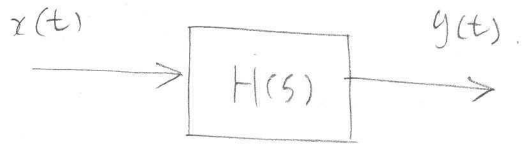
< Stability assessment of feedback Systems >.

- objective
  - Understand the Root Locus & Nyquist test.
  - Their relation with the loop transfer function : $L(s)$
  - phase margin & Nyquist plot.

- Stability Condition for LTI Systems



  - An LTI system with transfer function $H(s)$ is stable if and only if all of the poles of $H(s)$ are in the left-half plane. That is, $Re\{p_i\} < 0$.
  - In other words, no poles in the right-half plane (RHP).

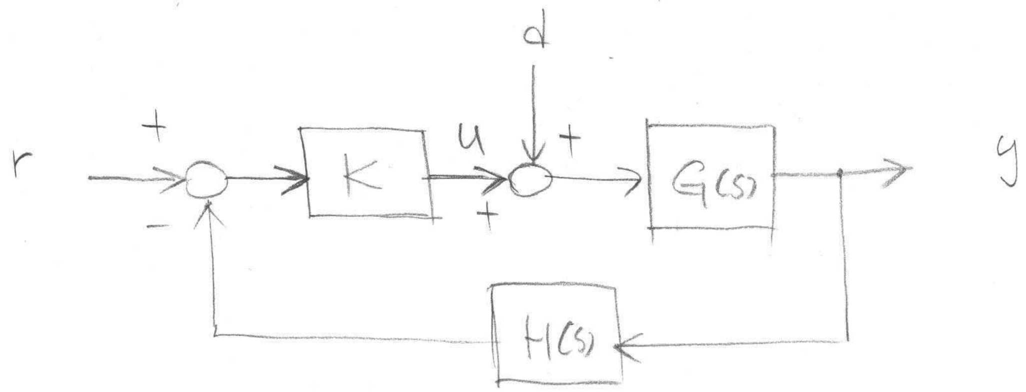
- Stability assessment of feedback systems.
  - Directly checking the closed-loop poles is difficult, and it does not provide guidance for controller design.
  - There exist methods that use the information about the loop transfer function to assess the closed-loop stability

① Root Locus : shows the RHP CL pole locations (explicit)

② Nyquist test : tells the number of RHP CL poles. (implicit)

∘ Characteristic Equation.

Consider a closed-loop system :



· Closed-loop transfer function "matrix" is

$$\begin{bmatrix} Y \\ U \end{bmatrix} = \begin{bmatrix} \frac{KG}{1+KGH} & \frac{G}{1+KGH} \\ \frac{K}{1+KGH} & \frac{-KGH}{1+KGH} \end{bmatrix} \begin{bmatrix} R \\ D \end{bmatrix}$$

$$= \underbrace{\frac{1}{1+KGH}}_{\triangleq S(s)} \begin{bmatrix} KG & G \\ K & -KGH \end{bmatrix} \begin{bmatrix} R \\ D \end{bmatrix}$$

· The system is stable if $\frac{1}{1+KGH}$ does not have RHP poles.

· This is equivalent to $f(s) \triangleq 1 + K\,G(s)\,H(s)$ not having zeros (or "roots") in the RHP.

· $f(s) \triangleq 1 + \underbrace{K\,G(s)\,H(s)}_{L(s)}$ "Characteristic equation"

$L(s)$: Loop transfer function.

◎. No RHP poles for $\frac{1}{f(s)}$ $\iff$ No RHP zeros for $f(s)$.

- Root Locus : infers the CL pole locations from $L(s)$

  - Shows how the roots of $f(s)$ move with respec to a parameter.

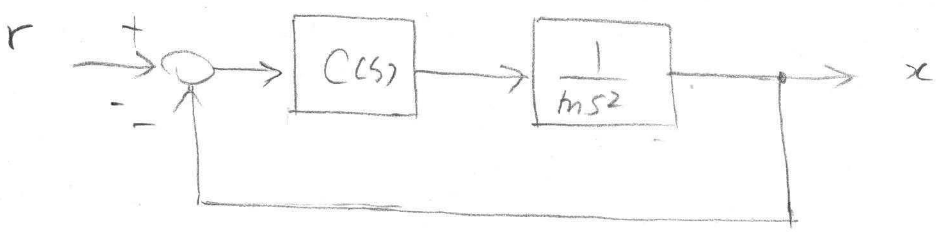
  - For control system, we mostly use "K" as the parameter.

$$f(s) = 1 + \underbrace{K\, G(s)\, H(s)}_{L(s)} = 0 \quad \rightarrow \quad G(s)\, H(s) = -\frac{1}{K}$$

$$\left(\measuredangle\, G(s)\, H(s) = 180^\circ \right)$$

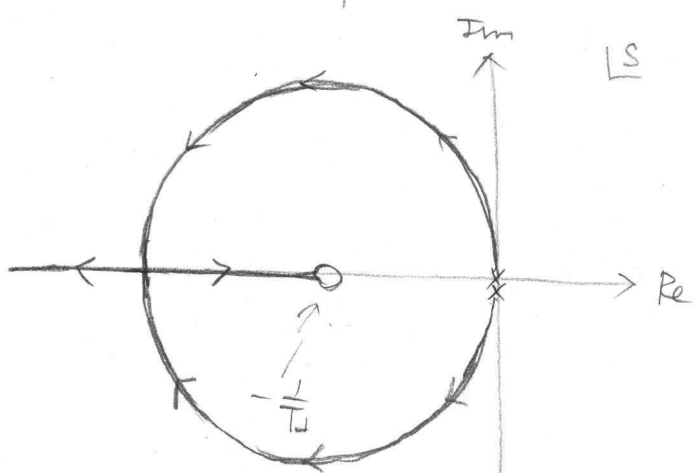
$$\begin{cases} \text{when } K \rightarrow \infty, & \text{roots of } f(s) \rightarrow \text{zeros of } L(s) \\ \text{when } K \rightarrow 0, & \text{roots of } f(s) \rightarrow \text{poles of } L(s) \end{cases}$$

ex). Free mass position control

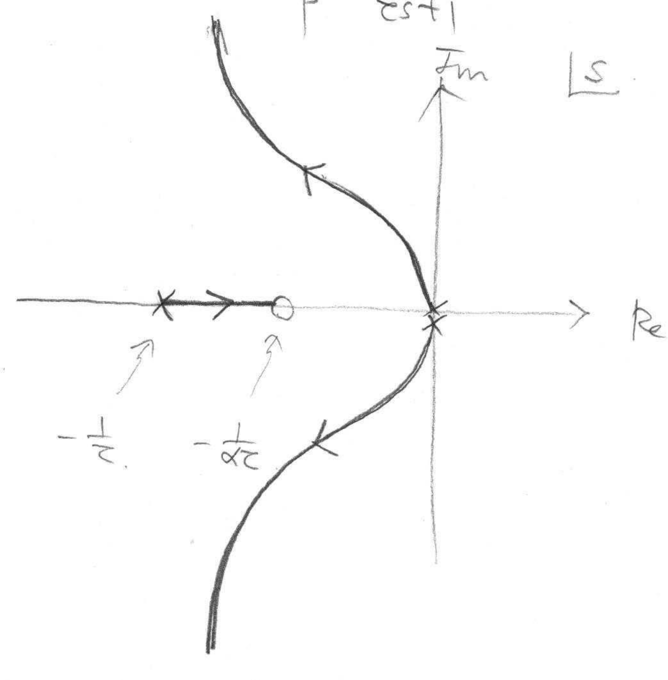


$$L(s) = C(s)\,\frac{1}{ms^2}$$

① $C(s) = K_p(1 + T_D s)$



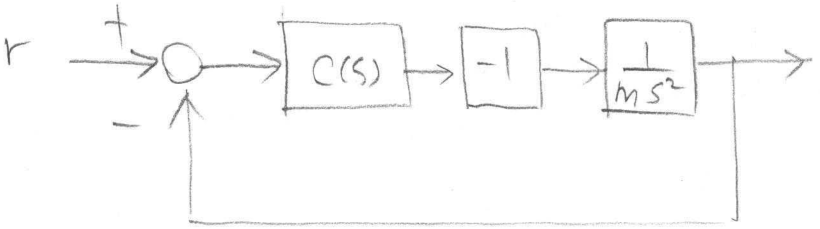
② $C(s) = K_p \dfrac{\alpha\tau s + 1}{\tau s + 1}$



O : zeros of $L(s)$ → : poles of CL

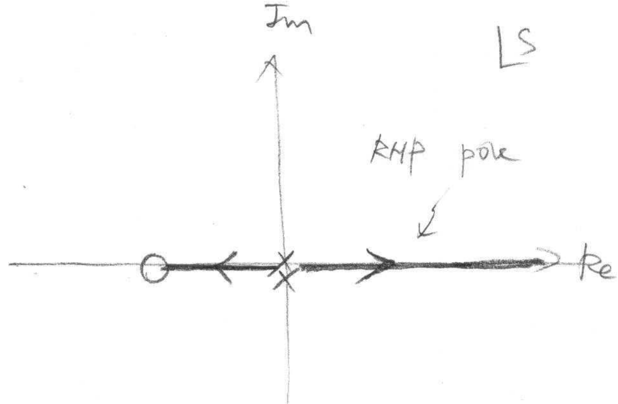
X : poles of $L(s)$. $K_p > 0$.

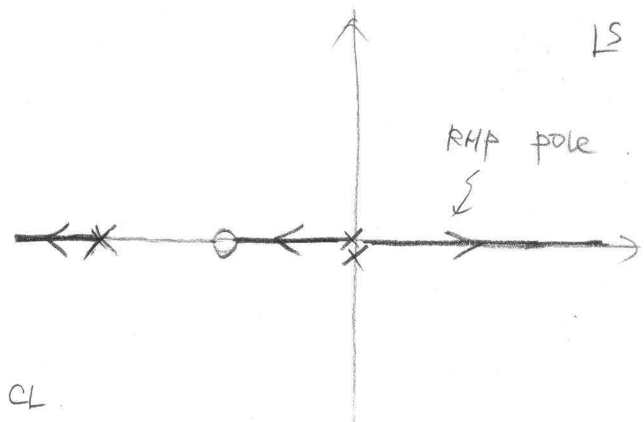
ex). Free mass with a negative sign



$L(s) = C(s)\dfrac{-1}{ms^2}$

① $C(s) = K_p(1 + T_D s)$　　② $C(s) = K_p \dfrac{\alpha\tau s + 1}{\tau s + 1}$

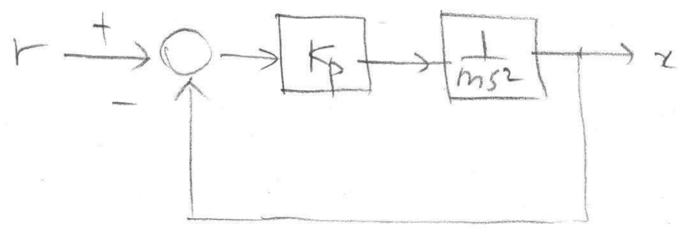


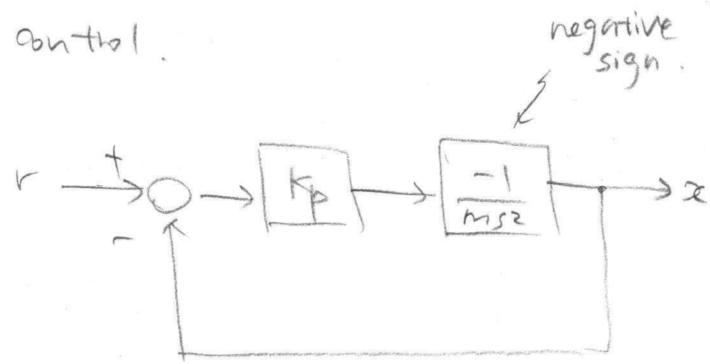


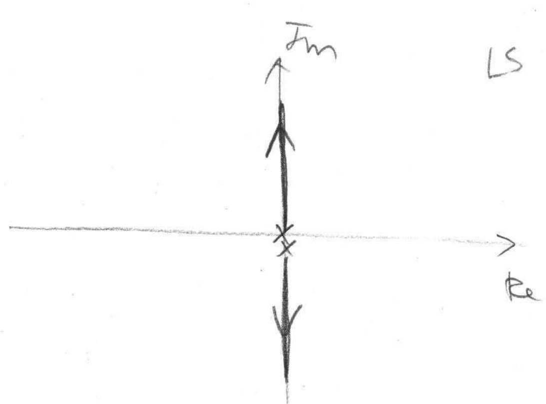
o : zeros of $L(s)$　　→ : poles of CL

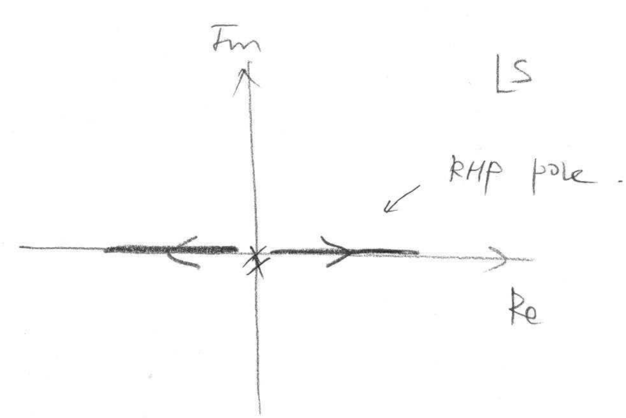
x : poles of $L(s)$　　$K_p > 0$.

ex) Free mass with proportional control.









x : poles of $L(s)$.　　$K_p > 0$.

→ : poles of CL.

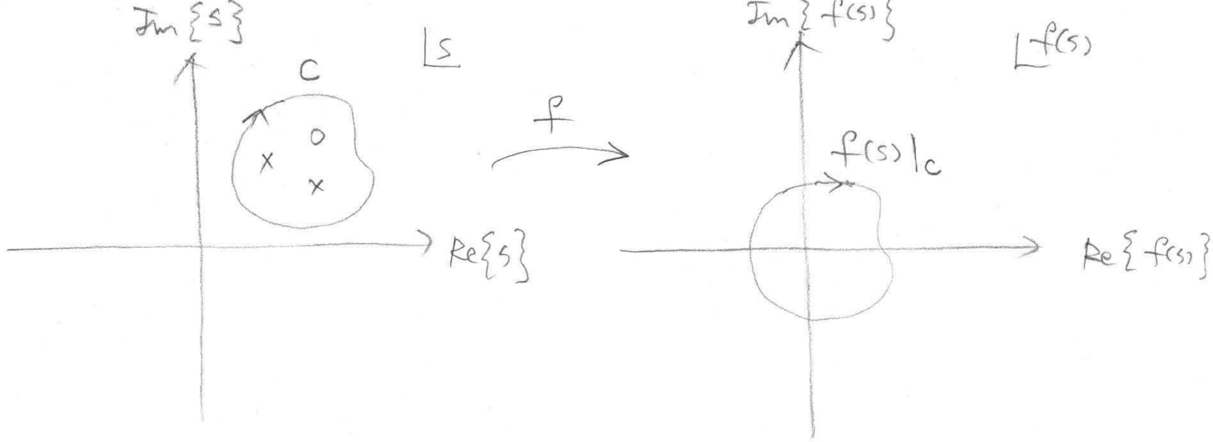
- Nyquist test : Infers the number of RHP CL poles from $L(s)$

- Root Locus Limitations.
  - Root Locus method becomes diffrinlt to use in practice if
    - There are too many poles & zeros, or
    - Delay in the loop.
  - We don't necessarily need to keep track of the exact closed-loop pole "locations" to check the stability.
  - We just need to check the existence of CL poles in the RHP.

- Nyquist test tells us the number of closed-loop poles in the RHP. → If the number is zero, the CL system is stable.
  - It is the mathematical foundation for loop-shaping design.
  - The concepts of gain margin & phase margin are derived from the Nyquist plot.
  - When we meet challenging control design problem, where the gain margin & phase margin methods fall apart, we need to go back to the Nyquist test. "First principle"
  - Nyquist plot consists of the loop Bode plot. $L(j\omega)$

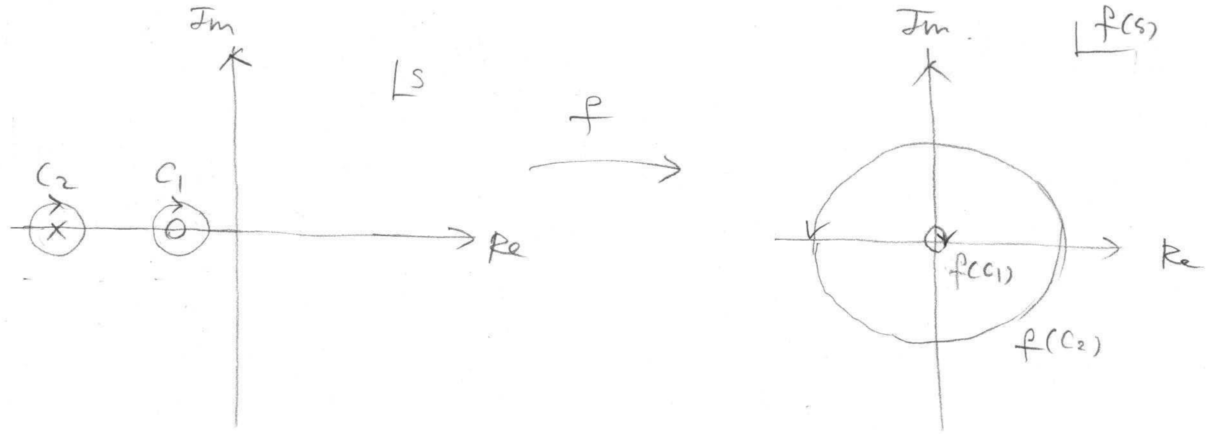
- Argument Principle

- Given a complex function $f(s)$ $\left(\text{i.e., } f: \mathbb{C} \to \mathbb{C}\right)$



. A contour C in the s-plane that captures Z number of zeros of $f(s)$ and P number of poles of $f(s)$ maps to an image $f(s)|_c$ that encircles the origin by N times, where $\boxed{N = Z - P}$.

ex) $f(s) = \dfrac{s+1}{s+10}$

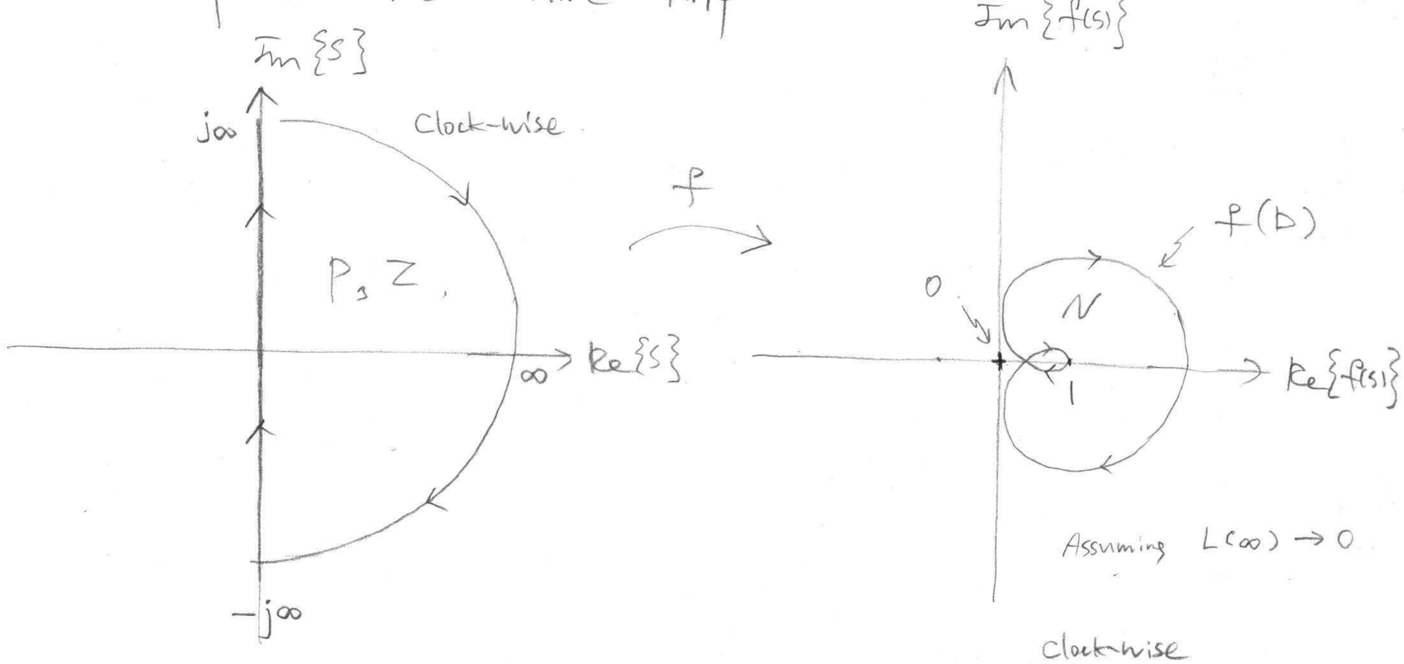


$C_1 : -1 + re^{j\theta}$ $\quad f(s)|_{C_1} \doteq \dfrac{re^{j\theta}}{9} = \dfrac{1}{90} e^{j\theta}$

$C_2 : -10 + re^{j\theta}$ $\quad f(s)|_{C_2} \doteq \dfrac{-9}{re^{j\theta}} = -90\, e^{-j\theta} = 90\, e^{j(\pi-\theta)}$

- Application to the Char. Eqn.

  - Let $f(s) = 1 + L(s)$: "characteristic equation"
  - Let the contour in the s-plane be a big "D" that captures the entire RHP.



- Z : # of zeros of $f(s)$ inside D on the s-plane

  P : # of poles of $f(s)$ inside D on the s-plane

N : # of the clockwise encirclement of the image $f(s)|_D$ about the origin of $f(s)$ plane.

- From the argument principle, $N = Z - P \iff \boxed{Z = N + P}$

- Z tells us $\left\{ \begin{array}{l} \text{\# of zeros of } f(s) \\ \text{\# of poles of } \frac{1}{f(s)} \end{array} \right\}$ in the RHP.

  ↳ $Z = 0$ means stable!

- N can be obtained by counting the # of encirclement about the origin.

- How about P? Since $f(s) = 1 + L(s)$,

  $\left\{ \begin{array}{l} \text{poles of } L(s) = \text{poles of } f(s) \\ L(s_0) \to \infty \iff 1 + L(s_0) \to \infty. \end{array} \right.$ So, we can count the RHP poles of $L(s)$ instead.

Nyquist Test.

- Slight modification on the previous → Nyquist test.

- $Z = N + P$

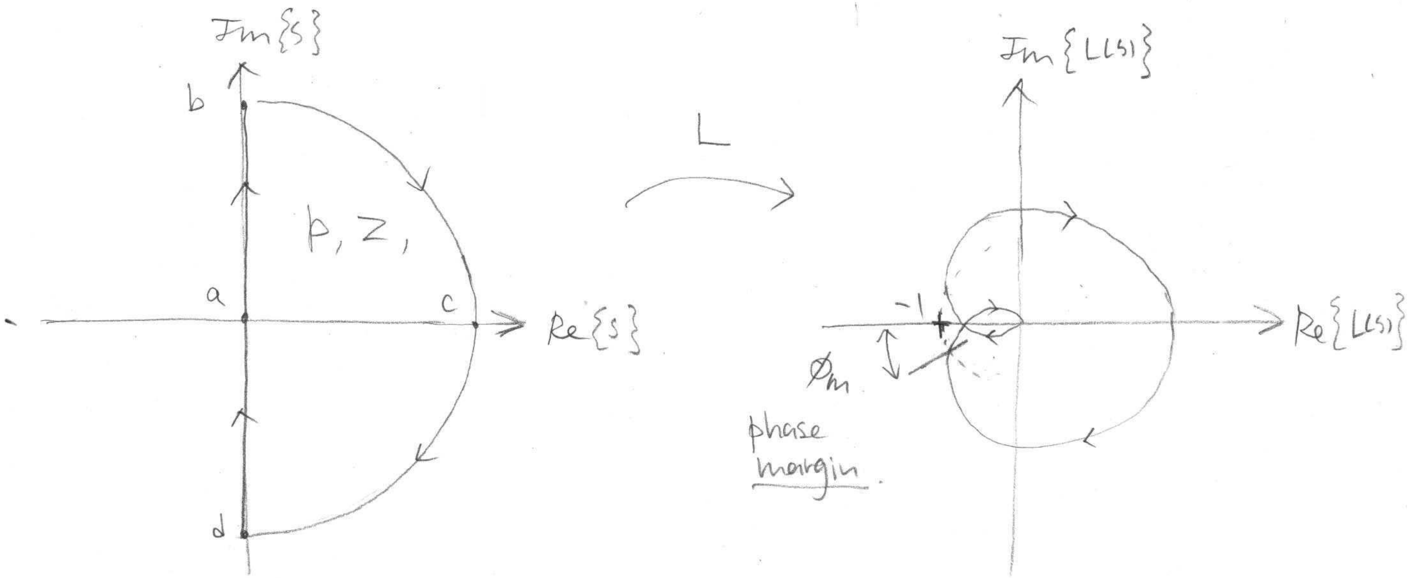
  ↑ End result: # of unstable CL poles

  P : # of unstable poles of $L(s)$

  N : can we obtain this number from $L(s)$ as well? Yes

  → # of clock-wise encirclement of $L(s)|_D$ about -1 point.

- Recall that $f(s) = 1 + L(s)$ → $L(s) = f(s) - 1$.



- The origin of the $f(s)$-plane → "-1 point" of the $L(s)$-plane
- The image $L(s)|_D$ is the shifted version of $f(s)|_D$ by -1.
- N can be obtained by counting the # of clock-wise encirclement of $\underbrace{L(s)|_D}_{\text{Nyquist plot}}$ about $\underbrace{\text{"}-1 \text{ point"}}_{\text{Nyquist point}}$

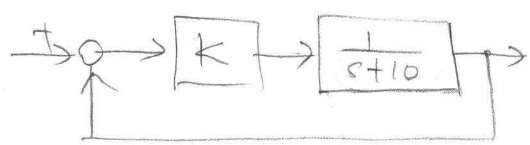
o Nyquist plot vs Loop Bode plot

- The contour D consists of theree segments $\overline{ab}$, $\overparen{bcd}$, $\overline{da}$
- $L(s)$ evaluated over $s \in \overline{ab}$ is the Bode plot.

  i.e.) $L(j\omega)$, $\omega > 0$.
- $L(s)$ evaluated over $s \in \overline{da}$ is the complex conjugate.

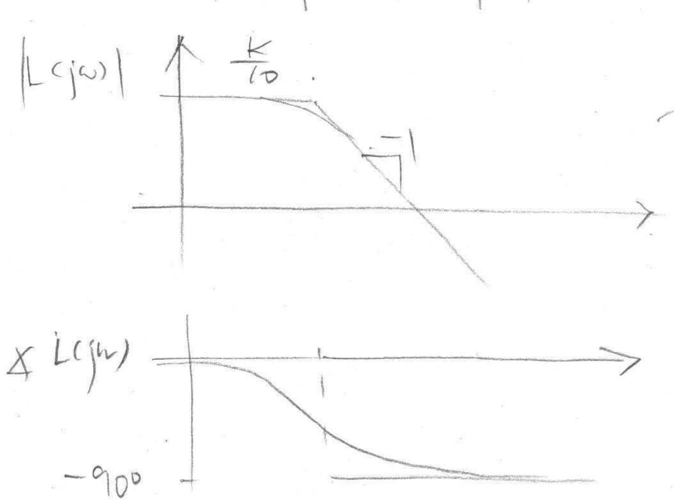
  i.e.) $L(-j\omega) = L(j\omega)^*$. $\quad Re\{L(-j\omega)\} = Re\{L(j\omega)\}$

  $Im\{L(-j\omega)\} = -Im\{L(j\omega)\}$
- $L(s)\big|_{s\to\infty} = 0$ for most practical systems.
- Thus, drawing the Nyquist plot is just re-drawing the loop Bode plot in a polar coordinate system.

ex) 

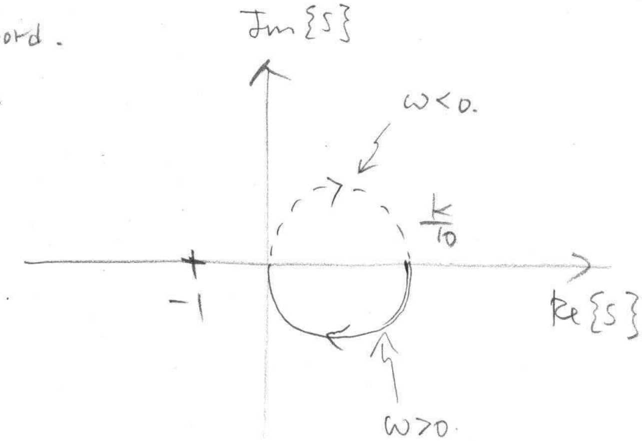
$L(s) = \frac{k}{s+10}$

< Loop Bode plot >

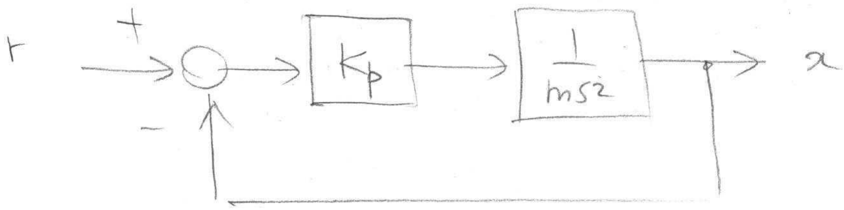


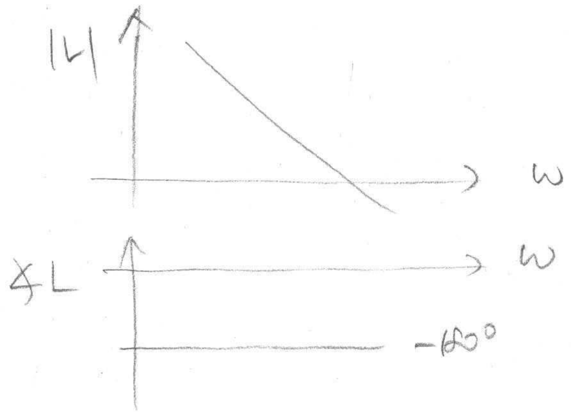
polar coord.

< Nyquist plot >

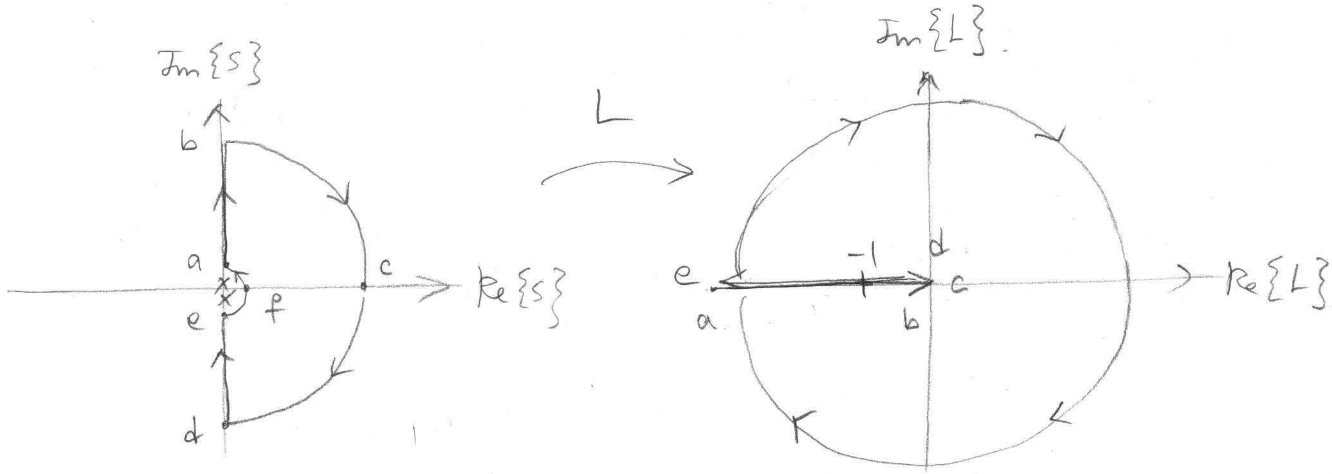


Example. Free mass + proportional control.





$$L(s) = \frac{K_p}{ms^2}$$

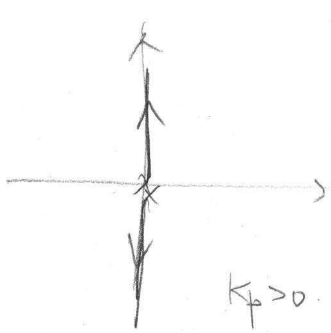


- $\widehat{efa}$: $re^{j\theta}$, $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$. ccw.

then, $L(s)\Big|_{\widehat{efa}} = \frac{K_p}{mr^2 e^{j2\theta}} = \frac{K_p}{mr^2} e^{j(-2\theta)}$ $\quad \measuredangle L \in (\pi, -\pi)$ cw.

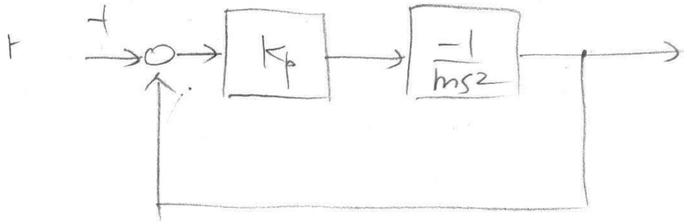
- $N = 0$, $P = 0 \rightarrow \underline{Z = 0}$ No unstable CL poles.

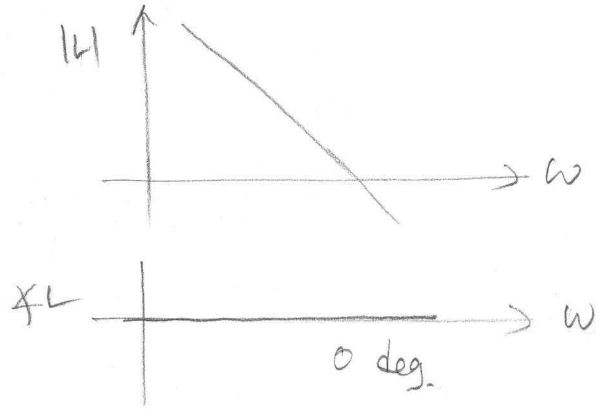
- It agrees with the Root Locus picture.



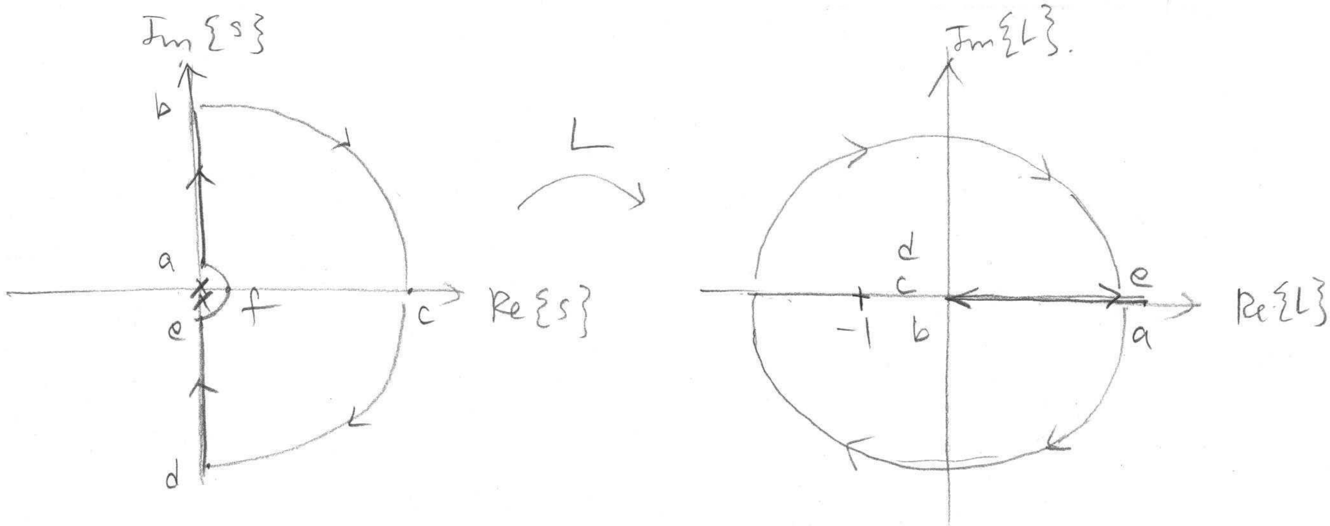
No unstable CL poles for $K_p > 0$.

Free mass with a negative sign.





$$L(s) = -\frac{K_p}{ms^2}$$



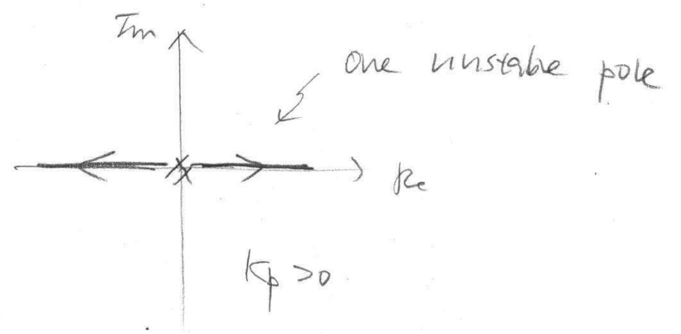
- $\widehat{efa}$: $r\cdot e^{j\theta}$ $\quad \theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$, ccw

$$L(s)\Big|_{\widehat{efa}} = -\frac{K_p}{mr^2}\cdot e^{j(-2\theta)} = \frac{K_p}{mr^2}\, e^{j(\pi-2\theta)}$$

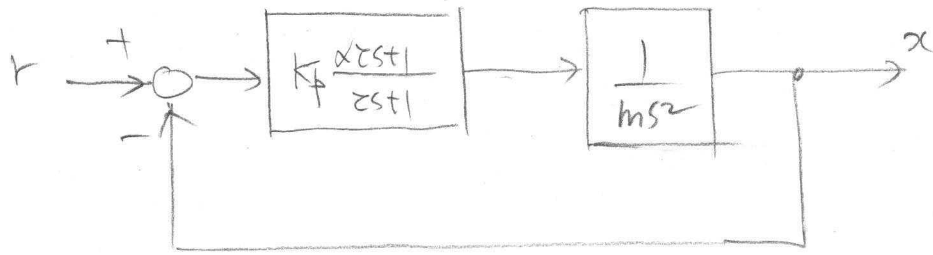
$\therefore \angle L \in (2\pi, 0)$, cw

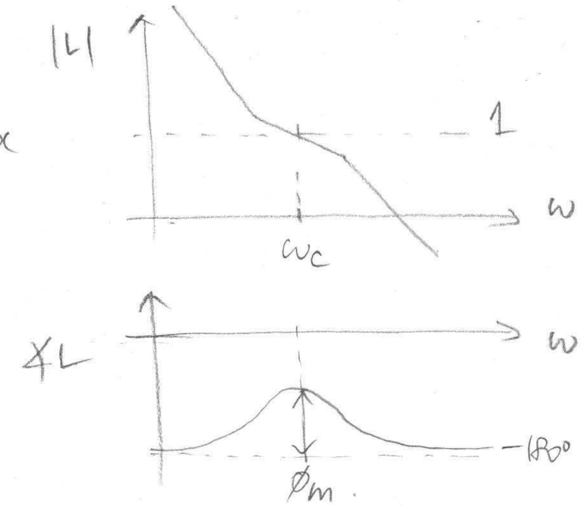
- $N = 1$, $P = 0$ $\rightarrow$ $Z = 1$ one RHP pole

- It agrees with the root locus.

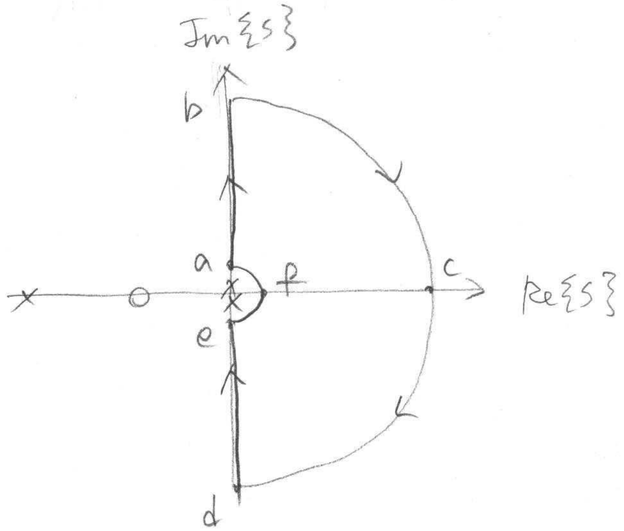


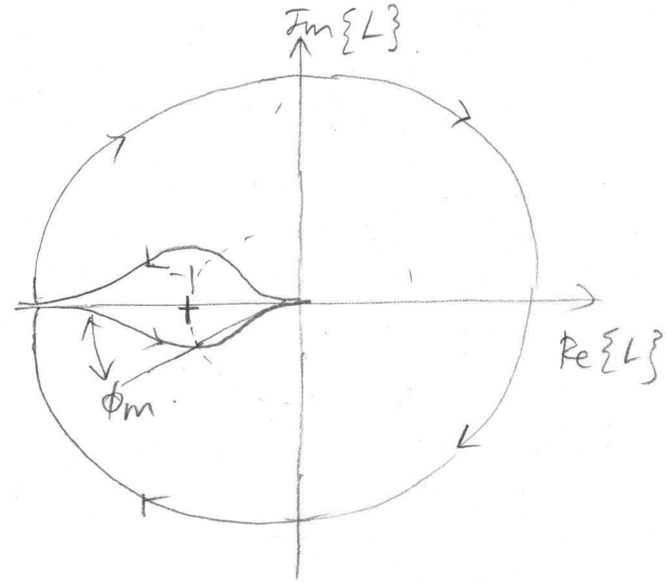
Free mass + Lead Compensator.





$$L(s) = K_p \frac{\alpha\tau s+1}{\tau s+1} \frac{1}{ms^2}$$





$N=0$, $p=0$ → $Z=0$ No unstable pole.