MECH 420
Sensors and Actuators

Presentation Part 11

Clarence W. de Silva, Ph.D., P.Eng.
Professor of Mechanical Engineering
The University of British Columbia
e-mail: desilva@mech.ubc.ca

Part 11: Data Analysis Considerations

- Data Analysis
- Parameter Estimation
- Least Squares Estimation (LSE)

PLAN

- The role of estimation in sensing
- Concepts of model error and measurement error
- Handling of randomness in error (mean, variance or covariance)
- Least squares point estimation
- Least squares line estimation (regression line)
- Parameters for representing the quality of an estimate

Sensing and Estimation

Measured Quantity

Categories:

1. **Constant parameter (e.g., moment of inertia of a robotic arm link)**
2. **Average property of a batch of items (e.g., average internal diameter and its variance of a batch of ball bearings)**
3. **Varying parameter (e.g., strain gauge resistance as the temperature changes)**
4. **Variable of a dynamic system (e.g., velocity of a vehicle; torque of a turbine)**

Need for Estimation

Reasons:

- Measured quantity is not the required quantity; has to be computed from the measured value/s using a suitable "model" **Examples?**
- The measured value itself has errors (e.g., noise, environmental effects, process disturbances) can be called "model errors"
- Sensor or sensing process is not perfect; will introduce "measurement error"

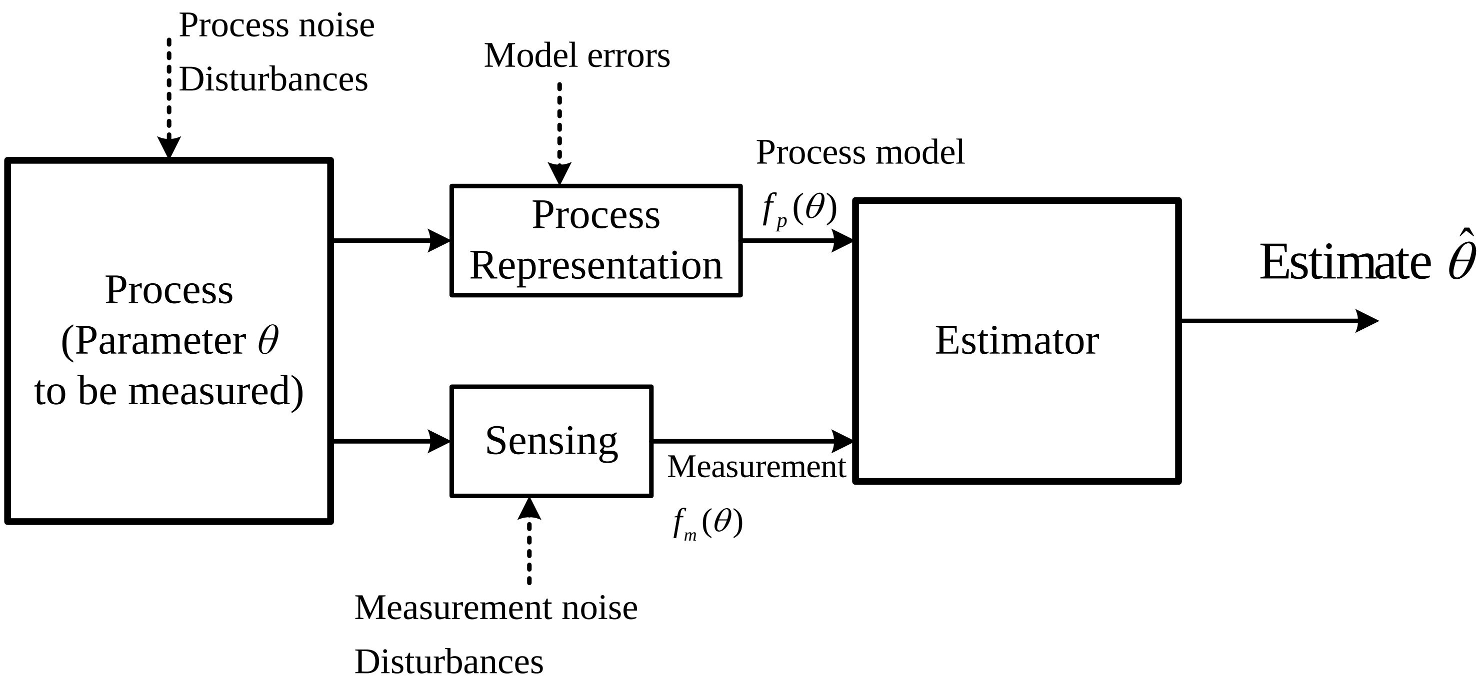
➔ **Required quantity or "true value" of measured quantity is "estimated" using the measured data**

Main Categories of Error: What category is loading effects of sensor?

1. Model error (e.g., from: product manufacturing process and mounting, analytical model, nonlinearities, unwanted inputs/disturbances into system, process noise)
2. Measurement error (e.g., from: sensor and its setup, data acquisition, measurement process, measurement noise)

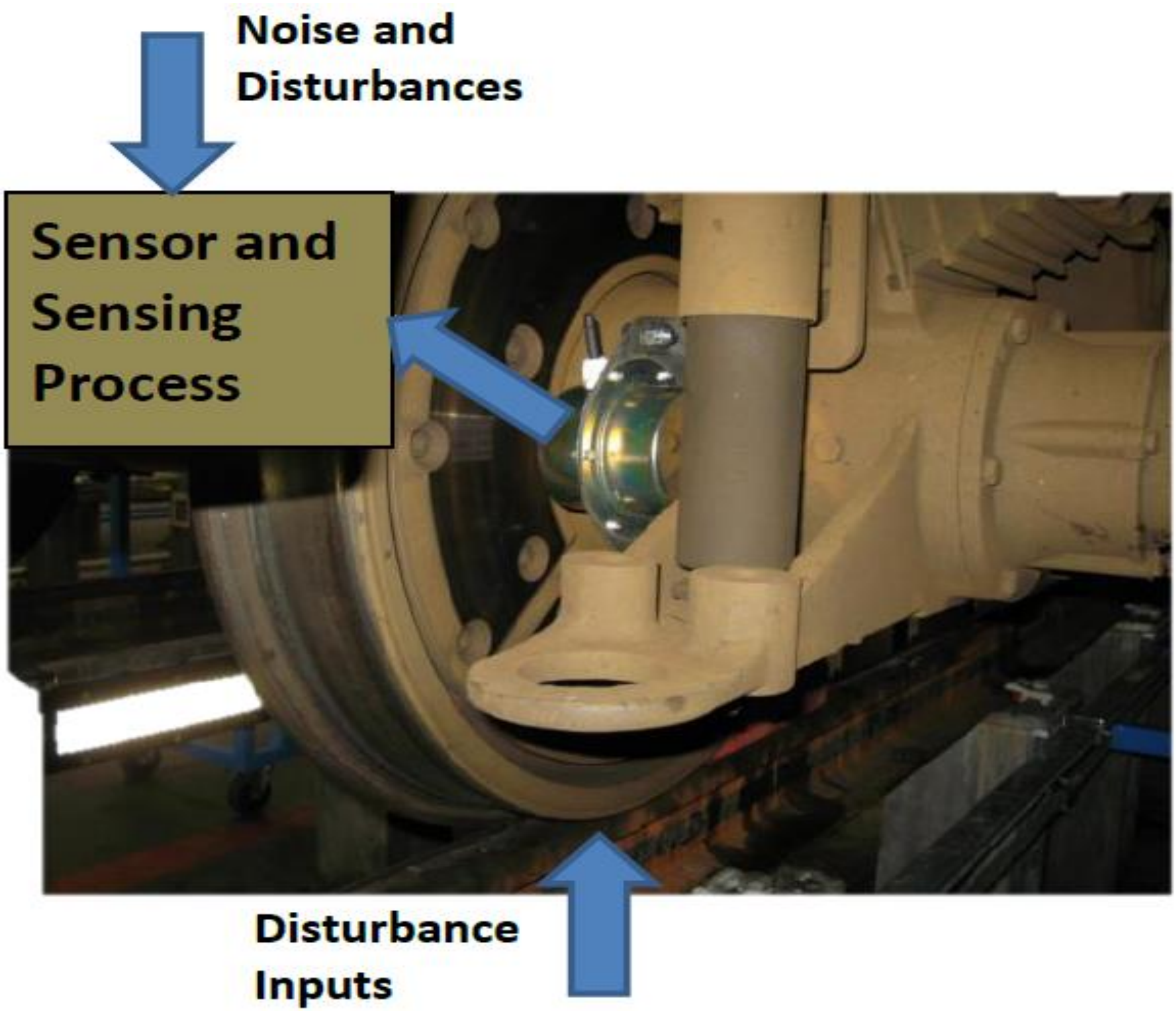
All these will affect the accuracy of the estimated result.

Model Error and Measurement Error in Estimation



Model Error and Measurement Error

Train wheel monitoring:



Other examples?

Methods of Estimation

Terminology of Estimation

Batch Estimation: All the measured data are used simultaneously (non-recursive)

Recursive Estimation: Measured data are used as they are generated, to “update” or “improve” the current estimate (Current estimate and new data are used to compute a new estimate at each sensing step)

Optimal Estimation: An “objective function” (typically in terms of error) is optimized to determine the estimated value

Methods of Estimation

1. **Least Squares Error Estimation (LSE, minimizes sum of squared error)** Our focus
2. **Maximum Likelihood Estimation (MLE, maximizes the likelihood of the estimated value, given the available set of data)**
3. **Four versions of Kalman filter (KF minimizes error covariance of estimate):**

(a) Scalar static Kalman filter

(b) Linear multi-variable dynamic Kalman filter (KF—applicable for linear systems; Gaussian assumption)

(c) Extended Kalman Filter (EKF—applicable for nonlinear situations; Gaussian assumption)

(d) Unscented Kalman Filter (UKF—applicable for nonlinear situations; takes into account the propagation of random characteristics through nonlinear process; non-Gaussian Okay)

Least Squares Estimation (LSE)

Least Squares Estimation (LSE)

Estimate unknown parameters by minimizing sum of squared error between data and a model of data

➔ This is an "optimal" method of estimation **Why?**

Estimated parameters are the model parameters

Linear LSE: Model is linear (2 parameters)

Nonlinear LSE: Model is nonlinear (more parameters)

Least Squares Point Estimate: 1. An unknown constant parameter is estimated using multiple measurements (containing measurement error) of the parameter; 2. A batch of items of a specific nominal parameter value is measured (both model error and measurement error are present) **Why?**

Least Squares Line Estimate: A line (linear or nonlinear) is fitted to "pairs" of data (input, output) ← linear regression

This approach was used in your labs

How do we extend this to relations of more than two variables (not model parameters)?

Least Squares Point Estimate

The parameter of unknown value *m* is measured:

1. **Repeatedly *N* times from the same object → measurement error**
2. **Once each from a batch of *N* (nominally identical) objects**

using a sensor (having random error) → combined model error and measurement error

Data set: {Y_1, Y_2, . . . , Y_N}

***Note*: Use the "uppercase" *Y* to represent data with "random" error**

Squared error: $e = \sum_{i=1}^{N} (Y_i - m)^2$

Determine estimate $\hat{m}$ of unknown constant *m*

Differentiate *e* wrt *m* and equate to zero:

→: Optimal estimate (LS point): "sample mean" $\hat{m} = \frac{1}{N}\sum_{i=1}^{N} Y_i$ **(batch)**

Recursive Scheme: $\hat{m}_1 = Y_1$

$$\hat{m}_{i+1} = \frac{1}{(i+1)}(i \times \hat{m}_i + Y_{i+1}), \quad i = 1, 2, ...$$

Estimation Error (Variance)

Assume: Each measurement is *independent* of any other measurement; measurements have *same probability distribution*

➔ Y_i are "*independent and identically distributed*" (*iid*)

***Note*: Measurement is a random variable ➔ estimate (function of measurement) is also a random variable** Why?

Variance of estimate: $\mathrm{Var}(\hat{m}) = \mathrm{Var}\left[\frac{1}{N}(Y_1 + Y_2 + \ldots + Y_N)\right] = \frac{1}{N^2}\mathrm{Var}(Y_1 + Y_2 + \ldots + Y_N)$

$$= \frac{1}{N^2}\left[\mathrm{Var}(Y_1) + \mathrm{Var}(Y_2) + \ldots + \mathrm{Var}(Y_N)\right] = \frac{N\sigma_m^2}{N^2}$$

➔ Std. deviation of estimate $\sigma_{\hat{m}} = \dfrac{\sigma_m}{\sqrt{N}}$

➔ Randomness of estimate decreases (precision improves) with:

1. **Number of data items (*N*) in the "measurement sample"**
2. **Precision (inverse of variance) of the data (including measurement error and possibly model error)**

Note: Std deviation represents random error
Systematic error cannot be estimated.

These two are intuitive?

Sample Mean and Sample Variance

Sample mean $\bar{Y} = \frac{1}{N}\sum_{i=1}^{N} Y_i$

Sample variance $S^2 = \frac{1}{(N-1)}\sum_{i=1}^{N}\left(Y_i - \bar{Y}\right)^2$

***Note*:**

1. These are *unbiased estimates*: $E\left(\bar{Y}\right) = \mu$ **;** $E\left(S^2\right) = \sigma^2$

2. For *N* = 1, *S* is indeterminate (0/0) ➔ logical (see *N*-1 in the denominator)

3. Typically $\bar{Y}$ **approaches *μ* as *N* increases**

4. Typically S^2 **approached** σ^2 **as *N* increases**

Least Squares Line Estimate

Least Squares Line Estimate

Sum of squared error between data set and a line is minimized

***Note*: Line is the "model" (represented by two or more parameters—a polynomial)**

Straight line: 2 parameters → Linear regression line (mean calibration curve)

What are the 2 parameters?

Quadratic function: 3 parameters

Etc.

Linearity (Nonlinearity): Measured by max deviation of input/output data (or actual calibration curve—nonlinear) from least squares straight-line fit (or, mean calibration curve)

Least Squares Linear Estimate

Data: $\{(X_1, Y_1), (X_2, Y_2), \ldots, (X_N, Y_N)\}$

Linear regression (linear model): $Y = mX + a$

Sum of squared error: $e = \sum_{i=1}^{N} (Y_i - mX_i - a)^2$

Differentiate wrt *m* and *a*: $\sum_{i=1}^{N} X_i (Y_i - mX_i - a) = 0$; $\sum_{i=1}^{N} (Y_i - mX_i - a) = 0$

$\Leftarrow$ **2 equations with 2 unknowns (*m* and *a*)**

$\Rightarrow$ $m = \left(\frac{1}{N} \sum_{i=1}^{N} X_i Y_i - \bar{X}\,\bar{Y} \right) \Big/ \left(\frac{1}{N} \sum_{i=1}^{N} X_i^2 - \bar{X}^2 \right)$; $Y - \bar{Y} = m(X - \bar{X})$ $\rightarrow$ $a = \bar{Y} - m\bar{X}$

(Value of *Y* when *X* = 0)

***Note*: All the data used simultaneously (i.e., "batch" or "non-recursive")**

Quality of Estimate

“Quality” (goodness) of an estimate depends on:

- **Accuracy of data**
- **Size of data set**
- **Method of estimation (what are you optimizing)**
- **Estimation model (e.g., linear fit, quadratic fit)**
- **Number of estimated parameters**

Measures of the Quality of Estimate

Sum of Squared Error (SSE): $\text{SSE} = \sum_{i=1}^{N}\left(Y_i - \hat{Y}_i\right)^2$

Mean Square Error (MSE): $\text{MSE} = \frac{1}{(N-M)}\sum_{i=1}^{N}\left(Y_i - \hat{Y}_i\right)^2$

M **= estimated number of parameters (of fitted curve)**

N-M **= "residual degrees of freedom"**

***Note*: For a line fit, *M* = 2.**

"hat" ➔ estimated value
"over-bar" ➔ average value

Root Mean Square Error (RMSE): Square root of MSE

R-Squared: $\text{R-Squared} = 1 - \frac{\text{SSE}}{\sum_{i=1}^{N}\left(Y_i - \bar{Y}\right)^2}$

(coefficient of determination)

Why is this a quality measure?
(Closer to 1 is better)
Value is in [0,1]. Why?

Adjusted R-Square: $\text{Adjusted R-Squared} = 1 - \frac{\text{MSE}}{\text{VAR}}$

VAR = sample variance

Why better?

***Note* 1: R-Squared numerator: Fit with model (estimate); denominator: Fit with average (Average is "one" value; Estimate is a set of values)**

***Note* 2: Accuracy decreases with number of estimated parameters**

***Note* 3: May include a weighting w_i for each data value Y_i**