a place of mind

UBC

MECH468 : Modern Control Engineering
MECH509 : Controls

L27 : Discrete-time LQR

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Zoom lecture to be recorded and posted on Canvas

Course plan

Topics	CT	DT
Modeling	✓	✓
Stability	✓	✓
Controllability/observability	✓	✓
Realization	✓	✓
State feedback/observer	✓	✓
LQR/Kalman filter		→

Review & today's topic

- Continuous-time LQR optimal control
  - State feedback
  - Finite-horizon case
    - Time-varying gain
    - Matrix Riccati equation
  - Infinite-horizon case
    - Constant gain
    - Algebraic Riccati Equation (ARE)
- Discrete-time LQR optimal control

DT finite-horizon LQR optimal control

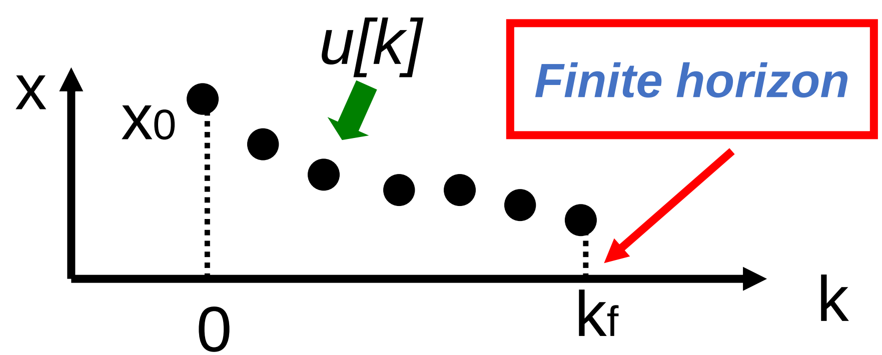
- Problem $\min_{u[\cdot]} J(u[\cdot])$ subj. to $\begin{cases} x[k+1] &= Ax[k]+Bu[k] \\ x[0] &= x_0 \text{ (given)} \end{cases}$

- J : *Quadratic* performance index (cost function)

$$J(u[\cdot]) := \sum_{k=0}^{k_f-1} \Big[\underbrace{x^T[k]Qx[k]}_{\text{For small state}} + \underbrace{u^T[k]Ru[k]}_{\text{For small input}} \Big] + \underbrace{x^T[k_f]Sx[k_f]}_{\text{For small final state}}$$

Design parameters

$$Q \geq 0, \;\; R > 0, \;\; S \geq 0$$



LQR optimal control law

- LQR optimal control is obtained as a state feedback

$$u[k] = -\left[R + B^T P[k+1] B\right]^{-1} B^T P[k+1] A x[k] \quad \boxed{\textit{Linear}}$$

  - *P[k]*: positive semidefinite solution to a *matrix Riccati difference equation* (A dual equation will appear in Kalman filter.)

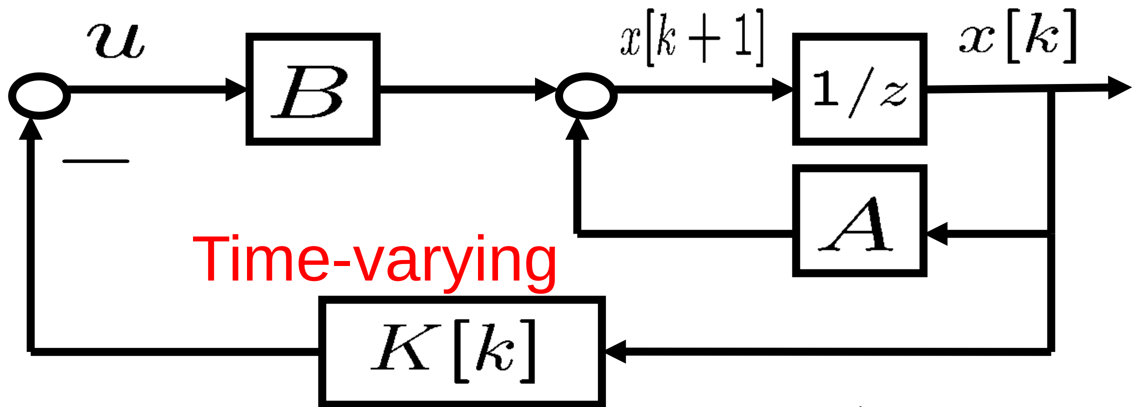
$$\begin{cases} P[k] &= A^T P[k+1] A + Q \\ & \quad - A^T P[k+1] B \left[R + B^T P[k+1] B\right]^{-1} B^T P[k+1] A \\ P[k_f] &= S \end{cases}$$

  - Optimal performance index $J(u) = x_0^T P[0] x_0$

(Proof given in the Appendix)

LQR optimal control law (cont'd)

- Block diagram



$$K[k] = \left[R + B^T P[k+1] B\right]^{-1} B^T P[k+1] A$$

- Q1: How to solve the matrix Riccati difference equation?

  Solve recursively!

- Q2: How to derive the LQR optimal control law? (Appendix)

Example

$$\min_{u[\cdot]} \sum_{k=0}^{1} \left(2x[k]^2 + u[k]^2\right) + x[2]^2$$

$$\text{subj. to} \begin{cases} x[k+1] &= x[k] + u[k] \\ x(0) &= x_0 \text{ (given)} \end{cases}$$

$\Rightarrow$ $A = 1, B = 1, Q = 2$
$R = 1, S = 1$

$$\Rightarrow \begin{cases} P[2] &= 1 \\ P[1] &= P[2] + 2 - \frac{P[2]^2}{1+P[2]} = \frac{5}{2} \\ P[0] &= P[1] + 2 - \frac{P[1]^2}{1+P[1]} = \frac{17}{14} \end{cases}$$

Compute these off-line!

$$\Rightarrow \begin{cases} u[0] = -\dfrac{P[1]}{1+P[1]} x_0 \\ u[1] = -\dfrac{P[2]}{1+P[2]} \underbrace{(x[0] + u[0])}_{x[1]} \\ J(u[\cdot]) = \dfrac{17}{14} x_0^2 \end{cases}$$

DT infinite-horizon LQR optimal control

- Problem $\min_{u[\cdot]} J(u[\cdot])$ subj. to $\begin{cases} x[k+1] = Ax[k] + Bu[k] \\ x[0] = x_0 \text{ (given)} \end{cases}$

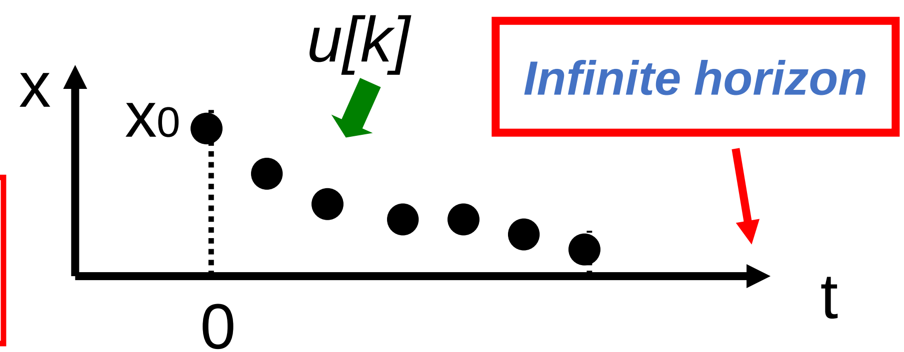
  - *J* : *Quadratic* performance index (cost function)

$$J(u[\cdot]) := \sum_{k=0}^{\infty} \Big[\underbrace{x^T[k]Qx[k]}_{\text{For small state}} + \underbrace{u^T[k]Ru[k]}_{\text{For small input}} \Big]$$

Design parameters

$$Q \geq 0, \;\; R > 0$$

Assumptions: (*A*,*B*) controllable & (*A*,*Q*) observable



LQR optimal control law

- LQR optimal control is obtained as a state feedback

$$u[k] = -\left[R + B^T P B\right]^{-1} B^T P A x[k] \quad \boxed{\textit{Linear}}$$

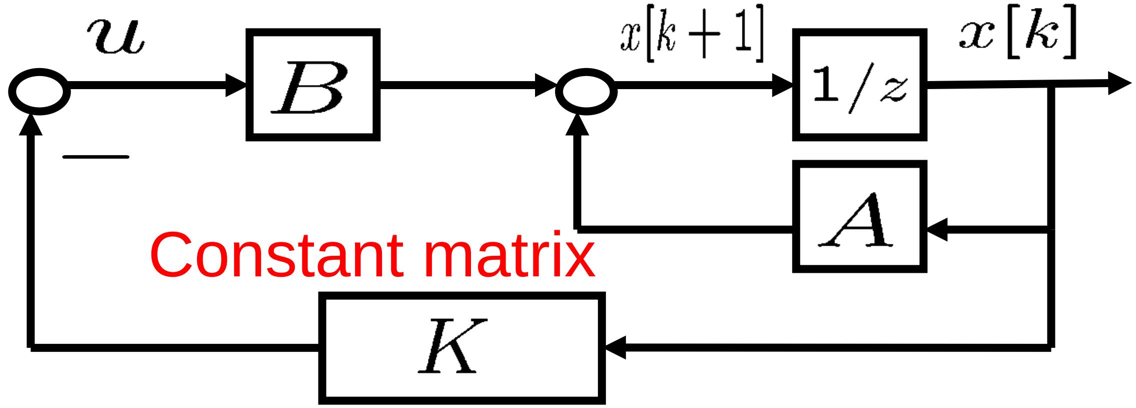
  - *P*: unique positive definite solution to a *discrete algebraic Riccati equation (DARE)*

$$A^T P A - P + Q - A^T P B\left[R + B^T P B\right]^{-1} B^T P A = 0$$

  - CL system is stable, i.e., |eig(*A*-*BK*)|<1
  - Optimal performance index $J(u) = x_0^T P x_0$

LQR optimal control law (cont'd)

- Block diagram



$$K = \left[R + B^T P B\right]^{-1} B^T P A$$

- DARE is obtained by setting *P[k]=P* in matrix Riccati difference equation for finite-horizon LQR problem.

$$\overset{P}{P[k]} = A^T \overset{P}{P[k+1]} A + Q - A^T \overset{P}{P[k+1]} B \left[R + B^T \overset{P}{P[k+1]} B\right]^{-1} B^T \overset{P}{P[k+1]} A$$

How to solve DARE

$$A^T PA - P + Q - A^T PB\left[R + B^T PB\right]^{-1} B^T PA = 0$$

1. Numerically in Matlab.

$$[P, \Lambda_-, K] = \mathrm{dare}(A, B, Q, R)$$

Closed-loop eigenvalues **Controller gain**

2. Brute force (Next slide)
3. Method with Hamiltonian matrix (not covered)

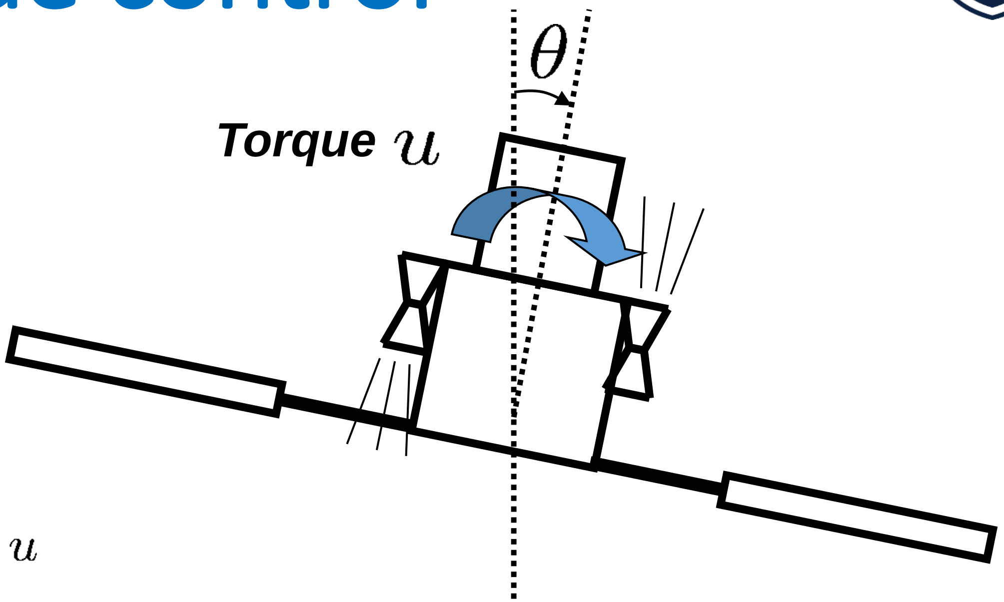
Note: The uniqueness of the positive definite solution to DARE is guaranteed by the assumptions "(*A*,*B*) is controllable and (*A*,*Q*) is observable."

Example of DT LQR

$$\min_{u[\cdot]} \sum_{k=0}^{\infty} \Big[\underbrace{2x[k]^2}_{Q=2} + \underbrace{u[k]^2}_{R=1}\Big] \quad \text{subj. to } x[k+1] = x[k] + u[k]$$

- DARE $P = A^T PA + Q - A^T PB(R + B^T PB)^{-1} B^T PA$ ➡ $P = 1 + \sqrt{3}$
- Control gain $K = (R + B^T PB)^{-1} B^T PA = \dfrac{1+\sqrt{3}}{2+\sqrt{3}}$
- Closed-loop *A*-matrix $A_{cl} = \dfrac{1}{2+\sqrt{3}}$

Satellite attitude control



- After normalization,

$$\ddot{\theta} = u$$

- SS model $x := \left[\theta, \dot{\theta}\right]^T$

$$\begin{cases} \dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \\ y = \begin{bmatrix} 1 & 0 \end{bmatrix} x \end{cases}$$

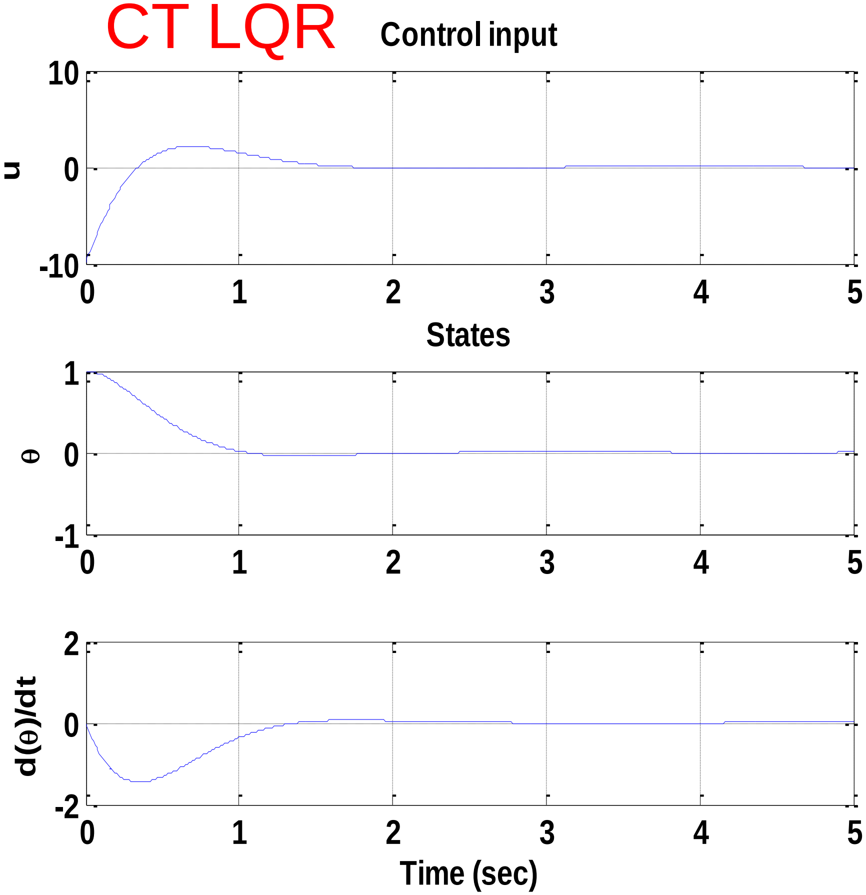
- Requirements
  - Small θ
  - Small *u*

Design LQR controller for the discretized system.

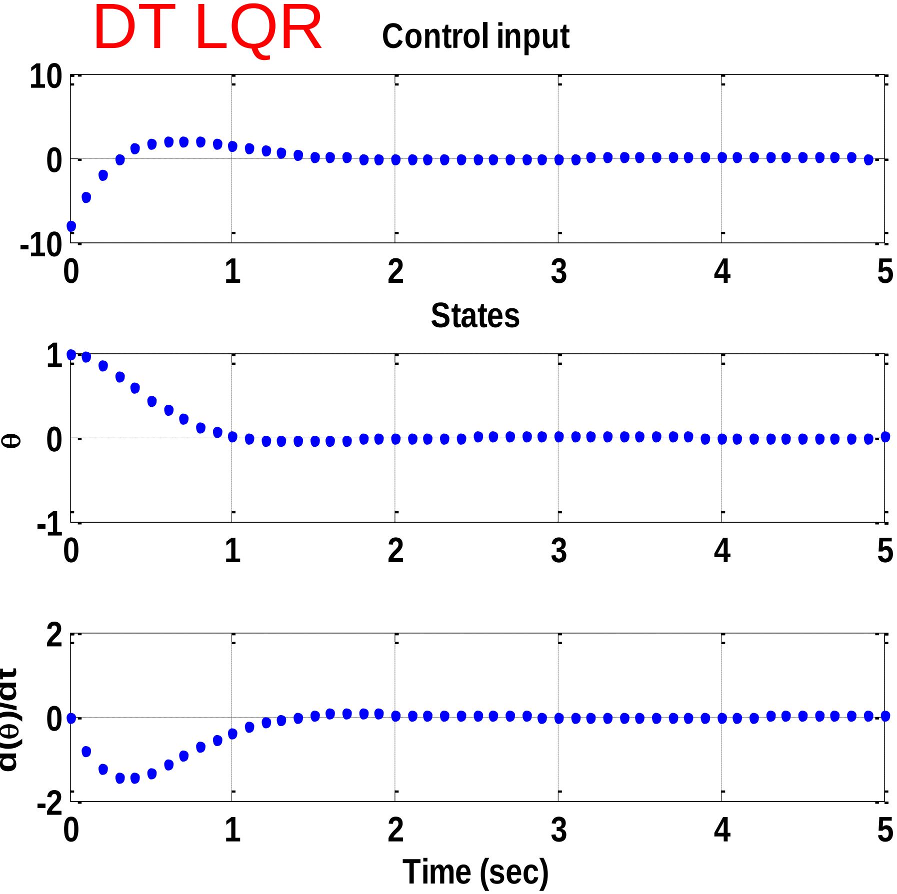
$$Ad = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix}, \quad Bd = \begin{bmatrix} T^2/2 \\ T \end{bmatrix}$$

Satellite attitude control (Finite-horizon, $R=0.01$, $T=0.1$ sec)

CT LQR

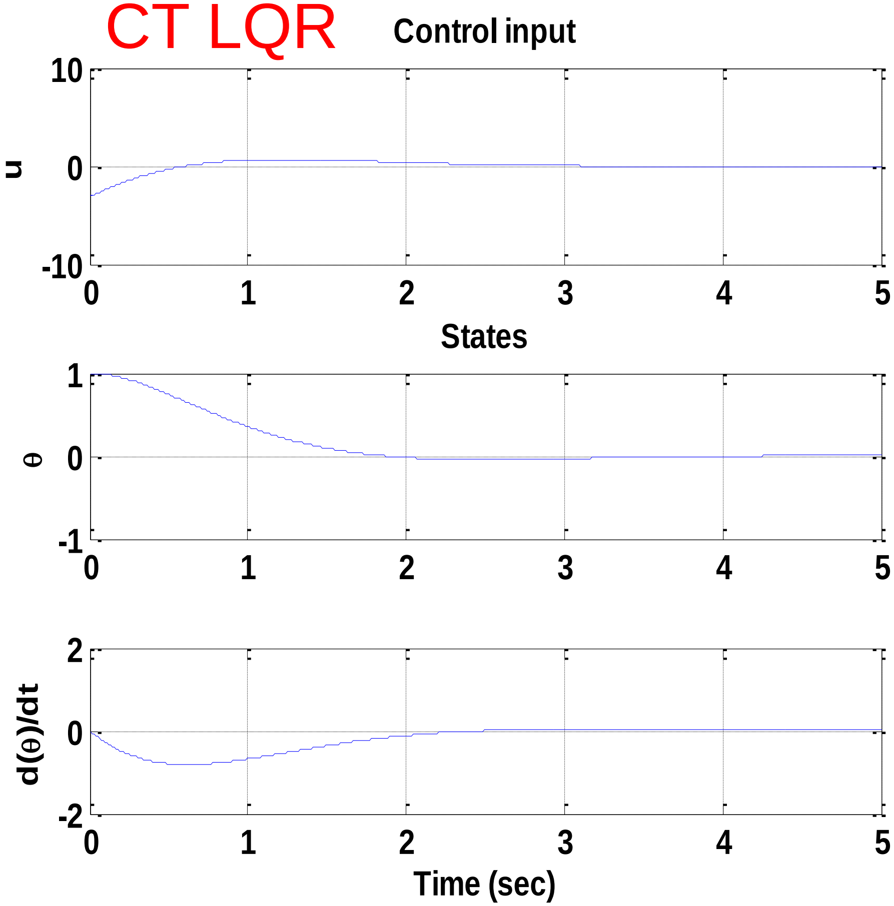


DT LQR

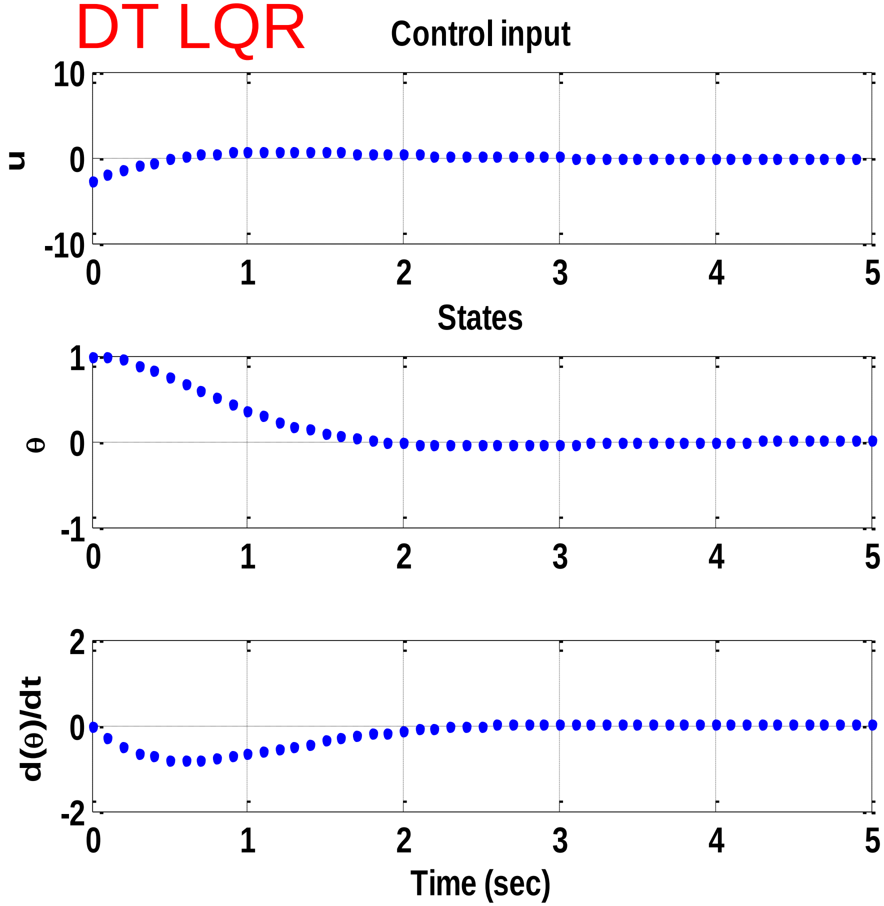


Satellite attitude control (Finite-horizon, *R=0.1*, *T=0.1* sec)

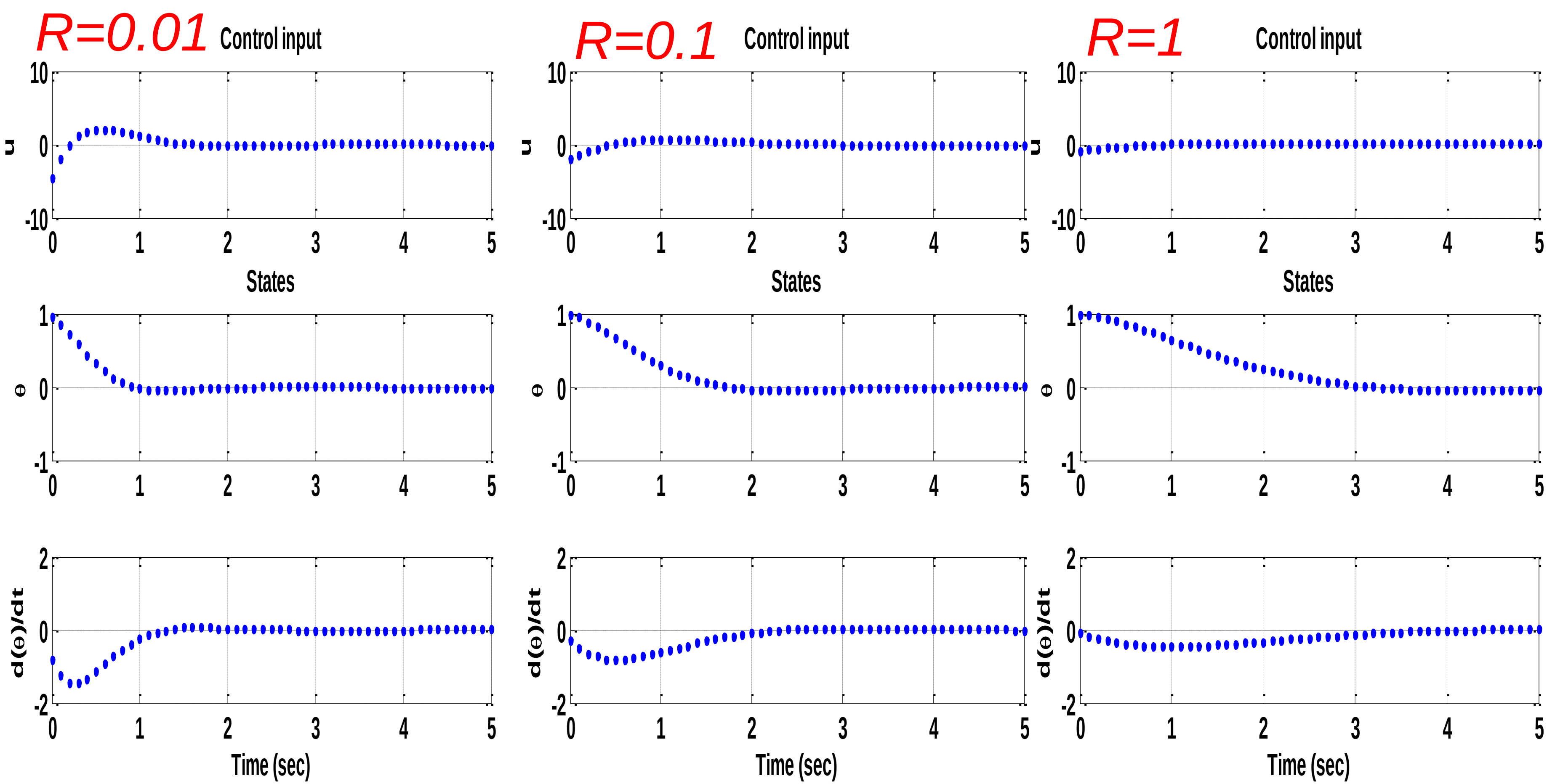
CT LQR



DT LQR



Satellite attitude control (Infinite-horizon, T=0.1 sec)



Matlab commands for DT LQR

- "dlqr.m": for

$$\min \sum_{0}^{\infty} \left[x^T[k]Qx[k] + u^T[k]Ru[k] + 2x^T[k]Nu[k]\right] dt$$
$$\text{subj. to } x[k+1] = Ax[k] + Bu[k]$$

$$[K, P, E] = \text{dlqr}(A, B, Q, R, N)$$

K feedback gain $(R + B^T PB)^{-1} B^T PA$
P solution to the DARE
E closed-loop eigenvalues eig$(A - BK)$

- "dlqry.m": for

$$\min \sum_{0}^{\infty} \left[y^T[k]Qy[k] + u^T[k]Ru[k]\right] dt$$
$$\text{subj. to } x[k+1] = Ax[k] + Bu[k], \;\; y[k] = Cx[k] + Du[k]$$

$$[K, P, E] = \text{dlqry}(A, B, C, D, Q, R)$$

Summary

- Discrete-time LQR
  - Finite horizon: Matrix Riccati difference equation
  - Infinite horizon: Discrete Algebraic Riccati Equation (DARE)
- Next, Kalman filter
  - We will see later that discrete-time LQR is “dual” of Kalman filter.

Optimality of DT LQR control law (optional)

1. For any n-by-n symmetric *P[k]* and *x[k]* satisfying

$$x[k+1] = Ax[k] + Bu[k], x[k] = x_0$$

we have

$$\begin{aligned}
& x^T[k_f]P[k_f]x[k_f] - x^T[0]P[0]x[0] \\
= & \sum_{k=0}^{k_f-1}\left[x^T[k+1]P[k+1]\underbrace{x[k+1]}_{Ax[k]+Bu[k]} - x^T[k]P[k]x[k]\right] \\
= & \sum_{k=0}^{k_f-1}\left[x^T[k]\left\{A^TP[k+1]A - P[k]\right\}x[k]\right. \\
& +u^T[k]B^TP[k+1]Ax[k] + x^T[k]A^TP[k+1]Bu[k] \\
& \left. +u^T[k]B^TP[k+1]Bu[k]\right]
\end{aligned}$$

See the next slide

Optimality of DT LQR control law

2. Select a special *P[k]* satisfying

$$\begin{cases} P[k] &= A^T P[k+1]A + Q \\ & \quad -A^T P[k+1]B\left[R + B^T P[k+1]B\right]^{-1} B^T P[k+1]A \\ P[k_f] &= S \end{cases}$$

Then,

$$\begin{aligned} 0 \;=\; & -x^T[k_f]Sx[k_f] + x^T[0]P[0]x[0] \\ & + \sum_{k=0}^{k_f-1}\Big[x^T[k]\left\{-Q + A^T P[k+1]B\left[R + B^T P[k+1]B\right]^{-1} B^T P[k+1]A\right\}x[k] \\ & \quad + u^T[k]B^T P[k+1]Ax[k] + x^T[k]A^T P[k+1]Bu[k] \\ & \quad + u^T[k]B^T P[k+1]Bu[k]\Big] \end{aligned}$$

Optimality of DT LQR control law

3. By adding the cost function below to both sides

$$J(u[\cdot]) := \sum_{k=0}^{k_f-1} \left[x^T[k]Qx[k] + u^T[k]Ru[k]\right] dt + x^T[k_f]Sx[k_f]$$

we have

$$\begin{aligned} J(u[\cdot]) &= x_0^T P[0] x_0 \\ &+ \sum_{k=0}^{k_f-1} x^T[k]\left\{A^T P[k+1]B\left[R+B^T P[k+1]B\right]^{-1} B^T P[k+1]A\right\} x[k] \\ &+ \sum_{k=0}^{k_f-1} u^T[k]B^T P[k+1]Ax[k] + x^T[k]A^T P[k+1]Bu[k] \\ &+ \sum_{k=0}^{k_f-1} u^T[k]\left\{R+B^T P[k+1]B\right\} u[k] \end{aligned}$$

Optimality of DT LQR control law

3. (cont'd) By completion of square

$$
\begin{aligned}
J(u[\cdot]) \;=\;& x_0^T P[0] x_0 \\
&+ \sum_{k=0}^{k_f-1} \left[u[k] + \left[R + B^T P[k+1] B\right]^{-1} B^T P[k+1] A x[k]\right]^T \\
&\qquad \times \left[R + B^T P[k+1] B\right] \\
&\qquad \times \left[u[k] + \left[R + B^T P[k+1] B\right]^{-1} B^T P[k+1] A x[k]\right]
\end{aligned}
$$

4. Since *R>0*, the function *J* achieves its minimum when

$$u[k] = -\left[R + B^T P[k+1] B\right]^{-1} B^T P[k+1] A x[k], \; k = 0, 1, \cdots, k_f - 1$$

DT LQR with an integrator

- Problem

$$\min_{u[\cdot]} J(u[\cdot]) \text{ subj. to } \left\{ \begin{array}{rcl} x[k+1] & = & Ax[k] + Bu[k], \;\; x[0] = 0 \\ y[k] & = & Cx[k] \end{array} \right.$$

  - J : Quadratic performance index (cost function)

$$J(u[\cdot]) := \sum_{k=0}^{\infty} \underbrace{(r - y[k])^T Q (r - y[k])}_{\text{For small deviation from reference r}} + \underbrace{(u[k+1] - u[k])^T R (u[k+1] - u[k])}_{\text{For small input rate of change}}$$

  - Design parameters $Q \geq 0, \;\; R > 0$

Reduction to standard DT LQR

- Consider an auxiliary state vector: $\tilde{x}[k] := \begin{bmatrix} x[k+1] - x[k] \\ e[k] \end{bmatrix}$

$$e = r - y$$

- Then,

$$\tilde{x}[k+1] = \begin{bmatrix} x[k+2] - x[k+1] \\ e[k+1] \end{bmatrix} = \underbrace{\begin{bmatrix} A & 0 \\ -C & I \end{bmatrix}}_{\tilde{A}} \tilde{x}[k] + \underbrace{\begin{bmatrix} B \\ 0 \end{bmatrix}}_{\tilde{B}} \underbrace{(u[k+1] - u[k])}_{\Delta u[k]}$$

$$J(u[\cdot]) := \sum_{k=0}^{\infty} \tilde{x}[k]^T \underbrace{\begin{bmatrix} 0 \\ I \end{bmatrix} Q \begin{bmatrix} 0 & I \end{bmatrix}}_{\tilde{Q}} \tilde{x}[k] + \Delta u[k]^T R \Delta u[k]$$

- Problem is reduced to a standard LQR problem:

$$\min_{u[\cdot]} \sum_{k=0}^{\infty} \tilde{x}[k]^T \tilde{Q} \tilde{x}[k] + \Delta u[k]^T R \Delta u[k] \text{ subj. to } \tilde{x}[k+1] = \tilde{A}\tilde{x}[k] + \tilde{B}\Delta u[k]$$

Reduction to DT LQR (cont'd)

- LQR optimal control is

$$\Delta u[k] = -(R + \tilde{B}^T \tilde{P} \tilde{B})^{-1} \tilde{B}^T \tilde{P} \tilde{A} \tilde{x}[k] = -K_x(x[k+1] - x[k]) - K_e e[k]$$

$$\Rightarrow \quad u[k] = \sum_{k=0}^{k-1} \Delta u[k] = -K_x(x[k] - \underbrace{x[0]}_{0}) - K_e \sum_{k=0}^{k-1} e[k]$$