MECH 463
Mechanical Vibrations

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Contents

To The Student

The topic of vibrations is very broad: both in its theoretical content and practical *mechanical* engineering applications. Almost all major engineering achievements have to contend with vibration problems —from everyday washing machines to the Man's first mission to the Moon. From the Tacoma Narrows bridge to the Millennium Bridge, vibration problems abound.

MECH 463 is an *introductory* course. This course is not about covering every nook and corner of the subject, but to give a fundamental understanding of the basic concepts. The purpose of this course is to guide the students to a point where they can explore this rich subject on their own and at their own pace.

The spirit of this course is *synthesis* of the concepts from Mathematics: (Linear Algebra; Fourier Series and Transforms) and Mechanics (Kinematic and Dynamics; Solid Mechanics) and *apply* them in a Mechanical Engineering context. A thorough understanding of SDOF systems is essential, which is provided here. Later this knowledge will be used in understanding MDOF systems: see Appendix B. The mathematical concepts are introduced *as needed*: the tools are presented as the job demands.

The five overarching course objectives and how they are fulfilled by different topics is given under the course outline, on page iv. These notes combined with the laboratory exercise shall equip you with necessary fundamental knowledge of vibration modelling, analysis, and design. You are expected to achieve these objectives after completing this course.

The extent to which we cover these notes depends on the pace of the course.
In order to derive maximum benefit you are encouraged to:

1. Read the notes and the associated textbook sections marked in the notes as (T X.X) before each lecture. For example, T 1.3 refers to Section 1.3 in Chapter 1 of the Textbook.
2. Do the weekly homework on your own to identify any gaps in knowledge sooner than later.
3. Do Assignments on your own.
4. Think critically about every idea introduced. Unless *you* understand and apply an idea it does not really matter if it exists in the course notes, formula sheet, or a textbook.
5. Enjoy the course. I will be glad to assist in your learning.

Comments, typos, mistakes and errors? e-mail srikanth@mech.ubc.ca.

MECH 463: Mechanical Vibrations

(September, 2013)

Prerequisites: MECH 221, MECH 260; a sound working knowledge in kinematics, solid mechanics, differential equations, complex numbers, and matrix algebra is essential for this course.

Course Objectives: i) *Develop* lumped parameter models of mechanical systems; ii) *Formulate* equations of motion using free-body-diagrams and energy methods; iii) *Solve* for vibration response; iv) *Design* counter-vibration measures: absorbers, isolators, and system modification; v) *Understand* working principles of vibration measurement devices; vi) *Apply* computational tools (MATLAB and MSC-ADAMS) in vibration analysis and design.

Course Outline: Read the sections indicated below from the course text book− Mechanical Vibrations by S.S. Rao (5th or 4th Edition)− before each lecture. Lectures: Tu & Th 8-9.30am in DMP 310; Tutorial (3 members per group): W 12-1pm in CEME 1202 & Fr 2-3 pm in ESB 2012; Lab (2 members per group): ICICS X039, times–TBA.

No	Topic	Read	Do	Objectives
1	**Introduction to Vibrations**	1.1−1.6	A1	i
2	**Single-Degree-of-Freedom Systems**			
2.1	Formulation of Equations of Motion	2.1; 2.2.1; 2.2.3;	A2	ii
2.2	Equivalent Systems	1.7; 1.8	A2	ii
2.3	Undamped SDOF response	1.10; 2.2; 3.3	A3	iii
2.4	Viscously Damped SDOF response	2.6;3.4	A3	iii
2.5	Vibration Isolation	9.10	A4	iv
2.6	Forced Vibration: General Excitation	4.2−4.5	A5	iii
3	**Vibration Measuring Instruments**	10.4; 10.5		v
4	**Spectral Analysis**	Class Notes	A6	v
4.1	Introduction to Fourier Series & Fourier Transform			
4.2	MATLAB Implementation			
4.3	Frequency Response Functions and Coupled Systems			ii
5	**Multi-Degree-of-Freedom Systems**			
5.1	Formulation of Matrix Equations of Motion; Coupling and Principal Coordinates	5.1−5.6 6.1−6.8		i–ii
5.2	Eigenvalue Problems & Orthogonality Conditions	6.9−6.10		iii
5.3	Free and Forced Vibration Response	6.13−6.15		iii
5.4	Vibration Absorbers	9.11	A7	iv
6*	**Continuous Systems**			ii–iii
6.1	String Vibrations; Normal Modes and Orthogonality	8.1−8.2	A8	

Assignments (A1−A8) will be issued periodically on CONNECT; solutions will appear a week after. Do the assignments on your own to test your course knowledge. Ignore this advice at your own risk!

Grading: Homework (5%); Tutorials (5%); ADAMS (5%); 3 'Midterm' exams (15%); Laboratory Report (10%); Final (60%). **You must secure at least 50% overall and at least 50% in the final to pass. If you score below 50% in the final exam, your final exam mark will be your course mark.**

Exam Dates: **26th Sep. (MT1); 31st Oct. (MT2); 19th Nov. (MT3); TBA (Final)**

Contacts: Dr.Srikanth (Instructor) srikanth@mech.ubc.ca; CEME 2061. TA information on the reverse.

All course related material (notes, announcements, e-mails, e-signup sheets) will be on CONNECT.

Course Information and Policies

TA Information

Name	Office hours	Contact
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1. **Course Load**: This is a 4 credit course and the course load is heavier compared to a normal 3 credit course. It integrates concepts from Dynamics & Solid Mechanics. Students find this material challenging and rewarding. **A 3-5 hours of study per week spent on this course outside the lectures, labs, and tutorials will keep you up to speed.** Please do not leave things to the last moment.
2. **Tutorial attendance is compulsory.** Tutorial problems are posted every Friday on CONNECT. We will solve these problems together, **in groups**, on the following Wednesday and Friday in the tutorials. **Please sign up via CONNECT in groups of 3.** Your group will solve the tutorial problems and submit the solution to the TA allocated for marking and feedback.
3. **Assignments**: Regular assignments will be posted on CONNECT. Solve assignment problems on your own to determine your understanding of the lecture material. Solutions will be posted on CONNECT, approximately a week after the assignments are issued.
4. **Homework**: One home-work problem per week will be posted on every Wednesday; your hand-written answers should be submitted before 3pm on the following Wednesday in CEME 1054. **Late submissions are not allowed under any circumstances**. Solving homework and assignment problems will help you prepare for the midterm and final examinations. **Marked homework can be picked up from CEME 1054, approximately a week after you submit them.**
5. **Labs**: Please come prepared: read the handout carefully and complete the pre-lab exercise. *You will be given a short quiz in the lab, the mark of which will count toward the pre-lab.* The report should be submitted within two weeks (including holidays, weekends) in the tutorial. **Late submissions will incur mark deduction ($\sim$ 2%).** Grading scheme for the lab reports is posted on CONNECT along with the lab handout. *Follow the report guidelines in the handout and ensure that your report addresses all points in the grading scheme.*
6. **ADAMS**: Tutorials on using ADAMS will be conducted in the PACE LAB in ICICS Building. Please sign up for these sessions on CONNECT.
7. **'Midterm' exams**: Three exams will be held in the lecture times on the dates announced (see page 1). **These dates are final and non-negotiable.** The exams are closed-book ; **your hand-written formula sheet (letter paper, both sides)** is allowed. The exam is of 45 minutes duration and will comprise one question (20 marks) with parts.
8. **Office hours**: **Open doors policy**. Please email to arrange individual appointments. I am here to help you learn. Let me know if you face any difficulties. Make most of the learning opportunities given to you and enjoy what you study. All the very best to you!

Topic 1: Introduction to Vibrations

This topic introduces you to mechanical vibrations, degrees of freedom, and vibration modelling. You are expected to be able to identify degrees of freedom of simple mechanical systems and develop simple spring-mass-damper models after completing this.

1.1 Learning Objectives

1. Understand the importance of vibrations and force and energy perspectives.
2. Identify degrees of freedom of a mechanical system.
3. Identify different types of vibrations and vibration analysis procedures.
4. Apply principle of superposition (in this and later topics).
5. Develop lumped parameter models (in this and later topics).

1.2 Why study vibrations? (T 1.3)

Knowledge of vibrations is essential for safe and efficient design and operation of many mechanical systems. Vibrations can be desirable or not depending on the application.

1. Vibrations can be pleasant and useful in the following circumstances: in the form of music; reduce pain

through ultrasound massaging; separate granular matter (sieves); listening (cochlear); speech (vocal chords), and so on.

2. Vibration is often associated with sound. Vibrating bodies act as sound sources which can cause noise.

3. Vibrations cause cyclic stresses: the stress changes its sign rapidly, resulting in sudden and unexpected *fatigue* failures. Vibrations cause unacceptable and large amplitude motions especially at critical frequencies called resonances, as we shall find in this course.

4. Unwanted Vibration is energy wasted. Reducing unwanted vibrations is good for the environment!

1.3 What is vibration? (T 1.4.1)

Vibration is any fluctuating motion of a mechanical system about its *equilibrium* position. The essential feature of a *periodic* vibration is that the system returns to its *stable* equilibrium position repeatedly in a well defined interval of time, called time period T. Some simple examples of periodic vibration are: pendulum clocks; transformer hum; roll oscillations of a BC ferry.

1.4 Why should a mechanical system vibrate? (T 1.4.2)

Consider the example of a simple pendulum, comprising a rigid mass suspended by a string or a metal wire, sketched below. **Fill in the class**

Given an initial displacement, a pendulum executes oscillations about its static equilibrium position. It is understandable that the initial disturbance causes the pendulum to depart from its equilibrium position.

Question: *What* brings the pendulum back to equilibrium?

Fill in the class

Question: Once it is brought to equilibrium why does the pendulum swing through to the other side? Why can't it just creep back to equilibrium and stay there?

Fill in the class

One can also consider the energies involved in this oscillatory motion. Let us write-down the potential and kinetic energy expressions.

Fill in the class

From this simple example we learn the following:

1. Vibration can be viewed as an energy exchange mechanism in which potential energy is continuously transformed into kinetic energy, and vice versa, in a perioidic manner.
2. Vibration is an interplay between inertial and restoring forces.
3. All *realistic* vibrations involve dissipative forces as well.
4. Free body diagrams are useful to understand vibration problems.

Returning to the question we started with, mechanical systems are made of solids, and solids are deformable under the application of applied forces as we know. They possess the property of *elasticity*, or the ability to regain their original configuration when the loads are removed. The elastic forces serve the role of a restoring force. Forces generated during their operation serve as disturbances. Friction in components such as bearings, and joints gives rise to dissipative forces. Because mechanical systems are subjected to all these forces, they vibrate!

1.5 Degrees of Freedom (T 1.4.3+Notes)

> The number of degrees of freedom (dof) of any mechanical system in motion is defined as the **minimum** number of **independent** co-ordinates required to determine completely the position of all its parts during motion.

You might have seen in earlier studies that a particle in space requires three independent co-ordinates: for example $x - y - z$ co-ordinates in a Cartesian reference. You might also have known that a rigid body requires six independent co-ordinates, since we need to specify not just the location of a typical point such as its centre of mass, but also its *spatial orientation*. Therefore, a particle and a rigid body in space possess 3 dofs and 6 dofs, respectively.

Question: What are the dofs a particle and rigid body, if we constrain their motion to a plane?

We see that **constraints on motion** play an important role in deciding the number of dofs. Strangely enough, the very word **rigid** body implies a constraint. Can you see what it is?

Let us explore the role of constraints through some more simple examples, noting constraints in each case explicitly. Consult your text book for additional examples.

Compound pendulum: **Fill in the class**

Double pendulum:

Four-bar mechanism:

Rolling of a rigid circular disc on a flat surface:

Rigid cantilever beam:

Flexible cantilever beam:

We learn the following from the above examples:

1. Kinematic constraints on motion—imposed by supports and other objects in contact—influence the number of degrees of freedom.
2. The choice of co-ordinates is not *unique* and it is *subjective*.
3. Rigid body assumption is essentially a constraint: the distance between two points on the body is always fixed during its motion.
4. Distributed parameter (continuous) systems such as beams and plates require infinite number of dofs to completely specify their motion.
5. Discrete systems require finite number of dofs.

Having seen some systems and their degrees of freedom, let us consider different types of vibration.

1.6 Classification of Vibration (T 1.5+Notes)

It is important to know the *character* of a vibration problem in order to resolve it. As we will see later, different types of vibrations respond differently to changes in system parameters.

Some classifications are sketched below: **Fill in the class**

1.6.1 Free and Forced

On the basis of force, vibration can be classified into two types.

A *free* or *natural* vibration takes place when there are no external, time dependent, forces acting on the system. The system is *free* to choose its *natural* time period of oscillations when it is given an initial disturbance– such as initial displacement, velocity, or both. For example, vibrations of a simple pendulum are *free* vibrations.

Vibrations in the presence of an external force (time dependent) are called *forced* vibrations. In this case, the system is not allowed to choose its preferred *natural* time period of oscillations, but is forced to choose some other time period(s) as dictated by the external force(s). An example of a forced vibration is the vibration of a washing machine operating at a certain speed.

1.6.2 Damped and Undamped

Vibrations can be classified into two types based on whether the mechanical system has any inherent damping (dissipative forces) or not. In *undamped* vibration the initial energy supplied will not be lost, but only converted from one form to the other. This means that the free vibrations of this system will last *for ever*! Of course, we know that all mechanical systems have friction and real vibrations are *damped*. In *damped vibration* energy is removed from the system in each cycle. Thus *damped-free* vibrations do not last for ever. However, when an external force supplies sufficient energy to compensate for the energy lost through damping, *damped-forced* vibration can last as long as the force is acting.

1.6.3 Steady and Transient

You will also hear *transient* vibrations and steady-state or *steady* vibrations. Transient vibrations are temporary, while steady vibrations are not. **It is important not to confuse transient vibrations with free vibrations**. Damped-free vibrations are transient as they last only for a small amount of time. Undamped-free vibrations are not transient. Transient vibrations need not necessarily be free vibrations. When a washing machine changes its operating speed, you observe a *transient* vibration from one steady state to the other.

1.6.4 Other classifications

There are other classifications too, such as linear and non-linear, deterministic and random.

1.6.5 Principle of Superposition

In this course we are concerned with linear vibrations. Linear vibrations have the property that the response is linearly proportional to the input/force. Consequently, their governing equations are linear differential equations. There are a range of mathematical tools that are based on principle of superposition—valid only for linear systems.

> Principle of superposition states that the response of a mechanical system to a set of forces is equal to the sum of the responses of the system subjected to *each force acting on its own.*

Let us see what this principle means using an example of a linear system. **Fill in the class**

1.7 Vibration Analysis

Since vibrating objects move, vibration is a part of dynamics. Some knowledge in dynamics is a pre-requisite. You have studied kinematics, rigid body dynamics earlier (MECH 221 for example) that you will find useful here.

Many *real world* vibrating systems are *complex* in detail but *simple* in behaviour. The goal for an engineer is to seek *simple models* first and then gradually add complexities.

It is common to represent the restoring forces by a linear elastic spring; inertia by a mass; dissipation by a viscous dashpot. Developing spring-mass-damper *model* of any mechanical system is a skill and a subjective choice of every engineer! The model is as good as it can describe the underlying phenomenon. A famous quote goes like this: "All models are wrong, some are useful."

The following steps summarise vibration analysis procedure, which we will follow in this course.

1. Develop a simple model of the system and gradually improve it. Simple models that we develop comprise spring-mass-damper elements.

2. Formulate equations of motion. Here we need some knowledge in dynamics: kinematics and kinetics. We will use Free-Body-Diagrams (FBDs) and energy methods to formulate equations of motion.

3. Solve the equations of motion. This requires some knowledge in ordinary differential equations, matrix algebra, complex numbers, Fourier methods *etc.* In an advanced course one will employ computers and numerical methods. But here we use simple analytical or hand calculations.

4. Interpretation of results with a view to fix the vibration problem. We will do this throughout the course, especially when we consider vibration isolators and absorbers.

Let us start with step 1 of developing a mathematical model. Developing models is an art and a science. The purpose of the following examples is to give you an indication of how the spring, mass, damper models arise. In the subsequent topics you will see lots of spring-mass-damper examples.

Example 1: A reciprocating engine is mounted on foundations as shown in Fig.(1.1). The unbalanced forces and moments developed in the engine are transmitted to the frame and the foundation. An elastic pad is placed between the engine and the foundation block to reduce the transmission of vibration. Develop two mathematical models of the system using a gradual refinement of the modelling process.

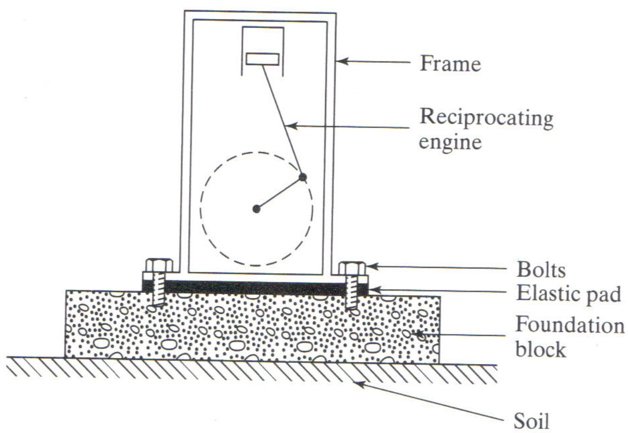


Figure 1.1: Figure for example 1.

Solution:

Fill in the class

Example 2: Consider an automobile on a rough road shown in Fig.(1.2) exhibits vibration caused by road roughness. Develop three models in the increasing order of complexity by considering (a) weight of the car body, passengers, seats, front wheels, and rear wheels; (b) elasticity of tires/suspension, main spring, and seats; and (c) damping of seats, shock absorbers, and tires.

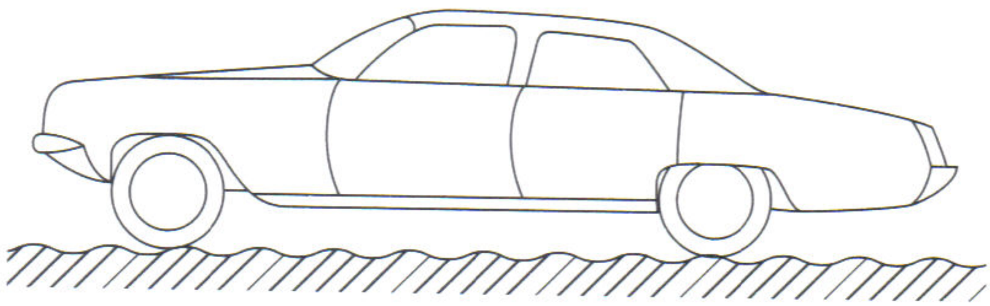


Figure 1.2: Figure for example 2.

Solution:

Fill in the class

From the above examples we learn:

1. Spring-mass-damper models can be used to model vibrations of real world systems.
2. Vibration models are not *unique*.
3. Developing spring-mass-damper models requires skill and judgement that comes with experience and practise.

A typical—but by no means unique—spring-mass-damper model of a human body is shown in Fig.(1.3) below. Vibration can cause human discomfort and injury: motion sickness and white finger are two examples.

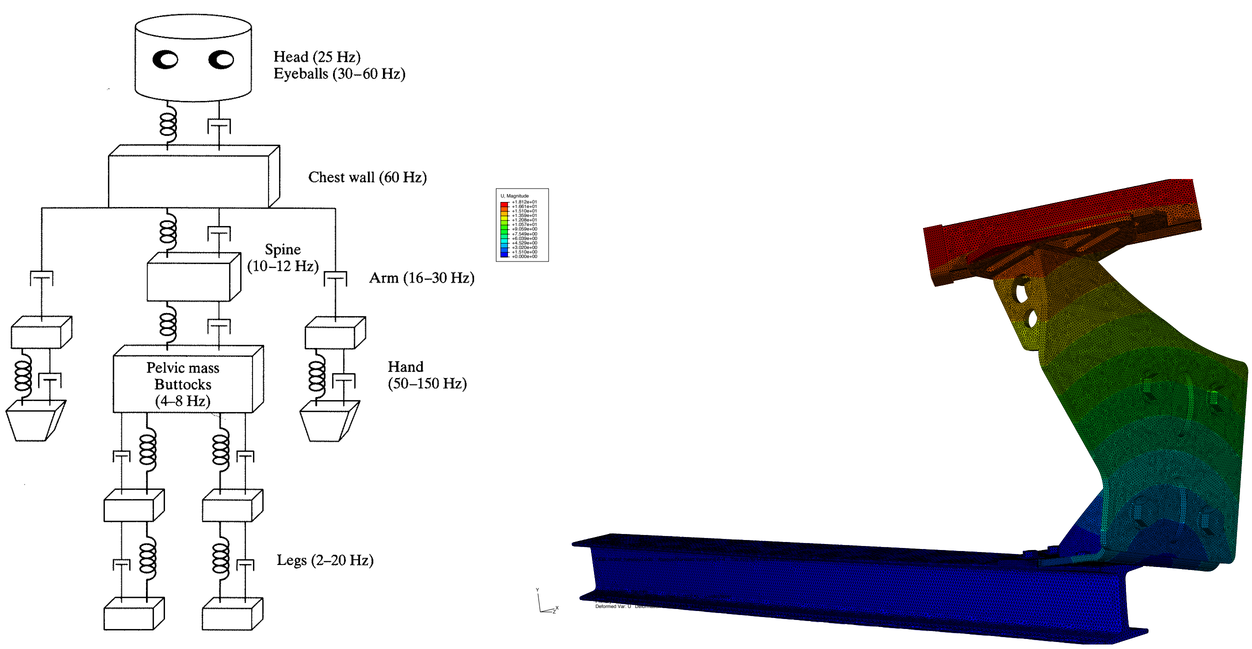


Figure 1.3: A spring-mass-damper model for a human body (left). The frequencies indicate the range in which different organs are sensitive. A Finite Element Model (right) of a typical industrial component with thousands of degrees of freedom. Ack.: Chandra Prakash Sharma.

1.8 Summary

In this topic we learnt the following:

1. A study of Vibrations is important to design safe and efficient products.
2. Vibration phenomenon can be viewed as an energy exchange mechanism in which potential energy is continuously transformed into kinetic energy, and vice versa.
3. Vibration is an interplay between inertial and restoring forces. All *realistic* vibrations involve dissipative forces as well.
4. Kinematic constraints play an important role in reducing the number of co-ordinates required to describe motion of mechanical systems.
5. Vibrations can be free or forced; damped or undamped; transient or steady; linear or nonlinear; deteministic or random; or any combination of these.
6. Spring-mass-damper models can be used to effectively describe vibrations of complex real world systems.

Topic 2.1: Single Degree of Freedom Systems Formulation of Equations of Motion

The principal learning objective for this topic is to formulate equations of motion by constructing free body diagrams.

We have seen some examples of single degree of freedom (sdof) systems in Topic 1. We have also seen how real-world systems like a car can be *modelled* as sdof systems. In this topic, we will develop equations of motion of sdof systems.

2.1 Introduction

There are two main groups of techniques to formulate equations of motion of mechanical systems. They are:

1. Force methods

   Newton's second law

   D'Alembert's principle

2. Work-Energy methods

   Principle of conservation of energy

   Principles of virtual displacements

   Lagrange equations

In this topic we will focus on force methods, which typically require the construction of a Free Body Diagram (FBD). This approach is appealing as we are visualising the system in terms of forces and displacements. On the other hand, work and energy methods are *analytical*.

Consider the simplest of all sdof systems, a spring-mass model sketched below. **Fill in the class**

Here, the spring represents a restoring force, the mass models inertial forces, and $f(t)$ denotes an externally applied force.

Let us consider each of the *elements* of vibrations in the above model.

2.2 Elements of Undamped Vibration

2.2.1 Springs (T 1.7, pp.20–22)

A linear elastic spring is *assumed* to have negligible mass and damping. A spring resists relative displacement x between its two ends. How do we find spring constants? Consider a spring with one end fixed and the other end subjected to an external force f_s as sketched below.

Fill in the class

A typical force-displacement diagram will look like the sketch above. When displacements are *small*, the force-displacement relation is given by

$$f_s = kx \quad \text{(Hooke's law)} \tag{1}$$

The work done by the force f_s as it moves through a displacement x, $W = \int_0^x f_s dx = \frac{1}{2} f_s x$, will be stored as the elastic strain energy or potential energy of the spring. When the force is removed, the strain energy is released and the spring *springs* back! The strain energy U is equal to the area under the load deflection curve (why?), which for a linear elastic spring is given by

$$U = \frac{1}{2}kx^2 \tag{2}$$

Question: *Suppose* you are given an expression for force f_s as a function of displacement x. How will you determine the spring constant, k using Eq.(1)?

Question: *Suppose* an expression for strain energy U as a function of displacement x is given: $U = 10x^2 + 2x^4$. How will you find k ?[1]

[1] You just saw an application of Castigliano's theorem. This is a powerful tool. See Shigley's Mechanical Engineering Design, Ninth edition, section $4-8$ for the theorem and section $10-3$ applied to a spring. In general, the elastic strain energy stored in a structure is due to axial, bending, and shear deformations: $U = U_{\text{axial}} + U_{\text{bending}} + U_{\text{shear}}$.

2.2.2 Mass (T 2.2.1 & T 2.2.2)

Newton's second law relates *net* force or moment acting on a mass to its *absoute acceleration*: that is, acceleration measured by a fixed observer, via

$$\boldsymbol{F} = m\ddot{\boldsymbol{x}}, \quad \ddot{\boldsymbol{x}} \equiv \frac{d^2\boldsymbol{x}}{dt^2} \quad \text{(Newton's second law)} \tag{3a}$$

$$\boldsymbol{M}_o = J_o\ddot{\boldsymbol{\theta}}, \quad \ddot{\boldsymbol{\theta}} \equiv \frac{d^2\boldsymbol{\theta}}{dt^2} \quad \text{(Newton-Euler's law)} \tag{3b}$$

Note that bold letters denote vectors. The symbols in the above equations are defined as follows. **Fill in the class**

Another view of the above equation of motion is given by D'Alembert's principle, which converts the above *equations of motion* into *equations of dynamic <u>equilibrium</u>*

$$\boldsymbol{F} - m\ddot{\boldsymbol{x}} = \boldsymbol{0} \tag{4a}$$

$$\boldsymbol{M}_o - J_o\ddot{\boldsymbol{\theta}} = \boldsymbol{0} \tag{4b}$$

The forces $-m\ddot{\boldsymbol{x}}$ and $-J_o\ddot{\boldsymbol{\theta}}$ are called *inertial* forces. The negative signs indicate that **inertial forces oppose motion; they act in a direction opposite to the absolute accelerations**. They act at the *centre of mass* (CM) of a rigid body so **the point o coincides with CM**. Some books call these forces as *pseudo forces* or *fictitious* forces. This may seem strange as these are the forces you *experience* in high speed racing! These are the forces acting at a cross section of a turbine blade causing failures when it spins at high speeds. But, when there is no acceleration there are no inertial forces! So do not be fooled by the word *pseudo* or *fictitious*. In general, objects in every non-inertial (not fixed but accelerating) frame experience this. Unlike in Physics, we almost always deal with mechanical components in a non-inertial frame, such as stresses in a turbine blade. A more familiar example is when you enter into 99 B line: you are entering into a non-inertial frame (the bus). We know what we experience if the bus starts, or stops, or accelerates in general.

Question: You fall *backwards* when a bus moves *forwards* from rest; you fall *forwards* when the bus halts *suddenly*. Explain this using FBD of a person standing in the bus? **Fill in the class**

Why D'Alembert's principle? Superficially, there seems very little difference between Eq.(3a) and Eq.(4a), for example. In fact there is every reason to question why a simple re-arranagement of an equation deserves a special name at all (it took at least a century in the development of human thought to recognise D'Alembert's principle).
While Eq.(3a) is an equation of *motion*, Eq.(4a) is an equation of dynamic *equilibrium*. We can apply the concepts associated with *equilibrium* if we use D'Alembert's principle. For example, you are used to the idea that for a system of forces to be in equilibrium in STATICS, they need to satisfy the following equations:

$$\sum \boldsymbol{F} = \boldsymbol{0}, \quad \text{or} \quad \sum F_x = 0, \sum F_y = 0, \sum F_z = 0 \tag{5a}$$

$$\sum \boldsymbol{M} = \boldsymbol{0}, \quad \text{or} \quad \sum M_x = 0, \sum M_y = 0, \sum M_z = 0 \tag{5b}$$

You are used to applying the above equilibrium equations in a FBD to enforce *static equilibrium* conditions to calculate forces in a truss, for example.

All you need to do is add extra inertial forces acting at the centre of mass and apply Eq.(5a) and Eq.(5b) to enforce *dynamic equilibrium*. This is powerful: we are using the same ideas from STATICS in DYNAMICS! **Through D'Alembert's principle we converted a dynamics problem into a statics problem**

Note that there are two *vector* inertial forces $-m\ddot{\boldsymbol{x}}$ and $-J_o\ddot{\boldsymbol{\theta}}$ acting at the centre of mass of a rigid body. Further more, for a *particle* all mass is concentrated at one point, so $J_o = 0$. $J_o \neq 0$ for rigid bodies with distributed mass. Be also aware that $\ddot{\boldsymbol{x}}$ and $\ddot{\boldsymbol{\theta}}$ are absolute accelerations. Determining these requires a knowledge in *kinematics* (Reviewed in Tutorial 1).

2.3 Force Methods

The following steps are to be followed when applying the force method:

1. Isolate the system you want to draw the FBD for.
2. Select an appropriate set of displacement co-ordinates.
3. Draw the FBD of the system for which you seek to determine equations of motion.

4. If you wish to apply Newton's second law, do not indicate inertial forces in the FBD. Use Eq.(3a) or Eq.(3b).
5. Determine absolute accelerations using kinematics
6. Do indicate inertial forces and inertial moments (as required) acting at the centre of mass. Remember that mass moment moments of inertia J_o is about centre of mass in the FBD. Use Eq.(4a) or Eq.(4b).

In this course, we prefer D'Alembert's principle as it not only gives the desired equations of motion but also gives *complete* FBDs that can be used for determining reactions and internal stresses. Application of Newton's second law does not give this complete picture of forces, though it does give equations of motion.

Let us formulate the equation of motion of an undamped spring-mass system using both the approaches.

<u>Newton' second law</u>: **Fill in the class**

D'Alembert's principle: **Fill in the class**

Question: Can you draw the FBDs when the displacement co-ordinate is chosen as positive in *opposite direction* to the above?

To fix ideas, let us now consider some further simple examples we have already seen in Topic 1.

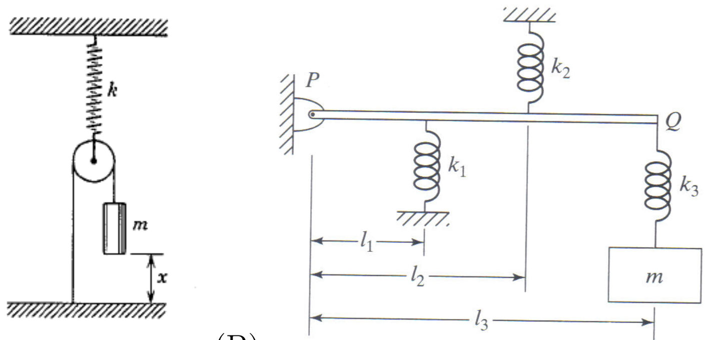
Point mass, or particle: **Fill in the class**

Simple pendulum:

Compound pendulum:

Example 3: Formulate the equations of motion of the following systems using

(i) Newton's second law and (ii) D'Alembert's principle.



(A) (B)

Figure 2.1: Figure for example 3.

Solution:

Fill in the class

Important learning points from this example:

- In problem (A) we used the kinematic condition of no slip. Without knowing this kinematic condition, we cannot solve the problem!
- In problem (B) we have two co-ordinates: x and θ. But, we eliminated θ using equilibrium conditions at the point Q where the rod is connected to the spring k_3. We call θ a *passive* co-ordinate since it can be related to the displacement of mass x via equilibrium condition. We will use this idea again when we derive the equivalent spring constant for two springs connected in series.

Let us summarise what we learnt thus far in this topic.

1. A linear elastic spring is governed by Hooke's law $f_s = kx$. For sdof case, the spring constant k can be determined from the energy expression: $k = \frac{\partial^2 U}{\partial x^2}$.
2. A mass is governed by Newton's second law.
3. Newton's second law and D'Alembert's principle are equivalent; both yield identical equations.
4. Inertial forces and moments act at the centre of mass of a rigid body in a direction opposite to the direction of the absolute acceleration.
5. Kinematics is essential to study vibrations.

2.4 Principle of Conservation of Energy

The principle states that the sum total of potential energy U and kinetic energy T of a *conservative* mechanical system is constant, at every instant in time.

$$T + U = \text{const, or} \quad \frac{d}{dt}[T + U] = 0 \tag{6}$$

It is important to remember that kinetic energy can be due to both translation and rotation. You need to choose a suitable datum to define the potential energy.

Let us apply the principle of conservation of energy to the familiar spring mass system to obtain equations of motion.

Fill in the class

Example 4: Write the expressions for potential and kinetic energies of a simple pendulum and apply the principle of conservation of energy to formulate it's equations of motion. Do the same for the inverted pendulum shown below.

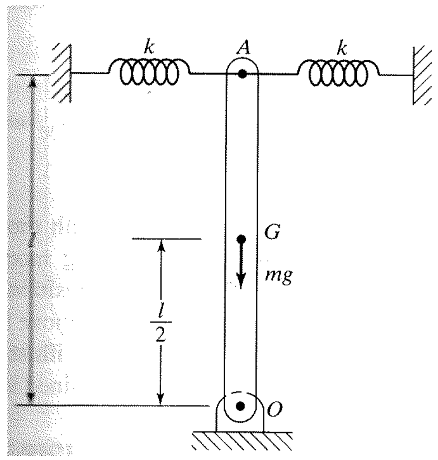


Figure 2.2: Figure for example 4.

Solution:

Fill in the class

Important learning points from this example:

- Gravitational weight tends to destabilise oscillations if the Centre of Mass (C.M.) where mg acts, is above the pivot point. It will stabilise the system if the C.M. is below the pivot point. Gravity cannot *always* be ignored! Beware of this!!

- Inertial forces act at C.M. Since the bar is a rigid body and has distributed mass, both inertial force and inertial moment act at C.M. Notice the difference with a simple pendulum on page 32, where there is no inertial moment acting on the pendulum mass: because, it is a point mass having no mass moment of inertia about it's C.M.

In this topic we learnt the following:

1. We learnt three techniques to arrive at equations of motion: Newton's second law, D'Alembert's principle, and principle of conservation of energy. All give same equations of motion through different routes.

2. A knowledge in Kinematics is essential to determine the absolute velocities and accelerations.

3. D'Alembert's principle gives a complete picture of forces acting in a mechanical system during its motion.

4. Inertial forces and moments act at the centre of mass of a rigid body in a direction opposite to the direction of the absolute acceleration.

5. By adding *inertial forces and moment acting the centre of mass* we can extend the equilibrium concepts of statics to study vibration problems.

Question: Compare the three techniques we studied in this topic based on the examples we covered. **Fill in the class**

Topic 2.2: Single Degree of Freedom Systems Equivalent Systems

The principal learning objective for this topic is to apply force and energy methods (learned in Topic 2.1) together with kinematics to find equivalent spring (translational and rotational) and mass (translational) or mass moment of inertia (rotational) of a multi-component mechanical system at a point of interest.

2.5 Introduction

We often find that mechanical systems comprise many spring-mass elements, but our interest may lie in the response at only one point. *Equivalent* systems capture the response, **at the point of interest**, without loosing accuracy.

The simplest example is the spring-mass system. We *assumed* that the spring is massless. In reality, all springs have some mass. How do we account for the distributed mass of a spring? We will answer this later in example 6.

In machine tools the vibration induced displacements at the tool central point (TCP), where all tools are connected, determine the quality of the machined product. Hence response at TCP is critical.

Another example is vibrations of a water tank induced by wind forces. Here, our interest is to determine the *sway* in the lateral direction of the water tank.

All the above motivate the need for equivalent systems which *simplify* modelling with acceptable *accuracy*.

We can find equivalent springs and masses using two approaches: force methods (FBDs) and energy methods (analytical). Force methods are suitable when we have fewer components (fewer FBDs). Energy methods can deal with multi spring-mass systems with ease. We will now consider both these methods.

2.6 Equivalent Spring (T 1.7.1+Notes)

Let us determine the equivalent spring constant of two springs arranged such that: (i) they have same displacements, but carry different loads (parallel configuration); (ii) they carry same load, but undergo different displacements (series configuration). We will use the force method first.

Parallel Configuration **Fill in the class**

Series Configuration

Fill in the class

In the energy approach we deduce the equivalent spring constant k_{eq} by expressing the TOTAL internal potential energy of the system in the form $U = \frac{1}{2}k_{eq}x^2$. **It is important to ensure that the final energy expression contains only *one* displacement co-ordinate.** This method is more suitable for multi-component systems where drawing FBD for each component is tedious.

Let us apply the energy method to the above two cases.

Parallel Configuration

Fill in the class

Series Configuration[1]

Fill in the class

[1]Notice that for the springs in series case, we used the relation between x_1 and x_2 obtained from the *equilibrium* conditions. This is a straight-forward approach. However, if you insist on using energy expressions only, we need to know that equilibrium is a *minimum* energy configuration and use this to relate x_1 and x_2. If you are keen to know, here is how we proceed.

The following results from elementary solid mechanics are also useful in formulating equivalent springs.

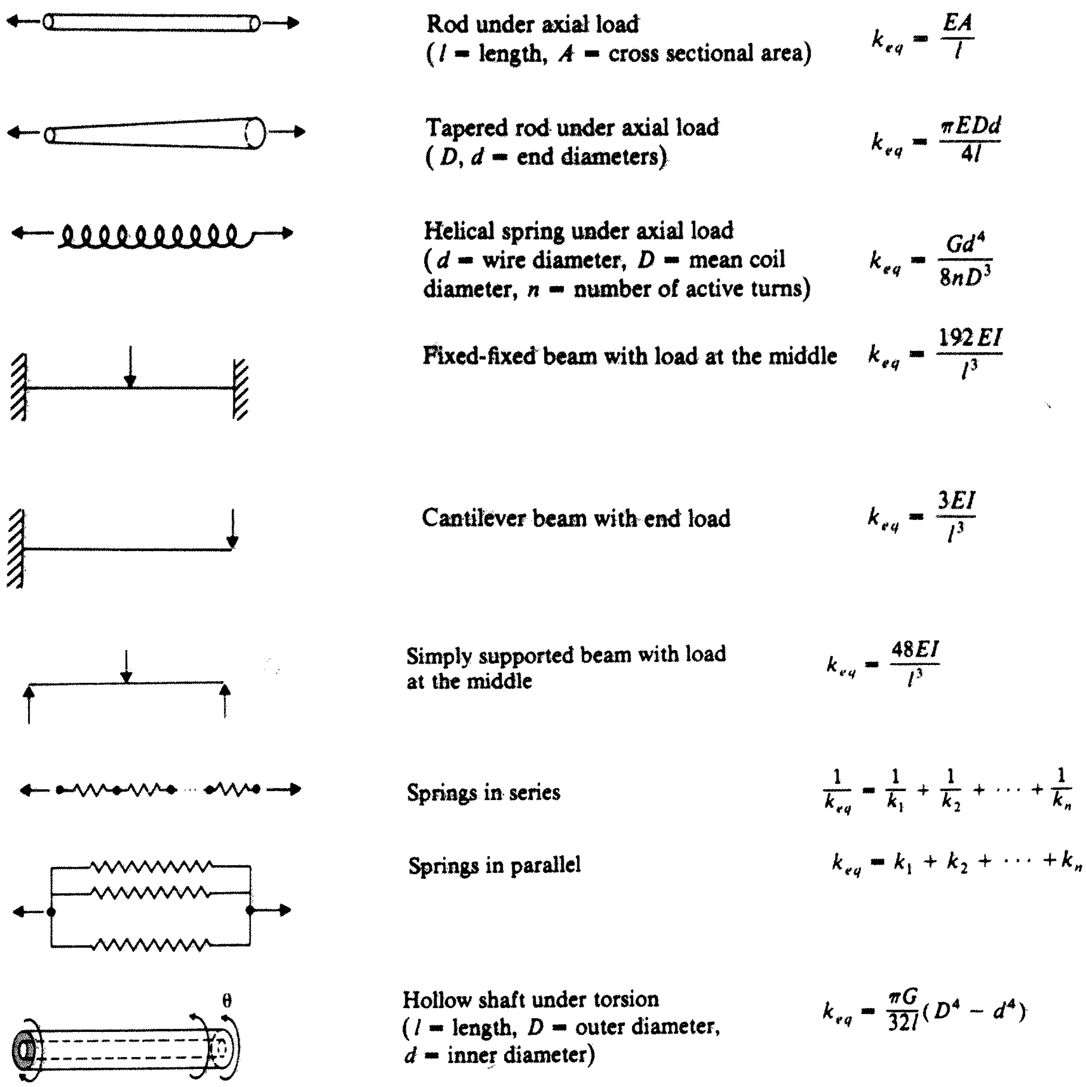


Figure 2.3: Equivalent spring constants.

This is advanced material! We need some mathematics here as we are talking about a *minimum* of a function of two variables. Mathematically, *minimum* means that the change in total energy $E = \frac{1}{2}k_1x_1^2 + \frac{1}{2}k_2\left[x_2 - x_1\right]^2 - fx_2$ for small *independent* changes in the co-ordinates x_1 by dx_1 and x_2 by dx_2 is zero. Recall your first year mathematics, where we learnt that a change in a function (energy here) of two variables (x_1 and x_2) is given by the total differential theorem $dE(x_1, x_2) = \frac{\partial E}{\partial x_1}dx_1 + \frac{\partial E}{\partial x_2}dx_2 = 0$ or $\frac{\partial E}{\partial x_1} = 0$ and $\frac{\partial E}{\partial x_2} = 0$, since dx_1 and dx_2 are *independent.* You can verify that these two equations are two *equilibirum* equations: one at the joint where two springs are connected; one at the point of application of force. This approach—a little involved mathematically —gives the same result!

2.7 Equivalent Mass or Equivalent Mass Moment of Inertia (T 1.7.1+Notes)

We find equivalent mass m_{eq} or equivalent mass moment of inertia J_{eq} by writing the TOTAL kinetic energy (translational and rotational) in the form $T = \frac{1}{2}m_{eq}\dot{x}^2$ or $T = \frac{1}{2}J_{eq}\dot{\theta}^2$. We find the **kinematic constraint of no slip** very useful in relating translations to rotations. Mechanisms, such as rack and pinion, gears and pulleys rely on no slip. We also find kinematics useful in relating velocities of different parts of a rigid body.

Consider the following cases:

Translation masses connected by a rigid bar **Fill in the class**

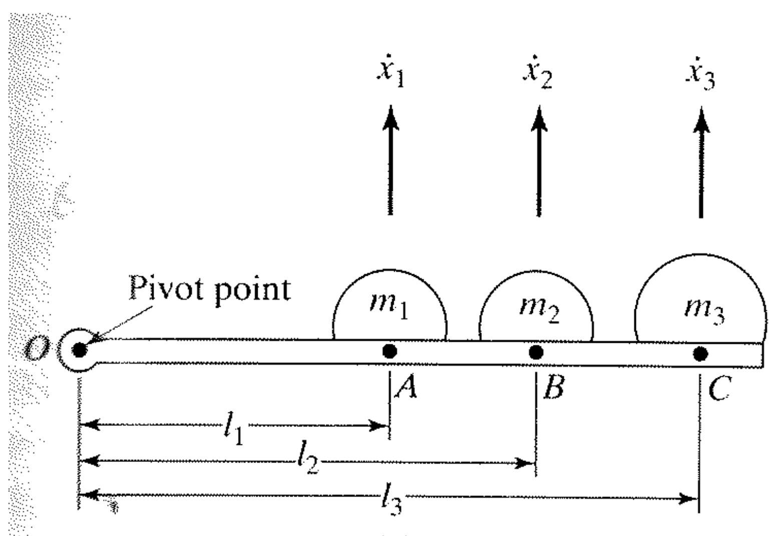


Figure 2.4: Translation masses connected by a rigid bar.

Question: Can you see how you can find equivalent mass moment of inertia of the above system? **Fill in the class**

Translational and rotational masses coupled

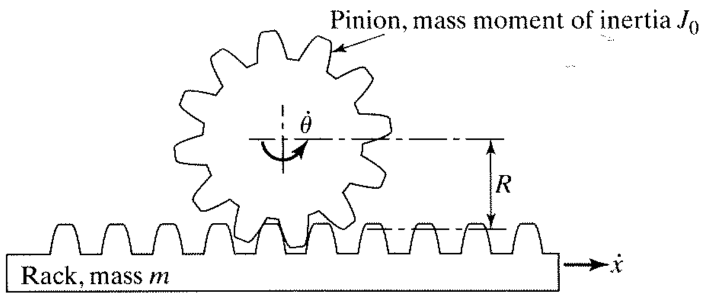


Figure 2.5: Rack and Pinion mechanism.

Example 5: Determine the equivalent mass and spring constant of the following systems.

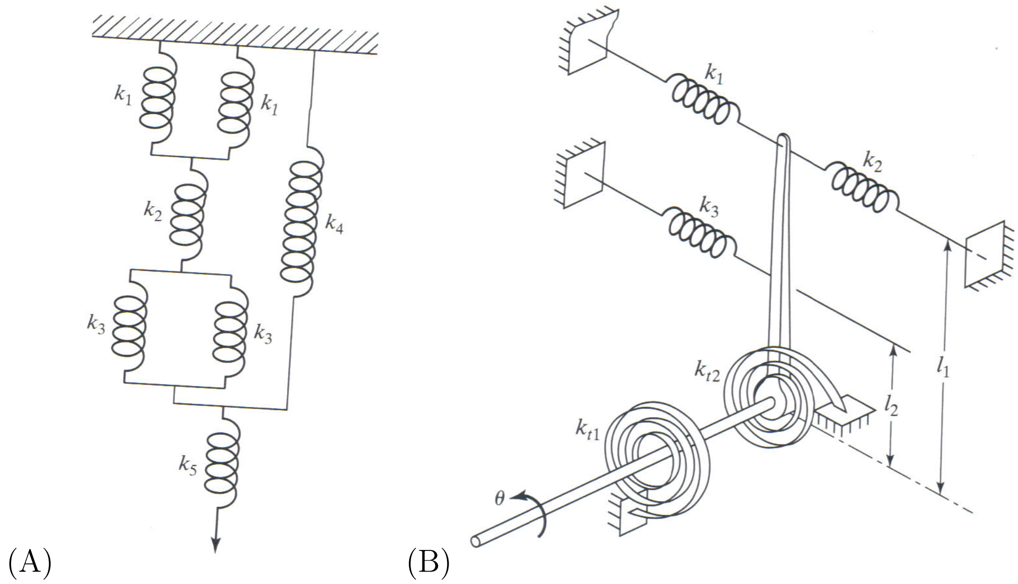
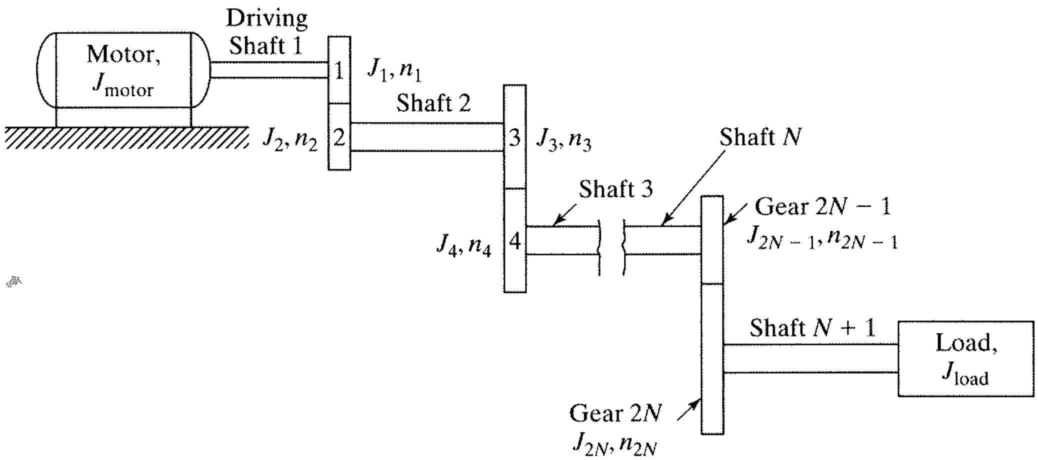


Figure 2.6: Figure for example 5.

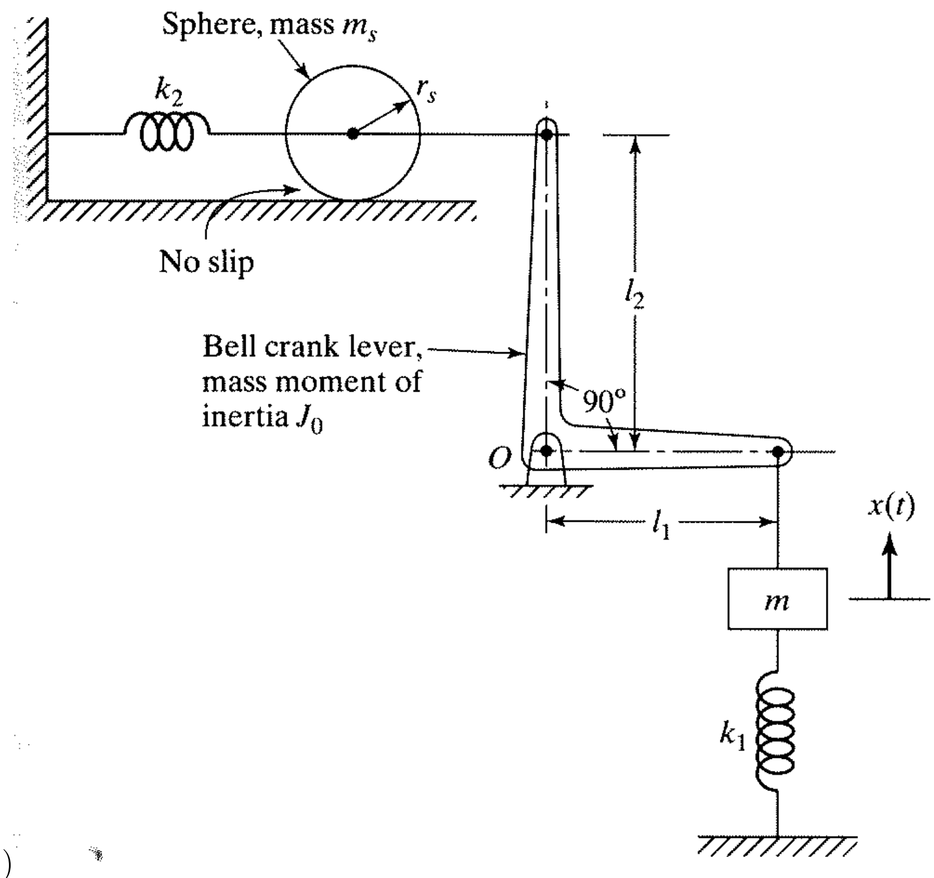
Fill in the class

Solution:

Example 6: Determine the equivalent mass of the spring-mass system when the spring has a total mass of m_s. Determine the equivalent mass moment of inertia /mass of the systems shown below



(A)



(B)

Figure 2.7: Figures for example 6.

Solution:

Fill in the class

We conclude the following from the preceding discussion and examples.

1. Equivalent systems simplify the modelling of multi-component systems.
2. Equivalent spring is determined from the potential energy and equivalent mass (or mass moment of inertia) is determined from the kinetic energy expression of the system expressed in terms of a **single displacement co-ordinate**.
3. A thorough understanding of planar kinematics is essential in relating translations and rotations.

Topic 2.3: Undamped SDOF Response

We have seen that the equations of motion for SDOF systems are differential equations. The fundamental differential equation that is common to all the undamped SDOF systems we encountered thus far is of the form $m\ddot{x} + kx = f$. In this topic, we will solve for the free vibration response, $f = 0$, and harmonically forced vibration $f = F\cos(\omega t)$. We will study the influence of forcing frequency, ω, on the vibration response and observe the phenomenon of resonance. After completing this topic, you are expected to be able to determine natural frequencies, and compute free and forced vibration response of undamped SDOF systems.

2.8 Introduction

The main goal for this topic is to evaluate the vibration response of undamped systems. Specifically, we are interested in determining the natural frequencies; response to initial perturbations, or free vibration; response to a harmonic force, or forced vibration.

So far, we know how to develop a single degree of freedom model and write its equation of motion. In topic 2.2, we learned to use three different techniques to formulate the equations of motion of mechanical systems.: (a) Newton's second law; (b) D'Alembert's principle; and (c) Principle of conservation of energy. All of them yielded equations of the form:

$$m\ddot{x} + kx = f \quad \text{Translatory vibrations} \tag{1a}$$
$$J_o\ddot{\theta} + k_\theta\theta = M_o \quad \text{Rotatory or torsional vibrations} \tag{1b}$$

It turned out that all undamped SDOF systems are governed by the above *universal* second order, linear, ordinary differential equations (ODEs).

In the following discussion, we shall be concerned with solving Eq.(1a) for vibratory motions involving translations. It is straight forward to extend this solution to torsional vibrations by making appropriate replacements: J_o for m and k_θ for k.

It can be verified that the equations of motion in Eq.(1a) and Eq.(1b) are **Linear, Second Order,Ordinary Differential Equations.** To find the vibration response we must solve these differential equations. You may recall from your previous courses that the response depends on both the initial conditions and the nature of the forcing function. In this topic **we focus on harmonic force of the form** $f(t) = F_0 \cos \omega t$. Most rotating systems are subjected to forces of this kind. **For general, non harmonic forcing, we make use of superposition principle using Fourier series or in the form of a convolution integral as we shall see in later topics.**

2.9 Undamped Vibration Response

We know from the principle of superposition (Topic 1 and Assignment 1) that the *total* response consists of two parts: the first part of the solution is called *homogeneous* solution, or *complementary function*; the second part is called a *particular* solution or *particular integral.* Since, we are considering a second order, linear, ordinary differential equation we expect no more than two *unknown* constants in the *total* solution.

$$m\ddot{x}_h + kx_h = 0 \quad \text{Homogeneous response/Free vibration} \tag{2a}$$

$$m\ddot{x}_h + kx_h = f \quad \text{Particular solution/Forced vibration.} \tag{2b}$$

Adding the above two equations we have the TOTAL response, from the principle of superposition

$$m\ddot{x} + kx = f, \quad x = x_h + x_p \quad \text{TOTAL response} \tag{3}$$

It is required to specify the initial conditions on the TOTAL response. They can be initial velocity, or initial displacement:

$$x(0) = x_0; \quad \dot{x}(0) = \dot{x}_0 \qquad \text{INITIAL conditions } \underline{\textbf{apply on the TOTAL solution}}. \tag{4}$$

2.10 Free Vibration Response (T 2.2.4, T 2.2.5+Notes)

Let us first determine the *homogeneous* solution or free vibration response.

$$m\ddot{x}_h + kx_h = 0 \quad \text{Homogeneous response/Free vibration} \tag{5}$$

To solve the above differential equation we assume a solution of the form $x_h = Xe^{st}$ where C and s are complex constants to be determined. Let us insert this *trial* solution into the equation of motion Eq.(5):

$$\begin{aligned} & m\ddot{x}_h + kx_h = 0, \quad x_h = Xe^{st}, \quad \ddot{x}_h = Xs^2e^{st} \\ & \Rightarrow m\left[Xs^2e^{st}\right] + k\left[Xe^{st}\right] = 0 \\ & \Rightarrow X\left[ms^2 + k\right]e^{st} = 0, X \neq 0 \text{ for non-trivial solution} \end{aligned} \tag{6}$$

to form the *auxiliary* or *characteristic* equation:

$$ms^2 + k = 0 \tag{7}$$

The two roots of of the above *quadratic* equation are

$$s_{1,2} = \pm\sqrt{-\frac{k}{m}} = \pm j\omega_n, \quad j \equiv \sqrt{-1} \tag{8}$$

where j is the complex number and ω_n is called the *natural frequency*, the meaning of which will become clear shortly.

$$j \equiv \sqrt{-1}, \omega_n \equiv \sqrt{\frac{k}{m}}, \quad \text{(Definitions)} \tag{9}$$

Each of the characteristic root gives one solution to the governing equation in Eq.(5). The *general solution* is thus given by

$$x_h(t) = Real\left[C_1 e^{s_1 t} + C_2 e^{s_2 t}\right] = Real\left[C_1 e^{j\omega_n t} + C_2 e^{-j\omega_n t}\right] \tag{10}$$

Making use of the Euler's identity and the real part of the solution is being understood to be relevant

$$e^{j\theta} = \cos\theta + j\sin\theta \tag{11}$$

we have[1]

$$\begin{aligned} x_h(t) &= C_1\left[\cos\omega_n t + j\sin\omega_n t\right] + C_2\left[\cos\omega_n t - j\sin\omega_n t\right] \\ &= \left[C_1 + C_2\right]\cos\omega_n t + j\left[C_1 - C_2\right]\cos\omega_n t \\ &\text{Let } A_1 = \left[C_1 + C_2\right] \text{ and } A_2 = j\left[C_1 - C_2\right] \\ &= A_1\cos\omega_n t + A_2\cos\omega_n t \end{aligned}$$

$$\therefore x_h(t) = A_1\cos\omega_n t + A_2\sin\omega_n t \tag{12}$$

Notice that we have two unknown constants A_1 and A_2. These can be found using:

1. Initial conditions only when the particular solution is zero, $x_p = 0$.

[1] This identity can be proved in many different ways. The algebraic way would be to use the power series expansions for the three functions. Thus, $e^{j\theta} = 1 + j\theta + \frac{1}{2!}(j\theta)^2 + \frac{1}{3!}(j\theta)^2 + \ldots$; $\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \ldots$; $\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \ldots$. Adding $\cos\theta + j\sin\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} + j\left[\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!}\right] = e^{j\theta}$. A geometric method would involve representing $e^{j\theta}$ as a vector in the complex plane (called Argand diagram) and adding its projection along the horizontal (real) and vertical (imaginary) axes to obtain Euler's identity! We will use the geometric method in the class.

2. Initial conditions on the TOTAL solution $x = x_h + x_p$, when there is an external force, $x_p \neq 0$.

In free vibration problems, there is no external force $f = 0 \Rightarrow x_p = 0$. Therefore $x = x_h + x_p = x_h$. Imposing initial conditions in Eq.(4) on the total solution, we have

$$x_h(t) = x_0 \Rightarrow A_1 \cos 0 + A_2 \sin 0 = x_0 \Rightarrow A_1 = x_0$$

$$\dot{x}_h(t) = \dot{x}_0 \Rightarrow [-\omega_n A \sin \omega_n t + \omega_n A_2 \cos \omega_n t]_{t=0} = \dot{x}_0$$

$$\omega_n A_2 = \dot{x}_0 \Rightarrow A_2 = \frac{\dot{x}_0}{\omega_n}$$

Thus, the free vibration response of an undamped SDOF system is obtained as given below.

$$\boxed{x = x_h = x_o \cos \omega_n t + \frac{\dot{x}_0}{\omega_n} \sin \omega_n t, \quad \omega_n = \sqrt{\frac{k}{m}} \qquad (13)}$$

In solving problems it is best not to memorize the above formula, but, instead use the form $x_h(t) = A_1 \cos \omega_n t + A_2 \sin \omega_n t$ **and determine** A_1 **and** A_2 **as appropriate. You will see in Section 2.11 different mathematical representations of the same response. <u>The real challenge is finding the initial conditions</u>. A good sense of judgement with regard to the choice of the co-ordinate and understanding of the physics of the problem is absolutely essential**. The following observations are worth making about free vibration:

1. **Free vibration takes place at the system's natural frequency**, irrespective of the initial conditions.

2. Natural frequencies depend only on the stiffness and mass properties of the system. Natural frequency increases with an increase in the stiffness or a *decrease* in the mass.

3. One can estimate natural frequencies from static deflections using the formula $\omega_n = \sqrt{\frac{g}{\delta_{st}}}$. Can you show this? **Fill in the class**

4. The S.I. unit of natural frequency is Hertz. One Hz is one cycle per second. In most computer programs, and in mathematics, frequencies are measured in rad/s. In engineering practise one also uses rpm. The following conversion may be useful $1Hz = 2\pi rad/s = \frac{1}{60}rpm$. 1200 rpm is thus 20 Hz or 40π rad/s.

Example 7: Determine the natural frequencies of: (a) simple pendulum; (b) compound pendulum; (c) Example 4; (d) Tutorial 1 problem on rotating spring-mass system. **Fill in the class**

Solution:

Example 8: An air-conditioning chiller unit weighs 600 kg is to be supported by four air springs as shown below. Design the air springs such that the natural frequency of vibration of the unit lies between 5 rad/s and 10 rad/s.

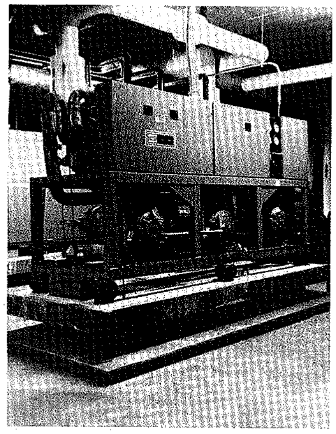


Figure 2.9: Figure for example 8.

Solution:

Fill in the class

Example 9: A rigid block of mass M is mounted on four elastic supports, as shown below. A mass m drops from a height l and adheres to the rigid block without rebounding. If the spring constant of each support is k, find the natural frequency of vibration of the system (a) without mass m, and (b) with the mass m. Also find the resulting motion of the system in case (b).

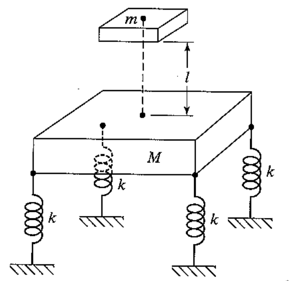


Figure 2.10: Figures for example 9.

Solution:

Fill in the class

2.11 Different Representations of Response (T 2.2.5, T 1.10+Notes)

It is possible to represent the free vibration response given in Eq.(13) in other equivalent *algebraic* and *geometric* forms. In particular, we shall see that there is a graphical method to represent each of the forces in the differential equations of motion in Eq.(1a) as a rotating vector, when the force is harmonic $f(t) = F\cos(\omega t)$. This is a powerful way of visualising harmonic vibrations that offers valuable *insight*. Let us consider these alternate forms.

2.11.1 Amplitude-Phase form

Two particular equivalent forms are

$$\text{Phase-lag form: } x(t) = A\cos(\omega_n t - \phi_0), \quad A = \sqrt{x_0^2 + \left(\frac{\dot{x}_0}{\omega_n}\right)^2}, \phi_0 = \tan^{-1}\left(\frac{\dot{x}_0}{x_0\omega_n}\right) \tag{14a}$$

$$\text{Phase-lead form:} x(t) = A\cos(\omega_n t + \phi_0) \quad A = \sqrt{x_0^2 + \left(\frac{\dot{x}_0}{\omega_n}\right)^2}, \phi_0 = \tan^{-1}\left(-\frac{\dot{x}_0}{x_0\omega_n}\right) \tag{14b}$$

It may be observed that there still are two unknown constants: amplitude A and phase ϕ_0, which are determined using the initial conditions. Which form of response is better? The form given in Eq.(13) is preferred in solving problems. This is because in the amplitude-phase form there are two unknowns to be determined: A and ϕ_0. But ϕ_0 is tucked inside a sine or cosine function, which causes problems when applying the initial conditions. We will illustrate the meaning of phase-lag and phase-lead, later in this topic.

2.11.2 Complex variable representation

It is also possible to represent the response of an undamped system in the following complex variable form:

$$x(t) = Re\left[Ce^{j\omega_n t}\right] \tag{15}$$

where $Re\,[]$ denotes the real part of the complex variable inside the brackets. C is a complex number in general. Again, this form is not directly useful when solving numerical problems. However, this form can be applied with ease to solve the response of damped systems and evaluate forced vibration response. Thus it is useful for analytical calculations.

Both the amplitude-phase forms and the complex variable representation of free vibration offers a graphical representation of the vibrations. This graphical method can be used to our great advantage to gain insight into vibration, which will be described next, and then applied later in example 11, to solve the forced response.

2.11.3 Rotating vector representation

A geometric description of vibration is readily obtained by visualising $A_1 \cos \omega_n t$ as the horizontal projection of a uniformly rotating vector of magnitude A_1, rotating at ω_n rad/s. Then $A_2 \sin \omega_n t$ is the vertical projection of rotating vector of a different magnitude A_2. This means that the free vibration response can be visualised as a sum of projections of two rotating vectors of amplitudes A_1 and A_2, respectively, as shown below.

Fill in the class

The response forms mentioned in Eq.(14a) and Eq.(14b) are particularly suitable to be visualised as rotating vectors. The response in Eq.(14a) can be visualised as the horizontal projection of a rotating vector of amplitude A and phase lag ϕ_0 as shown below

Fill in the class

Similarly, the response in Eq.(14b) can be visualised as the horizontal projection of a rotating vector of amplitude A and phase lead ϕ_0 as shown above.

Another visualisation is provided by a rotating vector in a complex plane. This rotating vector is called a phasor. Let us represent the displacement, velocity, and acceleration as complex numbers, and then as rotating vectors in the complex plane. We will take the form given Eq.(14a) for the displacement:

$$
\begin{aligned}
x(t) &= A\cos(\omega_n t - \phi_0) = Re\left[Ae^{j(\omega_n t - \phi_0)}\right] \\
\dot{x}(t) &= -A\omega_n \sin(\omega_n t - \phi_0) = A\omega_n \cos\left(\omega_n t - \phi_0 + 90^0\right) \\
&= Re\left[A\omega_n e^{j(\omega_n t - \phi_0 + 90^0)}\right] = Re\left[j\omega_n A e^{j(\omega_n t - \phi_0)}\right] \\
\ddot{x}(t) &= -A\omega_n^2 \cos(\omega_n t - \phi_0) = A\omega_n^2 \cos\left(\omega_n t - \phi_0 + 180^0\right) \\
&= Re\left[A\omega_n^2 e^{j(\omega_n t - \phi_0 + 180^0)}\right] = Re\left[(j\omega_n)^2 A e^{j(\omega_n t - \phi_0)}\right]
\end{aligned}
$$

we used $j = e^{j\frac{\pi}{2}}$ and $-1 = j^2 = e^{j\pi}$; Note $\frac{\pi}{2}rad = 90^0, \pi rad = 180^0$.

We can represent the above in the following graphical form

Fill in the class

We note that, it is sufficient to know the displacement amplitude to deduce the velocity and acceleration amplitudes for a harmonic motion. **You will find this result useful in the Shaky table experiment.**

Let us summarise the main conclusions from our study on undamped free vibrations.

1. Undamped free vibration is specified by the second order, linear, ODE: $m\ddot{x} + kx = 0$ along with the initial conditions: an initial displacement $x(0) = x_0$ and an initial velocity $\dot{x}(0) = \dot{x}_0$.

2. Undamped free vibration response is given by $x = x_h = x_o \cos \omega_n t + \frac{\dot{x}_0}{\omega_n} \sin \omega_n t$, where the *natural frequency*, ω_n, is given by $\omega_n = \sqrt{\frac{k}{m}}$.

3. Undamped free vibration response can also be represented in amplitude-phase form $x(t) = A \cos(\omega_n t - \phi_0)$, which lends itself into a rotating vector representation of harmonic motion.

4. In a harmonic motion at frequency ω rad/s and phase lag ϕ_0, whose displacement is given by $x(t) = A\cos(\omega t - \phi_0)$, the velocity and acceleration amplitudes are related to the displacement amplitude, A, via $A_{velocity} = \omega A$ and $A_{acceleration} = \omega^2 A$. The phase lags are related via $\phi_{0,velocity} = \phi_0 - 90^0$ and $\phi_{0,acceleration} = \phi_0 - 180^0$.

5. The essential features of the free-vibration of a undamped system are shown in the sketch below:

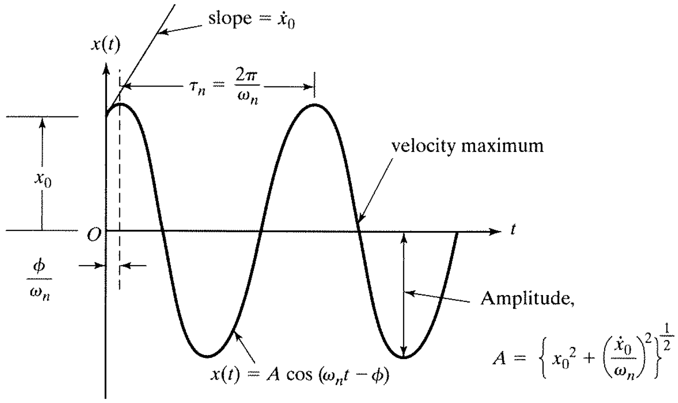


Figure 2.11: Free vibration response of an undamped SDOF system. Notice the phase-lag of the response $A\cos(\omega t - \phi_0)$ with reference to $\cos\omega t$. In our notes we replace ϕ with ϕ_0. We will reserve ϕ for the phase-lag of the forced vibration response response in the steady state.

Question: Can you explain the meaning of phase-lag from the above figure? How will the response sketch change if the phase is a lead?

Fill in the class

2.12 Forced Vibration Response (T 3.3+Notes)

The particular solution, x_p, when the spring-mass system is subjected to a harmonic force $f(t) = F_0 \cos \omega t$ is governed by

$$m\ddot{x}_p + kx_p = F_0 \cos \omega t \tag{17}$$

The particular solution can be obtained using two methods. In the first method, we use the theory of ODEs, while the second is a graphical method. We will use the ODE theory here, and present the graphical method in example 11 and when dealing with damped systems.

2.12.1 Ordinary Differential Equation Theory

Assuming

$$x_p(t) = X \cos \omega t \tag{18}$$

in Eq.(17), we find

$$x_p = \frac{F_0}{k - m\omega^2} \cos \omega t, \quad X = \frac{F_0}{k - m\omega^2} \tag{19}$$

Therefore, the total response is given by

$$x(t) = x_h + x_p = A_1 \cos \omega_n t + A_2 \sin \omega_n t + \frac{F_0}{k - m\omega^2} \cos \omega t, \tag{20}$$

The following observations can be made about the above solution:

1. The homogeneous solution is harmonic at *natural* frequency ω_n. The forced vibration or particular solution is harmonic at the *forcing* frequency ω.

2. **The two unknown constants, A_1 and A_2, are to be determined from the initial conditions applied to the total response.**

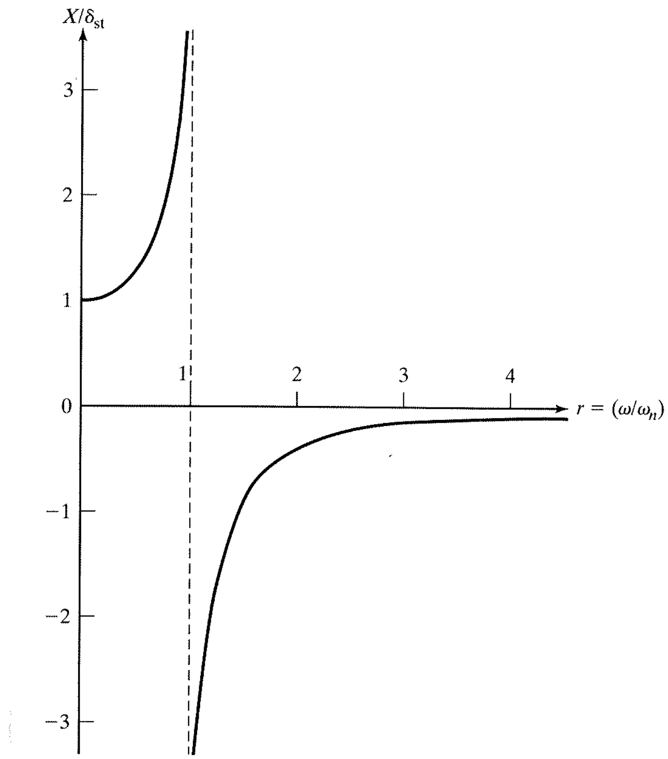
Substituting the initial conditions $x(0) = x_0$ and $\dot{x}(0) = \dot{x}_0$ in Eq.(20) gives the total response

$$x(t) = \left(x_0 - \frac{F_0}{k - m\omega^2}\right)\cos\omega_n t + \frac{\dot{x}_0}{\omega_n}\sin\omega_n t + \frac{F_0}{k - m\omega^2}\cos\omega t, \tag{21}$$

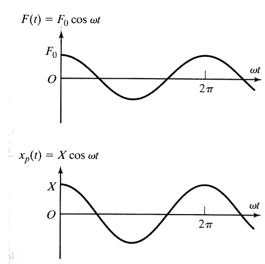
It is useful to represent $\frac{F_0}{k-m\omega^2}$ in terms of a non-dimensional parameter, called *Dynamic Magnification Factor* (DMF), which is defined as the ratio of the displacement amplitudes in the dynamic and static case as follows

$$\frac{X}{\delta_{st}} = \frac{X}{\frac{F_0}{k}} = \frac{\frac{F_0}{k-m\omega^2}}{\frac{F_0}{k}} = \frac{k}{k - m\omega^2} = \frac{1}{1 - \frac{m\omega^2}{k}} = \frac{1}{1 - \left(\frac{\omega}{\omega_n}\right)^2} \tag{22}$$

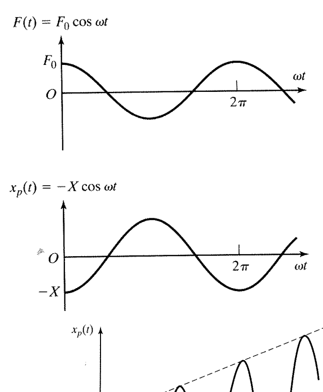
The DMF is the factor by which the static displacement needs to be multiplied with in order to obtain the dynamic displacement in the steady state, <u>ignoring the homogeneous solution.</u> A plot of the DMF as a function of the non-dimensional frequency ratio $r = \frac{\omega}{\omega_n}$ is shown below:



Case 1:



Case 2:



Case 3:

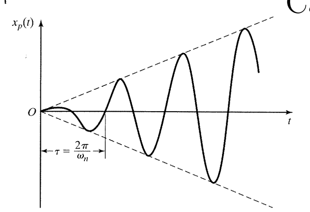


Figure 2.12: Forced vibration response of an undamped SDOF system: Dynamic Magnification Factor; Case 1: $\omega < \omega_n$; Case 2: $\omega > \omega_n$; Case 3: $\omega = \omega_n$.

Question: List the important features of the DMF curve?

Fill in the class

Three cases are of particular interest, and they are discussed below.

Case 0: $\omega = 0$

In this case $f(t) = F_0 \cos \omega t = F_0$ is a time-independent (STATIC) force. So we get the amplitude of the particular solution as $X = \delta_{st}$.

Case 1: $\omega < \omega_n$

In this case, DMF is positive and more than 1. So we get the amplitude of the particular solution as $X = \alpha\delta_{st}$, where $\alpha > 1$. The displacement associated with the particular solution is in-phase with the force signal. This means that both the force and displacement reach their maxima and minima simultaneously. They are in sync. This is sketched in Fig.(2.12) under Case 1.

Case 2: $\omega > \omega_n$

In this case, DMF is *negative*. In order to keep the DMF *positive*, we can absorb the negative sign into the phase of the response as follows: $X\cos(\omega t) = -DMF \times \delta_{st}\cos(\omega t) = DMF \times \delta_{st}\cos(\omega t - 180^0)$. This means that the displacement lags force by 180^0. When force is maximum positive, displacement is maximum negative, and so on. When the force is crossing zero upwards then the displacement is crossing zero downwards. They are out of sync. This is sketched in Fig.(2.12) under Case 2.

An important point to remember is that the displacement amplitude of the particular solution falls rapidly with increasing ω.

Question: Can you explain why X increases with frequency in Case 1, while X decreases with forcing frequency in Case 2? **Fill in the class**

Case 3: $\underline{\omega = \omega_n}$

This is least desired in practise. We do not want the forcing frequency equal or close to the resonant frequency. Any designer that makes this happen is bound to hire a vibration consultant!

In this case, DMF is *infinity*! Only if we are mathematicians. Real systems do not like infinities. They vibrate with such large amplitudes that either they fail, or go into a nonlinear range which anyway is not governed by linear ODES. How will the amplitude grow? This question can be answered by letting the forcing frequency approach the natural frequency. This is where the idea of limits learned in a calculus course is handy.

The total response in Eq.(21) can be expressed as follows

$$x(t) = x_0 \cos \omega_n t + \frac{\dot{x}_0}{\omega_n} \sin \omega_n t + \delta_{st} \left[\frac{\cos \omega t - \cos \omega_n t}{1 - \left(\frac{\omega}{\omega_n} \right)^2} \right] \quad (23)$$

In the limit $\omega \to \omega_n$ the factor in [] reduces to an indeterminate form $\frac{0}{0}$. Recall from your Calculus that in such cases we use L'Hospital rule. That is, we differentiate the numerator and denominator with respect to ω until such point where the limit is determinate. Let us do this **Fill in the class**

Thus the total response when the forcing frequency approaches the natural frequency is given by

$$x(t) = x_0 \cos \omega_n t + \frac{\dot{x}_0}{\omega_n} \sin \omega_n t + \delta_{st} \frac{\omega_n t}{2} \sin \omega_n t \qquad (24)$$

The above result says that the **response grows linearly with time at resonance. In other words, the system becomes unstable!**

Question: Can you tell the phase relationship between the force and displacement associated with the particular solution at resonance? **Fill in the class**

Another important phenomenon is observed as the forcing frequency is brought close to resonance leading to *beats*. In beat phenomenon the vibration amplitude builds up then decreases and then builds up again leading to a periodic *beating*. **You can observe this in the SHAKY TABLE experiment in the out-of-phase mode of operation.** The response for zero initial velocity and displacement is given by

$$x(t) = \frac{F_0/m}{\omega_n^2 - \omega^2}\left[2\sin\frac{\omega+\omega_n}{2}t\sin\frac{\omega-\omega_n}{2}t\right] = \frac{F_0/m}{2\epsilon\omega}\sin\epsilon t\sin\omega t,$$
$$\epsilon = \frac{\omega_n - \omega}{2}. \quad (25)$$

You are encouraged to derive the above expression. The beating response is sketched below

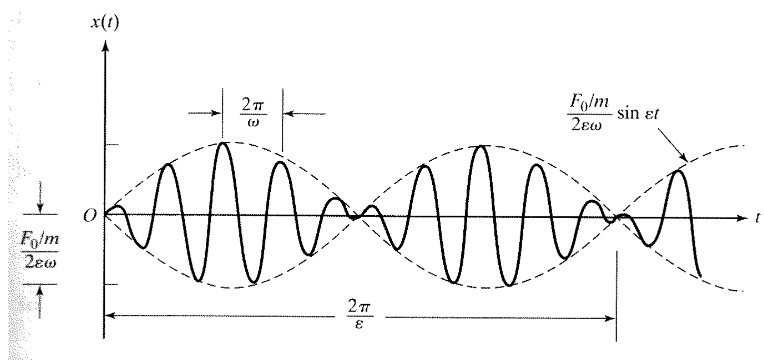


Figure 2.13: Beating phenomenon due to the interaction between the homogenous (free vibration) and particular (forced vibration) solutions. Here the forcing frequency is slightly below the natural frequency.

Let us summarise the main learning points pertaining to the forced vibration:

1. The total response of an undamped system subjected to a harmonic force $f(t) = F_0 \cos \omega t$ is given by $x(t) = \left(x_0 - \frac{F_0}{k-m\omega^2} \right) \cos \omega_n t + \frac{\dot{x}_0}{\omega_n} \sin \omega_n t + \frac{F_0}{k-m\omega^2} \cos \omega t$.
2. Free vibration takes place at the natural frequency ω_n while the forced vibration is at ω.
3. With increasing forcing frequency from zero, the response increases reaching an instability at resonance $\omega = \omega_n$ and then decreases for forcing frequencies above resonances. The forced vibration response grows linearly with time at resonance $\omega = \omega_n$.
4. The response is in-phase with the force for $\omega < \omega_n$; a phase lag of 90^0 at resonance $\omega = \omega_n$; and the response lags behind the force by 180^0 above resonance. The displacement is in exactly the opposite direction to the force. **This is the first counter-intuitive feature we observe in vibration!**
5. The amplitude of the forced vibration can be evaluated from $DMF = \frac{X}{\delta_{st}} = \frac{1}{1-\left(\frac{\omega}{\omega_n}\right)^2}$
6. Beating arises due to the interaction between the free and forced vibration.

Example 10: A portable shredder used to shred bark, tree branches, and shrub clippings, has a mass of 200 kg resting on tires and support system with an elastic constant of 460 N/mm. The amplitude of the vertical sinusoidal force shown below is 3 kN. Find the maximum vertical displacement, if the shredder operates at 1200 rpm.

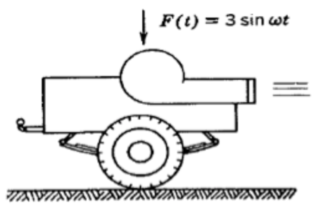


Figure 2.14: Figure for example 10.

Solution:

Fill in the class

Example 11: Deduce the expression for forced vibration amplitude X, by using the rotating vector representation. Which forces are dominant below, at, and above the resonant frequency in the vector diagram of forces?

Solution:

Fill in the class

We draw the following conclusions from our study in this topic:

1. Undamped free vibration is specified by the second order, linear, ODE: $m\ddot{x} + kx = 0$ along with the initial conditions: an initial displacement $x(0) = x_0$ and an initial velocity $\dot{x}(0) = \dot{x}_0$.

2. The undamped free vibration response is given by $x = x_h = x_o \cos\omega_n t + \frac{\dot{x}_0}{\omega_n} \sin\omega_n t$, where the *natural frequency*, ω_n, is given by $\omega_n = \sqrt{\frac{k}{m}}$.

3. Undamped free vibration response can also be represented in term of the amplitude-form $x(t) = A\cos(\omega_n t - \phi_0)$, which lends itself into a rotating vector representation of harmonic motion.

4. In a harmonic motion at frequency ω rad/s and phase lag ϕ_0, whose displacement is given by $x(t) = A\cos(\omega t - \phi_0)$, the velocity and acceleration amplitudes are related to the displacement amplitude, A, via $A_{velocity} = \omega A$ and $A_{acceleration} = \omega^2 A$. The phase lags are related via $\phi_{0,velocity} = \phi_0 - 90^0$ and $\phi_{0,acceleration} = \phi_0 - 180^0$.

5. The total response of an undamped system subjected to a harmonic force $f(t) = F_0 \cos\omega t$ is given by $x(t) = \left(x_0 - \frac{F_0}{k - m\omega^2}\right)\cos\omega_n t + \frac{\dot{x}_0}{\omega_n}\sin\omega_n t + \frac{F_0}{k - m\omega^2}\cos\omega t$. Free vibration takes place at the natural frequency ω_n while the forced vibration is at ω.

6. With increasing forcing frequency from zero, the response increases reaching an instability at resonance $\omega = \omega_n$ and then decreases for forcing frequencies above resonances. The forced vibration response grows linearly with time at resonance $\omega = \omega_n$.

7. The response is in-phase with the force for $\omega < \omega_n$; a phase lag of 90^0 at resonance $\omega = \omega_n$; and the response lags behind the force by 180^0 above resonance. The displacement is in exactly the opposite direction to the force. **This is the first counter-intuitive feature we observe in vibration!**

8. The amplitude of the forced vibration can be evaluated from $DMF = \frac{X}{\delta_{st}} = \frac{1}{1 - \left(\frac{\omega}{\omega_n}\right)^2}$

9. Beating arises due to the interaction between the free and forced vibration.

10. Elastic forces dominate below resonance while inertial forces dominate above the resonance. Thus, low frequency forced vibrations can be reduced by stiffening the system, while reducing high frequency forced vibration requires considerable addition of mass. **Adding stiffness has little influence on the DMF well above resonance!**

Additional Problems for Practice

Example 12: **(Modelling a drop test in packaging industry)** A 5-kg fragile glass vase is packed in chopped sponge rubber and placed in a cardboard box that has negligible mass. It is then accidentally dropped from a height of 1m. This particular sponge rubber exhibits the force-deflection curve sketched below. Determine the maximum acceleration experienced by the vase.

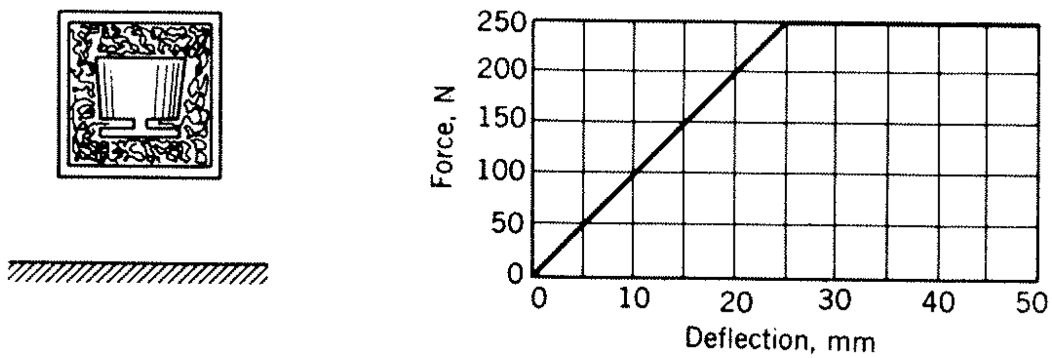


Figure 2.15: Figure for example 12.

Solution:

Fill in the class

Example 13: **(Vibration due to rotating unbalance)** Unbalanced masses are a common source of vibration problems in many rotating systems: turbo machinery; washing machines; and shaky table, to name a few. The simplest SDOF system that models the vibrations of these systems is sketched below. A mass m_u is mounted by a shaft and bearings to the mass m. The unbalance mass m_u follows a circular path of radius e (unbalance). *Ignoring damping*, determine the equations that govern the motion of the main mass m.

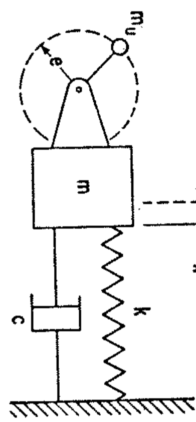


Figure 2.16: Figure for example 13. Ignore damping.

Solution:

Fill in the class

Example 14: **(Heavy Spring)** So far we assumed that springs are massless. This need not be so. How will the natural frequency change if we include the mass of the spring? Assume m_s is the total mass of the spring. **Recall Example 6 where we already determined the equivalent mass**.

TOPIC 2.4: RESPONSE OF VISCOUSLY DAMPED SDOF SYSTEMS

We have seen in Topic 2.3 that the response of an undamped system grows without a limit at resonance: when the forcing frequency matches with the natural frequency of the system. In practise, however, all mechanical systems possess damping or sources of energy dissipation. This topic is concerned with the free and forced vibration response of a viscously damped system. We will see that there are three types of damped free vibrations: underdamped, overdamped, and critically-damped. Among these three, we will focus on the underdamped case to which most mechanical systems belong. We will also see forced response depends on the forcing frequency: gradually increases up to resonance and decreases thereafter. **After completing this topic you are expected to calculate the free and forced vibration response of a viscously damped SDOF system and use in response design.**

2.13 Introduction

The main goal for this topic is to evaluate the vibration response of viscously damped SDOF systems. Specifically, we are interested in determining the response to initial perturbations, or free vibration; response to a harmonic force, or forced vibration.

A viscous damper applies an opposing force proportional to the relative velocity between its two ends. For a damper with one end fixed, the damping force is given by

$$f_d = -c\dot{x} \tag{1}$$

where c is called the viscous damping coefficient (units: N-s/m). The *negative* sign indicates that the damping force opposes vibration.

Thus for the spring-mass system with a viscous damper, we can obtain the following equations of motion: **Fill in the class**

$$m\ddot{x} + c\dot{x} + kx = f \tag{2}$$

2.14 Damped Vibration Response

We know from the principle of superposition (Topic 1 and Assignment 1) that the *total* response consists of two parts: the first part of the solution is called *homogeneous* solution, or *complementary function*; the second part is called a *particular* solution or *particular integral.* Since, we are considering a second order, linear, ordinary differential equation we expect no more than two *unknown* constants in the *total* solution.

$$m\ddot{x}_h + c\dot{x}_h + kx_h = 0 \quad \text{Homogeneous response/Free vibration} \tag{3a}$$

$$m\ddot{x}_p + c\dot{x}_p + kx_p = f \quad \text{Particular solution/Forced vibration.} \tag{3b}$$

Adding the above two equations we have the TOTAL response, from the principle of superposition

$$m\ddot{x} + c\dot{x} + kx = f, \quad x = x_h + x_p \quad \text{TOTAL response} \tag{4}$$

It is required to specify the initial conditions on the TOTAL response. They can be initial velocity, or initial displacement:

$$x(0) = x_0; \quad \dot{x}(0) = \dot{x}_0 \qquad \text{INITIAL conditions } \underline{\textbf{apply on the TOTAL solution}}. \tag{5}$$

In the subsequent sections we will determine the free vibration response x_h and the forced vibration response x_p. You will notice that this is purely an exercise in solving a second order linear ordinary differential equation. Hence, you may find it useful to revise your notes from previous years on Differential Equations and Calculus. In the class, we will examine the outline of solution procedures without getting into algebraic details.

2.15 Free Vibration Response (T 2.6+Notes)

Let us first determine the *homogeneous* solution or free vibration response of a viscously damped system.

$$m\ddot{x}_h + c\dot{x}_h + kx_h = 0 \quad \text{Homogeneous response/Free vibration} \tag{6}$$

To solve the above differential equation we assume a solution of the form $x_h = Xe^{st}$ where C and s are to be determined. Let us insert this *trial* solution into the equation of motion Eq.(6)

$$\begin{aligned} & mXs^2e^{st} + csXe^{st} + KXe^{st} = 0 \\ & \Rightarrow \left[ms^2 + cs + k\right] e^{st} = 0 \\ & \Rightarrow \left[ms^2 + cs + k\right] = 0, \quad \because e^{st} \neq 0 \end{aligned}$$

to form the *auxiliary* or *characteristic* equation:

$$ms^2 + cs + k = 0 \tag{7}$$

The two roots of of the above *quadratic* equation are

$$s_{1,2} = \frac{-c \pm \sqrt{c^2 - 4km}}{2m} \tag{8}$$

Each of the characteristic root gives one solution to the governing equation in Eq.(6). The *general solution* is thus given by

$$x_h(t) = C_1e^{s_1t} + C_2e^{s_2t} \tag{9}$$

It can be seen that the displacement solution associated with each of the roots decreases with time. After sufficiently long time $x_h = 0$.

$$x_h(t) = C_1 e^{\left(-\frac{c}{2m}+\frac{\sqrt{c^2-4km}}{2m}\right)t} + C_2 e^{\left(-\frac{c}{2m}-\frac{\sqrt{c^2-4km}}{2m}\right)t}$$
$$= e^{-\frac{c}{2m}t}\left[C_1 e^{\left(\frac{\sqrt{c^2-4km}}{2m}\right)t} + C_2 e^{\left(-\frac{\sqrt{c^2-4km}}{2m}\right)t}\right]$$

For the purposes of our discussion, it is useful to introduce the non-dimensional parameter, called the damping ratio, $\zeta = \frac{c}{c_c}$. Here, c_c is the critical damping associated with the condition $c_c^2 - 4km = 0$ or $c_c = 2\sqrt{km} = 2m\omega_n$, since $\omega_n = \sqrt{\frac{k}{m}}$. Thus, we have

$$\zeta = \frac{c}{c_c} = \frac{c}{2m\omega_n}, \quad \text{or } c = 2\zeta m\omega_n. \tag{10}$$

Three cases are of particular interest, depending on the sign of the term $c^2 - 4km$, or the value of ζ. These will be discussed now.

2.15.1 Underdamping $\zeta < 1$ or $c < c_c$

Many mechanical systems fall under the category of underdamped systems. The shock absorbers in a car gradually wear out and fall under the category of underdamped systems, eventhough they are critically damped when they are brand new. In the underdamped case the two roots can be expressed as:

$$
\begin{aligned}
s_{1,2} &= -\frac{c}{2m} \pm \frac{\sqrt{c^2 - 4km}}{2m} = -\frac{c}{2m} + \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{4km}{4m^2}} \\
&= -\frac{2\zeta m\omega_n}{2m} \pm \sqrt{\left(\zeta\omega_n\right)^2 - \omega_n^2}, \quad \because c = 2\zeta m\omega_n \text{ and } \omega_n = \sqrt{\frac{k}{m}} \\
&= -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1} \\
&= -\zeta\omega_n \pm j\omega_n\sqrt{1 - \zeta^2}, \quad j = \sqrt{-1}, \zeta < 1
\end{aligned}
$$

$$
s_1 = -\zeta\omega_n + j\omega_n\sqrt{1 - \zeta^2}, \quad s_2 = -\zeta\omega_n - j\omega_n\sqrt{1 - \zeta^2}, \tag{11}
$$

or in terms of the *damped natural frequency* ω_d as

$$
s_1 = -\zeta\omega_n + j\omega_d, \quad s_2 = -\zeta\omega_n - j\omega_d, \quad \omega_d = \omega_n\sqrt{1 - \zeta^2}, \tag{12}
$$

and the general solution is given by

$$
x_h(t) = C_1e^{s_1t} + C_2e^{s_2t} = C_1e^{-\zeta\omega_nt + j\omega_dt} + C_2e^{-\zeta\omega_nt - j\omega_dt} \tag{13}
$$

We can use the Euler's result

$$
e^{j\theta} = \cos\theta + j\sin\theta \tag{14}
$$

and simplify the response in Eq.(13)as follows.

$$
\begin{aligned}
x_h(t) &= e^{-\zeta\omega_n t}\left[C_1\cos\omega_d t + jC_1\sin\omega_d t + C_2\cos\omega_d t - jC_2\sin\omega_d t\right] \\
&= e^{-\zeta\omega_n t}\left[(C_1+C_2)\cos\omega_d t + j(C_1-C_2)\sin\omega_d t\right] \\
&= e^{-\zeta\omega_n t}\left[A_1\cos\omega_d t + A_2\sin\omega_d t\right], \text{ using rotating vector diagrams} \\
&= e^{-\zeta\omega_n t}A\cos(\omega_d t-\phi_0), \quad A=\sqrt{A_1^2+A_2^2}, \tan\phi_0=\frac{A_2}{A_1}
\end{aligned}
$$

$$
x_h(t) = e^{-\zeta\omega_n t}\left[A_1\cos\omega_d t + A_2\sin\omega_d t\right] = e^{-\zeta\omega_n t}A\cos(\omega_d t-\phi_0) \tag{15}
$$

Question: What can you say about the influence of different parameters on the underdamped free vibration response based on the above?

Fill in the class

Again, we see that there are two constants A_1 and A_2 in the general solution. They can be fixed by enforcing the initial conditions on the **TOTAL** response. Since we are dealing with *free* vibrations here $x_p = 0$ and $x = x_h$. The initial conditions yield:

$$x(0) = x_h(0) = x_0 \Rightarrow A\cos\phi_0 = x_0 \tag{16a}$$

$$\dot{x}(0) = \dot{x}_h(0) = \dot{x}_0 \Rightarrow -\zeta\omega_n A\cos\phi_0 + A\omega_d \sin\phi_0 = \dot{x}_0 \tag{16b}$$

$$\Rightarrow A\sin\phi_0 = \frac{\zeta\omega_n x_0 + \dot{x}_0}{\omega_d} \tag{16c}$$

Now we can solve for the two unknowns: amplitude A and phase lag ϕ_0. We obtain ϕ_o by dividing Eq.(16c) with Eq.(16a), and A by squaring and adding the two equations

$$\tan\phi_0 = \frac{\zeta\omega_n x_0 + \dot{x}_0}{\omega_d x_0} \tag{17a}$$

$$A = \sqrt{x_0^2 + \left[\frac{\zeta\omega_n x_0 + \dot{x}_0}{\omega_d}\right]^2} \tag{17b}$$

Therefore, the final solution for the displacement response is

$$
\boxed{
\begin{aligned}
x =& x_h = e^{-\zeta\omega_n t} A \cos(\omega_d t - \phi_0) \\
\tan\phi_0 =& \frac{\zeta\omega_n x_0 + \dot{x}_0}{\omega_d x_0}; A = \sqrt{x_0^2 + \left[\frac{\zeta\omega_n x_0 + \dot{x}_0}{\omega_d}\right]^2} \\
x =& x_h = e^{-\zeta\omega_n t}\left[x_0 \cos\omega_d t + \frac{\zeta\omega_n x_0 + \dot{x}_0}{\omega_d}\sin\omega_d t\right] \\
\omega_d =& \omega_n\sqrt{1-\zeta^2} \text{ (Undamped natural frequency)}
\end{aligned}
\qquad (18)
}
$$

The following observations are worth making about the free vibration:

1. **Free vibration takes place at the system's damped natural frequency, slightly below the undamped natural frequency**, irrespective of the initial conditions.

2. Undamped natural frequency depends only on the properties of the system: mass, stiffness, and damping. It increases with an increase in the stiffness or a *decrease* in the mass or damping.

2.15.2 Critical Damping $\zeta = 1$ or $c = c_c = 2\sqrt{km}$

In some applications, it is desired that the system be brought to its equilibrium position in the *least possible time.* Such applications include, recoil mechanisms in guns, door stoppers, and so on. We must realise that critical damping is a *finely balanced* damping. For example, if the mass or stiffness of the system changes, only slightly, then the system may no longer be critically damped.

We can let the damping approach the value of 1 in Eq.(18). In this case, $\omega_d = \omega_n\sqrt{1-\zeta^2} \to 0$ and $\cos\omega_d t \to 1$, $\sin\omega_d t \to \omega_d t$. Using these in Eq.(18) gives the free vibration response of a critically damped system as follows.

$$\boxed{x = x_h = e^{-\omega_n t}\left[x_0 + \left(\omega_n x_0 + \dot{x}_0\right)t\right] \qquad (19)}$$

2.15.3 Overdamping $\zeta > 1$ or $c > c_c$

It is a common practise to engineer overdamping in many systems. The cure for the wobbly Millenium bridge in London was to engineer overdamping[1]. In overdamped systems the vibration response does not show any oscilation, but a slow creeping back to equilibrium position.

The roots of the auxilary equation— Eq.(7)—are given by

$$s_1 = -\zeta\omega_n + \omega_d, \quad s_2 = -\zeta\omega_n - \omega_d, \quad \omega_d = \omega_n\sqrt{\zeta^2 - 1} \tag{20}$$

and the free vibration response of the overdamped system

[1]See the videos posted on VISTA under VIDEOS

is given by

$$x = x_h = e^{-\zeta\omega_n t}\left[C_1 e^{\omega_d t} + C_2 e^{-\omega_d t}\right]. \qquad (21)$$

The two constants can be evaluated from the initial conditions $x(0) = x_0$ and $\dot{x}(0) = \dot{x}_0$, to give:

$$C_1 = \frac{\omega_d x_o + \zeta\omega_n x_0 + \dot{x}_0}{2\omega_d}, \quad C_2 = \frac{\omega_d x_o - \zeta\omega_n x_0 - \dot{x}_0}{2\omega_d}, \qquad (22)$$

Therefore, the free vibration response of the overdamped system

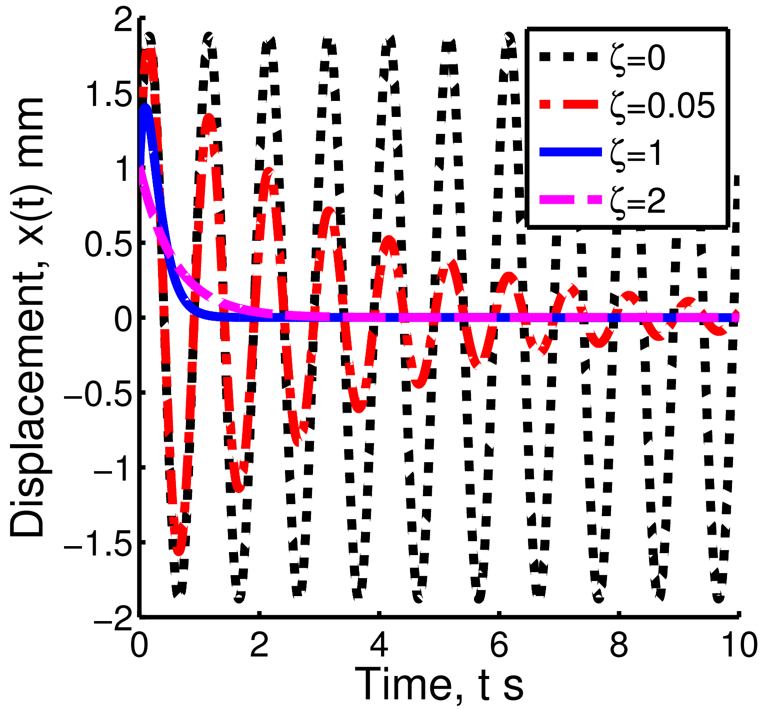
$$\boxed{x = x_h = e^{-\zeta\omega_n t}\left[\frac{\omega_d x_o + \zeta\omega_n x_0 + \dot{x}_0}{2\omega_d} e^{\omega_d t} + \frac{\omega_d x_o - \zeta\omega_n x_0 - \dot{x}_0}{2\omega_d} e^{-\omega_d t}\right]. \qquad (23)}$$

Note that the **definition of ω_d has changed for the overdamped case**. Furthermore, the vibrations of a overdamped system is aperiodic.

2.15.4 A Comparison of the Three Cases

The response for the three cases is sketched in Fig.(2.17). It is worthwhile to examine these critically and compare them.

(A)



(B)

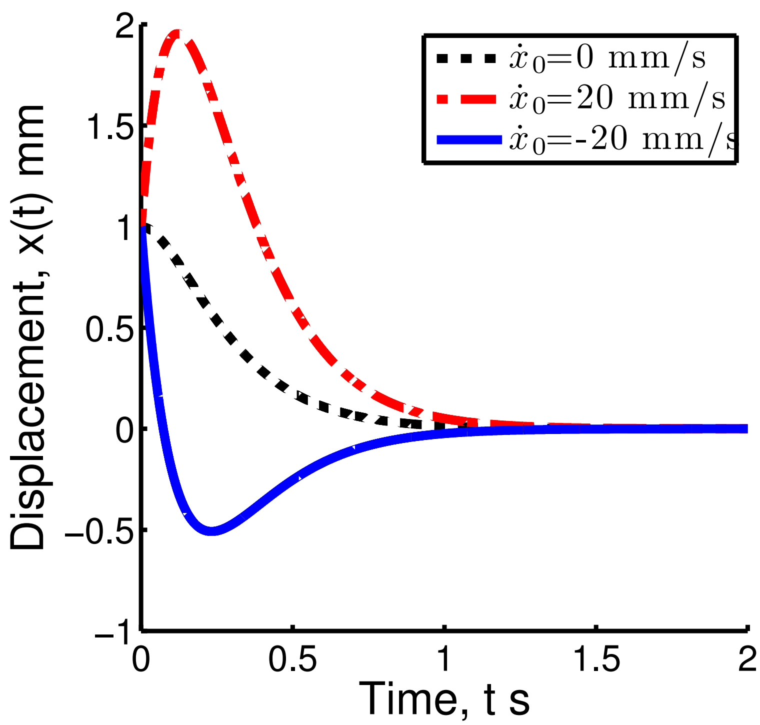


Figure 2.17: (A): A comparison of undamped ($\zeta = 0$), underdamped ($\zeta = 0.05$), critically damped ($\zeta = 1$), and overdamped ($\zeta > 1$) free vibration responses. The sdof system is subjected to an initial displacement $x_0 = 1$ mm and initial velocity $\dot{x}_0 = 10$ mm/s. (B): The response of a critically damped system subjected to different initial velocities, for the same initial displacement $x_0 = 1$ mm.

Question: What features do you observe in the responses of the damped systems sketched in Fig.(2.17)?

Fill in the class

2.15.5 Logarithmic Decrement

The salient feature of an underdamped system's response is that it decays exponentially with time, for viscous damping.[2] Larger is damping, faster is the decay. By measuring the displacement amplitude between two successive cycles (or N cycles apart) we can *measure* damping of a mechanical system. Logarithmic decrement is a time-domain measure of damping, which you will use in the Shaky table experiment. It is denoted by δ and can be defined as the natural logarithm of the ratio of two successive amplitudes, measured at one damped time period apart. The relation between δ and ζ can be obtained from Eq.(18) as follows.

$$
\begin{aligned}
x_1 &= x(t) = Ae^{-\zeta\omega_n t}\cos(\omega_d t - \phi_0) \\
x_2 &= x(t+T_d) = Ae^{-\zeta\omega_n (t+T_d)}\cos(\omega_d (t+T_d) - \phi_0)\,, \quad T_d = \frac{2\pi}{\omega_d} \\
\Rightarrow x_2 &= x(t+T_d) = Ae^{-\zeta\omega_n (t+T_d)}\cos(\omega_d t + 2\pi - \phi_0) = Ae^{-\zeta\omega_n (t+T_d)}\cos(\omega_d t - \phi_0) \\
\therefore \delta &= \ln\frac{x_1}{x_2} = \frac{Ae^{-\zeta\omega_n t}\cos(\omega_d t - \phi_0)}{Ae^{-\zeta\omega_n (t+T_d)}\cos(\omega_d t - \phi_0)} = \ln\frac{e^{-\zeta\omega_n t}}{e^{-\zeta\omega_n (t+T_d)}} \\
&= \ln e^{-\zeta\omega_n t} - \ln e^{-\zeta\omega_n (t+T_d)} = \zeta\omega_n T_d = \zeta\omega_n\frac{2\pi}{\omega_d} \\
&= \zeta\omega_n\frac{2\pi}{\omega_n\sqrt{1-\zeta^2}} = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}} \approx 2\pi\zeta
\end{aligned}
\tag{24}
$$

We can *measure* x_2 at any integer number of damped cycles after we measure x_1 *i.e.*, at $t + NT_d$ as x_{1+N}. In this case you can verify that $\delta = \frac{1}{N}\ln\left(\frac{x_1}{x_{1+N}}\right)$.

[2] Note that the decay is *exponential* for viscously damped systems only. For Coulomb damping the decay is *linear*. In general, we do not know *a priori* whether damping is viscous or not in any mechanical system. A useful check is to look at the decay envelope and see whether it is exponential or not!

Thus, we have the important result:

$$\boxed{\delta = \frac{1}{N} \ln\left(\frac{x_1}{x_{1+N}}\right) = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}} \approx 2\pi\zeta = \frac{2\pi c}{2m\omega_n}. \quad (25)}$$

Example 15 : From the consideration of work performed in harmonic motion show that viscous damper dissipates energy over one cycle while an elastic spring does not. Sketch the rotating vector representation of the underdamped free vibrations. **Fill in the class**

Solution:

Example 16: A shock absorber is to be designed to limit its overshoot to 15% of its initial displacement when released. Find the damping ratio ζ_0 required. What will be the overshoot if ζ is made equal to (a) $\frac{3}{4}\zeta_0$, and (b) $\frac{5}{4}\zeta_0$ **Fill in the class**

Solution:

Example 17: (T 2.91) A railroad car of mass 2000 kg travelling at a velocity of $v = 10$ m/s is stopped at the end of the tracks by a spring-damper system, as shown below. If the stiffness of the spring is $\frac{k}{2} = 40$ N/mm and the damping constant is $c = 20$ N-s/mm, determine (a) the maximum displacement of the car after engaging the springs and damper and (b) the time taken to reach the maximum displacement.

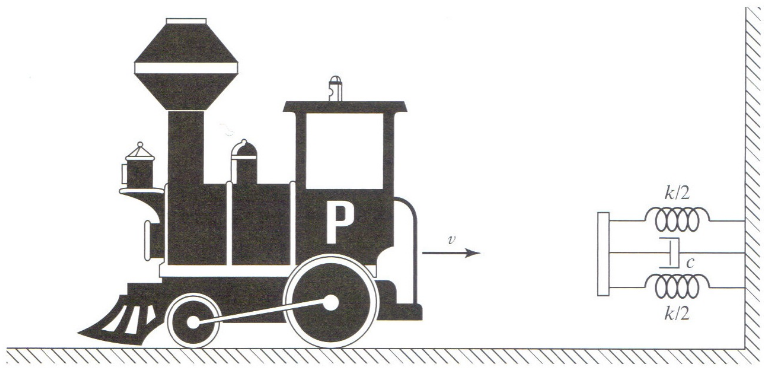


Figure 2.18: Figure for example 17.

Fill in the class

Solution:

Let us summarise the main learning points from the preceding discussions.

1. There are three categories of damped systems: underdamped, overdamped, and critically damped. Among these three, most of the mechanical systems belong to the underdamped category.

2. Critical damping is engineered when a quick return to the initial configuration in the shortest possible time is desired, such as in recoil mechanisms and shock absorbers.

3. With the passage of time over and critically damped systems gradually turn into underdamped systems.

4. Overdamping does not allow any oscillation but the return to equilibrium is a slow and creeping process.

5. Underdamped response is given by:
   $x = x_h = e^{-\zeta\omega_n t} A\cos(\omega_d t - \phi_0)$
   $\tan\phi_0 = \frac{\zeta\omega_n x_0 + \dot{x}_0}{\omega_d x_0}; A = \sqrt{x_0^2 + \left[\frac{\zeta\omega_n x_0 + \dot{x}_0}{\omega_d}\right]^2}$
   $x = x_h = e^{-\zeta\omega_n t}\left[x_0\cos\omega_d t + \frac{\zeta\omega_n x_0 + \dot{x}_0}{\omega_d}\sin\omega_d t\right]$
   $\omega_d = \omega_n\sqrt{1-\zeta^2}$ (Undamped natural frequency)

6. Logarithmic decrement is a time-domain measure of damping. It is given by:
   $\delta = \frac{1}{N}\ln\left(\frac{x_1}{x_{1+N}}\right) = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}} \approx 2\pi\zeta = \frac{2\pi c}{2m\omega_n}$.

2.16 Harmonically Forced Vibration Response (T 3.4+Notes)

Having seen the free vibration response characteristics, let us determine the forced vibration response of a viscously damped system. We restrict this discussion to harmonic forcing of the form $f(t) = f_0 \cos \omega t$. Our interest lies in the particular solution of the second oder ODE:

$$m\ddot{x}_p + c\dot{x}_p + kx_p = f = F_0 \cos \omega t. \tag{26}$$

We can determine the amplitude X and phase lag ϕ of the particular solution, taken to be

$$x_p(t) = X \cos(\omega t - \phi) \tag{27}$$

using the algebraic or graphical methods. Let us solve for the two unknowns using the rotating vector diagram methods.

Example 18: Use the rotating vector method to determine X and ϕ of the steady state or particular solution response of a system governed by Eq.(26). **Fill in the class**

Solution:

Thus we have the following result:

$$\boxed{\begin{aligned} x_p(t) &= X\cos(\omega t - \phi) \\ X &= \frac{F_0}{\sqrt{\left(k - m\omega^2\right)^2 + (c\omega)^2}}, \tan\phi = \frac{\omega c}{k - m\omega^2} \end{aligned} \quad (28)}$$

The total response is given by

$$\boxed{\begin{aligned} x(t) &= e^{-\zeta\omega_n t}\left[C_1\cos\omega_d t + C_2\sin\omega_d t\right] + X\cos(\omega t - \phi), \\ X &= \frac{F_0}{\sqrt{\left(k - m\omega^2\right)^2 + (c\omega)^2}}, \tan\phi = \frac{\omega c}{k - m\omega^2} \end{aligned} \quad (29)}$$

The two unknown constants C_1 and C_2 are to be determined from the initial conditions.

We can define the Dynamic Magnification Factor (DMF) for the damped system, similar to the undamped system, in the steady state as follows:

$$\begin{aligned} |DMF| = M = \left|\frac{X}{\delta_{st}}\right| &= \frac{F_0}{\delta_{st}\sqrt{\left(k - m\omega^2\right)^2 + (c\omega)^2}} \\ &= \frac{1}{\sqrt{(1 - r^2)^2 + (2\zeta r)^2}}, \quad r = \frac{\omega}{\omega_n} \\ \phi &= \tan^{-1}\left[\frac{2\zeta r}{1 - r^2}\right] \end{aligned} \quad (30)$$

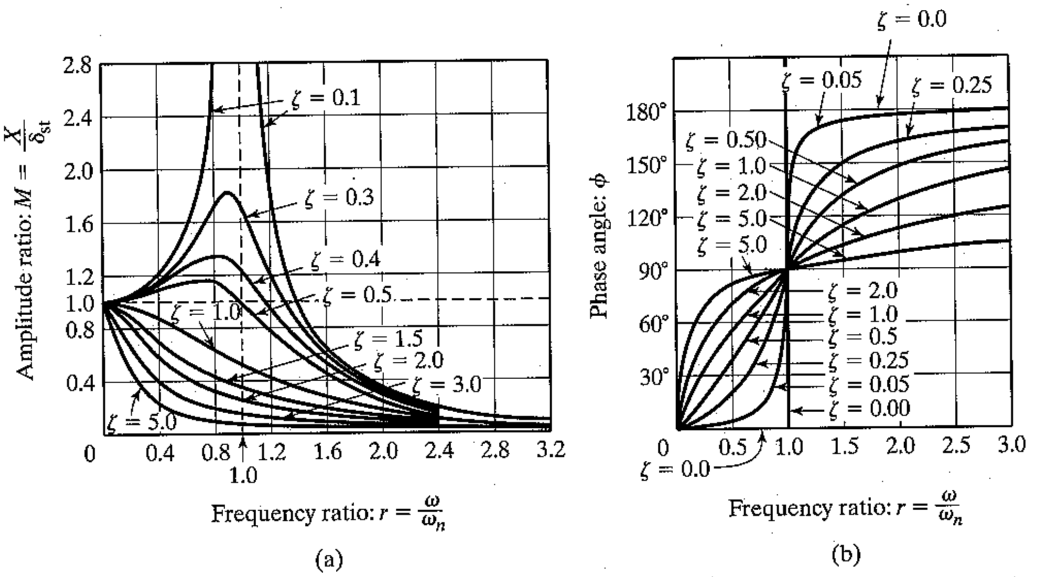


Figure 2.19: DMF curves for viscously damped SDOF system.

DMF curves for different levels of damping are portrayed in Fig.(2.19). The following are worth noting.

1. Damping has pronounced effect around resonance in reducing the value of the magnification factor M. It does decrease the response at other frequencies, but not as effectively.

2. For a constant, or static, force $r = 0$ and $M = 1$.

3. The maximum magnification occurs slightly below the undamped natural frequency $\omega < \omega_n$, or $r < 1$ at $r = \sqrt{1 - 2\zeta^2}$ and is given by $M_{max} = \left|\frac{X}{\delta_{st}}\right|_{max} = \frac{1}{2\zeta\sqrt{1-2\zeta^2}}$.

4. At the undamped natural frequency, $\left|\frac{X}{\delta_{st}}\right|_{\omega=\omega_n} = \frac{1}{2\zeta}$

5. For frequencies well above resonance the DMF curves for different levels of ζ for an underdamped system

show small but insignificant differences in M, suggesting that damping is most effective around resonance.

6. For overdamped systems M decreases monotonically with r.

7. The phase lag of the response starts at 0^0 (in-phase) for $r = 0$ and gradually increases to a value of 90^0 at $r = 1$. Eventually, well above resonance $r >> 1$, the phase lag is 180^0 (out-of-phase). Notice that the phase change around $r = 1$ is gradual and not as abrupt as in the case of an undamped system $\zeta = 0$.

Question: How do we measure DMF curve of a practical system such as a machine tool?

Fill in the class

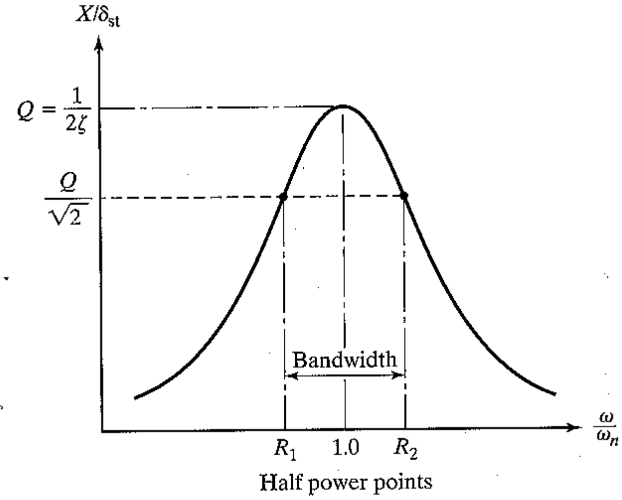


Figure 2.20: Q factor interms of half power bandwidth (HPBW).

We can also estimate damping from the DMF curves by measuring the half-power-bandwidth (HPBW) points, shown in Fig.(2.21). We are measuring the response now in the **frequency domain**, in the steady state. This is in contrast to logarithmic decrement, wherein we measured the transient, free vibration response in the **time domain.**
The half-power frequencies are the frequencies at which the vibration power is half the power observed at $r = 1$, see Fig.(2.21). Systems with low damping exhibit peaky and narrow DMF curves, while systems with high damping show flatter and wider DMF curves. Q factor or Quality factor is another measure of damping (commonly used by Electronics Engineers). Q is defined as follows:

$$Q \approx \frac{1}{2\zeta} \approx \frac{\omega_n}{\Delta\omega}, \quad \Delta\omega = \omega_2 - \omega_1 \text{ (HPBW).} \tag{31}$$

Example 19 (T 3.35 open problem) : The landing gear of an airplane can be idealised as the spring-mass-damper system shown below. If the runway surface is described by $y(t) = y_0 \cos \omega t$, determine the equations of motion and steady damped vibration response. What design criteria will you use to select the k and c?

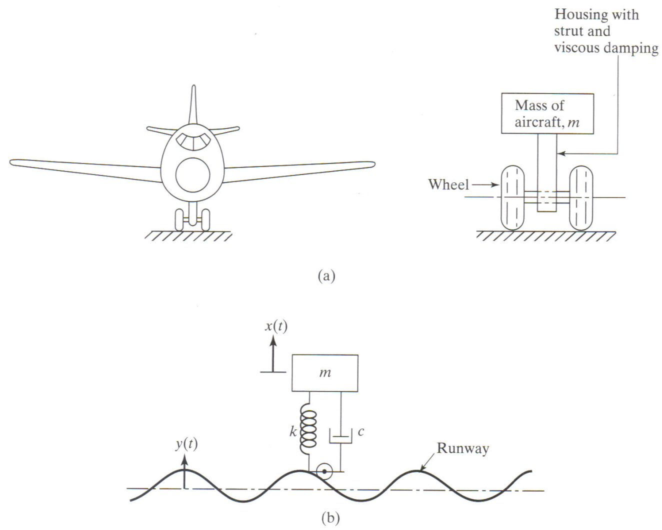


Figure 2.21: Figure for example 19.

Fill in the class

Solution:

Let us summarise the important findings of this topic:

1. There are three categories of damped systems: underdamped, overdamped, and critically damped. Among these three, most of the mechanical systems belong to the underdamped category. With the passage of time all damped systems gradually turn into underdamped systems.

2. Critical damping is engineered when a quick return to the initial configuration in the shortest possible time is desired, such as in recoil mechanisms and shock absorbers.

3. Overdamping does not allow any oscillation but the return to equilibrium is a slow and creeping process.

4. Underdamped free vibration response is given by:
   $x = x_h = e^{-\zeta\omega_n t} A\cos(\omega_d t - \phi_0)$
   $\tan\phi_0 = \frac{\zeta\omega_n x_0 + \dot{x}_0}{\omega_d x_0}; A = \sqrt{x_0^2 + \left[\frac{\zeta\omega_n x_0 + \dot{x}_0}{\omega_d}\right]^2}$
   $x = x_h = e^{-\zeta\omega_n t}\left[x_0\cos\omega_d t + \frac{\zeta\omega_n x_0 + \dot{x}_0}{\omega_d}\sin\omega_d t\right]$
   $\omega_d = \omega_n\sqrt{1-\zeta^2}$ (Damped natural frequency)

5. Logarithmic decrement is a time-domain measure of damping, obtained from transient, free vibration response. It is given by:
   $\delta = \frac{1}{N}\ln\left(\frac{x_1}{x_{1+N}}\right) = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}} \approx 2\pi\zeta = \frac{2\pi c}{2m\omega_n}$.

6. The total response of an underdamped SDOF system is given by:
   $x(t) = e^{-\zeta\omega_n t}\left[C_1\cos\omega_d t + C_2\sin\omega_d t\right] + X\cos(\omega t - \phi)$,
   $X = \frac{F_0}{\sqrt{(k-m\omega^2)^2 + (c\omega)^2}}, \tan\phi = \frac{\omega c}{k - m\omega^2}$

7. The DMF of a viscously damped system in the steady state is given by:
   $|DMF| = M = \left|\frac{X}{\delta_{st}}\right| = \frac{F_0}{\delta_{st}\sqrt{(k-m\omega^2)^2+(c\omega)^2}} = \frac{1}{\sqrt{(1-r^2)^2+(2\zeta r)^2}}, \quad r = \frac{\omega}{\omega_n}$
   $\phi = \tan^{-1}\left[\frac{2\zeta r}{1-r^2}\right]$

8. The maximum magnification occurs slightly below the undamped natural frequency $\omega < \omega_n$, or $r < 1$ at $r = \sqrt{1-2\zeta^2}$ and is given by $M_{max} = \left|\frac{X}{\delta_{st}}\right|_{max} = \frac{1}{2\zeta\sqrt{1-2\zeta^2}}$. At the undamped natural frequency, $\left|\frac{X}{\delta_{st}}\right|_{\omega=\omega_n} = \frac{1}{2\zeta}$

9. The phase lag of the response starts at 0^0 (in-phase) for $r = 0$ and gradually increases to a value of 90^0 at $r = 1$. Eventually, well above resonance $r >> 1$, the phase lag is 180^0 (out-of-phase).

10. Quality factor or Q-factor is a frequency-domain measure of damping, obtained from steady state forced vibration response. It is related to ζ via: $Q \approx \frac{1}{2\zeta} \approx \frac{\omega_n}{\Delta\omega}, \quad \Delta\omega = \omega_2 - \omega_1$ (HPBW).

Vibration Isolation

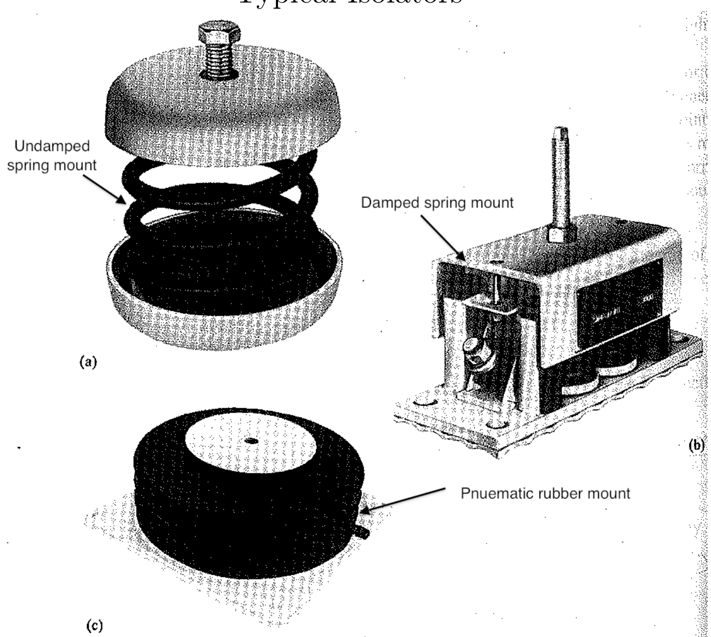
(Course Objective# 3)

Learning objectives

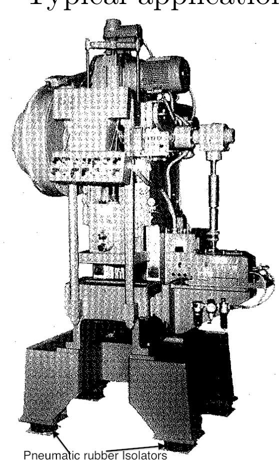
1. **Know** the working principles involved in vibration isolation.
2. **Apply** SDOF vibration theory to deduce vibration transmissibilities.
3. **Design** simple SDOF isolators.
4. **Appreciate** the design trade-offs.

SDOF: Single-Degree-Of-Freedom

Typical Isolators



Typical application



We have seen how single degree of freedom (SDOF) models are developed; how response of such systems to free and forced vibrations with or without viscous damping is determined. This topic will apply the SDOF theory to design the first vibration counter measure, namely, vibration isolation. We shall restrict to SDOF systems in a *steady state* vibration.

1 Introduction

The main goal for this topic is to understand and apply SDOF vibration isolation theory. So far, we have completed the following three steps in vibration *analysis*: developing a SDOF model; formulation of its equations of motion; and solving the equations of motion for response. The final, and perhaps the most crucial, step is concerned with using this knowledge to *fix* vibration problems or *design against* vibrations. As we have already seen, vibrations can be transient or steady. In transient vibration problems our interest may be to restrict the overshoot (**see Example 16**), which we achieved by adjusting the damping ratio ζ. Reducing steady state vibration is the focus of this topic. **We shall only be concerned with the steady state** (particular solution), under the assumption that *sufficient time has elapsed for transients to have decayed.*

Henceforth, we are dealing with steady-state vibration.

Broadly speaking, we can mitigate vibration problems by taking one or more of the following routes.

1. Appropriate design of restoring (spring) and inertial (mass) elements of a mechanical system. Here, our goal is to avoid resonant frequencies by suitably choosing the spring constants and mass.

2. **We can isolate the source of vibration from the sensitive components we wish to protect**. In the case of a car, vibration from unbalance in engines and from the roughness of the road can be prevented from being transmitted into the passenger cabin. We achieve this using vibration isolators. **We shall study isolation system design in this topic**.

3. Equally applicable is the notion of absorbing vibration energy from a vibrating component. This can be accomplished by channelling away the energy into a secondary device, such as a vibration absorber, effective at selected tuned frequencies. We shall study vibration absorbers later.

4. Somewhat related to the above ideas is the notion of applying an external force (using actuators) to counter vibrations. This method of active vibration control is outside our scope as a detailed knowledge of control theory and actuator and sensor technologies is needed.

Question: Can you list a few situations, other than the car example, where vibration isolation may be useful?

Fill in the class

1.1 SDOF Isolation (T 9.10+Notes)

The goal of vibration isolation is to reduce the transmitted vibration from a vibrating source to other parts of the system, by **decreasing** the natural frequency of the whole system compared to the disturbing force frequency. This is accomplished by inserting resilient elements (spring mounts, damped spring mounts, pneumatic rubber mounts) between the source of vibration and the object to be protected.

The effectiveness of an isolation system is quantified using a non-dimensional parameter called Transmissibility. It can be defined as a ratio of transmitted to applied force, or transmitted to applied displacement. Thus we have

$$TR = \frac{F_t}{F} \text{ (force transmissibility)}; \; TR_d = \frac{X}{Y} \text{ (displacement transmissibility)} \tag{1}$$

where F_t, F, X and Y are respectively, transmitted force, applied force, transmitted displacement, and applied displacement.

Note that we can obtain general expressions for the TR values of SDOF systems from what we have learned so far. We will do this in example 1, shortly. Before that, let us examine the TR critically.

Question: What are the assumptions inherent in the definition of TR or TR_d? Describe situations where one is a better measure than the other. What is the ideal value for a TR?

Fill in the class

Question: Which of the following designs *may* be better? Why?

Fill in the class

Example 20 : Show that the force and displacement transmissibility of a SDOF system is given by $TR = \frac{F_t}{F} = \frac{\sqrt{k^2+(c\omega)^2}}{\sqrt{(k-m\omega^2)^2+(c\omega)^2}} = \frac{\sqrt{1+(2\zeta r)^2}}{\sqrt{(1-r^2)^2+(2\zeta r)^2}}$ (force transmissibility) ; and $TR_d = \frac{X}{Y} = \frac{\sqrt{1+(2\zeta r)^2}}{\sqrt{(1-r^2)^2+(2\zeta r)^2}}$, where $r = \frac{\omega}{\omega_n}$. **Fill in the class**

Solution:

Example 21 : Using the result from example 20 discuss the transmissibility characteristics of an undamped spring mounting; damped spring mounting and a rigid connection of a machinery to a foundation. **Fill in the class**

Solution:

The key results from examples 20 and 21 can be summarised in the following picture.

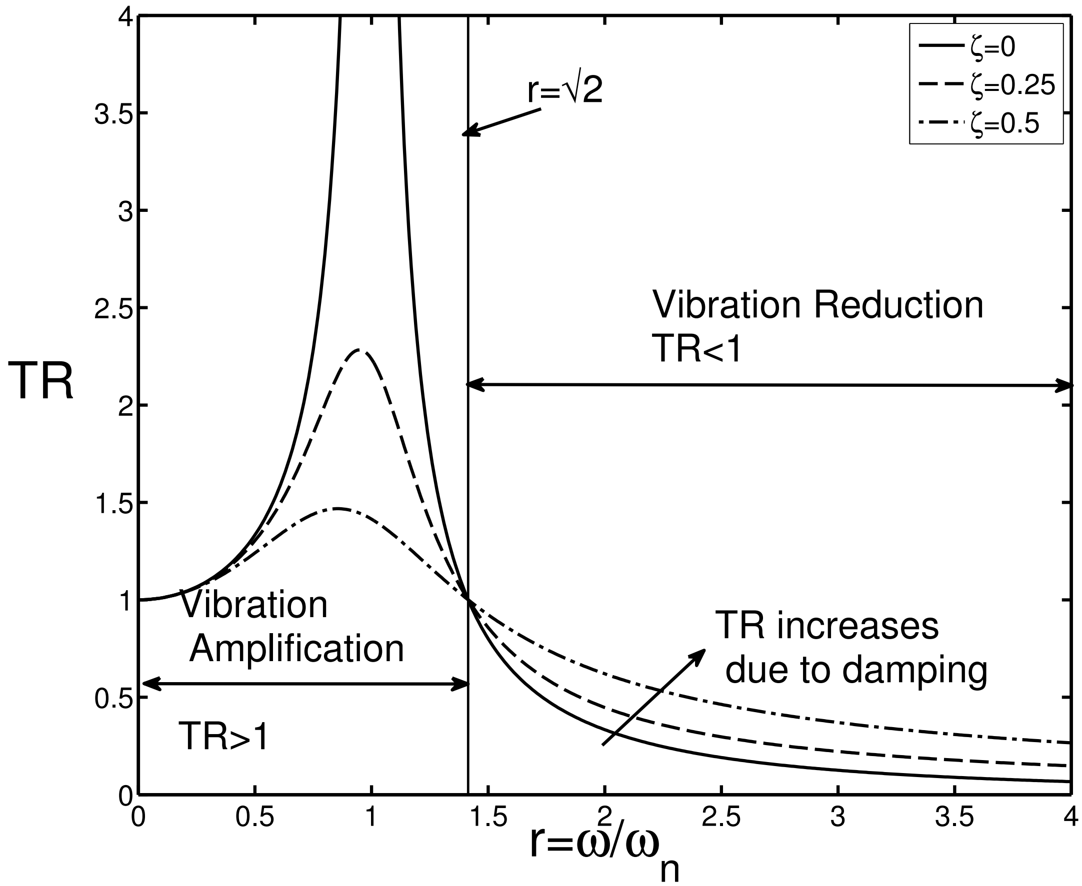


Figure 22: Transmissibility of a SDOF system.

The following are worth noting

1. For a given operating frequency, say ω, we can choose $r = \frac{\omega}{\omega_n}$ in the above curves by suitably choosing the isolator's spring constant k. If we choose k such that $r > \sqrt{2}$, that is, the natural frequency of the combined system is well below the forcing frequency, then we obtain $TR < 1$ or less force is transmitted. Thus we obtain isolation by making the dynamics of the whole

system with the isolator slow compared to the forcing frequency ω.

2. If spring constants are chosen such that $r < \sqrt{2}$, then vibration is amplified and hence more force is transmitted.

3. The above two points suggest that softer springs are better and stiffer springs are undesirable. Thus rigid mounting is the worst case in terms of force transmissibility.

4. Maximum transmissibility occurs around $r \approx 1$. A spring mounted system (or stiffness only isolator) has infinite force transmissibility around $r \approx 1$. Damping helps to reduce the TR dramatically around resonance.

5. There is an increase in the TR value above $r = \sqrt{2}$ due to the damper.

Question: Discuss the practical consideration in the choice of k and damping ζ of an isolator? How will you use the TR curve in practice? **Fill in the class**

Example 22 : (T 9.27 modified) An electronic instrument panel is to be isolated from a panel that vibrates at frequencies ranging from 25 Hz to 35 Hz. It is desired to have at least 80% vibration isolation in order to prevent damage to the instrument. If the instrument has a weight of 85 N, determine the necessary stiffness of a spring mounting. What is the limitation of this design? **Fill in the class**

Solution:

Example 23 : (T 9.28) An exhaust fan, having a small unbalance, weighs 800 N and operates at a speed of 600 rpm. It is desired to limit the maximum transmissibility to 2.5 throughout the operation of the fan including the start-up. In addition, an isolation of 90% is to be achieved at the operating speed. Design the isolator for the fan.

Fill in the class

Solution:

We draw the following conclusions from the study of this topic.

1. Isolation is a measure used to reduce the transmitted displacements and forces from a source of vibration in a mechanical system. It is quantified by a non-dimensional parameter called transmissibility (TR). An ideal isolator has $TR = 0$.

2. The force and displacement transmissibility of a SDOF system is given by $TR = \frac{F_t}{F} = \frac{\sqrt{k^2+(c\omega)^2}}{\sqrt{(k-m\omega^2)^2+(c\omega)^2}} = \frac{\sqrt{1+(2\zeta r)^2}}{\sqrt{(1-r^2)^2+(2\zeta r)^2}}$; and the displacement transmissibility is given by $TR_d = \frac{X}{Y} = \frac{\sqrt{1+(2\zeta r)^2}}{\sqrt{(1-r^2)^2+(2\zeta r)^2}}$, where $r = \frac{\omega}{\omega_n}$

3. Softer springs give lower ω_n and hence higher $r = \frac{\omega}{\omega_n}$ for a fixed forcing frequency ω. This takes the operating point above $r = \sqrt{2}$ giving isolation or $TR < 1$. However, the displacements are more. Thus the maximum allowed displacement sets the limit on how soft the spring in a isolator can be.

4. Damping increases TR above $r = \sqrt{2}$. However, this increase is tolerated as the system has to go through resonance in order to reach $r > \sqrt{2}$. Damping is desired to avoid resonance in the run-up to the operating point.

5. Note that $F = me\omega^2$ for a system subjected to a forcing due to a rotating unbalance.

Topic 2.6: General Excitation: Part 1

The learning objective for this topic is to evaluate the response of a SDOF system subjected to *any* type of forcing function. Our interest is in the particular solution, since the form of the homogeneous solution does not depend on the force. We start with harmonic response and extend it to step response, and impulse response. Finally, we will develop *convolution integral* method that can deal with any type of forcing, such as the roughness of a road that induces vibration in a car. In the second part, we shall study Fourier series to determine the response for a periodic forcing case.

2.13 Introduction

The objective of this topic is to evaluate the response of a SDOF system subjected to *any* type of forcing function. We have seen the total response of a SDOF system subjected to a **harmonic** forced vibration

$$m\ddot{x} + c\dot{x} + kx = f = F_0 \cos(\omega t) \tag{1}$$

is of the following form:

$$\begin{aligned} x(t) &= e^{-\zeta\omega_n t}\left[C_1 \cos\omega_d t + C_2 \sin\omega_d t\right] + X\cos(\omega t - \phi), \\ X &= \frac{F_0}{\sqrt{\left(k - m\omega^2\right)^2 + (c\omega)^2}}, \tan\phi = \frac{\omega c}{k - m\omega^2}, \quad \omega_d = \omega_n\sqrt{1-\zeta^2} \\ X &= \frac{\delta_{st}}{\sqrt{\left(1 - r\right)^2 + (2\zeta r)^2}}, \tan\phi = \frac{2\zeta r}{1 - r^2}, \quad r = \frac{\omega}{\omega_n}, \delta_{st} = \frac{F_0}{k} \end{aligned} \tag{2}$$

How do we determine the response if f is not harmonic?

Question: Which aspects of the solution will be different for a non-harmonic force?

Fill in the class

2.14 A Survey of Available Methods (Notes)

There are many routes to determine the particular solution. Let us survey these methods before getting into the details.

1. Approach 1: Use ODE theory. In particular, choose the particular solution depending on the force. In this

case we will resort to a dictionary of particular solutions depending on the force as shown in Fig.(2.24).

Particular integrals

For linear differential equations with constant coefficients:

Right-hand side	Trial P.I.
constant	a
x^n (n integer)	$a\,x^n + b\,x^{n-1} + \dots$
e^{kx}	$a\,e^{kx}$
$x\,e^{kx}$	$(a\,x + b)\,e^{kx}$
$x^n\,e^{kx}$	$(a\,x^n + b\,x^{n-1} + \dots)\,e^{kx}$
$\left.\begin{array}{l}\sin px\\ \cos px\end{array}\right\}$	$a \sin px + b \cos px$
$\left.\begin{array}{l}e^{kx}\sin px\\ e^{kx}\cos px\end{array}\right\}$	$e^{kx}(a \sin px + b \cos px)$

Figure 2.24: A dictionary of Particular Solutions for linear ODEs. Note that we replace x with t and k with ω for vibration problems, since time t is the independent variable.

2. Approach 2: Convert the force f into a sum of harmonic forces and use the principle of superposition from Topic 1. This requires harmonic analysis or Fourier analysis. In Fourier analysis we express the force in the following form $f(t) = \frac{a_0}{2} + \sum_{n=1}^{n=\infty} [a_n \cos n\omega t + b_n \sin n\omega t]$, $\omega = \frac{2\pi}{T}$, where T is the period of the force. This method works for periodic forces and periodisable forces. We treat each term in the Fourier series as a force and find

the associated response. **Using the principle of superposition:** $x_p(t) = x_{a_0} + \sum_{n=1}^{n=\infty} [x_{a_n} + x_{b_n}]$**.**

3. Approach 3: Use superposition principle in the most general possible way. This leads to the convolution integral. This can deal with any forcing. Here, we break up any time series representing the force (periodic or not) in to a sum of shifted impulse functions (or Dirac delta functions). The response of the system due to the original force is given by the sum of the responses of the system, calculated for each shifted impulse. If $h(t)$ is the impulse response then the response for any force $f(t)$ is: $x_p(t) = \int_{t=0^+}^{t} h(t-\tau) f(\tau) d\tau$

Question: List the advantages and drawbacks of each of the above methods?

Fill in the class

It is worth remembering the following map, which underscores the importance of harmonic response!

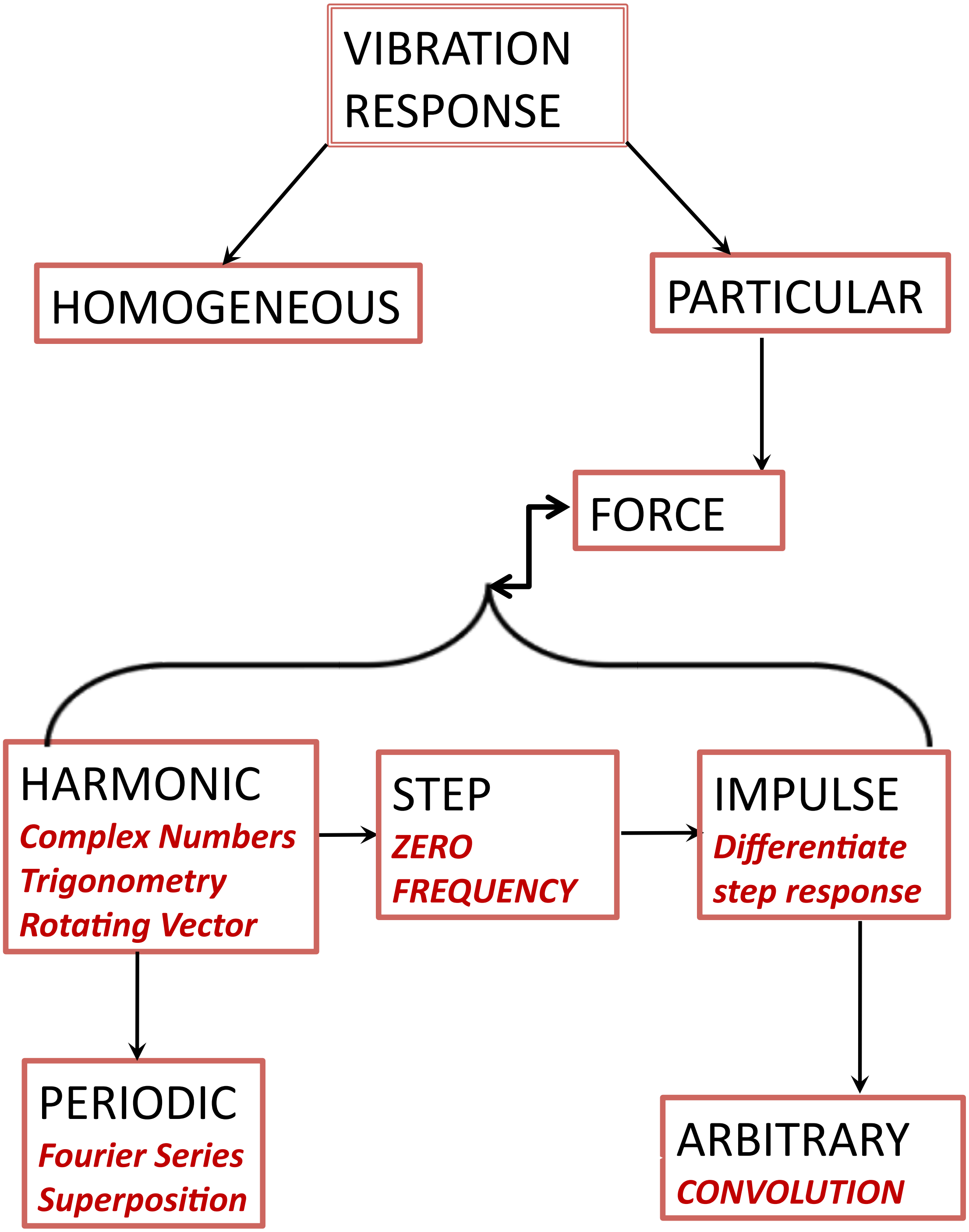


Figure 2.25: Relationship among different forced vibration responses.

The following diagrams illustrate the power of superposition principle. With some ingenuity, we can extend the step, and ramp responses to different cases, much like we extended the harmonic response to periodic forces using Fourier series and superposition principle!

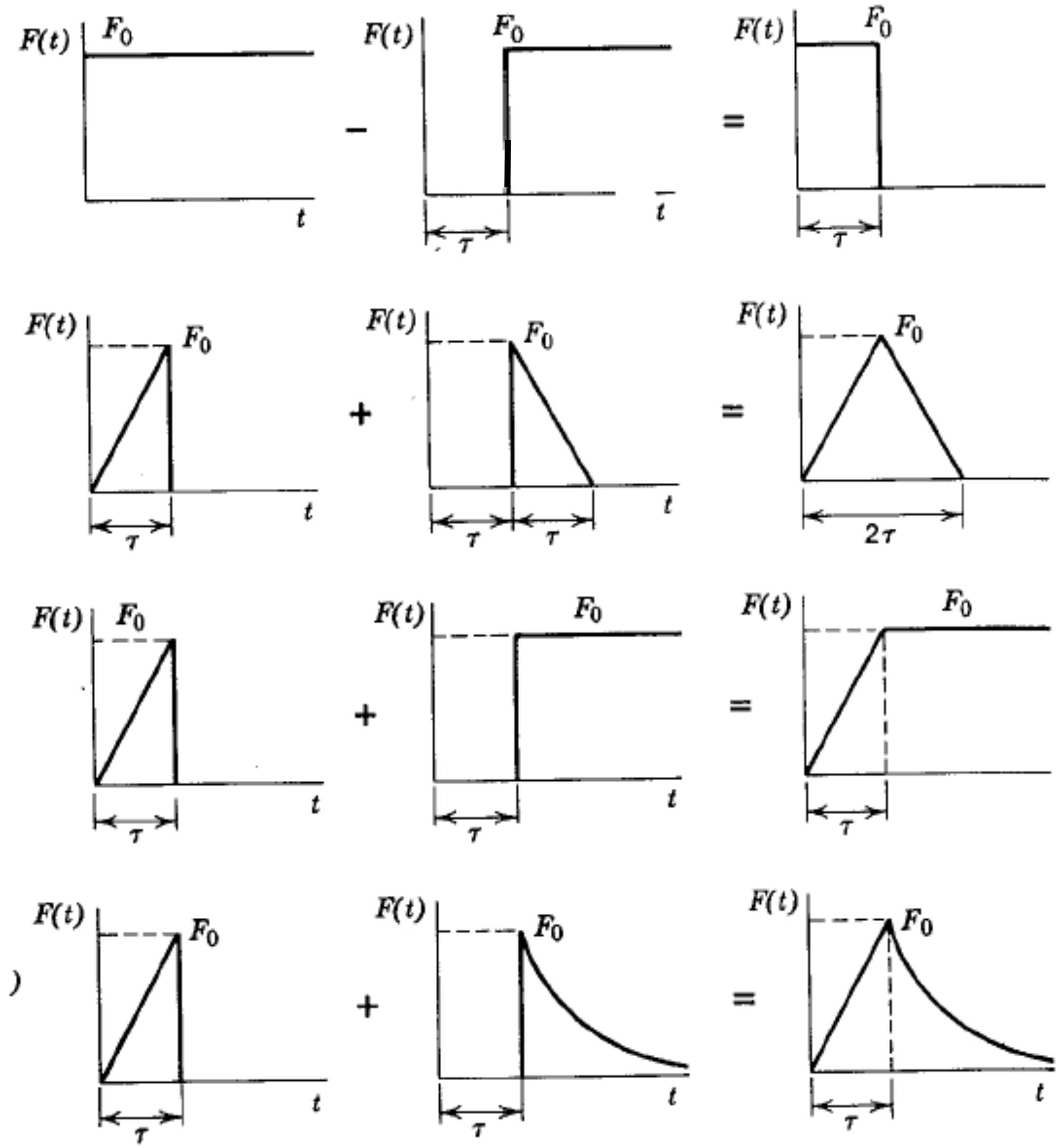


Figure 2.26: Superposition is a powerful principle for linear systems.

In the first part we shall elaborate on the third approach. We shall return to Fourier series, which can be applied to periodic or periodisable forcing functions, in the second part.

We shall focus on two responses of particular interest: (a) step response and (b) impulse response. These are the two essential non-harmonic forcing cases of wide interest in vibrations and control system design. A somewhat less common is the ramp response to the forcing of the form $f(t) = Ft$.

2.15 Step Response (T4.5+Notes)

Following the map presented earlier in Fig.(2.25), we can evaluate the step response from the harmonic response by setting $\omega = 0$:

$$\begin{aligned} m\ddot{x} + c\dot{x} + kx = f = F_0 = [F_0 \cos(\omega t)]_{\omega=0} \\ x(t) = e^{-\zeta\omega_n t}\left[C_1 \cos\omega_d t + C_2 \sin\omega_d t\right] + \frac{F_0}{k} \\ C_1 = x_0 - \frac{F_0}{k}; \quad C_2 = \frac{\dot{x}_0 + \zeta\omega_n\left[x_0 - \frac{F_0}{k}\right]}{\omega_d} \end{aligned} \tag{3}$$

Fill in the class

Note that C_1 and C_2 were found from the initial conditions imposed on the TOTAL response. This must always be followed.

The unit step response $F_0 = 1$ tells us three things:

(i) Rise time: Time taken from 10% to 90% of final value (0.1 to 0.9)

(ii) Settling time: Response reaches within 1% or 5% of final value (0.99 or 0.95).

(iii) Overshoot: % of final value by which the response rises initially.

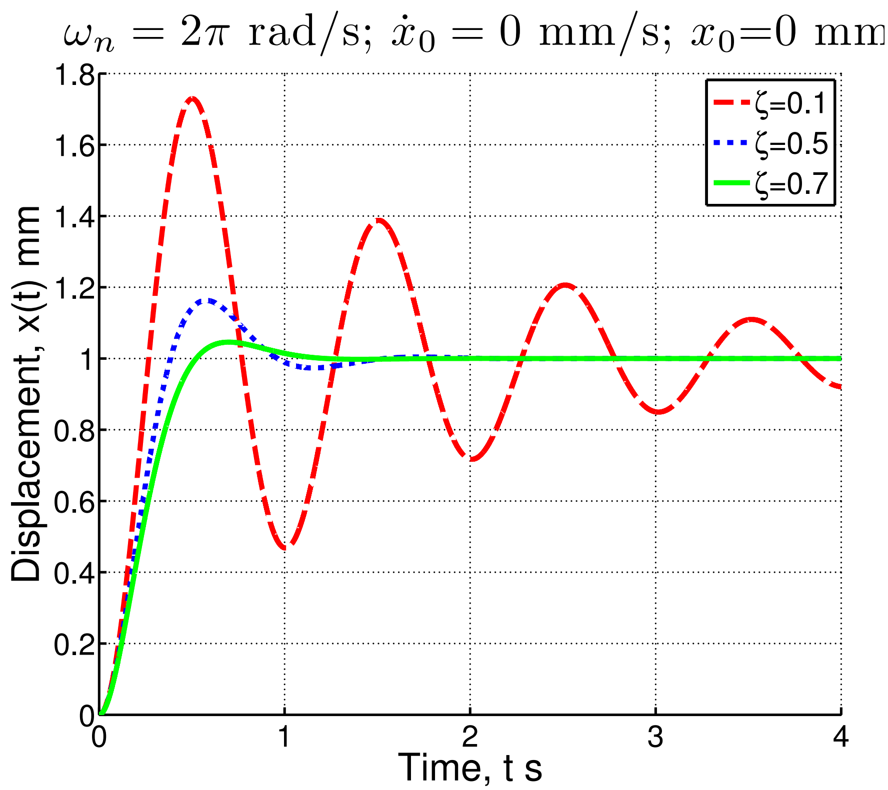
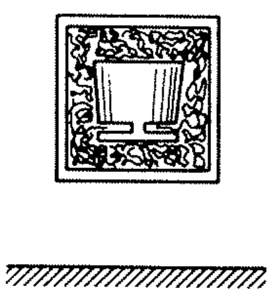


Figure 2.27: Unit step response of a SDOF system, initially at rest, for different damping ratios. Notice that higher is the damping ratio faster is the settling time, lower overshoot, but increased rise time!

It can be concluded from the unit step response that while high damping is desired in order to minimise settling time and overshoot, it may result in a lower rise time. Issues such as this are of interest in designing manipulators and feedback control systems.

Returning to vibrations, we have already seen step forces in Bungee jumping (Tutorial 3), Midterm 2 question (seat design), Cushion design (Example 12, Topic 2.3). There, we side stepped the questions of determining step response, by choosing Static equilibrium (compressed spring position) as our reference reference. We will solve one of these problems now with unstretched configuration as our reference.

Example 24 : (Example 12 revisited) A 5-kg fragile glass vase is packed in chopped sponge rubber and placed in a cardboard box that has negligible mass. It is then accidentally dropped from a height of 1m. This particular sponge rubber exhibits the force-deflection curve sketched below. Determine the maximum acceleration experienced by the vase.



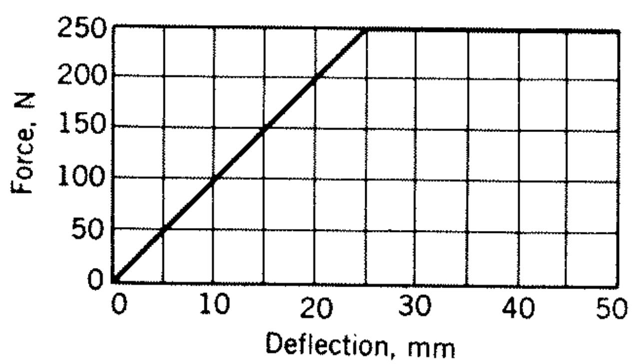


Figure 2.28: Figure for example 24.

Solution:

Fill in the class

Example 25 : In ejection seat experiments, the torso is modelled as a spring and mass system. The head is a single mass weighing 5.44 kg. It is supported by the spinal column, with an elastic modulus of 87.56 N/mm. If ejection follows the acceleration time curve shown, determine the peak acceleration of the head. Does it match the experimental result?

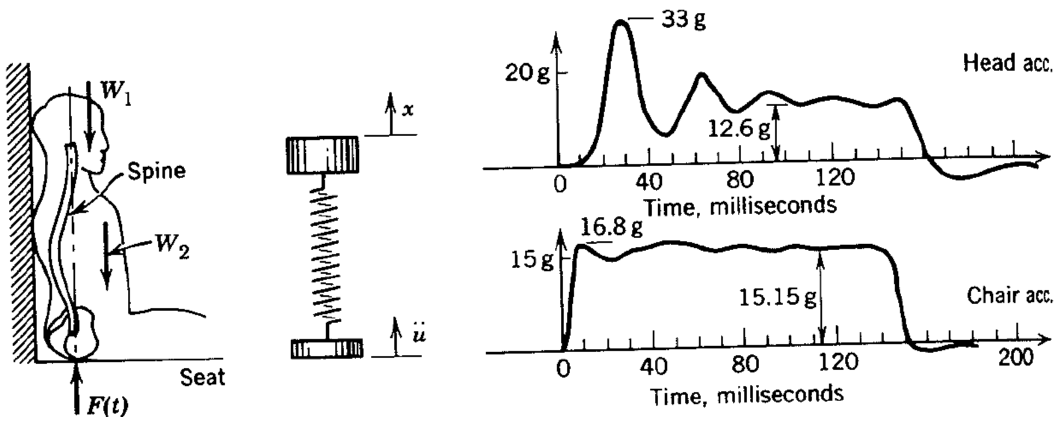


Figure 2.29: Figure for example 25.

Solution:

Fill in the class

2.16 Impulse Response (T4.5+Notes)

A large force applied over a small interval of time is impulse. Units of impulse are N-s or kg-m/s. A unit impulse has a magnitude of 1 N-s. Mathematically, an ideal impulse function is denoted by $\delta(t)$ and it has the following properties

$$\int_0^\infty \delta(t)dt = 1 \quad \int_0^\infty \delta(t-t_0)f(t)dt = f(t_0) \qquad (4)$$

An impulse forcing function can be defined by shrinking a rectangular pulse of width δt and height $\frac{1}{\delta t}$, in other words of unit area, to an infinitesimally small interval of time: $\delta t \to 0$.

Impulse response can be found using two approaches. In the first approach, we use impulse-momentum theorem that follows from Newton's second law. This gives us an SDOF system subjected to initial velocity. A second approach would be to follow the map in Fig.(2.25), differentiate step response for a system at rest: zero initial displacement and velocity. Do this on your own as an exercise. We will use the Impulse-momentum theorem now.

Example 26 : Using impulse-momentum theorem show that the response of a SDOF at rest subjected to unit impulse is of the form $h(t) = \frac{1}{m\omega_d} e^{-\zeta\omega_n t} \sin \omega_d t$. **Fill in the class**

Solution:

Example 27 : Using the unit impulse response $h(t) = \frac{1}{m\omega_d} e^{-\zeta\omega_n t} \sin \omega_d t$, and the principle of superposition, show that the response of a SDOF system subjected to arbitrary force $f(t)$ is given by the integral $x_p(t) = \int_{t=0^+}^{t} h(t-\tau) f(\tau) d\tau$. Discuss the role of initial conditions.

Solution:

Fill in the class

Let us summarise the main findings of this part:

1. Harmonic response is the fundamental response. Setting $\omega = 0$ in the harmonic response gives the step response. Differentiating step response gives impulse response. Convolution gives reponse for any force.

2. The step response is given by

$$m\ddot{x} + c\dot{x} + kx = f = F_0 = [F_0 \cos(\omega t)]_{\omega=0}$$

$$x(t) = e^{-\zeta\omega_n t}\left[C_1 \cos\omega_d t + C_2 \sin\omega_d t\right] + \frac{F_0}{k} \quad (5)$$

$$C_1 = x_0 - \frac{F_0}{k}; \quad C_2 = \frac{\dot{x}_0 + \zeta\omega_n\left[x_0 - \frac{F_0}{k}\right]}{\omega_d}.$$

Higher damping ratios ζ lead to lower overshoot, faster settling time, but result in slow rise time.

3. The impulse response of a SDOF system is given by $h(t) = \frac{1}{m\omega_d} e^{-\zeta\omega_n t} \sin\omega_d t$.

4. The response of a SDOF system to any arbitrary force is given by the convolution integral: $x_p(t) = \displaystyle\int_{t=0^+}^{t} h(t-\tau)f(\tau)d\tau$. This integral is usually evaluated numerically using a computer. Hand calculations are possible only for the simpler cases.

5. The homogenous response must be added to the above in order to determine the two unknown constants. Thus, we have for the most general case, the total response: $x(t) = e^{-\zeta\omega_n t}\left[C_1 \cos\omega_d t + C_2 \sin\omega_d t\right] + \int_{t=0^+}^{t} h(t-\tau)f(\tau)d\tau$. C_1 and C_2 must be found by imposing initial conditions on the total response $x(t)$.

In the second part, we shall use Fourier series for the periodic forcing.

Topic 2.6: General Excitation: Part 2 – Fourier Series

Periodic forces arise in many mechanical processes, such as a rotating machinery operating in steady state, a single cylinder reciprocating engine, and so on. Such forces need not be *harmonic*, or, pleasant to hear! Viewing a time dependent force of period T—with the property $f(t+T) = f(t)$—as a time series, we can decompose any complicated *periodic* force $f(t)$ into its consituent harmonics using harmonic analysis, or, Fourier analysis. The harmonics are evenly spaced with each harmonic separated from its neighbouring harmonic by a fixed frequency interval, given by $\omega_0 = \frac{2\pi}{T}$. Combining the Fourier series with the principle of superposition leads to determining the forced vibration response of SDOF system subjected to any periodic force. This, we do in this topic.

2.8 Introduction

The main goal for this topic is to evaluate the forced vibration response x_p of a viscously damped, SDOF system, subjected to a periodic force:

$$m\ddot{x} + c\dot{x} + kx = f(t), \quad f(t+T) = f(t), x_p(t) =? \tag{1}$$

If the forcing function is not periodic, but acts over a certain interval of time say $0 < t < t_0$, we can extend the forcing function as a periodic function *outside* the time interval. This is called periodising a function. We shall deal with this through an example.

The central idea for this topic is to use Fourier series *approx-*

imation in conjunction with the principle of superposition valid for linear systems. We begin with the Fourier series.

2.9 Fourier Series

Fourier series is concerned with approximating periodic, or *periodisable*, signals or functions over a finite interval. Periodicity can be in spatial domain, time domain, or both. Consider a function $f(t)$ of period T, it's Fourier series is given by

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{2n\pi t}{T}\right) + b_n \sin\left(\frac{2n\pi t}{T}\right)\right], \text{ or}$$

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\left(n\omega_0 t\right) + b_n \sin\left(n\omega_0 t\right)\right], \quad \omega_0 = \frac{2\pi}{T}$$

$$a_n = \frac{2}{T}\int_0^T f(t)\cos(n\omega_0 t), \quad b_n = \frac{2}{T}\int_0^T f(t)\sin(n\omega_0 t) \tag{2}$$

Question: Can you give a physical interpretation of Fourier coefficients? Can you decide which coefficients may be relevant for a given forcing function, say, a square pulse? If so, how? **Fill in the class**

Question: Suppose that you are given a Fourier series coefficients for a forcing function $f(t)$, how will you compute the particular solution? What is the influence of Fourier coefficients on the homogeneous solution?

Fill in the class

Having seen the *use* of Fourier series, logically, we ask the following questions:

1. How do we find the Fourier coefficients, a_0, a_n and b_n?
2. Under what conditions is the Fourier approximation valid?
3. How many terms must we retain in the series expansion?
4. What if the function is not periodic or periodisable?

Fortunately for us, scientists and engineers have grappled with the above questions. The outcome is Dirichelt conditions. Dirichlet conditions are particular conditions a function must satisfy in order that it may be expressed as a Fourier series. These are *sufficient* conditions.

1. the function must be periodic;
2. the function must be single-valued and continuous, except possibly at a finite number of finite discontinuities;
3. the function can only have a finite number of maxima and minima within one period;
4. the integral $\int_0^T |f(t)|\,dt$ over one time period must converge.

Forces arising in most of the engineering vibrations satisfy the above criteria. The following *facts*, given below without proof, are worth bearing in mind when we *apply* Fourier series.

1. Fourier series decomposes a periodic signal of period T into its constituent harmonics $\frac{2n\pi}{T}$.
2. It converges to $f(t)$ at all points where $f(t)$ is continuous.
3. At points of discontinuity $t = t_0$ the Fourier series converges to $\frac{1}{2}\lim\limits_{t\to t_0}[f(t+t_0) + f(t-t_0)]$.
4. The *spectrum* of Fourier series coefficients is *discrete*. Thus, we take $f(t)$ vs. t and turn it into Fourier coefficients vs. frequency, also known as spectrum.

Fourier series can be visualised in terms of rotating vector diagrams in a complex number plane (Argand diagram), also known as a phasor diagram. To do this, consider the real Fourier series, re-arranged as follows in the form of *complex* Fourier series using Euler's identity:

$$
\begin{aligned}
f(t) &= \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\left(n\omega_0 t\right) + b_n \sin\left(n\omega_0 t\right)\right], \quad \omega_0 = \frac{2\pi}{T} \\
&= \frac{1}{2}\left[a_0 e^{j0\times\omega_0 t} + a_0 e^{-j0\times\omega_0 t}\right] + \sum_{n=1}^{\infty} \left\{a_n \frac{1}{2}\left[e^{jn\times\omega_0 t} + e^{-jn\times\omega_0 t}\right]\right\} + \\
&\left\{b_n \frac{1}{2j}\left[e^{jn\times\omega_0 t} - e^{-jn\times\omega_0 t}\right]\right\} \quad \because \text{Euler's identity } e^{j\theta} = \cos\theta + j\sin\theta \\
&= \sum_{n=0}^{\infty} \frac{a_n - jb_n}{2} e^{jn\omega_0 t} + \frac{a_n + jb_n}{2} e^{-jn\omega_0 t}, \quad \because \frac{1}{j} = -j \\
&= \sum_{n=0}^{\infty} c_n e^{jn\omega_0 t} + c_{-n} e^{-jn\omega_0 t}, \quad \text{Define: } c_n = \frac{a_n - jb_n}{2}, c_{-n} = \frac{a_n + jb_n}{2} \\
&= \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t}, \quad c_n = \frac{1}{T}\int_0^T f(t) e^{-jn\omega_0 t}
\end{aligned}
\tag{3}
$$

The above Fourier series can be represented in the following phasor diagram **Fill in the class**

The complex Fourier coefficients, c_{-n} for negative values of n are complex conjugate (denoted by a $*$ as superscript) of c_n for positive n

$$c_{-n} = c_n^* \tag{4}$$

such that when we sum the complex Fourier series, for both negative and positive values of n, we get a real function. The following properties of the Fourier series are useful in relating Fourier series of one function to the other.

1. **Differentiation**:

$$g(t) = \frac{df(t)}{dt} \tag{5}$$

   Fourier coefficients $c_n^{'}$ of $g(t)$ are related to the Fourier coefficients c_n of $f(t)$ by

$$c_n^{'} = jn\omega_0 c_n \tag{6}$$

   Differentiation amplifies higher harmonics → Differentiating a noisy periodic signal amplifies high frequency noise

2. **Integration**:

$$g(t) = \int f(t)dt \tag{7}$$

$$c_n^{'} = \frac{c_n}{jn\omega_0}, \quad c_0^{'} = \frac{1}{T}\int_{t_0}^{t_0+T} g(t)dt. \tag{8}$$

   Integration amplifies lower harmonics → Integration of a noisy periodic signal amplifies low

frequency noise

3. *Parseval's theorem* is a conservation law: **Conservation of power** or average energy stored in a signal over one period is conserved whether we evaluate it in time domain or from the Fourier Spectrum. It states

$$\begin{aligned}\text{Power, or average energy over one period} &= \frac{1}{T}\int_{t_0}^{t_0+T} |f(t)|^2\, dt \\ &= \sum_{n=-\infty}^{\infty} |c_n|^2 \\ &= \frac{a_0^2}{2} + \sum_{n=1}^{\infty} \frac{a_n^2 + b_n^2}{2}\end{aligned} \tag{9}$$

Question: Can you relate the Fourier series of a square wave and a triangular wave? Which of these two series converges faster? Why?

Fill in the class

Example 28 (T4.8, 4th Edition): Determine the steady vibration response of the following SDOF system (ignoring the transient vibration), by expanding the base excitation $y(t)$ applied by the cam as a Fourier series.

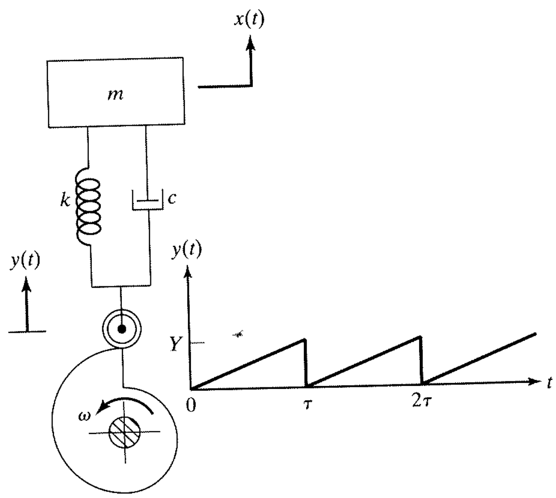


Figure 2.30: Figure for Homework 9.

Solution:

The general procedure is as follows:

1. Draw the FBD of the mass and formulate it's equations of motion. Note that the source of vibration is the displacements imparted by the cam at the base through the spring and the damper.

2. Since the base displacement is perioidic, expand it as a Fourier series.

3. Use the Fourier series for the displacement above to express the *force* acting on the mass as a Fourier series.

4. Workout the response for each force component in the Fourier series and add the responses, because principle of superposition is valid for our system.

Step 1: Equations of Motion

From the FBD of the mass the equation of motion is obtained as follows, ignoring gravity:

$$\uparrow \sum_{+ve} F_x = 0 \Rightarrow m\ddot{x}+c\left[\dot{x}-\dot{y}\right]+k\left[x-y\right] = 0 \Rightarrow m\ddot{x}+c\dot{x}+kx = c\dot{y}+ky \tag{10}$$

Notice how the displacement y appears as a force on the mass.

Step 2: Fourier Series for $y(t)$

$$y(t) = \frac{Yt}{\tau}, y(t+\tau) = y(t) \Rightarrow \text{Time Period} = T = \tau, \quad 0 \leq t \leq T \tag{11}$$

Now, Fourier Series for y is given by

$$y(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\left(n\omega_0 t\right) + b_n \sin\left(n\omega_0 t\right)\right], \quad \omega_0 = \frac{2\pi}{T} = \frac{2\pi}{\tau};$$

$$a_0 = \frac{2}{T}\int_0^T y(t)dt, \quad a_n = \frac{2}{T}\int_0^T y(t)\cos(n\omega_0 t)dt, \quad b_n = \frac{2}{T}\int_0^T y(t)\sin(n\omega_0 t)dt \tag{12}$$

$\boxed{\text{Bias/D.C. Term: } \frac{a_0}{2}}$

$$a_0 = \frac{2}{T}\int_0^T y(t)dt = \frac{2}{\tau}\int_0^\tau \frac{Yt}{\tau}dt = \frac{2}{\tau}\left[\frac{Yt^2}{2\tau}\right]_0^\tau = 2\tau\frac{Y\tau^2}{2\tau} = \frac{Y}{2} \tag{13}$$

Note that $\frac{a_0}{2}$ is the average value of the function—averaged over one time period. So we can obtain a_0 also from the average value of the function as follows:

$$
\begin{aligned}
\frac{a_0}{2} &= \frac{1}{T}\int_0^T y(t)dt \\
&= \frac{1}{\tau} \times \text{Area under the curve (triangle of base } \tau \text{ and height } Y\text{) over one time period} \\
&= \frac{1}{\tau}\frac{1}{2}Y\tau = \frac{Y}{2}
\end{aligned} \tag{14}
$$

$\boxed{a_n \text{ Term}}$

$$
\begin{aligned}
a_n &= \frac{2}{T}\int_0^T y(t)\cos n\omega_0 t dt = \frac{2}{\tau}\int_0^\tau \frac{Yt}{\tau}\cos n\omega_0 t dt \\
&= \frac{2}{\tau}\left[\frac{Yt}{\tau}\int_0^\tau \cos n\omega_0 t dt - \int_0^\tau \frac{Y}{\tau}\int_0^\tau \cos n\omega_0 t dt\right] \\
&= \frac{2}{\tau}\left[\left[\frac{Yt\sin n\omega_0 t}{\tau n\omega_0}\right]_0^\tau + \left[\frac{Y}{\tau}\frac{\cos n\omega_0 t}{n^2\omega_0^2}\right]_0^\tau\right] \\
&= \frac{2}{\tau}\left[\left[\frac{Y\tau\sin n\omega_0\tau}{\tau n\omega_0} - 0\right] + \left[\frac{Y}{\tau}\frac{\cos n\omega_0\tau}{n^2\omega_0^2} - \frac{Y}{\tau}\frac{\cos 0}{n^2\omega_0^2}\right]\right] \\
&= \frac{2}{\tau}\left[\left[\frac{Y\tau\sin 2n\pi}{\tau n\omega_0} - 0\right] + \left[\frac{Y}{\tau}\frac{\cos 2n\pi - \cos 0}{n^2\omega_0^2}\right]\right], \quad \because \omega_0\tau = \frac{2\pi}{\tau}\tau = 2\pi \\
&= 0, \quad \because \cos 2n\pi = \cos 0 = 1, \sin 2n\pi = 0
\end{aligned} \tag{15}
$$

$\boxed{b_n \text{ Term}}$

$$
\begin{aligned}
b_n &= \frac{2}{T}\int_0^T y(t)\sin n\omega_0 t dt = \frac{2}{\tau}\int_0^\tau \frac{Yt}{\tau}\sin n\omega_0 t dt \\
&= \frac{2}{\tau}\left[\frac{Yt}{\tau}\int_0^\tau \sin n\omega_0 t dt - \int_0^\tau \frac{Y}{\tau}\int_0^\tau \sin n\omega_0 t dt\right] \\
&= \frac{2}{\tau}\left[\left[\frac{-Yt\cos n\omega_0 t}{\tau n\omega_0}\right]_0^\tau + \left[\frac{Y}{\tau}\frac{\sin n\omega_0 t}{n^2\omega_0^2}\right]_0^\tau\right] \\
&= \frac{2}{\tau}\left[\left[\frac{-Y\tau}{\tau n\omega_0}\cos n\omega_0\tau - 0\right] + \left[\frac{Y}{\tau n^2\omega_0^2}(\sin n\omega_0\tau - 0)\right]\right] \\
&= \frac{2}{\tau}\left[\left[\frac{-Y\tau\cos 2n\pi}{\tau n\omega_0} - 0\right] + \left[\frac{Y}{\tau}\frac{\sin 2n\pi - 0}{n^2\omega_0^2}\right]\right], \quad \because \omega_0\tau = \frac{2\pi}{\tau}\tau = 2\pi \\
&= -\frac{2Y\tau}{n\omega_0\tau^2} = -\frac{Y}{n\pi}, \quad \because \cos 2n\pi = \cos 0 = 1, \sin 2n\pi = 0
\end{aligned}
\tag{16}
$$

Therefore the Fourier series for the displacement is given by

$$
\begin{aligned}
y(t) &= \frac{Y}{2} + \sum_{n=1}^{\infty}\left[-\frac{Y}{n\pi}\sin(n\omega_0 t)\right], \quad \omega_0 = \frac{2\pi}{T} = \frac{2\pi}{\tau}; \\
\dot{y} &= \frac{dy}{dt} = 0 + \sum_{n=1}^{\infty}\left[-\frac{Y}{n\pi}n\omega_0\cos(n\omega_0 t)\right] = -\sum_{n=1}^{\infty}\left[\frac{Y\omega_0}{\pi}\cos(n\omega_0 t)\right]
\end{aligned}
\tag{17}
$$

Step 3: Fourier Series for Force

Now the governing equation of motion for the forced vibrations response of the mass in Eq.(10) becomes

$$
m\ddot{x}+c\dot{x}+kx = c\dot{y}+ky = \frac{kY}{2} - \sum_{n=1}^{\infty}\left[\frac{kY}{n\pi}\sin(n\omega_0 t)\right] - \sum_{n=1}^{\infty}\left[\frac{cY\omega_0}{\pi}\cos(n\omega_0 t)\right]
\tag{18}
$$

Notice that the spring force diminishes with increasing values of n while damper force does not. Note also that we can find the displacement response for each Fourier series term and add, or simplify the $\cos n\omega_0 t$ and $\sin n\omega_0 t$ functions by adding them using **rotating**

vector diagram applied for each Fourier series component at forcing frequency $n\omega_0$. Or, we can evaluate the response for $\cos n\omega_0 t$ and $\sin n\omega_0 t$ force terms separately and add.

Step 4: Response

We have two equivalent methods. Each is described below.

$\boxed{\text{Summing Force for Each Term in the Series}}$

We use the formula $x = \frac{F\cos(\omega t-\phi)}{\sqrt{[k-m\omega^2]^2+[c\omega]^2}}$ associated with a harmonic force $f(t) = F\cos\omega t$ for each Fourier component of the force. We replace cos with sin if the force varies as $\sin n\omega_0 t$; we also must introduce a new phase lag θ_n for each $\sin n\omega_0 t$ force.

The D.C. term is a step force. The steady solution due to the force $\frac{kY}{2}$

$$x_0(t) = \frac{kY}{2}\frac{1}{k} = \frac{Y}{2} \quad \because \omega = 0 \text{ in our formula } x = \frac{F\cos(\omega t-\phi)}{\sqrt{[k-m\omega^2]^2+[c\omega]^2}} \tag{19}$$

The steady solution due to the force $\frac{kY}{n\pi}\sin n\omega_0 t$

$$x_{nk}(t) = X_n\sin(n\omega_0 t-\theta_n) \quad X_n = \frac{\frac{kY}{n\pi}}{\sqrt{[k-m(n\omega_0)^2]^2+[cn\omega_0]^2}}, \;\; \tan\theta_n = \frac{cn\omega_0}{k-m(n\omega_0)^2} \tag{20}$$

The steady solution due to the force $\frac{cY\omega_0}{\pi}\cos n\omega_0 t$

$$x_{nc}(t) = X_n\cos(n\omega_0 t-\phi_n) \quad X_n = \frac{\frac{cY\omega_0}{\pi}}{\sqrt{[k-m(n\omega_0)^2]^2+[cn\omega_0]^2}}, \; \tan\phi_n = \frac{cn\omega_0}{k-m(n\omega_0)^2} \tag{21}$$

Adding the above solutions for each $n = 1, 2, 3, \ldots \infty$ we have the steady-state forced vibration solution in the series form:

$$x_p(t) = \frac{Y}{2} - \sum_{n=1}^{\infty} \frac{\left[\frac{kY}{n\pi}\right]}{\sqrt{\left[k - m(n\omega_0)^2\right]^2 + \left[cn\omega_0\right]^2}} \sin(n\omega_0 t - \theta_n)$$
$$- \sum_{n=1}^{\infty} \frac{\left[\frac{cY\omega_0}{\pi}\right]}{\sqrt{\left[k - m(n\omega_0)^2\right]^2 + \left[cn\omega_0\right]^2}} \cos(n\omega_0 t - \phi_n) \tag{22}$$
$$\tan\theta_n = \frac{cn\omega_0}{k - m(n\omega_0)^2}, \quad \tan\phi_n = \frac{cn\omega_0}{k - m(n\omega_0)^2}$$

$\boxed{\text{Using Rotating Vector Diagram Method}}$

Here, we combine the spring and damper forces into one force. This is possible because both forces have the same frequency.

$$m\ddot{x} + c\dot{x} + kx = c\dot{y} + ky = \frac{kY}{2} - \sum_{n=1}^{\infty} F_n \cos(n\omega_0 t - \theta)$$
$$F_n = \sqrt{\left[\frac{kY}{n\pi}\right]^2 + \left[\frac{cY\omega_0}{\pi}\right]^2}, \tag{23}$$
$$\tan\theta_n = \frac{\left[\frac{kY}{n\pi}\right]}{\left[\frac{cY\omega_0}{\pi}\right]} = \frac{k}{n\omega_0 c}$$

We use the formula $x = \frac{F\cos(\omega t - \phi)}{\sqrt{[k - m\omega^2]^2 + [c\omega]^2}}$ associated with a harmonic force $f(t) = F\cos\omega t$ for each Fourier component of the force. The D.C. term is a step force.

The steady solution due to the force $\frac{kY}{2}$

$$x_0(t) = \frac{kY}{2}\frac{1}{k} = \frac{Y}{2} \qquad \because \omega = 0 \tag{24}$$

The steady solution due to the force $F_n\cos(\omega t - \theta_n)$

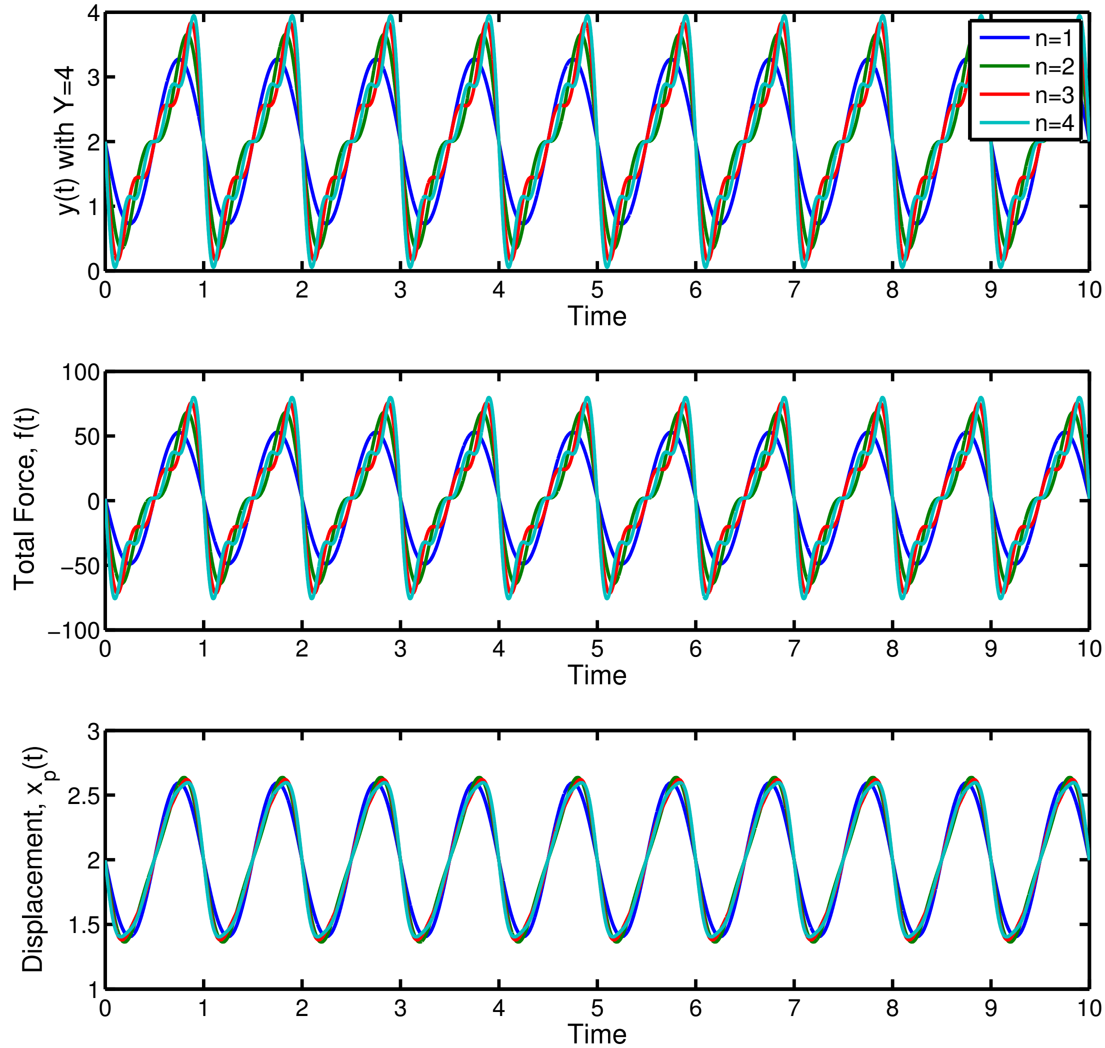
$$x_n(t) = X_n \cos(n\omega_0 t - \theta_n - \phi_n)$$
$$X_n = \frac{F_n}{\sqrt{\left[k - m(n\omega_0)^2\right]^2 + \left[cn\omega_0\right]^2}}, \ \tan\phi_n = \frac{cn\omega_0}{k - m(n\omega_0)^2} \tag{25}$$

Adding the two solutions for each $n = 1, 2, 3, \ldots \infty$ we have the steady-state forced vibration solution in the series form:

$$x_p(t) = \frac{Y}{2} - \sum_{n=1}^{\infty} \frac{\sqrt{\left[\frac{kY}{n\pi}\right]^2 + \left[\frac{cY\omega_0}{\pi}\right]^2}}{\sqrt{\left[k - m(n\omega_0)^2\right]^2 + \left[cn\omega_0\right]^2}} \cos(n\omega_0 t - \theta - \phi_n)$$
$$\tan\theta = \frac{k}{n\omega_0 c}, \quad \tan\phi_n = \frac{cn\omega_0}{k - m(n\omega_0)^2} \tag{26}$$

On the Accuracy of Fourier Series

What we have is a series solution. It is natural to ask, how many terms must one retain?
At the level of displacements we need a large number terms if we wish to approximate the discontinuity at $t = \tau$, $t = 2\tau$ *etc.*due to the Gibb's effect. However at the level of forces the spring force decreases with increasing n as $b_n \propto \frac{1}{n}$. The damper force however does not change with n.
What about displacements?
Let us see what we expect. When the spring is weak, it does not transmit y displacments to the mass. When the spring is infinitely rigid we expect x to be identicallly equal to y because a rigid spring does not deform and hence the two ends must have the same displacements. So for a rigid spring we need a large number of terms. But, we can avoid Fourier Series altogether, if we recognise that a rigid body does not deform in the very beginning of our formulation. So, Fourier series for rigid spring is useless and makes sense only for flexible springs. We must only look at the flexible spring case. Let us say $m = 10$ kg, $\tau = 1$ s, $k = 20$ N/m, $c = 0.1$ N-s/m. For this case, **if you run the MATLAB script FS.m posted on VISTA, you will find that 4 terms are adequate, as shown below!**



Concepts Used in This Problem: FBD, Fourier Series, Harmonic Response, Superposition Principle, Rotating Vector Diagram Method

To summarise:

1. Fourier series is a series approximation of a periodic, or a periodisable function. It converges in the mean to the original function. Dirichlet conditions stipulate the sufficient conditions.

2. The Fourier series decomposes a periodic function of period T into its constituent harmonics: $\omega_0 = \frac{2\pi}{T}$ (Fundamental harmonic); $2\omega_0$ (First harmonic), $3\omega_0$ (Second harmonic) and so on. The contribution of each harmonic is given by the amplitude of the complex Fourier coefficient.

3. Plotted as a function of frequency on $\omega-$axis the Fourier amplitudes are evenly spaced at discrete points $\omega = n\omega_0 = \frac{2\pi n}{T}$. The frequency interval is $\Delta\omega = \frac{2\pi}{T}$ rad.

4. Integration diminishes the contribution of higher harmonics or increases the contribution of lower harmonics.[1] Differentiation emphasises higher harmonics.

5. Please verify the above using the Fourier series applet provided on VISTA.

For aperiodic forces we generalise the Fourier series by extending the period to infinity leading to Fourier transform.

Topic4: Fourier Transforms

The main objective of this topic is to *introduce* you to Fourier transforms and *show* it's applications in engineering practise. The concept of Frequency Response Function (FRF) for steady state vibration analysis will be introduced. This allows us to extend the springs in series and parallel formulae to *systems* in series and parallel!

4.1 Introduction

Fourier transform is a generalisation of Fourier series for *aperiodic* functions. The central idea is to let the function repeat itself after infinite interval of time, and consider the Fourier series in this limiting case. Admittedly, this is a mathematical technique, but, it has immense practical applications in vibration analysis. **Let us consider two applications of the Fourier Transform in engineering practise.**

Application 1: Shaky Table

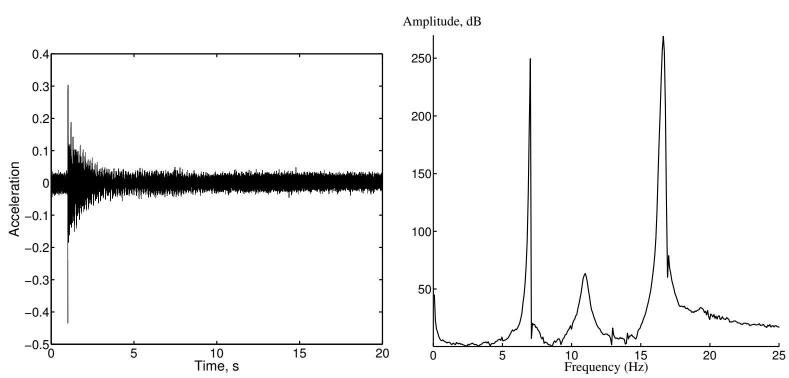


Figure 4.1: Transient response of shaky table and its spectral content obtained using Fast Fourier Transform (FFT). The peaks correspond to resonant frequencies. At least three well-spaced resonances can be seen in the frequency range 0 Hz–20 Hz.

Application 2: Response of Coupled Systems

Coupled, or, assembled systems frequently occur in engineering practice as they are modular, easy to repair and maintain. Multiple stages of a rocket are assembled, for example. It is of practical interest to assess the contribution of the vibration response of individual components towards the overall vibration response of the entire system. How can this be achieved in an efficient manner?

Fourier transform allows us to express the relation between the **steady state** displacement and the forcing function in **frequency domain**. By transforming the **steady state** displacement of a SDOF system $x_p(t)$ into the Fourier or Frequency domain using Fourier transform we obtain $X(j\omega)$. Similarly, the force $f(t)$ can be transformed into $F(j\omega)$. Now we can define the Frequency Response Function (FRF) denoted by $H(j\omega)$ as follows:

$$\boxed{H(j\omega) \equiv \frac{X(j\omega)}{F(j\omega)}, \quad \text{VALID IN STEADY-STATE ONLY} \qquad (1)}$$

Question: Can you give a physical interpretation of the FRF? Hint: Think about its units. How can FRFs be used in analysing the vibration of a connected system shown below?

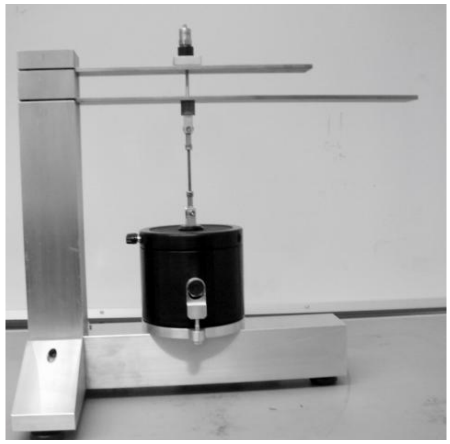


Figure 4.2: Coupled beams.

Fill in the class

Fourier transform allowed us to *predict* the response of the assembled system! Not only that, it showed us that the simple springs in series formula we obtained in Topic 2, holds true for *point* connected systems in general. We have the *powerful* formula $\frac{1}{H} = \frac{1}{H1} + \frac{1}{H2}$ that we can use to predict the response of ANY system to which another system (say a spring-mass system we studied earlier) is connected via the coupling of a *single* displacement co-ordinate. Notice the ease with which rapid design calculations can be performed in the frequency domain! Given the FRF of the system at the point of connection H_{system} we can *predict* the FRF of the system when we attach a spring mass system as shown below.

Fill in the class

Thus, **the frequency domain perspective provided by the Fourier transform is extremely useful and powerful**. It is for this and many other reasons that Fourier transform plays such an important role in understanding linear systems.

We develop the mathematical underpinnings of Fourier transform in the next section. We will return to FRF in example 31.

4.2 Fourier Transform

We have studied the Fourier series in Topic 2.6. Let us recall some of the essential ideas. A function $f(t)$ which satisfies Dirichlet conditions has it's Fourier series given by

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{2n\pi t}{T}\right) + b_n \sin\left(\frac{2n\pi t}{T}\right)\right] \quad (2)$$

1. Fourier series decomposes a periodic signal of period T into its constituent harmonics $\frac{2n\pi}{T}$
2. It converges to $f(t)$ at all points where $f(t)$ is continuous.
3. At points of discontinuity $t = t_0$ the Fourier series converges to $\frac{1}{2}\lim_{t\to t_0}[f(t+t_0) + f(t-t_0)]$
4. The spectrum of Fourier series coefficients is discrete.

4.2.1 From Fourier Series to Transform

In many vibration problems, Fourier series is adequate, if the time-domain solution is desired. For frequency domain analysis, with general input forces (without any periodicity), a generalisation of Fourier series in the form of Fourier transform is desirable.

Fourier transform can be obtained by imagining that an aperiodic signal is a periodic signal of infinite time period. In this case $T \to \infty$. We shall now consider the Fourier series in this limit. Necessary mathematics is given below.

$$
\begin{aligned}
f(t) &= \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t}, \quad j \equiv \sqrt{-1}, \quad \omega_0 \equiv \frac{2\pi}{T} \\
c_n &= \frac{\Delta\omega}{2\pi} \int_{t_0}^{t_0+T} f(t) e^{-jn\omega_0} dt, \quad \Delta\omega \equiv \frac{1}{T} \\
f(t) &= \sum_{n=-\infty}^{\infty} \left[\frac{\Delta\omega}{2\pi} \int_{\frac{-T}{2}}^{\frac{T}{2}} f(t) e^{-jn\omega_0 t} dt \right] e^{jn\omega_o t} \\
f(t) &= \sum_{n=-\infty}^{\infty} \frac{\Delta\omega}{2\pi} \left[\int_{\frac{-T}{2}}^{\frac{T}{2}} f(t) e^{-jn\omega_0 t} dt \right] e^{jn\omega_o t}
\end{aligned}
$$

Besides the obvious change in limits, the following things happen in the limit $T \to \infty$

$$T \to \infty \Rightarrow \Delta\omega \to d\omega$$

$$\sum \to \int, \text{ Integral is a limit of a sum}$$

$n\omega_0 \to \omega$, Discrete $n\omega_0$ are replaced by a continuous variable ω

Inserting the above in the Fourier series gives the Fourier inversion theorem:

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt\right] e^{j\omega t}d\omega, \text{ Fourier's inversion theorem}$$

The Fourier's inversion theorem forms the basis for the following *definitions* of Fourier transforms:

$$\begin{aligned} F(j\omega) &\equiv \int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt, \text{ Forward Transform} \\ f(t) &\equiv \frac{1}{2\pi}\int_{-\infty}^{\infty} F(j\omega)e^{j\omega t}d\omega, \text{ Inverse Transform} \end{aligned} \tag{3}$$

Fourier transform thus allows transition from a continuous time variable/domain to a continuous frequency variable/domain. The only requirement for Fourier transform is that $\int_{-\infty}^{\infty} |f(t)|\, dt$ converges. Thus the Fourier transform for $f(t) = t$ is not defined, since $f(t)$ does not tend to zero for large values of t, hence the integral defining $F(j\omega)$ does not converge.

Question: What are the differences between Fourier Series and Fourier Transform?

Fill in the class

Some properties of the Fourier Transform are summarised next.

$$\mathcal{F}[f(t)] \equiv \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt \equiv F(j\omega)$$

Property	**Example**
Proportionality	$\mathcal{F}[af(t)] = aF(j\omega)$
Superposition	$\mathcal{F}[f(t) + g(t)] = F(j\omega) + G(j\omega)$
Scaling	$\mathcal{F}[f(at)] = \frac{1}{a}F(\frac{j\omega}{a})$
Shifting	$\mathcal{F}[f(t \pm a)] = e^{\pm ja\omega}F(j\omega)$
Exponential multiplication (a can be real, imaginary or complex)	$\mathcal{F}[e^{\pm at}f(t)] = F(j\omega \mp a)$
Integration	$\mathcal{F}\left[\int^t f(\tau)d\tau\right] = \frac{1}{j\omega}F(j\omega) + 2\pi c\delta(\omega)$
Differentiation	$\mathcal{F}\left[\frac{df(t)}{dt}\right] = (j\omega)F(j\omega)$
Convolution	$\mathcal{F}\left[\int h(t-\tau)f(\tau)d\tau\right] = H(j\omega)F(j\omega)$
Parseval's theorem	$\int_{-\infty}^{\infty} \lvert F(j\omega)\rvert^2 d\omega = \int_{-\infty}^{\infty} \lvert f(t)\rvert^2 dt$
Heisenberg-Gabor Uncertainty principle: this is an important fact to remember when considering sophisticated signal analyses such as wavelets	Deviations or spreads in t and ω are inversely related: $\Delta\omega\Delta t = 1$

Example 29 : Cosnider a pulse of width $T = 1$ and amplitude $A = 1$ centred around origin, sketched below. Can you use Fourier series to approximate this function? If yes, obtain the Fourier series coefficients. Can you obtain the Fourier transform of this pulse?

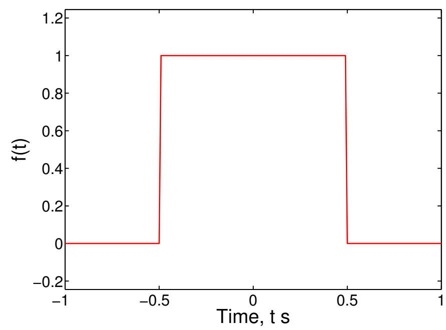


Figure 4.3: Figure for example 29.

Fill in the class

Solution:

Clearly, our function is not periodic. We have two options: (A) Periodise the function and workout the Fourier series, or (B) Use Fourier Transform. Let us do both, and, learn how Fourier transform is obtained in the process. Note that you can always use $\mathcal{F}[f(t)] \equiv \int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt \equiv F(j\omega)$ to find the Fourier transform. We will do this towards the end.

(A) Periodise and Apply Fourier Series

1. Periodise the function. Sketch in the class.
2. Find the Fourier Series coefficients.

Thus we have the following periodised function over one time interval

$$
\begin{aligned}
f(t) &= A, 0 \leq t \leq \frac{T}{2} \\
&= -A, \frac{T}{2} \leq t \leq \frac{3T}{2} \\
&= A, \frac{3T}{2} \leq t \leq 2T
\end{aligned}
$$

Therefore our new periodic function has a period $T_0 = 2T$ and hence a fundamental frequency $\omega_0 = \frac{2\pi}{T_0}$.

Let us find the Fourier series coefficients. Since the function is even $b_n = 0$.

$a_0 = \frac{2}{T_0} \displaystyle\int_0^{T_0} f(t)dt = \frac{2}{T_0} \left[\int_0^{\frac{T}{2}} Adt - \int_{\frac{T}{2}}^{\frac{3T}{2}} Adt + \int_{\frac{3T}{2}}^{2T} Adt \right] = 0$. Can you see why?

$$a_n = \frac{2}{T_0} \int_0^{T_0} f(t) \cos(n\omega_0 t)dt$$

$$\therefore a_n = \frac{2}{T_0} \left[\int_0^{\frac{T}{2}} A \cos(n\omega_0 t)dt - \int_{\frac{T}{2}}^{\frac{3T}{2}} A \cos(n\omega_0 t)dt + \int_{\frac{3T}{2}}^{2T} A \cos(n\omega_0 t)dt \right]$$

Evaluating the above integrals, we obtain

$a_n = \frac{2A}{n\omega_0 T_0} \left[2\sin(\frac{n\omega_0 T}{2}) - 2\sin(\frac{3n\omega_0 T}{2}) + \sin(2n\omega_0 T) \right]$. We obtain the approximation shown in Fig.(4.4), if we include the first 20 terms with the corresponding Fourier spectrum shown on the right.

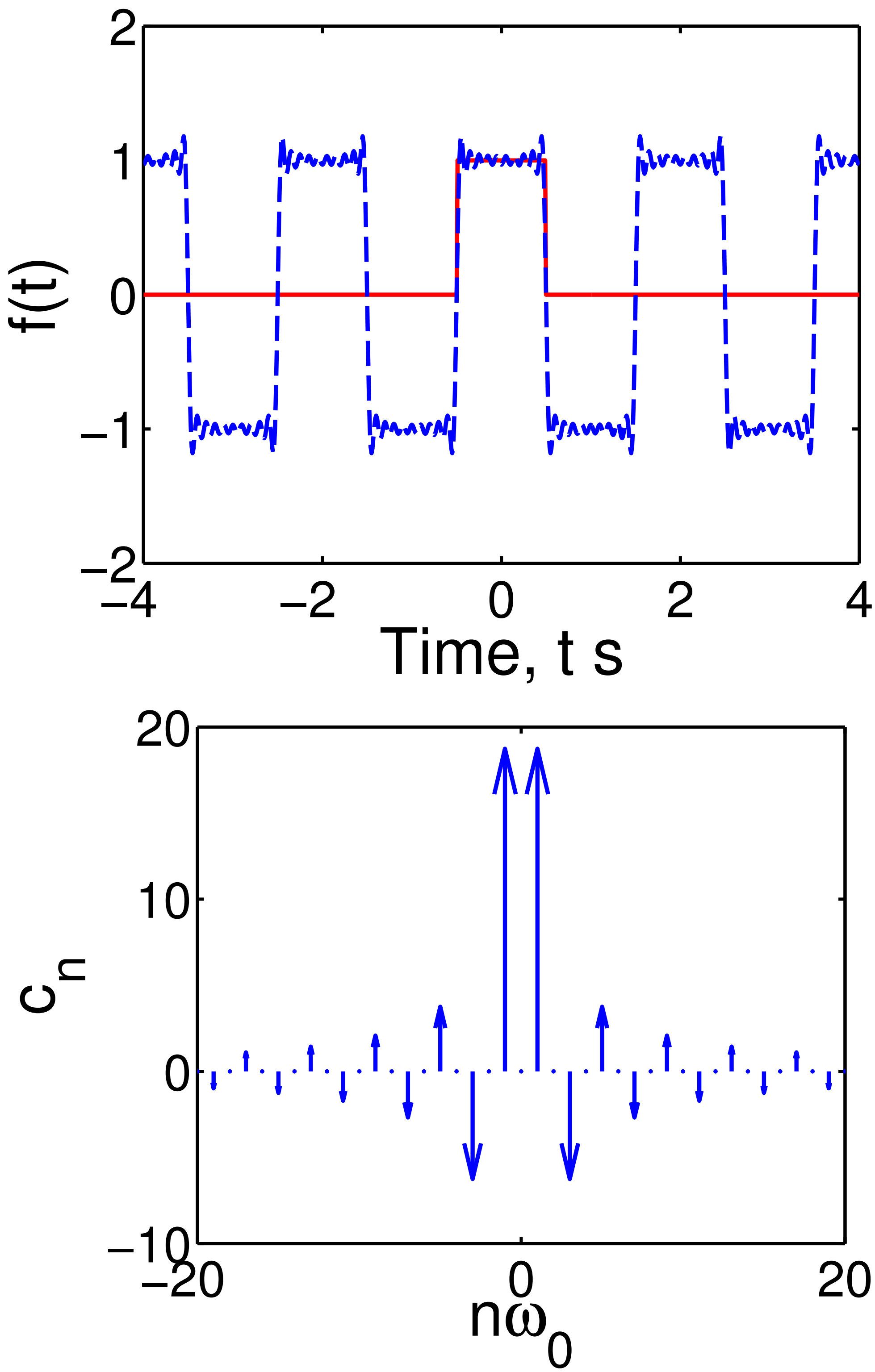


Figure 4.4: Fourier spectrum of a pulse truncated to $n = 20$. Notice the non-convergence at the points of discontinuity: this is also known as *Gibb's phenomenon*. *Use the MATLAB script 'Rectangular_Pulse_FS.m' posted on VISTA under Lecture Notes folder.*

(B) Fourier Transform

1. From Fourier series. We will do this first to gain insight into how Fourier transform works.
2. Use the integral formula for Fourier transform.

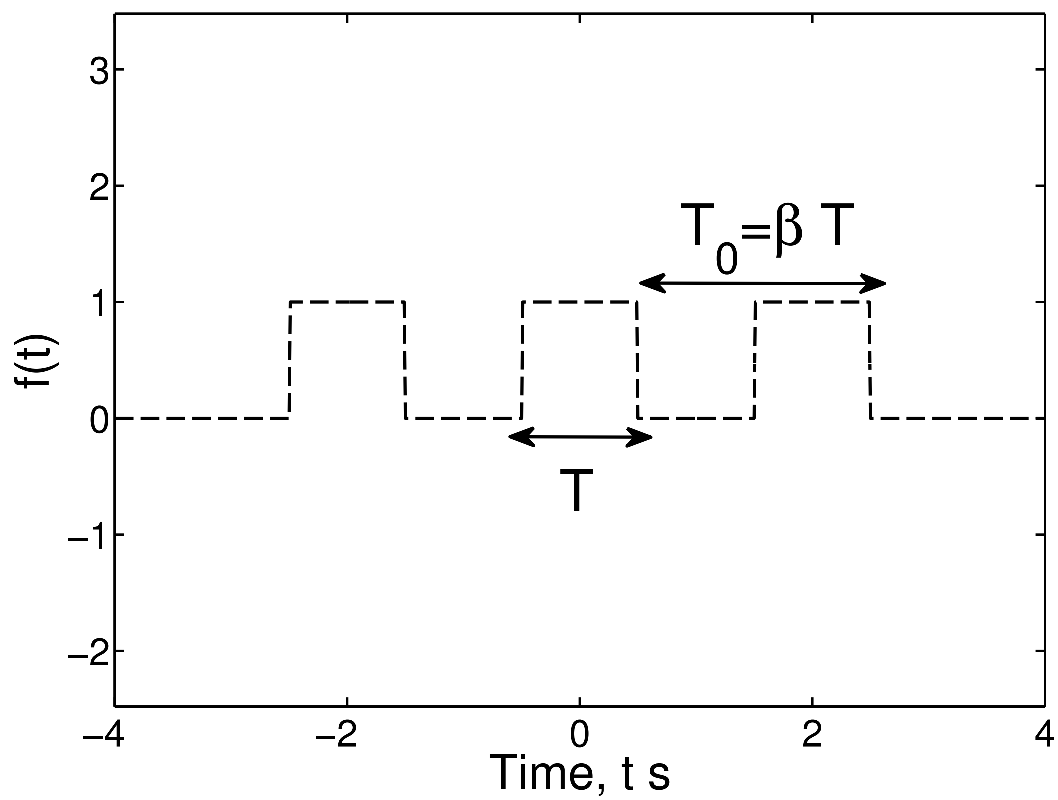


Figure 4.5: Periodising the pulse as a function that repeats after infinite time T_0. In the limit $\beta \to \infty$ we get the Fourier Transform.

Now our function is defined as follows within one "period" T_0.

$$
\begin{aligned}
f(t) &= A, 0 \leq t \leq \frac{T}{2} \\
&= 0, \frac{T}{2} \leq t \leq T_0 - \frac{T}{2} \\
&= A, T_0 - \frac{T}{2} \leq t \leq T_0
\end{aligned}
$$

The plan is to find the Fourier series and watch what happens to Fourier spectrum as we increase β to larger and larger numbers. **Note that we purposefully chose a different periodisation this time for the SAME PULSE which we periodised as an even function before in (A)**. In the limit $\beta \to \infty$ we get the Fourier transform as shown in Fig.(4.6) and Fig.(4.7)

We notice the following from these two figures.

1. As we increase the period of repetition $T_0 = \beta T$, by increasing β, the frequency spacing between the c_n components decreases because ω_0 decreases. That is why for the same number of Fourier series terms $n = 20$, we cover a less narrower frequency range in the spectrum for $\beta = 10$ when compared with the case $\beta = 2$.

2. We also see that the curve $\frac{AT}{T_0}F(\omega) = \frac{\sin(\frac{n\pi}{\beta})}{\frac{n\pi}{\beta}}$ in red color fits the envelop of the Fourier spectrum. In the limit of $\beta \to \infty$ we have $\frac{n\pi}{\beta} = \frac{n\pi T}{T_0} = \frac{n\pi T\omega_0}{2\pi} = \frac{n\omega_0 T}{2} = \frac{\omega T}{2}$, since the discrete $n\omega_0$ in Fourier series is replaced by the continuous variable ω. $F(\omega) = AT\frac{\sin(\omega\frac{T}{2})}{\omega\frac{T}{2}}$. This is the Fourier transform of the pulse of amplitude A and width T!! Let us verify this by evaluating the integral.

$$F(j\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt = \int_{-\frac{T}{2}}^{\frac{T}{2}} Ae^{-j\omega t}dt = A\left[\frac{e^{-j\omega t}}{-j\omega}\right]_{-\frac{T}{2}}^{\frac{T}{2}} = A\left[\frac{e^{-j\omega\frac{T}{2}}}{-j\omega} + \frac{e^{j\omega\frac{T}{2}}}{j\omega}\right]$$

Applying Euler's rule $e^{\pm j\theta} = \cos\theta \pm j\sin\theta$ we have

$$\boxed{F(j\omega) = \frac{2A}{\omega}\sin(\omega\tfrac{T}{2}) = AT\frac{\sin(\omega\frac{T}{2})}{\omega\frac{T}{2}}.}$$

Same as what we obtained from the Fourier series by extending the pulse as a periodic function of infinite time period!

This example illustrates that Fourier Transform is a generalised Fourier series in the limit of infinite time period!

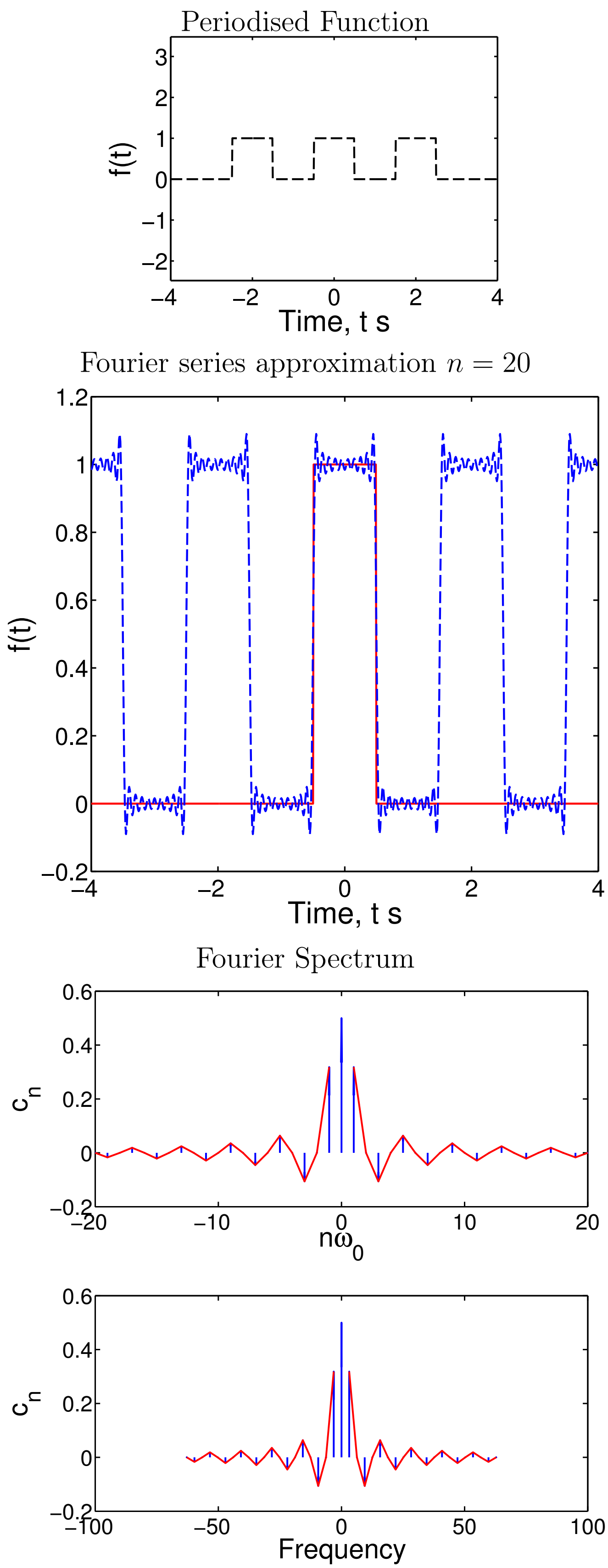


Figure 4.6: $\beta = 2$ and $n = 20$.

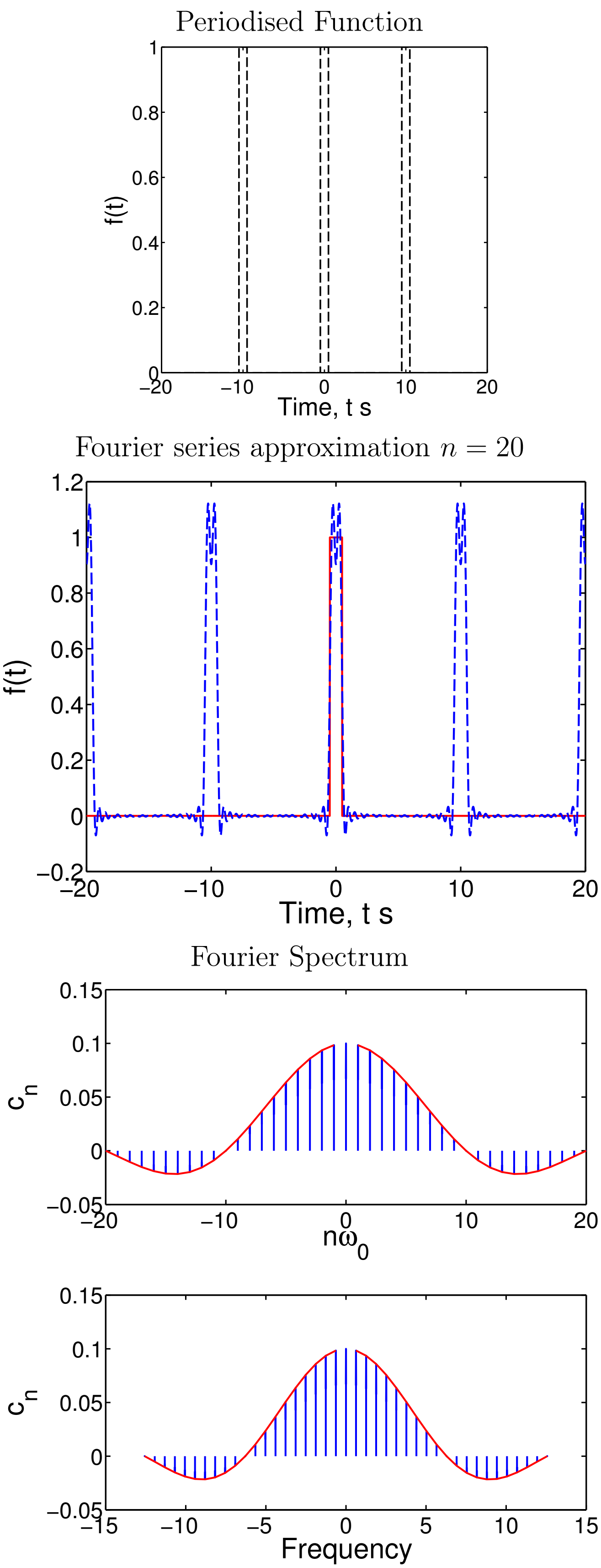


Figure 4.7: $\beta = 10$ and $n = 20$.

Use the MATLAB script 'Rectangular_Pulse_FS_FT.m' posted on VISTA under Lecture Notes folder.

Example 30 : Obtain the Fourier Transform of a unit impulse from the Fourier transform of a pulse.

Fill in the class

Solution:

This example illustrates that an ideal impulse excites all frequencies!

Example 31 : Show that the FRF of a SDOF system is given by $H = \frac{1}{k - m\omega^2 + jc\omega}$ using the properties of the Fourier transforms.

Solution: **Fill in the class**

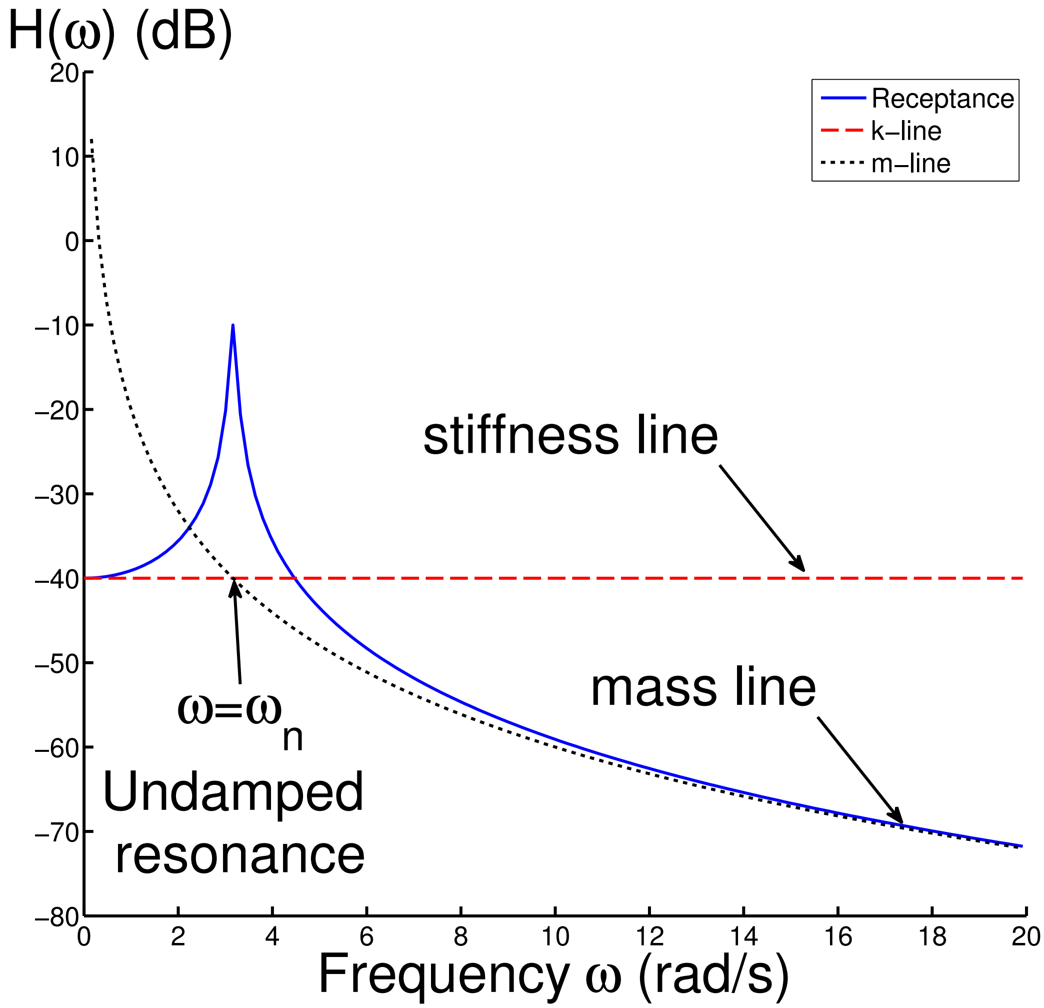


Figure 4.8: FRF for a damped system. The y-axis is the magnitude of H in decibels $20log_{10}|H|$. Notice that the resonance occurs when the mass and stiffness lines intersect. The low frequency is stiffness dominated, while the high frequency response if mass dominated.

This example demonstrates the following:

1. Below resonance, FRF is dominated by the stiffness. Well above resonance FRF is dominated by the mass.

2. Resonance occurs in the vicinity where mass and stiffness lines intersect

To summarise:

1. Fourier transform has immense practical applications. It can convert a time series into its constituent frequency components whether the signal is periodic or not.

2. Fourier transform is a generalisation of Fourier series by extending the time period to infinity. The transforms thus obtained are:

$$\begin{aligned} F(j\omega) &\equiv \int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt, \text{ Forward Transform} \\ f(t) &\equiv \frac{1}{2\pi}\int_{-\infty}^{\infty} F(j\omega)e^{j\omega t}d\omega, \text{ Inverse Transform} \end{aligned} \tag{4}$$

3. Frequency response related the force and displacement in the frequency domain $H(j\omega) = \frac{X(j\omega)}{F(j\omega)}$ in the steady state. This is extremely powerful for steady state vibration analysis.

4. The FRF is dominated by stiffness below resonance, mass above resonance. Resonance occurs in the vicinity where mass and stiffness lines intersect

5. The impulse response function $h(t)$ can be obtained by taking the inverse Fourier transform of the FRF $H(j\omega)$!

Topic 5: Two and Multi-Degree of Freedom (MDOF) Systems – Part 1

In this topic we shall consider the free and forced vibration analysis of two degree of freedom systems in particular, and multi degree of freedom systems in general. The notion of co-ordinate coupling and uncoupling matrix equations of motion through principal/normal co-ordinates will be discussed.

5.1 Introduction

Thus far, we have considered SDOF *models*. We have seen that a broad range of vibration problems can be effectively addressed using SDOF models, so long as the interest is at one spatial location and one displacement co-ordinate. You have seen the success and limitation of SDOF analysis applied to the Shaky table apparatus. While SDOF theory explained the frequency response for in-phase configuration, it failed to predict multiple resonances observed in out-of-phase arrangement of eccentric masses. Clearly, SDOF is a very useful but limited approximation of real systems.

A two degree of freedom system requires two independent co-ordinates to completely specify it's motion at any instant in time. A N-DOF system requires N independent co-ordinates to describe it's motion. Note that the number of degrees of freedom of a system has nothing to do with the number of masses.

A wilberforce pendulum, comprising a mass suspended by a helical spring is a good example to consider. This system can support two *independent* harmonic motions: vertical translatory motion of the mass and torsional rotation of the mass. When given an initial displacement, be it a translation or a twist, this system exhibits oscillations which cannot be explained by SDOF model. See the video at http://www.youtube.com/watch?v=S42lLTlnfZc. Here, a helical spring is the source of coupling between up-down and torsional motion. Energy is exchanged between these two *modes* of vibration due to the coupling induced by the spring.

Question: What is the 'coupling induced' by a helical spring?

Fill in the class

The goal of vibration analysis is to reduce a N-DOF system into N SDOF systems so that the response of each SDOF system can be evaluated using the concepts discussed in Topic 2. Thus, SDOF theory is the cornerstone of vibration analysis of MDOF systems.

The general flow of ideas is summarised below:

1. Formulate equations of motion. This can be achieved using force methods or energy methods.

2. Transform equations of motion into N SDOF systems in normal or principal co-ordinates. This requires knowledge of linear algebra concerned with transforming vectors.[1]

3. Solve the N SDOF systems in normal co-ordinates using ideas learned inTopic 2.

4. Transform our solution back into original co-ordinates for interpretation and design calculations.

A study of two DOF system reveals the essential differences in the response characteristics of the SDOF and the MDOF systems. This we shall pursue now.

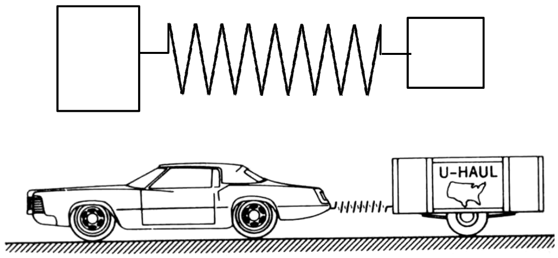
[1]You may have already seen this when dealing with Mohr's circle. There, the eigenvalues correspond to principal stresses. We find the idea of principal directions useful in vibration analysis too. Eigenvalues here represent the square of the undamped natural frequencies!

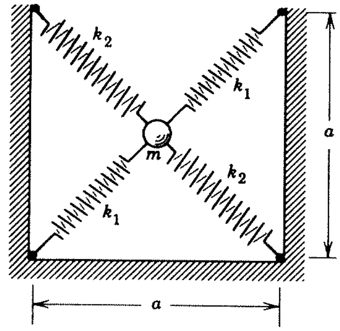
5.2 Formulation of Equations of Motion

The equations of motion can be formulated using force and energy methods. As we saw earlier, energy method is more powerful in that it reduces the number of FBDs. In the energy method we express the energies in the matrix form to obtain the stiffness and mass matrices: $V = \frac{1}{2}\boldsymbol{x}^T\boldsymbol{K}\boldsymbol{x}$, and $T = \frac{1}{2}\dot{\boldsymbol{x}}^T\boldsymbol{M}\dot{\boldsymbol{x}}$, where $\boldsymbol{x}$ contain the degrees of freedom.
Question: What are the sizes for the vector $\boldsymbol{x}$ and matrices $\boldsymbol{M}$ and $\boldsymbol{M}$ for a N degree of freedom system?

Fill in the class

Example 32 : Formulate the equations of motion of the following systems using the force and energy methods by choosing appropriate co-ordinates.





Solution:

Fill in the class

We saw that the two systems considered in Example 32 can give equations that are coupled. This implies that the displacement along one co-ordinate also leads to displacement along other co-ordinates. In the case of car-trailer problem the spring couples the displacement $x_1(t)$ of the car with the displacement $x_2(t)$ of the trailer. This coupling from elastic forces is called elastic or **static coupling**. The coupled governing equations are:

$$m_1\ddot{x}_1(t) + k\left[x_1(t) - x_2(t)\right] = 0$$

$$m_2\ddot{x}_2(t) + k\left[x_2(t) - x_1(t)\right] = 0$$

or in matrix form

$$\boldsymbol{M}\ddot{\boldsymbol{x}} + \boldsymbol{K}\boldsymbol{x} = \boldsymbol{0}, \quad \boldsymbol{M} = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \quad \boldsymbol{K} = \begin{bmatrix} k & -k \\ -k & k \end{bmatrix}$$

Question: Can you give a physical interpretation of entries in the stiffness matrix? Why is it symmetric?

Fill in the class

In contrast to the car-trailer problem, for the 4 spring-mass system in Example 32, we saw that the governing equations of motion can be coupled or not depending on our choice of co-ordinates. This leads us to ask the question following question. **How do we determine co-ordinates which give uncoupled equations of motion?**

It turns out that there are co-ordinates, called principal co-ordinates, in which the equations of motion are uncoupled. These principal co-ordinates are related to the eigenvalue problem that emerges for free vibration response, which plays an important role in the forced vibration response. We shall see this in the next section.

5.3 Vibration Response

Assuming a harmonic motion of the form $\boldsymbol{x} = \boldsymbol{u}e^{j\omega t}$, where the real part only is intended to be our solution, and substituting this in the equations of motion

$$\boldsymbol{M}\ddot{\boldsymbol{x}} + \boldsymbol{K}\boldsymbol{x} = \boldsymbol{f}, \quad \boldsymbol{x}(t=0) = \boldsymbol{x}_0, \quad \dot{\boldsymbol{x}}(t=0) = \dot{\boldsymbol{x}}_0 \quad (1)$$

we obtain the eigenvalue problem as follows. **Fill in the class**

$$\boxed{\boldsymbol{K}\boldsymbol{u} = \lambda \boldsymbol{M}\boldsymbol{u}, \quad \lambda \equiv \omega^2 \qquad (2)}$$

Question: Can you give a physical interpretation of the above eigenvalue problem? How many free vibration modes are possible for a N DOF system? **Fill in the class**

An important point to realise is that $\boldsymbol{u}_i$, $c_i\boldsymbol{u}_i$ are both eigenvectors or modes. c_i here is a scaling factor that can be real or complex. This implies that we cannot define the absolute displacements for a mode but relative displacements of all other co-ordinates with respect to a fixed amplitude (of unity in general) set for a reference co-ordinate. Some books use the word *modal fractions* to denote these amplitude ratios. Thus a typical eigenvector $\boldsymbol{u}_i$ can be normalised as $\boldsymbol{u}_i = \begin{bmatrix} 1 & r_{i2} & r_{i3} & r_{i4} \ldots \end{bmatrix}, \quad r_{i2} = \frac{u_{i2}}{u_{i1}}$ where r_{ij}s denote the modal fractions associated with mode i. The first subscript denote the mode number, and the second denotes the co-ordinate or DOF number.

It is convenient to scale the eigenvectors, such that they give unit modal mass

$$\boldsymbol{\phi}_i^T \boldsymbol{M} \boldsymbol{\phi}_i = 1, \boldsymbol{\phi}_i = c_i \boldsymbol{u}_i \tag{3}$$

By this scaling procedure the arbitrary constant c_i is obtained as

$$c_i = \frac{1}{\sqrt{\boldsymbol{u}_i^T \boldsymbol{M} \boldsymbol{u}_i}} \tag{4}$$

For real and symmetric matrices, which fall under the category of normal matrices, the eigenvectors satisfy the following orthogonality conditions

$$\begin{aligned} &\boldsymbol{u}_i^T \boldsymbol{M} \boldsymbol{u}_j = m_i \delta_{ij} \\ &\boldsymbol{u}_i^T \boldsymbol{K} \boldsymbol{u}_j = k_i \delta_{ij} \\ &\delta_{ij} = 1, \quad i = j, \quad \delta_{ij} = 0, \quad i \neq j \end{aligned} \tag{5}$$

where k_i and m_i are called modal stiffness and mass respectively. Provided all modes are scaled consistently, the modal stiffness tells how easy or hard it is to deform the system into a particular mode shape. For unit modal mass scaling, the above properties can be re-stated as

$$\begin{aligned} \boldsymbol{\phi}_i^T \boldsymbol{M} \boldsymbol{\phi}_j &= \delta_{ij} \\ \boldsymbol{\phi}_i^T \boldsymbol{K} \boldsymbol{\phi}_j &= \omega_i^2 \delta_{ij} \\ \delta_{ij} &= 1, \quad i = j \\ \delta_{ij} &= 0, \quad i \neq j \end{aligned} \tag{6}$$

Question: Can you give a physical interpretation of orthogonality conditions? **Fill in the class**

Because of the orthogonality conditions, we can decompose a set of N-coupled O.D.Es represented by

$$\boldsymbol{M}\ddot{\boldsymbol{x}} + \boldsymbol{K}\boldsymbol{x} = \boldsymbol{f}, \quad \boldsymbol{x}(t=0) = \boldsymbol{x}_0, \quad \dot{\boldsymbol{x}}(t=0) = \dot{\boldsymbol{x}}_0 \tag{7}$$

in chosen (physical) generalised co-ordinates into N independent SDOF equations in *normal* co-ordinates $q_i(t)$ (these are a special class of principal co-ordinates that give unit mass matrix in the transformed co-ordinates. They are more restrictive.)

$$\ddot{q}_i + \omega_i^2 q_i = Q_i, \quad i = 1, 2, \ldots N \tag{8}$$

via the transformation

$$\boxed{\boldsymbol{x} = \boldsymbol{\Phi q}} \tag{9}$$

Question: What is the physical meaning of the above equation? How do you find $\boldsymbol{\Phi}$? **Fill in the class**

The initial conditions and the generalised force expressions are obtained via

$$\boldsymbol{q}(t=0) = \boldsymbol{\Phi}^{-1}\boldsymbol{x}_0, \quad \dot{\boldsymbol{q}}(t=0) = \boldsymbol{\Phi}^{-1}\dot{\boldsymbol{x}}_0, \quad \boldsymbol{Q} = \boldsymbol{\Phi}^T\boldsymbol{f} \tag{10}$$

We can give the following physical interpretation for the modal (normal) co-ordinate transformation in Eq.(9): $q_j(t)$ are *modal participation* factors quantifying the amount of each mode present in a given vibration.

The solution to

$$\boxed{\begin{aligned} &\ddot{q}_i + \omega_i^2 q_i = Q_i, \quad i = 1, 2, \ldots N \\ &\dot{\boldsymbol{q}}(t=0) = \boldsymbol{\Phi}^{-1}\dot{\boldsymbol{x}}_0, \quad \boldsymbol{Q} = \boldsymbol{\Phi}^T\boldsymbol{f} \end{aligned}} \tag{11}$$

for the general forcing $\boldsymbol{f}$ is (see Topic 2.6)

$$\boxed{\begin{aligned} &q_i(t) = C_1\cos(\omega_i t) + C_2\sin(\omega_i t) + \int_0^t h_i(t-\tau)Q_i(\tau)d\tau \\ &h_i(t) = \frac{1}{\omega_i}\sin(\omega_i t) \end{aligned}} \tag{12}$$

Having solved for the response in modal co-ordinates q_i we obtain the response in original coupled co-ordinates as

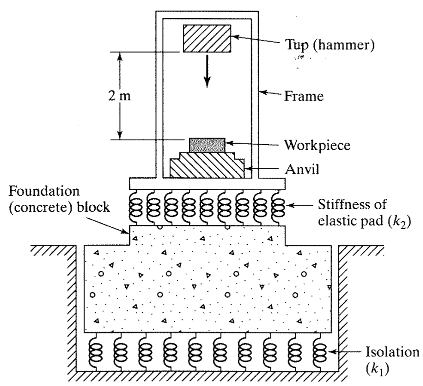
$$\boxed{\begin{aligned}&\boldsymbol{x} = \boldsymbol{\Phi q}\\ &x_1(t)\\ &= \Phi_{11}\left[C_1\cos(\omega_1 t) + C_2\sin(\omega_1 t) + \int_0^t h_1(t-\tau)Q_1(\tau)d\tau\right]\\ &+ \Phi_{12}\left[C_1\cos(\omega_2 t) + C_2\sin(\omega_2 t) + \int_0^t h_2(t-\tau)Q_2(\tau)d\tau\right]\\ &+ \ldots\end{aligned}} \tag{13}$$

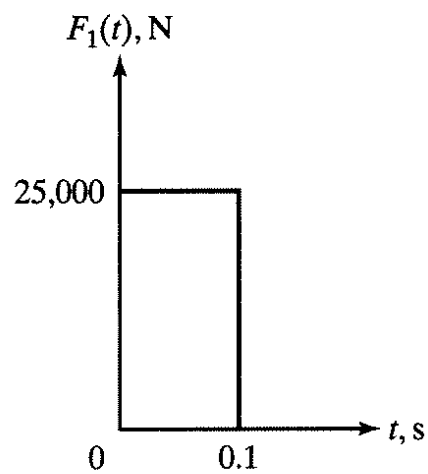
Question: Can you list the differences you see in the equations of motion when expressed in $\boldsymbol{x}$ and $\boldsymbol{q}$ co-ordinates? **Fill in the class**

Question: What are the differences between physical, principal, and normal co-ordinates? **Fill in the class**

Example 33 : Consider forging hammer shown below. The force acting on the work piece is due to the impact of the hammer and can be approximated as a rectangular pulse as shown. Find the resulting vibrations with the following data: the mass of the work piece, anvil and frame combined is 200 Mg; mass of the foundation block is 250 Mg; stiffness of the elastic pad is 150 MN/m; stiffness of isolation system is 75 MN/m.

Assume that the initial displacements and velocities are zero.





Solution: General Procedure:

Fill in the class

1. Formulate equations of motion: $\boldsymbol{M}\ddot{\boldsymbol{x}} + \boldsymbol{K}\boldsymbol{x} = \boldsymbol{f}$
2. Solve for natural frequencies (eigenvalue) and modeshapes (eigenvector): $\boldsymbol{K}\boldsymbol{u}_i = \omega_i^2 \boldsymbol{M}\boldsymbol{u}_i$
3. Normalise the modeshapes for unit modal mass: $\boldsymbol{\phi}_i = \frac{\boldsymbol{u}_i}{\sqrt{\boldsymbol{u}_i^T \boldsymbol{M}\boldsymbol{u}_i}}$
4. Apply the modal transformation $\boldsymbol{x} = \boldsymbol{\Phi}\boldsymbol{q}$ to the governing equations
5. Solve the uncoupled equations in normal(modal) co-ordinates $\ddot{q}_i + \omega_i^2 q_i = Q_i, \quad i = 1, 2, \ldots N$ with $\boldsymbol{q}(t=0) = \boldsymbol{\Phi}^{-1}\boldsymbol{x}_0$, $\dot{\boldsymbol{q}}(t=0) = \boldsymbol{\Phi}^{-1}\dot{\boldsymbol{x}}_0$, $\quad \boldsymbol{Q} = \boldsymbol{\Phi}^T \boldsymbol{f}$ for $q_i(t)$
6. Transform back to original co-ordinates to complete the solution $\boldsymbol{x} = \boldsymbol{\Phi}\boldsymbol{q}$

See the MATLAB code EX33.m is posted under lecture notes on vista.

Stiffness: $\boldsymbol{K} = \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} = 10^6 \begin{bmatrix} 225 & -150 \\ -150 & 150 \end{bmatrix}$

Mass: $\boldsymbol{M} = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} = 10^3 \begin{bmatrix} 250 & 0 \\ 0 & 200 \end{bmatrix}$

We can deduce the above matrices from the energy expressions or influence coefficients as shown in the class.

The eigenvalues problem associated with the above matrices reads as follow:

$$10^6 \begin{bmatrix} 225 & -150 \\ -150 & 150 \end{bmatrix} \boldsymbol{u}_i = \omega_i^2 10^3 \begin{bmatrix} 250 & 0 \\ 0 & 200 \end{bmatrix} \boldsymbol{u}_i$$

We obtain the eigenvalue ω_i^2 and associated eigenvector $\boldsymbol{u}_i$ by calling the MATLAB command *eig*: $[\boldsymbol{U}, \boldsymbol{D}] = eig(\boldsymbol{K}, \boldsymbol{M})$

This gives us the eigenvector matrix:

$\boldsymbol{U} = \begin{bmatrix} -0.0013 & -0.0015 \\ -0.0017 & 0.0015 \end{bmatrix}$.

The columns of $\boldsymbol{U}$ denote the eigenvectors. It is useful to insist on positive sign for the first component of each eigenvector or any physically meaningful component. Note that the column vectors of $\boldsymbol{U}$ satisfies orthogonality conditions, but not necessarily unit modal mass normalisation. If we want normal co-ordinates then we need to mass normalise the eigenvectors.

The mass normalised eigenvectors are given by: $\boldsymbol{\phi}_i = \frac{\boldsymbol{u}_i}{\sqrt{\boldsymbol{u}_i^T \boldsymbol{M} \boldsymbol{u}_i}}$. The matrix $\boldsymbol{\Phi}$ whose columns denote the mass normalised eigenvectors is obtained as: $\boldsymbol{\Phi} = \begin{bmatrix} 0.0013 & 0.0015 \\ 0.0017 & -0.0015 \end{bmatrix}$

With this we can now write the modal co-ordinate transformation as:

$\boldsymbol{x}(t) = \boldsymbol{\Phi q}(t)$ or $\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} 0.0013 & 0.0015 \\ 0.0017 & -0.0015 \end{bmatrix} \begin{bmatrix} q_1(t) \\ q_2(t) \end{bmatrix}$

Because of the orthogonality conditions, we have:

$\boldsymbol{\Phi}^T \boldsymbol{M\Phi} = \boldsymbol{I}$ and $\boldsymbol{\Phi}^T \boldsymbol{K\Phi} = diag\boldsymbol{\omega}_i^2 = 10^3 \begin{bmatrix} 0.1500 & 0 \\ 0 & 1.5000 \end{bmatrix}$.

The initial conditions and force vector in modal co-ordinates are

$$\begin{bmatrix} q_1(0) \\ q_2(0) \end{bmatrix} = \begin{bmatrix} 0.0013 & 0.0015 \\ 0.0017 & -0.0015 \end{bmatrix}^{-1} \begin{bmatrix} x_1(0) = 0 \\ x_2(0) = 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} \dot{q}_1(0) \\ \dot{q}_2(0) \end{bmatrix} = \begin{bmatrix} 0.0013 & 0.0015 \\ 0.0017 & -0.0015 \end{bmatrix}^{-1} \begin{bmatrix} \dot{x}_1(0) = 0 \\ \dot{x}_2(0) = 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} Q_1(t) \\ Q_2(t) \end{bmatrix} = \begin{bmatrix} 0.0013 & 0.0015 \\ 0.0017 & -0.0015 \end{bmatrix}^{T} \begin{bmatrix} 0 \\ F(t) \end{bmatrix} = \begin{bmatrix} 0.0017F(t) \\ -0.0015F(t) \end{bmatrix}$$

Note: even though there is no force acting on co-ordinate $x_1(t)$ in physical domain, there is a force acting on $q_1(t)$ in modal domain. This applies to initial conditions as well: initial conditions on x_2 can lead to initial conditions on $q_1(t)$, and similarly for other co-ordinates.

Equations of motion for $q_1(t)$:

$\ddot{q}_1(t) + \omega_1^2 q_1(t) = 0.0017F(t), \quad q_1(t) = 0, \dot{q}_1(t) = 0$

Total solution is:

$$\boxed{q_1(t) = \underbrace{C_1 \cos(\omega_1 t) + C_2 \sin(\omega_1 t)}_{homogeneous} + \underbrace{\int_0^t h_1(t-\tau) 0.0017F(\tau) d\tau}_{particular}}$$

Note that the impulse response of an undamped SDOF equation associated with $q_1(t)$ is given by

$$\boxed{h_1(t) = \frac{1}{\omega_1} \sin(\omega_1 t)}.$$

Now the particular solution is
$q_{1p} = \int_0^t h_1(t-\tau)0.0017F(\tau)d\tau = \int_0^t \sin\left[\omega_1(t-\tau)\right]0.0017\times 25000 d\tau = 0.2778\left[1-\cos(\omega_1 t)\right], \quad \omega_1 = 12.2474 rad/s$

We must always impose the initial conditions on the total solution to find C_1 and C_2. Zero initial displacement condition on the <u>total</u> response $q_1(0) = 0$ gives $C_1 = 0$. Similarly zero initial velocity gives $C_2 = 0$. Thus the total response of $q_1(t)$ is

$q_1(t) = 0.2778\left[1-\cos(12.2474t)\right], 0 \leq t \leq 0.1$

Equations of motion for $q_2(t)$:
$\ddot{q}_2(t) + \omega_2^2 q_2(t) = -0.0015F(t), \quad q_2(t) = 0, \dot{q}_2(t) = 0.$
Following the same procedure as for $q_1(t)$ the response is obtained as
$q_2(t) = -0.0248\left[1-\cos(38.7298t)\right], 0 \leq t \leq 0.1$

We obtain the response in physical co-ordinates using the transformation
$\boldsymbol{x} = \boldsymbol{\Phi q}$ or

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} == \begin{bmatrix} 0.0013 & 0.0015 \\ 0.0017 & -0.0015 \end{bmatrix} \begin{bmatrix} 0.2778\left[1-\cos(12.2474t)\right] \\ -0.0248\left[1-\cos(38.7298t)\right] \end{bmatrix}.$$

Note that the above solution is valid only for the time interval $0 \leq t \leq 0.1s$. After $t = 0.1s$ the vibration is a free vibration with initial conditions given by the displacements and velocities <u>at time $t = 0.1s$</u>. The response plots are shown below.

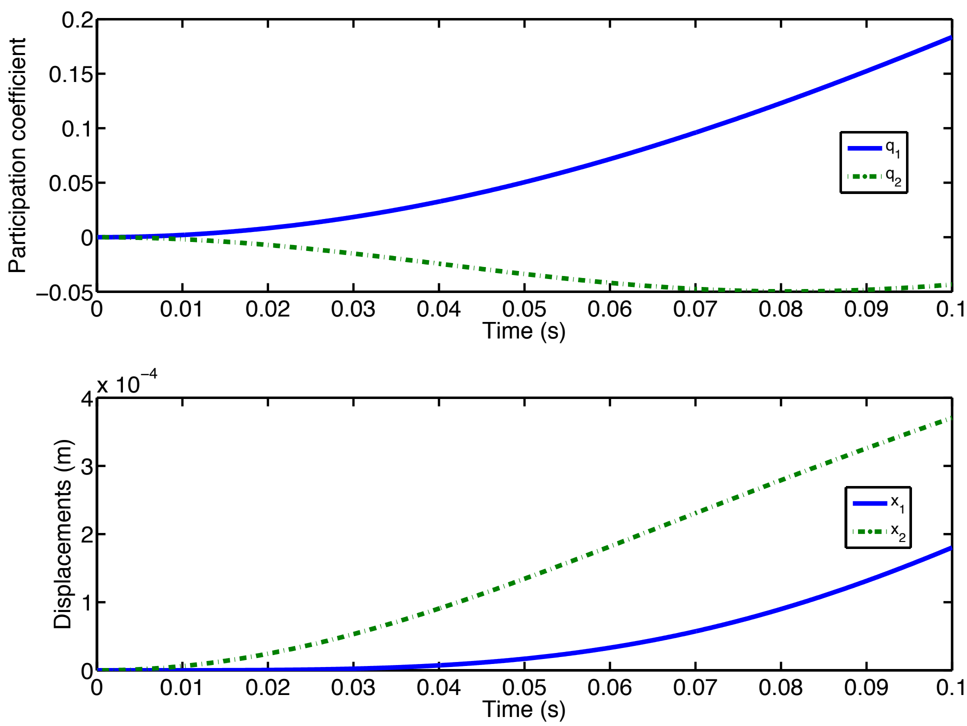


Figure 5.1: Displacement response in normal and physical co-ordinates

To summarise:

1. The free and forced vibration response of two DOF and MDOF systems can be understood in terms of natural frequencies and natural modes.
2. The shape of the deformation patttern associated with a natural frequency ω_i is called its natural mode $\boldsymbol{u}_i$.
3. The natural frequencies and natural modes can be determined by solving the symmetric, linear, algebraic, eigenvalue problem: $\boldsymbol{K}\boldsymbol{u}_i = \omega_i^2 \boldsymbol{M}\boldsymbol{u}_i$.
4. With the aid of the transformation $\boldsymbol{x} = \boldsymbol{\Phi}\boldsymbol{q}$ we can uncouple the equations of motion.
5. Determining principal co-ordinates is non-trivial. It requires the solution of a linear algebraic eigenvalue problem. So principal co-ordinates are not immediately obvious!
6. Coupling leads to an exchange of energy among the two modes. Initial conditions can also lead to coupled vibrations. Coupling is to be expected in general, even though we uncouple the equations of motion for computational purposes.
7. We can interpret the coefficients of stiffness matrix as *influence coefficients*. k_{ij} is the force requied at point i to cause a unit deflection at point j and zero deflections at all other points. k_{ij} can be found from statics. By reciprocity the stiffness matrix $\boldsymbol{K}$ is symmetric.

Topic 5: Two and Multi-Degree of Freedom (MDOF) Systems – Part 2

The main objective of this topic is to extend the SDOF analysis of vibration absorbers in frequency domain already discussed in Topic 2. A 2 DOF analysis will be presented using matrix methods. It will be seen that the design of an absorber involves an optimisation in which the range of operation is maximised while keeping the mass of the absorber as small as possible.

5.1 Introduction

Vibration absorbers are simple devices used to channel away vibration energy from the main system (a machine, helicopter, power transmission lines) into an auxiliary absorber unit. See Fig.(5.1) below.

Figure 5.1: Typical applications of vibration absorbers. In the helicopter (top-right) and transmission tower (bottom-left) the absorbers are highlighted in a circle.

A vibration absorber works by applying a large counter force on the main system. This makes the main system not move **at the point of attachment** of the absorber. This reduction in vibration of the main system is achieved at the expense of the increased displacements of the absorber mass, since the absorber is now subjected to resonance in order to extract energy from the main system.

Previously, we have analysed the response of a system to which an absorber is connected in frequency domain by enforcing equilibrium and compatibility conditions at the point of attachment. Let us recall the SDOF analysis we discussed earlier. Let us consider how each FRF can be obtained.

The design problem is to reduce the vibrations of a system at a point where vibrations are not desired. For example, if you want to get a high quality photograph in a situation where a camera is mounted at the end of a long arm, we want less vibration at the point where the camera is mounted. We are given the frequency response function (FRF) at the point of interest. Denote this as H_{system}. How do we find this? Typically H_{system} can be measured: by placing a sensor at the point of interest and applying a force at different frequencies. You have done this in Shaky Table experiment. Note that, there, you have measured each point of the FRF (or DMF depending on what you plot on $Y-$ axis). This may seem a cumbersome process. In reality, an ideal impulse gives all frequencies equal energies (see example 30 from Fourier Transforms). Thus, by using the Fourier transform we can measure H_{system} of shaky table by applying a single impulse at the point of interest, using an impulse hammer. This hammer testing takes less than a minute to obtain the FRF, whereas exciting the shaky table at each frequency will take at least 2 hours. This is another practical use of Fourier transform!

Given H_{system}, our goal is to modify the response at one frequency (forcing frequency of the steady harmonic force, ω) by attaching an absorber unit. We can think of this as combining two systems: the main system whose FRF is given by H_{system} and the absorber unit whose FRF is given by H_a. At the point of connection, forces add while displacements are the same. This gave us the simple rule to predict the FRF of the combined system using the following relation:

$$\frac{1}{H_{combined}} = \frac{1}{H_{system}} + \frac{1}{H_a}, \quad H_a = -\frac{k_a - m_a\omega^2}{k_a m_a \omega^2} \tag{1}$$

where H_{system} is given, H_a **can be calculated by choosing** k_a **and** m_a. If we select the parameters k_a and m_a such that forcing frequency, ω, matches the absorber's natural frequency $\omega_a = \sqrt{\frac{k_a}{m_a}}$ then the absorber applies an infinitely large force on the main system because $H_a = 0$!

Recall that H_a is the ratio of the displacement to force in the frequency domain (see page 182 in Topic 4) which is valid in the steady state. Thus in designing an absorber we are reducing the **steady state** response! **A system's perspective would be to see the absorber as a device which imposes a zero in the transfer function at the natural frequency of the absorber,** ω_a. By varying ω_a we can reduce vibrations at different target frequencies.

Here is the summary of SDOF analysis:

1. Given H_{system} **at a point** and the frequency ω where vibration needs to be suppressed.

2. Design the absorber by varying k_a and m_a such that vibration is suppressed at the frequency of interest ω. A simple way will be to match the absorber's resonant frequency $\omega_a = \sqrt{\frac{k_a}{m_a}}$ to the external forcing frequency $\omega_a = \omega$.

3. In practise, one would like to keep the secondary resonances well apart and hence increase the effective frequency range of the absorber.

4. Any damping in the absorber unit will reduce the effectiveness of the absorber in the sense that the steady vibration of the main system will not be zero at the point of attachment, there will be *some residual* vibration. Thus, damping in the absorber unit is undesired.

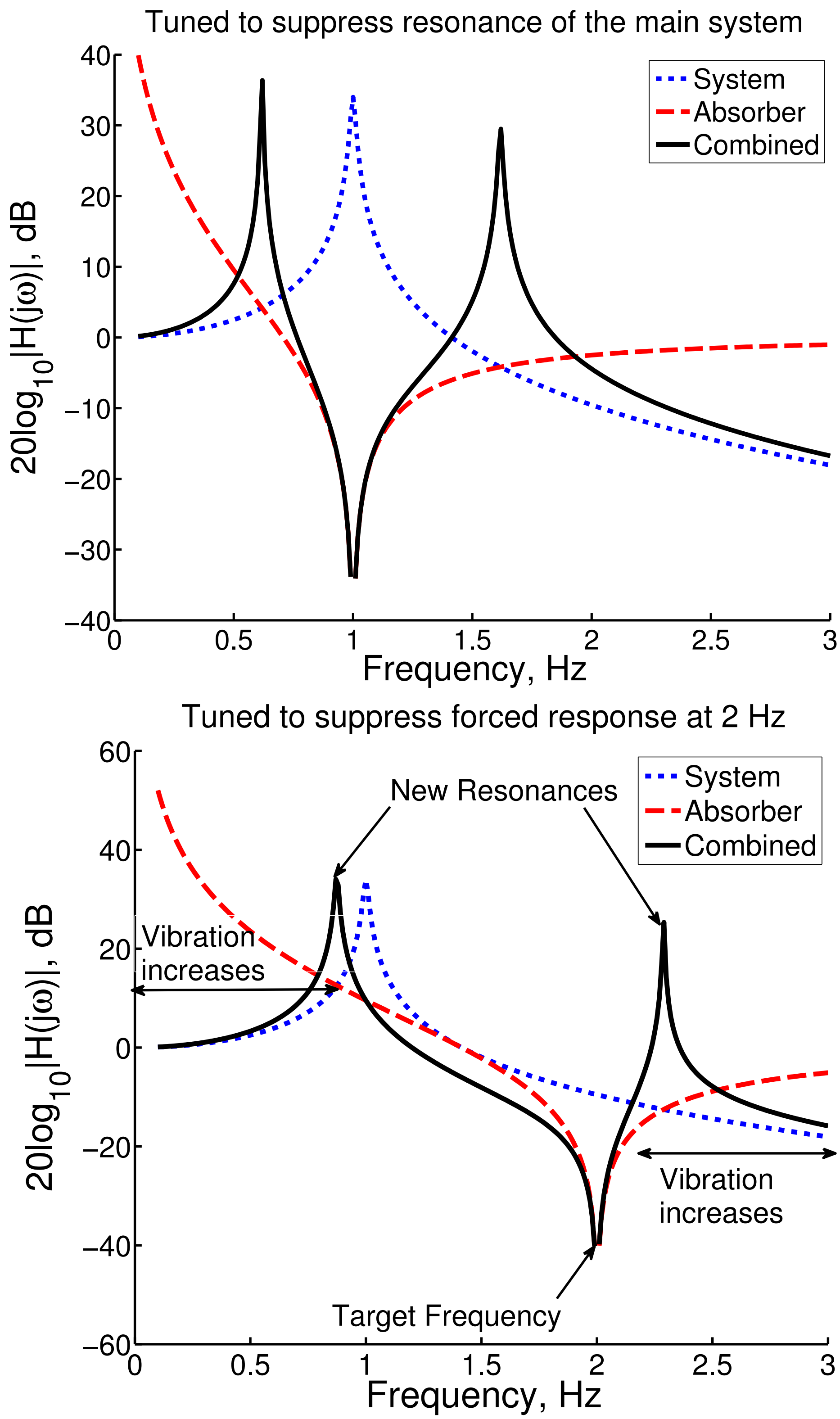


Figure 5.2: FRFs of the original system (dotted line), absorber (dashed line), and combined system (solid line). Notice that there are two resonance peaks in the combined system. **Vibration is suppressed at the target frequency at the expense of increased vibration levels in other frequency regions.** X**dB**$= 10^{X/20}$ **in linear scale.** Vibration is reduced in the range where the black curve (solid) is below the blue curve (dotted).

Clearly, there are two resonance peaks in the combined system's response. We wish to avoid forcing frequencies matching these resonances. In order to do that, we need to find the frequencies of the resonances first. How can we predict resonant frequencies? What are the displacements of the main system and the absorber mass? To answer these, we need to perform a MDOF analysis.

5.2 2 DOF Analysis (T 9.11)

Consider the system shown in Fig.(5.3). An absorber of mass m_2 and stiffness k_2 is connected to a machine of mass m_1 resting on an isolation system of effective stiffness k_1.

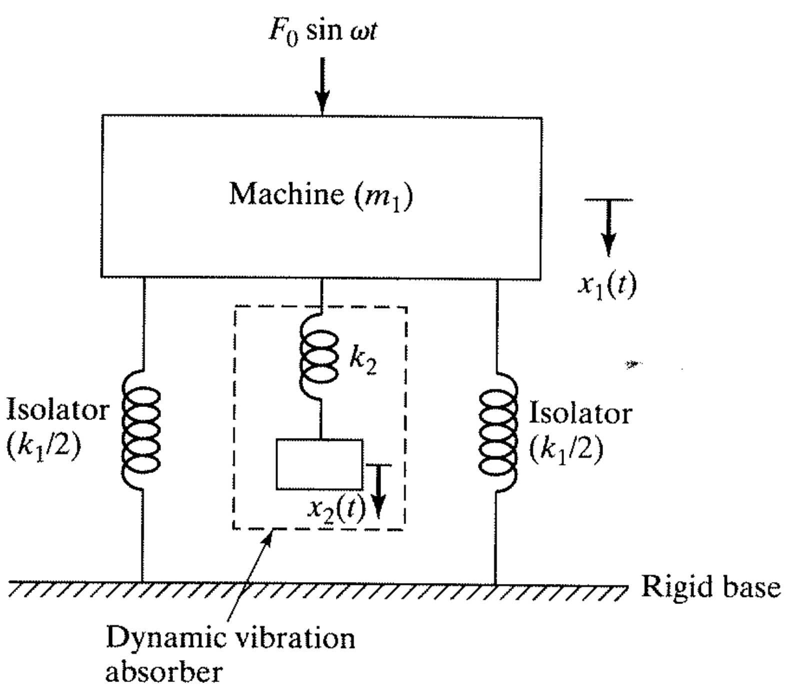


Figure 5.3: An undamped vibration absorber connected to a machine resting on an isolation system.

5.2.1 Equations of Motion

We can formulate the equations of motion using energy and force methods. Let us use the energy method: Potential Energy: $V = \frac{1}{2}2\frac{k_1}{2}x_1^2 + \frac{1}{2}k_2\left[x_2 - x_1\right]^2$. Writing V in terms of the displacement vector $\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$V = \frac{1}{2}k_1x_1^2 + \frac{1}{2}k_2\left[x_2 - x_1\right]^2 = \frac{1}{2}\boldsymbol{x}^T\boldsymbol{K}\boldsymbol{x} = \frac{1}{2}\begin{bmatrix} x_1 & x_2 \end{bmatrix}\begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$\therefore \boldsymbol{K} = \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix}$.

Similarly, kinetic energy is given by $V = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2 = \frac{1}{2}\dot{\boldsymbol{x}}^T\boldsymbol{M}\dot{\boldsymbol{x}} = \frac{1}{2}\begin{bmatrix} \dot{x}_1 & \dot{x}_2 \end{bmatrix}\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix}$

$\therefore \boldsymbol{M} = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}$.

Therefore, the equations of motion of the system shown in Fig.(5.3) in matrix format are:

$$\boldsymbol{M}\ddot{\boldsymbol{x}} + \boldsymbol{K}\boldsymbol{x} = \boldsymbol{f} \tag{2}$$

Or,

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} f_1(t) \\ f_2(t) \end{bmatrix}, \tag{3a}$$

$$f_1 = F_0\sin\omega t = imag(F_0e^{j\omega t}), f_2 = 0. \tag{3b}$$

We do not consider the initial conditions since our interest is in controlling the forced vibration response, or particular solution only.

5.2.2 Natural Frequencies

Assuming a harmonic displacement solution of the form $\boldsymbol{x}(t) = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \boldsymbol{X} e^{j\omega t}$ where $\boldsymbol{X} = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$ and substituting this in the governing equations in Eq.(3) we find that

$$\begin{bmatrix} -m_1\omega^2 & 0 \\ 0 & -m_2\omega^2 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} + \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} F_1 = 0 \\ F_2 = 0 \end{bmatrix}. \tag{4}$$

We can express the above in a more succinct form as follows:

$$\boldsymbol{D}\boldsymbol{X} = \boldsymbol{F} \tag{5a}$$

$$\boldsymbol{D} \equiv \text{Dynamic Stiffness Matrix} = \begin{bmatrix} k_1 + k_2 - m_1\omega^2 & -k_2 \\ -k_2 & k_2 - m_2\omega^2 \end{bmatrix} \tag{5b}$$

$$\boldsymbol{F} = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} \tag{5c}$$

An alternative route to obtain the above from Eq.(3) will be to apply Fourier Transform to governing equations in Eq.(3).

In this regard, we can see that the Frequency Response Function matrix is the inverse of the Dynamic Stiffness Matrix. So FRF matrix is a Dynamic Compliance Matrix. Notice the similarities with SDOF system. Here, instead of one FRF we have an FRF matrix.

$$\boldsymbol{H} = \boldsymbol{D}^{-1}, \quad \boldsymbol{H}\boldsymbol{F} = \boldsymbol{X}, \quad \boldsymbol{H}\text{: FRF matrix in the steady state} \tag{6a}$$

$$X_1 = H_{11}F_1 + H_{12}F_2, \quad X_2 = H_{21}F_1 + H_{22}F_2, \tag{6b}$$

The natural frequencies are obtained by setting $\boldsymbol{f} = \boldsymbol{0}$ in time domain or equivalently $\boldsymbol{F} = \boldsymbol{0}$ in Eq.(5). Doing so results in the characteristic equation as follows:

$$\boldsymbol{D}\boldsymbol{X} = \boldsymbol{0} \tag{7a}$$

$$\Rightarrow |\boldsymbol{D}| = 0, \quad \text{if } \boldsymbol{X} \neq \boldsymbol{0} \tag{7b}$$

$$\Rightarrow \left[k_1 + k_2 - m_1\omega^2\right]\left[k_2 - m_2\omega^2\right] - k_2^2 = 0 \tag{7c}$$

where $|\boldsymbol{D}|$ is the determinant of the matrix $\boldsymbol{D}$. To simplify the analysis let us introduce the following parameters:

$$\omega_n^2 = \frac{k_1}{m_1} = \text{Natural frequency of the main systems on it's own.} \tag{8a}$$

$$\omega_a^2 = \frac{k_2}{m_2} = \text{Natural frequency of the absorber systems on it's own.} \tag{8b}$$

$$\mu = \frac{m_2}{m_1} = \text{Ratio of absorber mass to main mass.} \tag{8c}$$

$$\beta = \frac{\omega_a}{\omega_n} = \text{Ratio of natural frequency of the absorber to the main system.} \tag{8d}$$

$$r = \frac{\omega}{\omega_n} \tag{8e}$$

With the above parameter definitions the characteristic equation in Eq.(7c) can be simplified as follows:

$$\left[k_1 + k_2 - m_1\omega^2\right]\left[k_2 - m_2\omega^2\right] - k_2^2 = 0 \tag{9a}$$

$$\left[m_1\omega_n^2 + m_2\omega_a^2 - m_1\omega^2\right]\left[m_2\omega_a^2 - m_2\omega^2\right] - m_2^2\omega_a^4 = 0 \tag{9b}$$

$$\text{Divide with } m_1 m_2 \omega_n^2 \omega_a^2 \tag{9c}$$

$$\left[1 + \frac{m_2}{m_1}\frac{\omega_a^2}{\omega_n^2} - \frac{\omega^2}{\omega_n^2}\right]\left[1 - \frac{\omega^2}{\omega_n^2}\right] - \frac{m_2}{m_1}\frac{\omega_a^2}{\omega_n^2} = 0 \tag{9d}$$

$$\left[1 + \mu\beta^2 - \frac{\omega^2}{\omega_n^2}\right]\left[1 - \frac{\omega^2}{\omega_n^2}\right] - \mu\beta^2 = 0 \tag{9e}$$

$$\left[1 + \mu\beta^2 - r^2\right]\left[1 - r^2\right] - \mu\beta^2 = 0 \tag{9f}$$

The mass ratio μ and the frequency ratio β de-

cide the roots $r_{1,2}$ of the characteristic equation, and hence the location of the natural frequencies along the frequency axis. Thus for a given value of β, we can design for as wide a separation of the natural frequencies as possible for reasonable values of μ. We can construct a design chart of μ vs. r for different values of β.

For $\beta = 1$, that is the absorber is tuned to natural frequency of the main system such that $\omega_a = \omega_n$ then the two resonant frequencies are given by the roots r_1 and r_2 of the above characteristic equation which read as

$$r_{1,2}^2 = \frac{2+\mu}{2} \pm \frac{1}{2}\sqrt{(2+\mu)^2 - 4} \tag{10}$$

and a plot of μ vs. r in Fig.(5.4) shows that wider separation is possible for higher mass ratios. In practise we do not want the absorber mass to be too heavy as it defeats the purpose of lightweight design. **The practical design requires optimisation** which is beyond the scope of this introductory course. Suffice it to say that an optimum tuning occurs when

$$\boxed{\beta = \frac{1}{1+\mu} \quad \text{Optimal tuning condition} \qquad (11)}$$

As mentioned earlier damping in the absorber defeats the purpose as it does not reduce the main system's response

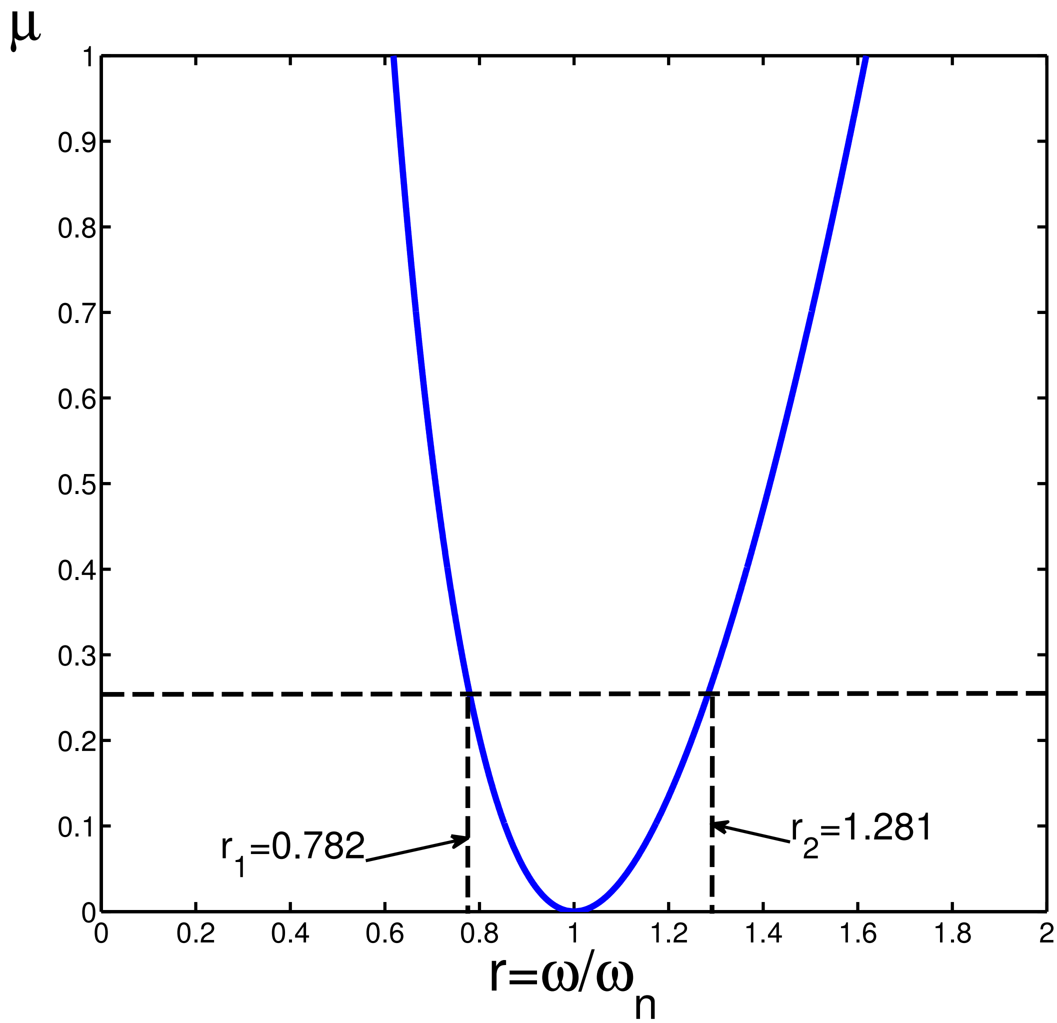
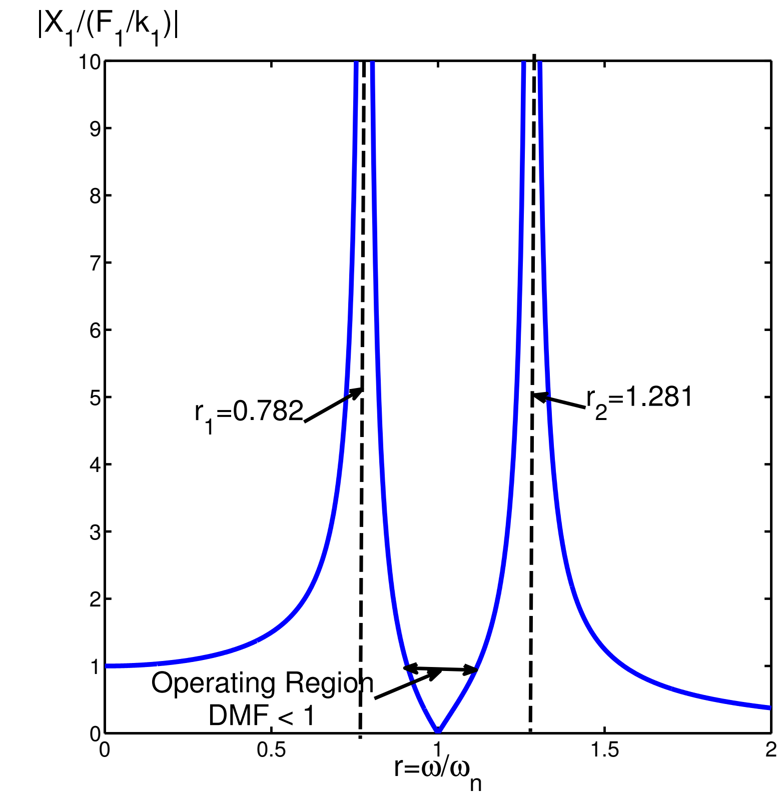


Figure 5.4: Influence of mass ratio $\mu = \frac{m_2}{m_1}$ on the location of natural frequencies. Higher mass ratio is preferred for wider spacing. But higher mass ratio means that the absorber is heavy! An optimal mass ratio needs to be obtained according to optimal tuning $\beta = \frac{1}{1+\mu}$.

to zero. Optimal design of the absorber in the presence of damping is beyond the scope of this course.

The effective range of operation is summarised in Fig.(5.5).



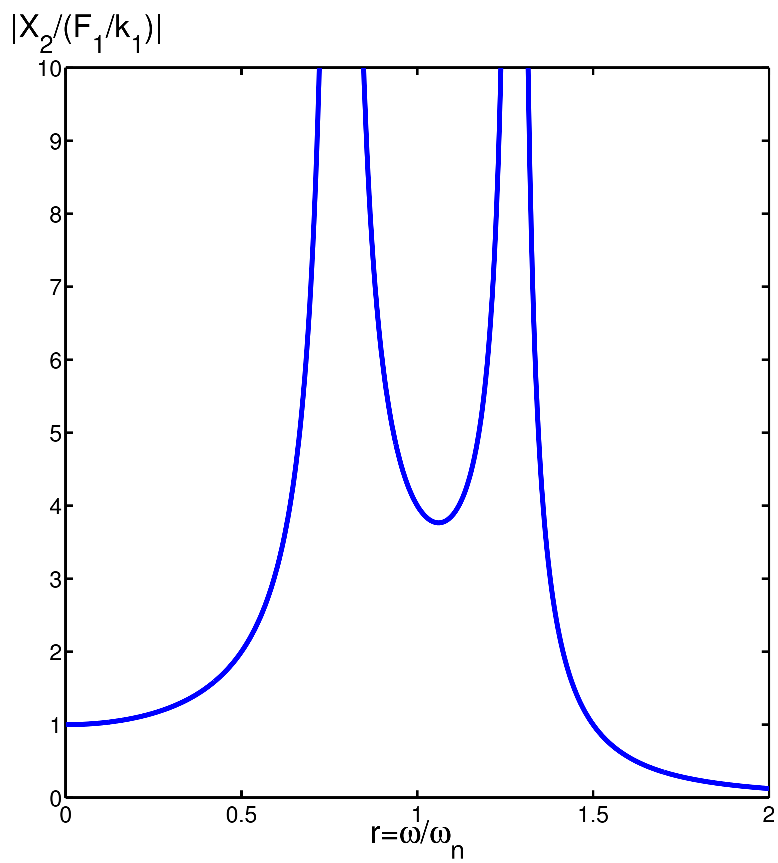


Figure 5.5: DMF curves for the undamped vibration absorber tuned to suppress the resonances of the main system $\omega_a = \omega_n$ for a mass ratio $\mu = \frac{1}{4}$. The operating region for forcing frequencies is set by the frequency region where DMF <1 for the main mass (top curve).

5.2.3 Displacements

We can obtain the displacements of each mass by inverting the Dynamics Stiffness Matrix $\boldsymbol{D}$ and multiplying with the force vector $\boldsymbol{F}$.

$$\boldsymbol{X} = \boldsymbol{D}^{-1}\boldsymbol{X} \tag{12a}$$

$$\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} k_1 - m_1\omega^2 & -k_2 \\ -k_2 & k_2 - m_2\omega^2 \end{bmatrix}^{-1} \begin{bmatrix} F_1 \\ F_2 = 0 \end{bmatrix} \tag{12b}$$

$$\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} \frac{F_1(k_2 - m_2\omega^2)}{\left[k_1+k_2-m_1\omega^2\right]\left[k_2-m_2\omega^2\right]-k_2^2} \\ \frac{F_1 k_2}{\left[k_1+k_2-m_1\omega^2\right]\left[k_2-m_2\omega^2\right]-k_2^2} \end{bmatrix} \tag{12c}$$

It can be imemdiately observed that at the resonant frequency of the absorber $\omega = \omega_a$, we have $k_2 - m_2\omega^2 = k_2 - m_2\omega_a^2 = k_2 - m_2\frac{k_2}{m_2} = 0$. This suggests that $X_1 = 0$ and $X_2 = -\frac{F_1}{k_2}$. The main system does not move and the displacement amplitude of the absorber mass is the static deflection of the absorber subjected to the force of magnitude F_1. **It is as if the force F_1 is applied to the absorber unit!** We can construct the DMF curves for each displacement in terms of the non-dimensional parameters as follows:

$$X_1 = \frac{F_1}{k_1} \frac{\left[1 - r^2\beta^2\right]}{\left[1 + \mu\beta^2 - r^2\right]\left[1 - r^2\right] - \mu\beta^2} \tag{13a}$$

$$\Rightarrow \frac{X_1}{\frac{F_1}{k_1}} = \frac{\left[1 - r^2\beta^2\right]}{\left[1 + \mu\beta^2 - r^2\right]\left[1 - r^2\right] - \mu\beta^2} \tag{13b}$$

$$X_2 = \frac{F_1}{k_1} \frac{1}{\left[1 + \mu\beta^2 - r^2\right]\left[1 - r^2\right] - \mu\beta^2} \tag{13c}$$

$$\tag{13d}$$

$$\Rightarrow \frac{X_2}{\frac{F_1}{k_1}} = \frac{1}{\left[1 + \mu\beta^2 - r^2\right]\left[1 - r^2\right] - \mu\beta^2} \tag{13e}$$

Note that the DMF curve goes to infinity at the roots of the characteristic equation $\left[1 + \mu\beta^2 - r^2\right]\left[1 - r^2\right] - \mu\beta^2 = 0$.

In practise, we desire a wider spacing between the two resonances while keeping the mass ratio small. Wider spacing implies large frequency range of operation. An optimisation is required in practise. However, simple design calculation based on $\omega = \omega_a$ and restrictions on maximum excursions of the absorber mass can be performed. This will be illustrated next in Example 34.

Example 34 : A diesel engine weighing 3000 N, is supported on a pedestal mount. it has been observed that the engine induces vibration into the surrounding area through its pedestal mount at an operating speed of 6000 rpm. Determine the parameters of the vibration absorber that will reduce the vibration when mounted on the pedestal. The magnitude of the exciting force is 250 N, and the amplitude motion of the auxiliary mass is to be limited to 2 mm. See Fig.(5.1) for a practical application of absorber.

Solution:

Fill in the class

Given: $\omega = 6000\frac{2\pi}{60} = 628.32$ rad/s.

$F_0 = m_2\omega^2 X_2 = 250$ N; $X_2 = 2 \times 10^{-3}$ m.

$\Rightarrow m_2 =$ Absorber mass $= \frac{F_0}{\omega^2 X_2} = \frac{250}{628.32^2 \times 2 \times 10^{-3}} = 0.3165 kg.$

$\Rightarrow k_2 =$ Absorber stiffness $= m_2\omega^2 = 0.3165 \times 628.32^2 = 125009$ N/m.

If the absorber is a cantilever with a tip mass as in Fig.(5.1) on Page 224, then $k_2 = \frac{3EI}{L^3}$. For a given Young's modulus E and given moment of inertia I, we can decide the length of the beam L.

To summarise:

1. A vibration absorber is a device that applies a large counter force at a single frequency, called tuning frequency, which can be adjusted by varying the absorber's resonant frequency through changing it's mass and stiffness. Typically, one sets absorber's resonance to match forcing frequency: $\omega_a = \omega$.

2. Mostly, an absorber is a single frequency device. It's operating range is limited to a narrow band surrounded by secondary resonances. Compare this with an isolator!

3. Damping in the absorber reduces it's effectiveness and hence not desired.

4. **SDOF analysis is more general as the main system is allowed to have any number of degrees of freedom. This generality and efficiency is due to <u>frequency domain</u> analysis which is possible because of Fourier Transform! A matrix MDOF analysis is possible but cumbersome!!**

5. Further insight was provided by the MDOF analyis based on matrix methods.

6. Both the mass ratio μ and frequency ratio β decide the location of the secondary resonances and hence the effective frequency region of operation for the absorber.

7. The design objective of wide separation for resonances with minimal added mass is achieved using optimal tuning.

Appendix A: Eigenvalue Problems

Mathematics is an expression of an idea in an efficient form. It is abstract and difficult — only if we ignore the idea and concentrate on the form.

In mechanics and elsewhere we often encounter eigenvalue problems. It is no exaggeration to say that *all* stability problems have some relation to eigenvalue problems: buckling loads are eigenvalues and buckling modes are eigenvectors.

Eigenvalues and eigenvectors also assume an important place in co-ordinate transformations. For example, Mohr's transformation of stresses naturally leads to principal directions and principal stresses, when cast as an eigenvalue problem. In vibration problems eigenvectors define the principal coordiantes in which the equations of motion of a N DOF system are uncoupled into N SDOF systems.

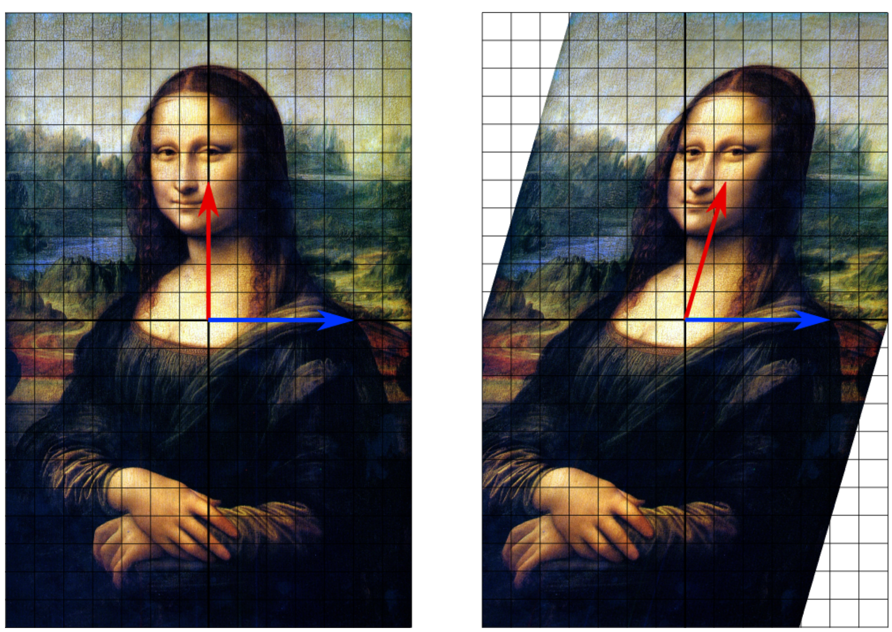


Figure 1.1: An operation of simple shear on Mona Lisa.

Given the diversity of applications, it is prudent to use the language of mathematics.

We can think of

$$\boldsymbol{A}\boldsymbol{x} = \boldsymbol{y} \tag{1}$$

as a mapping or operation. Here the operator $\boldsymbol{A}$ operates on a vector $\boldsymbol{x}$ (which can be thought as input to $\boldsymbol{A}$) and transforms into another vector $\boldsymbol{y}$ (which can be seen as the output).

Consider Fig.(1.1). Here all vectors contained in the image of Mona Lisa are sheared. Thus the operation of deforming Mona Lisa by subjecting her to simple shear can be represented by

$$\boldsymbol{A}\boldsymbol{x} = \boldsymbol{y} \tag{2}$$

where $\boldsymbol{x}$ is a vector in the image on the left which is mapped into a vector $\boldsymbol{y}$ in the image on the right *after* subjecting it to the shear operator defined by[1]

$$\boldsymbol{A} = \begin{bmatrix} 1 & \tan\gamma \\ 0 & 1 \end{bmatrix} \tag{3}$$

where γ is the angle of shear measured with respect to vertical plane. Observing the two vectors (blue and red) in Fig.(1.1) shows that some vectors do not change their direction. For example, the blue vector remains horizontal in the undeformed and deformed configurations. This

[1] A point (x,y) in the left image maps to (x+y$\tan\gamma$, y) in the right image in Fig.(1.1).

vector which does not change it's direction when a matrix operates on it is called the eigenvector of **that** matrix. The red vector is not an eigenvector since it changes it's direction when $\boldsymbol{A}$ acts on it. How does one find eigenvectors? Simply by requiring that the vector should not change it's direction (or line of action to be precise), thus $\lambda\boldsymbol{x}$ is allowed but not $\boldsymbol{x}+\boldsymbol{x}_0$ where $\boldsymbol{x}_0$ is another constant vector of same size. So we want $\boldsymbol{y} = \boldsymbol{A}\boldsymbol{x}$ to be along $\boldsymbol{x}$, that is, $\boldsymbol{y} = \lambda\boldsymbol{x}$, thus giving us the eigenvalue problem

$$\boldsymbol{A}\boldsymbol{x} = \boldsymbol{y} = \lambda\boldsymbol{x}. \tag{4}$$

What about the meaning of λ? Consider $\lambda = 0$ here the vector $\boldsymbol{x}$ is mapped into a null vector which is the origin. If $\lambda < 0$ then the vector $\boldsymbol{y} = \boldsymbol{A}\boldsymbol{x}$ flips it's direction along the same line (which is ok). If $\lambda > 0$ the vector $\boldsymbol{y} = \boldsymbol{A}\boldsymbol{x}$ points in the same direction as $\boldsymbol{x}$. Furthermore: $\lambda > 1$ stretches $\boldsymbol{x}$ while $\lambda < 1$ shortens $\boldsymbol{x}$. Thus λ can be seen as a measure of the amount by which an eigenvector changes it's magnitude when operated upon by the matrix.

Finding λ and $\boldsymbol{x}$ requires us to solve the eigenvalue problem. We will do this separately, since it depends on the type of the matrix: square symmetric or not.

Question: If $\boldsymbol{A} = \boldsymbol{K}$ is the stiffness matrix of a structure/machine what does $\lambda = 0$ physically mean and what do the corresponding eigenvectors represent?

Appendix B: MDOF Systems to SDOF Systems

Multi-Degree-of-Freedom (MDOF) systems require more than one co-ordinate to describe their motion. Applying Newton's second law to each co-ordinate gives one equilibrium equation. Thus, a N-DOF system is governed by N Ordinary Differential Equations (ODEs). In general these N ODEs are coupled via stiffness (elastic coupling) or inertial (inertial coupling) forces.

The coupled equations can be represented in a succinct matrix form as follows:

$$\boldsymbol{M}\ddot{\boldsymbol{x}} + \boldsymbol{C}\dot{\boldsymbol{x}} + \boldsymbol{K}\boldsymbol{x} = \boldsymbol{f}; \text{and} \quad \boldsymbol{x}(0) = \boldsymbol{x}_0, \quad \dot{\boldsymbol{x}}(0) = \dot{\boldsymbol{x}}_0, \text{ Initial conditions} \tag{1}$$

The mass matrix $\boldsymbol{M}$, the stiffness matrix $\boldsymbol{K}$, and the damping matrix $\boldsymbol{C}$ are not diagonal matrices. The off-diagonal entries in any of these matrices can couple the N ODEs.

By uncoupling the equations we turn the equations in Eq.(1) into N uncoupled SDOF systems. **This uncoupling is achieved using the co-ordinate transformations obtained by solving an eigenvalue problem**. The undamped ($\boldsymbol{C} = \boldsymbol{0}$) free-vibrations of the form $\boldsymbol{x}(t) = \boldsymbol{u}e^{j\omega t}$ define an eigenvalue problem of the following form:

$$\boldsymbol{K}\boldsymbol{u} = \omega^2\boldsymbol{M}\boldsymbol{u}, \quad \text{or } \boldsymbol{K}\boldsymbol{u} = \lambda\boldsymbol{M}\boldsymbol{u} \tag{2}$$

There are N undamped natural frequencies (or eigenvalues) given by the above eigenvalue problem. Associated with each eigenvalue $\lambda_n = \omega_n^2$, there is a modeshape (or eigenvector) $\boldsymbol{u}_n$. $\boldsymbol{u}_n$ represents the deformation *pattern* in each natural mode at frequency ω_n. It turns out that the elastic forces associated with $\boldsymbol{u}_i$, $\boldsymbol{K}\boldsymbol{u}_i$, do no work on any other mode $\boldsymbol{u}_j$. Similarly, the inertial forces associated with $\boldsymbol{u}_i$, $\boldsymbol{M}\boldsymbol{u}_i$, do no work on any other mode $\boldsymbol{u}_j$. These two conditions define the orthogonality conditions:

$$\begin{aligned}
\boldsymbol{u}_i^T\boldsymbol{M}\boldsymbol{u}_j &= m_{ij} = 0 \text{ if } i \neq j \\
\boldsymbol{u}_i^T\boldsymbol{K}\boldsymbol{u}_j &= k_{ij} = 0 \text{ if } i \neq j \\
\boldsymbol{u}_i^T\boldsymbol{M}\boldsymbol{u}_i &= m_{ii} \text{ if } i = j \\
\boldsymbol{u}_i^T\boldsymbol{K}\boldsymbol{u}_i &= k_{ii} \text{ if } i = j
\end{aligned} \tag{3}$$

Thus, introducing the co-ordinate transformation

$$\begin{aligned}
&\boldsymbol{x}(t) = \boldsymbol{U}\boldsymbol{q}(t) \\
&\Rightarrow \boldsymbol{x}(t) = q_1(t)\boldsymbol{u}_1 + q_2(t)\boldsymbol{u}_2 + q_3(t)\boldsymbol{u}_3 + \dots q_N(t)\boldsymbol{u}_N \\
&\text{Note: } \boldsymbol{U} \text{ contains } \boldsymbol{u}_n \text{ as nth column}
\end{aligned} \tag{4}$$

in Eq.(1) gives

$$\boldsymbol{M}\boldsymbol{U}\ddot{\boldsymbol{q}} + \boldsymbol{C}\boldsymbol{U}\dot{\boldsymbol{q}} + \boldsymbol{K}\boldsymbol{U}\boldsymbol{q} = \boldsymbol{f}. \tag{5}$$

Premultiplying the above with $\boldsymbol{U}^T$ and using the orthogonality conditions in Eq.(3) uncouples the equations, since $\boldsymbol{U}^T\boldsymbol{M}\boldsymbol{U} = diag(m_{ii})$, $\boldsymbol{U}^T\boldsymbol{K}\boldsymbol{U} = diag(k_{ii})$ and $\boldsymbol{U}^T\boldsymbol{C}\boldsymbol{U} = diag(c_{ii})$. Denoting $\boldsymbol{Q} = \boldsymbol{U}^T\boldsymbol{f}$, we have

$$\boldsymbol{U}^T\boldsymbol{M}\boldsymbol{U}\ddot{\boldsymbol{q}} + \boldsymbol{U}^T\boldsymbol{C}\boldsymbol{U}\dot{\boldsymbol{q}} + \boldsymbol{U}^T\boldsymbol{K}\boldsymbol{U}\boldsymbol{q} = \boldsymbol{U}^T\boldsymbol{f} = \boldsymbol{Q}; \text{and} \quad \boldsymbol{q}(0) = \boldsymbol{U}^{-1}\boldsymbol{x}_0, \quad \dot{\boldsymbol{q}}(0) = \boldsymbol{U}^{-1}\dot{\boldsymbol{x}}_0, \text{ initial conditions in new co-ordinates} \tag{6}$$

which can be written explicitly as

$$\boxed{m_{ii}\ddot{q}_i + c_{ii}\dot{q}_i + k_{ii}q_i = Q_i, \text{initial conditions: } \boldsymbol{q}(0) = \boldsymbol{U}^{-1}\boldsymbol{x}_0, \quad \dot{\boldsymbol{q}}(0) = \boldsymbol{U}^{-1}\dot{\boldsymbol{x}}_0, \text{with force: } Q_i = \boldsymbol{u}_i^T\boldsymbol{f}} \tag{7}$$

The above represent N SDOF systems for each co-ordinate $q_i(t)$ $i = 1\dots N$, which can be solved using techniques developed in Topic 2. We obtain the displacements in original co-ordinates through the transformation $\boldsymbol{x} = \boldsymbol{U}\boldsymbol{q}$. Note that when we scale the eigenvectors such that $m_{ii} = 1$ we call them mass normalised modes and denote them by $\boldsymbol{\phi}$. In this case $k_{ii} = \omega_i^2$ and $c_{ii} = 2\zeta_i\omega_i$ and we use $\boldsymbol{\Phi}$ instead of $\boldsymbol{U}$ to define the co-ordinate transformation (see notes on MDOF systems).

MECH 463: Summary

Degrees of Freedom: Minimum number of independent co-ordinates required to completely specify a system's motion. A single degree of freedom system requires one co-ordinate, two DOF system requires 2 co-ordinates *etc.*

Vibration Analysis: Develop a model → Formulate equations of motion using Newton/D'Alembert (FBD) or Energy methods → Solve for response → Design calculations. **Equivalent systems are valid at a single spatial location where the response is sought.** For springs in series $\frac{1}{k_{eq}} = \frac{1}{k_1} + \frac{1}{k_2}$ for springs in parallel $k_{eq} = k_1 + k_2$. In general equivalent mass is obtained from kinetic energy expression: $KE = \frac{1}{2} m_{eq}\dot{x}^2$ and equivalent spring constant is obtained from potential energy expression: $PE = \frac{1}{2}k_{eq}x^2$

Planar Kinematics: $\boldsymbol{r} = r\boldsymbol{e}_1$ (displacement); $\frac{d\boldsymbol{r}}{dt} = \dot{\boldsymbol{r}} = \dot{r}\boldsymbol{e}_1 + r\dot{\theta}\boldsymbol{e}_2$ (velocity); $\frac{d^2\boldsymbol{r}}{dt^2} = \ddot{\boldsymbol{r}} = \left[\ddot{r} - r\dot{\theta}^2\right]\boldsymbol{e}_1 + \left[2\dot{r}\dot{\theta} + r\ddot{\theta}\right]\boldsymbol{e}_2$ (accln.) $\boldsymbol{e}_2$ is $\boldsymbol{e}_1$ rotated 90^0 in the direction of the angular velocity vector $\boldsymbol{\omega}$

Harmonic Response of a Viscously Damped SDOF System (Underdamped): Equation of motion: $m\ddot{x}+c\dot{x}+kx = F\cos(\omega t)$; $x = x_h + x_p$ $x_h = e^{-\zeta\omega_n t}\left[C_1\cos(\omega_d t) + C_2\sin(\omega_d t)\right]$; $x_p = X\cos(\omega t - \phi)$, $\omega_d = \omega_n\sqrt{1-\zeta^2}$, $X = \frac{F}{\sqrt{[k-m\omega^2]^2+[c\omega]^2}} = \frac{F}{k}\frac{1}{\sqrt{[1-r^2]^2+[2\zeta r]^2}} = \delta_{st}\frac{1}{\sqrt{[1-r^2]^2+[2\zeta r]^2}}$, $\delta_{st} \equiv \frac{F}{k}$, $\tan\phi = \frac{c\omega}{k-m\omega^2} = \frac{2\zeta r}{1-r^2}$, $r \equiv \frac{\omega}{\omega_n}$, $\zeta = \frac{c}{2m\omega_n}$

Note: (a) $F = m_u e\omega^2$ for an unbalanced eccentric mass m_u at a distance e from center of rotation; (b) Set $\zeta = 0$ and $\omega_d = \omega_n$ to obtain undamped SDOF response; (c) Initial conditions ALWAYS apply on the TOTAL response.

Natural Frequency: Free vibration of an undamped SDOF system takes place at it's natural frequency $\omega_n = \sqrt{\frac{k_{eq}}{m_{eq}}} = \sqrt{\frac{g}{\delta_{st}}}$ for linear vibrations and $\omega_n = \sqrt{\frac{k_{eq}}{J_{eq}}} = \sqrt{\frac{g}{\theta_{st}}}$ for torsional vibrations. MDOF systems have more than one natural frequency and each natural frequency has a characteristic modeshape associated with it. They are obtained by solving the eigenvalue problem $\boldsymbol{Ku} = \omega^2\boldsymbol{Mu}$.

Damping Measures: $\delta = \frac{1}{N}\ln\left(\frac{x_1}{x_{1+N}}\right) = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}} \approx 2\pi\zeta$, $\zeta = \frac{c}{c_c} = \frac{c}{2m\omega_n} = \frac{c}{2\sqrt{km}}$, $Q = \frac{\text{Dynamic displacement at } \omega = \omega_n}{\text{Static displacement, } \delta_{st}} = \frac{1}{2\zeta}$

Isolation System Design

$TR = \frac{F_t}{F} = \frac{\sqrt{k^2+(c\omega)^2}}{\sqrt{(k-m\omega^2)^2+(c\omega)^2}} = \frac{\sqrt{1+(2\zeta r)^2}}{\sqrt{(1-r^2)^2+(2\zeta r)^2}}$; and the displacement transmissibility is given by $TR_d = \frac{X}{Y} = \frac{\sqrt{1+(2\zeta r)^2}}{\sqrt{(1-r^2)^2+(2\zeta r)^2}}$, where $r = \frac{\omega}{\omega_n}$. Maximum value occurs at $r = 1$. $TR < 1$ above $r = \sqrt{2}$. For rotating unbalance replace $F = m_u e\omega^2$

Forced Response (General): Use the look-up table→Use Fourier series if the force is periodic→ Use convolution integral $x_p = \int_0^t h(t-\tau)f(\tau)d\tau$, where $h(t) = \frac{1}{m\omega_d}e^{-\zeta\omega_n t}\sin(\omega_d t)$. Convolution integral is the most general method available for arbitrary forces which is suitable for *computer* implementation. Harmonic Response → Set $\omega = 0$ to get step response → Differentiate step response to get impulse response.

Fourier Series: $f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n\cos(n\omega_0 t) + b_n\sin(n\omega_0 t)\right] = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t}, \quad \omega_0 = \frac{2\pi}{T}$;

$a_0 = \frac{2}{T}\int_0^T f(t)dt, \quad a_n = \frac{2}{T}\int_0^T f(t)\cos(n\omega_0 t)dt, \quad b_n = \frac{2}{T}\int_0^T f(t)\sin(n\omega_0 t)dt, c_n = \frac{a_n - jb_n}{2}, j = \sqrt{-1}$

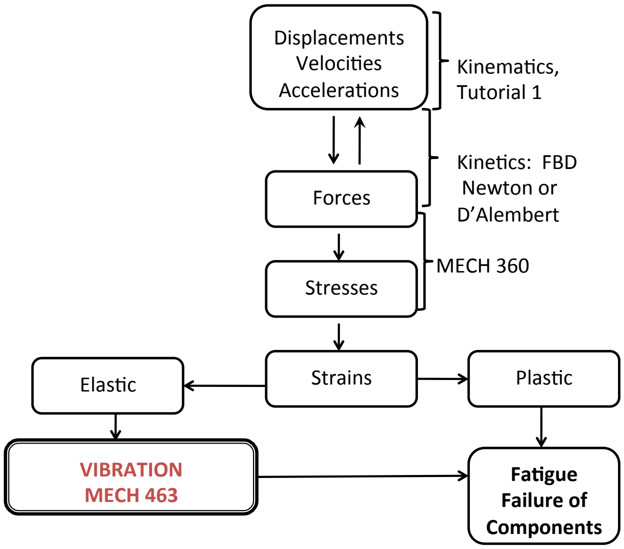
Fourier Transform : $F(j\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt$, Forward Transform $\quad f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(j\omega)e^{j\omega t}d\omega$, Inverse Transform

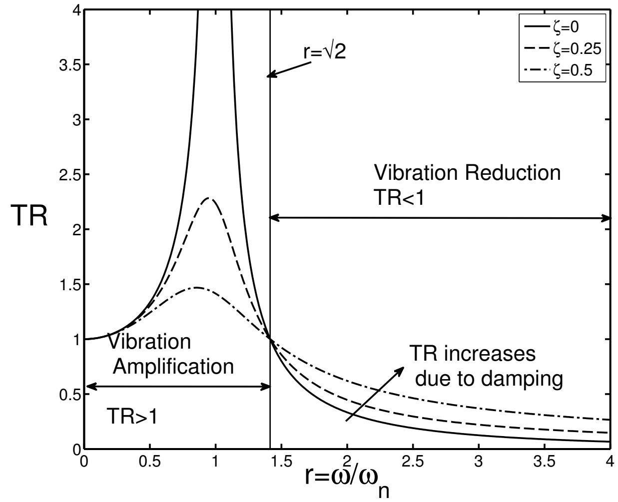
Frequency Response Function: Ratio of output to input of a linear system in frequency (Fourier) domain. Displacement FRF (Receptance)$=\frac{X(j\omega)}{F(j\omega)} = \frac{1}{k-m\omega^2+j\omega c}$. This allows coupling **systems** in frequency domain much like springs coupled in series or parallel.

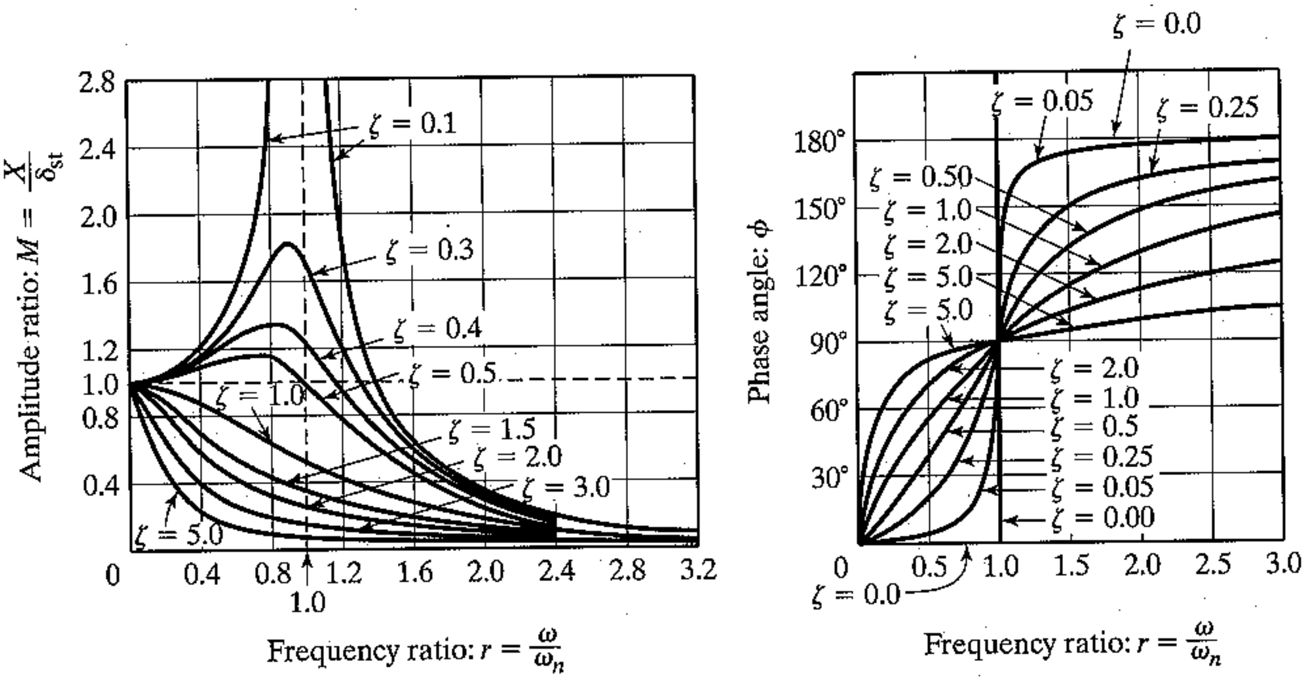
Absorber Design: Select the spring constant k_a and mass m_a of the absorber such that the natural frequency of the absorber unit $\omega_a = \sqrt{\frac{k_a}{m_a}}$ is tuned to the forcing frequency ω. Given H_{sys} of ANY system, design the absorber H_a such that the combined system's frequency response $\frac{1}{H} = \frac{1}{H_{sys}} + \frac{1}{H_a}$ has the desired response characteristics. The absorber introduces two resonances which limit it's operating range. The maximum displacement of an undamped absorber is $X_a = \frac{F_0}{k_a}$ where F_0 is the amplitude of the applied harmonic force on the main system. Damping reduces absorber's effectiveness. m_a and X_a are design constraints.

Orthogonality Conditions: The eigenvectors (modes) of the undamped system obtained from $\boldsymbol{Ku} = \lambda\boldsymbol{Mu}$, $\lambda = \omega^2$ obey the orthogonality conditions $\boldsymbol{u}_i^T\boldsymbol{Mu}_j = m_i\delta_{ij}$ $\quad \boldsymbol{u}_i^T\boldsymbol{Ku}_j = k_i\delta_{ij}$ $\quad \delta_{ij} = 1$ if $i = j$, $\quad \delta_{ij} = 0$ if $i \neq j$. These conditions can be interpreted in terms of work. For stiffness orthogonality, the work done by elastic forces associated with mode j do no work on displacements u_i. These orthogonality conditions enable the transformation $\boldsymbol{x} = \boldsymbol{\Phi q}$ from physical co-ordinates to normal/principal co-ordinates which uncouple MDOF matrix equations.

Principal and Normal Co-ordinates: Both co-ordinates uncouple equations of motion by transforming mass and stiffness matrices into diagonal matrices. In normal co-ordinates, mass matrix is identity matrix and stiffness matrix contains squared undamped natural frequencies as the diagonal entries.







(a) (b)

Note: $DMF = \frac{X}{\delta_{st}} = \frac{1}{\sqrt{[1-r^2]^2+[2\zeta r]^2}}$