1. Consider the following continuous-time system:

$$\begin{cases} \dot{x}(t) &= \begin{bmatrix} -1 & a \\ 1 & -2 \end{bmatrix} x(t) + \begin{bmatrix} b \\ 1 \end{bmatrix} u(t), \\ y(t) &= \begin{bmatrix} c & 1 \end{bmatrix} x(t), \end{cases}$$

where a, b, c are constants.

(a) Obtain the condition for asymptotic stability in terms of a. (2pt)

Write your answer here.

Eigenvalues of A-matrix are computed as

$$\det(\lambda I - A) = \det \begin{bmatrix} \lambda+1 & -a \\ -1 & \lambda+2 \end{bmatrix}$$

$$= (\lambda+1)(\lambda+2) - a$$

$$= \lambda^2 + 3\lambda + 2 - a = 0$$

The condition for $Re[\lambda] < 0$ is that all coefficients have the same sign, because this is 2-nd order equation.

Thus $2 - a > 0$, i.e. $a < 2$ //

(b) Obtain the condition for controllability in terms of a and b. (2pt)

(c) Obtain the condition for observability in terms of a and c. (2pt)

Write your answer here.

(b) Controllability matrix

$$\mathcal{C} = \begin{bmatrix} b & -b+a \\ 1 & b-2 \end{bmatrix}$$

For $\mathcal{C}$ to be full rank, the condition is $\det \mathcal{C} \neq 0$

$$\det \mathcal{C} = b(b-2) - (a-b) \neq 0$$

$$b^2 - b - a \neq 0$$

(c) Observability matrix

$$\mathcal{O} = \begin{bmatrix} c & 1 \\ -c+1 & ac-2 \end{bmatrix}$$

For $\mathcal{O}$ to be full rank, the condition is $\det \mathcal{O} \neq 0$

$$\det \mathcal{O} = c(ac-2) - (1-c) \neq 0$$

$$ac^2 - c - 1 \neq 0$$

(d) Consider the case when $a = 0$ and $b = c = 1$, i.e.,

$$\begin{cases} \dot{x}(t) &= \begin{bmatrix} -1 & 0 \\ 1 & -2 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(t), \\ y(t) &= \begin{bmatrix} 1 & 1 \end{bmatrix} x(t). \end{cases} \tag{1}$$

i. Obtain the Kalman decomposition. (2pt)

ii. Write explicitly which state is controllable / uncontrollable and observable / unobservable. (1pt)

iii. Verify that the "controllable-and-observable part" (A_{co}, B_{co}, C_{co}) is actually controllable and observable. (1pt)

Write your answer here.

(i)

$$\mathcal{C} = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \Rightarrow \operatorname{rank} \mathcal{C} = 1 < 2 \Rightarrow \text{NOT controllable}$$

$$\mathcal{O} = \begin{bmatrix} 1 & 1 \\ 0 & -2 \end{bmatrix} \Rightarrow \operatorname{rank} \mathcal{O} = 2 \Rightarrow \text{observable}$$

$$\operatorname{Im} \mathcal{C} = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$$

$$T^{-1} = \begin{bmatrix} \underbrace{\begin{matrix} 1 \\ 1 \end{matrix}}_{T_c} & \underbrace{\begin{matrix} 0 \\ 1 \end{matrix}}_{T_{\bar{c}}} \end{bmatrix} \Rightarrow T = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$$

$$TAT^{-1} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}$$

$$TB = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$CT^{-1} = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \end{bmatrix}$$

(iii) $(A_{co}, B_{co}, C_{co}) = (-1, 1, 2)$

$\mathcal{C} = 1 \neq 0$ ∴ controllable

$\mathcal{O} = 2 \neq 0$ ∴ observable

$$\begin{cases} \begin{bmatrix} \dot{z}_{co} \\ \dot{z}_{\bar{c}o} \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} z_{co} \\ z_{\bar{c}o} \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u \\ y = \begin{bmatrix} 2 & 1 \end{bmatrix} \begin{bmatrix} z_{co} \\ z_{\bar{c}o} \end{bmatrix} \end{cases}$$

z_{co} : controllable & observable

$z_{\bar{c}o}$: uncontrollable & observable

(ii)

(e) For the system (1) in question (d), compute the A_d-matrix (A-matrix of a discrete-time system) of the discretized system (by zero-order-hold) with sampling period $T = 1$. (2pt)

Write your answer here.

$$A_d = e^{AT} = \mathcal{L}^{-1}\left\{(sI - A)^{-1}\right\}\Big|_{t=T}$$

$$= \mathcal{L}^{-1}\left\{\begin{bmatrix} s+1 & 0 \\ -1 & s+2 \end{bmatrix}^{-1}\right\}\Bigg|_{t=T}$$

$$= \mathcal{L}^{-1}\left\{\frac{1}{(s+1)(s+2)}\begin{bmatrix} s+2 & 0 \\ 1 & s+1 \end{bmatrix}\right\}\Bigg|_{t=T}$$

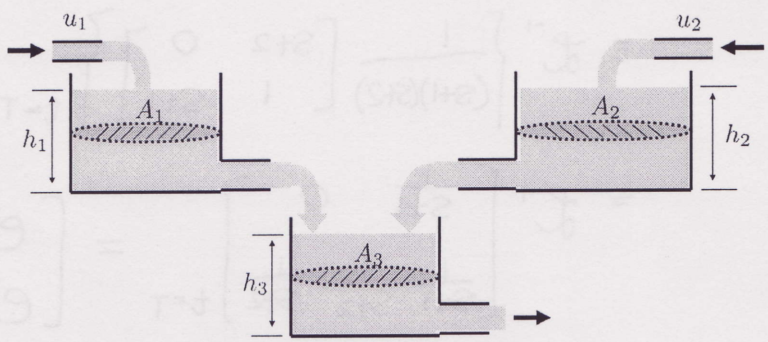
$$= \mathcal{L}^{-1}\begin{bmatrix} \frac{1}{s+1} & 0 \\ \frac{1}{s+1} - \frac{1}{s+2} & \frac{1}{s+2} \end{bmatrix}\Bigg|_{t=T} = \begin{bmatrix} e^{-T} & 0 \\ e^{-T} - e^{-2T} & e^{-2T} \end{bmatrix}$$

$$T = 1: \quad A_d = \begin{bmatrix} e^{-1} & 0 \\ e^{-1} - e^{-2} & e^{-2} \end{bmatrix}$$

2. Consider a three-water-tank system in the figure below. Here, A_i, $i = 1, 2, 3$, are tank section areas, h_i, $i = 1, 2, 3$, are the water heights of the tanks, and u_i, $i = 1, 2$, are input flow rates u_1 and u_2. The nonlinear state equation of this system is assumed to be expressed as

$$\begin{aligned}
\dot{h}_1(t) &= \frac{1}{\rho A_1}\left(-K\sqrt{h_1(t)} + u_1(t)\right), \\
\dot{h}_2(t) &= \frac{1}{\rho A_2}\left(-K\sqrt{h_2(t)} + u_2(t)\right), \\
\dot{h}_3(t) &= \frac{1}{\rho A_3}\left(-K\sqrt{h_3(t)} + K\sqrt{h_1(t)} + K\sqrt{h_2(t)}\right),
\end{aligned}$$

where ρ and K are given positive constants.



We would like to linearize the nonlinear state equation around the situation when we maintain the water heights at $h_1(t) = h_{10}$ and $h_2(t) = h_{20}$, where h_{10} and h_{20} are given positive constant heights.

(a) Obtain the corresponding constant input flow rates $u_1(t) = u_{10}$ and $u_2(t) = u_{20}$ in terms of given constants h_{10} and h_{20}. (2pt)

(b) Obtain the corresponding constant water height $h_3(t) = h_{30}$ in terms of given constants h_{10} and h_{20}. (2pt)

(c) Derive a linearized state equation $\dot{\delta h}(t) = A\delta h(t) + B\delta u(t)$ around the equilibrium point $(h_1, h_2, h_3) = (h_{10}, h_{20}, h_{30})$ and $(u_1, u_2) = (u_{10}, u_{20})$. To answer this question, you do not need to use solutions obtained in (a) and (b); just use (h_{10}, h_{20}, h_{30}) and (u_{10}, u_{20}). (2pt)

Write your answer here.

(a) $\dot{h}_1 = 0 \Rightarrow u_1(t) = K\sqrt{h_1(t)}$ Since $h_1(t) = h_{10}$, $u_{10} = K\sqrt{h_{10}}$

$\dot{h}_2 = 0 \Rightarrow u_2(t) = K\sqrt{h_2(t)}$ $h_2(t) = h_{20}$, $u_{20} = K\sqrt{h_{20}}$

(b) $\dot{h}_3 = 0 \Rightarrow K\sqrt{h_3(t)} = K\sqrt{h_1(t)} + K\sqrt{h_2(t)}$

With $h_1(t) = h_{10}$, $h_2(t) = h_{20}$, $h_{30} = (\sqrt{h_{10}} + \sqrt{h_{20}})^2$

Write your answer here.

(c) $\dot{\delta h} = A\delta h + B\delta u$

$$\delta h := \begin{bmatrix} h_1 - h_{10} \\ h_2 - h_{20} \\ h_3 - h_{30} \end{bmatrix}, \quad \delta u := \begin{bmatrix} u_1 - u_{10} \\ u_2 - u_{20} \end{bmatrix}$$

$$A = \left.\frac{\partial f}{\partial h}\right|_{\substack{h=h_0 \\ u=u_0}} = \begin{bmatrix} -\frac{k}{\rho A_1}\frac{1}{2\sqrt{h_{10}}} & 0 & 0 \\ 0 & -\frac{k}{\rho A_2}\frac{1}{2\sqrt{h_{20}}} & 0 \\ \frac{k}{\rho A_3}\frac{1}{2\sqrt{h_{10}}} & \frac{k}{\rho A_3}\frac{1}{2\sqrt{h_{20}}} & -\frac{k}{\rho A_3}\frac{1}{2\sqrt{h_{30}}} \end{bmatrix}$$

$$B = \left.\frac{\partial f}{\partial u}\right|_{\substack{h=h_0 \\ u=u_0}} = \begin{bmatrix} 1/\rho A_1 & 0 \\ 0 & 1/\rho A_2 \\ 0 & 0 \end{bmatrix}$$

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3. Consider the following controllable discrete-time system:

$$x[k+1] = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} x[k] + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u[k],$$

Compute the minimum energy control $u[k]$, $k = 0, 1, 2$, which transfers state vector from $x[0] = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ to $x[3] = \begin{bmatrix} 6 \\ 6 \end{bmatrix}$. (2pt)

——— (End of Midterm Exam) ———

Write your answer here.

Min. energy control

$$\begin{bmatrix} u[2] \\ u[1] \\ u[0] \end{bmatrix} = C_d[k_f]^T \left(C_d[k_f]\, C_d[k_f]^T \right)^{-1} \left(\underbrace{x[k_f]}_{\begin{bmatrix} 6 \\ 6 \end{bmatrix}} - \underbrace{A^{k_f} x[0]}_{\begin{bmatrix} 0 \\ 0 \end{bmatrix}} \right), \quad k_f = 3$$

$$C_d[3] = \begin{bmatrix} B & AB & A^2B \end{bmatrix} = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix}$$

$$\left(C_d[3]\, C_d[3]^T \right)^{-1} = \left(\begin{bmatrix} 0 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{bmatrix} \right)^{-1} = \begin{bmatrix} 5 & 3 \\ 3 & 3 \end{bmatrix}^{-1} = \frac{1}{6} \begin{bmatrix} 3 & -3 \\ -3 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{bmatrix} \frac{1}{6} \begin{bmatrix} 3 & -3 \\ -3 & 5 \end{bmatrix} \begin{bmatrix} 6 \\ 6 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$$

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