1. Answer the following true-or-false questions. Write (T) (meaning *true*) or (F) (meaning *false*). **No need to motivate your answers.** (0.5pt each)

Below, x, u and y denote respectively state, input and output vectors.

(a) The system $y(t) = \sin(t) \cdot u(t)$ is a nonlinear system.

(b) The system $y(t) = \sin(t) \cdot u(t)$ is a memoryless system.

(c) The system $y(t) = \sin(t) \cdot u(t)$ is a time-varying system.

(d) Kernel space of a matrix $\begin{bmatrix} 1 & -1 & 1 \\ 2 & -2 & 2 \end{bmatrix}$ is of one-dimensional (i.e., the basis of the kernel space consists of one vector).

(e) An uncontrollable and unobservable system is always unstable.

(f) For a discrete-time system $x[k+1] = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} x[k] + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u[k]$, it is possible to transfer state from $x[0] = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ to $x[2] = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$.

(g) It is possible to asymptotically stabilize an unstable system without feedback control.

(h) If all the element of a symmetric matrix is positive, then the matrix is always positive definite.

(i) If we apply a state coordinate transformation ($z = Tx$) to an unstable system, then the resulting new state-space model is always unstable.

(j) If a state-space model is asymptotically stable, then it is always BIBO stable.

Question	**Write your answer here**
(a)	False
(b)	True
(c)	True
(d)	False
(e)	False
(f)	True
(g)	False
(h)	False
(i)	True
(j)	True

2. For the continuous-time system $\dot{x} = Ax$ and the discrete-time system $x[k+1] = Ax[k]$ with the following A matrices, determine if it is asymptotically stable, marginally stable, or unstable. Fill out the following table, with abbreviations:

- "AS" meaning "asymptotically stable",
- "MS" meaning "marginally stable", or
- "UN" meaning "unstable".

No need to motivate your answers. (0.5pt each)

A	Continuous-time $\dot{x} = Ax$	Discrete-time $x[k+1] = Ax[k]$
$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$	UN	AS
$\begin{bmatrix} 0 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$	MS	MS
$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$	UN	MS
$\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix}$	AS	UN
$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$	MS	AS

3. Consider the following continuous-time state-space model:

$$\begin{cases} \dot{x} &= \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} x + \begin{bmatrix} 2 \\ -3 \end{bmatrix} u, \\ y &= \begin{bmatrix} 1 & 1 \end{bmatrix} x. \end{cases}$$

Below, you need to motivate your answers. Do not just write "Yes" or "No".

(a) Verify that the system is asymptotically stable, using:

- eigenvalue criteria (1pt)
- Lyapunov theorem (2pt)

(b) Is the system BIBO stable? (1pt)

(a) eigenvalues of A

$$\det(\lambda I - A) = 0 \Rightarrow \lambda(\lambda+2)+1 = (\lambda+1)^2 = 0 \Rightarrow \lambda = -1, -1$$

$Re(\lambda) < 0$

$\therefore$ asymptotically stable.

Lyapunov equation

$$\begin{bmatrix} 0 & -1 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} P_1 & P_2 \\ P_2 & P_3 \end{bmatrix} + \begin{bmatrix} P_1 & P_2 \\ P_2 & P_3 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$(1,1)$: $-2P_2 = -1 \Rightarrow P_2 = \frac{1}{2}$

$(1,2)$: $-P_3 + P_1 - 2P_2 = 0 \Rightarrow P_1 = \frac{3}{2}$

$(2,2)$: $2(P_2 - 2P_3) = -1 \Rightarrow P_3 = \frac{1}{2}$

$$P = \frac{1}{2}\begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix} > 0$$

$\therefore$ asymptotically stable

(b) System is BIBO stable, because it is asymptotically stable.

(c) Is the system controllable? (1pt)

(d) Is the system observable? (1pt)

(e) Obtain Kalman decomposition. Indicate which state is controllable / uncontrollable and observable / unobservable. (2pt)

(c) $\mathcal{C} = \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} 2 & -3 \\ -3 & 4 \end{bmatrix}$: full rank $\therefore$ controllable

(d) $\mathcal{O} = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$: Not full rank $\therefore$ NOT observable

(e) $\ker \mathcal{O} = \operatorname{span}\left\{ \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$

$$T^{-1} = \begin{bmatrix} T_o & T_{\bar{o}} \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \Rightarrow T = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$$TAT^{-1} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix}$$

$$TB = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ -3 \end{bmatrix} = \begin{bmatrix} -1 \\ -3 \end{bmatrix}, \quad CT^{-1} = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

$$\begin{cases} \begin{bmatrix} \dot{z}_{co} \\ \dot{z}_{c\bar{o}} \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} z_{co} \\ z_{c\bar{o}} \end{bmatrix} + \begin{bmatrix} -1 \\ -3 \end{bmatrix} u \\ y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} z_{co} \\ z_{c\bar{o}} \end{bmatrix} \end{cases}$$

(f) Compute the A-matrix of the zero-order-hold discretized system, with the sampling period $T > 0$. (2pt)
Hint: You may want to use the Laplace transform:

$$\mathcal{L}\{e^{-t}\} = \frac{1}{s+1}, \ \mathcal{L}\{te^{-t}\} = \frac{1}{(s+1)^2}.$$

$$A_d = e^{AT} = \mathcal{L}^{-1}\{(s\cancel{I}-A)^{-1}\} = \mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2}\begin{bmatrix} s+2 & 1 \\ -1 & s \end{bmatrix}\right\}$$

$$\frac{1}{(s+1)^2}\begin{bmatrix} s+2 & 1 \\ -1 & s \end{bmatrix} = \frac{1}{s+1}k_1 + \frac{1}{(s+1)^2}k_2$$

partial fraction expansion

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} s + \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix} = (s+1)k_1 + k_2$$

$$\Rightarrow \begin{cases} k_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ k_2 = \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \end{cases}$$

$$\therefore e^{AT} = \mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\}k_1 + \mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2}\right\}k_2$$

$$= e^{-T}k_1 + Te^{-T}k_2$$

$$= e^{-T}\begin{bmatrix} 1+T & T \\ -T & 1-T \end{bmatrix}$$

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