University of British Columbia
Department of Mechanical Engineering

MECH468 Modern Control Engineering
MECH522 Foundations in Control Engineering
Midterm exam

Examiner: Dr. Ryozo Nagamune
February 26 (Wednesday), 2020, 1-1:50pm

Last name, First name

Name: Student #:

Signature:

Exam policies

- Allowed: One-page letter-size hand-written cheat sheet (both front side and back side). Do not hand in your cheat sheet.
- Not-allowed: PC, calculators.
- Write all your answers on this booklet. No extra sheet will be provided.
- Motivate your answers properly. (No chance to defend your answers orally.)
- 30 points in total.

Before you start ...

- Use washroom before the exam.
- Turn off your mobile phone.
- No eating.
- Questions are NOT allowed.

If you finish early ...

- Please stay at your seat until the end of exam, i.e., 1:50pm. (You are not allowed to leave the room before the end of exam, except going to washroom.)

To be filled in by the instructor/marker

Problem #	Expected duration	Mark	Full mark
1	about 10 min		10
2	about 5 min		6
3	about 5 min		4
4	about 25 min		10
Total	about 45 min		30

Extra page. Write the problem number before writing your answer.

1. Answer the following 'true' or 'false' questions. **You don't need to motivate your answers**; Mark will be given solely based on your answer 'true' or 'false'. (1pt each)

   (a) A system $y(t+1) = \sqrt{2} \cdot u(t)$ is a linear system.
   (b) A system $y(t+1) = \sqrt{2} \cdot u(t)$ is a causal system.
   (c) A system $y(t+1) = \sqrt{2} \cdot u(t)$ is a lumped system.
   (d) Any linear time-invariant (LTI) state-space model with order $n \geq 1$ has at least one state which is both controllable and observable.
   (e) If an LTI system is not BIBO stable, then the system is not asymptotically stable.
   (f) For single-input-single-output (SISO) transfer function $G(s)$, it is always the case that the controllable canonical realization and the observable canonical realization have the same number of states.
   (g) A nonlinear model $\dot{x} = -x^3 - xu$ can be linearized around the equilibrium point $(x_0, u_0) = (1, 1)$.
   (h) If all the elements of a symmetric matrix are positive, then the matrix is positive definite.
   (i) For a controllable discrete-time LTI system with order 3, 2-step state transfer from any initial state to any final state is always possible.
   (j) If a continuous-time LTI system is observable, then the corresponding zero-order-hold discretized system is always observable.

Question	'True' or 'False'
(a)	**True**
(b)	**True**
(c)	**False**
(d)	**False**
(e)	**True**
(f)	**True**
(g)	**False**
(h)	**False**
(i)	**False**
(j)	**False**

2. For the **continuous-time** linear time-invariant systems $\dot{x} = Ax$ with the following A-matrices, judge whether the system is asymptotically stable ('AS'), marginally stable ('MS'), or unstable ('US').

   You don't need to motivate your answers; Mark will be given solely based on your answer 'AS', 'MS' or 'US'. (2pt each)

   (a) $A = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix}$

   (b) $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

   (c) $A = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & 0 & -1 \end{bmatrix}$

Question	'AS', 'MS' or 'US'
(a)	**AS**
(b)	**US**
(c)	**MS**

3. For transfer matrices $G(s)$ below, obtain its realization (in any form, i.e., you can use either controllable, or observable canonical form, or any other form).

You will get full-mark if your final answers (A, B, C, D)-matrices are correct. If they are incorrect, partial mark may be given based on the derivation process. (2pt each)

$$\text{(a) } G(s) = \frac{1}{s} + \frac{1}{s+1} \qquad \text{(b) } G(s) = \begin{bmatrix} \dfrac{s+2}{s+1} \\ 2 \end{bmatrix}$$

Write your answer here.

(a): $G(s) = \dfrac{2s+1}{s^2+s}$

Controllable canonical form

$$A = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}, \ B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \ C = \begin{bmatrix} 1 & 2 \end{bmatrix}, \ D = 0$$

Observable canonical form

$$A = \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix}, \ B = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \ C = \begin{bmatrix} 0 & 1 \end{bmatrix}, \ D = 0$$

Parallel connection

$$A = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}, \ B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \ C = \begin{bmatrix} 1 & 1 \end{bmatrix}, \ D = 0$$

(b): $G(s) = \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \dfrac{1}{s+1} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

Controllable canonical form

$$A = -1, \ B = 1, \ C = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \ D = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Observable canonical form

$$A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \ B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \ C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \ D = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

4. Let us consider the following **discrete-time** system.

$$\begin{cases} x[k+1] &= \underbrace{\begin{bmatrix} 1/2 & 1 \\ 0 & -1 \end{bmatrix}}_{A} x[k] + \underbrace{\begin{bmatrix} 2 \\ 1 \end{bmatrix}}_{B} u[k] \\ y[k] &= \underbrace{\begin{bmatrix} 0 & 1 \end{bmatrix}}_{C} x[k] \end{cases}$$

(a) Check the controllability and the observability. (2pt)

(b) Obtain the Kalman decomposition. Explain which state is controllable or uncontrollable, and which state is observable or unobservable. (2pt)

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Write your answer here.

(a) Controllability analysis

$$\mathcal{C} = \begin{bmatrix} 2 & 2 \\ 1 & -1 \end{bmatrix} \Rightarrow \text{rank } \mathcal{C} = 2 \Rightarrow \text{ Controllable}$$

Observability analysis

$$\mathcal{O} = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} \Rightarrow \text{rank } \mathcal{O} = 1 \Rightarrow \text{ Not observable}$$

(b) Decomposition for observability

$$\ker \mathcal{O} = \text{span}\left\{T_{c\bar{o}}\right\}, \quad T_{c\bar{o}} := \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

The coordinate transformation matrix is

$$T^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Write your answer here.

Thus, Kalman decomposition becomes as follows.

$$\begin{cases} \begin{bmatrix} z_{co}[k+1] \\ z_{c\bar{o}}[k+1] \end{bmatrix} = \underbrace{\begin{bmatrix} -1 & 0 \\ 1 & 1/2 \end{bmatrix}}_{TAT^{-1}} \begin{bmatrix} z_{co}[k] \\ z_{c\bar{o}}[k] \end{bmatrix} + \underbrace{\begin{bmatrix} 1 \\ 2 \end{bmatrix}}_{TB} u[k] \\ y[k] = \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{CT^{-1}} \begin{bmatrix} z_{co}[k] \\ z_{c\bar{o}}[k] \end{bmatrix} \end{cases}$$

Here, z_{co} is controllable and observable, and $z_{c\bar{o}}$ is controllable but unobservable.

(c) For the **discrete-time** system above, let us consider the minimum energy control which solves the following optimization problem.

$$\min_{u[\cdot]} \sum_{k=0}^{2} (u[k])^2 \quad \text{subject to} \quad \begin{cases} x[k+1] = \underbrace{\begin{bmatrix} 1/2 & 1 \\ 0 & -1 \end{bmatrix}}_{A} x[k] + \underbrace{\begin{bmatrix} 2 \\ 1 \end{bmatrix}}_{B} u[k] \\ x[0] = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \; x[3] = \begin{bmatrix} 4 \\ 3 \end{bmatrix} \end{cases}$$

i. Compute the **discrete-time** minimum energy control $u^*[0]$, $u^*[1]$, $u^*[2]$, and corresponding states $x[1]$ and $x[2]$. (2pt)

ii. Briefly explain the reason why the minimum energy control is not practically useful. (2pt)

iii. Obtain another (non-minimum energy) control input sequence $u[0]$, $u[1]$, $u[2]$ which achieves the state transfer from $x[0]$ to $x[3]$ given above, and confirm that the following inequality is satisfied by calculating values of left-hand side and right-hand side: (2pt)

$$\sum_{k=0}^{2} (u^*[k])^2 \le \sum_{k=0}^{2} (u[k])^2.$$

— (END OF MIDTERM EXAM) —

Write your answer here.

i. The minimum energy control can be obtained as follows.

$$\begin{aligned} \begin{bmatrix} u^*[2] \\ u^*[1] \\ u^*[0] \end{bmatrix} &= \mathcal{C}_d[3]^T (\mathcal{C}_d[3]\mathcal{C}_d[3]^T)^{-1}(x[3] - A_d^3 x[0]) \\ &= \begin{bmatrix} 2 & 1 \\ 2 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 8 & 0 \\ 0 & 3 \end{bmatrix}^{-1} \begin{bmatrix} 4 \\ 3 \end{bmatrix} \\ &= \begin{bmatrix} 2 & 1 \\ 2 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1/2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \end{aligned}$$

$$\mathcal{C}_d[3] = [B \; AB \; A^2B] = \begin{bmatrix} 2 & 2 & 0 \\ 1 & -1 & 1 \end{bmatrix}$$

$$\begin{aligned} x[1] &= Ax[0] + Bu[0] = A\begin{bmatrix} 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \end{bmatrix} \cdot 1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \\ x[2] &= Ax[1] + Bu[1] = A\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix} \end{aligned}$$

Write your answer here.

ii. Since it is open-loop control, the desired state transfer will not be achievable when there are uncertainties such as modeling errors and disturbances, and the system cannot be stabilized when the plant is unstable.

iii. We find a solution to the following linear system of equation.

$$\underbrace{\begin{bmatrix} 4 \\ 3 \end{bmatrix}}_{x[3]} - A^3 \underbrace{\begin{bmatrix} 0 \\ 0 \end{bmatrix}}_{x[0]} = \underbrace{\begin{bmatrix} 2 & 2 & 0 \\ 1 & -1 & 1 \end{bmatrix}}_{\mathcal{C}_d[3]} \begin{bmatrix} u[2] \\ u[1] \\ u[0] \end{bmatrix}$$

One such solution can be

$$\begin{bmatrix} u[2] \\ u[1] \\ u[0] \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}$$

Thus, we can confirm that:

$$\sum_{k=0}^{2} (u^*[k])^2 = 1^2 + 0^2 + 2^2 = 5 \leq \sum_{k=0}^{2} (u[k])^2 = 1^2 + 1^2 + 3^2 = 11.$$

Extra page. Write the problem number before writing your answer.