vec{y}$$

$$\vec{x} = \frac{\text{adjoint}[A]}{\det[A]}\vec{y} \Rightarrow \det[A]\,\vec{x} = \text{adjoint}[A]\,\vec{y}$$

consider $\vec{y} = 0 \Rightarrow \det[A]\,\vec{x} = 0$

$$\Rightarrow \det[A] = 0$$

For our matrix equation of motion:

$$\det \begin{bmatrix} 2k-\omega^2 m & -k \\ -k & 2k-\omega^2 m \end{bmatrix} = 0$$

$$\Rightarrow (2k-\omega^2 m)(2k-\omega^2 m) - k^2 = 0$$

$$\Rightarrow \omega^4 m^2 - 4mk\omega^2 + 3k^2 = 0 \qquad \text{(Characteristic eq.)}$$

$$\Rightarrow \omega_1^2 = \frac{k}{m} \quad \text{and} \quad \omega_2^2 = \frac{3k}{m} \quad \text{as before.}$$

Now:

$$\begin{bmatrix} 2k-\omega^2 m & -k \\ -k & 2k-\omega^2 m \end{bmatrix} \begin{bmatrix} 1 \\ u \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

First line: $2k - \omega^2 m - uk = 0$

$$\Rightarrow u = 2 - \omega^2 \frac{m}{k}$$

$$\Rightarrow u_1 = 1 \quad \text{where } \omega^2 = \frac{k}{m}$$

$$u_2 = -1 \quad \text{where } \omega^2 = \frac{3k}{m}$$