MECH 421: Mechatronics System Instrumentation
2019/20 Winter Session – Term 2

Mid-term Exam - Solution

Date: Feb 24, 2020
Time: 3:00 – 4:00pm

Problem 1 (130 points)

Let us consider an op-amp circuit in Figure 1. We assume that the op-amp has infinite input impedance, zero output impedance, and open-loop transfer function $A(s)$. Figure 3 shows the Bode plot of $A(s)$.

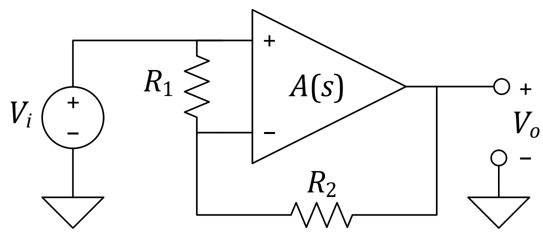

Figure 1: Op-amp circuit for Problem1.

(a) (20 pt.) Draw a block diagram that shows the feedback relation between the input voltage $V_i(s)$ and output voltage $V_o(s)$.

Solution
The op-amp input terminal voltages are

$$\begin{aligned} V_+ &= V_i \\ V_- &= \frac{R_2}{R_1+R_2}V_i + \frac{R_1}{R_1+R_2}V_o. \end{aligned}$$

Therefore, the op-amp output terminal voltage is

$$\begin{aligned} V_o &= A(s)(V_+ - V_-) \\ &= A(s)\left(V_i - \frac{R_2}{R_1+R_2}V_i - \frac{R_1}{R_1+R_2}V_o\right) \\ &= A(s)\left(\frac{R_1}{R_1+R_2}V_i - \frac{R_1}{R_1+R_2}V_o\right). \end{aligned}$$

The corresponding block diagram is

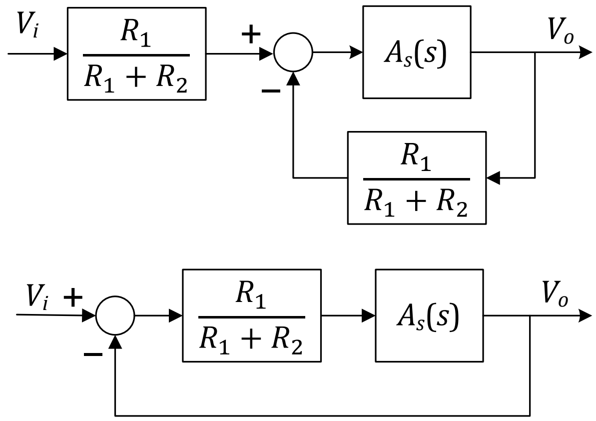

Figure 2: Op-amp circuit for Problem1.

(b) (10 pt.) Find the expression for the loop transfer function $L(s)$ in terms of R_1, R_2, and $A(s)$.

Solution

$$L(s) = A(s)\frac{R_1}{R_1 + R_2}$$

(c) (20 pt.) For $R_1 \to \infty$, $R_2 = 1\,\text{k}\Omega$, and $A(s)$ given in Figure 3, find the gain crossover frequency ω_c and phase margin ϕ_m of $L(s)$.

Solution

Physically this means that we remove R_1 and turns the op-amp circuit into a voltage follower configuration. As $R_1 \to \infty$, the attenuation factor becomes $\frac{R_1}{R_1+R_2} = 1$, and therefore the loop transfer function becomes $L(s) = A(s)$.

By definition, the gain crossover frequency ω_c satisfies $|L(j\omega_c)| = 1$. Since $|L(j\omega)| = |A(j\omega)|$, and $|A(j\omega)| = 1$ at $\omega = 10^7\,\text{Hz}$,

$$\omega_c = 10^7\,\text{Hz}.$$

By definition, the phase margin is $\phi_m = \angle L(j\omega_c) - (-180°)$. Since $\angle L(j\omega_c) = -110°$,

$$\phi_m = 70°.$$

(d) (30 pt.) For $R_1 = 1\,\text{k}\Omega$ and $A(s)$ given in Figure 3, find the resistance value R_2 that makes the closed-loop transfer function $G(s) = V_o(s)/V_i(s)$ achieve a $-3\,\text{dB}$ bandwidth of $100\,\text{kHz}$.

Intuitive solution

The $-3\,\text{dB}$ bandwidth ω_h of the closed-loop $G(s)$ is approximately the same as the crossover frequency ω_c of the loop $L(s)$. In particular, $\omega_h = \omega_c$ if the phase margin of the loop $L(s)$ is $90°$.

Therefore, we aim for $\omega_c = 100\,\mathrm{kHz}$ to achieve $\omega_h = 100\,\mathrm{kHz}$.

$$|L(j\omega)|_{\omega=100\,\mathrm{kHz}} = \frac{R_1}{R_1+R_2}|A(j\omega)|_{\omega=100\,\mathrm{kHz}} = 1.$$

Since $|A(j\omega)|_{\omega=100\,\mathrm{kHz}} = 100$ from the Bode plot, the required attenuation is $\frac{R_1}{R_1+R_2} = \frac{1}{100}$ for the loop to achieve a unity-gain crossing at $100\,\mathrm{kHz}$. Given $R_1 = 1\,\mathrm{k\Omega}$,

$$R_2 = 99\,\mathrm{k\Omega}.$$

The above is the intended process for the problem. The key idea of loop shaping is that we design the loop (i.e., set the crossover frequency ω_c and phase margin ϕ_m) for the closed-loop system to achieve desired performances (i.e., bandwidth ω_h and resonance peak M_r).

Formal solution

By definition

$$|G(j\omega)|_{\omega=100\,\mathrm{kHz}} = -3\,\mathrm{dB} = \frac{1}{\sqrt{2}}.$$

From the Black's formula,

$$\begin{aligned} G(j\omega) &= \frac{L(j\omega)}{1+L(j\omega)} \\ &= \frac{A(j\omega)f}{1+A(j\omega)f}, \end{aligned}$$

where $f = \frac{R_1}{R_1+R_2}$.

We want to find f that satisfies

$$\frac{1}{\sqrt{2}} = \left|\frac{A(j\omega)f}{1+A(j\omega)f}\right|_{\omega=100\,\mathrm{kHz}}.$$

From the Bode plot, we know that $A(j\omega)_{\omega=100\,\mathrm{kHz}} = 100\angle{-90^\circ} = -100j$. Substituting this into the above equation leads to

$$\frac{1}{\sqrt{2}} = \left|\frac{-100jf}{1-100jf}\right|_{\omega=100\,\mathrm{kHz}}.$$

Solving the above equation leads to

$$f = \frac{1}{100}.$$

Therefore, given $R_1 = 1\,\mathrm{k\Omega}$ and $f = \frac{R_1}{R_1+R_2}$,

$$R_2 = 99\,\mathrm{k\Omega}.$$

(e) (20 pt.) What is the dc gain of $G(s)$ designed in part (d)?

Solution

$$
\begin{aligned}
|G(j\omega)|_{\omega=0} &= \left| \frac{A(j\omega) \times 10^{-2}}{1 + A(j\omega) \times 10^{-2}} \right|_{\omega=0} \\
&= \left| \frac{10^4}{1 + 10^4} \right| \\
&= 0.9999 \approx 1
\end{aligned}
$$

(f) (30 pt.) Suppose $G(s)$ designed in part (d) is excited with an input voltage

$$V_i(t) = \cos(2\pi \times 10^5\ t),$$

which is a persistent sinusoid defined for all time including $t < 0$. Find the magnitude M_o and phase ϕ_o of the output voltage

$$V_o(t) = M_o \cos(2\pi \times 10^5\ t + \phi_o).$$

Solution

The frequency of the input voltage is $2\pi \times 10^5$ rads $= 100\,\text{kHz} = \omega_h = \omega_c$. Therefore, the output voltage becomes a sinusoid of the same frequency, where the magnitude is scaled by $M_o = |G(j\omega_c)|$ and the phase is shifted by $\phi_o = \angle G(j\omega_c)$.

At $\omega_c = 100\,\text{kHz}$, the closed-loop frequency response is

$$G(j\omega_c) = \frac{L(j\omega_c)}{1 + L(j\omega_c)},$$

where $L(j\omega_c) = 1\angle{-90^\circ} = -j$. Therefore, the magnitude is

$$
\begin{aligned}
M_o &= |G(j\omega_c)| \\
&= \left| \frac{-j}{1 - j} \right| = \frac{1}{\sqrt{2}} = -3\,\text{dB}.
\end{aligned}
$$

Note that this could have been answered directly without going through the math because we designed the closed loop as such. The phase is

$$
\begin{aligned}
\phi_o &= \angle G(j\omega_c) \\
&= \angle \frac{-j}{1 - j} \\
&= -90^\circ - (-45^\circ) = -45^\circ
\end{aligned}
$$

Here, the last step can be carried out either arithmetically or graphically.

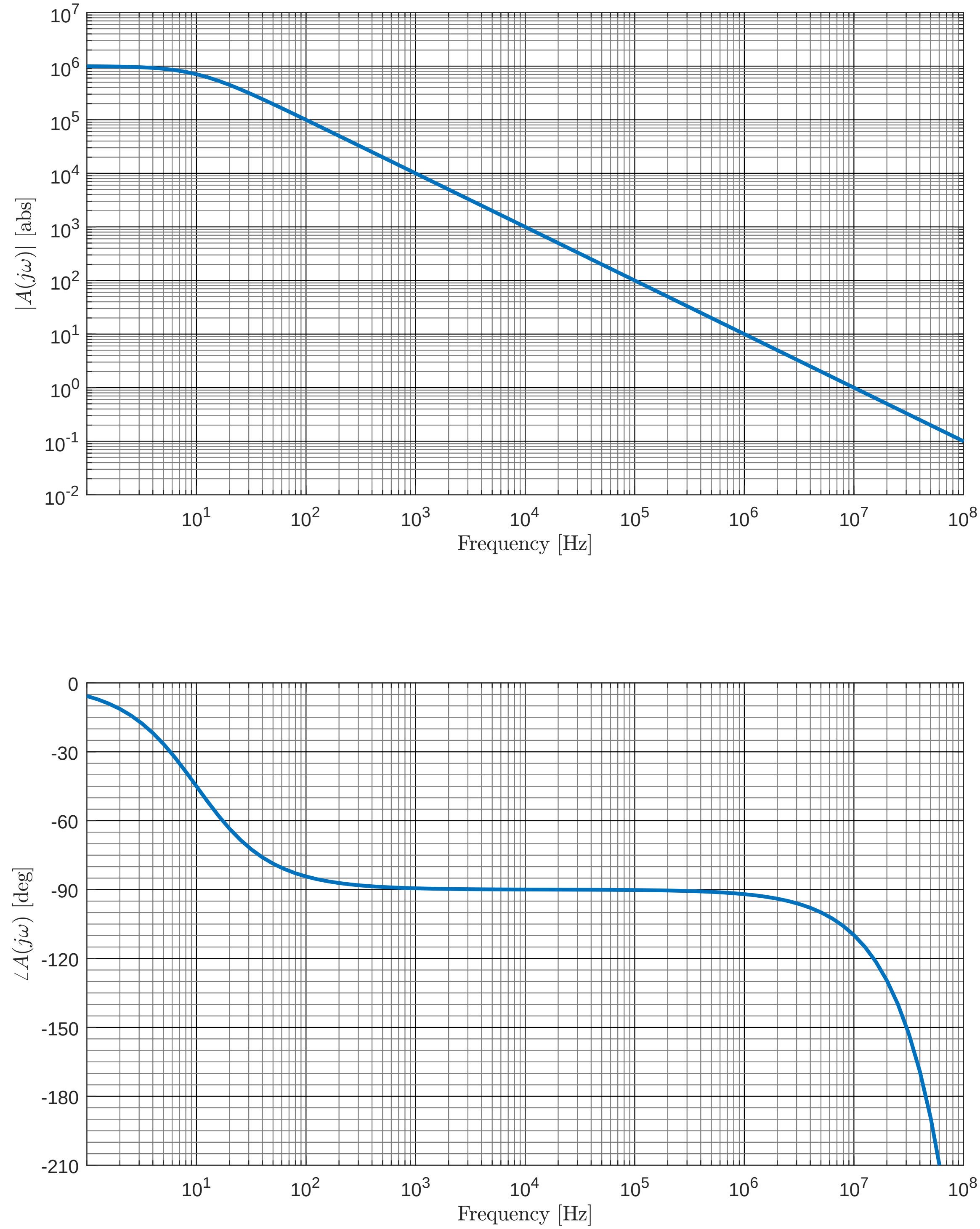

Figure 3: Bode plot of $A(s)$.

Problem 2 (70 points)

Let us consider a full-bridge strain gauge circuit in Figure 4. Here, $V_s = 5\,\text{V}$ is the supply voltage, R is the nominal resistance, and r is the resistance change due to the strain.

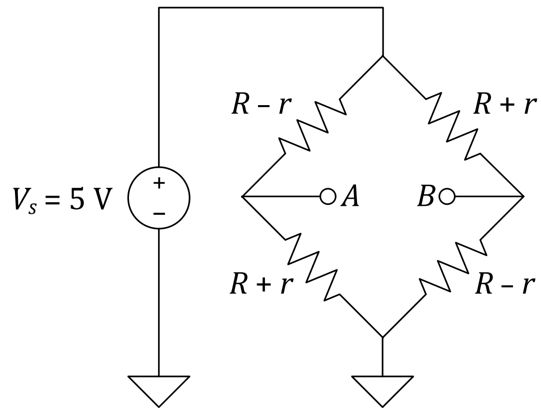

Figure 4: Full-bridge strain gauge circuit.

(a) (10 pt.) Find the expression for the voltage between the output terminals A and B in terms of R and r.

Solution

The terminal voltages V_A and V_B can be derived by applying the voltage divider rule.

$$V_A = \frac{R+r}{R-r+R+r}V_s = \frac{R+r}{2R}V_s$$
$$V_B = \frac{R-r}{R=r+R-r}V_s = \frac{R-r}{2R}V_s.$$

The difference voltage between the two terminal is

$$V_{AB} = \frac{r}{R}V_s = \frac{r}{R}5\,\text{V}.$$

(b) (30 pt.) The output terminals A and B of the bridge circuit are connected to the input terminals A and B of the op-amp circuit in Figure 5. The grounds of the two systems are also connected together. Suppose the op-amps are ideal (i.e., infinite input impedance, zero output impedance, and infinite open-loop gain) and do not saturate, and r varies such that $-0.01R < r < 0.01R$. Find the range of the output voltage V_o.

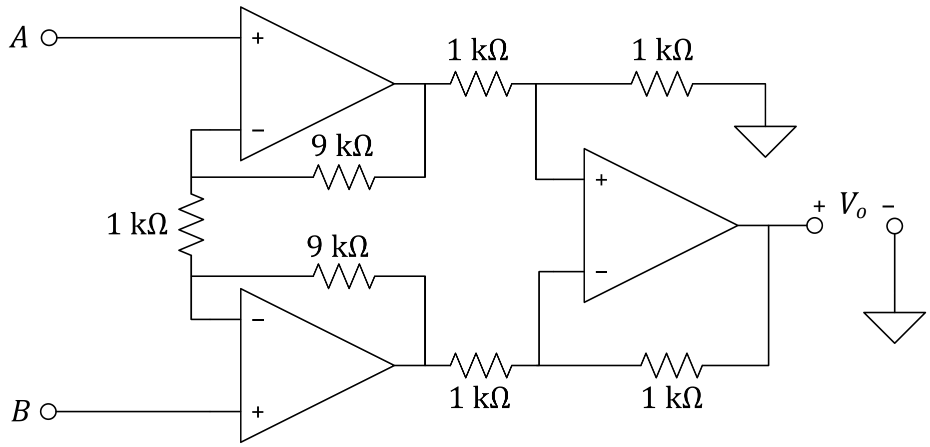

Figure 5: Instrumentation amplifier.

Solution

An ideal instrumentation, as in this problem, can measure differential voltages without loading down the signal source. This is due to the high impedances at the input terminals.

Let us define node voltages V_A, V_B, V_1, and V_2 as in Figure 6, and use virtual-short approximation to quickly identify node voltages at the op-amp input terminals. The circuit can be analyzed in two modular stages: amplification stage and subtraction stage.

Amplification stage: using KCL at the negative input terminals of A_1 and A_2 leads to

$$\frac{V_A - V_B}{1\,\text{k}\Omega} + \frac{V_A - V_1}{9\,\text{k}\Omega} = 0$$
$$\frac{V_B - V_A}{1\,\text{k}\Omega} + \frac{V_B - V_2}{9\,\text{k}\Omega} = 0$$

Subtracting the bottom from the the top equation leads to

$$2\frac{V_A - V_B}{1\,\text{k}\Omega} + \frac{V_A - V_B}{9\,\text{k}\Omega} = \frac{V_1 - V_2}{9\,\text{k}\Omega} \quad \rightarrow \quad V_1 - V_2 = 19(V_A - V_B).$$

Subtraction stage: using the superposition method at the negative input terminal of A_3 leads to

$$\frac{1}{2}V_1 = \frac{1}{2}V_2 + \frac{1}{2}V_o \quad \rightarrow \quad V_o = V_1 - V_2.$$

Therefore, the output voltage is

$$V_o = V_1 - V_2 = 19(V_A - V_B).$$

Given $-0.01R < r < 0.01R$, the range of the differential voltage $V_{AB} = \frac{r}{R} 5\,\mathrm{V}$ is $-0.05\,\mathrm{V} < V_{AB} < 0.05\,\mathrm{V}$. With the differential gain of 19, the output voltage range is

$$-0.95\,\mathrm{V} < V_o < 0.95\,\mathrm{V}.$$

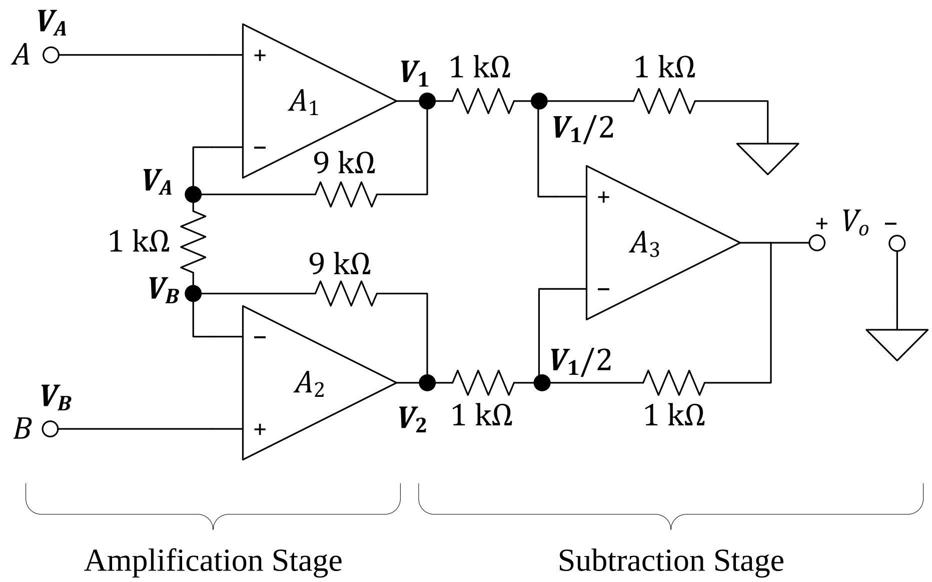

Figure 6: Instrumentation amplifier circuit with annotations.

(c) (30 pt.) The output terminals A and B of the bridge circuit are connected to the input terminals A and B of the op-amp circuit in Figure 7. The grounds of the two systems are also connected together. Suppose the op-amp is ideal (i.e., infinite input impedance, zero output impedance, and infinite open-loop gain) and does not saturate, and r varies such that $-0.01R < r < 0.01R$. Find the range of the output voltage V_o.

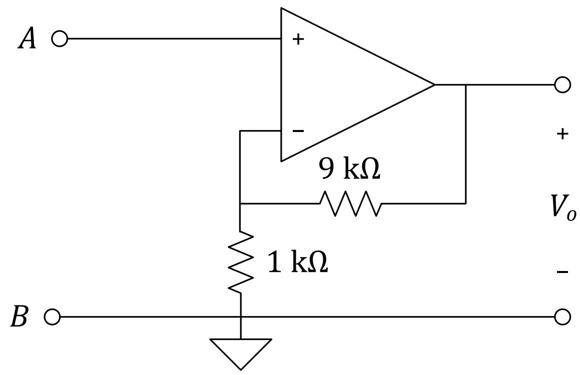

Figure 7: Non-inverting amplifier.

Solution

An amplifier with a single-ended input, as in this problem, may not properly measure differential voltages because it could load down the signal source. Particularly in this problem, the terminal B of the amplifier is grounded, and therefore it makes the terminal B voltage of the bridge circuit zero once connected. On the other hand, the terminal A voltage of the bridge circuit remains intact because it is connected to the amplifier's high-impedance input terminal.

$$
\begin{aligned}
V_B &= 0 \qquad \text{grounded} \\
V_A &= \frac{R+r}{2R} V_s = \left(1 + \frac{r}{R}\right) 2.5\,\mathrm{V}
\end{aligned}
$$

Given $-0.01R < r < 0.01R$, the range of V_A is $2.475\,\mathrm{V} < V_A < 2.525\,\mathrm{V}$. Since the amplifier gain is $\frac{1\,\mathrm{k\Omega}+9\,\mathrm{k\Omega}}{1\,\mathrm{k\Omega}} = 10$, the range of the output voltage is

$$24.75\,\mathrm{V} < V_o < 25.25\,\mathrm{V}.$$

Be sure to check the port impedances when connecting two systems together.

Grading Criteria

(a) Correct answer: full marks

(b) Correct process but wrong answer due to simple arithmetic mistakes: (-2)

(c) Correct process but wrong answer due to serious algebraic mistakes: (-5)

(d) Correct process but wrong answer due to the dependency of the previous wrong answer: (-5)

(e) On-track process but incomplete answer: (-10)

(f) Erroneous process but some reasonable efforts: $(+5)$, $(+10)$, or $(+15)$ from zero

(g) No answer: zero

The criteria (c) and (d) are additive.