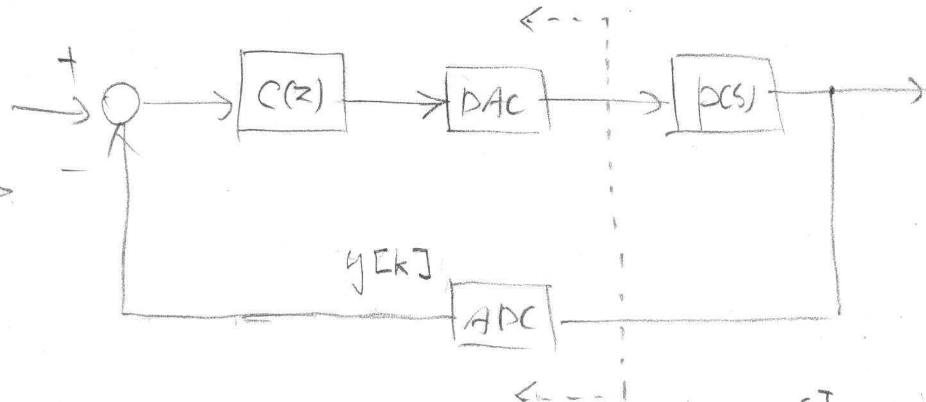
< Digital Control Design >

2020.

- Objective.
  - Understand the difference between DT approximation methods.
  - Direct DT Controller design with DT Equivalents.

<Review>



$\simeq C(s)e^{-s\frac{T}{2}}$

"Indirect Design"

Design $C(s)$ for $P(s)e^{-s\frac{T}{2}}$ and map $C(s) \rightarrow C(z)$

- Discrete-time Approximation Methods

① Forward Rectangular Method (Euler Method)

$$\frac{1}{s} = T\frac{z^{-1}}{1-z^{-1}} \rightarrow s = \frac{z-1}{T} \rightarrow z = 1+T\cdot s$$

"Substitution Rule" "Mapping Rule"

② Backward Rectangular Method

$$\frac{1}{s} = T\frac{1}{1-z^{-1}} \rightarrow s = \frac{z-1}{T\cdot z} \rightarrow z = \frac{1}{1-T\cdot s}$$

③ Tustin method.

$$\frac{1}{s} = \frac{T}{2}\frac{1+z^{-1}}{1-z^{-1}} \rightarrow s = \frac{2}{T}\frac{z-1}{z+1} \rightarrow z = \frac{1+2T\cdot s}{1-2T\cdot s}$$

- These are essentially mapping rules that map points in the s-plain to points in the z-plain.

- Once $C(z)$ is obtained via approximate mapping : $C(s) \to C(z)$

i) Directly implement it using, for example, Simulink or LabVIEW.

ii) Convert it into the difference equation and implement it using text-base programing language.

Example : DT Approximation of CT Lead Compensator.

$$C(s) = \frac{10s+1}{s+1}, \quad T = 0.1$$

- Find the approximate DT system using the backward rect.

Substitute $s = \frac{z-1}{Tz} = \frac{10(z-1)}{z}$

$$C(z) = \frac{\frac{100z-100}{z}+1}{\frac{10z-10}{z}+1} = \frac{101z-100}{11z-10}$$

- Find the difference equation between $e[k]$ and $u[k]$, where

$$C(z) = \frac{U(z)}{E(z)}, \quad E(z) = \mathcal{Z}\{e[k]\}, \quad U(z) = \mathcal{Z}\{u[k]\}$$

$$C(z) = \frac{101-100z^{-1}}{11-10z^{-1}} = \frac{U(z)}{E(z)}$$

$$\to E(z)(101-100z^{-1}) = U(z)(11-10z^{-1})$$

$$\to 101\,e[k] - 100\,e[k-1] = 11\,u[k] - 10\,u[k-1]$$

$$\to u[k] = \frac{10}{11}u[k-1] + \frac{101}{11}e[k] - \frac{100}{11}e[k-1]$$

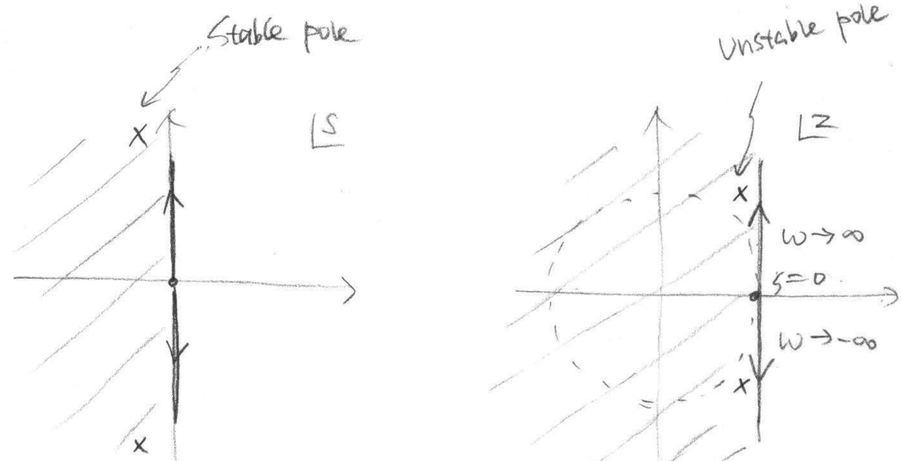
- Effect on Stability

  - Each method maps the left half plane (LHP) of the s-plane to a different region in the z-plane.
  - This affects the stability of the approximate DT system.

① Forward Rect. Method

$z = 1 + Ts$

$\begin{cases} s=0 \rightarrow z=1 \\ s=j\omega \rightarrow z = 1 + jT\omega \end{cases}$

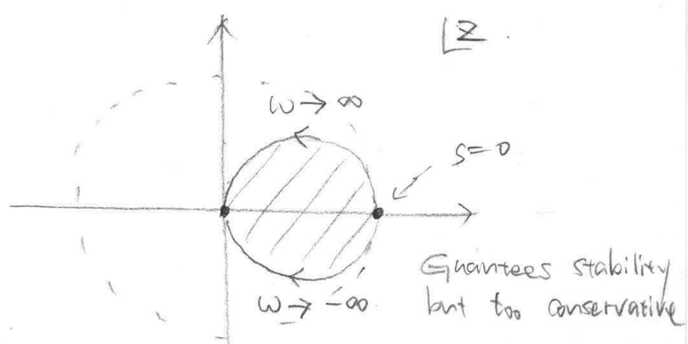


- The LHP of the s-plane is scaled by T and shifted by +1.
- Stable H(s) can turn into unstable H(z)

② Backward Rect. Method

$z = \dfrac{1}{1-Ts}$ $\begin{cases} s=0 \rightarrow z=1 \\ s=j\omega \rightarrow z = \dfrac{1}{1-jT\omega} \end{cases}$

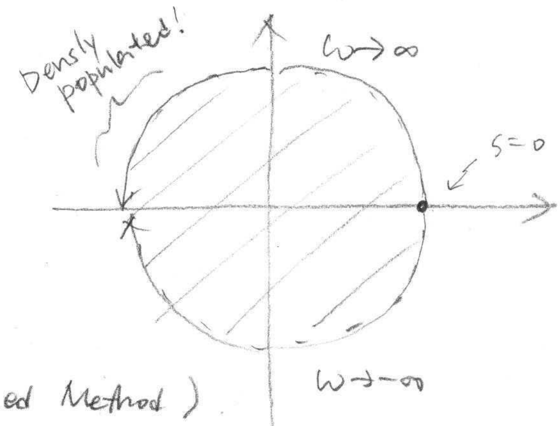
As $\omega \rightarrow \infty$, $z = \dfrac{1}{-jT\omega}$ $\begin{cases} |z| = 0 \\ \measuredangle z = \frac{\pi}{2} \end{cases}$



③ Tustin Method

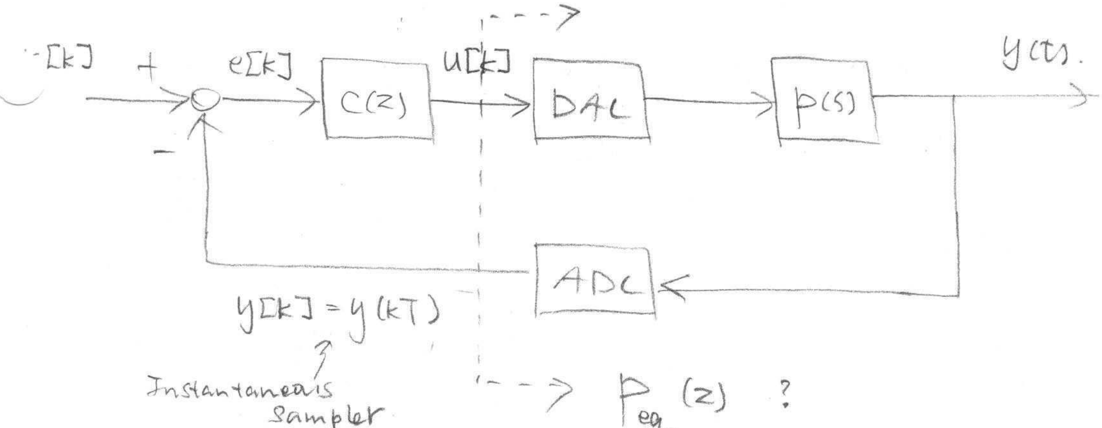
$z = \dfrac{1+\frac{T}{2}s}{1-\frac{T}{2}s}$ $\begin{cases} s=0 \rightarrow z=1 \\ s=j\omega \rightarrow z = \dfrac{1+j\frac{T}{2}\omega}{1-j\frac{T}{2}\omega} \end{cases}$

As $\omega \rightarrow \infty$, $z = \dfrac{j\frac{T}{2}\omega}{-j\frac{T}{2}\omega}$ $\begin{cases} |z| = 1 \\ \measuredangle z = \pi \end{cases}$



- Stability guanteed and exact. (Recommeded Method)
- But high-frequency distortion

○ Discrete-time Equivalents & Direct DT Design.

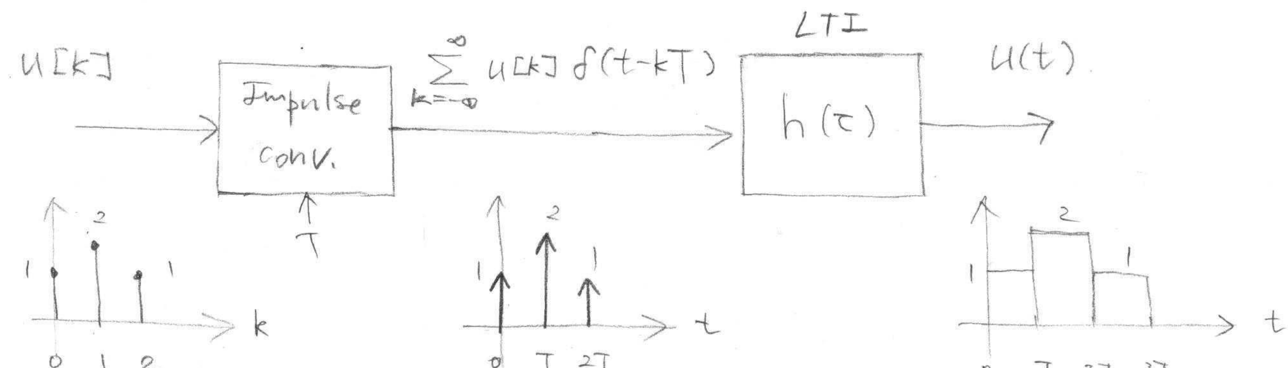


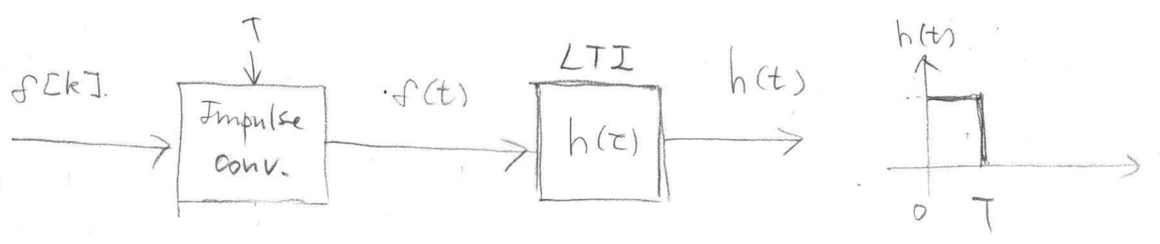
· Seen from the DT system through DAC & ADC, the CT system looks like a DT system that takes u[k] and outputs y[k].

· Let us define $P_{eq}(z) = \frac{Y(z)}{U(z)}$ as the DT equivalent of p(s)

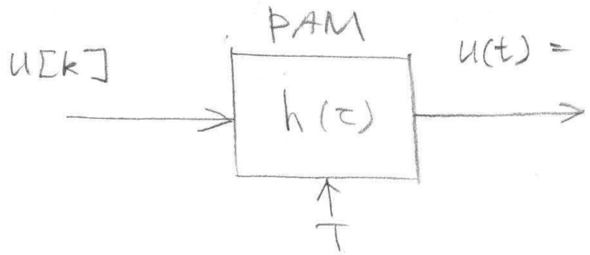
· For the same p(s), there can exist different $P_{eq}(z)$ depending on the characteristics of DAC, (e.g. Hold types)

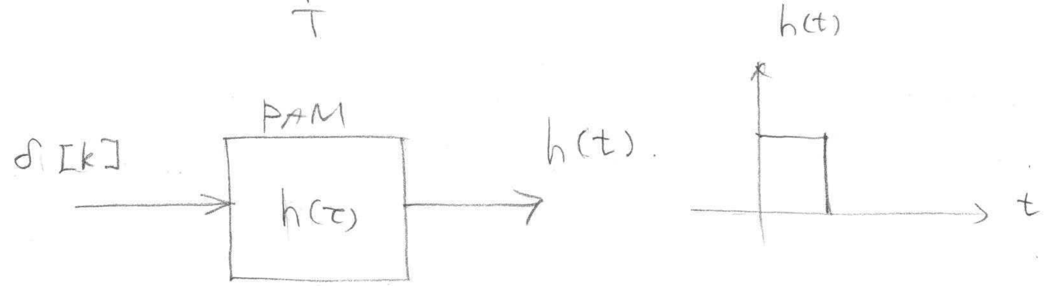
· DAC process can be mathematically modeled as a two-step process. : Impulse converter + LTI filter.





- Alternatively, it can be modeled as a single-step process called pulse amplitude modulation (PAM).



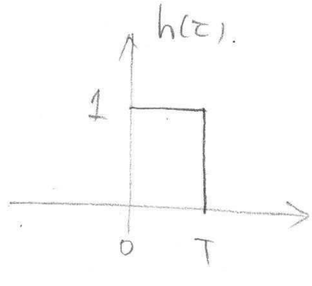


- The shape of $h(\tau)$ characterizes a DAC, which leads to different $P_{eq}(z)$. (Note that $\int_{-\infty}^{\infty} h(\tau)\, d\tau = T$ for all cases.)

① $h(\tau) = T\delta(\tau)$ $\quad$ $P_{eq}(z) = T \cdot \mathcal{Z}\{P(s)\}$

- Impulse-Invariant Equivalent

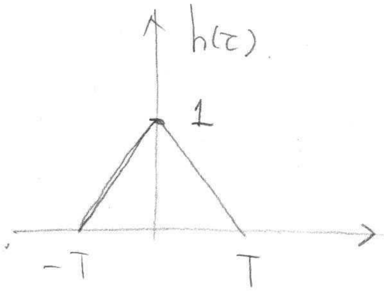
②



$$P_{eq}(z) = (1-z^{-1})\, \mathcal{Z}\left\{\frac{P(s)}{s}\right\}$$

- ZOH Equivalent
(Step Invariant)

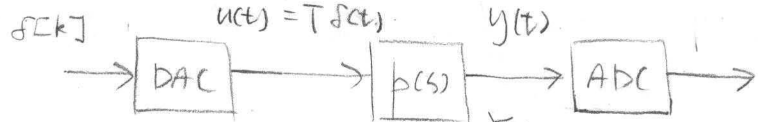
③



$$P_{eq}(z) = \frac{(z-1)^2}{Tz}\, \mathcal{Z}\left\{\frac{P(s)}{s^2}\right\}$$

- FOH Equivalent
(Ramp Invariant)

- Proofs.
  - Impulse-invariant Equivalent



$Y(s) = T \cdot p(s)$

$$\therefore P_{eq}(z) = \mathcal{Z}\{y(kT)\} = T\,\mathcal{Z}\{p(s)\} = T\,\mathcal{Z}\{\underbrace{\mathcal{L}^{-1}\{p(s)\}\big|_{t=kT}}\}$$

$\triangleq \mathcal{Z}\{p(s)\}$ "short-hand" notation.

  - ZOH Equivalent

unit step, shifted unit step

$u(t) = 1(t) - 1(t-T)$

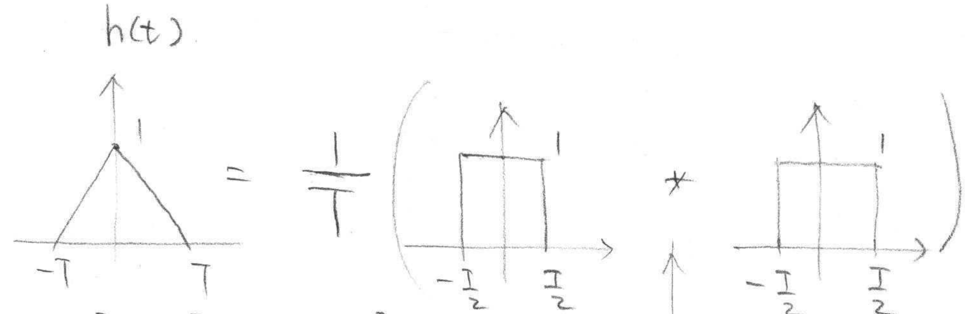
$$Y(s) = \frac{p(s)}{s} - \frac{p(s)}{s}e^{-sT} = \frac{p(s)}{s}(1-e^{-sT})$$

$$\therefore P_{eq}(z) = \mathcal{Z}\{y(kT)\} = \mathcal{Z}\left\{(1-e^{-sT})\frac{p(s)}{s}\right\} = (1-z^{-1})\,\mathcal{Z}\left\{\frac{p(s)}{s}\right\}$$

step response of $p(s)$

  - FOH Equivalent

Recognizing that



$$\therefore H(s) = \frac{1}{T}\left(\frac{e^{s\frac{T}{2}} - e^{-s\frac{T}{2}}}{s}\right) \times \left(\frac{e^{s\frac{T}{2}} - e^{-s\frac{T}{2}}}{s}\right)$$

Convolution $\overset{\mathcal{L}}{\longleftrightarrow}$ Multiplication

$$= \frac{e^{sT} - 2 + e^{-sT}}{Ts^2}$$

$$\therefore Y(s) = \frac{e^{sT} - 2 + e^{-sT}}{T}\,\frac{p(s)}{s^2}$$

Ramp response of $p(s)$.

$$\therefore P_{eq}(z) = \mathcal{Z}\left\{\frac{e^{sT}-2+e^{-sT}}{T}\,\frac{p(s)}{s^2}\right\} = \frac{z-2+z^{-1}}{T}\,\mathcal{Z}\left\{\frac{p(s)}{s^2}\right\} = \frac{(z-1)^2}{T\cdot z}\,\mathcal{Z}\left\{\frac{p(s)}{s^2}\right\}$$