University of British Columbia
Department of Mechanical Engineering

MECH468 Modern Control Engineering
MECH522 Foundations in Control Engineering
Final exam

Examiner: Dr. Ryozo Nagamune
December 9 (Friday), 2016, noon-2:30pm

Last name, First name

Name: Student #:

Signature:

Exam policies

- Allowed: Lecture note print-outs, hand-written notes, homework, books.
- Not-allowed: PC, calculators, mobile phones.
- Write all your answers on this booklet. No extra sheet will be provided.
- Motivate your answers properly. (No chance to defend your answers orally.)
- 100 points in total. Mark will be scaled later.

Before you start ...

- Use washroom before the exam.
- Turn off your mobile phone.
- No eating.
- Questions are NOT allowed.

If you finish early ...

- If you would like to leave the room **before 2:20pm**, **raise your hand with this booklet**, and **wait at your seat** until the invigilator comes to you and collects your exam booklet.

To be filled in by the instructor/marker

Problem #	Mark	Full mark
1		40
2		20
3		20
4		20
Total		100

1. Consider the following continuous-time state-space model:

$$\begin{cases} \dot{x}(t) &= \underbrace{\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}}_{A} x(t) + \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{B} u(t), \\ y(t) &= \underbrace{\begin{bmatrix} 0 & 1 \end{bmatrix}}_{C} x(t). \end{cases} \tag{1}$$

(a) Is this system asymptotically stable, marginally stable, or unstable? You do **not** need to motivate your answer for this question. (5pt)

(b) Linearize the system (1) at equilibrium point $x = [0, 1]^T$ and $u = 0$. (5pt)

(c) From the state-space model above, compute the transfer function $G(s)$ from the input u to the output y. (5pt)

(d) Compute the matrix exponential e^{At}. (5pt)

(You will find Questions 1-(e) and 1-(f) in the next pages.)

Write your answer here for Question 1.

Write your answer here for Question 1.

(e) For the state equation in (1), compute the minimum energy control $u(t)$ which transfers the state from $x(0)$ to $x(1)$, where $x(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and $x(1) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. (10pt)

Write your answer here for Question 1.

(f) Obtain the continuous-time infinite-horizon LQR optimal control law $u(t)$ which solves the following optimization problem: (10pt)

$$\min_{u(\cdot)} \int_0^\infty \left\{y^2(t) + u^2(t)\right\} dt, \text{ subject to the state-space model (1).}$$

Write your answer here for Question 1.

2. Consider the transfer matrix

$$G(s) = \begin{bmatrix} \dfrac{1}{s+1} & \dfrac{s+\alpha}{s+1} \\ \dfrac{1}{s+1} & \dfrac{1}{s+1} \end{bmatrix},$$

where α is a positive constant.

(a) Obtain the realization of $G(s)$ in the controllable canonical form. (5pt)

(b) Obtain the realization of $G(s)$ in the observable canonical form. (5pt)

(c) Find α such that the minimal realization of $G(s)$ has only one state (i.e., the size of A-matrix becomes 1-by-1). For that α, obtain the minimal realization of $G(s)$. (10pt)

Write your answer here for Question 2.

Write your answer here for Question 2.

3. Consider the following continuous-time state-space model:

$$\begin{cases} \dot{x}(t) &= \underbrace{\begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix}}_{A} x(t) + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{B} u(t), \\ y(t) &= \underbrace{\begin{bmatrix} 1 & 1 \end{bmatrix}}_{C} x(t) \end{cases}$$

Answer the following questions with proper explanations.

(a) Is this system stabilizable? (5pt)

(b) Is this system detectable? (5pt)

(c) If possible, design a state feedback controller $u(t) = -Kx(t)$ (i.e., obtain a matrix K) so that the closed-loop system has an A-matrix (i.e., $A - BK$) with eigenvalues at -1 and -2. If that is not possible, explain the reason why. (5pt)

(d) If possible, design an observer gain L so that the eigenvalues of $A - LC$ are -1 and -2. If that is not possible, explain the reason why. (5pt)

Write your answer here for Question 3.

Write your answer here for Question 3.

4. Consider the following discrete-time system:

$$\begin{cases} x[k+1] &= 2x[k] + 2w[k], \\ y[k] &= x[k] + v[k], \end{cases}$$

where w and v are noise terms with:

- expected values $E\{w[k]\} = 0$ and $E\{v[k]\} = 0$ for any k, and
- variances $R_w := E\{w^2[k]\} = 1/2$ and $R_v := E\{v^2[k]\} = 1/2$ for any k.

(a) Design the (two-step) time-varying Kalman filter. (10pt)

(b) Using the designed time-varying Kalman filter, for initial *a priori* estimate $\hat{x}[0|-1] = 0$ and its error variance $P[0|-1] = 1$, as well as for measurements

$$y[0] = 1/2, \; y[1] = 2/3,$$

compute the state estimates and their error variances, and complete the table below. (10pt)
(**Hint**: Compute all the variances <u>before</u> computing state estimates.)

k	*a priori* estimate $\hat{x}[k\|k-1]$	variance $P[k\|k-1]$	*a posteriori* estimate $\hat{x}[k\|k]$	variance $P[k\|k]$
0	0	1		
1				

<u>Write your answer here for Question 4.</u>

Write your answer here for Question 4.

Extra page. Write the question number before writing your answer.

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