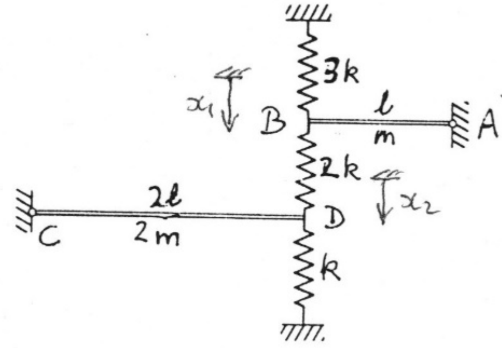
MECH 463 -- Homework 5

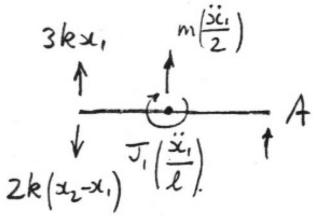
1. The diagram shows an idealization of a system of machine parts. Slender bars AB (length ℓ, mass m) and CD (length 2ℓ and mass 2m) pivot freely at A and C respectively. The bars are fastened to the springs by pins at B and D. The three springs have stiffnesses 3k, 2k and k, as shown. Determine the natural frequencies and mode shapes for small vibrations. Comment on any interesting features of the system.



Take moments about A and C

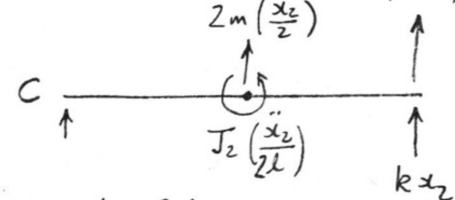
$$m\frac{\ddot{x}_1}{2}\cdot\frac{\ell}{2} + J_1\frac{\ddot{x}_1}{\ell} + 3kx_1\cdot\ell - 2k(x_2-x_1)\cdot\ell = 0$$

$$2m\frac{\ddot{x}_2}{2}\cdot\ell + J_2\frac{\ddot{x}_2}{2\ell} + 2k(x_2-x_1)\cdot 2\ell + kx_2\cdot 2\ell = 0$$



where $J_1 = \frac{1}{12}m\ell^2$

$J_2 = \frac{1}{12}\cdot 2m\cdot(2\ell)^2 = \frac{2}{3}m\ell^2$



Divide 1st equ by ℓ and 2nd equ by 2ℓ.

$$\frac{m}{3}\ddot{x}_1 + 5kx_1 - 2kx_2 = 0$$
$$\frac{2m}{3}\ddot{x}_2 - 2kx_1 + 3kx_2 = 0$$

$$\longrightarrow \begin{bmatrix}\frac{m}{3} & 0\\ 0 & \frac{2m}{3}\end{bmatrix}\begin{bmatrix}\ddot{x}_1\\ \ddot{x}_2\end{bmatrix} + \begin{bmatrix}5k & -2k\\ -2k & 3k\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} = \begin{bmatrix}0\\ 0\end{bmatrix}$$

"ℓ" does not appear in the equations of motion. Therefore, only the masses of the rods are significant, not their lengths.

Assume a solution $\underline{x} = \underline{X}\cos\omega t \longrightarrow (-\omega^2\underset{\sim}{M} + \underset{\sim}{K})\underline{X}\cos\omega t = \underline{0}$

For a non-trivial solution valid for all t $\longrightarrow |-\omega^2\underset{\sim}{M} + \underset{\sim}{K}| = 0$

$$\longrightarrow \begin{vmatrix}5k - \frac{m}{3}\omega^2 & -2k\\ -2k & 3k - \frac{2m}{3}\omega^2\end{vmatrix} = 0$$

$$\longrightarrow \left(5k - \frac{m}{3}\omega^2\right)\left(3k - \frac{2m}{3}\omega^2\right) - 4k^2 = 0$$

$$= \frac{2m^2}{9}\omega^4 - \frac{13}{3}km\,\omega^2 + 11k^2 = 0$$

$$\longrightarrow \omega^2 = \frac{\frac{13}{3}km \pm \sqrt{\frac{169}{9}k^2m^2 - \frac{88}{9}k^2m^2}}{\frac{4}{9}m^2} = \frac{3}{4}\cdot\frac{k}{m}\left(13 \pm 9\right)$$

$$\omega^2 = 3\,k/m \text{ and } \frac{33}{2}\,k/m$$

For mode shape, try $\underset{-}{X} = C \begin{bmatrix} 1 \\ u \end{bmatrix}$

$$\left(-\omega^2 \underset{\sim}{M} + \underset{\sim}{K}\right) \underset{-}{X} \cos \omega t = 0 \quad \longrightarrow \quad \begin{bmatrix} 5k - \frac{m}{3}\omega^2 & -2k \\ -2k & 3k - \frac{2m}{3}\omega^2 \end{bmatrix} \begin{bmatrix} 1 \\ u \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

From first equation $\longrightarrow 5k - \frac{m}{3}\omega^2 - 2ku = 0$

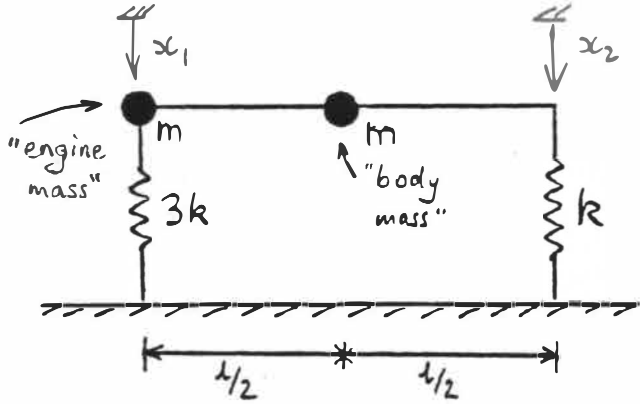
$$\longrightarrow u = \frac{5k - \frac{m}{3}\omega^2}{2k}$$

When $\omega_1^2 = 3 k/m \longrightarrow u_1 = \dfrac{5k - \frac{m}{3} \cdot 3 k/m}{2k} = 2$

When $\omega_2^2 = \frac{33}{2} k/m \longrightarrow u_2 = \dfrac{5k - \frac{m}{3} \cdot \frac{33}{2} k/m}{2k} = -\frac{1}{4}$

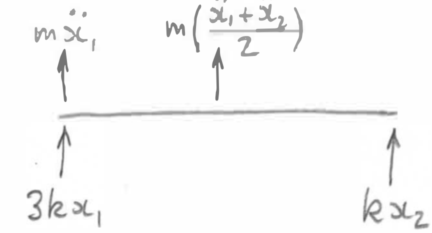
$\rightarrow$ Mode shapes are $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -1/4 \end{bmatrix}$

2. (a) The diagram show a very idealized 2-DOF model of an automobile. A beam of length ℓ, whose mass m is concentrated at its centre, represents the body of the vehicle. A concentrated mass m at one end of the beam represents the engine. Springs of stiffness 3k and k represent the front and rear suspensions respectively. Choose a coordinate system that will give no static coupling, formulate the equations of motion, and solve for the two natural frequencies.



(b) Repeat part (a) using a coordinate system based on the displacement and rotation of the midpoint between the two masses, i.e., the overall mass centre. Formulate the equations of motion for this coordinate system. What type of coupling is present ? Solve for the natural frequencies in the most direct and simple way, and confirm that the results are the same as found in part (a).

(a)



Choose spring-based coordinates x_1, x_2 to get no static coupling.

Take moments about each end of FBD.

$$m\ddot{x}_1 l + m(\ddot{x}_1+\ddot{x}_2)l/4 + 3kx_1 l = 0$$

$$m(\ddot{x}_1+\ddot{x}_2)l/4 + kx_2 l = 0$$

no static coupling.

Multiply by $4/l$ $\longrightarrow$ $$\begin{bmatrix} 5m & m \\ m & m \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} 12k & 0 \\ 0 & 4k \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Try solution $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}\cos(\omega t+\phi)$

$$\longrightarrow \left(-\omega^2\begin{bmatrix} 5m & m \\ m & m \end{bmatrix} + \begin{bmatrix} 12k & 0 \\ 0 & 4k \end{bmatrix}\right)\begin{bmatrix} X_1 \\ X_2 \end{bmatrix}\cos(\omega_f t+\phi) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

For a solution valid for all time:

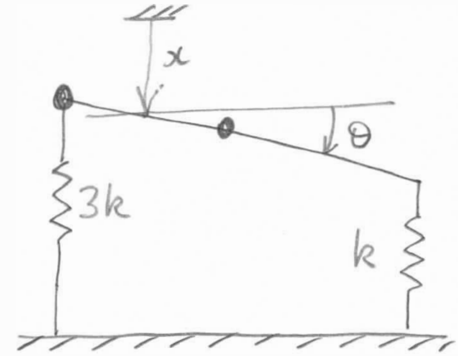
$$\begin{bmatrix} 12k-5\omega^2 m & -\omega^2 m \\ -\omega^2 m & 4k-\omega^2 m \end{bmatrix}\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \xrightarrow[\text{solution}]{\text{non-trivial}} \begin{vmatrix} 12k-5\omega^2 m & -\omega^2 m \\ -\omega^2 m & 4k-\omega^2 m \end{vmatrix} = 0$$

$$\longrightarrow (12k-5\omega^2 m)(4k-\omega^2 m) - \omega^4 m^2 = 0$$

$$= 4m^2\omega^4 - 32km\omega^2 + 48k^2 = 0$$

$$\longrightarrow \omega^2 = \frac{32 \pm \sqrt{1024-768}}{8}\cdot\frac{k}{m} \longrightarrow \boxed{\begin{aligned}\omega_1^2 &= 2\,k/m \\ \omega_2^2 &= 6\,k/m\end{aligned}}$$

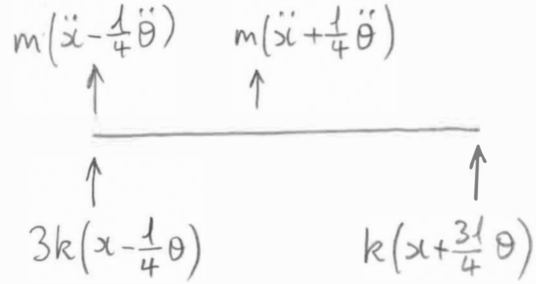
(b) Use coordinates based on the overall mass centre, x, θ



Force and moment balance about mass centre

$$m(\ddot{x} \cancel{- \tfrac{\ell}{4}\ddot{\theta}}) + m(\ddot{x} \cancel{+ \tfrac{\ell}{4}\ddot{\theta}})$$
$$+3k(x \cancel{- \tfrac{\ell}{4}\theta}) + k(x \cancel{+ \tfrac{3\ell}{4}\theta}) = 0$$

$$-m(\cancel{\ddot{x}} - \tfrac{\ell}{4}\ddot{\theta})\cdot\tfrac{\ell}{4} + m(\cancel{\ddot{x}} + \tfrac{\ell}{4}\ddot{\theta})\cdot\tfrac{\ell}{4}$$
$$- 3k(\cancel{x} - \tfrac{\ell}{4}\theta)\cdot\tfrac{\ell}{4} + k(\cancel{x} + \tfrac{3\ell}{4}\theta)\cdot\tfrac{3\ell}{4} = 0$$



$$\longrightarrow 2m\ddot{x} + 4kx = 0$$

and $2m\ddot{\theta} + 12k\theta = 0$ (after $\times \frac{16}{\ell^2}$)

These equations are entirely uncoupled (no dynamic or static coupling). x and θ happen to be the principal coordinates in this case (not true in general for coords based on overall mass centre). From the uncoupled equations

$$\underline{\omega_1^2 = \frac{4k}{2m} = 2\,k/m} \qquad \underline{\omega_2^2 = \frac{12k}{2m} = 6\,k/m}$$