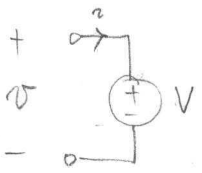
< Linear Circuits Review >

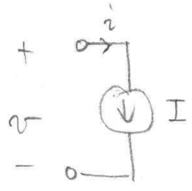
- One-port Elements

  - Voltage Source



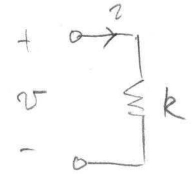
$Z = 0$

  - Current Source



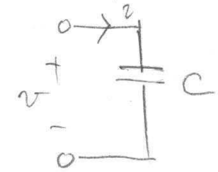
$Z = \infty$

  - Resistor



$Z = R$

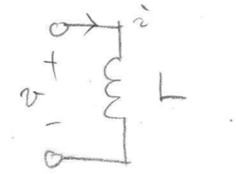
  - Capacitor



$Z = \frac{1}{Cs}$

$\left(v = \frac{1}{C} \underbrace{\int i\,dt}_{q} \right)$

  - Inductor

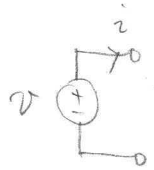


$Z = Ls$

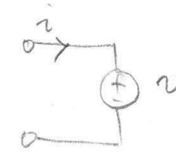
$\left(i = \frac{1}{L} \underbrace{\int v\,dt}_{\lambda} \right)$

- port
  - A pair of terminals that must carry the same current through the element.
  - Power through a port : $P = v \cdot i$ (Check the polarity) :

ex)



$P_{in} = -vi$



$P_{in} = +vi$

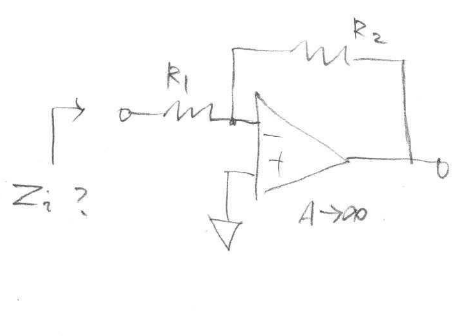
  - Impedance seen "looking into" a port

    It means the apparent impedance between the two terminals of a port.

  - Impedance seen "looking into" a terminal

    It means the apparent impedance between a particular terminal of interest and the common (ground) of the circuit.

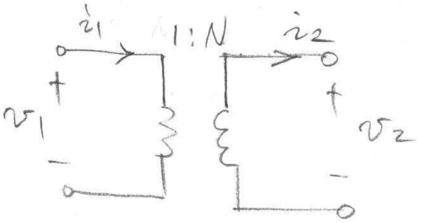
ex)

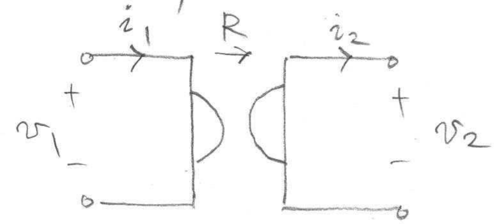


Q. What is the impedance looking into the input terminal ?

- Two-port Elements. (Passive, Lossless)

- Transformer
- Gyrator





Constitutive Relation:

Transformer: $v_2 = N \cdot v_1$, $i_2 = \frac{1}{N} \cdot i_1$

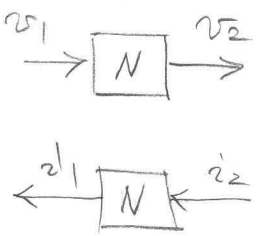
Gyrator: $v_2 = R\, i_1$, $i_2 = \frac{1}{R} v_1$.

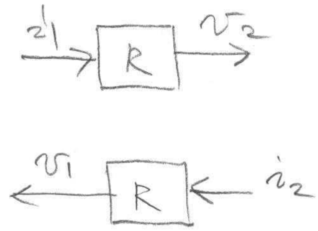
Power Relation:

Transformer: $v_2 i_2 = v_1 i_1$

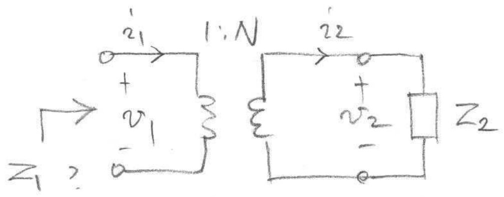
Gyrator: $v_2 i_2 = v_1 i_1$

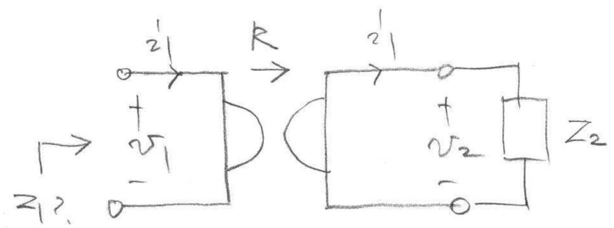
Block Diagram:

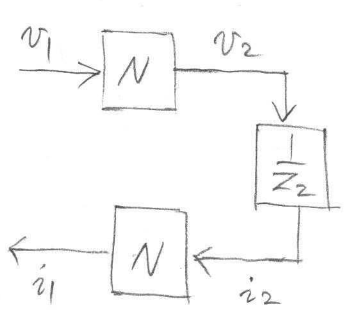


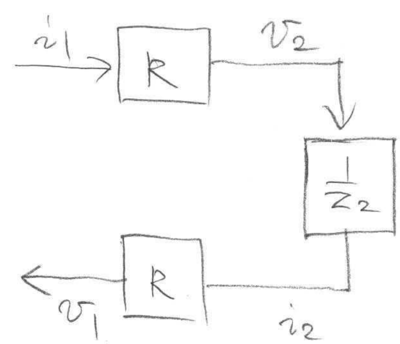


- Impedance Transformation.









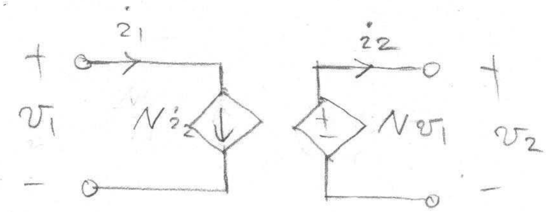
$$Y_1 \triangleq \frac{i_1}{v_1} = \frac{N^2}{Z_2}$$

$$Z_1 \triangleq \frac{v_1}{i_1} = \frac{Z_2}{N^2}.$$

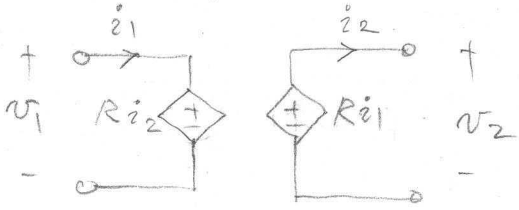
$$\therefore Z_1 \triangleq \frac{v_1}{i_1} = \frac{R^2}{Z_2}$$

Alternative Representations. (using dependant sources)

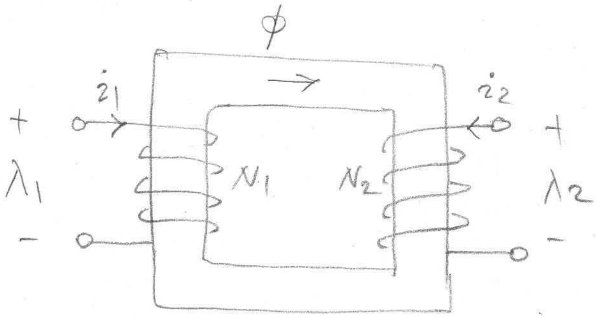
transformer



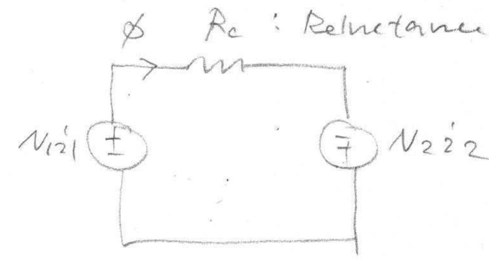
Gyrator



- Transformer Modeling



$\Rightarrow$



Magnetic circuit

- Terminal variables $\begin{cases} \text{Flux linkage } \lambda \\ \text{Current } i \end{cases}$

- Terminal Relations : $\begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = \begin{bmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \end{bmatrix}$

- Votage Relation : $\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \frac{d}{dt} \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = \begin{bmatrix} N_1 \frac{d\phi}{dt} \\ N_2 \frac{d\phi}{dt} \end{bmatrix} \Rightarrow \boxed{\frac{v_1}{N_1} = \frac{v_2}{N_2}}$

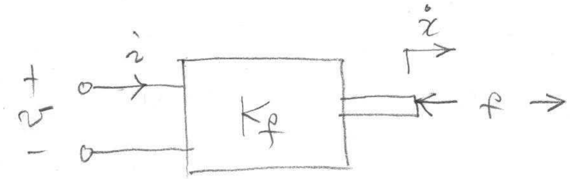
☆ assuming no leakage flux.

- Current Relation : $R_c \phi = N_1 i_1 + N_2 i_2$

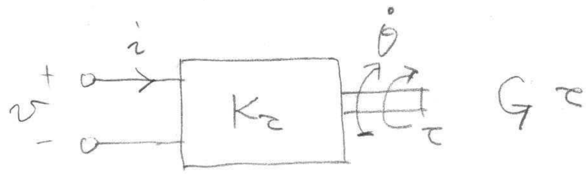
☆ As $\mu \to \infty$ $R_c \to 0$ $\Rightarrow \boxed{N_1 i_1 + N_2 i_2 = 0}$

- Two-port Element (Ideal Lorenz-type transducer)

  - Voice Coil



  - Brushed DC Motor



Constitutive Relation :

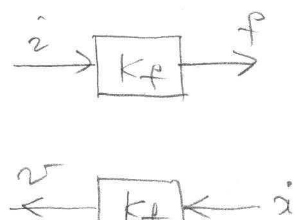
$$f = K_f \cdot i \qquad\qquad \tau = K_\tau \cdot i$$

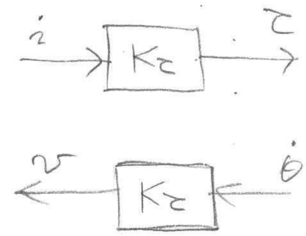
$$\dot{x} = \frac{1}{K_f} v \qquad\qquad \dot{\theta} = \frac{1}{K_\tau} v$$

power Relation

$$\underset{(P_{out})}{f \cdot \dot{x}} = \underset{(P_{in})}{v i} \qquad\qquad \underset{(P_{out})}{\tau \dot{\theta}} = \underset{(P_{in})}{v i}$$

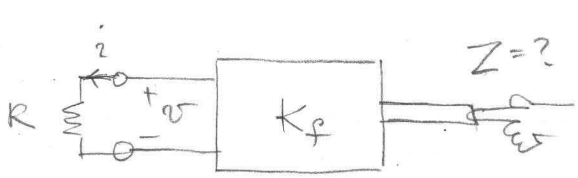
Block Diagram :



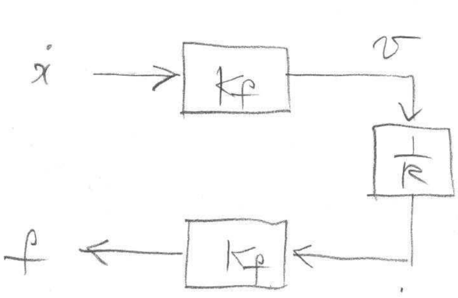


- Note that these elements behave like "multi-domain gyrator".

- Apparent damping.



Suppose you connected a resistor across the two terminals of a voice coil.
Ignoring its internal inductance ($L = 0$).
What is the mechanical impedance you "feel" at the mechanical port?



$$Z = \frac{F}{\dot{x}} = \frac{K_f^2}{R} \Rightarrow \text{"Apparent damping".}$$

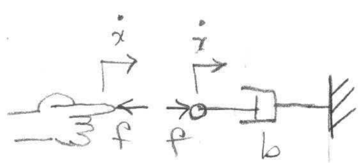
As $R \downarrow$, you feel more damping.

○ Mechanical Impedance.

$$Z = \frac{\text{effort}}{\text{flow}} \quad \begin{cases} \dfrac{\text{voltage}}{\text{current}} & \text{in electric circuits} \\ \\ \dfrac{\text{force}}{\text{velocity}} & \text{in mechanical systems.} \end{cases}$$

- We can define mechanical impedance at a "mechanical port".

- "port" is a channel of power flow.



When you push a dashpot with your finger (with force f & velocity $\dot{x}$) the area of contact becomes a mechanical port.

Power into your finger : $-f\dot{x}$

power into the dashpot : $f\dot{x}$

- The ratio between the force and velocity, i.e. $\frac{f}{\dot{x}}$ can be defined as mechanical impedance.

Here, $Z = \dfrac{f}{\dot{x}} = \dfrac{b\dot{x}}{\dot{x}} = b$.

○ Dynamic Stiffness.

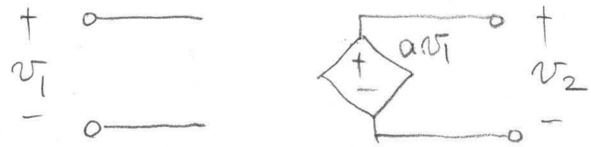
For mechanical systems where we care more about "position" than velocity, it is more convinient to use the concept of dynamic stiffness $K \triangleq \dfrac{F}{X}$

Here, $K = \dfrac{b\dot{x}}{x} = bs$, $s \in \mathbb{C}$.

<u>Note</u> : many people use the word "Mechanical Impedance" for "Dynamic stiffness", we should check the meaning from the context.

Two-port Elements (Active; Amplifier)

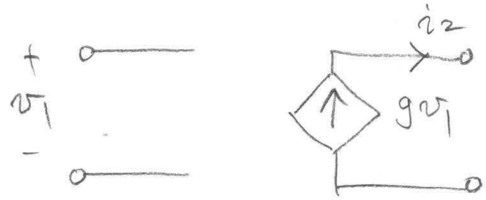
- Voltage-controlled Voltage Source.



$$v_2 = a \cdot v_1$$

a : voltage gain [−].

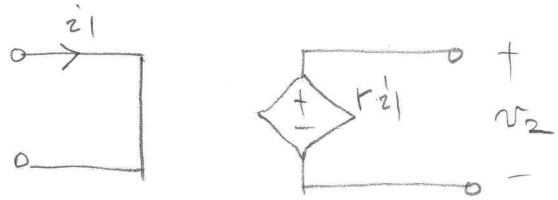
- Voltage-controlled Current Source.



$$i_2 = g\, v_1$$

g : transconductance [Ω^{-1}].

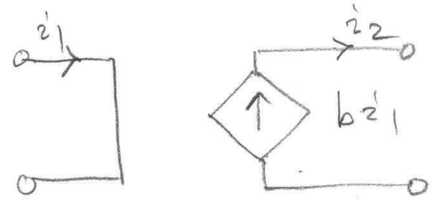
- Current-controlled Voltage Source



$$v_2 = r \cdot i_1$$

r: trans impedance [Ω].

- Current-controlled Current Source

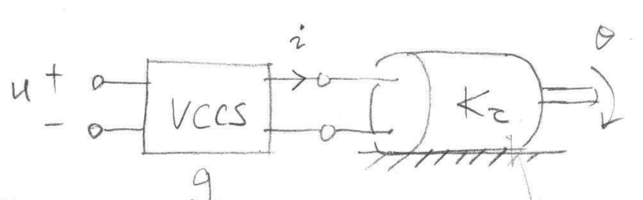


$$i_2 = b \cdot i_1$$

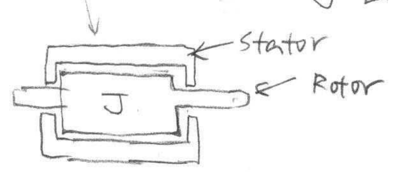
b : current gain [−]

Q1. What is the output impedance of VCCS when the input terminals are shorted? (Answer : ∞)

Q2. A brushed DC motor is driven with a VCCS.



- u [V] : control effort
- K_τ [Nm/A] : torque constant
- J [kg·m²] : rotational inertia



Find the transfer function from u [V] to θ [rad].

(Answer : $\frac{K_\tau g}{J s^2}$)

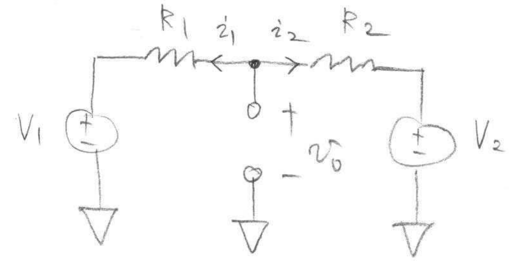
- Interconnection Constraints

  - Kirchhoff's Voltage Law (KVL) : $\sum_{k=1}^{n} v_k = 0$
  - Kirchhoff's Current Law (KCL) : $\sum_{k=1}^{n} i_k = 0$

- Node Method

  - Identify all the nodes whose voltages are to be solved, and apply KCL to each of the nodes.
  - Collect all the equations and solve for the unknown voltages.

ex)



$$i_1 = \frac{v_o - V_1}{R_1}, \quad i_2 = \frac{v_o - V_2}{R_2}$$

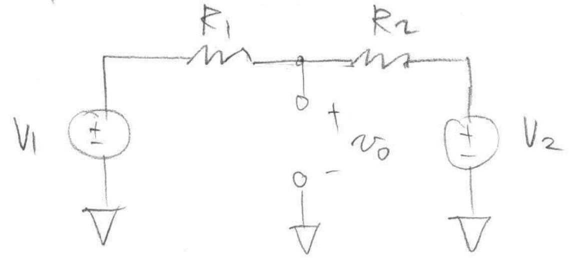
$$\text{KCL} : i_1 + i_2 = 0 \Rightarrow \frac{v_o - V_1}{R_1} + \frac{v_o - V_2}{R_2} = 0$$

$$\therefore \left(\frac{1}{R_1} + \frac{1}{R_2}\right) v_o = \frac{1}{R_1} V_1 + \frac{1}{R_2} V_2$$

$$\therefore v_o = \frac{R_2}{R_1 + R_2} V_1 + \frac{R_1}{R_1 + R_2} V_2$$

- It is a sure-fire way to solve linear circuit, **BUT** the complexity in algebra builds up rapidly & it does not help you build intuition.

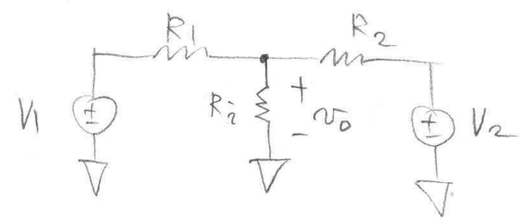
- Superposition Method



$$v_o = \underbrace{\frac{R_2}{R_1+R_2} V_1}_{\substack{\text{response when} \\ V_2=0}} + \underbrace{\frac{R_1}{R_1+R_2} V_2}_{\substack{\text{response when} \\ V_1=0}}$$

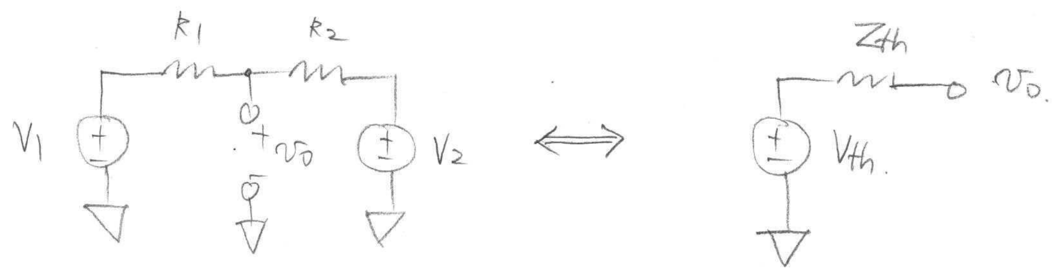
- Thevenin Equivalent Circuit

  . You are asked to solve for v_o.



Would you solve this circuit from scratch ?

<u>OR</u> is there a more elegant way to re-use the previous result ?



- Thevenin voltage (V_{th})
  : the voltage when the port is open-circuited.

$$V_{th} = \frac{R_2}{R_1+R_2} V_1 + \frac{R_1}{R_1+R_2} V_2$$

⇒ which we already know.

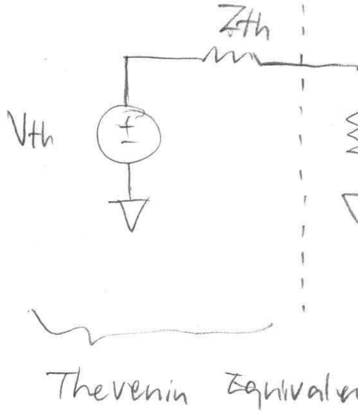
- Thevenin Impedance (Z_{th})

: the impedance looking into the port when all the independent sources are turned off

$$R_{th} = R_1 \parallel R_2 = \frac{R_1 + R_2}{R_1 R_2}.$$

$\Rightarrow$ which we can quickly figure out via inspection.

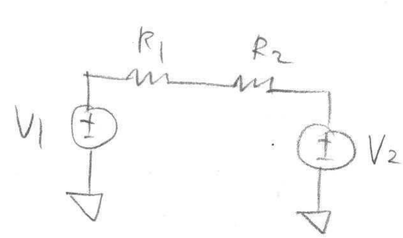
Thus, the original circuit problem can be simplified to



$$v_o = \frac{Z_i}{Z_i + Z_{th}} v_{th}$$

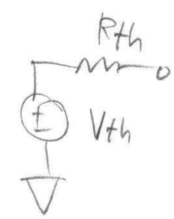
$$= \frac{Z_i}{Z_i + R_1 \parallel R_2}\left(\frac{R_2}{R_1+R_2} V_1 + \frac{R_1}{R_1+R_2} V_2\right).$$

- Note that Thevenin equivalent tells us the equivalent system only in terms of the port voltage and current.
- For example. Power dissipation from the original circuit and that from the Thevenin equivalent can be different.



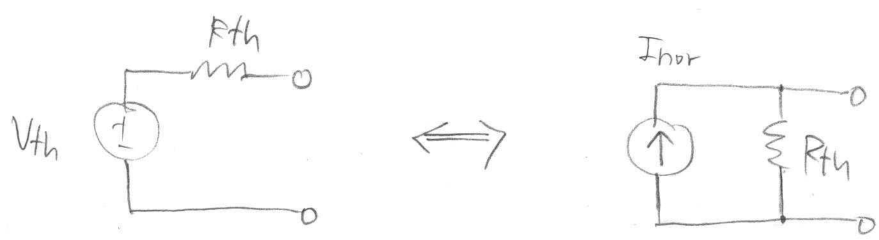
$$P_{diss} = \frac{(V_1 - V_2)^2}{R_1 + R_2}.$$

Vs.



$P_{diss} = 0.$

- Norton Equivalent Circuit.

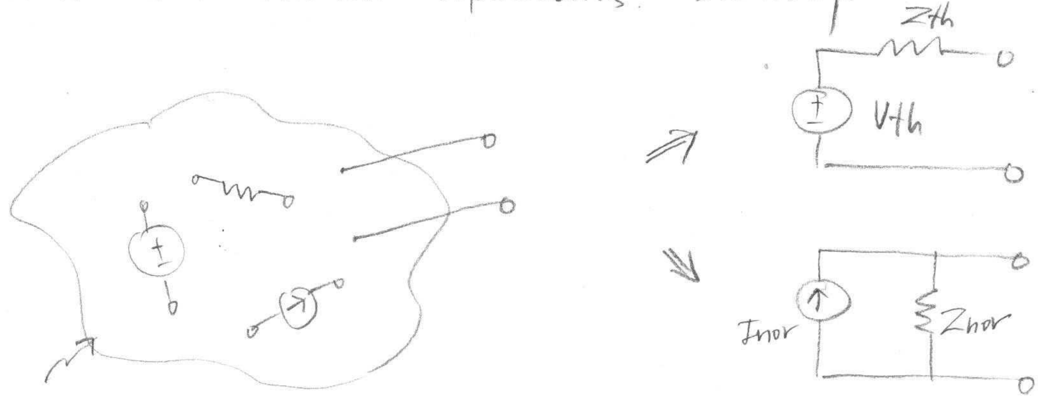


- $I_{nor} = \dfrac{V_{th}}{R_{th}}$

"Short-circuited current"

- Useful in magnetic circuits

Thevenin and Norton Equivalents Summary.



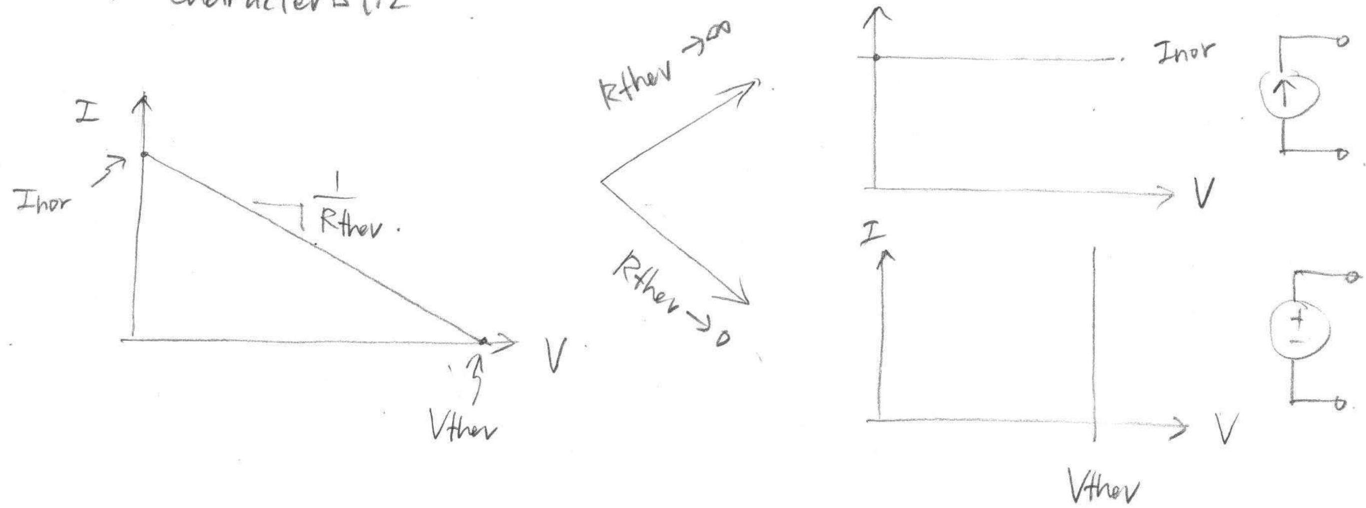
A bag of linear circuit elements.

- Two tests
  1. Open-circuit voltage : V_{thev}
  2. Short-circuit current : I_{nor}

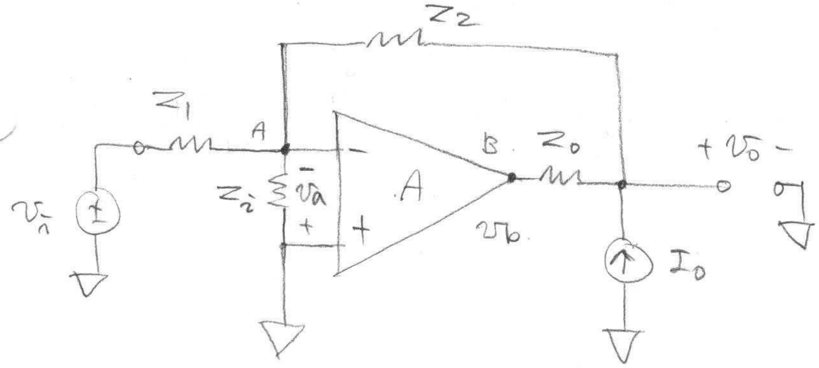
- $Z_{thev} = Z_{nor} \triangleq \dfrac{V_{thev}}{I_{nor}}$ (a.k.a source impedance or output impedance).

- Z_{thev} physical meaning : Impedance "seen" when you are "looking into" the output port

- I-V characteristic

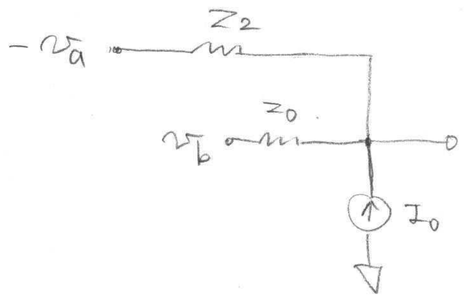


Example I: Inverting Amplifier with finite Z_i, Z_o, and A.



Z_i and Z_o are pulled out of the op-amp.

① Find v_o via superposition.

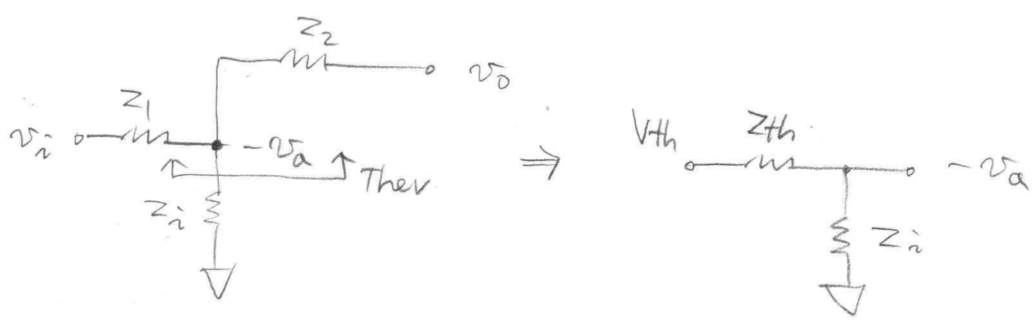


$v_b = A \cdot v_a$

$$v_o = \frac{Z_o}{Z_2+Z_o}(-v_a) + \frac{Z_2}{Z_2+Z_o} v_b + (Z_o \| Z_2) I_o$$

$$= \left(\frac{A Z_2 - Z_o}{Z_2+Z_o} \right) v_a + (Z_o \| Z_2) I_o$$

② Find v_a via Thevenin method

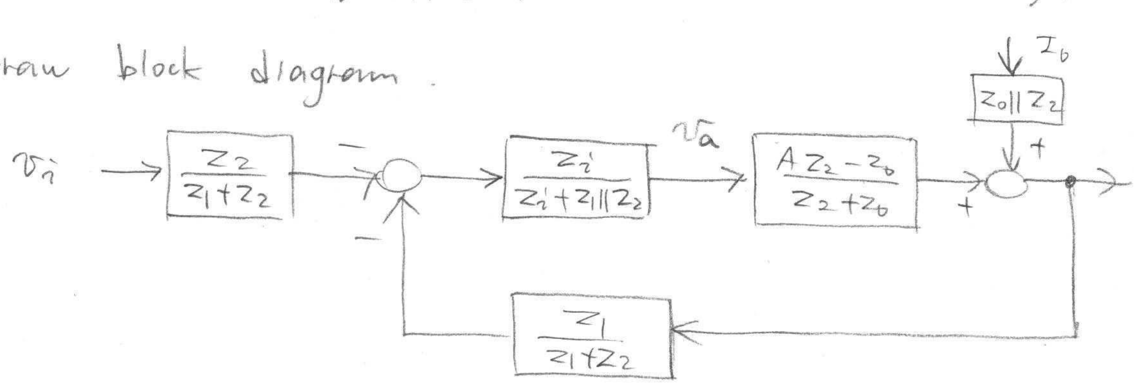


$$V_{th} = \frac{Z_2}{Z_1+Z_2} v_i + \frac{Z_1}{Z_1+Z_2} v_o$$

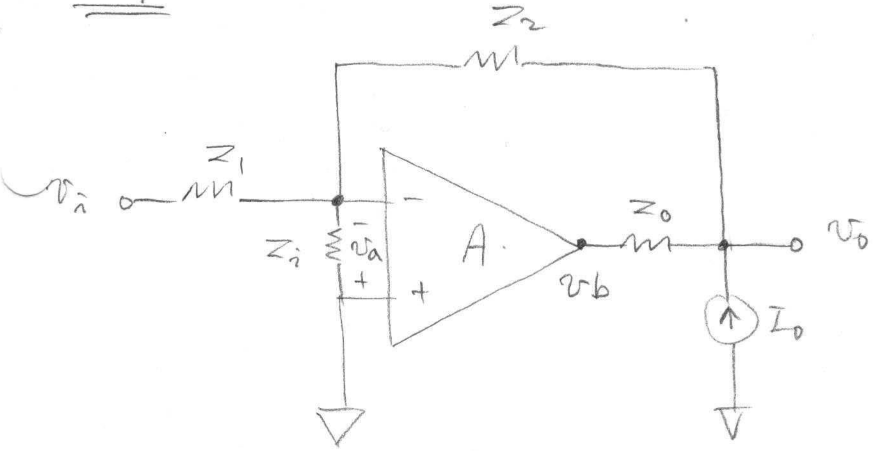
$$Z_{th} = Z_1 \| Z_2$$

$$\therefore \; -v_a = \frac{Z_i}{Z_i + Z_1\|Z_2} \left(\frac{Z_2}{Z_1+Z_2} v_i + \frac{Z_1}{Z_1+Z_2} v_o \right).$$

③ Draw block diagram.



Example I   solved with Node Method.



$$v_b = A v_a$$

① Find v_o via Node Method.

$$\frac{v_o - v_b}{Z_o} + \frac{v_o + v_a}{Z_2} = I_o$$

$$\left(\frac{1}{Z_o} + \frac{1}{Z_2}\right) v_o = \frac{A}{Z_o} v_a - \frac{1}{Z_2} v_a + I_o$$

$$\therefore v_o = \frac{Z_2 A}{Z_o + Z_2} v_a - \frac{Z_o}{Z_o + Z_2} v_a + \frac{Z_o Z_2}{Z_o + Z_2} I_o.$$

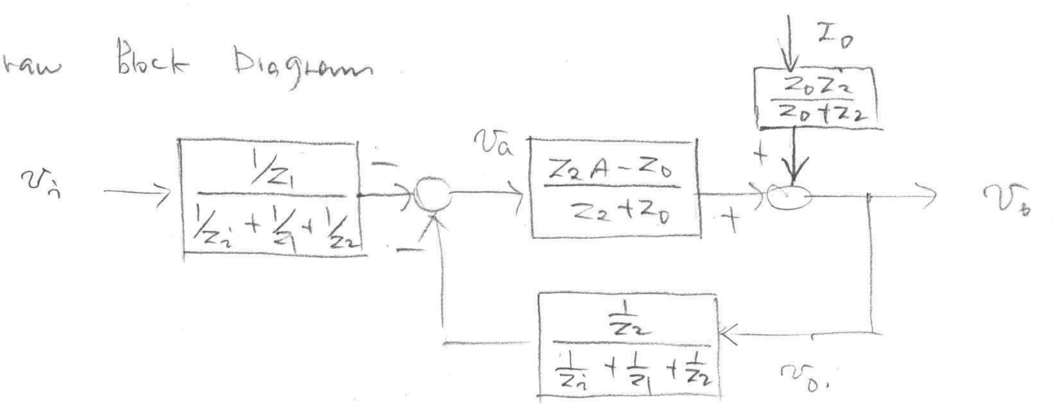
② Find v_a via Node Method.

$$\frac{v_a}{Z_i} + \frac{v_i + v_a}{Z_1} + \frac{v_o + v_a}{Z_2} = 0$$

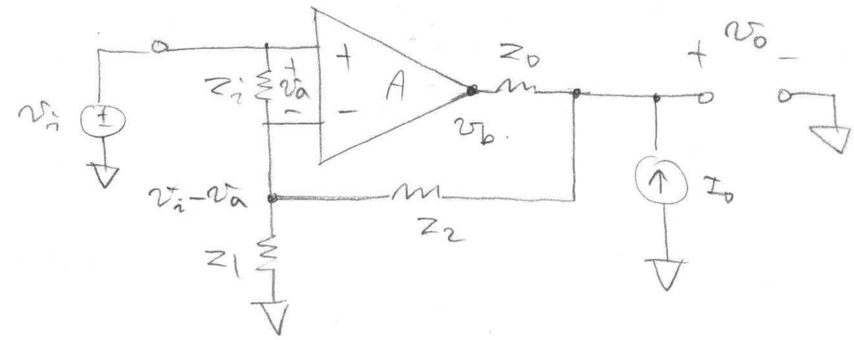
$$\therefore \left(\frac{1}{Z_i} + \frac{1}{Z_1} + \frac{1}{Z_2}\right) v_a = -\frac{1}{Z_1} v_i - \frac{1}{Z_2} v_o$$

$$\therefore v_a = \left(\frac{-1/Z_1}{1/Z_i + 1/Z_1 + 1/Z_2}\right) v_i + \left(\frac{-1/Z_2}{1/Z_i + 1/Z_1 + 1/Z_2}\right) v_o$$

③ Draw Block Diagram



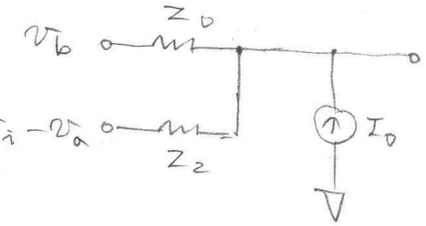
Example II: Non-inverting Amplifier with finite Z_i, Z_o, and A.



Z_i and Z_o are pulled out of the op-amp.

① Find v_o via superposition.

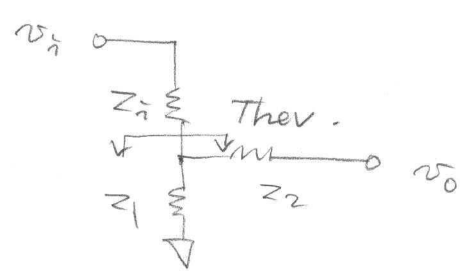
$v_b = A \cdot v_a$



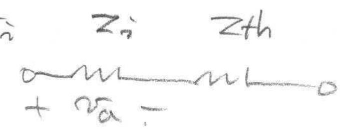
$$v_o = \frac{Z_2}{Z_2+Z_o} v_b + \frac{Z_o}{Z_2+Z_o}(v_i - v_a) + (Z_o \| Z_2) I_o$$

$$= \left(\frac{AZ_2 - Z_o}{Z_2+Z_o}\right) v_a + \frac{Z_o}{Z_2+Z_o} v_i + (Z_o \| Z_2) I_o.$$

② Find v_a via Thevenin method.



$\Rightarrow$



$$V_{th} = \frac{Z_1}{Z_1+Z_2} v_o.$$

$$Z_{th} = Z_1 \| Z_2.$$

$$\therefore v_a = \frac{Z_i}{Z_i + Z_{th}}(v_i - V_{th})$$

$$= \frac{Z_i}{Z_i + Z_1\|Z_2}\left(v_i - \frac{Z_1}{Z_1+Z_2} v_o\right)$$

③ Draw block diagram.