On the Direct Kinematics of Spherical Three-Degree-of-Freedom Parallel Manipulators with a Coplanar Platform

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This paper presents a polynomial solution to the direct kinematic problem of a class of spherical three-degree-of-freedom parallel manipulators. This class is defined as the set of manipulators for which the axes of the three revolute joints attached to the gripper link are coplanar and symmetrically arranged. It is shown that, for these manipulators, the direct kinematic problem admits a maximum of 8 real solutions. A polynomial of degree 8 is obtained here to support this result and cases for which all the roots of the polynomial lead to real configurations are presented. Finally, the spherical parallel manipulator with collinear actuators, which received some attention in the literature, is also treated and is shown to lead to a minimal polynomial of the same degree. Examples of the application of the method to manipulators of each category are given and solved.

1 Introduction

Although complex kinematic chains have been used in the design of mechanical systems for a very long time, their use in the context of robotic systems is more recent—see for instance MacCallion and Pham (1979), Hunt (1983), Fichter (1986), and Merlet (1987). In the latter references, it has been proposed to use manipulators with a parallel or hybrid architecture to provide an alternative to current serial-type robotic manipulators, which have their own limitations. Potential applications of parallel manipulators arise whenever there is a need for large structural stiffness and when it is desirable to bring the actuators as close as possible to the base. Current applications of parallel devices include flight simulators (Stewart, 1965; Dieudonne et al., 1972) and robotic applications requiring force control (Reboulet and Berthomieu, 1987; Merlet, 1987; Kim and Tesar, 1990).

Part of the work on parallel manipulators has been devoted to spherical robots (Cox and Tesar, 1989; Craver, 1989, Asada and Cro Granito,1985, Gosselin and Angeles, 1989). A spherical three-degree-of-freedom parallel manipulator could be used, for instance, as a stiff orientation wrist in robotics or as a mechanism for the orientation of machine tool beds. Prototypes of such manipulators have been developed (Craver, 1989; Asada and Cro Granito, 1985).

It is now well known (Hunt, 1983) that the inverse kinematic problem of parallel manipulators is in general much easier to solve than the direct one, as opposed to the result obtained for serial manipulators. In fact, the inverse kinematic problem of a parallel manipulator can be solved using formulations developed for serial manipulators (Gosselin, 1988) whereas the direct kinematic problem associated with these manipulators is, in general, rather involved and its solution requires additional derivations. The complexity of the problem appears clearly when a planar three-degree-of-freedom parallel manipulator is considered: despite the apparently simple architecture of the manipulator, no closed-form solution can be found for its direct kinematics (Gosselin and Sefrioui, 1991). For some time, researchers have considered only numerical solutions (Dieudonne et al., 1972; Merlet, 1987; Gosselin, 1988) because of the complexity of the problem. Although numerical solutions can be efficient and reliable under certain conditions, they don't provide any answer to theoretical questions such as the number of assembly modes and they rely on a certain initial guess of the solution which might not be very close to the solution. Hence, as a complement to these methods and as a way of obtaining additional theoretical results, polynomial solutions have been recently investigated. Several authors (Merlet, 1989; Griffis and Duffy, 1989; Charentus and Renaud, 1989; Nanua et al., 1990; Gosselin and Sefrioui, 1991) have studied the direct kinematics of spatial and planar parallel manipulators, obtaining polynomial solutions of different orders.

In this paper, a polynomial solution for the direct kinematics of a spherical three-degree-of-freedom manipulator is derived. The case for which the three gripper axes are coplanar and symmetrically arranged is investigated. It is shown to lead to a polynomial of degree 8. Moreover, this polynomial is shown to be minimal since 8 real solutions to the problem can be found for some configurations. This result also applies to manipulators whose legs are not symmetric and to manipulators whose actuators are collinear, as demonstrated here. An

Contributed by the Mechanisms Committee for publication in the JOURNAL OF MECHANICAL DESIGN. Manuscript received Nov. 1990; revised July 1993. Associate Technical Editor: C. F. Reinholtz.

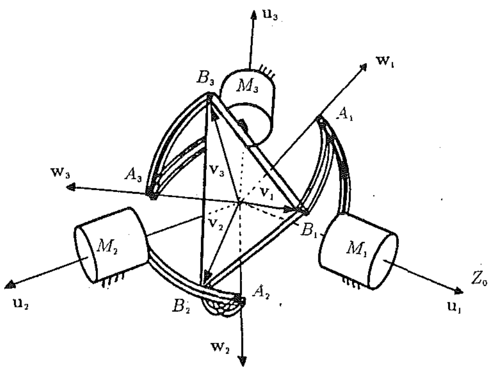

Fig. 1 Spherical three-degree-of-freedom parallel manipulator

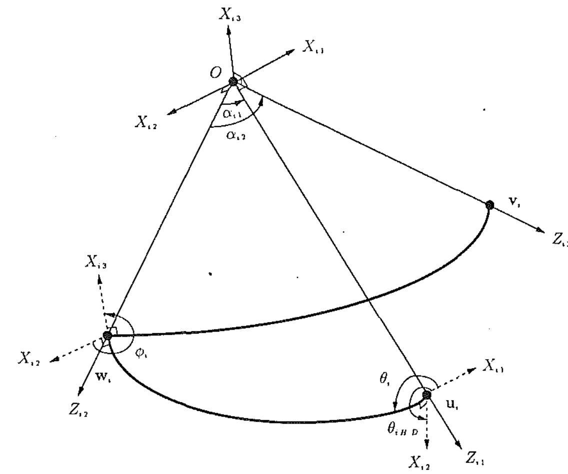

Fig. 2 Kinematic description of the ith leg of the manipulator using the Denavit-Hartenberg convention

example taken from Gosselin (1988) and for which 6 solutions had been found numerically is revisited. The 6 solutions mentioned above are reproduced and 2 additional solutions are found. Finally, the case for which the gripper axes are not coplanar will be the subject of a future report.

2 Formulation of the Kinematic Equations

A spherical three-degree-of-freedom parallel manipulator is shown in Fig. 1. The kinematic design of this type of manipulator has been studied in detail by a few authors (Craver, 1989; Gosselin and Angeles, 1989). However, in the latter references, the direct kinematic problem has not been addressed. The structure of the manipulator is such that the axes of all the revolute joints intersect at one common point which we will henceforth call the *center* of the wrist. The three motors of the manipulator, M_1, M_2, and M_3, are fixed to the base. Moreover, the design presented in Gosselin and Angeles (1989) is now assumed, i.e., it is assumed that the axes of the actuators are coplanar and that they are placed at 120 degrees from one another. Additionally, the three axes of the revolute joints attached to the gripper link are also coplanar and at 120 degrees from each other. Referring to Fig. 1, we will call $\mathbf{u}_i$, i = 1, 2, 3, the unit vectors directed along the motor axes and $\mathbf{v}_i$, i = 1, 2, 3, the unit vectors directed along the axes of the revolute joints on the gripper link. The reference configuration is the one in which we have:

$$\mathbf{u}_i = \mathbf{v}_i, \quad i = 1,2,3 \tag{1}$$

and the actuator angles, θ_i, i = 1, 2, 3, are measured from the plane formed by the motor axes. Hence we will have:

$$\mathbf{v}_i = \mathbf{Q}\mathbf{u}_i, \quad i = 1,2,3 \tag{2}$$

where $\mathbf{Q}$ is the rotation matrix representing the orientation of the gripper with respect to the fixed coordinate frame. Moreover, if the fixed coordinate frame is as shown in Fig. 1, we will have:

$$\mathbf{u}_i = \begin{bmatrix} \sin\eta_i \\ 0 \\ \cos\eta_i \end{bmatrix}, \quad i = 1,2,3 \tag{3}$$

and, according to the symmetry that has been imposed above, we can write

$$\eta_1 = 0, \quad \eta_2 = 2\pi/3, \quad \eta_3 = 4\pi/3 \tag{4}$$

Furthermore, each of the legs of the manipulator can be analyzed as an open loop spherical manipulator using the Denavit-Hartenberg notation (Denavit and Hartenberg, 1955). This is depicted in Fig. 2, where we have defined three coordinate frames associated with the ith leg, according to the Denavit-Hartenberg notation, and which we denote $OX_{i1}Y_{i1}Z_{i1}$, $OX_{i2}Y_{i2}Z_{i2}$, and $OX_{i3}Y_{i3}Z_{i3}$, respectively. Since all the frames have a common origin, the a_i's and the b_i's of the Denavit-Hartenberg notation are all equal to zero whereas the rotation matrices involved are:

Table 1 Denavit-Hartenberg parameters of the ith leg of spherical manipulator, as shown in Fig. 2

i	a_i	b_i	α_i	θ_i
1	0	0	α_{i1}	$\theta_i + \pi/2$
2	0	0	α_{i2}	ϕ_i

$$\mathbf{Q}_{i1} = \begin{bmatrix} -\sin\theta_i & -\cos\alpha_{i1}\cos\theta_i & \sin\alpha_{i1}\cos\theta_i \\ \cos\theta_i & -\cos\alpha_{i1}\sin\theta_i & \sin\alpha_{i1}\sin\theta_i \\ 0 & \sin\alpha_{i1} & \cos\alpha_{i1} \end{bmatrix}, \quad i = 1,2,3 \tag{5}$$

and

$$\mathbf{Q}_{i2} = \begin{bmatrix} \cos\phi_i & -\cos\alpha_{i2}\sin\phi_i & \sin\alpha_{i2}\sin\phi_i \\ \sin\phi_i & \cos\alpha_{i2}\sin\phi_i & -\sin\alpha_{i2}\cos\phi_i \\ 0 & \sin\alpha_{i2} & \cos\alpha_{i2} \end{bmatrix}, \quad i = 1,2,3 \tag{6}$$

The Denavit-Hartenberg parameters of the ith leg are given in Table 1 where α_{i1} and α_{i2} are the link angles for that leg. Finally, the rotation matrix carrying the original fixed frame $OXYZ$ into the $OX_{i1}Y_{i1}Z_{i1}$ frame—which is also fixed—is denoted by $\mathbf{R}_i$ and is given as

$$\mathbf{R}_i = \begin{bmatrix} \cos\eta_i & 0 & \sin\eta_i \\ 0 & 1 & 0 \\ -\sin\eta_i & 0 & \cos\eta_i \end{bmatrix}, \quad i = 1,2,3 \tag{7}$$

Now, we define another set of unit vectors $\mathbf{w}_i$, i = 1, 2, 3, which are directed along the axis of the intermediate revolute joint of each of the legs, i.e., which are directed along the Z_{i2} axes. Using the notation adopted above, vector $\mathbf{w}_i$ can be written, in the fixed reference frame $OXYZ$, as:

$$w_i = \mathbf{R}_i\mathbf{Q}_{i1}\mathbf{e}, \quad i = 1,2,3 \tag{8}$$

where

$$\mathbf{e} = [0\ 0\ 1]^T \tag{9}$$

which leads to

$$\mathbf{w}_i=\begin{bmatrix} \cos\eta_i \sin\alpha_{i1}\cos\theta_i+\sin\eta_i\cos\alpha_{i1} \\ \sin\alpha_{i1}\sin\theta_i \\ -\sin\eta_i\sin\alpha_{i1}\cos\theta_i+\cos\eta_i\cos\alpha_{i1}\end{bmatrix},\quad i=1,2,3 \tag{10}$$

As it is shown in Craver (1989), and Gosselin and Angeles (1989), the solution of the inverse kinematic problem of this manipulator is straightforward and leads to a maximum of 8 solutions. The derivation of the solution to this problem will now be shortly recalled, for quick reference. Once the orientation of the gripper-link, i.e., matrix $\mathbf{Q}$, is given, Eq. (2) can be used to compute the unit vector $\mathbf{v}_i$ and the following holds:

$$\mathbf{w}_i \cdot \mathbf{v}_i=\cos\alpha_{i2},\quad i=1,2,3 \tag{11}$$

The substitution of Eq. (10) in the latter equation leads to a quadratic equation in T_i for each of the legs, i.e.,

$$A_iT_i^2+2B_iT_i+C_i=0,\quad i=1,2,3 \tag{12}$$

where

$$T_i=\tan\left(\frac{\theta_i}{2}\right)\quad i=1,2,3 \tag{13}$$

and

$$A_i=x_i\sin\eta_i\cos\alpha_{i1}+z_i\cos\eta_i\cos\alpha_{i1} \\ -\cos\alpha_{i2}-x_i\cos\eta_i\sin\alpha_{i1}+z_i\sin\eta_i\sin\alpha_{i1} \tag{14}$$

$$B_i=y_i\sin\alpha_{i1} \tag{15}$$

$$C_i=x_i\cos\eta_i\sin\alpha_{i1}-z_i\sin\eta_i\sin\alpha_{i1} \\ +x_i\sin\eta_i\cos\alpha_{i1}+z_i\cos\eta_i\cos\alpha_{i1}-\cos\alpha_{i2} \tag{16}$$

where x_i, y_i, z_i are the components of vector $\mathbf{v}_i$. Hence, 2 solutions for angle θ_i are obtained from Eq. (12), for a given orientation of the platform, thereby completing the solution of the inverse kinematic problem.

In the next section, the kinematic notation presented above will be used to derive a polynomial solution to the direct kinematic problem for the manipulator at hand.

3 Polynomial Solution for the Direct Kinematics of the Spherical Parallel Manipulator

As mentioned in the introduction, the direct kinematic problem of a parallel manipulator is more involved than the inverse one. The direct problem can be formulated as follows: given the actuator coordinates—the θ_i's in the notation presented above—find the Cartesian coordinates of the gripper link. In the case of a spherical manipulator, the Cartesian coordinates are the parameters that describe the orientation of the gripper link. These can be a set of Euler angles, linear invariants, Euler parameters, or any other representation that uniquely defines the orientation. For instance, the 3 unit vectors noted $\mathbf{v}_i$, i = 1, 2, 3, could be used here.

However, since we want to obtain a system of equations whose solution leads to the orientation of the gripper link, it is probably better to minimize the number of parameters used in the description of the orientation in order to reduce, from the outset, the number of unknowns—and hence the number of equations—in the problem. Referring to the notation given in the preceding section and keeping in mind that, for this problem, the actuator angles are specified, we will then use the three angles ϕ_i, i = 1, 2, 3, which represent the rotation of the intermediate unactuated joints of the manipulator, as the unknowns for this problem. Indeed, if these angles can be found, the problem becomes trivial since the unit vectors $\mathbf{v}_i$, i = 1, 2, 3, can be written as:

$$\mathbf{v}_i=\mathbf{R}_i\mathbf{Q}_{i1}\mathbf{Q}_{i2}\mathbf{e}\quad i=1,2,3 \tag{17}$$

where $\mathbf{e}$ is defined as in Eq. (9). The substitution of Eqs. (5), (6), (7), and (9) in the above equation leads to an expression of vector $\mathbf{v}_i$ in terms of angle ϕ_i, i.e.,

$$\mathbf{v}_i=\begin{bmatrix} a_i\sin\phi_i+b_i\cos\phi_i+c_i \\ d_i\sin\phi_i+e_i\cos\phi_i+f_i \\ g_i\sin\phi_i+h_i\cos\phi_i+i_i\end{bmatrix},\quad i=1,2,3 \tag{18}$$

where the coefficients $\{a_i, \ldots, i_i\}$ are functions of the manipulator's dimensional parameters and of the imposed actuator angles. These coefficients are not given here because of space limitation but they are available from the authors upon request.

Now, the expressions of vectors $\mathbf{v}_i$, i = 1, 2, 3 given above are used to write the closure equations of the three kinematic loops of the manipulator. To this end, the following fact is used: *if the sum of three vectors of equal magnitude embedded in a three-dimensional space is equal to the zero vector, then these vectors are coplanar and are disposed symmetrically at* 120 deg *from one another.* The proof of this fact is straightforward and will be left to the reader. Now, since the three unit vectors $\mathbf{v}_i$, i = 1, 2, 3 on the gripper link are coplanar and symmetrically disposed and given the above statement, we can write the three kinematic closure equations of the manipulator as:

$$\sum_{i=1}^{3}\mathbf{v}_i=\mathbf{0} \tag{19}$$

where $\mathbf{0}$ stands for the three-dimensional zero vector. The substitution of Eqs. (18) in the above closure conditions will lead to the three following equations:

$$a_1\sin\phi_1+b_1\cos\phi_1+a_2\sin\phi_2+b_2\cos\phi_2+a_3\sin\phi_3 \\ +b_3\cos\phi_3+A_1=0 \tag{20}$$

$$d_1\sin\phi_1+e_1\cos\phi_1+d_2\sin\phi_2+e_2\cos\phi_2+d_3\sin\phi_3 \\ +e_3\cos\phi_3+A_2=0 \tag{21}$$

$$g_1\sin\phi_1+h_1\cos\phi_1+g_2\sin\phi_2+h_2\cos\phi_2+g_3\sin\phi_3 \\ +h_3\cos\phi_3+A_3=0 \tag{22}$$

with

$$A_1=c_2+c_2+c_3 \tag{23}$$

$$A_2=f_1+f_2+f_3 \tag{24}$$

$$A_3=i_1+i_2+i_3 \tag{25}$$

Equations (20)–(22) represent in fact a system of three nonlinear algebraic equations whose solution for the ϕ_i's will lead to the solution of the direct kinematics of the manipulator at hand. In order to obtain a polynomial solution for this problem, we will try to eliminate two out of the three variables ϕ_1, ϕ_2, ϕ_3 in the above equations. To begin with, Eqs. (20) and (21) are solved for $\sin\phi_1$ and $\cos\phi_1$. They are rewritten as,

$$a_1\sin\phi_1+b_1\cos\phi_1+A_1+B_1=0 \tag{26}$$

$$d_1\sin\phi_1+e_1\cos\phi_1+A_2+B_2=0 \tag{27}$$

which gives

$$\sin\phi_1=(A_4+b_1B_2-e_1B_1)/A_6 \tag{28}$$

$$\cos\phi_1=(A_5-a_1B_2+d_1B_1)/A_6 \tag{29}$$

with

$$A_4=b_1A_2-e_1A_1 \tag{30}$$

$$A_5=-a_1A_2+d_1A_1 \tag{31}$$

$$A_6=-b_1d_1+a_1e_1 \tag{32}$$

$$B_1 = a_2 \sin\phi_2 + b_2 \cos\phi_2 + a_3 \sin\phi_3 + b_3 \cos\phi_3 \quad (33)$$

$$B_2 = d_2 \sin\phi_2 + e_2 \cos\phi_2 + d_3 \sin\phi_3 + e_3 \cos\phi_3 \quad (34)$$

Then Eqs. (28) and (29) are substituted in Eq. (22) to yield:

$$M_1 \sin\phi_2 + M_2 \cos\phi_2 + M_3 = 0 \quad (35)$$

In the latter equation, which doesn't depend on ϕ_1, the coefficients M_1 and M_2 depend only on design parameters and on the input angles, whereas M_3 also depends on angle ϕ_3. Again, the detailed expressions of the coefficients are not given because of space limitation.

Additionally, Eqs. (28) and (29) are substituted in the following trigonometric identity:

$$\sin^2\phi_1 + \cos^2\phi_1 - 1 = 0 \quad (36)$$

This leads to a second equation, which can be written as,

$$N_4 \sin^2\phi_2 + N_3 \sin\phi_2 \cos\phi_2 + N_2 \sin\phi_2 + N_1 \cos\phi_2 + N_0 = 0 \quad (37)$$

where the coefficients are functions of the kinematic parameters and of angle ϕ_3. Hence, Eqs. (35) and (37) form a system of 2 nonlinear algebraic equations in ϕ_2 and ϕ_3. Then, Eq. (35) is solved for $\cos\phi_2$, i.e., we write:

$$\cos\phi_2 = -(M_3 + M_1 \sin\phi_2)/M_2 \quad (38)$$

The latter expression is substituted in Eq. (37), which results in an equation containing terms in $\sin\phi_2$ but not in $\cos\phi_2$. This is written as,

$$K_3 \sin^2\phi_2 + K_2 \sin\phi_2 + K_1 = 0 \quad (39)$$

Furthermore, Eq.(38) is substituted in the following trigonometric identity:

$$\sin^2\phi_2 + \cos^2\phi_2 - 1 = 0 \quad (40)$$

which yields an equation of the form,

$$H_3 \sin^2\phi_2 + H_2 \sin\phi_2 + H_1 = 0 \quad (41)$$

Again, the coefficients in Eqs. (39) and (41) are functions of the kinematic parameters and of angle ϕ_3.

Now, Eqs. (39) and (41) must be satisfied simultaneously. A necessary and sufficient condition for that is that the resultant of these equations, when considered as polynomials in $\sin\phi_2$, be equal to zero (Lang, 1984). This is written as the following determinant:

$$\begin{vmatrix} K_3 & K_2 & K_1 & 0 \\ 0 & K_3 & K_2 & K_1 \\ H_3 & H_2 & H_1 & 0 \\ 0 & H_3 & H_2 & H_1 \end{vmatrix} = 0 \quad (42)$$

i.e.,

$$H_3^2K_1^2 - H_2H_3K_1K_2 + H_1H_3K_2^2 + H_2^2K_1K_3 - 2H_1H_3K_1K_3 - H_1H_2K_2K_3 + H_1^2K_3^2 = 0 \quad (43)$$

which depends only on ϕ_3 and on the kinematic parameters (including the input angles, which are imposed for this problem). It is then easy to convert the above equation into a polynomial by using the following trigonometric identities:

$$\sin\phi_3 = \frac{2T}{1+T^2}, \ \cos\phi_3 = \frac{1-T^2}{1+T^2} \quad (44)$$

where

$$T = \tan\left(\frac{\phi_3}{2}\right) \quad (45)$$

and hence, the final result is a polynomial in T which turns out to be of degree 8, i.e.,

$$F_3T^8 + F_7T^7 + F_6T^6 + F_5T^5 + F_4T^4 + F_3T^3 + F_2T^2 + F_1T + F_0 = 0 \quad (46)$$

where the coefficients are functions of the kinematic parameters and of the input angles only. The detailed expressions for these coefficients take several pages. Therefore, they are not given here but can be obtained from the authors in machine readable form.

From the above expression, it is clear that the direct kinematics of the spherical manipulator for which the axes of the revolute joints attached to the gripper link are coplanar and symmetrically arranged leads to a maximum of 8 configurations. Moreover, as will be shown in Section 6, cases for which all the 8 roots of the polynomial lead to distinct real configurations have been found, thereby proving that the above polynomial is minimal. Hence, one has the following theorem:

Theorem: *The direct kinematic problem of the spherical parallel manipulator for which the axes of the three revolute pairs attached to the gripper link are coplanar and symmetrically arranged has at most 8 real solutions.*

4 Symmetric Manipulators

In addition to the symmetry imposed above on the arrangement of the motor axes and of the gripper link axes, one might consider building a fully symmetric manipulator (Gosselin and Angeles, 1989). In fact, since the tasks to be performed by the manipulator are not known *a priori*, there is no reason for the manipulator to have a preferred orientation in which it would have better properties. Therefore, symmetry can be justified. For the case at hand, since we have already placed the motor axes and the gripper link axes symmetrically, all that needs to be imposed to complete the symmetry of the manipulator is that the corresponding link angles on each of the legs be the same. In other words, the symmetric manipulator is defined as in Section 2 with the following additional constraints:

$$\alpha_{11} = \alpha_{21} = \alpha_{31} \equiv \alpha_1 \quad (47)$$

and

$$\alpha_{12} = \alpha_{22} = \alpha_{32} \equiv \alpha_2 \quad (48)$$

For such a manipulator, the derivation of the polynomial solution to the direct kinematics is identical to the one presented in the preceding section, and hence also leads to a 8th order polynomial. The only change is that the coefficients of the expressions given in Eq. (18) take on a simpler form.

5 Symmetric Manipulators with Collinear Actuators

Another type of spherical parallel manipulator that has attracted the attention of researchers is the one that has collinear actuators as shown in Fig. 3 (Asada and Cro Granito, 1985).

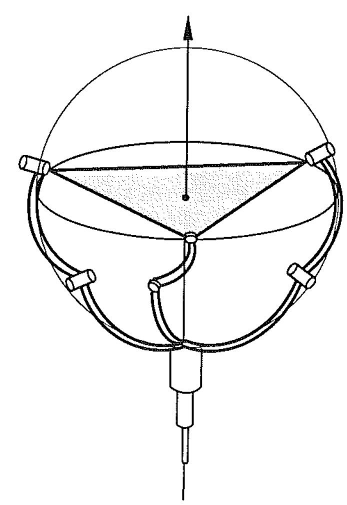

Fig. 3 Spherical three-degree-of-freedom parallel manipulator with collinear actuators

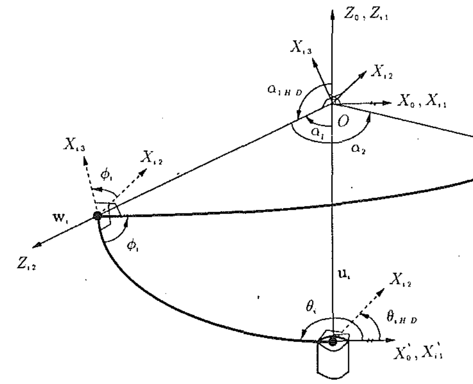

Fig. 4 Kinematic description of the ith leg of the manipulator with collinear actuators using the Denavit-Hartenberg convention

Table 2 Denavit-Hartenberg parameters of the ith leg of the spherical manipulator with collinear actuators, as shown in Fig. 4

i	a_i	b_i	α_i	θ_i
1	0	0	π-α_1	θ_i
2	0	0	α_2	ϕ_i

In the latter reference, it is mentioned that this kinematic arrangement is suitable for an orientation wrist. Moreover, a prototype in which the transmission of the power between the actuators and the manipulator is done through concentric driving shafts is briefly described. The kinematic analysis of such a manipulator is very similar to the one presented above (Gosselin 1988) and will now be summarized.

To begin with, since all the actuators are collinear, the Denavit-Hartenberg coordinate frame associated with the first revolute pair of each of the legs will be the same for all three legs. Hence, frames $OX_0Y_0Z_0$ and $OX_{i1}Y_{i1}Z_{i1}$, for $i = 1, 2, 3$, are all identical. This is shown in Fig. 4, where the ith leg is represented. The corresponding Denavit-Hartenberg parameters are given in Table 2. As a result, the rotation matrices involved can be written as:

$$\mathbf{R}_i = \mathbf{1}, \quad i = 1,2,3 \tag{49}$$

where $\mathbf{1}$ stands for the 3×3 identity matrix, and

$$\mathbf{Q}_{i1} = \begin{bmatrix} \cos\theta_i & \cos\alpha_1 \sin\theta_i & \sin\alpha_1 \sin\theta_i \\ \sin\theta_i & -\cos\alpha_1 \cos\theta_i & -\sin\alpha_1 \cos\theta_i \\ 0 & \sin\alpha_1 & -\cos\alpha_1 \end{bmatrix}, \quad i = 1,2,3 \tag{50}$$

$$\mathbf{Q}_{i2} \begin{bmatrix} \cos\phi_i & -\cos\alpha_2 \sin\phi_i & \sin\alpha_2 \sin\phi_i \\ \sin\phi_i & \cos\alpha_2 \cos\phi_i & -\sin\alpha_2 \cos\phi_i \\ 0 & \sin\alpha_2 & \cos\alpha_2 \end{bmatrix}, \quad i = 1,2,3 \tag{51}$$

Therefore, the unit vectors associated with the intermediate joints become:

$$\mathbf{w}_i = \begin{bmatrix} \sin\alpha_1 \sin\theta_i \\ -\sin\alpha_1 \cos\theta_i \\ -\cos\alpha_1 \end{bmatrix}, \quad i = 1,2,3 \tag{52}$$

Again, the inverse kinematic problem is solved by writing the closure equations as

$$\mathbf{w}_i \cdot \mathbf{v}_i = \cos\alpha_2, \quad i = 1,2,3 \tag{53}$$

Table 3 Link angles used for the example of Section 6.1

$\theta_1 = \pi/6$	$\theta_2 = \pi/6$	$\theta_3 = \pi/6$
$\alpha_{11} = 2\pi/9$	$\alpha_{21} = 2\pi/7$	$\alpha_{31} = \pi/4$
$\alpha_{12} = \pi/2$	$\alpha_{22} = 15\pi/29$	$\alpha_{32} = \pi/2$

Table 4 The 8 solutions of the direct kinematic problem for the example of Section 6.1

i	1	2	3	4	5	6	7	8
v_{1x}	-0.5634	0.7865	0.4730	0.5040	0.5876	-0.7981	-0.1954	-0.7396
v_{1y}	0.8237	0.0930	0.6354	0.5961	-0.8041	-0.0490	-0.8456	-0.2283
v_{1z}	0.0638	-0.6106	-0.6103	-0.6163	-0.0896	0.6005	0.4968	0.6332
v_{2x}	0.9578	-0.4969	-0.6663	-0.6998	-0.9790	0.5458	0.3647	0.9160
v_{2y}	0.0259	0.8133	-0.6470	-0.6146	-0.0723	-0.8196	0.7934	0.1949
v_{2z}	0.2863	0.3028	-0.3708	-0.3642	-0.1908	-0.1740	0.4874	0.3505
v_{3x}	-0.3944	-0.2896	0.1933	0.1958	0.3913	0.2523	-0.1693	-0.1765
v_{3y}	-0.8496	-0.9063	0.0116	0.0185	0.8765	0.8686	0.0522	0.0331
v_{3z}	-0.3501	0.3077	0.9811	0.9805	0.2804	-0.4265	-0.9842	-0.9837

which leads to three quadratic equations similar to the ones shown in Eq. (12) and which will give up to 8 solutions.

The derivation of the polynomial solution of the direct kinematic problem in that case parallels the one presented in Section 3. Indeed, using the rotation matrices given above, expressions similar to those of Eqs. (20)–(22) can be obtained for the unit vectors associated with the revolute pairs of the platform, i.e., vectors $\mathbf{v}_i$, $i = 1, 2, 3$. Then, condition (19) is invoked and the rest of the derivation is identical and finally leads to a polynomial of degree 8. As will be shown in the examples, the above theorem also applies to this type of manipulator.

6 Examples

In this section, we will present examples of the application of the polynomial solution to manipulators of the 3 types discussed above. Configurations that lead to the largest possible number of solutions have been chosen to clearly illustrate that the polynomials derived here are minimal.

6.1 General Manipulator with Coplanar Gripper Axes. As a first example, a manipulator of the type discussed in Section 3 and whose link angles are given in Table 3 is considered. Moreover, the actuator angles shown in this table are used. Under these conditions, the polynomial of Eq. (46) has 8 real roots, corresponding to the 8 possible configurations listed in Table 4 and shown in Fig. 5. In this figure, only the distal links and the platform are shown for purposes of clarity.

6.2 Fully Symmetric Manipulator. A manipulator of the type discussed in Section 4 and whose link angles are given by $\alpha_1 = \pi/3$ and $\alpha_2 = 7\pi/18$ is now considered. Moreover, the following motor angles, i.e., $\theta_1 = \theta_2 = \theta_3 = \pi/6$ are imposed. This example was taken from Gosselin (1988) where 6 solutions for the direct kinematic problem—in that particular case—were found using a numerical procedure and different initial guesses. The solution of this problem is revisited here using the polynomial of Eq. (46) which turned out to have 8 real roots corresponding to 8 real configurations of the manipulator. Six of these configurations are identical to the ones given in Gosselin (1988) and 2 additional orientations of the platform that satisfy the closure equations are revealed. The admissible configurations are represented in Fig. 6, where the same convention as in Fig. 5 is adopted, and they are listed in Table 5.

6.3 Manipulator with Collinear Axes. This example pertains to the manipulator of Section 5. The link angles are given as $\alpha_1 = \pi/3$ and $\alpha_2 = 7\pi/18$ and the actuator angles are specified as: $\theta_1 = 0$, $\theta_2 = 2\pi/3$, and $\theta_3 = 4\pi/3$. In that case,

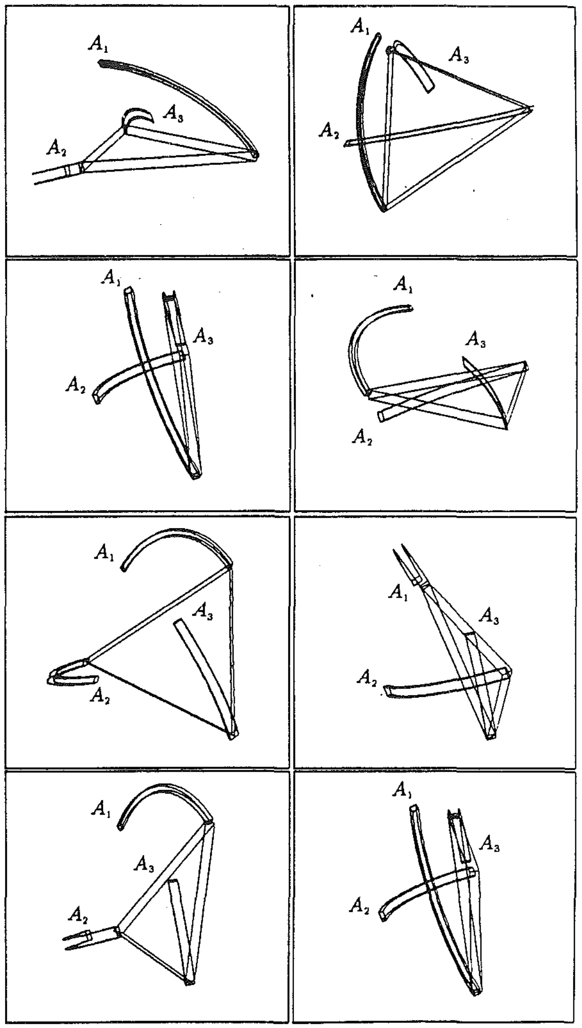

Fig. 5 The 8 solutions of the direct kinematic problem for the example of Section 6.1, whose link angles are given in Table 3

Table 5 The 8 solutions of the direct kinematic problem for the example of Section 6.2

i	1	2	3	4	5	6	7	8
v_{1x}	0.5881	-0.2023	0.8769	0.0599	0.8289	0.6020	-0.0304	-0.1975
v_{1y}	-0.6989	0.9679	-0.269	0.9679	0.0000	-0.6989	-0.2690	0.0000
v_{1z}	0.4071	0.1492	-0.3983	-0.2441	-0.5594	0.3863	0.9626	0.9803
v_{2x}	0.2304	0.8489	-0.2414	0.0335	-0.8989	-0.7834	0.0585	0.9477
v_{2y}	0.9679	-0.2690	0.9679	-0.6989	0.0000	-0.2690	-0.6989	0.0000
v_{2z}	0.1006	-0.4550	0.0701	-0.7145	-0.4382	-0.5603	-0.7129	-0.3191
v_{3x}	-0.8185	-0.6466	-0.6355	-0.0935	0.0699	0.1814	-0.0281	-0.7502
v_{3y}	-0.2690	-0.6989	-0.6989	-0.2690	0.0000	0.9679	0.9679	0.0000
v_{3z}	-0.5077	0.3058	0.3282	0.9586	0.9976	0.1740	-0.2498	-0.6612

the polynomial has 8 real roots which therefore leads to 8 possible configurations, as represented in Fig. 7. The solutions are listed in Table 6.

7 Extension to other Topologies

The polynomial solution derived above can be used for mechanical devices other than the parallel manipulators for which it was devised. Indeed, the kinematics of a spherical complex kinematic chain such as the double-input/single-output spherical two-degree-of-freedom mechanism shown in Fig. 8 can be solved using the above procedure. Once the orientation of the platform is determined, the solution for the output variable is trivial. This mechanism could also be used as an orientation device for tasks that require only two degrees of freedom such as the orientation of solar panels or radar antennas.

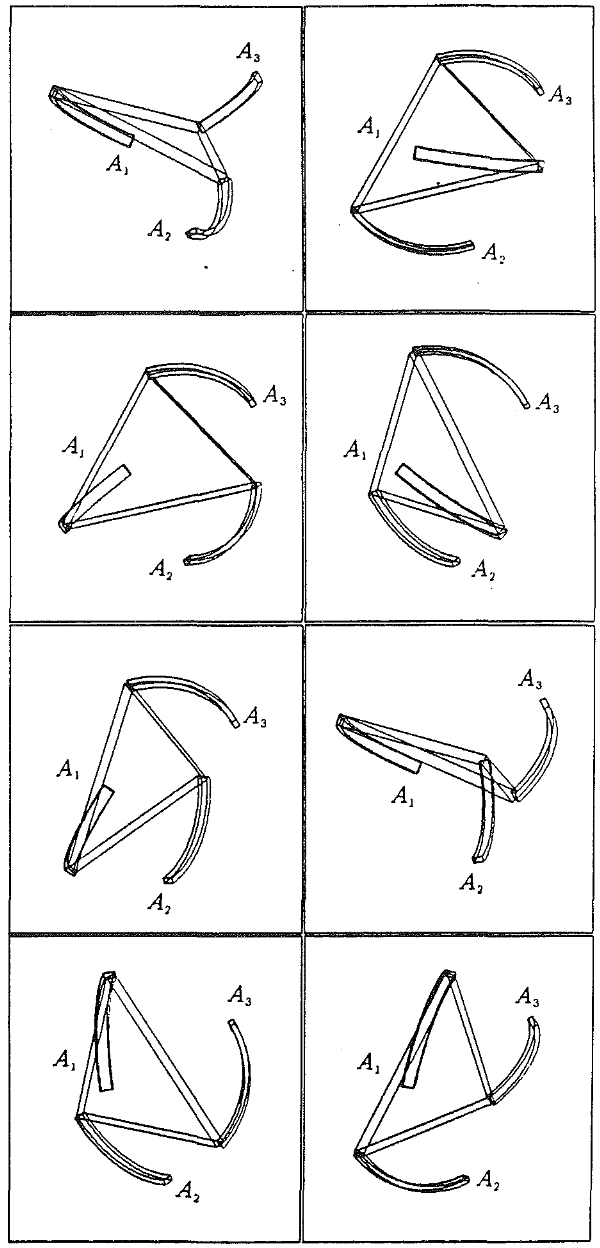

Fig. 6 The 8 solutions of the direct kinematic problem for the example of Section 6.2

Table 6 The 8 solutions of the direct kinematic problem for the example of Section 6.3

i	1	2	3	4	5	6	7	8
v_{1x}	0.6948	0.2862	-0.9187	-0.6948	-0.2862	0.9187	0.0907	-0.0907
v_{1y}	-0.6125	-0.7437	-0.3949	-0.6125	-0.7437	-0.3949	0.1715	0.1715
v_{1z}	0.3769	0.6041	0.0000	0.3769	0.6041	0.0000	-0.9810	-0.9810
v_{2x}	-0.1938	-0.1031	0.8014	0.5010	0.1830	-0.1173	0.7872	0.8779
v_{2y}	-0.0072	-0.1643	-0.5982	0.6197	0.9080	0.9931	0.1240	-0.2955
v_{2z}	-0.9810	-0.9810	0.0000	0.6041	0.3769	0.0000	0.6041	0.3769
v_{3x}	-0.5010	-0.1830	0.1173	0.1938	0.1031	-0.8014	-0.8779	-0.7872
v_{3y}	0.6197	0.9080	0.9931	-0.0072	-0.1643	-0.5982	-0.2955	0.1240
v_{3z}	0.6041	0.3769	0.0000	-0.9810	-0.9810	0.0000	0.3769	0.6041

8 Conclusion

A polynomial solution for the direct kinematics of a class of spherical parallel manipulators has been derived in this

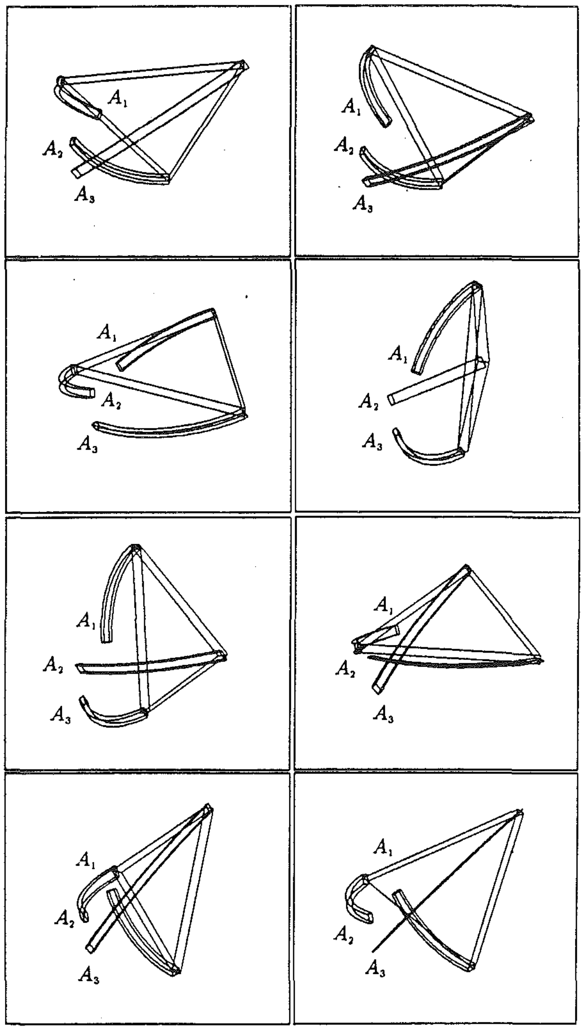

Fig. 7 The 8 solutions of the direct kinematic problem for the example of Section 6.3

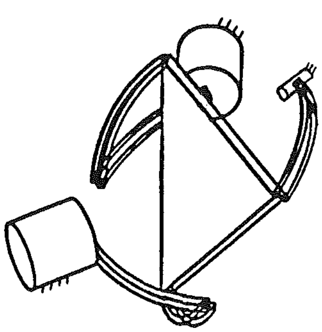

Fig. 8 Two-input/single output spherical two-degree-of-freedom mechanism

paper. It has been shown that all cases for which the axes of the three revolute joints attached to the gripper are coplanar and symmetrically arranged lead to a maximum of 8 solutions. A minimal polynomial has been derived for each of these cases. The coefficients of these polynomials have been obtained using a symbolic manipulator and complete analytical expressions (in machine readable form) of these coefficients are available upon request from the authors. Examples having 8 real solutions have been presented to clearly illustrate the results. The direct kinematics of spherical parallel manipulators which do not fall in the class studied here is studied in a companion paper.

9 Acknowledgments

This work was completed under research grants from the Natural Sciences and Engineering Research Council of Canada (NSERC) and from the Fonds pour la Formation des Chercheurs et l'aide à la Recherche (FCAR) du Québec.

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