MECH468 : Modern Control Engineering
MECH509 : Controls

L30 : Steady-state Kalman filter
Course summary

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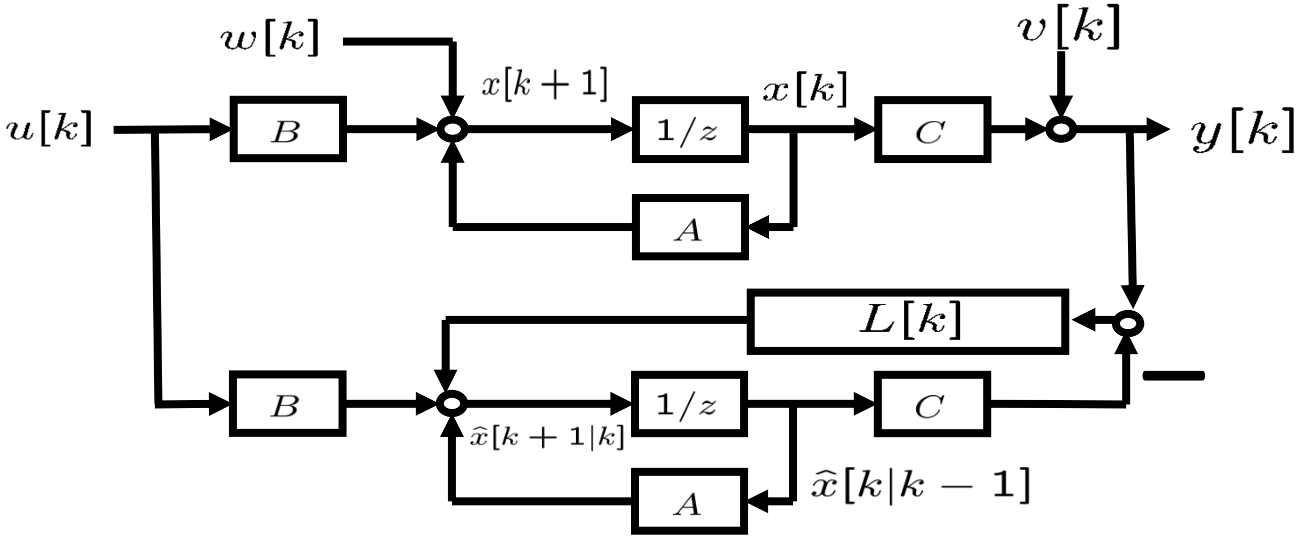
Zoom lecture to be recorded and posted on Canvas

Outline

- Duality between LQR and Kalman filter
- Steady-state Kalman filter
- Linear Quadratic Gaussian (LQG)
- Summary of the course

One-step Kalman filter (review)

$$\hat{x}[k+1|k] = A\hat{x}[k|k-1] + Bu[k] + L[k]\left(y[k] - C\hat{x}[k|k-1]\right)$$



$$L[k] = AP[k|k]C^T R_v^{-1} = AP[k|k-1]C^T\left[R_v + CP[k|k-1]C^T\right]^{-1}$$

$$\begin{cases} P[k+1|k] & = & AP[k|k-1]A^T + R_w \\ & & -AP[k|k-1]C^T\left[R_v + CP[k|k-1]C^T\right]^{-1} CP[k|k-1]A^T \\ P[0|-1] & = & P_0 \end{cases}$$

Duality between LQR and KF

- DT LQR $K[k] = \left[R + B^T P[k+1] B\right]^{-1} B^T P[k+1] A$

$$\begin{cases} P[k] &= A^T P[k+1] A + Q \\ & \quad - A^T P[k+1] B \left[R + B^T P[k+1] B\right]^{-1} B^T P[k+1] A \\ P[k_f] &= S \end{cases}$$

Backward computation

LQR		KF
$K[k]$	$\leftrightarrow$	$L[k]^T$
A	$\leftrightarrow$	A^T
B	$\leftrightarrow$	C^T
R	$\leftrightarrow$	R_v
Q	$\leftrightarrow$	R_w

- Kalman filter $L[k] = AP[k|k-1]C^T \left[R_v + CP[k|k-1]C^T\right]^{-1}$

$$\begin{cases} P[k+1|k] &= AP[k|k-1]A^T + R_w \\ & \quad - AP[k|k-1]C^T \left[R_v + CP[k|k-1]C^T\right]^{-1} CP[k|k-1]A^T \\ P[0|-1] &= P_0 \end{cases}$$

Forward computation

Mathematically dual!

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Remarks on Kalman filter (KF)

- The gain $L[k]$ is time-varying, but it typically reaches the steady state quickly, because $P[k|k-1]$ reaches steady-state quickly.
- To simplify implementation of KF, it is often preferable to use a constant-gain KF.
- In many cases, this does not degrade the filter performance so much.
- How to obtain such steady-state Kalman filter?
  - We need an assumption "(A,C) observable" for DARE in next slide to have a unique positive definite solution.

DARE for steady-state KF

- For time-varying gain *L[k]*, we solve an equation recursively to obtain the error covariance.

$$\begin{cases} P[k+1|k] = AP[k|k-1]A^T + R_w - AP[k|k-1]C^T\left[R_v + CP[k|k-1]C^T\right]^{-1} CP[k|k-1]A^T \\ P[0|-1] = P_0 \end{cases}$$

- To find the equation for steady-state, set

$$M = P[k+1|k] = P[k|k-1] > 0$$

$$\Rightarrow AMA^T - M + R_w - AMC^T\left[R_v + CMC^T\right]^{-1} CMA^T = 0$$

Discrete Algebraic Riccati Equation (DARE)

Gain computation for KF in Matlab

- dare.m
  - Steady-state *a priori* error covariance
    (Steady-state of *P[k+1|k]*) $M = \text{dare}(A^T, C^T, R_w, R_v)$
  - Steady-state *a posteriori* error covariance
    (Steady-state of *P[k|k]*) $P = M - MC^T\left(CMC^T + R_v\right)^{-1} CM$
  - Observer gain $L = A \underbrace{PC^T R_v^{-1}}_{=:K} = AMC^T(CMC^T + R_v)^{-1}$ Kalman gain
- dlqe.m $[K, M, P] = \text{dlqe}(A, B_w, C, R_w, R_v)$

(coefficient matrix of *w*)

Steady-state Kalman filter

- Initial conditions $\hat{x}[0|-1]$

① Measurement update

$$\hat{x}[k|k] = \hat{x}[k|k-1] + \underline{P[k|k]} C^T R_v^{-1} (y[k] - C\hat{x}[k|k-1])$$

$$\boxed{P := P[k|k]}$$

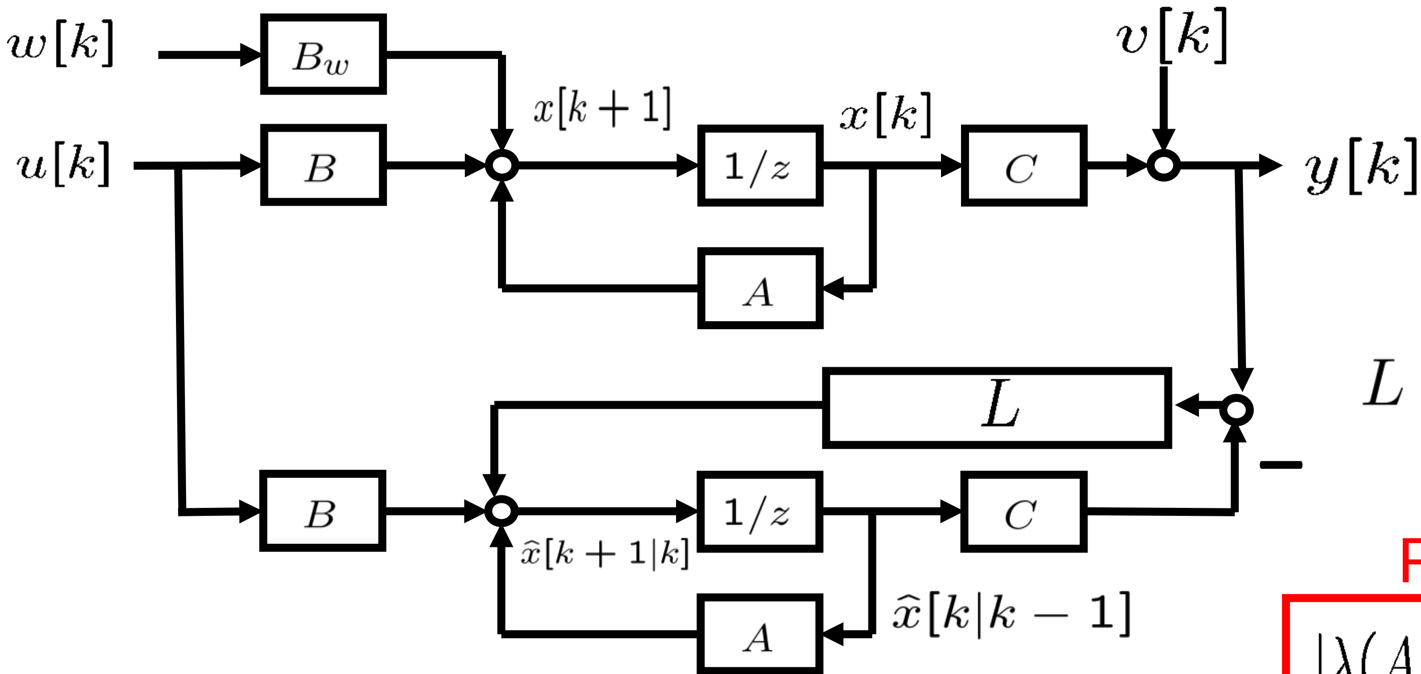
Kalman gain for steady-state Kalman filter

$$K_{ss} := PC^T R_v^{-1}$$

② Time update $\hat{x}[k+1|k] = A\hat{x}[k|k] + Bu[k]$

One-step steady-state KF

- One-step steady-state Kalman filter design can be seen as a special way of designing observer gain *L*.

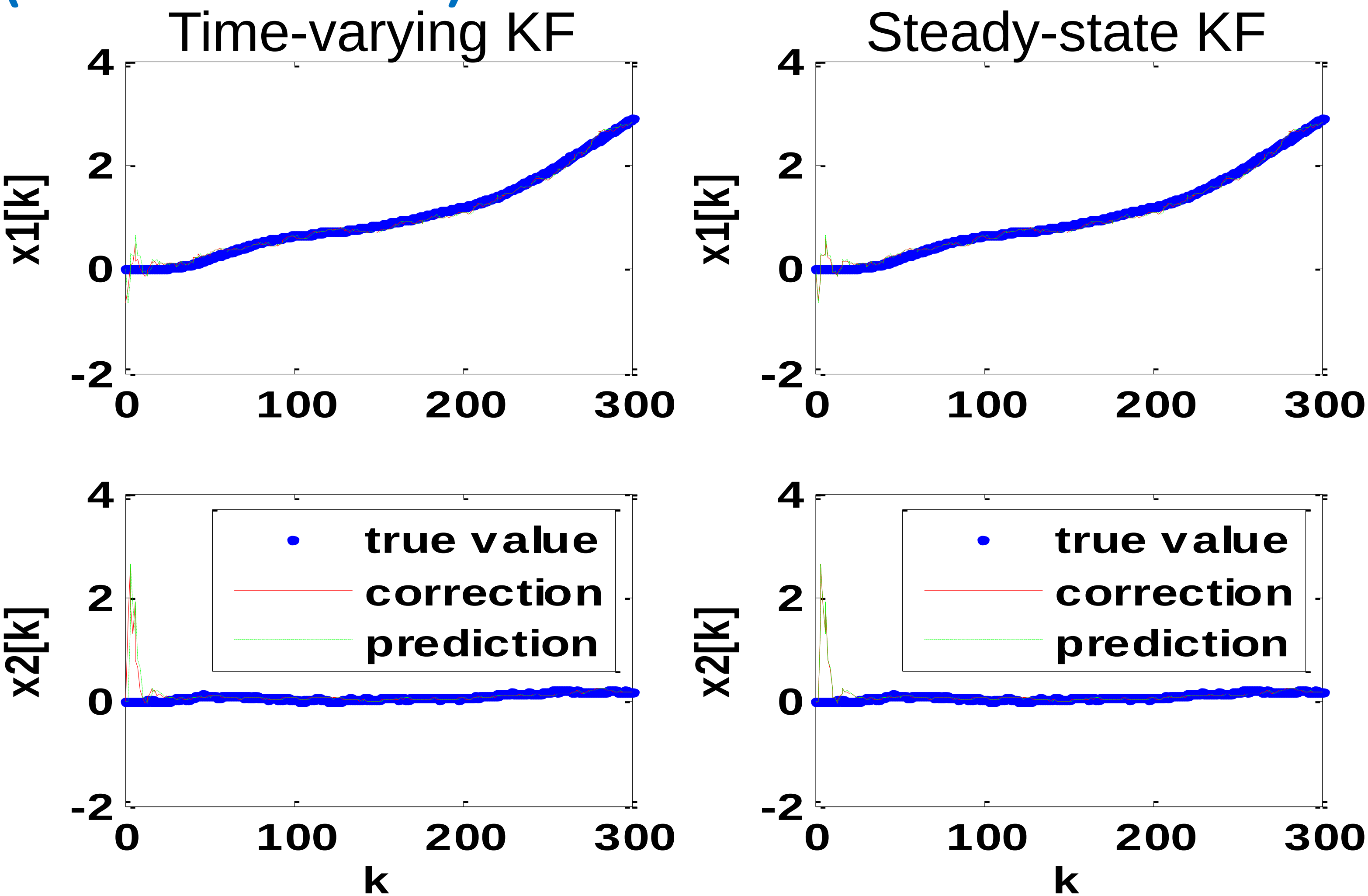


$$L = AK_{ss}$$

FACT

$$\boxed{|\lambda(A - LC)| < 1}$$

Satellite attitude estimation (KFsatellite.m)



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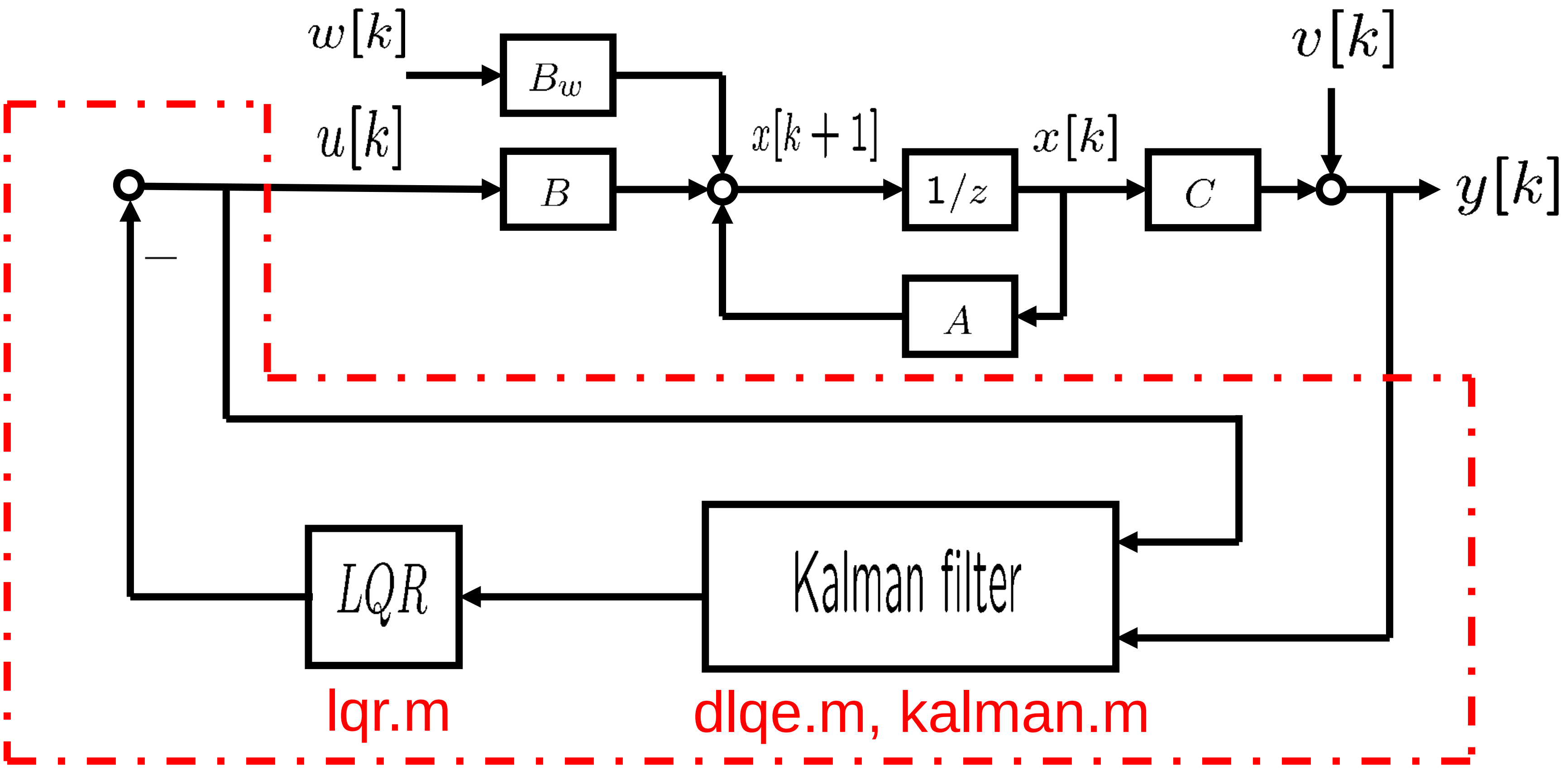
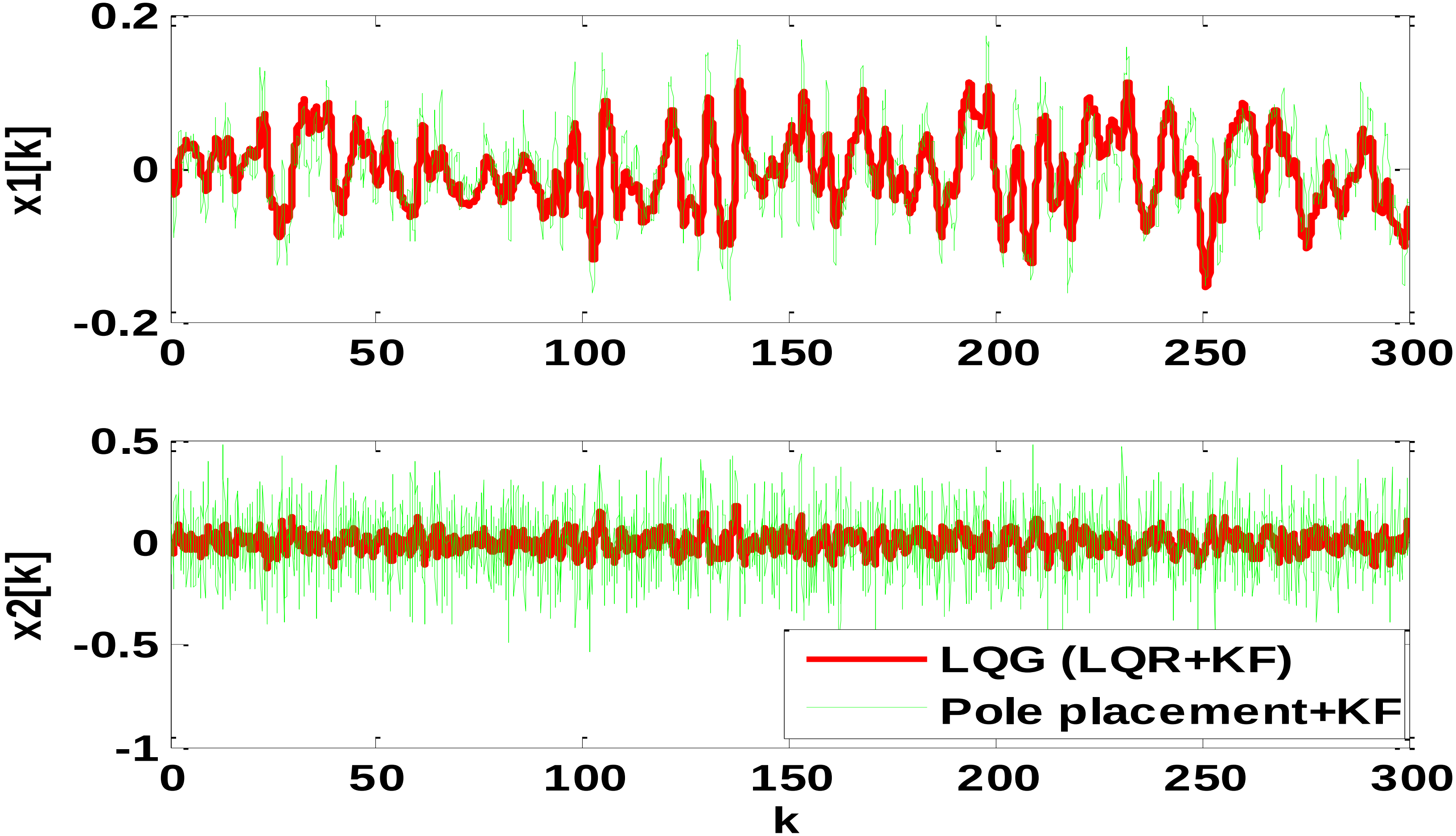
LQG control (LQR + Kalman filter)



LQG satellite attitude control (KFsatellite.m & LQGsatellite.slx)



More on Kalman filter

- Books
  - D. Simon, *"Optimal State Estimation"*, John Wiley & Sons, 2006
  - R. F. Stengel, "*Optimal Control and Estimation*", Dover Publications, 1994
  - B. D. O. Anderson and J. Moore, *"Optimal Filtering"*, Dover Publications, 2005
  - F. Lewis, L. Xie and D. Popa, "*Optimal and Robust Estimation*", 2nd ed., CRC Press, 2007
- Websites
  - Greg Welch and Gary Bishop http://www.cs.unc.edu/~welch/kalman/index.html

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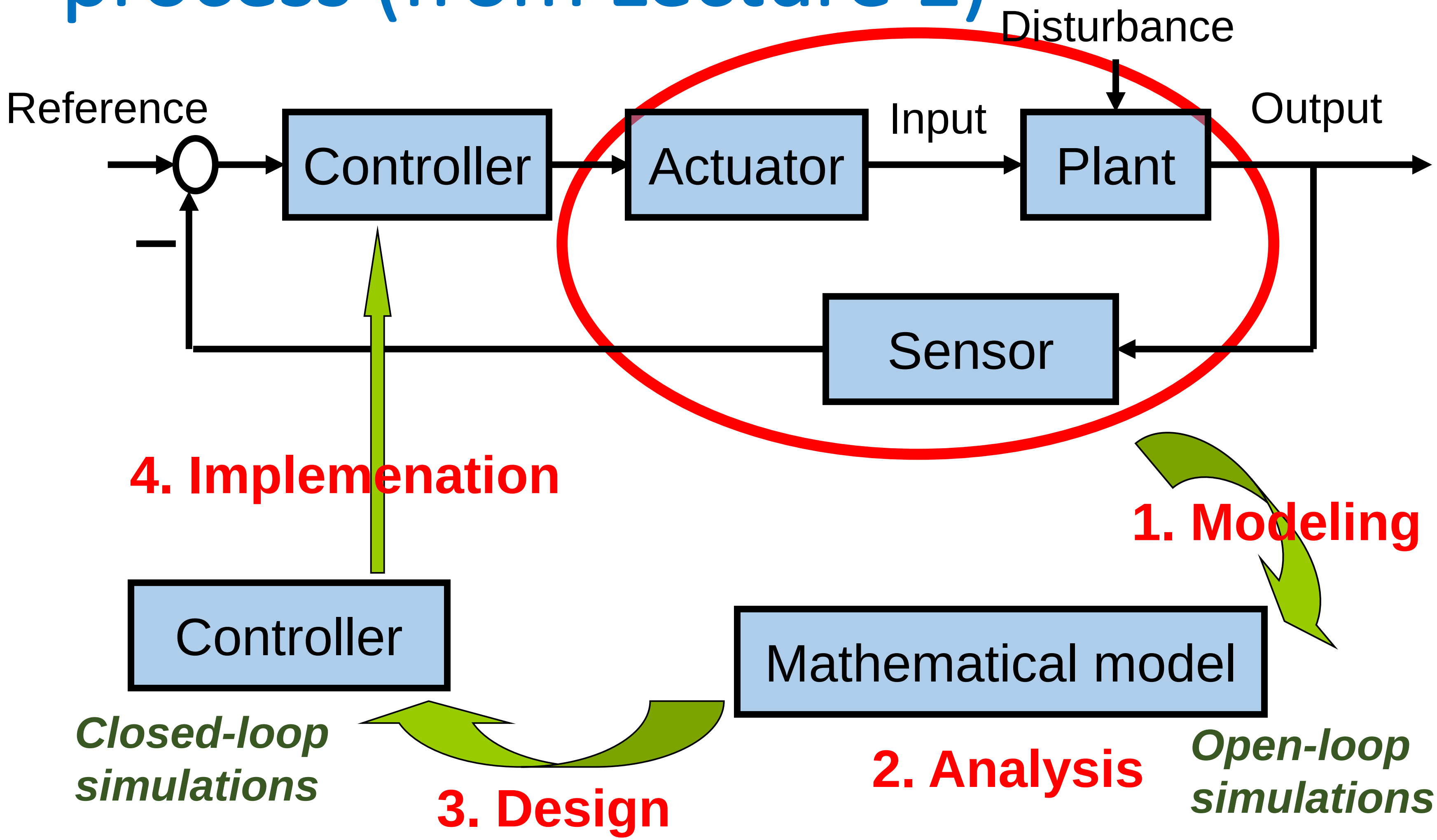
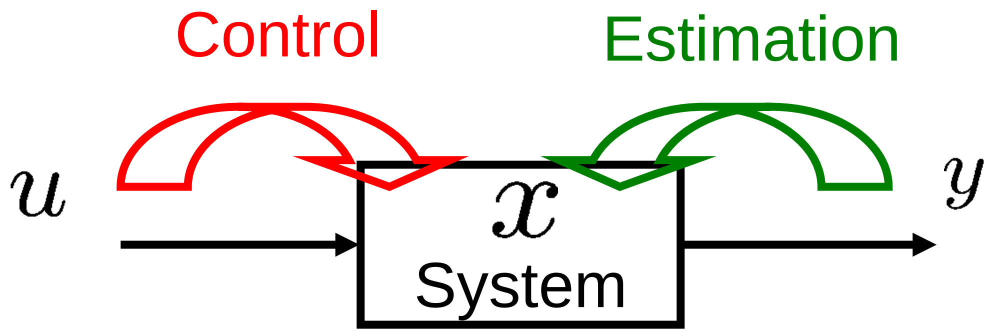
Model-based controller design process (from Lecture 1)



Control and estimation of states

- State has been the key concept in this course!



- Controllability & Observability
- State feedback & Observer
- Linear quadratic regulator & Kalman filter
- Mathematical duality between control and estimation

Goal of this course (from Lec. 1)

To learn control theory with linear state-space models

- Modeling as a state-space model
    - Differential or difference equation (instead of TF)
    - Linear algebra (instead of Laplace transform)
- Analysis
    - Stability, controllability, observability
    - Realization, minimality
- Design
    - State feedback, observer
    - Linear Quadratic Regulator (LQR), Kalman Filter
- Matlab simulation

Course plan

Topics	CT	DT
Modeling	✓	✓
Stability	✓	✓
Controllability/observability	✓	✓
Realization	✓	✓
State feedback/observer	✓	✓
LQR/Kalman filter	✓	✓

Brief history of control engineering (from Lecture 1)

- Classical control (-1950)
  - Transfer function
  - Frequency domain
- Modern control (1960-) (contents in this course)
  - State-space model
  - Time domain
  - Optimality
- Post-modern control (1980-)
  - Robust control
  - Hybrid control, etc.

What is next?

- Advanced control theory
  - Nonlinear control
  - Robust and optimal control
  - Adaptive control
  - Digital control, sampled-data control
  - Hybrid control
  - System identification
- In this course, you learned basic control theory:
  - not only for its immediate engineering applications,
  - but also for further study on control engineering.

Summary

Control engineering supports various disciplines!

Mechanical engineering
Electrical engineering
Chemical engineering
Civil engineering
Aerospace engineering

Environmental engineering
Computer engineering
Mechatronics
Nanotechnology
Medicine, Economics, Biology

Model

Control Engineering

Computer
Mathematics
Hardware
Physics
Chemistry