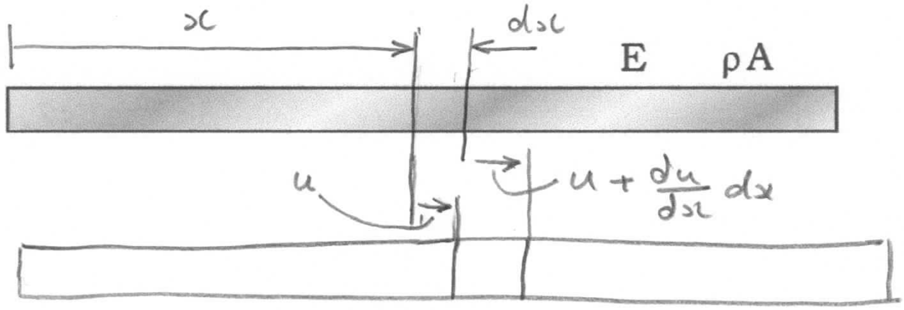
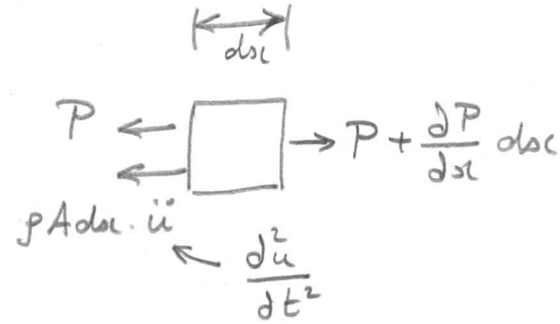
MECH 463 -- Tutorial 12

1. A uniform rod of cross-section area A is made of a material with Young's modulus E and mass density ρ. Formulate the wave equation for longitudinal vibrations of the rod.



Let u = longitudinal vibrational displacement

Consider a small element of length dx



Original length = dx

Displaced length = $dx + \frac{\partial u}{\partial x} dx$

$$\rightarrow \text{strain} = \frac{dx + \frac{\partial u}{\partial x} dx - dx}{dx} = \frac{\partial u}{\partial x}$$

$$\text{stress} = E \times \text{strain} = E \frac{\partial u}{\partial x} = \frac{P}{A} \quad \rightarrow P = AE\frac{\partial u}{\partial x}$$

$$\text{From FBD: } \cancel{P} + \frac{\partial P}{\partial x} dx - \cancel{P} - \rho A\, dx \frac{\partial^2 u}{\partial t^2} = 0$$

$$\div dx \qquad AE \frac{\partial^2 u}{\partial x^2} - \rho A \frac{\partial^2 u}{\partial t^2} = 0$$

$$\div \rho A \qquad \frac{E}{\rho} \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial t^2} = 0$$

Rearranging: $$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

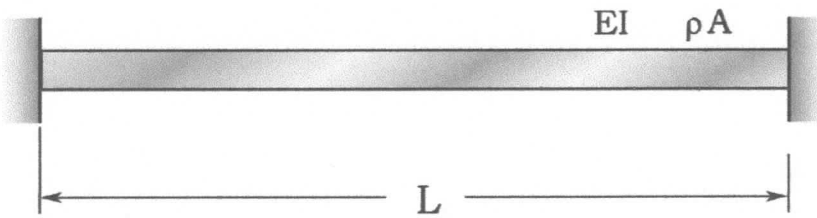
where wave speed $c = \sqrt{\frac{E}{\rho}}$

Wave speed is independent of A

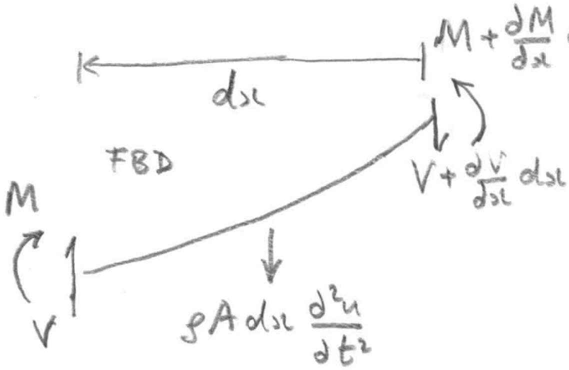
2. A uniform beam of length L, flexural rigidity EI, mass density ρ and cross-section area A is rigidly fixed at both ends. Using a method parallel to the method used in class for a stretched string, solve the wave equation for a beam undergoing transverse vibrations:

$$\frac{\partial^2 u}{\partial t^2} + c^2 \frac{\partial^4 u}{\partial x^4} = 0$$

where $c = \sqrt{(EI/\rho A)}$. Derive an equation to determine the natural frequencies of vibration. (*Hint: the solution for this 4th-order equation has both trigonometric and hyperbolic terms.*)



(The question does not ask for this derivation. → freebie.)



Moment equilibrium about centre of FBD:

$$\left(\cancel{M} + \frac{\partial M}{\partial x}dx\right) - \cancel{M} - V\frac{dx}{2} - \left(V + \frac{\partial V}{\partial x}dx\right)\frac{dx}{2} = 0$$

$$\rightarrow \frac{\partial M}{\partial x}dx - V\,dx - \frac{1}{2}\frac{\partial V}{\partial x}dx^2 = 0$$

$$\div dx \qquad \frac{\partial M}{\partial x} = V + \frac{1}{2}\frac{\partial V}{\partial x}dx \simeq V \quad \text{for } dx \rightarrow 0$$

Vertical force equilibrium:

$$\cancel{V} - \left(\cancel{V} + \frac{\partial V}{\partial x}dx\right) - \rho A\,dx\frac{\partial^2 u}{\partial t^2} = 0$$

$$\div dx \qquad \rho A\frac{\partial^2 u}{\partial t^2} = -\frac{\partial V}{\partial x} = -\frac{\partial^2 M}{\partial x^2} \quad \text{using } V = \frac{\partial M}{\partial x}$$

$$= -EI\frac{\partial^4 u}{\partial x^4} \quad \text{using } M = EI\frac{\partial^2 u}{\partial x^2} \text{ from beam theory}$$

Rearranging: $\frac{\partial^2 u}{\partial t^2} + c^2\frac{\partial^4 u}{\partial x^4} = 0$ where

not wave speed (different dimensions) → $c = \sqrt{\frac{EI}{\rho A}}$

As with the wave equation, try a separable solution:

$$u(x,t) = X(x)\, T(t)$$

$X(x)$ = mode shape

$T(t)$ = vibration

$$\rightarrow \quad X\ddot{T} + c^2 X'''' T = 0$$

$$\rightarrow \quad \frac{X''''}{X} = \frac{-1}{c^2}\frac{\ddot{T}}{T} = \text{a constant} = \beta^4 \text{ say}$$

$$\ddot{T} + \beta^4 c^2 T = 0 \quad \text{and} \quad X'''' - \beta^4 X = 0$$

Solving: $T = A\cos\omega t - B\sin\omega t$ where $\omega = \beta^2 c$

$$X = C\cos\beta x - D\sin\beta x + G\cosh\beta x + H\sinh\beta x$$

There are six constants to be determined: β and five of A, B, C, D, G, H (the other is not independent). Four of these constants are determined from the geometrical boundary conditions, and two from the initial conditions.

Geometrical boundary Conditions: $X(0) = 0$ $\quad X'(0) = 0$

$X(L) = 0$ $\quad X'(L) = 0$

$$X(0) = C - 0 + G + 0 = 0 \quad \rightarrow \quad G = -C$$

$$X'(x) = -\beta C\sin\beta x - \beta D\cos\beta x + \beta G\sinh\beta x + \beta H\cosh\beta x$$

$$X'(0) = \beta(0 - D + 0 + H) = 0 \quad \rightarrow \quad H = D$$

$$X(L) = C\cos\beta L - D\sin\beta L + G\cosh\beta L + H\sinh\beta L = 0$$

$$X'(L) = \beta(-C\sin\beta L - D\cos\beta L + G\sinh\beta L + H\cosh\beta L) = 0$$

Putting in matrix form with $G = -C$ and $H = D$

$$\begin{bmatrix} (\cos\beta L - \cosh\beta L) & (-\sin\beta L + \sinh\beta L) \\ (-\sin\beta L - \sinh\beta L) & (-\cos\beta L + \cosh\beta L) \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

For a non-trivial solution, the determinant of the matrix $= 0$

$$\rightarrow (\cos\beta L - \cosh\beta L)(-\cos\beta L + \cosh\beta L) - (-\sin\beta L - \sinh\beta L)(-\sin\beta L + \sinh\beta L) = 0$$

$$= \underbrace{-\cos^2\beta L - \sin^2\beta L}_{=-1} + 2\cos\beta L\cosh\beta L \underbrace{- \cosh^2\beta L - \sinh^2\beta L}_{=-1} = 0$$

$$\rightarrow \cos\beta L \cosh\beta L = 1$$

The roots of this equation give βL and hence ω

Roots are 4.73, 7.85, 11.00, 14.14 ....

$$\omega = \beta^2 c = \frac{(\beta L)^2}{L^2}\sqrt{\frac{EI}{\rho A}}$$

$$\frac{C}{D} = -\frac{G}{H} = \frac{\sin\beta L - \sinh\beta L}{\cos\beta L - \cosh\beta L} = \frac{-\cos\beta L + \cosh\beta L}{\sin\beta L + \sinh\beta L}$$

from 1st equation   from 2nd equation

$\div D$

$$\rightarrow X(x) = \frac{C}{D}(\cos\beta x - \cosh\beta x) - (\sin\beta x - \sinh\beta x)$$

mode shape

Too much algebra!