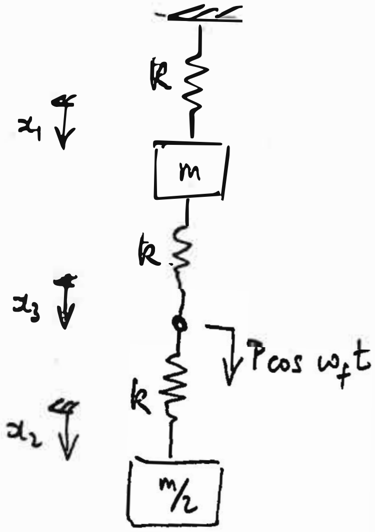
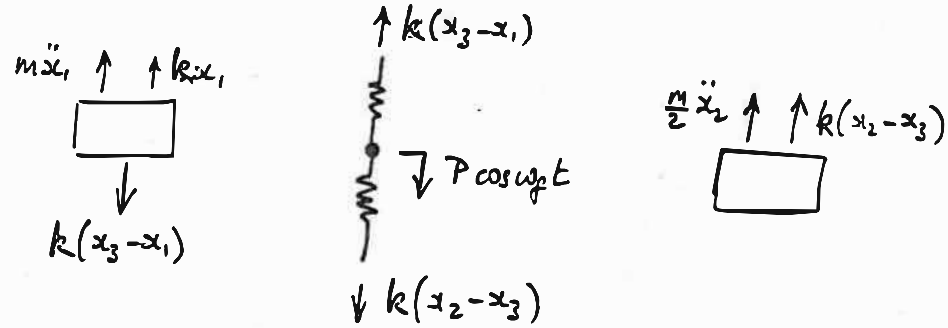
MECH463 -- Tutorial 7





Free body diagrams

Vertical force balances:

$$m\ddot{x}_1 + kx_1 - k(x_3 - x_1) = 0$$

$$(x_3 - x_1) - P\cos\omega_f t - k(x_2 - x_3) = 0$$

$$\frac{m}{2}\ddot{x}_2 + k(x_2 - x_3) = 0$$

From 2nd equation $\rightarrow$ $x_3 = \dfrac{x_1 + x_2}{2} + \dfrac{P\cos\omega_f t}{2k}$

Substitute into 1st + 3rd equations $\rightarrow$

$$m\ddot{x}_1 + \tfrac{3}{2}kx_1 - \tfrac{1}{2}kx_2 = \tfrac{P}{2}\cos\omega_f t$$

$$\tfrac{m}{2}\ddot{x}_2 - \tfrac{1}{2}kx_1 + \tfrac{1}{2}kx_2 = \tfrac{P}{2}\cos\omega_f t$$

In matrix form $\rightarrow$

$$\begin{bmatrix} m & 0 \\ 0 & \frac{m}{2} \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} \frac{3}{2}k & -\frac{1}{2}k \\ -\frac{1}{2}k & \frac{1}{2}k \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} P/2 \\ P/2 \end{bmatrix}\cos\omega_f t$$

Try a harmonic solution $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}\cos\omega_f t$

$$\rightarrow \begin{bmatrix} \frac{3}{2}k - m\omega_f^2 & -\frac{1}{2}k \\ -\frac{1}{2}k & \frac{1}{2}k - \frac{m}{2}\omega_f^2 \end{bmatrix}\begin{bmatrix} X_1 \\ X_2 \end{bmatrix}\cos\omega_f t = \begin{bmatrix} P/2 \\ P/2 \end{bmatrix}\cos\omega_f t$$

Solution must be true for all t $\rightarrow$ $\cos\omega_f t \neq 0$

By Cramer's rule $\quad X_1 = \dfrac{-P/2\left(\frac{m\omega_f^2}{2}-k\right)}{\Delta}, \qquad X_2 = \dfrac{-P\left(\frac{m}{2}\omega_f^2-k\right)}{\Delta}$

where $\Delta = \left(\tfrac{3}{2}k - m\omega_f^2\right)\left(\tfrac{1}{2}k - \tfrac{m}{2}\omega_f^2\right) - \tfrac{1}{4}s^2$

$$= \frac{m^2}{2}\omega_f^4 - \frac{5}{4}mk\omega_f^2 + \tfrac{1}{2}k^2$$

$$= \left(\frac{m}{2}\omega_f^2 - k\right)\left(m\omega_f^2 - k/2\right)$$

$$\rightarrow X_1 = \frac{-P/2\,\cancel{\left(\frac{m}{2}\omega_f^2-k\right)}}{\cancel{\left(\frac{m}{2}\omega_f^2-k\right)}\left(m\omega_f^2-k/2\right)} \qquad X_2 = \frac{-P\,\cancel{\left(\frac{m}{2}\omega_f^2-k\right)}}{\cancel{\left(\frac{m}{2}\omega_f^2-k\right)}\left(m\omega_f^2-k/2\right)}$$

$$\rightarrow \boxed{x_1 = \tfrac{1}{2}x_2 = \frac{P\cos\omega_f t}{k-2m\omega_f^2} \qquad \text{when } \left(\frac{m}{2}\omega_f^2-k\right)\neq 0 \;\rightarrow \omega_f^2 \neq \frac{2k}{m}}$$

For natural frequency calculation, consider free vibrations
$\rightarrow$ set right hand side $=0$

$$\rightarrow \begin{bmatrix} m & 0 \\ 0 & m/2 \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} 3/2\,k & -\tfrac{1}{2}k \\ -\tfrac{1}{2}k & \tfrac{1}{2}k \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Try a harmonic solution $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}\cos\omega t$

$$\Rightarrow \begin{bmatrix} 3/2\,k - m\omega^2 & -\tfrac{1}{2}k \\ -\tfrac{1}{2}k & \tfrac{1}{2}k - \tfrac{m}{2}\omega \end{bmatrix}\begin{bmatrix} X_1 \\ X_2 \end{bmatrix}\cos\omega t = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

True for all t $\rightarrow \cos\omega t \neq 0$

For non-trivial solution $\quad \begin{vmatrix} 3/2\,k - m\omega^2 & -\tfrac{1}{2}k \\ -\tfrac{1}{2}k & \tfrac{1}{2}k - \tfrac{m}{2}\omega \end{vmatrix} = 0$

$$\rightarrow \left(\frac{m}{2}\omega^2 - k\right)\left(m\omega^2 - k/2\right) = 0 \qquad \text{(from } \Delta\text{)}$$

$$\boxed{\rightarrow \omega^2 \quad \frac{k}{2m} \quad \text{and} \quad \frac{2k}{m}}$$

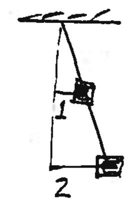
Let the mode shape be $\begin{bmatrix} 1 \\ u \end{bmatrix}$

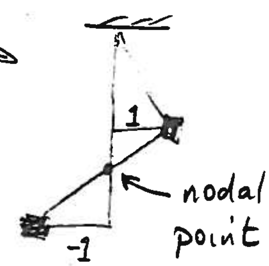
$$\rightarrow \begin{bmatrix} \frac{3}{2}k - m\omega^2 & -\frac{1}{2}k \\ -\frac{1}{2}k & \frac{1}{2}k - \frac{m}{2}\omega^2 \end{bmatrix} \begin{bmatrix} 1 \\ u \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

From first equation $\rightarrow$ $\frac{3}{2}k - m\omega^2 - \frac{1}{2}ku = 0$

$$\rightarrow u = 3 - \frac{2m}{k}\omega^2$$

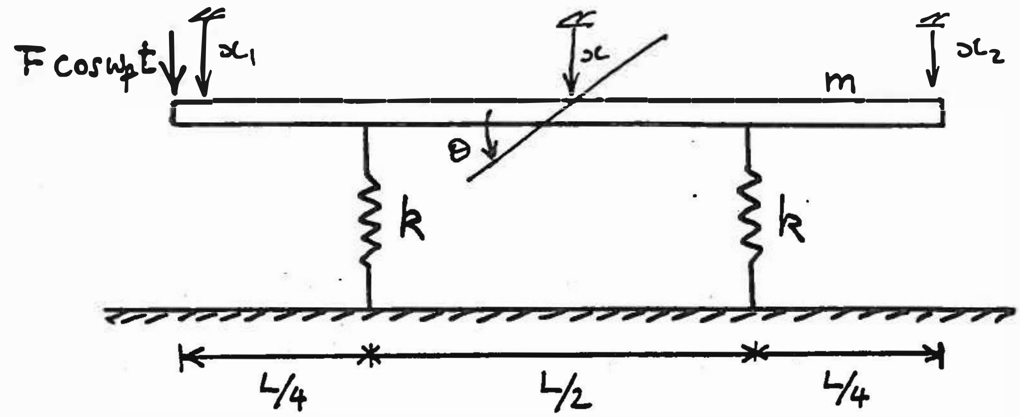
$$\boxed{\begin{array}{ll} \text{For } \omega^2 = \dfrac{k}{2m} & \rightarrow \begin{bmatrix} 1 \\ u \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \\ \text{For } \omega^2 = \dfrac{2k}{m} & \rightarrow \begin{bmatrix} 1 \\ u \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \end{array}}$$





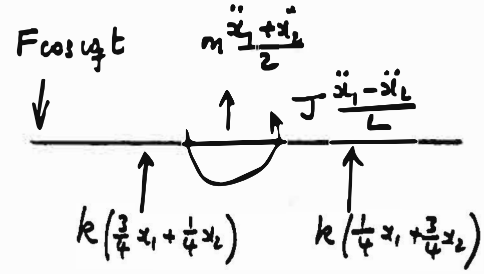
The second mode has a nodal point at the junction of the two lower springs. Thus, a harmonic force applied at this point cannot excite this mode. The force can only excite the first mode because this mode does not have a nodal point where the force is applied.

2. A very idealized model of an automobile consists of a uniform slender rod of mass m and length L. The rod is supported at its quarter points by two springs, each of stiffness k. A force $f(t) = F \cos \omega_f t$ is applied at one end of the rod. Derive an expression for the vibrational displacement at that point. (Hint: The centroidal moment of inertia of a slender rod is $J = mL^2/12$)



There are many possible choices of coordinates that are reasonable here. Of course, any choice will work, but it may be more or less convenient.

Coordinate choice 1: – ends of rod.



This choice gives the required displacement directly as X_1. However it involves both dynamic and static coupling, as well as awkward expressions for the inertia and spring force.

$$\Sigma F = 0 \quad \rightarrow \quad m\frac{\ddot{x}_1 + \ddot{x}_2}{2} + k\left(\tfrac{3}{4}x_1 + \tfrac{1}{4}x_2\right) + k\left(\tfrac{1}{4}x_1 + \tfrac{3}{4}x_2\right) - F\cos\omega_f t = 0$$

$$\Sigma M = 0 \quad \rightarrow \quad J\frac{\ddot{x}_2 - \ddot{x}_1}{L} + k\frac{L}{4}\left(\tfrac{1}{4}x_1 + \tfrac{3}{4}x_2 - \tfrac{3}{4}x_1 - \tfrac{1}{4}x_2\right) + F\cos\omega_f t \cdot \frac{L}{2} = 0$$

$$\rightarrow \begin{bmatrix} \frac{m}{2} & \frac{m}{2} \\ -\frac{J}{L} & \frac{J}{L} \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} k & k \\ -\frac{kL}{8} & \frac{kL}{8} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} F\cos\omega_f t \\ -F\cos\omega_f t \cdot \frac{L}{2} \end{bmatrix}$$

Put $J = \frac{1}{12}mL^2$ and divide second equation by $\frac{-L}{2}$

$$\rightarrow \begin{bmatrix} \frac{m}{2} & \frac{m}{2} \\ \frac{m}{6} & -\frac{m}{6} \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} k & k \\ k/4 & -k/4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} F \\ F \end{bmatrix}\cos\omega_f t$$

Try solution $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \cos \omega_f t$ for the steady state solution (particular integral)

$$\rightarrow \begin{bmatrix} k - \frac{m}{2}\omega_f^2 & k - \frac{m}{2}\omega_f^2 \\ \frac{k}{4} - \frac{m}{6}\omega_f^2 & -\frac{k}{4} + \frac{m}{6}\omega_f^2 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} F \\ F \end{bmatrix}$$

Solving using Cramer's rule

$$X_1 = \frac{\begin{vmatrix} F & k - \frac{m}{2}\omega_f^2 \\ F & -\frac{k}{4} + \frac{m}{6}\omega_f^2 \end{vmatrix}}{\begin{vmatrix} k - \frac{m}{2}\omega_f^2 & k - \frac{m}{2}\omega_f^2 \\ \frac{k}{4} - \frac{m}{6}\omega_f^2 & -\frac{k}{4} + \frac{m}{6}\omega_f^2 \end{vmatrix}}$$

$$= \frac{F\left(-\frac{k}{4} + \frac{m}{6}\omega_f^2\right) - F\left(k - \frac{m}{2}\omega_f^2\right)}{\left(k - \frac{m}{2}\omega_f^2\right)\left(-\frac{k}{4} + \frac{m}{6}\omega_f^2\right) - \left(\frac{k}{4} - \frac{m}{6}\omega_f^2\right)\left(k - \frac{m}{2}\omega_f^2\right)}$$

$$\boxed{X_1 = \frac{F\left(-\frac{5}{4}k + \frac{2}{3}m\omega_f^2\right)}{\left(k - \frac{m}{2}\omega_f^2\right)\left(-\frac{k}{4} + \frac{m}{6}\omega_f^2\right)}}$$

$\leftarrow$ vibration amplitude at force application point.

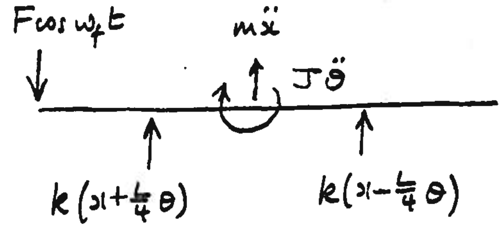
Static displacement $X_{01} = X_1$ when $\omega_f = 0$

$$\rightarrow X_{01} = 5F/k$$

$$\rightarrow \boxed{X_1 = \frac{X_{01}\left(k - \frac{8}{15}\frac{m}{k}\omega_f^2\right)}{\left(1 - \frac{1}{2}\frac{m}{k}\omega_f^2\right)\left(k - \frac{2}{3}\frac{m}{k}\omega_f^2\right)}}$$

We can see from the denominator that the two natural frequencies are $\omega_1^2 = \frac{3k}{2m}$ and $\omega_2^2 = \frac{2k}{m}$

Coordinate choice 2: – displacement and rotation at centre of mass



This choice gives no dynamic coupling. For this symmetrical system, we also get no static coupling. x and θ are the principal coordinates.

$$\Sigma F=0 \rightarrow m\ddot{x} + k(x+\tfrac{L}{4}\theta) + k(x-\tfrac{L}{4}\theta) - F\cos\omega_f t = 0$$

$$\Sigma M=0 \rightarrow J\ddot{\theta} + k\tfrac{L}{4}\left(x+\tfrac{L}{4}\theta - x + \tfrac{L}{4}\theta\right) - F\cos\omega_f t\cdot\tfrac{L}{2} = 0$$

$$\rightarrow \begin{bmatrix} m & 0 \\ 0 & J \end{bmatrix}\begin{bmatrix} \ddot{x} \\ \ddot{\theta} \end{bmatrix} + \begin{bmatrix} 2k & 0 \\ 0 & k\frac{L^2}{8} \end{bmatrix}\begin{bmatrix} x \\ \theta \end{bmatrix} = \begin{bmatrix} F\cos\omega_f t \\ F\cos\omega_f t\cdot\frac{L}{2} \end{bmatrix}$$

Put $J=\frac{1}{12}mL^2$ and divide second equation by $\frac{L}{2}$

$$\rightarrow \begin{bmatrix} m & 0 \\ 0 & \frac{1}{6}mL \end{bmatrix}\begin{bmatrix} \ddot{x} \\ \ddot{\theta} \end{bmatrix} + \begin{bmatrix} 2k & 0 \\ 0 & \frac{kL}{4} \end{bmatrix}\begin{bmatrix} x \\ \theta \end{bmatrix} = \begin{bmatrix} F \\ F \end{bmatrix}\cos\omega_f t$$

For convenience, define $y=\theta\cdot\frac{L}{2}$ = displacement of L.H. end relative to centre of mass

$$\rightarrow \begin{bmatrix} m & 0 \\ 0 & \frac{1}{3}m \end{bmatrix}\begin{bmatrix} \ddot{x} \\ \ddot{y} \end{bmatrix} + \begin{bmatrix} 2k & 0 \\ 0 & \frac{1}{2}k \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} F \\ F \end{bmatrix}\cos\omega_f t$$

Try solution $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} X \\ Y \end{bmatrix}\cos\omega_f t$ for the steady state solution.

$$\rightarrow \begin{bmatrix} 2k-m\omega_f^2 & 0 \\ 0 & \frac{1}{2}k-\frac{1}{3}m\omega_f^2 \end{bmatrix}\begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} F \\ F \end{bmatrix} \rightarrow X = \frac{F}{2k-m\omega_f^2} \rightarrow Y = \frac{F}{\frac{1}{2}k-\frac{1}{3}m\omega_f^2}$$

Displacement amplitude at LH end $= X+Y$

$$X+Y = \frac{F(\frac{1}{2}k-\frac{1}{3}m\omega_f^2) + F(2k-m\omega_f^2)}{(2k-m\omega_f^2)(\frac{1}{2}k-\frac{1}{3}m\omega_f^2)} = \boxed{\frac{F(\frac{5}{2}k-\frac{4}{3}m\omega_f^2)}{(2k-m\omega_f^2)(\frac{1}{2}k-\frac{1}{3}m\omega_f^2)} = X+Y}$$

$\rightarrow$ same as before.