Q1.

(a) iv (ii & iii are also correct)

(b) iii

(c) i

(d) ii

(e) iii

(f) iv

(g) i

(h) i

(i) ii

(j) iv

Q2

(a) Kirchhoff voltage law

$$\begin{cases} u = Ri + L\dfrac{di_L}{dt} & \cdots\cdots ① \\ L\dfrac{di_L}{dt} = \dfrac{1}{C}\displaystyle\int (i - i_L)\,dt & \cdots ② \end{cases}$$

Take $x_1 \triangleq i_L$

$x_2 \triangleq \frac{1}{C}\int (i - i_L)\,dt$

$$\dot{x}_1 = \frac{1}{L} x_2 \quad (\text{by } ②)$$

$$\dot{x}_2 = \frac{1}{C}(i - i_L) = \frac{1}{C}\Big(\underbrace{\frac{1}{R}(u - x_2) - x_1}_{(\text{by } ① \& ②)}\Big)$$

$\boxed{\text{SS model}}$

$$\begin{cases} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{L} \\ -\frac{1}{C} & -\frac{1}{CR} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{CR} \end{bmatrix} u \\ y = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \end{cases}$$

(b) Newton's 2nd law

$$m\ddot{z} = -k(z - w) - b(\dot{z} - \dot{w})$$

Take $x_1 \triangleq z$, $x_2 \triangleq \dot{z}$, $x_3 \triangleq w$

Then, $\dot{x}_1 = x_2$

$$\dot{x}_2 = \frac{1}{m}\left\{-k(x_1 - x_3) - b(x_2 - u)\right\}$$

$$\dot{x}_3 = u$$

$\boxed{\text{SS model}}$

$$\begin{cases} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -\frac{k}{m} & -\frac{b}{m} & \frac{k}{m} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{b}{m} \\ 1 \end{bmatrix} u \\ \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} & \frac{k}{m} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \frac{b}{m} \end{bmatrix} u \end{cases}$$

Q3. (a)

$$A = \begin{bmatrix} 2 & 1 \\ 0 & -2 \end{bmatrix}$$

$$e^{At} = \mathcal{L}^{-1}\left\{(sI-A)^{-1}\right\} = \mathcal{L}^{-1}\left\{\begin{bmatrix} s-2 & -1 \\ 0 & s+2 \end{bmatrix}^{-1}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{1}{(s+2)(s-2)}\begin{bmatrix} s+2 & 1 \\ 0 & s-2 \end{bmatrix}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s-2}\begin{bmatrix} 1 & \frac{1}{4} \\ 0 & 0 \end{bmatrix} + \frac{1}{s+2}\begin{bmatrix} 0 & -\frac{1}{4} \\ 0 & 1 \end{bmatrix}\right\}$$

$$= e^{2t}\begin{bmatrix} 1 & 1/4 \\ 0 & 0 \end{bmatrix} + e^{-2t}\begin{bmatrix} 0 & -1/4 \\ 0 & 1 \end{bmatrix}$$

(b) $\mathcal{C} = [B \; AB] = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$, $\operatorname{rank}\mathcal{C} = 1 < 2 \Rightarrow$ Not controllable

(c) $A = \begin{bmatrix} 2 & 1 \\ 0 & -2 \end{bmatrix}$, $B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is already a decomposed form.

$A_{\bar{c}} = -2$ which is a stable matrix. $\therefore$ Stabilizable.

(d) Select, for example, $s = -1$ & -2 as closed-loop pole locations.

~~Then, desired characteristic equ polynomial is $(s+1)(s+2) =$~~

$$\begin{bmatrix} 2 & 1 \\ 0 & -2 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \end{bmatrix}[k_1 \; k_2] = \begin{bmatrix} 2-k_1 & 1-k_2 \\ 0 & -2 \end{bmatrix}$$

We want $2 - k_1 = -1 \Rightarrow k_1 = 3$

k_2 is arbitrary.

(e) $Q = \begin{bmatrix} 0 & 0 \\ 0 & 3 \end{bmatrix} (= q C^T C)$, $R = 1$

ARE
$$\begin{bmatrix} 2 & 0 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} P_1 & P_2 \\ P_2 & P_3 \end{bmatrix} + \begin{bmatrix} P_1 & P_2 \\ P_2 & P_3 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 0 & -2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 3 \end{bmatrix} - \overbrace{\begin{bmatrix} P_1 & P_2 \\ P_2 & P_3 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix}}^{\begin{bmatrix} P_1 \\ P_2 \end{bmatrix}} 1^{-1} [1 \; 0]\begin{bmatrix} P_1 & P_2 \\ P_2 & P_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$(1,1)$: $2P_1 + 2P_1 - P_1^2 = 0$

$(1,2)$: $2P_2 + P_1 - 2P_2 - P_1 P_2 = 0$

$(2,2)$: $2(P_2 - 2P_3) + 3 - P_2^2 = 0$

$\Rightarrow P_1 = \cancel{2}\,4$, $P_2 = 1$, $P_3 = 1$

$$P = \begin{bmatrix} \overset{4}{\cancel{2}} & 1 \\ 1 & 1 \end{bmatrix} > 0 \quad \left(\text{due to Sylvester criterion}\right)$$

$$K_{LQR} = R^{-1}B^T P = [1 \; 0]\begin{bmatrix} \overset{4}{\cancel{2}} & 1 \\ 1 & 1 \end{bmatrix} = [\overset{4}{\cancel{2}} \; 1]$$

$$A - BK_{LQR} = \begin{bmatrix} 2 & 1 \\ 0 & -2 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \end{bmatrix}[\overset{4}{\cancel{2}} \; 1] = \begin{bmatrix} -2 & -3 \\ 0 & -2 \end{bmatrix} \Rightarrow \operatorname{eig}(A - BK_{LQR}) = \{-2, -2\}$$

$\Rightarrow$ CL system is stable.

Q4. (a) $P[k+1|k] = AP[k|k]A^T + R_w = P[k|k] + 1$

$$P[k|k] = P[k|k-1] - \frac{P[k|k-1]^2}{P[k|k-1]+2} = \frac{2P[k|k-1]}{P[k|k-1]+2}$$

$$\hat{x}[k+1|k] = A\hat{x}[k|k] = \hat{x}[k|k]$$

$$\hat{x}[k|k] = \hat{x}[k|k-1] + P[k|k]C^T R_v^{-1}\left(y[k] - C\hat{x}[k|k-1]\right)$$

$$= \hat{x}[k|k-1] + \frac{P[k|k]}{2}\left(y[k] - \hat{x}[k|k-1]\right)$$

(b) [k=1]

$$P[1|0] = \underbrace{P[0|0]}_{=0} + 1 = 1$$

$$P[1|1] = \frac{2P[1|0]}{P[1|0]+2} = \frac{2}{3}$$

$$\hat{x}[1|0] = \hat{x}[0|0] = 0$$

$$\hat{x}[1|1] = \underbrace{\hat{x}[1|0]}_{=0} + \frac{P[1|1]}{2}\left[y[1] - \underbrace{\hat{x}[1|0]}_{=0}\right] = \frac{1}{3}y[1]$$

[k=2]

$$P[2|1] = \underbrace{P[1|1]}_{2/3} + 1 = \frac{5}{3}$$

$$P[2|2] = \frac{2\cdot\frac{\cancel{2}5}{3}}{\frac{5}{3}+2} = \frac{\cancel{4}10}{11}$$

$$\hat{x}[2|1] = \hat{x}[1|1] = \frac{1}{3}y[1]$$

$$\hat{x}[2|2] = \hat{x}[2|1] + \frac{P[2|2]}{2}\left(y[2] - \hat{x}[2|1]\right)$$

$$= \frac{1}{3}y[1] + \frac{5}{11}\left(y[2] - \frac{1}{3}y[1]\right)$$

$$= \frac{1}{11}\left(2y[1] + 5y[2]\right)$$