MECH468 : Modern Control Engineering
MECH509 : Controls

L18 : Minimal realization

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Zoom lecture to be recorded and posted on Canvas

Course plan

Topics	CT	DT
Modeling	✓	✓
Stability	✓	✓
Controllability/observability	✓	✓
Realization	→	
State feedback/observer		
LQR/Kalman filter		

Review

- In the last two lectures, we considered the realizations of the following transfer matrix:

$$G(s) = \begin{bmatrix} \frac{1}{s^2+4s+3} & \frac{1}{s+3} \end{bmatrix} = \frac{1}{s^2+4s+3}\left\{ \begin{bmatrix} 0 & 1 \end{bmatrix} s + \begin{bmatrix} 1 & 1 \end{bmatrix} \right\}$$

Controllable canonical form with order 4

$$\dot{x} = \left[\begin{array}{cc|cc} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \hline -3 & 0 & -4 & 0 \\ 0 & -3 & 0 & -4 \end{array}\right] x + \left[\begin{array}{cc} 0 & 0 \\ 0 & 0 \\ \hline 1 & 0 \\ 0 & 1 \end{array}\right] u$$

$$y = \left[\begin{array}{cc|cc} 1 & 1 & 0 & 1 \end{array}\right] x$$

Observable canonical form with order 2

$$\dot{x} = \left[\begin{array}{c|c} 0 & -3 \\ \hline 1 & -4 \end{array}\right] x + \left[\begin{array}{cc} 1 & 1 \\ \hline 0 & 1 \end{array}\right] u$$

$$y = \left[\begin{array}{c|c} 0 & 1 \end{array}\right] x$$

Today's topic

- Realization of the smallest order
  - How to characterize such realization?
  - How to obtain such realization?
- Some terminologies:
  - *Minimal realization* of *G(s)*:

    Realization (*A,B,C,D*) of *G* that has the smallest dimension of *A*-matrix.
  - *McMillan degree* of *G(s)*:

    The dimension of *A* of the minimal realization. (This indicates the "complexity" of a system *G(s)*.)

Why minimal realization?

- Easy to …
  - Analyze (understand) the system
  - Design a controller
  - Implement a controller
- Computationally less demanding in both design and implementation
- Higher reliability
  - Few parts to go wrong in the hardware
  - Few bugs to fix in the software

Two important facts on minimal realization

- *Fact 1: A realization (A,B,C,D) is minimal*

  if and only if

  (A,B) is controllable and (A,C) is observable.

- *Fact 2: All minimal realizations of G(s) are related by coordinate transformations.*

(Proofs are given in the lecture note.)

Non-minimal realization example

$$G(s) = \frac{s^2 - 1}{s^3 + 1}$$

- Controllable canonical form

$$\begin{cases} \dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u \\ y = \begin{bmatrix} -1 & 0 & 1 \end{bmatrix} x \end{cases}$$

Is it observable? ➡ No!

$$\operatorname{rank} \begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix} = \operatorname{rank} \begin{bmatrix} -1 & 0 & 1 \\ -1 & -1 & 0 \\ 0 & -1 & -1 \end{bmatrix} = 2$$

- Observable canonical form

$$\begin{cases} \dot{x} = \begin{bmatrix} 0 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} x + \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} u \\ y = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} x \end{cases}$$

Is it controllable? ➡ No!

$$\operatorname{rank} \begin{bmatrix} B & AB & A^2B \end{bmatrix} = \operatorname{rank} \begin{bmatrix} -1 & -1 & 0 \\ 0 & -1 & -1 \\ 1 & 0 & -1 \end{bmatrix} = 2$$

How to obtain minimal realization

SISO (Single-Input-Single-Output) case

- Remove common factors from numerator and denominator of *G(s)*. Then, realize *G* in a controllable (or observable) canonical form.

$$G(s) = \frac{s^2-1}{s^3+1} = \frac{(s+1)(s-1)}{(s+1)(s^2-s+1)} = \frac{s-1}{s^2-s+1}$$

➡ C.C.F. $\begin{cases} \dot{x} = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \\ y = \begin{bmatrix} -1 & 1 \end{bmatrix} x \end{cases}$

➡ Observable! ➡ Minimal! McMillan degree 2

How to obtain minimal realization

- SIMO case (*G(s)* is a column vector)
  - Use the controllable canonical form.

$$G(s) = \frac{1}{s^2+4s+3}\left\{\begin{bmatrix} 0 \\ 1 \end{bmatrix} s + \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\} \Rightarrow \quad \dot{x} = \left[\begin{array}{c|c} 0 & 1 \\ \hline -3 & -4 \end{array}\right] x + \left[\begin{array}{c} 0 \\ \hline 1 \end{array}\right] u$$

$$y = \left[\begin{array}{c|c} 1 & 0 \\ 1 & 1 \end{array}\right] x$$

- MISO case (*G(s)* is a row vector)
  - Use the observable canonical form.

$$G(s) = \frac{1}{s^2+4s+3}\left\{\begin{bmatrix} 0 & 1 \end{bmatrix} s + \begin{bmatrix} 1 & 1 \end{bmatrix}\right\} \Rightarrow \quad \dot{x} = \left[\begin{array}{c|c} 0 & -3 \\ \hline 1 & -4 \end{array}\right] x + \left[\begin{array}{cc} 1 & 1 \\ \hline 0 & 1 \end{array}\right] u$$

$$y = \left[\begin{array}{c|c} 0 & 1 \end{array}\right] x$$

How to obtain minimal realization

MIMO (Multiple-Input-Multiple-Output) case

1. Realize *G(s)* in some non-minimal canonical form.
2. Use the Kalman decomposition to remove uncontrollable/unobservable parts. (“minreal.m”)

Remark: Unfortunately, it is generally hard to compute the Kalman decomposition by hand.

Remark: There is another famous algorithm, called *Ho’s algorithm*, to compute a minimal realization. (Not covered in this course.)

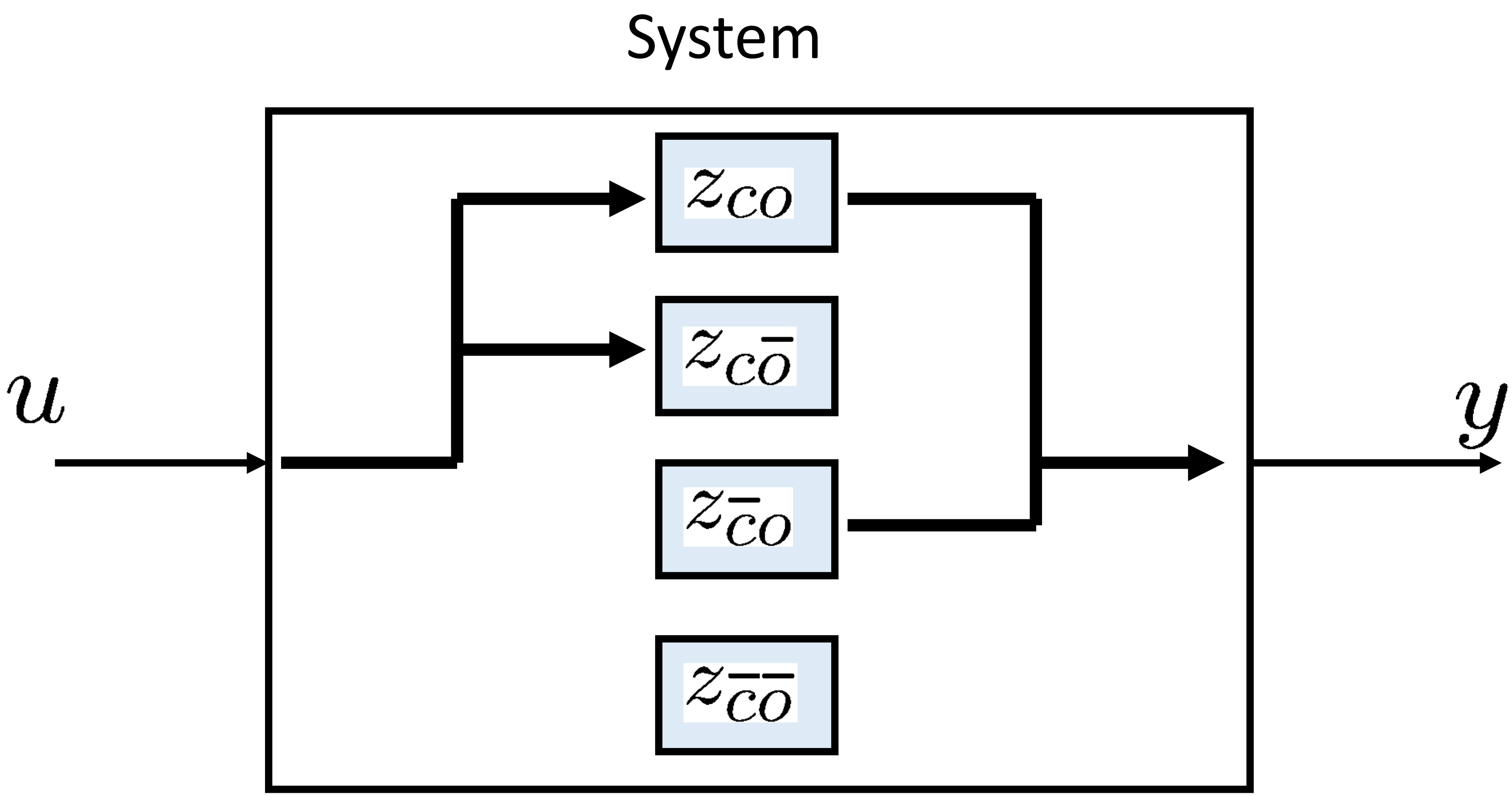
Kalman decomposition (review)

- Every SS model can be transformed by *z=Tx* for some appropriate *T* into a canonical form:

$$
\begin{cases}
\begin{bmatrix} \dot{z}_{co} \\ \dot{z}_{c\bar{o}} \\ \dot{z}_{\bar{c}o} \\ \dot{z}_{\bar{c}\bar{o}} \end{bmatrix} = \underbrace{\begin{bmatrix} A_{co} & 0 & A_{13} & 0 \\ A_{21} & A_{c\bar{o}} & A_{23} & A_{24} \\ 0 & 0 & A_{\bar{c}o} & 0 \\ 0 & 0 & A_{43} & A_{\bar{c}\bar{o}} \end{bmatrix}}_{TAT^{-1}} \begin{bmatrix} z_{co} \\ z_{c\bar{o}} \\ z_{\bar{c}o} \\ z_{\bar{c}\bar{o}} \end{bmatrix} + \underbrace{\begin{bmatrix} B_{co} \\ B_{c\bar{o}} \\ 0 \\ 0 \end{bmatrix}}_{TB} u \\
y = \underbrace{\begin{bmatrix} C_{co} & 0 & C_{\bar{c}o} & 0 \end{bmatrix}}_{CT^{-1}} \begin{bmatrix} z_{co} \\ z_{c\bar{o}} \\ z_{\bar{c}o} \\ z_{\bar{c}\bar{o}} \end{bmatrix} + Du
\end{cases}
$$

$$\boxed{(A_{co}, B_{co}) \text{ is controllable \& } (A_{co}, C_{co}) \text{ is observable}}$$

Kalman decomposition (review) Conceptual figure (Not block-diagram)



Example

$$G(s) = \begin{bmatrix} \frac{4s-10}{2s+1} & \frac{3}{s+2} \\ \frac{1}{(2s+1)(s+2)} & \frac{1}{(s+2)^2} \end{bmatrix}$$

```
sys11 = tf([4 -10],[2 1]);
sys12 = tf(3,[1 2]);
sys21 = tf(1,[2 5 2]);
sys22 = tf(1,[1 4 4]);
sysG = [sys11 sys12; sys21 sys22];
G = ss(sysG);
```

```
>> G

a =
          x1     x2     x3     x4     x5     x6
   x1   -0.5      0      0      0      0      0
   x2      0   -2.5     -1      0      0      0
   x3      0      1      0      0      0      0
   x4      0      0      0     -2      0      0
   x5      0      0      0      0     -4     -2
   x6      0      0      0      0      2      0
```

Order 6

Example (cont'd)

- Minimal realization

```
>> [Gmin,P] = minreal(G);
>> Gmin

a =
```

	x1	x2	x3
x1	-0.7192	0.4738	-0.2042
x2	0.5926	-1.781	0.5519
x3	-9.714e-016	-6.106e-016	-2

Order 3

```
%   Check the minimality
Ctr = ctrb(Gmin.a,Gmin.b);
Obs = obsv(Gmin.a,Gmin.c);

>> rank(Ctr)
ans =
     3
```

Controllable!

```
>> rank(Obs)
ans =
     3
```

Observable!

Minimal!

Example (cont'd)

- After realization, check the correctness by recovering the original transfer matrix.

$$G(s) = \begin{bmatrix} \frac{4s-10}{2s+1} & \frac{3}{s+2} \\ \frac{1}{(2s+1)(s+2)} & \frac{1}{(s+2)^2} \end{bmatrix}$$

```
>> zpk(Gmin)

Zero/pole/gain from input 1 to output...
      2 (s-2.5) (s+2)^2
 #1:  -----------------
       (s+0.5) (s+2)^2

         0.5 (s+2)
 #2:  ---------------
      (s+0.5) (s+2)^2

Zero/pole/gain from input 2 to output...
      3 (s+2) (s+0.5)
 #1:  ---------------
      (s+0.5) (s+2)^2

          (s+0.5)
 #2:  ---------------
      (s+0.5) (s+2)^2
```

(1,1)

(2,1)

(1,2)

(2,2)

Summary

- Minimal realization
- Realization for DT systems is exactly the same as that for CT systems.
- Next, design for control and estimation
  - State feedback
  - Observer
  - Linear Quadratic Regulator (LQR)
  - Kalman filter