1.1 To check BIBO stability, assume $x(0) = 0$ and find $G(s) = Y(s)/U(s)$

from SS, $G(s) = C(sI - A)^{-1}B + D$

$$A = \begin{pmatrix} 0 & 1 & 0 \\ -1 & -2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$B = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$C = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix}$$

$$D = (0)$$

Therefore,

$$G(s) = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix} \left[\begin{pmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & s \end{pmatrix} - \begin{pmatrix} 0 & 1 & 0 \\ -1 & -2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right]^{-1} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} s & -1 & 0 \\ 1 & s+2 & 0 \\ 0 & 0 & s \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

MATLAB

$$= \begin{pmatrix} 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} \frac{s+2}{s^2+2s+1} & \frac{1}{s^2+2s+1} & 0 \\ \frac{-1}{s^2+2s+1} & \frac{s}{s^2+2s+1} & 0 \\ 0 & 0 & \frac{1}{s} \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{s^2+2s+1} \\ \frac{s}{s^2+2s+1} \\ 0 \end{pmatrix}$$

$$= \frac{s+1}{s^2+2s+1} \longrightarrow \text{poles of } G(s) \text{ are } -1 \text{ \& } -1 \rightarrow \text{BIBO stable}$$

To check internal stability, assume $u(t)=0$ (no input)

1.2 $A = \begin{bmatrix} 0 & 1 & 0 \\ -1 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$\det(\lambda I - A) = 0$

$\det \begin{bmatrix} \lambda & -1 & 0 \\ 2 & \lambda+2 & 0 \\ 0 & 0 & \lambda \end{bmatrix} = 0$

$\lambda(\lambda+2)(\lambda) - (-1)2\lambda = 0$

$\lambda^2(\lambda+2) \quad +2\lambda = 0$

$\lambda^2 + 2\lambda + 2 = 0 \quad \rightarrow \lambda_1 = 0$

$\lambda = \frac{-2 \pm \sqrt{4-4(2)}}{2}$

$\lambda_{2,3} = -1 \pm i$

$Re[\lambda_2]$ & $Re[\lambda_3]$ are both <0, so we only check λ_1

$\left.\begin{array}{l} \text{rank}[\lambda_1 I - A] = \text{rank}[-A] = 2 \\ n - m_1 = 3 - 1 = 2 \end{array}\right\} 2 = 2$

So, This system is marginally stable for $x(t) \neq 0$

1.3 To check controllability, check rank $C = n$ where $C = [B, AB, A^2B]$

$$A = \begin{bmatrix} 0 & 1 & 0 \\ -1 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$AB = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} \qquad A^2B = \begin{bmatrix} 0 & 1 & 0 \\ -1 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ 3 \\ 0 \end{bmatrix}$$

$$C = [B, AB, A^2B] = \begin{bmatrix} 0 & 1 & -2 \\ 1 & -2 & 3 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \text{rank } C = 2$$

$n = 3 \rightarrow n = 3 > \text{rank}(C) = 2$

Therefore, not controllable.

$$\text{Im}(C) = \text{span}\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} \right\} \rightarrow T_C = \begin{bmatrix} 0 & 1 \\ 1 & -2 \\ 0 & 0 \end{bmatrix}$$

$$T_{\bar{C}} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \rightarrow T^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

1.4 To check observability, check rank $\mathcal{O} = n$

where $\mathcal{O} = \begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix}$

$C = [1 \quad 1 \quad 1]$

$A = \begin{bmatrix} 0 & 1 & 0 \\ -1 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$CA = [-1 \quad -1 \quad 0]$

$CA^2 = (-1 \quad -1 \quad 0)\begin{pmatrix} 0 & 1 & 0 \\ -1 & -2 & 0 \\ 0 & 0 & 0 \end{pmatrix} = [1 \quad 1 \quad 0]$

$\mathcal{O} = \begin{bmatrix} 1 & 1 & 1 \\ -1 & -1 & 0 \\ 1 & 1 & 0 \end{bmatrix} \rightarrow \text{rank}(\mathcal{O}) = 2 < n = 3$

Therefore, system not observeable.

Similar to controllability, we can use kernel space to get controllable subspace of this system.

$\text{Ker}(\mathcal{O}) = \text{span}\left\{\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}\right\} \Rightarrow T_{\bar{o}} = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$

$\rightarrow T_o = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \rightarrow T^{-1} = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$

1.5 $\mathcal{C} = \begin{bmatrix} 0 & 1 & -2 \\ 1 & -2 & 3 \\ 0 & 0 & 0 \end{bmatrix}$ $\quad \mathcal{O} = \begin{bmatrix} 1 & 1 & 1 \\ -1 & -1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$

$$\downarrow \qquad\qquad \downarrow$$

$\operatorname{Im}(\mathcal{C}) = \operatorname{span}\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} \right\}$ $\quad \operatorname{Ker}(\mathcal{O}) = \operatorname{span}\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \right\}$

$T_{c\bar{o}} = \operatorname{Im}(\mathcal{C}) \cap \ker(\mathcal{O})$

$$0 = a\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + b\begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} - c\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

$$0 = \begin{bmatrix} 0 & 1 & +1 \\ 1 & -2 & -1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = t\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, t \in \mathbb{R}$$

$$\left.\begin{aligned} \begin{bmatrix} 0 & 1 \\ 1 & -2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} &= \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \\ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}\begin{bmatrix} -1 \end{bmatrix} &= \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \end{aligned}\right\} T_{c\bar{o}} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

$T_{co} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ pick simplest $\quad T_{\bar{c}\bar{o}} =$ none, since $\ker(o)$ is a line

$T_{\bar{c}o} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ $\quad \to T^{-1} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = T$

$$z = Tx \rightarrow \begin{array}{c} z_{co} \\ z_{c\bar{o}} \\ z_{\bar{c}o} \end{array} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} [x]$$

$$A_{new} = TAT^{-1} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ -1 & -2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} -1 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$B_{new} = TB = \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

$$C_{new} = CT^{-1} = (1 \quad 1 \quad 1) \begin{pmatrix} 1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = [1 \quad 0 \quad 1]$$

$$\dot{z} = \begin{bmatrix} -1 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} z + \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} u$$

$$y = [1 \quad 0 \quad 1] z$$

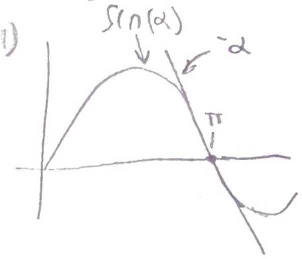
2.1 → For inv pedulum, we flip sign of all terms with g (gravity change direction).

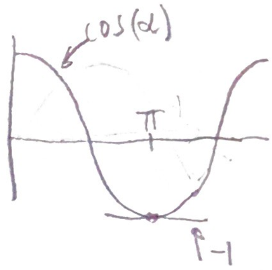
→ since we're now linearizing around $\alpha = \pi$, instead of $\alpha = 0$:

$\sin\alpha \approx \sin(\pi) + \cos(\pi) \times (\alpha - \pi)$
$\approx \pi - \alpha$

$\cos\alpha \approx -1$

$\sin^2\alpha \approx 0$





→ From these changes, we get

$$J_r\ddot{\theta} + -m_p r l \ddot{\alpha} = T - b_r\dot{\theta}$$

$$-m_p r l\ddot{\theta} + J_p\ddot{\alpha} = -b_p\dot{\alpha} - m_p g l(\pi - \alpha)$$

→ we notice that only r terms have flipped signs, so we just flip signs of terms with r in final matrix to get:

→ we notice that $(\pi - \alpha)$ takes place of α, so we can redefine x_3 as $\pi - \alpha$

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & \frac{-J_p b_r}{J_t} & \frac{(m_p l)^2 r g}{J_t} & -\frac{m_p r l b_p}{J_t} \\ 0 & 0 & 0 & -1 \\ 0 & -\frac{m_p r l b_r}{J_t} & -\frac{J_r m_p g l}{J_t} & -\frac{J_r b_p}{J_t} \end{bmatrix} \qquad B = \frac{1}{J_t}\begin{bmatrix} 0 \\ J_p \\ 0 \\ m_p r l \end{bmatrix}$$

added negative because $x_3 = \pi - \alpha$, $\dot{x}_3 = x_4 = -\dot{\alpha}$

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

where $x = \begin{bmatrix} \theta \\ \dot{\theta} \\ \pi - \alpha \\ \dot{\alpha} \end{bmatrix}$ $\quad u = [T]$ $\quad y = \begin{bmatrix} \theta \\ \alpha \end{bmatrix}$

To plot α, take $y_2 = \pi - \alpha \rightarrow \alpha = \pi - y_2$

2b.

```
mr = 0.095;
r = 0.085;
br = 0.001;
mp = 0.024;
Lp = .129;
bp = .00005;
g = 9.81;
Jr = mr*r*r/3;
Jp = mp*Lp*Lp/3;
l = Lp/2;
Jt = Jr*Jp - (mp*r*l)^2;
```

Fig 1: Matlab code

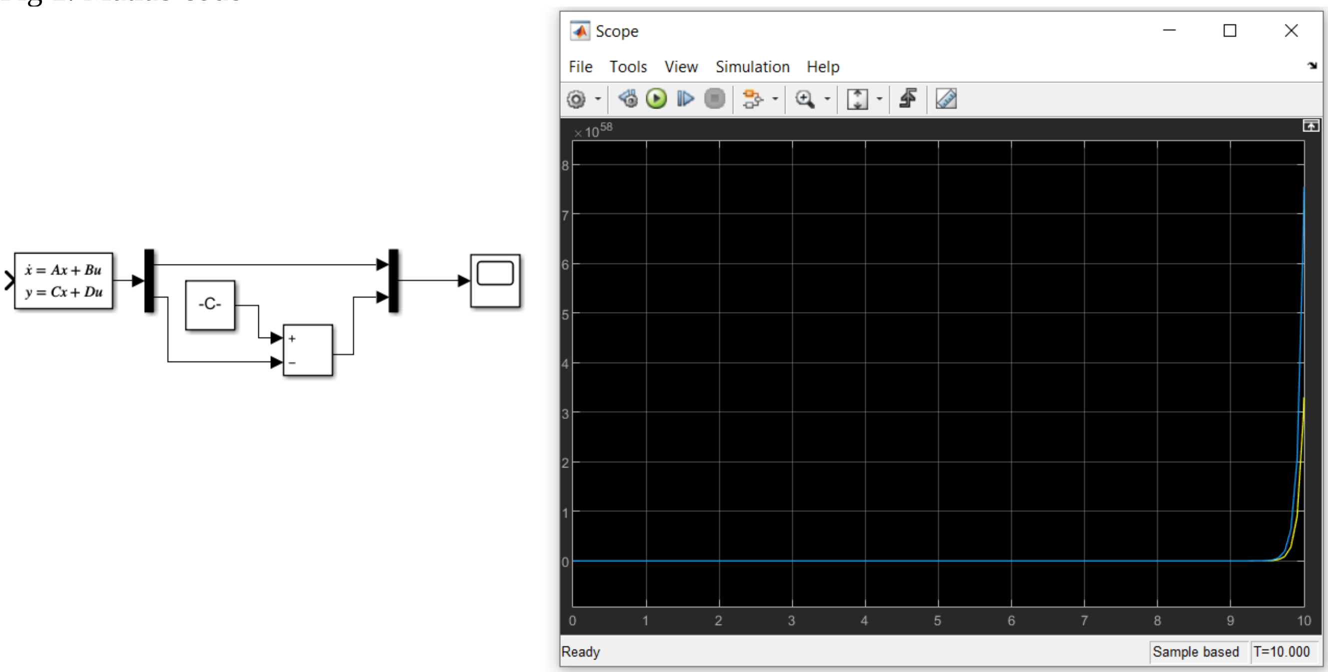


Fig 2: Simulink model, and plot (theta over time (yellow) and alpha over time (blue))

2c.

q2_vars.m

```
mr = 0.095;
r = 0.085;
br = 0.001;
mp = 0.024;
Lp = .129;
bp = .00005;
g = 9.81;
Jr = mr*r*r/3;
Jp = mp*Lp*Lp/3;
l = Lp/2;
Jt = Jr*Jp - (mp*r*l)^2;
A = [0 1 0 0; 0 -Jp*br/Jt ((mp*l)^2)*r*g/Jt mp*r*l*bp/Jt; 0 0 0 1; 0 mp*r*l*br/Jt -Jr*mp*g*l/Jt -Jr*bp/Jt];
B = [0; Jp/Jt; 0; -mp*r*l/Jt];
A_inv = [0 1 0 0; 0 -Jp*br/Jt -((mp*l)^2)*r*g/Jt -mp*r*l*bp/Jt; 0 0 0 -1; 0 -mp*r*l*br/Jt -Jr*mp*g*l/Jt -Jr*bp/Jt];
B_inv = [0; Jp/Jt; 0; mp*r*l/Jt];
C = [1 0 0 0; 0 0 1 0];
D = [0; 0];
[~,eigval] = eig(A);
[~,eigval_inv] = eig(A_inv);
disp("normal pendulum's eigen values:");
disp(eigval);
disp("an eigenvalue is zero, so we need to check rank(eigval*I-A) =? n-m");
disp("we know n=4, and m=1 (no repeated eig vals), so n-m=3");
disp("normal pendulum's rank(0*I - A):");
disp(rank(-A));
disp("since rank is 3 which is equal to n-m=3, normal pendulum is marginally stable");
disp("inverted pendulum's eigen values:");
disp(eigval_inv);
disp("one of the eigenvalues is >0, so inverted pendulum is unstable");
```

```
>> q2_vars
normal pendulum's eigen values:
   0.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i
   0.0000 + 0.0000i  -4.8479 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i
   0.0000 + 0.0000i   0.0000 + 0.0000i  -3.0749 +15.1276i   0.0000 + 0.0000i
   0.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i  -3.0749 -15.1276i

an eigenvalue is zero, so we need to check rank(eigval*I-A) =? n-m
we know n=4, and m=1 (no repeated eig vals), so n-m=3
normal pendulum's rank(0*I - A):
     3

since rank is 3 which is equal to n-m=3, normal pendulum is marginally stable
inverted pendulum's eigen values:
         0         0         0         0
         0  -20.8385         0         0
         0         0   -4.0042         0
         0         0         0   13.8450

one of the eigenvalues is >0, so inverted pendulum is unstable
```

Fig 3: Matlab code and computed values