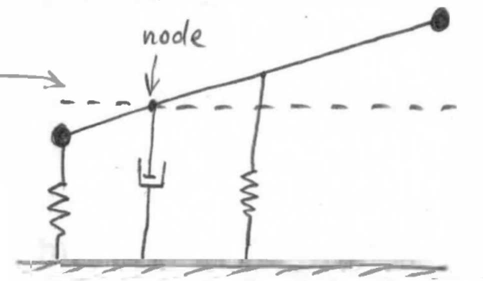
MECH 463 -- Tutorial 8

1. The diagram shows an idealized damped vibrating system. The rod supporting the two masses may be assumed to be rigid and have negligible mass. Find the natural frequencies and damping factors of the system.

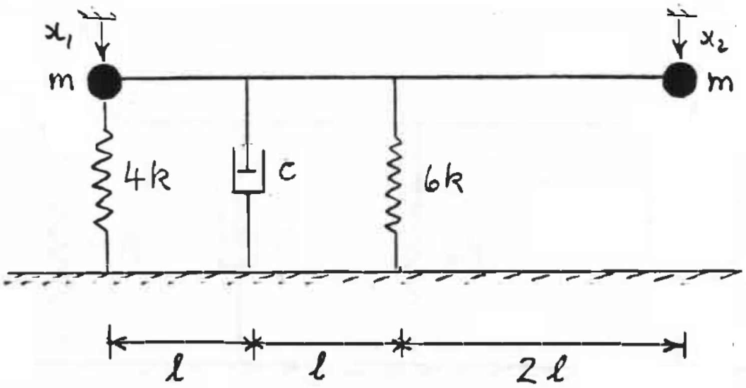
   Hint: Use the trial solution $\underline{x} = \underline{X}\, e^{\lambda t}$. Two of the roots of your characteristic equation are $\lambda^2 = -k/m$. You may therefore use the factor $(\lambda^2 + k/m)$ to reduce your characteristic equation to manageable form.

   Interpret the meaning and significance of the solutions $\lambda^2 = -k/m$.

   Ans. $\omega_n = \surd(6k/m)$. $\zeta = 5c/(16\surd(6km))$. $\omega_d = \omega_n \surd(1-\zeta^2)$



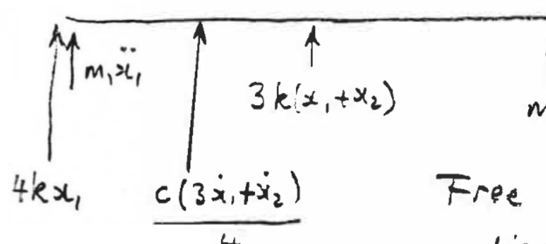
The given result $\lambda^2 = -k/m$ indicates a vibration mode without damping. For this to happen, the vibration mode must have a nodal point at the damper.



Take moments about the two masses

$$\ell\left(4m\ddot{x}_1 + 16kx_1 + \frac{3c(3\dot{x}_1+\dot{x}_2)}{4} + 6k(x_1+x_2)\right) = 0$$

$$\ell\left(4m\ddot{x}_1 + 6k(x_1+x_2) + \frac{c(3\dot{x}_1+\dot{x}_2)}{4}\right) = 0$$



Free body diagram

In matrix form:

$$\begin{bmatrix} 4m & 0 \\ 0 & 4m \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} \frac{9}{4}c & \frac{3}{4}c \\ \frac{3}{4}c & \frac{1}{4}c \end{bmatrix}\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} + \begin{bmatrix} 22k & 6k \\ 6k & 6k \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Try solution $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} e^{\lambda t} \longrightarrow \left(\lambda^2\begin{bmatrix} 4m & 0 \\ 0 & 4m \end{bmatrix} + \lambda\begin{bmatrix} \frac{9}{4}c & \frac{3}{4}c \\ \frac{3}{4}c & \frac{1}{4}c \end{bmatrix} + \begin{bmatrix} 22k & 6k \\ 6k & 6k \end{bmatrix}\right)\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} e^{\lambda t} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

For a non-trivial solution valid for all t

$$\rightarrow \begin{vmatrix} 4m\lambda^2 + \frac{9}{4}c\lambda + 22k & \frac{3}{4}c\lambda + 6k \\ \frac{3}{4}c\lambda + 6k & 4m\lambda^2 + \frac{c}{4}\lambda + 6k \end{vmatrix} = 0$$

The characteristic equation is:

$$\left(4m\lambda^2 + \tfrac{9}{4}c\lambda + 22k\right)\left(4m\lambda^2 + \tfrac{1}{4}c\lambda + 6k\right) - \left(\tfrac{3}{4}c\lambda + 6k\right)\left(\tfrac{3}{4}c\lambda + 6k\right) = 0$$

$$= 16m^2\lambda^4 + 10mc\lambda^3 + \left(24mk + 88mk + \cancel{\tfrac{9}{16}c^2} - \cancel{\tfrac{9}{16}c^2}\right)\lambda^2$$

$$+ \left(\tfrac{27}{2}ck + \tfrac{11}{2}ck - \tfrac{9}{2}ck - \tfrac{9}{2}ck\right)\lambda + (132 - 36)k^2 = 0$$

$$= 16m^2\lambda^4 + 10mc\lambda^3 + 112mk\lambda^2 + 10ck\lambda + 96k^2 = 0$$

Given that $\lambda = \pm i\sqrt{\frac{k}{m}}$ are two roots, then $\left(\lambda^2 + \frac{k}{m}\right)$ is a factor of the characteristic equation.

Divide by this factor to reduce the characteristic equation

$$\begin{array}{r|l}
 & 16m^2\lambda^2 + 10mc\lambda + 96mk \\
\lambda^2 + \frac{k}{m} & 16m^2\lambda^4 + 10mc\lambda^3 + 112mk\lambda^2 + 10ck\lambda + 96k^2 = 0 \\
 & 16m^2\lambda^4 + 16mk\lambda^2 \\
 & 10mc\lambda^3 + 96mk\lambda^2 + 10ck\lambda + 96k^2 \\
 & 10mc\lambda^3 + 10ck\lambda \\
 & 96mk\lambda^2 + 96k^2 \\
 & 96mk\lambda^2 + 96k^2
\end{array}$$

Dividing by $2m$, the remaining roots of the characteristic equation are the roots to

$$8m\lambda^2 + 5c\lambda + 48k = 0$$

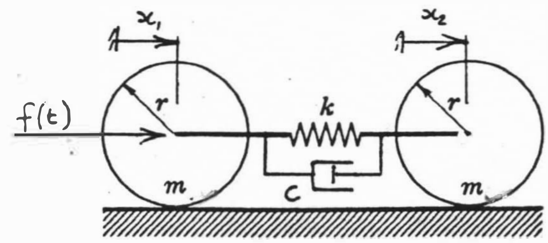
or

$$\lambda^2 + 2\cdot\zeta\omega_n\lambda + \omega_n^2 = 0$$

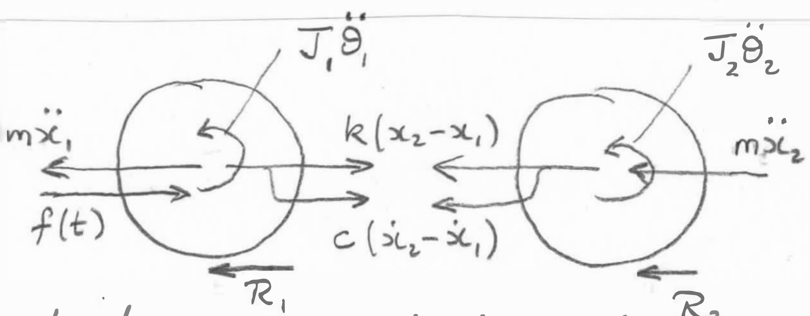
where $\boxed{\omega_n = \sqrt{6k/m} \qquad \zeta = \frac{5}{16}\frac{c}{\sqrt{6km}} \qquad \omega_d = \omega_n\sqrt{1-\zeta^2}}$

$\rightarrow \lambda_3, \lambda_4 = -\zeta\omega_n \pm i\omega_d$ (assuming underdamping)

2. Two solid cylinders of mass m rest on a rough horizontal surface. The cylinders, which both have radius r, are connected together by a spring of stiffness k and a damper of rate c. A horizontal force f(t) = F cos $\omega_f t$ is applied at the centre of cylinder 1. Derive an expression for the vibrational displacement at that point. Note: The polar moment of inertia of a cylinder J = ½mr².



Rotations of the cylinders: $\theta_1 = \frac{x_1}{r}$ $\theta_2 = \frac{x_2}{r}$



Take moments about the contact points with the surface (and avoid needing the friction forces R_1 and R_2)

$$m\ddot{x}_1 . r + J_1\ddot{\theta}_1 - c(\dot{x}_2-\dot{x}_1)r - k(x_2-x_1)r = f(t).r$$

$$m\ddot{x}_2 . r + J_2\ddot{\theta}_2 + c(\dot{x}_2-\dot{x}_1)r + k(x_2-x_1)r = 0$$

Put in matrix form with $J = \frac{mr^2}{2}$, $\theta = \frac{x}{r}$, $f(t) = F\cos\omega_f t$

$\div r$

$$\begin{bmatrix} \frac{3}{2}m & 0 \\ 0 & \frac{3}{2}m \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} c & -c \\ -c & c \end{bmatrix}\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} + \begin{bmatrix} k & -k \\ -k & k \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} F \\ 0 \end{bmatrix}\cos\omega_f t$$

For steady state response, try solution $\underline{x} = Re(\underline{X}e^{i\omega_f t})$

and note that $\underline{F}\cos\omega_f t = Re(\underline{F}e^{i\omega_f t})$, where $\underline{F}$ is real and $\underline{X}$ is complex.

Substitute both real and imaginary parts

$$\rightarrow (-\omega_f^2 \underset{\sim}{M} + i\omega_f \underset{\sim}{C} + \underset{\sim}{K})\underline{X}e^{i\omega_f t} = \underline{F}e^{i\omega_f t}$$

$$\rightarrow \begin{bmatrix} -\frac{3}{2}m\omega_f^2 + ic\omega_f + k & -(ic\omega_f + k) \\ -(ic\omega_f + k) & -\frac{3}{2}m\omega_f^2 + ic\omega_f + k \end{bmatrix}\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} F \\ 0 \end{bmatrix}$$

Solving by Cramer's rule

$$X_1 = \frac{F\left(-\frac{3}{2}m\omega_f^2 + ic\omega_f + k\right)}{\left(-\frac{3}{2}m\omega_f^2 + ic\omega_f + k\right)\left(-\frac{3}{2}m\omega_f^2 + ic\omega_f + k\right) - \left(ic\omega_f + k\right)^2}$$

$$= \frac{F\left(-\frac{3}{2}m\omega_f^2 + k + ic\omega_f\right)}{3m\omega_f^2\left(\frac{3}{4}m\omega_f^2 - k - ic\omega_f\right)}$$

$$= \frac{F\left(-\frac{3}{2}m\omega_f^2 + k + ic\omega_f\right)\left(\frac{3}{4}m\omega_f^2 - k + ic\omega_f\right)}{3m\omega_f^2\left(\left(\frac{3}{4}m\omega_f^2 - k\right)^2 + (c\omega_f)^2\right)}$$

$$= \frac{F\left(-\frac{9}{8}m^2\omega_f^4 + \left(\frac{9}{4}mk - c^2\right)\omega_f^2 - k^2 + i\frac{3}{4}mc\omega_f^3\right)}{3m\omega_f^2\left(\frac{9}{4}m^2\omega_f^4 + \left(\frac{3}{2}mk + c^2\right)\omega_f^2 + k^2\right)}$$

Let $X_1 = A + iB$ and recall $x_1 = \mathrm{Re}\left(X_1 e^{i\omega_f t}\right)$

$\rightarrow x_1 = \mathrm{Re}\left((A+iB)(\cos\omega_f t + i\sin\omega_f t)\right)$ using Euler formula

$$= \mathrm{Re}\left(A\cos\omega_f t - B\sin\omega_f t + i\left(B\cos\omega_f t + A\sin\omega_f t\right)\right)$$

$$= A\cos\omega_f t - B\sin\omega_f t$$

where $$A = \frac{F\left(-\frac{9}{8}m^2\omega_f^4 + \left(\frac{9}{4}mk - c^2\right)\omega_f^2 - k^2\right)}{3m\omega_f^2\left(\frac{9}{4}m^2\omega_f^4 + \left(\frac{3}{2}mk + c^2\right)\omega_f^2 + k^2\right)}$$

and $$B = \frac{F\left(\frac{1}{4}c\omega_f\right)}{\left(\frac{9}{4}m^2\omega_f^4 + \left(\frac{3}{2}mk + c^2\right)\omega_f^2 + k^2\right)}$$

The vibration amplitude $= |X_1| = \sqrt{A^2 + B^2}$. Phase $= \mathrm{atan}\left(\frac{B}{A}\right)$

The algebra soon gets ugly, even for a simple system.
In practice we would solve the matrix equation numerically.