University of British Columbia
Department of Mechanical Engineering

MECH468 Modern Control Engineering
MECH509 Controls
Midterm exam, Solutions

Examiner: Dr. Ryozo Nagamune
February 24 (Wednesday), 2021, 1-1:50pm (PST)

Exam policies

- Allowed: Open-book. Any distributed material and any textbook. You can see course materials on your computer.
- Not-allowed: Matlab. Calculators. Web-browsing.
- Write all your answers on **your own sheets**.
- Motivate your answers properly. (No chance to defend your answers orally.)
- **No questions are allowed.**
- 30 points in total.

Before you start ...

- Use washroom before the exam.
- Turn off your mobile phone, or make in in the airplane mode.
- No eating.

After you finish the exam ...

- Scan, or take a photo of, your answer sheets.
- Upload the pdf (or jpg) files of your answer sheets on Canvas "Assignments".
- **Make sure that you have uploaded all your answer sheets.** You cannot add some sheets later even if you somehow missed uploading them.

Marking scheme

Question #	Expected duration	Full mark
Q1	about 15 min	10 %
Q2	about 15 min	10 %
Q3	about 20 min	10 %
Total	about 50 min	30 %

1. Answer the following questions.

   (a) For a matrix $A = \begin{bmatrix} -1 & 0 \\ 1 & -1 \end{bmatrix}$, obtain the matrix exponential e^{At}. (2pt)

   Hint: If you want, you can use the formula $e^{M+N} = e^M e^N$ if $MN = NM$.

   <u>**Solution:**</u>

$$e^{At} = e^{\left(\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\right)t} = \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{-t} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ t & 1 \end{bmatrix} = \begin{bmatrix} e^{-t} & 0 \\ te^{-t} & e^{-t} \end{bmatrix}$$

   (b) For an equation of motion $m\ddot{p} = f$ where f is the force input and p is the position output of a mass m, obtain a state equation (output equation is not necessary) with the following two states: (4pt)

$$z_1 := p + \dot{p}, \quad z_2 := p - \dot{p}$$

   <u>**Solution:**</u> For the state vector $x := \begin{bmatrix} p & \dot{p} \end{bmatrix}$, we have a state equation:

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} f.$$

   Now, by the coordinate transformation

$$z := \underbrace{\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}}_{T} x,$$

   we can get another state equation as

$$\dot{z} = T \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} T^{-1} z + T \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} f \Rightarrow \dot{z} = \frac{1}{2} \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} z + \frac{1}{m} \begin{bmatrix} 1 \\ -1 \end{bmatrix} f.$$

   (c) Linearize the following nonlinear state equation with your selected linearization point. (4pt)

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -x_1(1 - x_2^3) + u + 1 \\ -x_1^2 - x_2^2 + 1 \end{bmatrix}$$

Solution: Select any combination of (x_{10}, x_{20}, u_0) (equilibrium point, linearization point) which satisfies the differential equation. Then, the linearized model around that point is

$$\delta\dot{x} = \left.\begin{bmatrix} -(1-x_2^3) & 3x_1x_2^2 \\ -2x_1 & -2x_2 \end{bmatrix}\right|_{(x_{10},x_{20},u_0)} \delta x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \delta u$$

2. Consider the following state-space model.

$$\begin{aligned} \dot{x} &= \begin{bmatrix} 0 & 2 \\ -1 & -3 \end{bmatrix} x + \begin{bmatrix} 2 \\ -1 \end{bmatrix} u, \\ y &= \begin{bmatrix} 1 & 2 \end{bmatrix} x \end{aligned}$$

(a) Check the internal stability. (2pt)

Solution: Compute the eigenvalues of A-matrix.

$$\det(\lambda I - A) = \det \begin{bmatrix} \lambda & -2 \\ 1 & \lambda+3 \end{bmatrix} = \lambda^2 + 3\lambda + 2 = (\lambda+1)(\lambda+2)$$

Thus, the eigenvalues of A-matrix are -1 and -2, and hence the system is asymptotically stable.

(b) Check the controllability. (2pt)

Solution:

$$\mathcal{C} = \begin{bmatrix} 2 & -2 \\ -1 & 1 \end{bmatrix} \Rightarrow \text{rank}(\mathcal{C}) = 1 \Rightarrow \text{Not controllable}$$

(c) Check the observability. (2pt)

Solution:

$$\mathcal{O} = \begin{bmatrix} 1 & 2 \\ -2 & -4 \end{bmatrix} \Rightarrow \text{rank}(\mathcal{O}) = 1 \Rightarrow \text{Not observable}$$

(d) Obtain the coordinate transformation matrix T^{-1} for the Kalman decomposition. State which column of T^{-1} corresponds to T_{co}, $T_{c\bar{o}}$, $T_{\bar{c}o}$, or $T_{\bar{c}\bar{o}}$. (4pt)

Solution:

$$\text{Im}(\mathcal{C}) = \text{Ker}(\mathcal{O}) = \text{span}\left\{ \begin{bmatrix} 2 \\ -1 \end{bmatrix} \right\} \Rightarrow T^{-1} = \begin{bmatrix} T_{c\bar{o}} & T_{\bar{c}o} \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ -1 & 1 \end{bmatrix}$$

3. Let us consider the minimum energy control for the velocity of a car. The car's dynamics is assumed to be represented by a discrete-time model:

$$v[k+1] = \frac{v[k]}{2} + f[k],$$

where v [m/s] is the velocity of a car and f [N] is the force applied to the car.

(a) Check the controllability of this system. (2pt)

<u>**Solution:**</u> The controllability matrix is $\mathcal{C} = 1$, and full rank. So, it is controllable.

(b) Obtain the minimum energy control to transfer the velocity from the initial velocity $v[0] = 8$ [m/s] to the final velocity $v[2] = 12$. (4pt)

<u>**Solution:**</u>

$$\begin{aligned} \begin{bmatrix} f[1] \\ f[0] \end{bmatrix} &= \begin{bmatrix} 1 \\ 1/2 \end{bmatrix} \left(\begin{bmatrix} 1 & 1/2 \end{bmatrix} \begin{bmatrix} 1 \\ 1/2 \end{bmatrix} \right)^{-1} \left(v[2] - (1/2)^2 v[0] \right) \\ &= \begin{bmatrix} 1 \\ 1/2 \end{bmatrix} \frac{4}{5} \left(12 - \frac{1}{4} \cdot 8 \right) = \begin{bmatrix} 8 \\ 4 \end{bmatrix} \end{aligned}$$

(c) Using the minimum energy control in (b), compute the velocity $v[1]$. Make a plot with the time step k in x-axis and the car velocity v in y-axis. (2pt)

<u>**Solution:**</u>

$$v[1] = \frac{v[0]}{2} + f[0] = \frac{8}{2} + 4 = 8$$

(d) Obtain one **non-minimum energy control** (i.e., control other than minimum energy control) to transfer the velocity from the initial velocity $v[0] = 8$ [m/s] to the final velocity $v[2] = 12$. (2pt)

<u>**Solution:**</u>

$$v[2] - (1/2)^2 v[0] = \begin{bmatrix} 1 & 1/2 \end{bmatrix} \begin{bmatrix} f[1] \\ f[0] \end{bmatrix} \Rightarrow 10 = f[1] + \frac{f[0]}{2}$$

Any pair of $f[0]$ and $f[1]$ satisfying this equation is fine (except the minimum energy control pair).