Operational Amplifier

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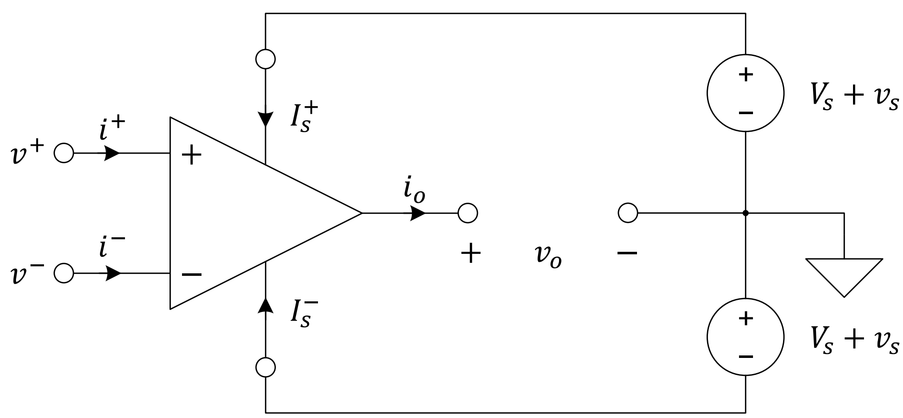
1 Op-amp Abstraction

An op-amp is an electronic device that consists of transistors, resistors, capacitors, etc.

In this course, we will treat it as an encapsulated object that interacts with the environment (external circuits) only through *five terminals*.

The five terminals form *three ports*: two input ports (signal and power) and one output port. The output port and input power port share the same reference potential, or *common*.



Terminal variables

Current:

$$i_o \quad = \quad I_s^+ \quad + \quad I_s^- \quad + \quad i^+ \quad + \quad i^- \qquad \text{(KCL)}$$

Voltage:

$$v_o \quad = \quad \underbrace{A(v^+ - v^-)}_{v_e} \quad + \quad \underbrace{A_c\left(\frac{v^+ + v^-}{2}\right)}_{v_{cm}} \quad + \quad A_s v_s \quad - \quad Z_o i_o$$

Terminal Relations

Open-loop gain: A

Common-mode gain: A_c

Supply-disturbance gain: A_s

Common-mode rejection ratio: $\dfrac{A}{A_c}$

Power-supply rejection ratio: $\dfrac{A}{A_s}$

Input impedance: Z_i

Output impedance: Z_o

Ideal op-amps

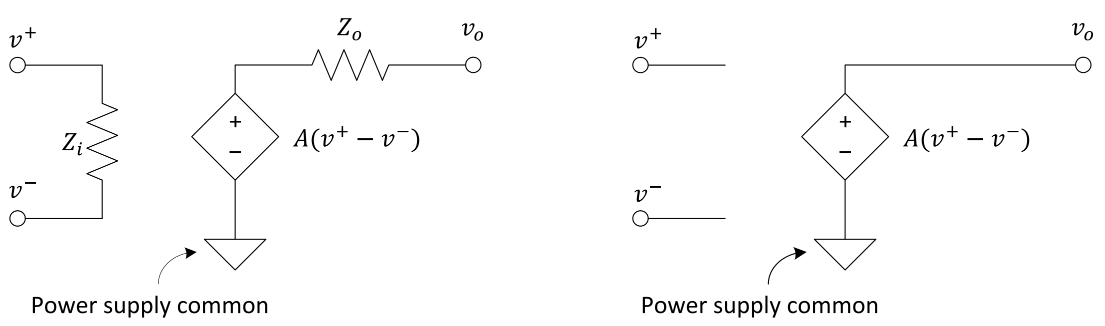
Infinite input impedance: $Z_i \to \infty \quad \Longrightarrow \quad i^+ = 0$ and $i^- = 0$

Zero output impedance: $Z_o = 0$

Zero common-mode gain: $A_c = 0$

Zero supply-disturbance gain: $A_s = 0$

Equivalent circuit model



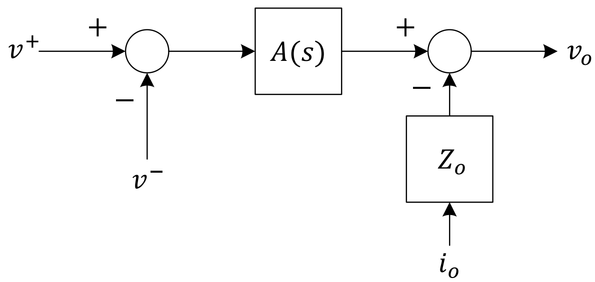
(a) Op-amp with finite I/O impedance (b) Ideal op-amp

Figure 1: Op-amp equivalent circuit.

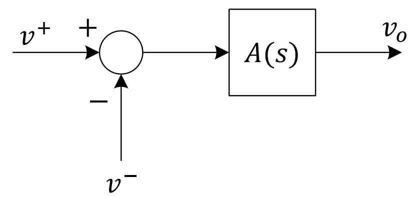
An op-amp can be modeled as a dependent voltage source whose output voltage only depends on the input *differential* voltage.

When $Z_o \to 0$ and $Z_i \to \infty$, i.e., an ideal op-amp, the circuit model simplifies to Figure 1b.

Block-diagram representation



(a) Finite output impedance



(b) Ideal op-amp

The open-loop gain $A(s)$ is a frequency-dependent complex number, i.e., *transfer function.*

We will start from a simple model for A and proceed toward more realistic models.

(1) $A = \text{constant}$

(2) $A = \text{constant but very large: } A \to \infty$

(3) $A = \dfrac{\omega_o}{s}$

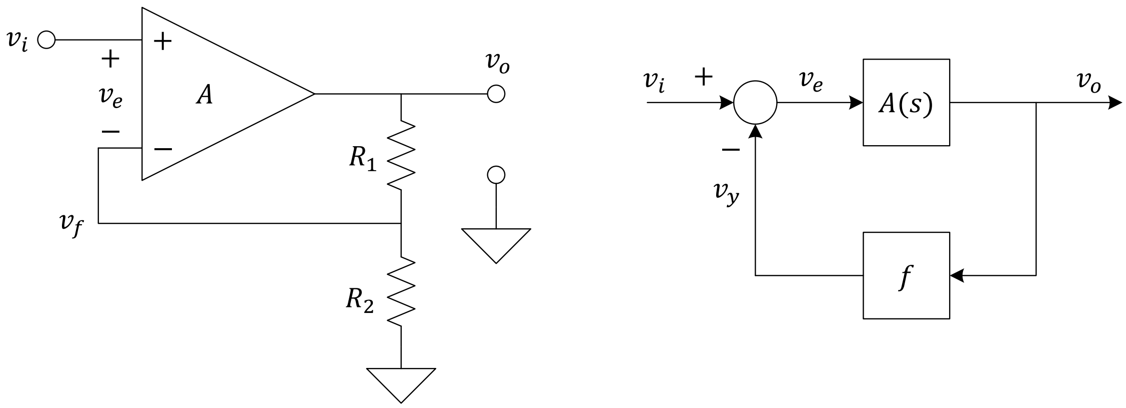
(4) $A = A(s)$

Note that A is unit-less [V/V].

2 Non-inverting amplifier

It is difficult to make the open-loop gain A precise. However, it is easy to make it large.

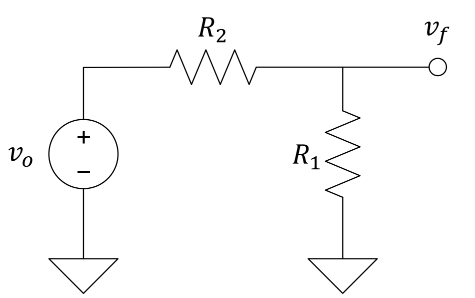
With large but imprecise A, how can we make a precise amplifier? $\implies$ **Feedback**



(a) Circuit diagram

(b) Block diagram

We measure the output voltage with a resistive voltage divider.



$$v_f = \underbrace{\frac{R_1}{R_2 + R_1}}_{f} v_o$$

The resistance of the voltage-sensing circuit should be much larger than R_o ($\sim 50\,\Omega$) and much smaller than R_i ($\sim 1\,\mathrm{M}\Omega$) of the op-amp. For example, in the order of $10\,\mathrm{k}\Omega$.

With Black's formula, we obtain the closed-loop gain

$$G \triangleq \frac{v_o}{v_i} = \frac{A}{1 + Af}$$

Let's consider two cases:

(1) $A = 10^5 \quad R_2 = 9\,\mathrm{k}\Omega \quad R_1 = 1\,\mathrm{k}\Omega \quad \implies f = \frac{1}{10}$

$$G = \frac{10^5}{1 + 10^4} = \frac{10}{1 + 10^{-4}} \approx 10(1 - 10^{-4}) = 0.999$$

(2) $A \to \infty \quad R_2 = 9\,\mathrm{k}\Omega \quad R_1 = 1\,\mathrm{k}\Omega \quad \Longrightarrow f = \frac{1}{10}$

$$G = \frac{A}{1+Af} \approx \frac{A}{Af} = \frac{1}{f} = 10 \qquad (\text{as long as } Af \gg 1)$$

The precision of G is determined by the feedback gain f not by the open-loop gain A. The tolerance of f can be as low as 0.005% with precision resistors.

Gain sensitivity

Feedback makes the closed-loop gain G insensitive to the change of the open-loop gain A. Suppose the open-loop gain varies by 10%. That is, $\frac{\mathrm{d}A}{A} = 10\ \%$.

(1) Without feedback ($f = 0 \implies G = A$)

$$\mathrm{d}G = \mathrm{d}A \quad \Longrightarrow \quad \frac{\mathrm{d}G}{G} = \frac{\mathrm{d}A}{A} = 10\%$$

(2) With feedback ($f \neq 0 \implies G = \dfrac{A}{1+Af}$)

$$\begin{aligned}\mathrm{d}G &= \frac{\mathrm{d}A\,(1+Af) - A(\mathrm{d}A\,f)}{(1+Af)^2}\\ &= \frac{\mathrm{d}A}{(1+Af)^2}\end{aligned}$$

Therefore,

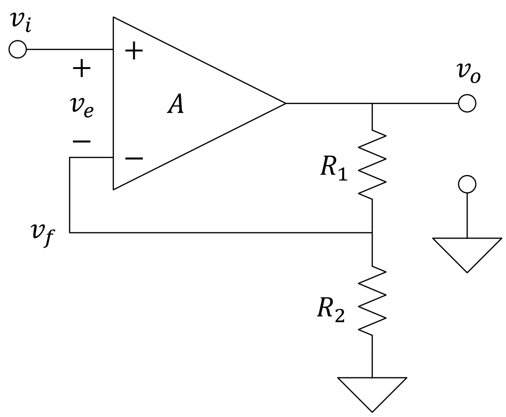
$$\begin{aligned}\frac{\mathrm{d}G}{G} &= \frac{1+Af}{A}\frac{\mathrm{d}A}{(1+Af)^2}\\ &= \underbrace{\frac{1}{1+Af}}_{S}\frac{\mathrm{d}A}{A} \approx 0.001\%\end{aligned}$$

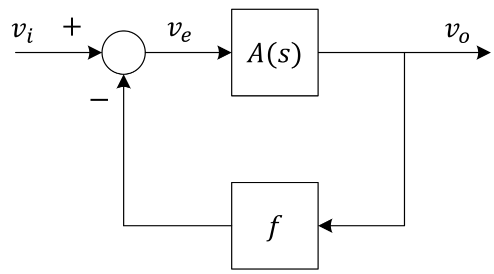
We define the sensitivity function as

$$S(s) \triangleq \frac{1}{1+L(s)} = \frac{1}{1+Af}.$$

The sensitivity function has two meanings. First, it means the sensitivity of the closed-loop gain to the open-loop gain variation. Feedback makes the closed-loop system insensitive (or robust) to the open-loop gain variation.

Second, it means the sensitivity of a signal inside the loop to an external disturbance. Feedback makes the closed-loop system insensitive to external disturbances. In other words, feedback rejects external disturbances. Below is an example.





Virtual short

Consider the transfer function from the input voltage v_i to the error voltage v_e

$$v_e = \underbrace{\frac{1}{1 + Af}}_{S} \, v_i$$

As $L = Af \to \infty$, $S \to 0$ and therefore

$$v_e \approx 0.$$

This means the external signal v_i, which can be arbitrary, is rejected by the feedback.

The concept of *virtual short* helps us quickly (but approximately) solve op-amp circuits.

Example   As $A \to \infty$, $v_f \approx v_i$

$$\begin{aligned} v_o &= v_f \left(\frac{1}{R_1}\right)(R_1 + R_1) \\ &= \left(\frac{R_2}{R_1} + 1\right) v_f \\ &\approx \left(\frac{R_2}{R_1} + 1\right) v_i \end{aligned}$$

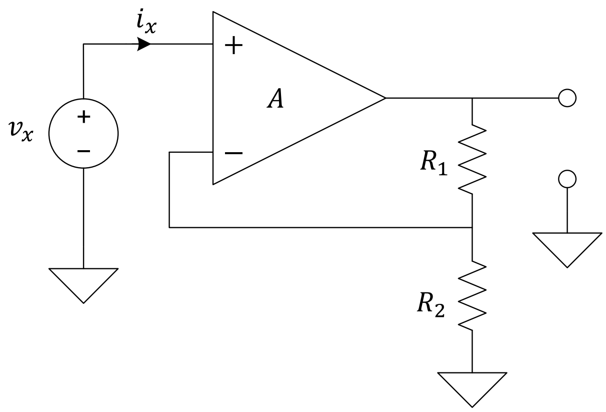
Power gain

For an op-amp driving a load resistance R_l

$$\begin{aligned} &\text{Input power:} && P_{\text{in}} = v_i \cancel{i_i} \approx 0 \\ &\text{Output power:} && P_{\text{out}} = v_o i_o = \frac{v_o^2}{R} \end{aligned} \quad \Longrightarrow \quad \frac{P_{\text{in}}}{P_{\text{out}}} \to \infty$$

The output power comes from the power supply.

Input impedance

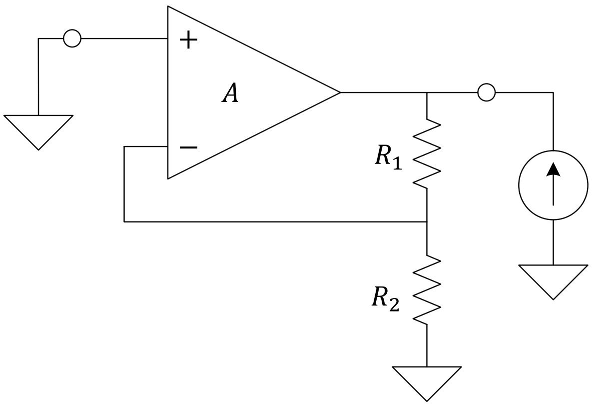


Put a test voltage v_x and see how much current i_x flows.

$i_x \approx 0$ regardless of v_x

$$Z_{\text{in}} = \frac{v_x}{i_x} \to \infty$$

Output impedance



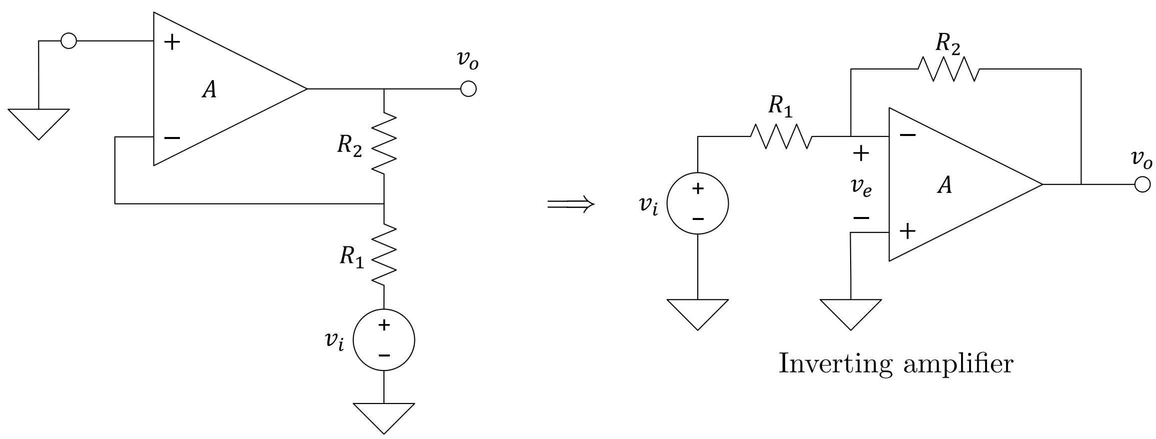
Put a test current i_x and see how much voltage change v_x occurs.

$v_x \approx 0$ regardless of i_x

$$Z_{\text{out}} = \frac{v_x}{i_x} \approx 0$$

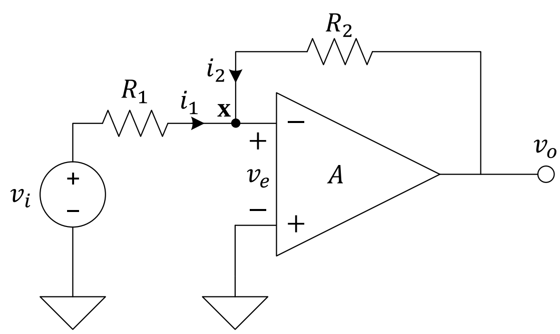
3 Inverting Amplifier

Inverting amplifier is another popular op-amp circuit topology. It is essentially the same as the non-inverting amplifier but driven from a different input node.



Inverting amplifier

Analysis using virtual short



Virtual short: $v_e \approx 0 \implies v_x \approx 0.$

KCL at $\mathbf{x}$: $\quad i_1 + i_2 = 0 \implies i_2 = -i_1$

$$v_i(\frac{1}{R_1})(-1)(R_2) = v_0 \implies G = -\frac{R_2}{R_1}$$

Output impedance: $\quad Z_{\text{out}} \approx 0$

Input impedance: $\quad \boxed{Z_{\text{in}} = R_1}$

Analysis for finite open-loop gain A

(1) Solve for v_e via superposition

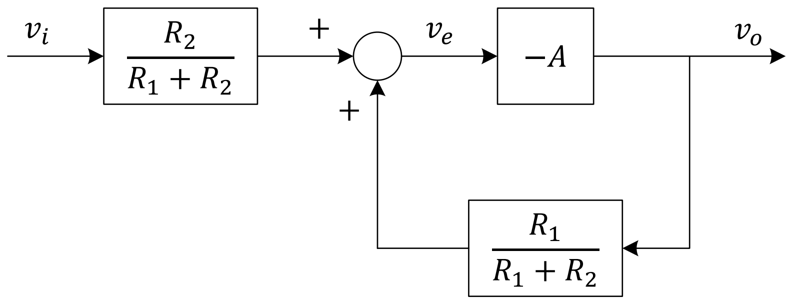
$$v_e = \frac{R_2}{R_1 + R_2} v_i + \frac{R_1}{R_1 + R_2} v_o$$

(2) Solve for v_o

$$v_o = -A\ v_e$$

Note that the polarity of v_e is the opposite of the previous v_e in non-inverting amplifier analysis.

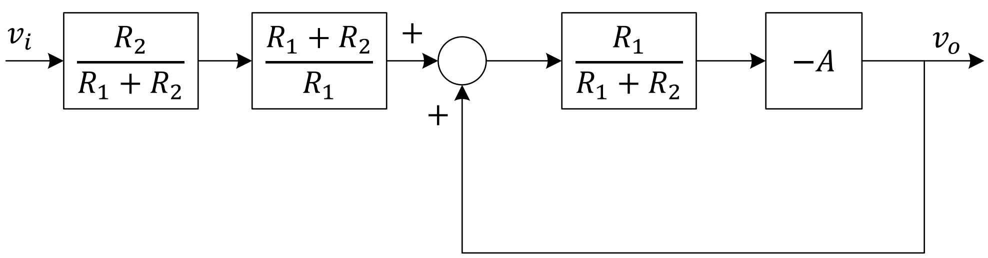
(3) Draw a block diagram



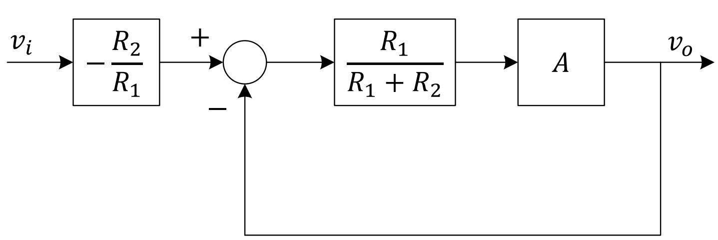
$$\text{L.T.} = -A\frac{R_1}{R_1 + R_2} \quad\Longrightarrow\quad L = -\text{L.T.} = A\frac{R_1}{R_1 + R_2}$$

Note that the loop transmission and loop return ratio are the same as that of non-inverting amplifier.

(4) Manipulate the block diagram for a unity-feedback configuration.



(5) Combine the fist two blocks and pull out the minus sign.



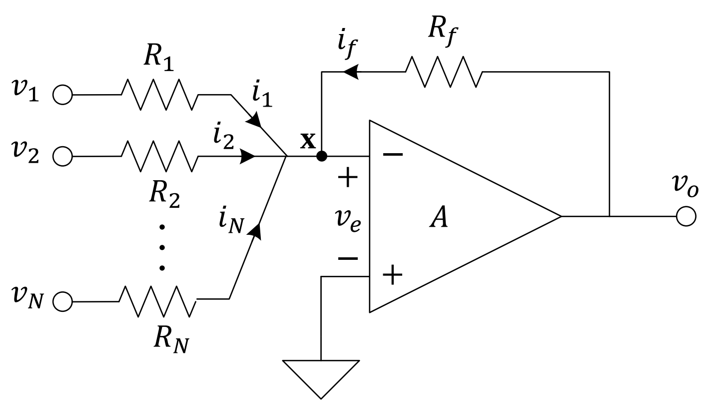
Therefore,

$$G \triangleq \frac{v_o}{v_i} = -\frac{R_2}{R_1}\left(\frac{L}{1+L}\right), \qquad \text{where} \quad L = A\frac{R_1}{R_1+R_2}$$

$$\text{As } L \to \infty, \frac{L}{1+L} \to 1, \text{and therefore } G \to -\frac{R_2}{R_1}$$

Summing Amplifier

KCL at the node **x** and the virtual short to the ground help us design a summing amplifier.



KCL and virtual short:

$$-i_f = \sum_{n=1}^{N} i_n \qquad i_n = \frac{v_n}{R_n} \qquad \implies i_f = -\sum_{n=1}^{N}\frac{v_n}{R_n}$$

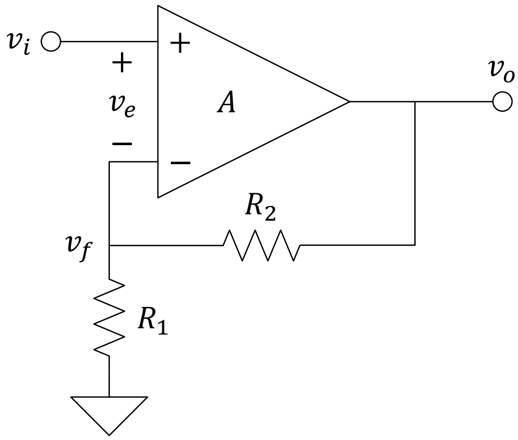
Therefore,

$$\begin{aligned} v_o &= R_f i_f \\ &= -R_f \sum_{n=1}^{N}\frac{v_n}{R_n} \end{aligned}$$

This topology is useful to implement a summing junction for feedback control.

Input impedance: R_n

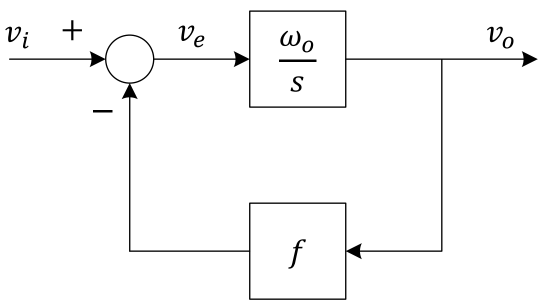
4 Op-amp Dynamics



Assume:

$Z_i \to \infty$
$Z_o \to 0$

Single-integrator model



$$A(s) = \frac{\omega_o}{s}$$

$A(s)$ is called the open-loop gain.

$L(s) = A(s)f$ is called the loop gain, or loop return ratio, or loop transfer function.

Note that $|A| \to \infty$ as $\omega \to 0$

For the OP27, $|A(f)|_{f=10\,\mathrm{kHz}} \approx 60\,\mathrm{dB} = 10^3$

$$\left|\frac{\omega_o}{2\pi \times 10^4}\right| = 10^3 \quad \Longrightarrow \quad \omega_o \approx 2\pi \times 10^7\,\mathrm{rad/s}$$

$$f_o \approx 10\,\mathrm{MHz}$$

Closed-loop transfer function is

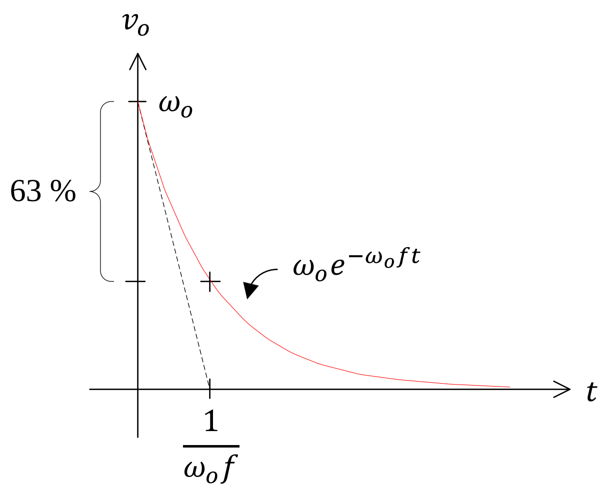
$$G(s) = \frac{\frac{\omega_o}{s}}{1 + \frac{\omega_o}{s} f} = \frac{\omega_o}{s + \omega_o f} \qquad \text{“Evans form”}$$

$$= \frac{1}{f}\left(\frac{1}{\frac{1}{\omega_o f} s + 1}\right) \qquad \text{“Bode Form”}$$

Evans form directly shows the pole location.

Bode form directly shows the DC gain and time constant.

Impulse response



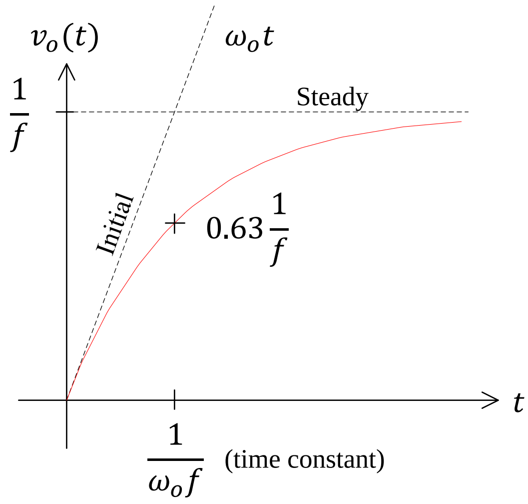
The block diagram helps us understand the initial response, e.g., impulse and step.

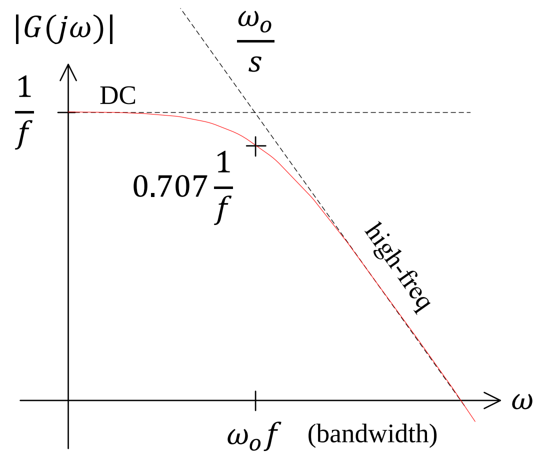
“Launch” an impulse at $t = 0$. The response is $v_o(0^+) = \omega_o$ then exponentially decays.

Step response vs. Bode plot

Note the “duality” between the step response & frequency response.

Time constant:	$\frac{1}{\omega_o f}$	Bandwidth:	$\omega_o f$
Initial response:	$\omega_o t$	High-freq response:	$\frac{\omega_o}{s}$
Final value:	$\frac{1}{f}$	DC gain:	$\frac{1}{f}$





Note the trade-off between DC gain and bandwidth

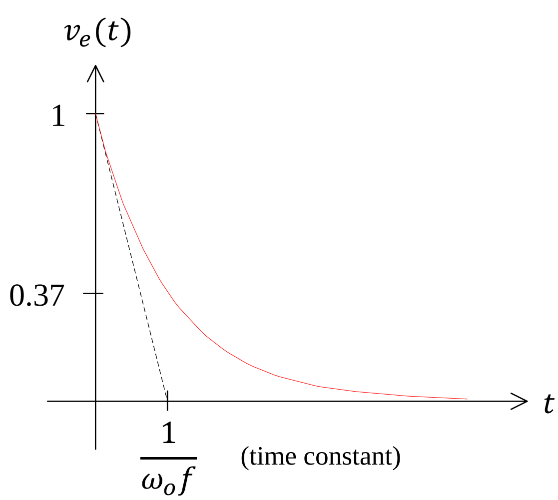
Gain-bandwidth product $= (\frac{1}{f}) \times (\omega_o f) = \omega_o$ for non-inverting amplifier.

Error dynamics

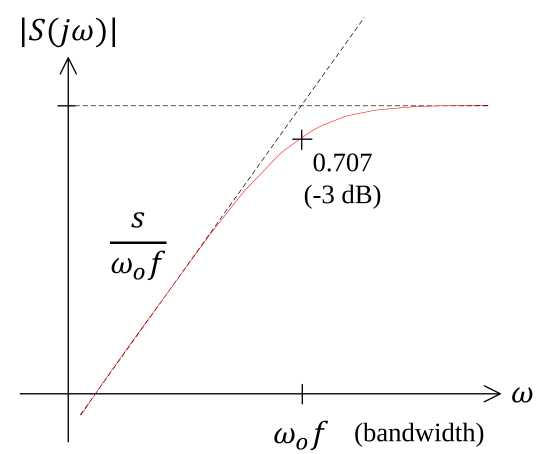
The error dynamics is governed by the sensitivity function $S(s)$.

$$\frac{v_e}{v_i} = S(s) = \frac{1}{1 + \frac{\omega_o}{s} f} = \frac{s}{s + \omega_o f}$$

The response of v_e to the unit-step v_i is



$v_e \neq 0$ during "transient"



DC gain $= 0$

Note that the virtual short *approximation* is valid only when we can ignore the transient.

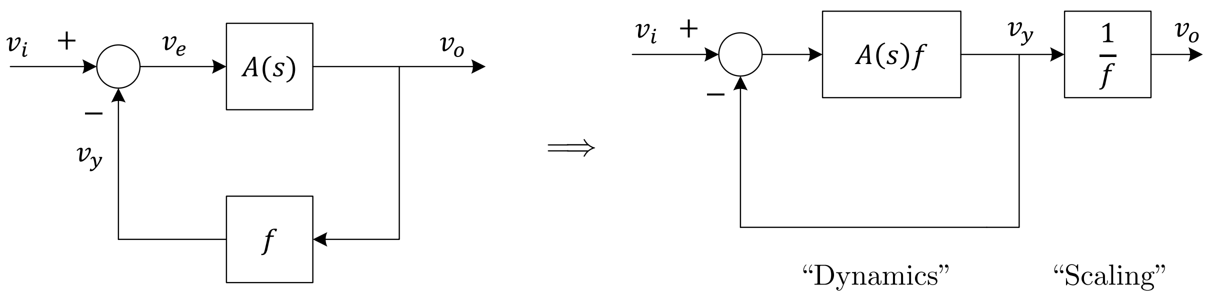
For example, when the time constants of external systems are much larger.

"Bandwidth" means

1. The frequency up to which a closed-loop system can track reference signals.
2. The frequency up to which a closed-loop system can reject disturbance signals.

General model

Note that an actual op-amp is not a single-integrator inside.
We will study this later - with loop shaping design.



$$L(s) = A(s)f \qquad \text{Loop return ratio}$$

$$S(s) \triangleq \frac{1}{1+L} = \frac{v_e}{v_i} \qquad \text{Sensitivity (Disturbance rejection)}$$

$$T(s) \triangleq \frac{L}{1+L} = \frac{v_y}{v_i} \qquad \text{Complementary Sensitivity (Tracking)}$$