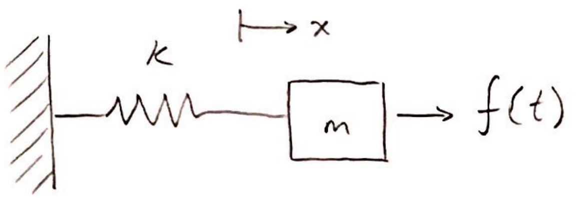
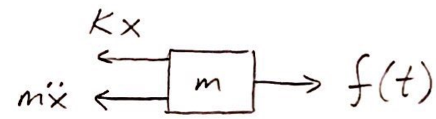
Lecture 5

Harmonic Forced Vibration



$\Rightarrow$



$\underline{\Sigma F_x}$: $m\ddot{x} + kx = f(t)$ Let $f(t) = F\cos(\omega_F t)$

$$m\ddot{x} + kx = F\cos(\omega_F t)$$ Forcing frequency ω_F

<u>General solution</u> : complementary + particular

$$\Rightarrow x = x_c + x_p$$

<u>Complementary</u>: $x_c = C\cos(\omega_N t + \phi)$

<u>Particular</u>: Try $x_p = X\cos(\omega_F t) \Rightarrow \ddot{x}_p = -\omega_F^2 X\cos(\omega_F t)$

Sub. into Eq. of motion:

$$\Rightarrow (-\omega_F^2 m + k) X\cos(\omega_F t) = F\cos(\omega_F t)$$

$$\Rightarrow X = \frac{F}{k - \omega_F^2 m} \Longleftrightarrow \frac{F/k}{1 - \frac{\omega_F^2 m}{k}}$$

$$\Rightarrow X = \frac{F/k}{1 - \left(\frac{\omega_F}{\omega_N}\right)^2} = \frac{X_0}{1 - r^2}$$

<u>Static deflection</u>: $X_0 = F/k$

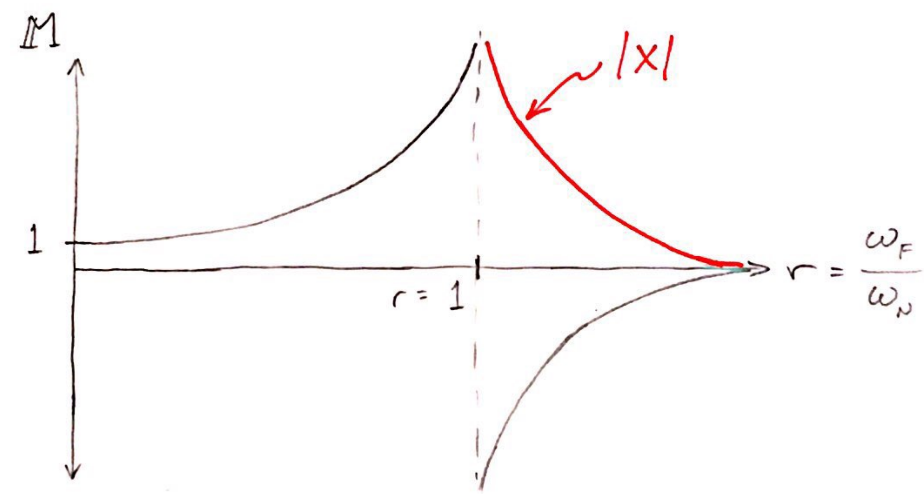
Frequency ratio: $r = \dfrac{\omega_F}{\omega_N}$

Full solution: $x = x_c + x_p$

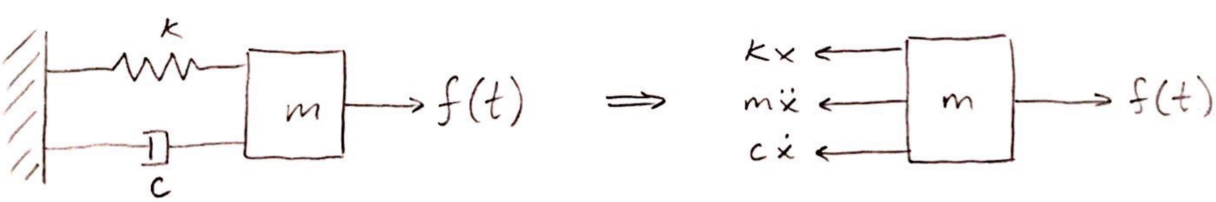
$$x = \underbrace{A\cos(\omega_N t) - B\sin(\omega_N t)}_{\substack{\text{"Free response"} \\ \text{"Transient"}}} + \underbrace{\frac{X_0}{1-r^2}\cos(\omega_F t)}_{\substack{\text{"Forced response"} \\ \text{"steady-state"}}}$$

Magnification Factor: $M = \dfrac{X}{X_0} = \dfrac{1}{1-r^2}$

The (+) and (-) M imply a change of direction and phase



Damped System w/ Forcing



ΣF_x: $m\ddot{x} + c\dot{x} + kx = F\cos(\omega_F t)$

Particular: Try $x_p = A\cos(\omega_F t) - B\sin(\omega_F t)$

Substitute x_p into equation of motion:

$$\Rightarrow m\left[-\omega_F^2 A\cos(\omega_F t) + \omega_F^2 B\sin(\omega_F t)\right] + C\left[-\omega_F A\sin(\omega_F t) - \omega_F B\cos(\omega_F t)\right] + \ldots$$

$$\ldots + k\left[A\cos(\omega_F t) - B\sin(\omega_F t)\right] = F\cos(\omega_F t)$$

Equate sin & cos terms on both sides of equation:

$$\begin{cases} (-\omega_F^2 mA - \omega_F cB + kA)\cos(\omega_F t) = F\cos(\omega_F t) \\ (\omega_F^2 mB - \omega_F cA - kB)\sin(\omega_F t) = 0 \end{cases}$$

Matrix:
$$\begin{bmatrix} k-\omega_F^2 m & -\omega_F C \\ -\omega_F C & -(k-\omega_F^2 m) \end{bmatrix}\begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} F \\ 0 \end{bmatrix}$$

Solve w/ Cramer's Rule:

$$A = \frac{\det\begin{bmatrix} F & -\omega_F C \\ 0 & -(k-\omega_F^2 m) \end{bmatrix}}{\det\begin{bmatrix} k-\omega_F^2 m & -\omega_F C \\ -\omega_F C & -(k-\omega_F^2 m) \end{bmatrix}}$$

determinant

$$\Rightarrow \begin{cases} A = \dfrac{(k-\omega_F^2 m)F}{(k-\omega_F^2 m)^2 + (\omega_F c)^2} \\ \\ B = \dfrac{-(\omega_F c)F}{(k-\omega_F^2 m)^2 + (\omega_F c)^2} \end{cases}$$

Notation: $\omega_N^2 = \dfrac{k}{m}$ $\quad r = \dfrac{\omega_F}{\omega_N}$ $\quad \zeta = \dfrac{c}{2\sqrt{km}}$ $\quad X_o = F/k$

$$\Rightarrow A = \frac{X_o(1-r^2)}{(1-r^2)^2 + (2\zeta r)^2} \qquad B = \frac{-X_o(2\zeta r)}{(1-r^2)^2 + (2\zeta r)^2}$$

Solution: $$x = \frac{X_o}{(1-r^2)^2 + (2\zeta r)^2}\left[(1-r^2)^2 \cos(\omega_F t) + (2\zeta r)\sin(\omega_F t)\right]$$

Change form of solution: $$C = \sqrt{A^2 + B^2}$$

$$\tan\phi = \frac{B}{A}$$

$$x = C\cos(\omega_F t + \phi_F)$$

Solution type 2: $$x = \frac{X_o}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}} \cos(\omega_F t + \phi_F)$$

$$\tan\phi_F = \frac{-2\zeta r}{1-r^2}$$

Amplitude: $$\frac{X_o}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}}$$