Brushed DC Motor

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1 Modeling

A schematic of a brushed dc motor is shown in Figure 1. Brushed dc motors have intricate internal construction, e.g., mechanical commutator, but here we will treat it as an encapsulated object that interacts with the environment only via its terminals.

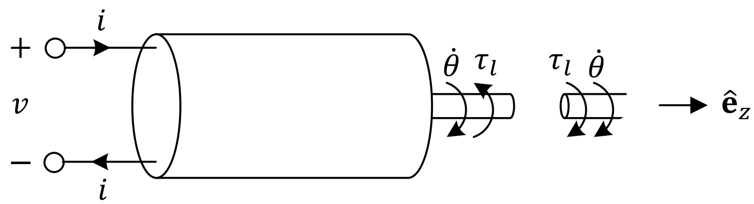


Figure 1: Brushed dc motor.

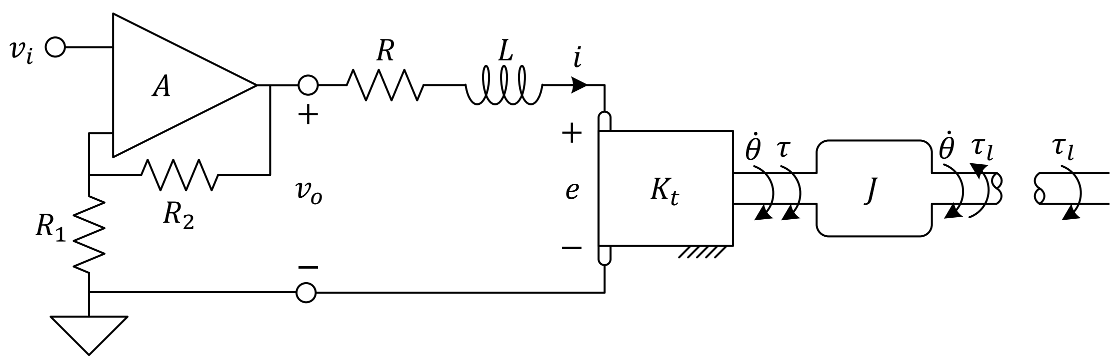


Figure 2: Brushed dc motor (exploded view) driven by a voltage amplifier.

Figure 2 shows an exploded view of brushed dc motor where the electrical and mechanical elements are conceptually pulled out of a motor. The core element is an ideal transducer labeled with an single parameter K_t.

1.1 Terminal variables

Let us define terminal variables for each port: the voltage v and current i for the electrical port, and the torque τ and rotational speed ω_r for the mechanical port. The rotor shaft in Figure 1 and Figure 2 is cut to better illustrate the the torque and reaction torque. The four terminal variables are

(1) Torque: the output torque transmitted to the mechanical load is

$$\boldsymbol{\tau}_l = \tau_l \hat{\mathbf{e}}_z$$

Note that the reaction torque $-\tau_l \hat{\mathbf{e}}_z$ is applied back to the motor.

(2) Speed: the rotor and the mechanical load share the same angular velocity

$$\boldsymbol{\omega}_r = \omega_r \hat{\mathbf{e}}_z,$$

where ω_r is called the rotor angular speed, or simply the rotor speed.

(3) Current: the current i goes into the positive terminal comes out of the negative terminal, and therefore satisfies the KCL.

(4) Voltage: the voltage v is the electrical potential difference between the two terminals.

In addition, there are two internal variables

(1) Rotor torque: the torque τ applied on the rotor by the stator via magnetic fields.

(2) Back emf: the voltage e induced on the motor winding due to time-varying flux linkage.

These internal variables are related with the terminal variables via the *torque constant*.

1.2 Torque constant

The rotor torque is proportional to the current

$$\tau = K_t\ i$$

and the back-emf is proportional to the rotor speed

$$e = K_t\ \omega_r.$$

Here, K_t is called the *torque constant*. This relationship holds for all time, i.e., $\tau(t) = K_t i(t)$.

Note that the rotor torque τ and speed ω_r, and the current i and back emf e are the terminal variables of the ideal transducer in Figure 2. The transducer behavior is governed by a pair of equations

$$\begin{aligned} \tau &= K\ i \\ e &= K\ \omega_r. \end{aligned}$$

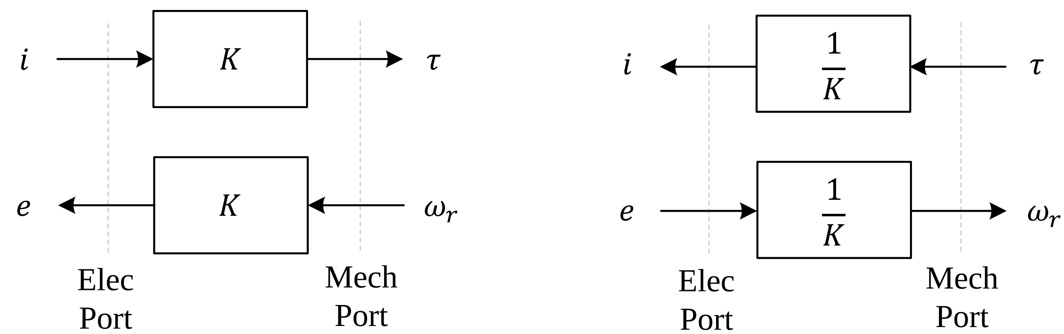
We drop the subscript 't' from the torque constant as it is identically equal to the back-emf constant due to the reciprocity principle.

Note that the ideal transducer is a lossless power converter.

$$P_{\text{in}} = ei = (K_t\dot{\theta})\left(\frac{\tau}{K_t}\right) = \tau\dot{\theta} = P_{\text{out}}$$

It can be understood as an electromechanical "gyrator" that converts the flow (current or speed) in one domain to the effort (torque or voltage) in the other domain.

The transducer relation can be represented as a pair of block diagrams.



Incorporating this to a block diagram of a brushed DC motor leads to

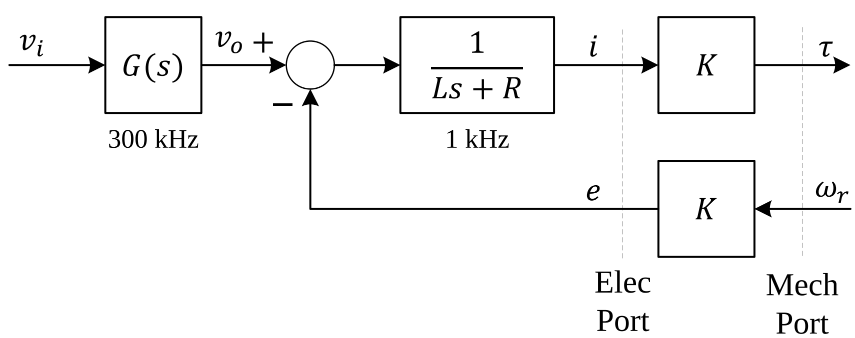


Figure 4: Brushed DC motor – block diagram.

1.3 Equivalent mechanical model

Let us find the equivalent mechanical model of a brushed dc motor via the Thevenin analysis. This helps us better understand the back-emf effect in terms of mechanical elements.

The key is to identify the Thevenin torque and impedance.

(1) Thevenin torque: rotor torque when the rotor speed is set to zero ($\omega_r = 0$).

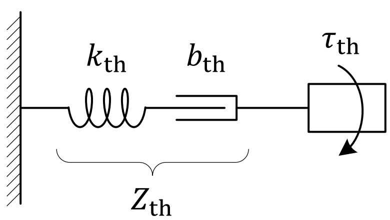
$$\tau_{\text{th}} = \frac{K}{Ls + R} v_o \qquad \text{(a.k.a. stall torque)}$$

(2) Thevenin impedance: impedance looking into the rotor shaft when all the other independent sources are turned off.

$$Z_{\text{th}} = \frac{\tau}{\omega_r} = \frac{K^2}{Ls+R} = \frac{\frac{K^2}{Ls}\frac{K^2}{R}}{\frac{K^2}{Ls}+\frac{K^2}{R}} = \frac{K^2}{Ls} \parallel \frac{K^2}{R}$$

The parallel operator $\parallel$ for mechanical elements means they are in series.

Figure 5 shows the equivalent mechanical model, where all the electrical elements are referred to the mechanical domain.



$$\tau_{\text{th}} = \frac{K}{Ls+R}\, v_o \qquad \text{Thevenin torque}$$

$$k_{\text{th}} = \frac{K^2}{L} \qquad \text{Apparent stiffness}$$

$$b_{\text{th}} = \frac{K^2}{R} \qquad \text{Apparent damping}$$

Figure 5: Equivalent mechanical model of a brushed dc motor.

The effect of back emf appears as a parasitic mechanical impedance attached between the rotor and the stator.

Apparent damping

At low frequencies, the mechanical impedance looking into the motor shaft is damping

$$\frac{K^2}{Ls+R} \approx \frac{K^2}{R} \triangleq b_{\text{th}}.$$

This can be directly read off from the block diagram in Figure 4. The apparent damping b_{th} is what you can actually feel by rotating the rotor shaft with the electrical terminals short-circuited. Note that the damping is inversely proportional to the winding resistance. The apparent damping is one of the measures of goodness of a motor, which is commonly reported in the datasheets as the *gradient* or *steepness*. A good motor has a high torque constant yet low winding resistance.

Apparent stiffness

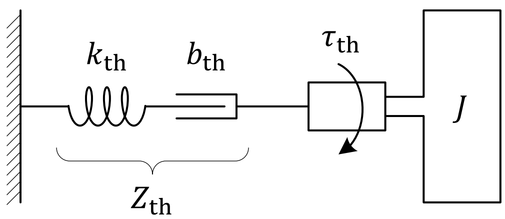
The apparent stiffness k_{th} does not have significant meaning in practice, as it is hidden behind the rotor inertia. The net mechanical impedance including the rotor inertia is

$$Z_m = Js + \frac{K^2}{Ls+R}$$

Note that the net impedance is dominated by the apparent damping in low frequencies, whereas by the rotor inertia in high frequencies (drawing the Bode plot will show it clearly).

This does not mean that the effect of inductance is ignorable in practice. The winding inductance plays an important role when analyzing the motor in electrical domain, e.g., current controller design.

Example: DC motor driving free inertia.



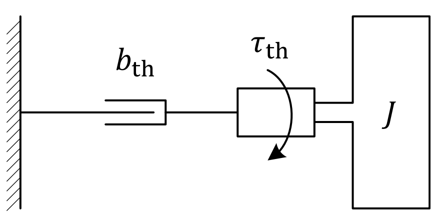
$$\tau_{\text{th}} = \frac{K}{Ls+R}\, v_o \qquad \text{Thevenin torque}$$
$$k_{\text{th}} = \frac{K^2}{L} \qquad \text{Apparent stiffness}$$
$$b_{\text{th}} = \frac{K^2}{R} \qquad \text{Apparent damping}$$

The torque transmitted to the inertia can be derived as follows (torque divider rule).

$$\begin{aligned}
\tau &= \frac{Js}{Js + Z_{\text{th}}}\, \tau_{\text{th}} \\
&= \frac{Js}{Js + \frac{K^2}{Ls+R}} \cdot \frac{K}{Ls+R}\, v_o \\
&= \frac{Js \cdot K}{Js(Ls+R) + K^2}\, v_o \\
&= \frac{K}{Ls + R + \frac{K^2}{Js}}\, v_o
\end{aligned}$$

This leads to the same result as the block diagram.

In practice, the apparent stiffness $\frac{K^2}{L}$ is hidden behind the load inertia, and we can simply the free-body diagram by ignoring it.



$$\tau_{\text{th}} = \frac{K}{Ls+R}\, v_o \qquad \text{Thevenin torque}$$
$$b_{\text{th}} = \frac{K^2}{R} \qquad \text{Apparent damping}$$

This leads to almost the same result.

$$\begin{aligned}
\tau &\approx \frac{Js}{Js + K^2/R}\, \tau_e \\
&= \frac{1}{1 + \frac{\frac{K^2}{R}}{Js}} \cdot \frac{K}{Ls+R}\, v_o
\end{aligned}$$

1.4 Equivalent electrical model

We can alternatively refer every mechanical element to electrical domain to find an equivalent circuit model. For example, if the mechanical load is just inertia J, the electrical impedance looking into the transducer becomes a capacitor (e.g., flywheel energy storage).

However, if you are comfortable with both electrical and mechanical domains, you don't need to refer one to the other. The main purpose of studying the equivalent mechanical model and equivalent electrical model is to develop our intuition. When analyzing multi-domain systems in general, I would recommend to handle each domain separately and couple them via a block diagram.

1.5 Power conversion

A brushed dc motor is a two-port system: one electrical port (a pair of electrical terminals) and one mechanical port (a shaft). It converts some of the electrical input power to the mechanical output power, or the other way around (generator).

The power input is

$$P_{\mathrm{in}} = vi$$

The power output is

$$P_{\mathrm{out}} = \tau\omega_r.$$

Substituting the voltage equation

$$v = Ri + L\frac{\mathrm{d}i}{\mathrm{d}t} + e$$

the input power is

$$\begin{aligned} P_{\mathrm{in}} &= Ri^2 + L\frac{\mathrm{d}i}{\mathrm{d}t}i + ei \\ &= \underbrace{Ri^2}_{P_{\mathrm{loss}}} + \underbrace{\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}Li^2\right)}_{W_m} + \underbrace{ei}_{P_{\mathrm{out}}}. \end{aligned}$$

Here, W_m is the energy stored in inductance. The power into the back-emf element is converted to mechanical power

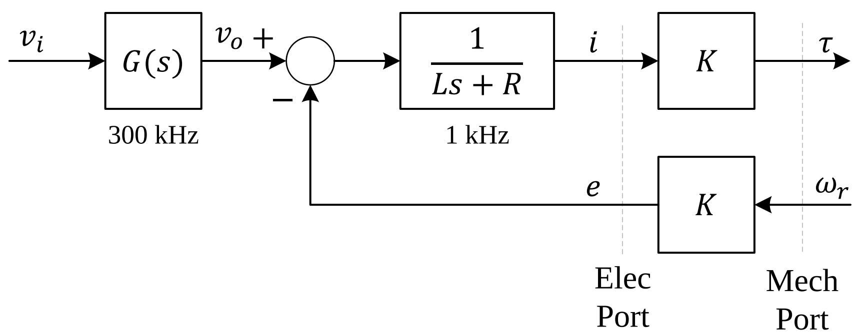
$$ei = K\omega_r\, i = \tau\omega_r.$$

2 Torque Control

For the analysis and control of mechanical systems (e.g., robots) we usually treat torques and forces as the driving variables.

Torque control is important because many of mechanical system control problems are formulated with torques being independent "driving" variables (e.g., robot manipulators).

2.1 Voltage drive



If we use a voltage source to drive a motor, the rotor torque is

$$\tau = \frac{K}{Ls+R}v_o - \frac{K^2}{Ls+R}\omega_r$$

There are two issues.

(1) Slow response to v_o

Even if we use a high-bandwidth voltage amplifier $G(s)$, e.g., 300 kHz, the output torque rise time is bottlenecked by the electrical time constant $\tau_e = L/R$.

For $L = 1\,\mathrm{mH}, R = 6\,\Omega \longrightarrow \frac{1}{\tau_e} = \frac{R}{L} = 6000\,\mathrm{rad/s} \approx 1\,\mathrm{kHz}.$

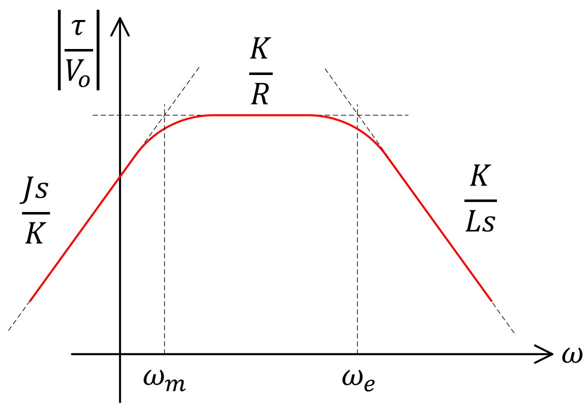
(2) Back-emf effect

The back-emf decreases the rotor torque. We can understand this in two ways. From the mechanical point of view, the back-emf effect can be understood as an apparent damping b_{th} dragging the rotor. From the electrical point of view, the back-emf effect can be understood as an external low-frequency disturbance voltage.

Example: Voltage-controlled dc motor driving free inertia.

The voltage-to-torque transfer function is

$$\frac{\tau}{v_o} = \frac{\frac{K}{Ls+R}}{1 + \frac{K^2}{Js(Ls+R)}} = \frac{K}{Ls + R + \frac{K^2}{Js}}$$
$$\approx \frac{Js}{Js + \frac{K^2}{R}} \frac{K}{Ls + R} \qquad (\text{if } \tfrac{K^2/R}{J} \ll \tfrac{R}{L})$$



The parameters of an actual motor in the lab are

$$L = 1\,\text{mH} \qquad R = 6\,\Omega \qquad K = 200\,\text{mNm/A} \qquad J = 2\,\text{kg}\cdot\text{cm}^2$$

(1) $\omega_m = \dfrac{K^2/R}{J} \approx 5\,\text{Hz}.$

$\tau_m = \frac{J}{K^2/R}$ is called the mechanical time constant.

This is the ratio between the inertia and the apparent damping.

(2) $\omega_e = \dfrac{R}{L} \approx 1\,\text{kHz}.$

$\tau_e = \frac{L}{R}$ is the electrical time constant.

2.2 Current drive

The issues of voltage-controlled dc motors can be addressed by using a current drive as in Figure 8.

The rotor torque can be directly controlled via

$$\tau = Ki = KG(s)\ u$$

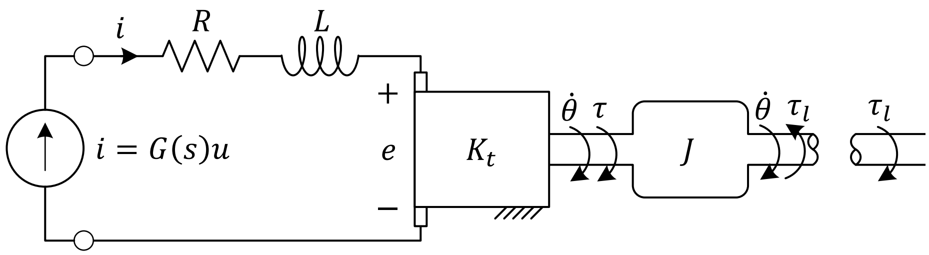


Figure 8: A dc motor driven by a current source.

This drive scheme assumes that we have a controlled current source. Such devices are available in the market with various names, such as power amplifiers, current amplifiers[1], current drives, servo amplifiers, and transconductance amplifiers, or motor drives with integrated motion-control capabilities.

They are commonly built upon a voltage amplifier with feedback current control.

[1]This is somewhat misleading as it can mean a current-controlled current source.

3 Electromagnetic Interference

Brushed dc motors have issues due to the mechanical contact-type commutator, which typically consists of a graphite brush on the stator and a circular array of electrodes on the rotor. It passively routes (commutate) the input current to different conductors wound on the rotor core (armature).

One long-term issue of the commutator is the mechanical wear of the brush. Another issue, which can even exist in a brand-new dc motor, is the electromagnetic interference. The switching action of the commutator modulates the impedance looking into the motor terminal – imagine when the brush slides over two adjacent electrodes – and causes intermittent current spikes in the motor winding. This transmits electromagnetic interference to the environment via two mechanisms.

(1) Conductive EMI: Switching current generates switching voltage at the amplifier output due to its finite output impedance.

(2) Inductive EMI: Switching current generates switching magnetic flux density **B** around the wire, which propagates through air to corrupt other circuits and electronics.

EMI reduction technique

(1) Twist the two wires. This makes the net current into the motor look like zero when observed from a distance. Therefore, this effectively reduces the inductive EMI. However, this does not reduce conductive EMI.

(2) Treat the motor as an ac noise source, and implement a differential low-pass filter from the motor to the amplifier, e.g., using ferrite beads and feed-through capacitors. Refer to Figure 5-29 in Ott [1, p. 230]. Ferrite beads are better than resistor because they have essentially zero resistance in low frequencies, i.e., ac-coupled resistors.

References

[1] H. W. Ott, *Electromagnetic Compatibility Engineering*. Wiley, 2009.