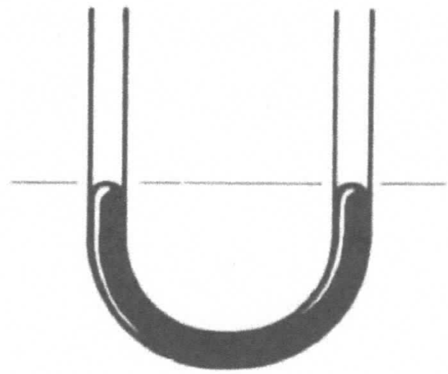
MECH 463 -- Homework 2

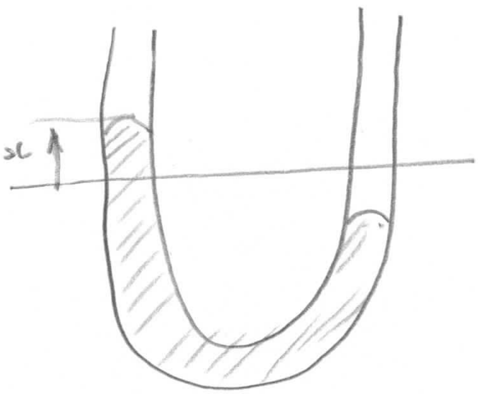
1. A manometer tube 15mm in diameter is filled with oil to a length $\ell = 250$mm. The specific gravity of the oil is 0.8 and its dynamic viscosity μ is 0.035 Pa.s. When the oil is displaced from its equilibrium position it oscillates with decaying amplitude. Determine the equivalent quantities m, c and k for the oil in the manometer and hence determine the damping factor ζ and damped natural frequency ω_d. *(Hint: Poisseuille's Law for the steady-state flow through a circular tube is $v_{avg} = \Delta p\, A / (8\pi\mu\ell)$, where Δp is the pressure differential across the length of the liquid).*



Mass of oil = density × volume

$m = \rho \ell A = \underset{\uparrow\ kg/m^3}{800} \times 0.25 \times \underbrace{\pi \frac{0.015^2}{4}}_{A = 1.76\times10^{-4}\ m^2}$

$m = 0.0353$ kg



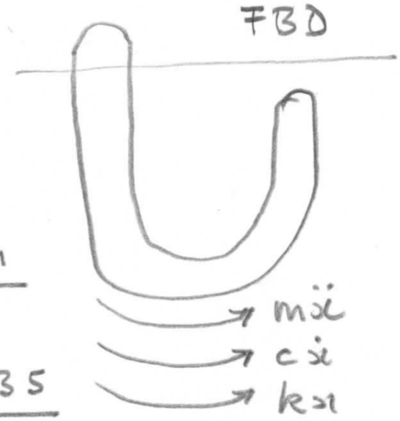
Poisseuille's Law $V_{avg} = \Delta p A / (8\pi\mu l)$

where here $V_{avg} = \dot{x}$ $\quad \Delta p = 2\rho g x$

Viscous force $= \Delta p A = 8\pi\mu l \dot{x} = c\dot{x}$

$c = 8\pi \times 0.035 \times 0.25 = 0.22$ Ns/m

FBD



Restoring force $= \Delta p A = 2\rho g A x = kx$

Spring constant $k = 2\rho g A$

$k = 2 \times 800 \times 9.81 \times 1.76\times10^{-4} = 2.77$ N/m

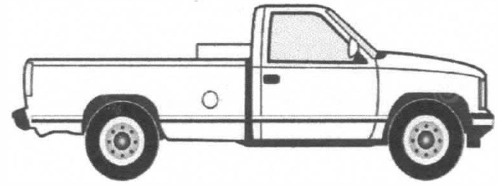
Damping factor $\zeta = \sqrt{\frac{c^2}{4km}} = \sqrt{\frac{0.22^2}{4\times 2.77\times 0.0353}} = 0.35$

Undamped natural freq. $\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{2.77}{0.0353}} = 8.86$ rad/s $\quad f_n = 1.4$ Hz

Damped natural freq. $\omega_d = \omega_n\sqrt{1-\zeta^2} = 8.86\sqrt{1-0.35^2} = 8.3$ rad/s

$f_d = 1.3$ Hz

2. A student of mass 75kg stepped onto the back of a small pickup truck, causing a steady state displacement of the truck body of 2.5cm. The student then stepped off and the truck body started to oscillate with a frequency of 1 Hz around the original position it had before the student stepped on. The first overshoot (in the opposite direction of the 2.5cm) was measured to be 1.5cm. What are the equivalent quantities m, c and k for the truck body ? *(Hint: the logarithmic decrement concept may be useful)*.



Load of $75 \times 9.81 = 736N$ causes $0.025m$ displacement

$$F = kx \rightarrow k = \frac{F}{x} = \frac{736}{0.025} = \underline{29.4\ kN/m}$$

Logarithmic decrement $\delta = \frac{1}{n} \ln \left| \frac{x_0}{x_n} \right| = \frac{1}{1/2} \ln \left| \frac{x_0}{x_{1/2}} \right|$

where $n = ½$ for the first opposite side overshoot.

$$\delta = 2 \ln \left| \frac{0.025}{-0.015} \right| = 1.02$$

$$\delta = \frac{2\pi\xi}{\sqrt{1-\xi^2}} \simeq 2\pi\xi \text{ for small } \xi \rightarrow \xi = \frac{\delta}{2\pi} = \underline{0.16}$$

Damped natural freq. $\omega_d = 2\pi f_d = 2\pi \times 1.0 = 6.28\ rad/s$

Undamped natural freq. $\omega_n = \frac{\omega_d}{\sqrt{1-\xi^2}} = \frac{3.14}{\sqrt{1-0.16^2}} = 6.36\ rad/s$

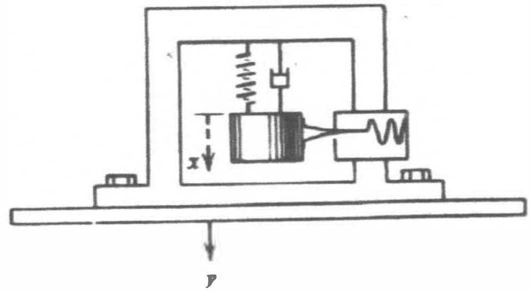
$$\omega_n = \sqrt{\frac{k}{m}} \rightarrow m = \frac{k}{\omega_n^2} = \frac{29400}{6.36^2} = \underline{727\ kg}$$

Damping factor $\xi = \sqrt{\frac{c^2}{4mk}} \rightarrow c = 2\xi\sqrt{mk}$

$$c = 2 \times 0.16 \sqrt{727 \times 29400}$$

$$c = \underline{1480\ Ns/m}$$

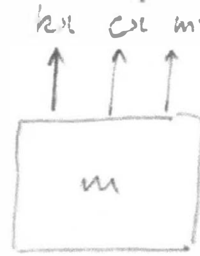
3. The seismic vibrometer schematically shown in the diagram has a mass m = 1 kg, a spring of stiffness k = 2 N/m and a damper that provides a damping factor $\zeta = 0.1$. An earthquake occurs where the dominant frequency of vibration is 0.5Hz. If thet peak-to-peak vibration indicated by the vibrometer is 10mm, what is the peak-to-peak vibration of the ground ?



y = absolute motion of base

x = motion of mass relative to base

z = absolute motion of mass



$$z = x + y \longrightarrow \ddot{z} = \ddot{x} + \ddot{y}$$

Spring and damper depend on x

Mass inertia depends on z

From FBD: $m\ddot{z} + c\dot{x} + kx = 0 \quad = m(\ddot{x}+\ddot{y}) + c\dot{x} + kx$

$$\longrightarrow m\ddot{x} + c\dot{x} + kx = -m\ddot{y}$$

where $y = Y\cos\omega t$

$\ddot{y} = -\omega^2 Y \cos\omega t$

Use solution type 4: $x = Re\left[De^{i\omega t}\right]$

Sub in equation of motion:

$$Re\left[(-m\omega^2 + ic\omega + k)De^{i\omega t}\right] = Re\left[-\omega^2 m Y e^{i\omega t}\right]$$

True for all $t \rightarrow e^{i\omega t} \neq 0$

$$\rightarrow D = \frac{-\omega^2 m Y}{k - m\omega^2 + ic\omega} = \frac{-\omega^2 \frac{m}{k} Y}{1 - \frac{m}{k}\omega^2 + i\frac{c}{k}\omega}$$

dividing through by k

$$D = \frac{-\frac{\omega^2}{\omega_n^2} Y}{1 - \frac{\omega^2}{\omega_n^2} + i\,2\sqrt{\frac{c^2}{4km}}\frac{\omega}{\omega_n}} = \frac{-r^2 Y}{(1-r^2) + i2\xi r}$$

where $\omega_n^2 = \frac{k}{m}$ $\quad \xi = \sqrt{\frac{c^2}{4km}}$

$r = \frac{\omega}{\omega_n}$

Magnification factor $\dfrac{|D|}{Y} = \dfrac{r^2}{\sqrt{(1-r^2) + (2\xi r)^2}}$

$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{2}{1}} = 1.41$ rad/s

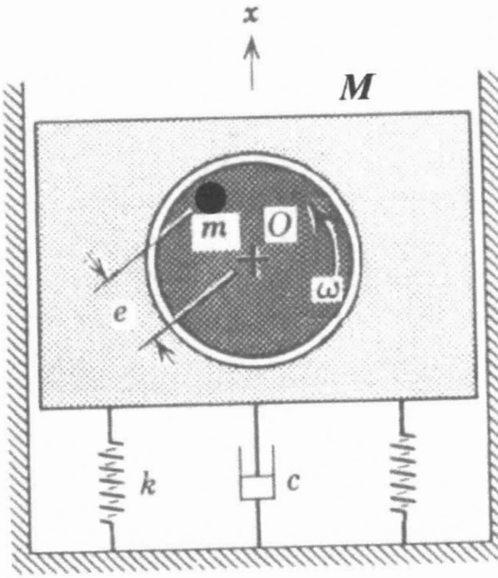
$\omega = 2\pi f = 2\pi \times 0.5 = 3.14 \frac{rad}{sec}$

$$MF = \frac{2.2^2}{\sqrt{(1-2.2^2)^2 + (2\times 0.1 \times 2.2)^2}} = 1.25$$

$r = \frac{3.14}{1.41} = 2.2 \quad \xi = 0.1$

Peak-Peak ground motion $= 2Y = \dfrac{2|D|}{MF} = \dfrac{10\text{ mm}}{1.25} = \underline{8\text{ mm}}$

4. A machine of total mass M contains a rotor of mass m and eccentricity e. It is supported on springs of combined stiffness k and has a damper of rate c. Derive a formula for the vibration response of the machine as a function of rotation frequency.



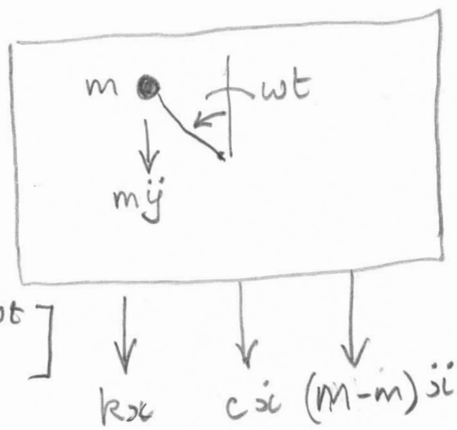
x = position of machine

y = position of eccentric mass

$$y = x + e\cos\omega t$$

$$\ddot{y} = \ddot{x} - \omega^2 e\cos\omega t$$

From FBD



$$(M-m)\ddot{x} + m\ddot{y} + c\dot{x} + kx = 0$$

$$(M-m)\ddot{x} + m\left(\ddot{x} - \omega^2 e\cos\omega t\right) + c\dot{x} + kx = 0$$

$$\rightarrow M\ddot{x} + c\dot{x} + kx = \omega^2 me\cos\omega t$$

Use solution type 4: $x = \mathrm{Re}\left[De^{i\omega t}\right]$

Sub in equation of motion:

$$\mathrm{Re}\left[\left(-M\omega^2 + ic\omega + k\right)De^{i\omega t}\right] = \mathrm{Re}\left[\omega^2 me\, e^{i\omega t}\right]$$

True for all t $\rightarrow$ $e^{i\omega t} \neq 0$

$$\rightarrow D = \frac{\omega^2 me}{k - M\omega^2 + ic\omega} = \frac{\omega^2 \frac{m}{M}\frac{M}{k}e}{1 - \frac{M}{k}\omega^2 + i\frac{c}{k}\omega}$$

dividing through by k

$$D = \frac{\frac{m}{M}\frac{\omega^2}{\omega_n^2}e}{\left(1 - \frac{\omega^2}{\omega_n^2}\right) + i2\sqrt{\frac{c^2}{4kM}}\frac{\omega}{\omega_n}} = \frac{\frac{m}{M}r^2 e}{(1-r^2) + i2\zeta r}$$

where $\omega_n^2 = \frac{k}{M}$ $\zeta = \sqrt{\frac{c^2}{4kM}}$

$r = \frac{\omega}{\omega_n}$

$$\text{Magnification factor} = \frac{|D|}{\frac{m}{M}e} = \frac{r^2}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}}$$

$$\text{Phase lead} = \angle D = \tan^{-1}\left(\frac{-2\zeta r}{1-r^2}\right)$$

actually a phase lag