MECH 420 SENSORS AND ACTUATORS

Solutions to Assignment 2

Sol-Problem 1 (Problem 2.4 from Textbook)

In open circuit, the voltage at the output port is given simply by the voltage-divider equation (in the frequency domain):

$$v_{oc} = \frac{\left[R_2 + \frac{1}{j\omega C}\right]}{\left[R_1 + R_2 + j\omega L + \frac{1}{j\omega C}\right]} v = v_{eq} \qquad \text{(i)}$$

Note: Equivalent source v_{eq} is expressed here as a function of frequency. Its corresponding time function $v_{eq}(t)$ is obtained by using inverse Fourier transform. Alternatively, first replace $j\omega$ by the Laplace variable *s*: $v_{eq}(s) = \frac{\left[R_2 + \frac{1}{sC}\right]}{\left[R_1 + R_2 + sL + \frac{1}{sC}\right]} v(s)$. Then obtain the inverse Laplace transform, for a given $v(s)$, using Laplace transform tables.

Now, in order to determine Z_{eq}, note from Figure P2.4(c) that when the voltage source is shorted, the resulting circuit has the two branches of impedance $(R_1 + j\omega L)$ and $\left[R_2 + \frac{1}{j\omega C}\right]$ in parallel. Their equivalent impedance is given by: $\frac{1}{Z_{eq}} = \frac{1}{(R_1 + j\omega L)} + \frac{1}{\left[R_2 + \frac{1}{j\omega C}\right]}$

Or,

$$Z_{eq} = \left[\frac{\left[R_2 + \frac{1}{j\omega C}\right]\left[R_1 + j\omega L\right]}{\left[R_1 + R_2 + j\omega L + \frac{1}{j\omega C}\right]}\right]$$

Sol-Problem 2 (Problem 2.7 from Textbook)

(a) The input impedance of the amplifier = 500 MΩ.

Estimated error $= \frac{10}{(500+10)} \times 100\% = 2\%$

(b) Impedance of the speaker = 4 Ω.

Estimated error $= \dfrac{0.1}{(4+0.1)} \times 100\% = 2.4\%$

Sol-Problem 3 (Problem 2.10 from Textbook)

For the given system, $\omega_n = \sqrt{\dfrac{1\times 10^6}{100}}$ rad/s = 100 rad/s and $\omega \geq 200$ rad/s. Hence, we have the frequency ratio $r \geq 2.0$.

For r = 2.0 and $\left|T_f\right|$ = 0.5 we have $0.5 = \sqrt{\dfrac{1+16\zeta^2}{9+16\zeta^2}}$ or, $\zeta = \sqrt{\dfrac{5}{48}}$. Hence,

$$b = 2\zeta\omega_n m = 2\sqrt{\frac{5}{48}} \times 100 \times 100 \text{ N.s/m} \rightarrow b = 6.455\times 10^3 \text{ N.s/m}.$$

With this damping constant, for $r \geq 2$, we will have $\left|T_f\right| \leq 0.5$. Decreasing b will decrease $\left|T_f\right|$ in this frequency range.

To plot the Bode diagram using MATLAB, first note that:

$$2\zeta\omega_n = b/m = 6.455\times 10^3/100 = 64.55 \text{ rad/s} \text{ and } \omega_n^2 = 10^4 \text{ (rad/s)}^2$$

The corresponding transmissibility function is $T_f = \dfrac{64.55s + 10^4}{s^2 + 64.55s + 10^4}$ with $s = j\omega$

The following MATLAB script will plot the required Bode diagram:

```
% Plotting of transmissibility function
clear;
m=100.0;
k=1.0e6;
b=6.455e3;
sys=tf([b/m k/m],[1 b/m k/m]);
bode(sys);
```

The resulting Bode diagram is shown in Figure S2.10. A transmissibility magnitude of 0.5 corresponds to $20\log_{10} 0.5$ dB = -6.02 dB.

Note from the Bode magnitude curve in Figure S2.10.4 that at the frequency 200 rad/s the transmissibility magnitude is less than -6 dB and it decreases continuously for higher frequencies. This confirms that the designed system meets the design specification.

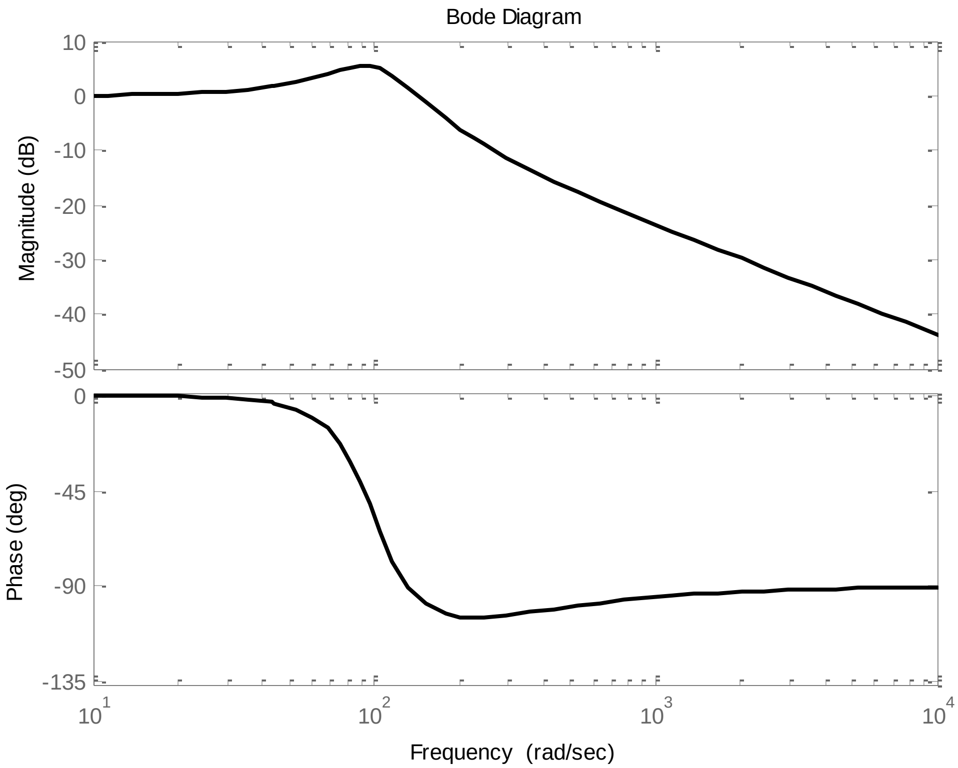

Figure S2.10: Transmissibility magnitude and phase curves of the designed system.

Sol-Problem 4 (Problem 2.17 from Textbook)

Slew rate: $s = 2\pi f_b a$ (i)

where, a = output amplitude, f_b = bandwidth (Hz).

The rise time T_r is inversely proportional to f_b. Hence, $f_b = \dfrac{k}{T_r}$ where, k = constant.

Substitution gives: $s = \dfrac{2\pi k a}{T_r}$ (ii)

From (i): For constant s, bandwidth decreases as a is increased.

For a sine signal, substitute the given values in (i): $f_b = \dfrac{0.5}{2\pi \times 2.5}\,\text{MHz} = 31.8\ \text{kHz}$

Next, for a step input, use $s = \dfrac{\Delta y}{\Delta t}$

where, Δy = final output value, Δt = time to reach the final output value

Substitute numerical values: $\Delta t = \dfrac{\Delta y}{s} = \dfrac{5.0}{0.5}\ \mu\text{s} = 10\ \mu\text{s}$.

Sol-Problem 5 (Problem 2.25 from Textbook)

(a)

Op-amp properties: 1. Voltages at input leads are equal; 2. Currents through input leads = 0

Op-amp property: $v_B = v_P = v_o$ (i)

Current Balance at Node A: $\dfrac{(v_i - v_A)}{Z_c} = \dfrac{(v_A - v_B)}{Z_c} + \dfrac{(v_A - v_P)}{R}$ (ii)

Current Balance at Node B: $\dfrac{(v_A - v_B)}{Z_c} = \dfrac{v_B}{R}$ (iii)

Note: $Z_c = \dfrac{1}{Cs}$ = impedance of capacitor

Substitute (i) and (iii) in (ii): $\dfrac{(v_i - v_A)}{Z_c} = \dfrac{v_o}{R} + \dfrac{(v_A - v_o)}{R} = \dfrac{v_A}{R} \rightarrow v_i = (1 + \dfrac{1}{\tau s})v_A$ (iv)

Substitute (i) in (iii): $\dfrac{(v_A - v_o)}{Z_c} = \dfrac{v_o}{R}$ $\rightarrow v_A = (1 + \dfrac{1}{\tau s})v_o$ (v)

Note: $\tau = RC$ = time constant

Substitute (iv) in (v): $G(s) = \dfrac{v_o}{v_i} = \dfrac{(\tau s)^2}{(\tau s + 1)^2}$

This is a 2nd order transfer function ➔ 2-pole filter

(b)

With $s = j\omega$ in $G(s)$, we have $G(j\omega) = \dfrac{-\tau^2\omega^2}{(1 + \tau j\omega)^2}$

Filter magnitude $|G(j\omega)| = \dfrac{\tau^2\omega^2}{(1 + \tau^2\omega^2)}$

The magnitude of the filter transfer function is sketched in Figure S2.25. This represents a high-pass filter.

(c)

When, $\omega << \dfrac{1}{\tau}$: $|G(j\omega)| \cong \tau^2\omega^2$

When, $\omega >> \dfrac{1}{\tau}$: $|G(j\omega)| \cong \dfrac{\tau^2\omega^2}{\tau^2\omega^2} = 1$

Hence, we may use $\omega_c = \dfrac{1}{\tau}$ as the cutoff frequency.

Note: $|G(j\omega)| \rightarrow 1$ as $\omega \rightarrow \infty$

For small ω: Roll-up slope of $|G(j\omega)|$ curve is = $20\log_{10}(\omega^2) = 40$ dB/decade

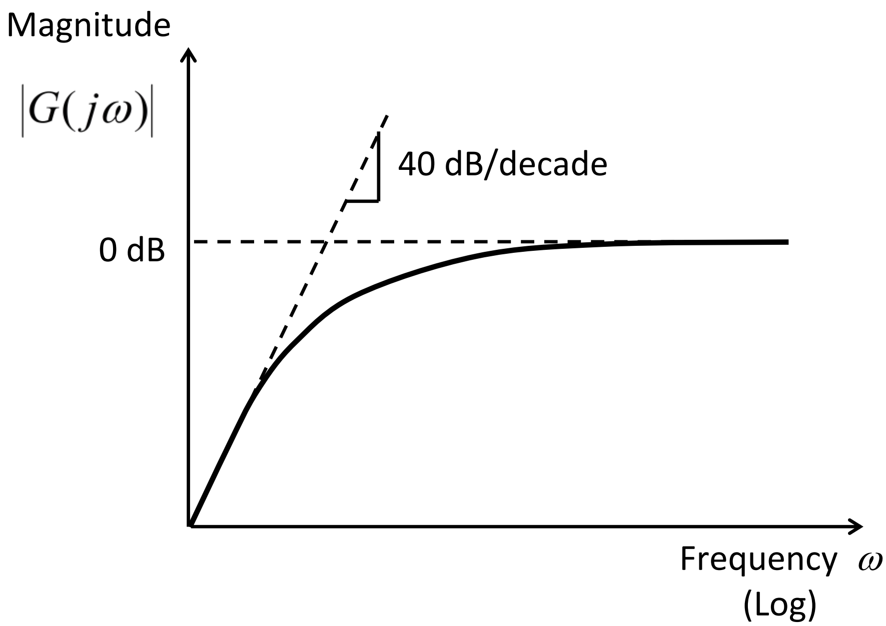

Figure S2.25: Filter transfer function magnitude.

Sol-Problem 6 (Problem 2.40 from Textbook)

From equation (2.80) we get $\delta v_o = \dfrac{\left[(R+\delta R)R - R(R-\delta R)\right]}{(R+\delta R+R)(R-\delta R+R)} v_{ref} - 0$

This simplifies to $\dfrac{\delta v_o}{v_{ref}} = \dfrac{2\delta R/R}{4-(\delta R/R)^2}$ which is nonlinear.

Similarly, it can be shown from equation (2.80) that the pair of changes: $R_2 \rightarrow R+\delta R$ and $R_4 \rightarrow R-\delta R$ will result in a nonlinear relation that is the same as before, except for the change in sign: $\dfrac{\delta v_o}{v_{ref}} = \dfrac{-2\delta R/R}{4-(\delta R/R)^2}$

Sol-Problem 7 (Problem 2.41 from Textbook)

From equation (2.87) we get $\delta v_o = \dfrac{\left[(R+\delta R)R - (R-\delta R)R\right]}{(R+\delta R+R-\delta R+R+R)} i_{ref} - 0$

On simplification we get the linear relation: $\dfrac{\delta v_o}{R i_{ref}} = \dfrac{\delta R/R}{2}$

If R_4 and R_3 are the active elements, with R_4 in tension and R_3 in compression, it is clear from equation (2.87) that we get an identical linear result (not even a sign change).

Sol-Problem 8 (Problem 2.48 from Textbook)

From the bridge circuit we have: $v_o = \left[\dfrac{R_1}{(R_1+R_2)} - \dfrac{R_3}{(R_3+R_4)}\right] v_{ref}$ (neglect load current)

If R_1 is changed to $R_1 + \delta R_1$ we have

$$\delta v_o = \left[\frac{R_1 + \delta R_1}{\left(R_1 + \delta R_1 + R_2\right)} - \frac{R_3}{\left(R_3 + R_4\right)} - \frac{R_1}{\left(R_1 + R_2\right)} + \frac{R_3}{\left(R_3 + R_4\right)} \right] v_{ref}$$

$$= \left[\frac{R_1 + \delta R_1}{\left(R_1 + R_2 + \delta R_1\right)} - \frac{R_1}{\left(R_1 + R_2\right)} \right] v_{ref} \quad = \left[\frac{R + \delta R}{2R + \delta R} - \frac{1}{2} \right] v_{ref} \qquad \{ R_1 = R_2 = R$$

Hence, $\delta v_o = \dfrac{\delta R}{2\left(2R + \delta R\right)} v_{ref}$

We can write this in Taylor series expansion as:

$$\delta v_o = \frac{\delta R}{4R\left(1 + \frac{\delta R}{2R}\right)} v_{ref} = \frac{\delta R}{4R} v_{ref} \left(1 + \frac{\delta R}{2R}\right)^{-1} = \frac{\delta R}{4R} v_{ref} \left(1 - \frac{\delta R}{2R} + \mathrm{O}(2)\right)$$

If we neglect O(2) terms, which are small compared to $\dfrac{\delta R}{2R}$, we have: $\delta v_o' = \dfrac{\delta R}{4R} v_{ref}$

➔ % error $= \dfrac{\left(\delta v_o' - \delta v_o\right)}{\delta v_o} \times 100 = \dfrac{\left[\dfrac{\delta R}{4R} - \dfrac{\delta R}{2\left(2R + \delta R\right)} \right]}{\dfrac{\delta R}{2\left(2R + \delta R\right)}} \times 100$

$$= \left[\frac{4R + 2\delta R}{4R} - 1 \right] \times 100 = \frac{\delta R}{2R} \times 100$$

For $\dfrac{\delta R}{R} = 0.05$: % error $= \dfrac{0.05}{2} \times 100\% = 2.5\%$