MECH 463 Revision Notes

Vibration Characteristics of a n-dof system

1. there are n natural frequencies

2. when vibrating at a given natural frequency, all parts of the system vibrate in phase (or exactly out of phase)

3. At each natural frequency, there is a definite ratio of the vibration amplitudes of each part of the system (negative ratio for out-of-phase parts). This is the mode shape.

4. 2n initial conditions are required to specify the motion.

In view of these characteristics, we can choose a trial solution of the form:

$$\underline{x} = \underset{\uparrow \, (3)}{\underline{u}} \, C \cos(\omega t + \underset{\uparrow \, (2)}{\phi})$$

The general solution for the 2-dof case is

$$\underline{x} = \underline{u}_1 C_1 \cos(\omega_1 t + \phi_1) + \underline{u}_2 C_2 \cos(\omega_2 t + \phi_2)$$

$n = 2$ natural frequencies — (1)
$2n = 4$ integration constants — (4)

Matrix Formulations

1-dof case $\quad m\ddot{x} + c\dot{x} + kx = f$

n-dof case $\quad \underset{\sim}{M}\,\ddot{\underline{x}} + \underset{\sim}{C}\,\dot{\underline{x}} + \underset{\sim}{K}\,\underline{x} = \underline{f}$

In general, n-dof solutions follow much the same procedure as 1-dof solutions, except that they involve matrix and vector quantities, rather than scalar quantities. The 1-dof case is just the specific case for which $n=1$.

Coupling

For no dynamic coupling (diagonal $\underset{\sim}{M}$), choose a coordinate system based on the mass centres

For no static coupling (diagonal $\underset{\sim}{K}$), choose a coordinate system based on the springs.

General Solution

general solution = complementary solution + particular integral

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general solution = complementary solution + particular solution
(free vibration. transient solution) (forced vibration. steady state solution)

The complementary solution is the solution to the homogeneous equation $\underset{\sim}{M}\ddot{\underline{x}} + \underset{\sim}{C}\dot{\underline{x}} + \underset{\sim}{K}\underline{x} = \underline{0}$. This part of the solution describes the <u>free vibration</u> behaviour of the system (involving natural frequencies, mode shapes, etc.) Since the free vibration of a damped system eventually dies away, the complementary solution is also called the <u>transient solution.</u>

For a general n-dof system, we can solve by using the trial solution $\underline{x} = C\underline{u}\,e^{\lambda t}$

$$\longrightarrow \left(\lambda^2 \underset{\sim}{M} + \lambda \underset{\sim}{C} + \underset{\sim}{K}\right)\underline{u} = 0$$

This is a generalized eigenvalue problem, where
$\lambda = i\omega$ = eigenvalue, $\underline{u}$ = eigenvector.

For the undamped case, we could use a simpler trial solution $\underline{x} = C\underline{u}\cos(\omega t + \phi)$

$$\longrightarrow \left(-\omega^2 \underset{\sim}{M} + \underset{\sim}{K}\right)\underline{u} = 0$$

or $\underset{\sim}{H}\underline{u} = \omega^2 \underline{u}$ where $\underset{\sim}{H} = \underset{\sim}{M}^{-1}\underset{\sim}{K}$

standard eigenvalue problem.

The particular solution is the additional solution that includes the forcing function on the right of the equation. This part of the solution describes the forced vibration behaviour of the system. (involving the forcing frequencies, magnification factors, etc.) Since the forced vibration persists as long as the forcing function is applied, the particular solution is also called the steady state solution

For a general n-dof system with harmonic excitation $\underline{f} = \mathrm{Re}\left[\underline{\bar{F}}\, e^{i\omega_f t}\right]$, using a trial solution $\underline{x} = \mathrm{Re}\left[\underline{\bar{X}}\, e^{i\omega_f t}\right]$

$$\longrightarrow \left(-\omega_f^2 \underset{\sim}{M} + i\,\omega_f \underset{\sim}{C} + \underset{\sim}{K}\right) \underline{\bar{X}} = \underline{\bar{F}}$$

$$= \underset{\sim}{\bar{A}}\, \underline{\bar{X}} = \underline{\bar{F}}$$

This is a regular linear equation, except that all the quantities are complex.

For the undamped case, we could use a simpler trial solution when $\underline{f} = \underline{F} \cos \omega_f t$, $\quad \underline{x} = \underline{X} \cos \omega_f t$

$$\longrightarrow \left(-\omega_f^2 \underset{\sim}{M} + \underset{\sim}{K}\right) \underline{X} = \underline{F}$$

$$= \underset{\sim}{A}\, \underline{X} = \underline{F}$$

regular linear equation.

Lagrange's Equations

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} + \frac{\partial R}{\partial \dot{q}_i} + \frac{\partial V}{\partial q_i} = Q_i$$

where q_i = generalized coordinate
Q_i = generalized force $= \sum F \frac{\partial y}{\partial q_i} + \sum M \frac{\partial \phi}{\partial q_i}$
T = kinetic energy
V = potential energy
R = dissipation function

T, R and V are quadratic functions of q_i and $\dot{q}_i$

$\longrightarrow$ $\underset{\sim}{M}$, $\underset{\sim}{C}$ and $\underset{\sim}{K}$ are <u>symmetric</u>

$$T = \frac{1}{2}\underline{\dot{q}}^T \underset{\sim}{M}\, \underline{\dot{q}} \qquad R = \frac{1}{2}\underline{\dot{q}}^T \underset{\sim}{C}\, \underline{\dot{q}} \qquad V = \frac{1}{2}\underline{q}^T \underset{\sim}{K}\, \underline{q}$$

$\underset{\sim}{M}$ is positive definite always
$\underset{\sim}{C}$ is at least positive semi-definite
$\underset{\sim}{K}$ is positive definite for a <u>stable</u> system.

Orthogonal relations

For mode shapes "r" and "s"

$$\underline{u}_r^T \underset{\sim}{M}\, \underline{u}_s \begin{cases} = 0 & \text{if } r \neq s \\ > 0 & \text{if } r = s \end{cases}$$

$$\underline{u}_r^T \underset{\sim}{K}\, \underline{u}_s \begin{cases} = 0 & \text{if } r \neq s \\ > 0 & \text{if } r = s \end{cases}$$

Principal coordinates (Expansion theorem)

The principal coordinates $\underline{p}$ describe how much of the corresponding mode shapes $\underline{u}$ are contained in some general coordinate $\underline{x}$

$$\underline{x} = p_1 \underline{u}_1 + p_2 \underline{u}_2 + p_3 \underline{u}_3 + \cdots$$

$$\underline{x} = \underset{\sim}{U}\, \underline{p}$$

modal matrix

When the equation of motion for an undamped system $\underset{\sim}{M}\,\underline{\ddot{x}} + \underset{\sim}{K}\,\underline{x} = \underline{f}$ is rewritten in terms of the principal coordinates $\underline{p}$ instead of $\underline{x}$, the resulting mass and stiffness matrices are diagonal. The equations become uncoupled

$$\underset{\sim}{M}^* \underline{\ddot{p}} + \underset{\sim}{K}^* \underline{p} = \underset{\sim}{U}^T \underline{f}$$

where $\underset{\sim}{M}^* = \underset{\sim}{U}^T \underset{\sim}{M}\, \underset{\sim}{U} = \text{diagonal}$

$\underset{\sim}{K}^* = \underset{\sim}{U}^T \underset{\sim}{K}\, \underset{\sim}{U} = \text{diagonal}$

Proportional Damping

In the particular case when $\underset{\sim}{C} = \alpha \underset{\sim}{M} + \beta \underset{\sim}{K}$ where α and β are constants, then the principal coordinates $\underline{p}$ also diagonalize $\underset{\sim}{C}^* = \underset{\sim}{U}^T \underset{\sim}{C}\, \underset{\sim}{U}$. The mode shapes of the damped system are the same as those of the same system without damping.

Rayleigh Quotient

If we know an exact mode shape $\underline{u}$, we can find the corresponding natural frequency from the Rayleigh Quotient

$$\omega^2 = \frac{\underline{u}^T \underset{\sim}{K} \underline{u}}{\underline{u}^T \underset{\sim}{M} \underline{u}} = \frac{V_{max}}{T^*_{max}}$$

If we only know a crude approximation of the mode shape $\underline{v}$, we can still get quite a good approximation for the natural frequency

$$\omega_R^2 = \frac{\underline{v}^T \underset{\sim}{K} \underline{v}}{\underline{v}^T \underset{\sim}{M} \underline{v}} = \frac{V_{max}}{T^*_{max}}$$

We can use the same procedure for continuous systems:

e.g. for strings

$$V_{max} = \frac{1}{2} \int \overset{\leftarrow \text{string tension}}{P} \left(X'(x) \right)^2 dx$$

$$T^*_{max} = \frac{1}{2} \int \rho A \left(X(x) \right)^2 dx$$

for beams

$$V_{max} = \frac{1}{2} \int EI \left(X''(x) \right)^2 dx$$

$$T^*_{max} = \frac{1}{2} \int \rho A \left(X(x) \right)^2 dx$$

Continuous Systems

Stretched string:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

wave equation

where $c = \sqrt{P/\rho A}$ = wave speed

solution is separable $\quad \omega = \beta c$

$$u(x,t) = \left(C\cos\beta x - D\sin\beta x\right)\left(A\cos\omega t - B\sin\omega t\right)$$

mode shape $\qquad$ vibration

Beam:

$$\frac{\partial^2 u}{\partial t^2} + c^2 \frac{\partial^4 u}{\partial x^4} = 0$$

where $c = \sqrt{\dfrac{EI}{\rho A}}$ (not wave speed)

solution is separable $\quad \omega = \beta^2 c$

$$u(x,t) = \left(C\cos\beta x - D\sin\beta x + G\cosh\beta x + H\sinh\beta x\right)\left(A\cos\omega t - B\sin\omega t\right)$$

mode shape $\qquad$ vibration.

The natural frequencies ω and the integration constants C, D, G, H come from the boundary conditions.
The constants A and B come from the initial conditions.