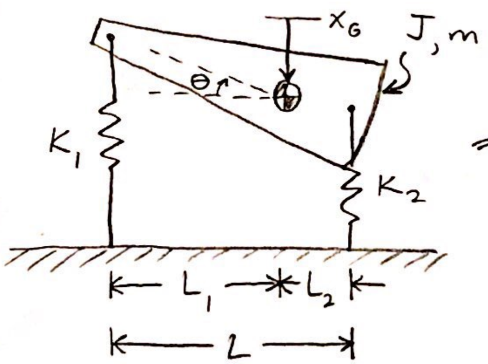
Lecture 9

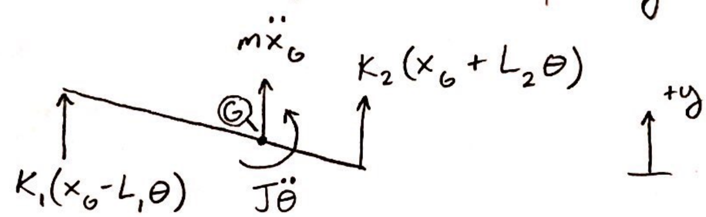
Coordinate Coupling

Ideal Car:



$\Rightarrow$

FBD: No Dynamic Coupling



Displace downward and C.W.

Equation of motion: $+\uparrow \Sigma F_y = 0$

$$\Rightarrow m\ddot{x}_G + k_1(x_G - L_1\theta) + k_2(x_G + L_2\theta) = 0$$

$$\circlearrowright{+}\ \Sigma M_G = 0$$

$$\Rightarrow J\ddot{\theta} - k_1(x_G - L_1\theta)L_1 + k_2(x_G + L_2\theta)L_2 = 0$$

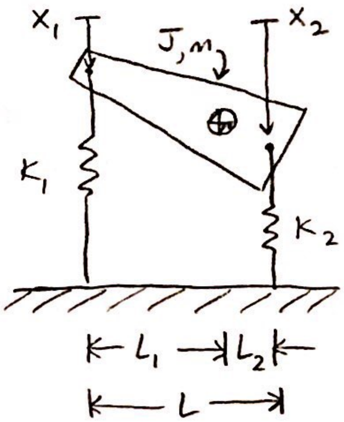
Matrix:

$$\begin{bmatrix} m & 0 \\ 0 & J \end{bmatrix}\begin{bmatrix} \ddot{x}_G \\ \ddot{\theta} \end{bmatrix} + \begin{bmatrix} k_1 + k_2 & k_2L_2 - k_1L_1 \\ k_2L_2 - k_1L_1 & k_1L_1^2 + k_2L_2^2 \end{bmatrix}\begin{bmatrix} x_G \\ \theta \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

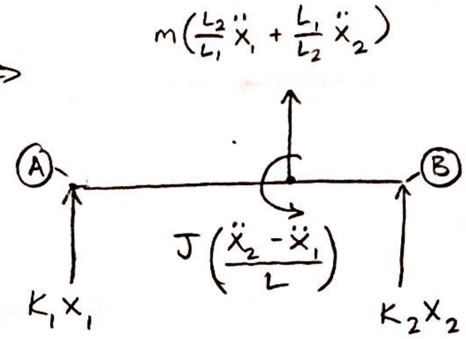
note: [M] is diagonal as expected from mass-based coordinates

[K] is generally not except when $k_1L_1 = k_2L_2$
↳ gives mode shapes of purely vertical oscillation, or pure rotation about G. Not generally.

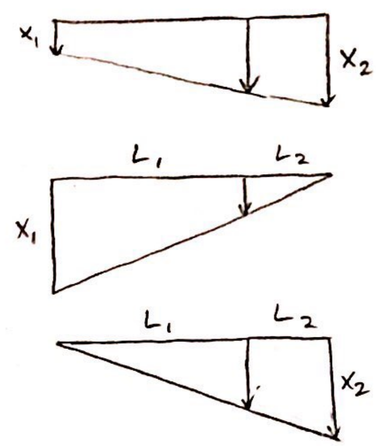
For No Static Coupling (Spring Based Coordinates)



$\Rightarrow$



Displacements; (Similar Δ)



E.O.M $\circlearrowright^{+} \Sigma M_A = 0$

$$\Rightarrow J\left(\frac{\ddot{x}_2 - \ddot{x}_1}{L}\right) + m\left(\ddot{x}_1 L_2 + \ddot{x}_2 L_1\right)\left(\frac{L_1}{L}\right) + k_2 x_2 L = 0$$

$\circlearrowright^{+} \Sigma M_B = 0$

$$\Rightarrow J\left(\frac{\ddot{x}_2 - \ddot{x}_1}{L}\right) - m\left(\ddot{x}_1 L_2 + \ddot{x}_2 L_1\right)\left(\frac{L_2}{L}\right) - k_1 x_1 L = 0$$

Matrix:

$$\begin{bmatrix} m\frac{L_2^2}{L^2} + \frac{J}{L^2} & \frac{mL_1L_2}{L^2} - \frac{J}{L^2} \\ \frac{mL_1L_2}{L^2} - \frac{J}{L^2} & \frac{mL_1^2}{L^2} + \frac{J}{L^2} \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Coordinate transformation: $x_G = \frac{L_2}{L}x_1 + \frac{L_1}{L}x_2$

$$\theta = \frac{x_2 - x_1}{L}$$

$$\Rightarrow \begin{bmatrix} x_G \\ \theta \end{bmatrix} = \begin{bmatrix} \frac{L_2}{L} & \frac{L_1}{L} \\ -\frac{1}{L} & \frac{1}{L} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \Rightarrow \vec{x} = [T]\vec{x}'$$

From mass matrix: $[M]\begin{bmatrix}\ddot{x}_G\\ \ddot{\theta}\end{bmatrix} + [K]\begin{bmatrix}x_G\\ \theta\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$

$$\Rightarrow [M]\begin{bmatrix}\frac{L_2}{L} & \frac{L_1}{L}\\ -\frac{1}{L} & -\frac{1}{L}\end{bmatrix}\begin{bmatrix}\ddot{x}_1\\ \ddot{x}_2\end{bmatrix} + [K]\begin{bmatrix}\frac{L_2}{L} & \frac{L_1}{L}\\ -\frac{1}{L} & -\frac{1}{L}\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$$

$$\Rightarrow \begin{bmatrix}m\frac{L_2}{L} & m\frac{L_1}{L}\\ -\frac{J}{L} & -\frac{J}{L}\end{bmatrix}\begin{bmatrix}\ddot{x}_1\\ \ddot{x}_2\end{bmatrix} + \begin{bmatrix}k_1 & k_2\\ -k_1L_1 & -k_2L_2\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$$

Not quite the same as for spring eq$^{\underline{n}}$.

Matrix Algebra: $[M]\ddot{\vec{x}} + [K]\vec{x} = \vec{0}$

$\vec{x}' = \begin{bmatrix}x_1\\x_2\end{bmatrix}$

$\ddot{\vec{x}}' = \begin{bmatrix}\ddot{x}_1\\ \ddot{x}_2\end{bmatrix}$

$$[M][T]\ddot{\vec{x}}' + [K][T]\vec{x}' = \vec{0}$$

$$\underbrace{[T]^T[M][T]}\ddot{\vec{x}}' + \underbrace{[T]^T[K][T]}\vec{x}' = \vec{0}$$

$$\Rightarrow [M']\ddot{\vec{x}}' + [K']\vec{x}' = \vec{0}$$

Both $[M']$ and $[K']$ are symmetric.

Summary:
1) Coupling depends on coordinates
2) Natural frequencies & mode shapes are independent of coordinate choice.
3) Mass coords. $\rightarrow$ diagonal $[M]$
   Spring coords. $\rightarrow$ diagonal $[K]$
4) Convert coords w/ $\vec{x} = [T]\vec{x}'$, $[M' \text{ or } K'] = [T]^T[M \text{ or } K][T]$

$\boxed{\text{Principal Coordinates}}$ - Gives both diagonal matrices

<u>What we want</u>: $\begin{bmatrix} m_{11} & 0 \\ 0 & m_{22} \end{bmatrix}\begin{bmatrix} \ddot{P}_1 \\ \ddot{P}_2 \end{bmatrix} + \begin{bmatrix} k_{11} & 0 \\ 0 & k_{22} \end{bmatrix}\begin{bmatrix} P_1 \\ P_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

<u>note</u>: $m_{11,22}$ and $k_{11,22}$ may be combinations of masses and springs, not individual

<u>Equations</u>: 1) $m_{11}\ddot{P}_1 + k_{11}P_1 = 0 \Rightarrow P_1 = C_1\cos(\omega_1 t + \phi_1)$

$$\Rightarrow \omega_1 = \sqrt{\frac{k_{11}}{m_{11}}}$$

2) $m_{22}\ddot{P}_2 + k_{22}P_2 = 0 \Rightarrow P_2 = C_2\cos(\omega_2 t + \phi_2)$

$$\Rightarrow \omega_2 = \sqrt{\frac{k_{22}}{m_{22}}}$$

<u>From before</u>: $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ u_1 \end{bmatrix} C_1\cos(\omega_1 t + \phi_1) + \begin{bmatrix} 1 \\ u_2 \end{bmatrix} C_2\cos(\omega_2 t + \phi_2)$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ u_1 \end{bmatrix} P_1 + \begin{bmatrix} 1 \\ u_2 \end{bmatrix} P_2$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ u_1 & u_2 \end{bmatrix}\begin{bmatrix} P_1 \\ P_2 \end{bmatrix} \iff \vec{x} = [U]\vec{P}$$

<u>Transformations</u>: $\left.\begin{matrix} [M^*] = [U]^T[M][U] \\ [K^*] = [U]^T[K][U] \end{matrix}\right\}$ Diagonal

$$\Rightarrow [M^*]\ddot{\vec{P}} + [K^*]\vec{P} = \vec{0}$$