MECH 463 Mechanical Vibrations

Midterm Exam 2 -- October 2018

1. Explain what coordinate coupling means. What are the two main types of coupling and how would you choose coordinates that would separately eliminate each of them? How could you eliminate both types of coupling simultaneously ? (Keep your answers focused on the main ideas).

Coordinate coupling occurs when there is more than one coordinate present in each equation of motion. If only one coordinate were present, the equations could be solved independently. The set of uncoupled equations produces diagonal $\underset{\sim}{M}$ and $\underset{\sim}{K}$ matrices. However, if mixed coordinates are present, the equations must be solved simultaneously. Mixed acceleration terms gives dynamic coupling and non-diagonal $\underset{\sim}{M}$. This type of coupling can be eliminated using mass-based coordinates. Mixed displacement terms gives static coupling and non-diagonal $\underset{\sim}{K}$. This type of coupling can be eliminated using spring-based coords. Both types of coupling can be eliminated simultaneously by working in terms of the principal coordinates $\underset{\sim}{p}$.

e.g. for 2-DOF $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ u_1 & u_2 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \end{bmatrix}$ modal matrix, whose columns contain the mode shapes.

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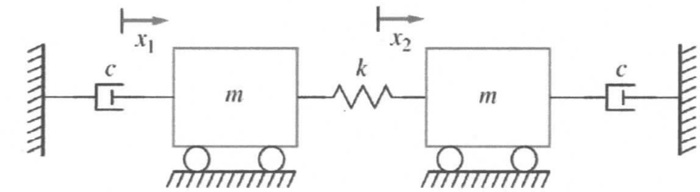
2. You are asked to determine the undamped and damped natural frequencies of the vibrating system shown in the diagram, also the mode shapes and corresponding damping factors.



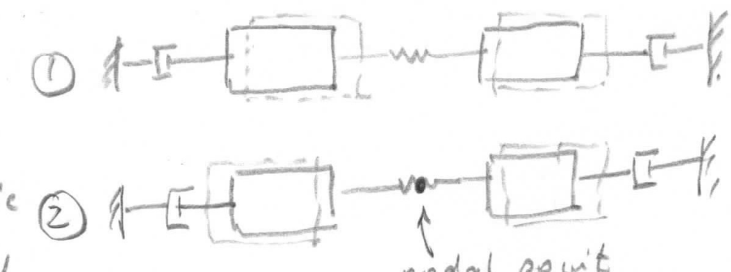
   (a) What features of the vibrating system do you notice that could allow you to greatly simplify your analysis ?

   (b) Determine the undamped and damped natural frequencies, the mode shapes and corresponding damping factors of the vibrating system.

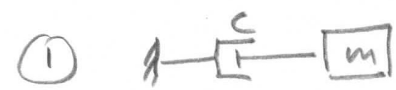
   (c) Explain any notable features of your results.

   (Hint: if you get stuck in a lot of algebra, go back to question (a)).

(a) This vibrating system is very similar to the one often discussed in class. It is geometrically symmetric and so has symmetric and anti-symetric vibration modes.



(b) Mode shape ① is $\begin{bmatrix}1\\1\end{bmatrix}$ and so the central spring is displaced but not deformed. It can be removed without effect.

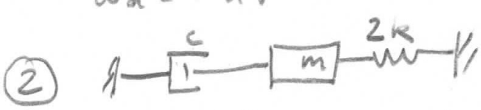


$\omega_n = \sqrt{\frac{0}{m}} = 0$

$\zeta = \frac{c}{2\sqrt{0m}} = \infty$ (both meaningless)

$\omega_d = \omega_n\sqrt{1-\zeta^2}$

Mode shape ② is $\begin{bmatrix}1\\-1\end{bmatrix}$ and so the centre of the central spring is a nodal point. The half spring has stiffness 2k



$\omega_n = \sqrt{\frac{2k}{m}}$ $\quad \zeta = \frac{c}{\sqrt{8km}}$

$\omega_d = \omega_n\sqrt{1-\zeta^2}$

(c) The first vibration mode corresponds to a rigid-body motion because both masses could be moved together and would be in equilibrium at a new place.

Hence $\omega_n = 0$

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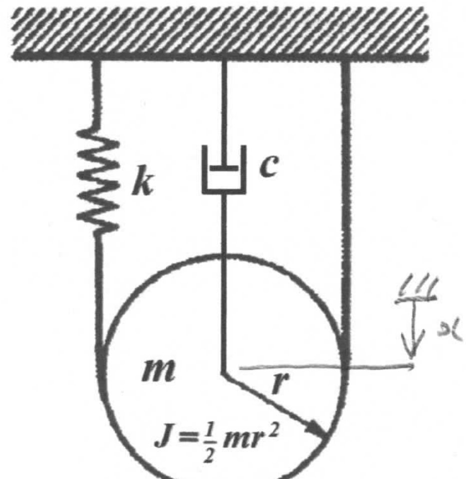
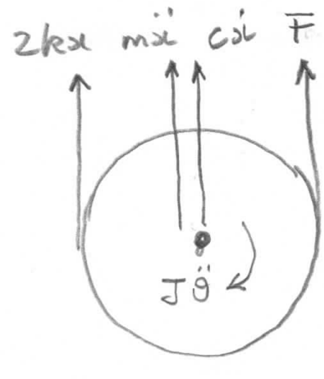
3. A part of a machine consists of a solid cylinder of mass m and radius r, supported by a spring if stiffness k. Draw a free-body diagram of the system and determine the undamped and damped natural frequencies and the damping factor. *(Hint: You need consider only the vertical motion of the cylinder.)*



x = downward displacement of centre of cylinder. Spring stretches by 2x. Consider vibrations around the equilibrium position
⇒ no need to include gravity,



No need to guess the right side force F by taking moments around right side:

$$J\ddot{\theta} + mr\ddot{x} + cr\dot{x} + 4krx = 0$$

where $\theta = \frac{x}{r}$ and $J = \frac{1}{2}mr^2$

$\div r$ $$\frac{3}{2}m\ddot{x} + c\dot{x} + 4kx = 0$$

$$\omega_n = \sqrt{\frac{4k}{\frac{3}{2}m}} = \sqrt{\frac{8k}{3m}}$$

using standard formulas

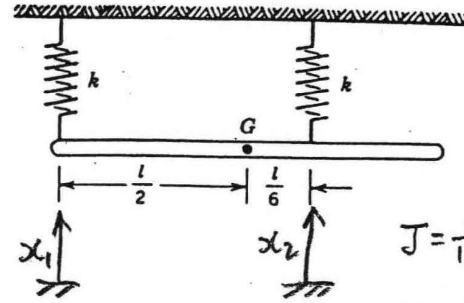
$$\omega_n = \sqrt{\frac{k}{m}} \quad \zeta = \frac{c}{2\sqrt{km}}$$

$$\zeta = \frac{c}{2\sqrt{4k\cdot\frac{3}{2}m}} = \frac{c}{\sqrt{24km}}$$

$$\omega_d = \omega_n\sqrt{1-\zeta^2}$$

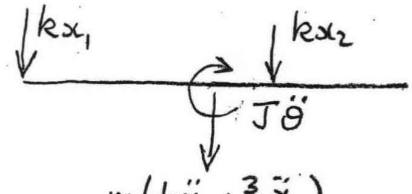
4. Part of a machine contains a uniform slender rod of length ℓ, mass m, and centroidal $J = m\ell^2/12$. The rod is supported by two springs, each of stiffness k. One spring secures the rod at its left end, and the other spring secures the rod one third the way from the right end.

Choose coordinates that will give no static coupling. Formulate the equations of motion and determine the natural frequencies and corresponding mode shapes. Sketch the mode shapes and comment on any interesting features of the system.



$J = \frac{1}{12} m l^2$

$\theta = \frac{x_2 - x_1}{2/3\, l}$



For no static coupling, choose coordinates x_1 and x_2 based on the springs. From FBD

$$\Sigma M_2 = \tfrac{1}{12} m l^2 \left(\frac{\ddot{x}_2 - \ddot{x}_1}{2/3\, l}\right) - \tfrac{2}{3} l k x_1 - \tfrac{1}{6} m \left(\tfrac{1}{4}\ddot{x}_1 + \tfrac{3}{4}\ddot{x}_2\right) = 0$$

$$\Sigma M_1 = \tfrac{1}{12} m l^2 \left(\frac{\ddot{x}_2 - \ddot{x}_1}{2/3\, l}\right) + \tfrac{2}{3} l k x_2 + \tfrac{1}{2} m \left(\tfrac{1}{4}\ddot{x}_1 + \tfrac{3}{4}\ddot{x}_2\right) = 0$$

$$\rightarrow -\tfrac{1}{8} m l \ddot{x}_1 + \tfrac{1}{8} m l \ddot{x}_2 - \tfrac{2}{3} l k x_1 - \tfrac{ml}{24}\ddot{x}_1 - \tfrac{1}{8} m l \ddot{x}_2 = 0$$

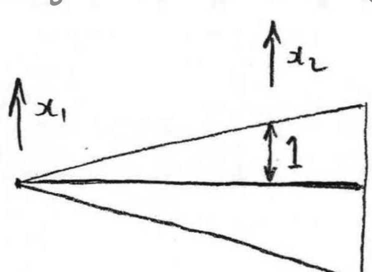
$$-\tfrac{1}{8} m l \ddot{x}_1 + \tfrac{1}{8} m l \ddot{x}_2 + \tfrac{2}{3} l k x_2 + \tfrac{ml}{8}\ddot{x}_1 + \tfrac{3ml}{8}\ddot{x}_2 = 0$$

$$\div \tfrac{l}{6} \quad \begin{bmatrix} m & 0 \\ 0 & 3m \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} 4k & 0 \\ 0 & 4k \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

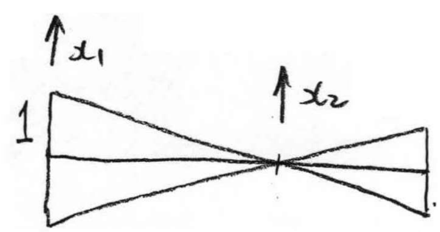
We've hit the jackpot! Having both $\underset{\sim}{M}$ and $\underset{\sim}{K}$ diagonal means that our spring-based coordinates are also the principal coordinates. (not usual). The equations of motion are uncoupled and separate out to:

$$\left.\begin{aligned} m\ddot{x}_1 + 4k x_1 &= 0 \\ 3m\ddot{x}_2 + 4k x_2 &= 0 \end{aligned}\right\} \rightarrow \begin{aligned} \omega_2^2 &= \frac{4k}{m} \\ \omega_1^2 &= \frac{4k}{3m} \end{aligned}$$

For principal coords., the mode shapes are: $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$



mode 1: $\omega_1^2 = \frac{4k}{3m}$



mode 2: $\omega_2^2 = \frac{4k}{m}$