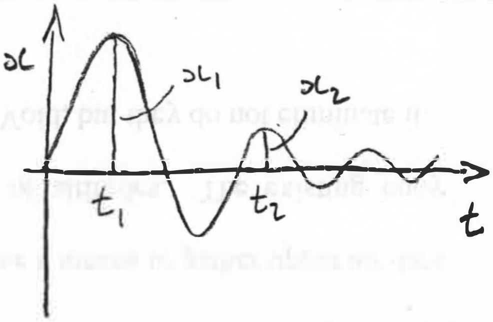
MECH 463 -- Tutorial 2

1. Measurement of the logarithmic decrement provides a common experimental way to determine the damping factor of a lightly damped 1-DOF vibrating system. The logarithmic decrement δ is the logarithmic ratio of the amplitudes of successive same-sign vibration peaks, $\delta = \ln(x_n/x_{n+1})$. Given that the response of a damped 1-DOF vibrating system is $x = e^{-\zeta\omega t}(A \cos \omega_d t - B \sin \omega_d t)$, determine the relationship between damping factor and logarithmic decrement.



Given solution for the transient response is:

$$x = e^{-\zeta\omega_n t}(A\cos\omega_d t - B\sin\omega_d t)$$

To simplify the algebra, we shall choose the initial time datum, $t=0$ such that $A=0$ ($x=0$ at $t=0$). This means that we need only work with one constant, B.

$$\rightarrow \quad x = -Be^{-\zeta\omega_n t}\sin\omega_d t$$

A vibration peak occurs when $\frac{dx}{dt} = 0$

$$\rightarrow \quad \frac{dx}{dt} = Be^{-\zeta\omega_n t}(\zeta\omega_n \sin\omega_d t - \omega_d\cos\omega_d t) = 0$$

In general, $Be^{-\zeta\omega_n t} \neq 0 \rightarrow \zeta\omega_n\sin\omega_d t - \omega_d\cos\omega_d t = 0$

$$\rightarrow \tan\omega_d t = \frac{\sin\omega_d t}{\cos\omega_d t} = \frac{\omega_d}{\zeta\omega_n} \rightarrow$$ repeats every π, or 2π for same-sign.

$$\rightarrow \omega_d t_2 = \omega_d t_1 + 2\pi \quad \rightarrow \quad t_2 = t_1 + \frac{2\pi}{\omega_d}$$

$$x_1 = -Be^{-\zeta\omega_n t_1}\sin\omega_d t_1$$

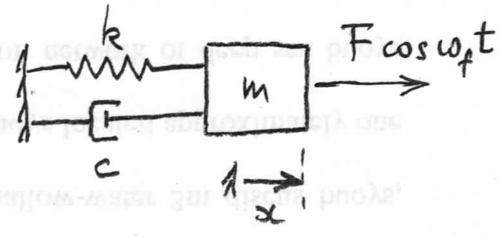
$$x_2 = -Be^{-\zeta\omega_n\left(t_1 + \frac{2\pi}{\omega_d}\right)}\sin(\omega_d t_1 + 2\pi) \quad \longrightarrow \quad \frac{x_1}{x_2} = e^{\frac{\zeta\omega_n 2\pi}{\omega_d}}$$

Recall $\omega_d = \omega_n\sqrt{1-\zeta^2} \quad \longrightarrow \quad \frac{x_1}{x_2} = e^{\frac{2\pi\zeta}{\sqrt{1-\zeta^2}}}$

Logarithmic decrement $\delta = \ln\left(\frac{x_1}{x_2}\right) = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}} \simeq 2\pi\zeta$ for small ζ

$$\longrightarrow \text{Damping factor } \zeta = \frac{\delta}{2\pi}$$

2. The equation of motion of a 1-DOF damped vibrating system with harmonic forced excitation is $m\ddot{x} + c\dot{x} + kx = F\cos(\omega_f t)$. Using the fourth of the four solution forms discussed in class (the one involving the constant D), determine the steady state response of the system.



Using complex notation:

$$f(t) = F\cos\omega_f t = Re\left[F(\cos\omega_f t + i\sin\omega_f t)\right] = Re\left[Fe^{i\omega_f t}\right]$$

Substitute $x = De^{i\omega_f t}$ to get particular solution:

$$m\ddot{x} + c\dot{x} + kx = Fe^{i\omega_f t} \qquad \text{(using whole solution)}$$

$$\rightarrow (-m\omega_f^2 + ic\omega_f + k)De^{i\omega_f t} = Fe^{i\omega_f t}$$

This must be true for all t $\rightarrow e^{i\omega_f t} \neq 0$

$$\rightarrow D = \frac{F}{k - m\omega_f^2 + ic\omega_f} = \frac{F/k}{(1-r^2) + i\,2\zeta r}$$

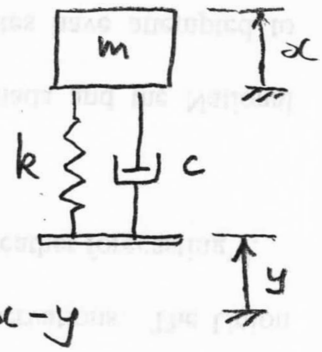
where $\omega_n^2 = \frac{k}{m}$ $\quad r = \frac{\omega_f}{\omega_n}$ $\quad \zeta = \frac{c}{2\sqrt{km}}$

Response amplitude $= |D|$ $\qquad$ phase $= \measuredangle D$

$$\text{Magnification factor} = \frac{|D|}{F/k} = \frac{1}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}}$$

$$\text{Phase lead} = \measuredangle D = \tan^{-1}\left(\frac{-2\zeta r}{1-r^2}\right)$$

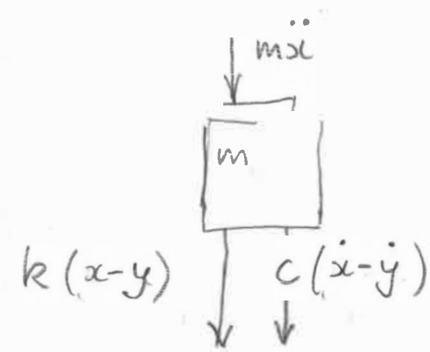
3. The diagram shows an accelerometer. Use the result from Q2 to determine an equation for the frequency response curve of the accelerometer (= spring stretch amplitude per unit acceleration amplitude vs. normalized frequency).



From FBD:

$$\Sigma F = m\ddot{x} + c(\dot{x}-\dot{y}) + k(x-y) = 0 \qquad z = x - y$$

$$\rightarrow m\ddot{z} + c\dot{z} + kz = -m\ddot{y}$$



For sinusoidal motion: $y = Y \cos \omega_f t$

Acceleration $\ddot{y} = -\omega_f^2 Y \cos \omega_f t = \ddot{Y} \cos \omega_f t$

Substitute in equation of motion:

$$m\ddot{z} + c\dot{z} + kz = -m\ddot{Y} \cos \omega_f t$$

This is the same form as in Q2 with $x \rightarrow z$, $F \rightarrow -m\ddot{Y}$

$$\text{Accel. response} = \frac{|D|}{\ddot{Y}} = \frac{|D|}{-F/m} = \frac{-m}{k} \frac{|D|}{F/k} = \frac{-1/\omega_n^2}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}}$$