MECH468 : Modern Control Engineering
MECH509 : Controls

L20 : State feedback
Canonical form method

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Zoom lecture to be recorded and posted on Canvas

Course plan

Topics	CT	DT
Modeling	✓	✓
Stability	✓	✓
Controllability/observability	✓	✓
Realization	✓	✓
→ State feedback/observer		
LQR/Kalman filter		

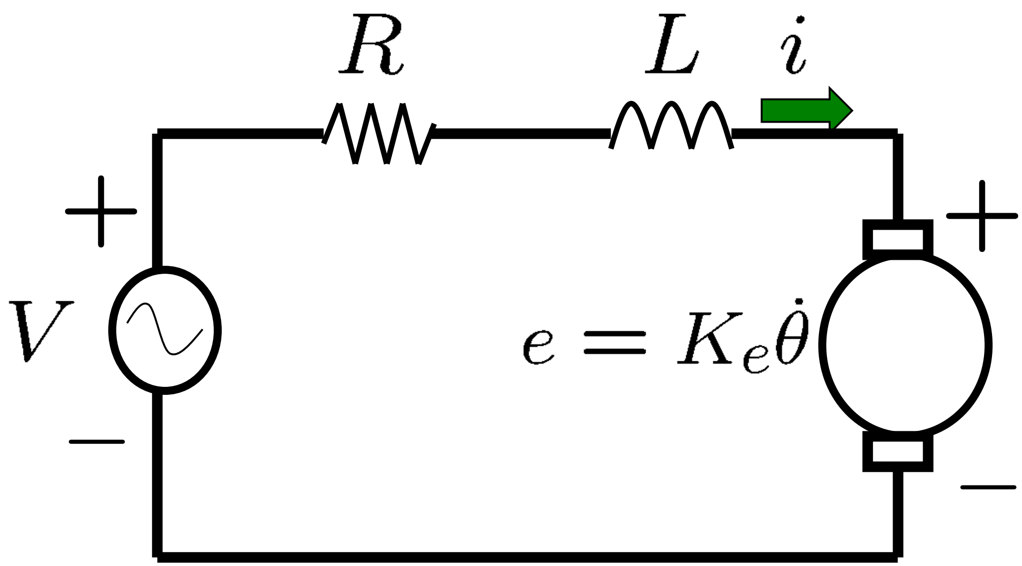
Review & today's topic

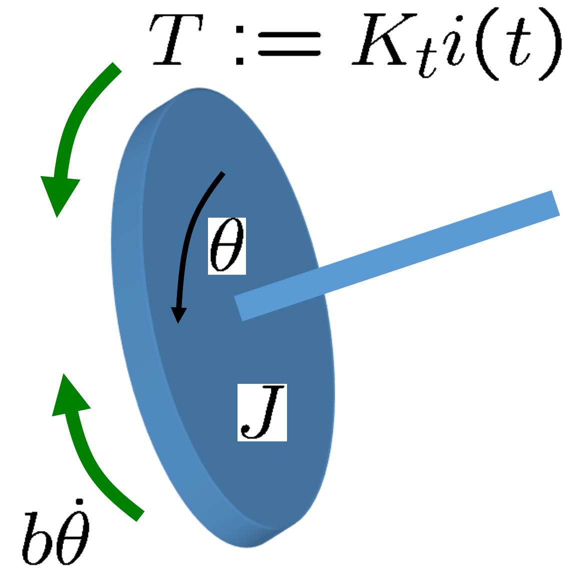
- In the last lecture
  - State feedback
  - Pole placement theorem

  Arbitrary pole placement is possible by a state feedback $u=-Kx$ if and only if (A,B) is controllable
  - A direct method to compute the feedback gain K (applicable to only problems of small sizes)
- In today's lecture
  - A canonical form method for state feedback with a scalar input (Extension to multi-input cases is possible, but complicated.)

Example: DC motor speed control

ctms.engin.umich.edu





$$
\begin{aligned}
J\ddot{\theta}(t) &= K_t i(t) - b\dot{\theta}(t) \\
V(t) &= Ri(t) + L\frac{d}{dt}i(t) + K_e\dot{\theta}(t)
\end{aligned}
$$

DC motor speed control (cont'd)

- State-space model

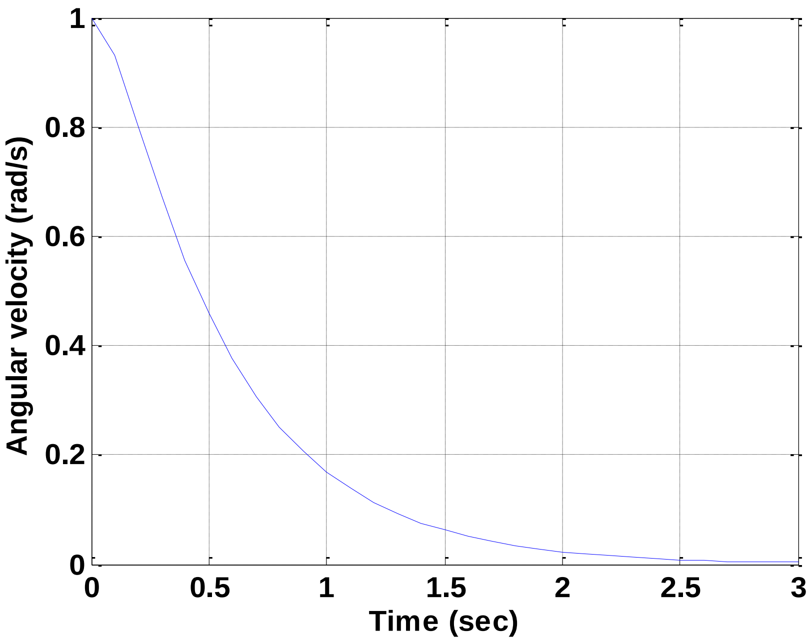
$$\begin{cases} \frac{d}{dt}\begin{bmatrix} \dot{\theta} \\ i \end{bmatrix} = \begin{bmatrix} -b/J & K_t/J \\ -K_e/L & -R/L \end{bmatrix}\begin{bmatrix} \dot{\theta} \\ i \end{bmatrix} + \begin{bmatrix} 0 \\ 1/L \end{bmatrix} V(t) \\ \dot{\theta}(t) = \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} \dot{\theta} \\ i \end{bmatrix} \end{cases}$$

J	moment of inertia	0.01	$\mathrm{kg \cdot m^2}$
b	damping coefficient	0.1	$\mathrm{kg \cdot m^2/s}$
$K_t = K_e$	emf constant	0.01	$\mathrm{N \cdot m/Amp}$

$$R = 1\Omega \quad L = 0.5H$$

DC motor speed control (cont'd)

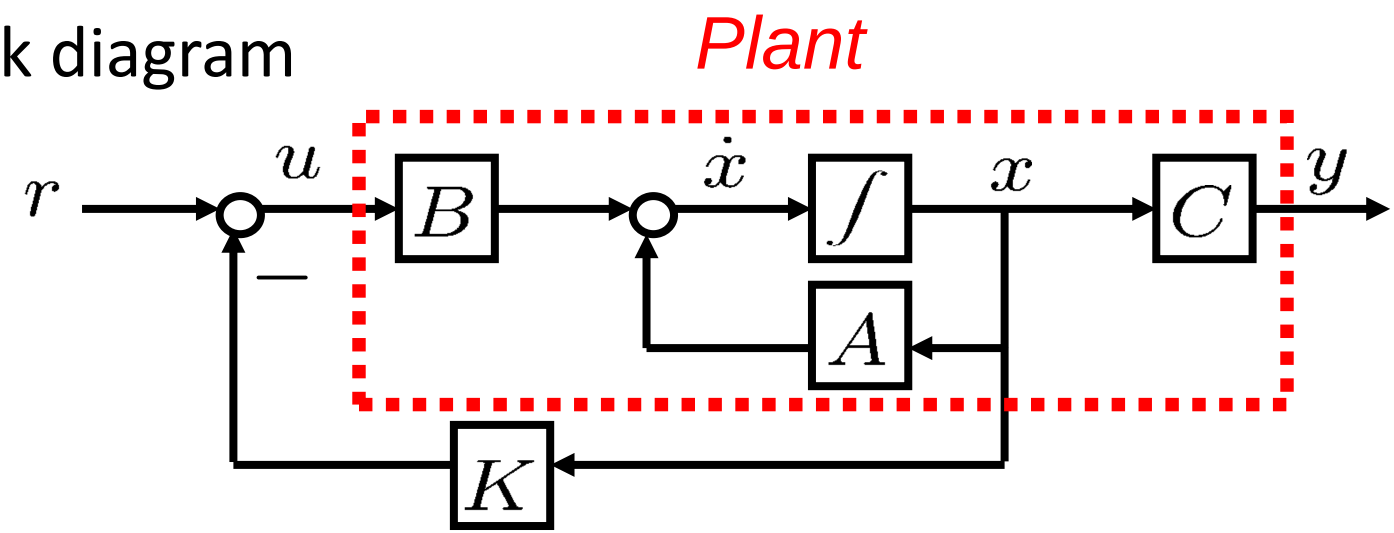
- Specifications: For initial condition $\begin{bmatrix} \dot{\theta}(0) \\ i(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 10 \end{bmatrix}$
  - *r(t)=0*
  - Settling time < 1 sec
  - Overshoot < 5 %
  - Steady state error < 1 %
- Open-loop system
  - Poles = -9.9975, -2.0025
  - Too slow



Feedback control for performance improvement!

State feedback (review)

- Block diagram

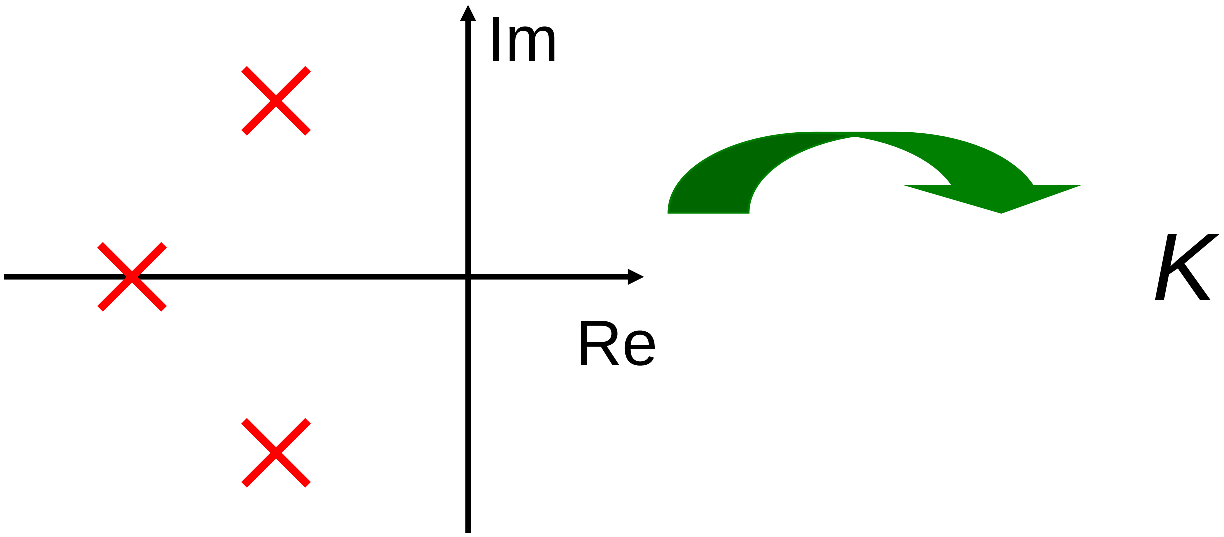


- Open-loop and closed-loop systems

$$\left.\begin{array}{l}\left\{\begin{array}{lcl}\dot{x}(t) & = & Ax(t) + Bu(t)\\ y(t) & = & Cx(t)\end{array}\right.\\[2ex] u(t) = -Kx(t) + r(t)\end{array}\right\} \Rightarrow \left\{\begin{array}{lcl}\dot{x}(t) & = & (A - BK)x(t) + Br(t)\\ y(t) & = & Cx(t)\end{array}\right.$$

Pole placement theorem (review)

- *If (A,B) is controllable, the eigenvalues of (A-BK) can be placed arbitrarily (provided that they are symmetric with respect to the real axis).*



: Closed-loop poles (design parameters)

State feedback design
Canonical form method

Step 0: Check whether (*A*,*B*) is controllable. If it is, go to Step 1.

Step 1: Compute the characteristic polynomial of the open-loop system:

$$\det(sI - A) = s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n$$

Step 2: Set $T^{-1} := \mathcal{C} W$

$$\mathcal{C} := \left[B, AB, \cdots, A^{n-1}B\right] \quad W := \begin{bmatrix} a_{n-1} & a_{n-2} & \cdots & a_1 & 1 \\ a_{n-2} & a_{n-3} & \cdots & 1 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_1 & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{bmatrix}$$

Canonical form method (cont'd)

Step 3: Specify the desired closed-loop poles, i.e., the desired characteristic polynomial

$$s^n + \alpha_1 s^{n-1} + \cdots + \alpha_{n-1} s + \alpha_n$$

Step 4: State feedback gain is computed by

$$K := \underbrace{[\alpha_n - a_n, \cdots, \alpha_1 - a_1]}_{\textit{(desired CL) – (OL)}} T$$

The more you want to move the poles,
the larger *K*, as well as the input, you will get.

Idea for canonical form method

- Suppose (*A*,*B*) is in a controllable canonical form.

$$A := \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 \\ -a_n & -a_{n-1} & \cdots & -a_1 \end{bmatrix} \quad B := \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \qquad K := \begin{bmatrix} k_n & k_{n-1} & \cdots & k_1 \end{bmatrix}$$

$$\begin{aligned} \det(sI-(A-BK)) &= \det\left(sI - \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 \\ -(a_n+k_n) & -(a_{n-1}+k_{n-1}) & \cdots & -(a_1+k_1) \end{bmatrix} \right) \\ &= s^n + \underbrace{(a_1+k_1)}_{\alpha_1} s^{n-1} + \cdots + \underbrace{(a_{n-1}+k_{n-1})}_{\alpha_{n-1}} s + \underbrace{(a_n+k_n)}_{\alpha_n} \end{aligned}$$

$$K = \begin{bmatrix} k_n & k_{n-1} & \cdots & k_1 \end{bmatrix} = [\alpha_n - a_n, \alpha_{n-1} - a_{n-1}, \cdots, \alpha_1 - a_1]$$

Idea (cont'd)

- If (A,B) is not in a controllable canonical form, the matrix T in Step 2 transforms A & B into C.C.F.

$$TAT^{-1} = \bar{A} := \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 \\ -a_n & -a_{n-1} & \cdots & -a_1 \end{bmatrix} \qquad TB = \bar{B} := \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

(Provable by using Cayley-Hamilton Theorem, but not covered)

- As explained, for a controllable canonical form,

$$\bar{K} = \begin{bmatrix} k_n & k_{n-1} & \cdots & k_1 \end{bmatrix} = [\alpha_n - a_n, \alpha_{n-1} - a_{n-1}, \cdots, \alpha_1 - a_1]$$

- Note that $\bar{A} - \bar{B}\bar{K} = TAT^{-1} - TB\bar{K} = T(A - B\underbrace{\bar{K}T}_{K})T^{-1}$

DC motor speed control: revisited

- SS model $\begin{cases} \frac{d}{dt}\begin{bmatrix} \dot\theta \\ i \end{bmatrix} = \begin{bmatrix} -10 & 1 \\ -0.02 & -2 \end{bmatrix}\begin{bmatrix} \dot\theta \\ i \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \end{bmatrix} V(t) \\ \dot\theta(t) = \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} \dot\theta \\ i \end{bmatrix} \end{cases}$

Step 0: (A,B) controllable

Step 1: Ch. Polynomial $\det(sI - A) = s^2 + \overbrace{12}^{a_1} s + \overbrace{20.02}^{a_2}$

Step 2: Set $T^{-1} := \mathcal{C}W = \begin{bmatrix} 0 & 2 \\ 2 & -4 \end{bmatrix}\begin{bmatrix} 12 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 20 & 2 \end{bmatrix}$

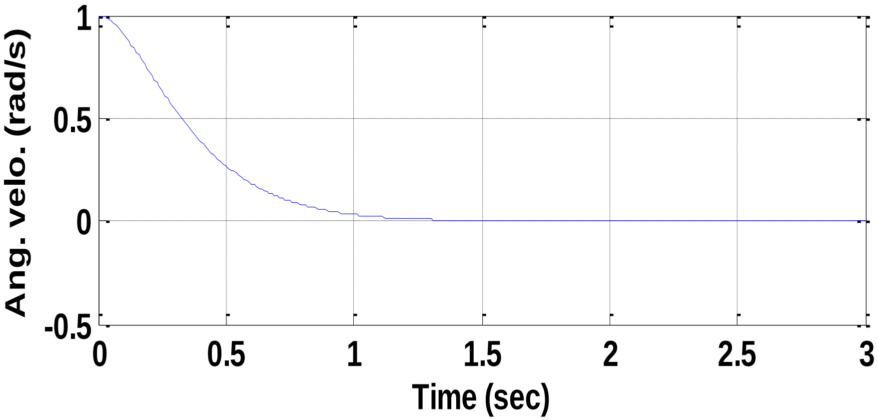
Step 3: Specify the desired CL poles at p_1 & p_2

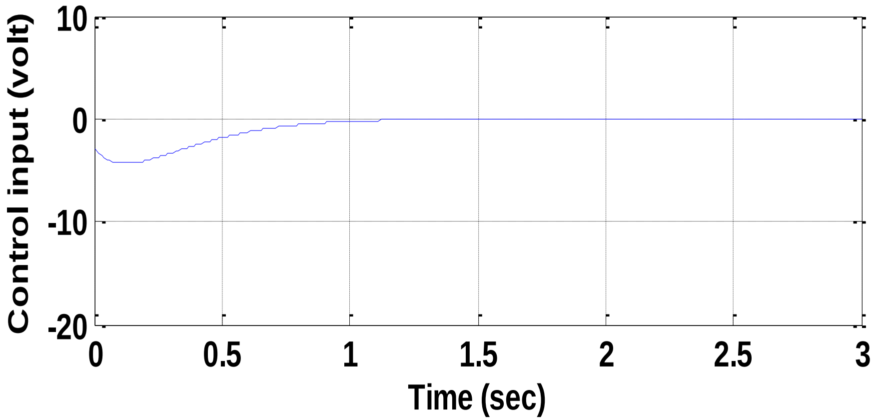
$$(s-p_1)(s-p_2) = s^2 \underbrace{-(p_1+p_2)}_{+\alpha_1} s + \underbrace{p_1 p_2}_{\alpha_2}$$

Step 4: $K := [\alpha_2 - a_2, \alpha_1 - a_1]\, T$

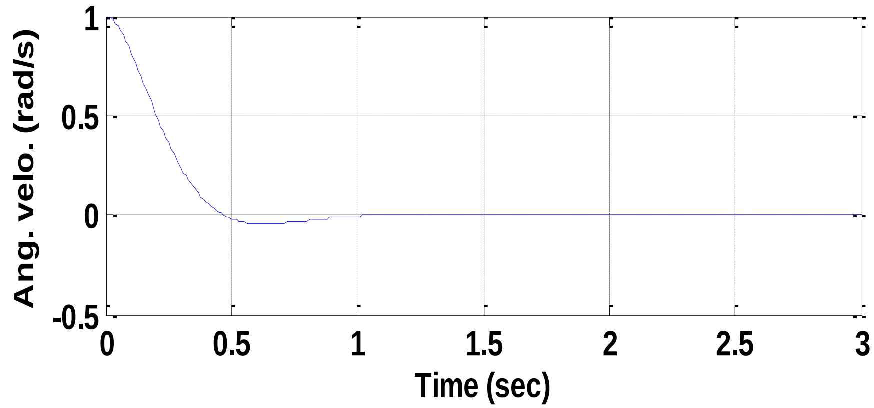
DC motor speed control (cont’s)

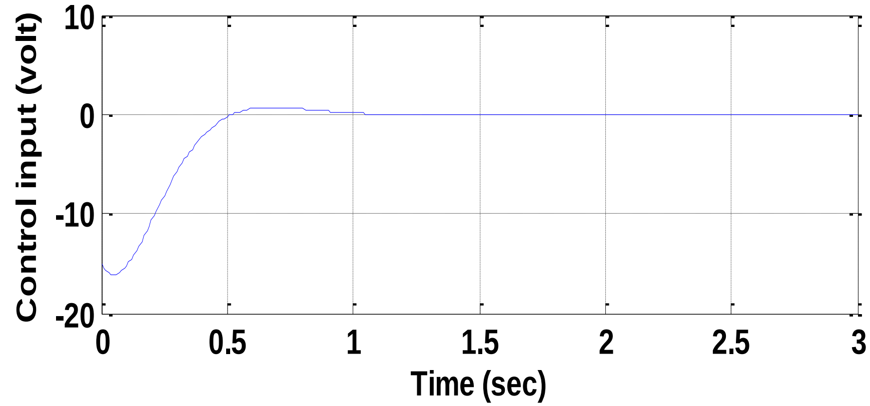
pole = -5+j, -5-j





pole = -5+5j, -5-5j





Exercise

- Try the simulation by yourselves! (Matlab code "motorspeed.m" is on Canvas.) Change the pole locations, and get a feeling how responses are affected by the pole location.
- Design a state feedback *u=-Kx* so that the closed-loop system has *-1* and *-2* as its eigenvalues, by using the canonical form method for:

$$\dot{x}(t) = \underbrace{\begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix}}_{A} x(t) + \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{B} u(t)$$

Multi-input example

- Place all the poles to *-1* for the system

$$A := \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \quad B := \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \quad K := \begin{bmatrix} k_{11} & k_{12} & k_{13} & k_{14} \\ k_{21} & k_{22} & k_{23} & k_{24} \end{bmatrix}$$

$$\Rightarrow \quad A - BK = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 - k_{11} & 1 - k_{12} & 1 - k_{13} & 1 - k_{14} \\ 0 & 0 & 0 & 1 \\ 1 - k_{21} & 1 - k_{22} & 1 - k_{23} & 1 - k_{24} \end{bmatrix}$$

Multi-input example (cont'd)

- Aim at companion forms!

$$A - BK = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1-k_{11} & 1-k_{12} & 1-k_{13} & 1-k_{14} \\ 0 & 0 & 0 & 1 \\ 1-k_{21} & 1-k_{22} & 1-k_{23} & 1-k_{24} \end{bmatrix}$$

$$K = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 2 & 5 & 7 & 5 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & -4 & -6 & -4 \end{bmatrix}$$

$$(s+1)^4 = s^4 + 4s^3 + 6s^2 + 4s + 1$$

$$K = \begin{bmatrix} 2 & 3 & * & * \\ 1 & 1 & 2 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & -2 & * & * \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & -2 \end{bmatrix}$$

$$(s+1)^4 = (s^2 + 2s + 1)^2$$

Summary

- State feedback
  - Canonical form method to design state feedback gain
  - “place.m” in Matlab
  - DC motor speed control example
    - As the pole is moved away from the real axis, the overshoot becomes larger.
  - Multi-input example
- Next,
  - Stabilizability
  - How to select desired pole locations
  - (Lyapunov method to design state feedback gain)