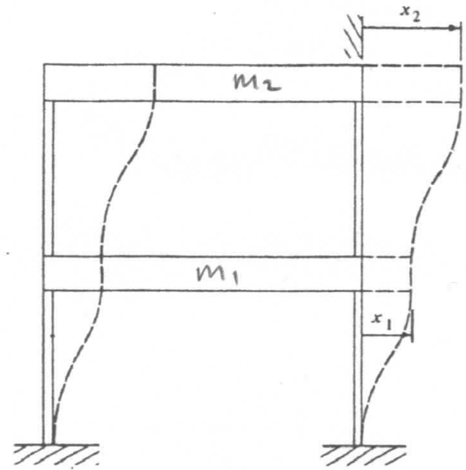
MECH 463 -- Homework 6

1. A 2-storey building is constructed of two rectangular concrete slabs, the lower one of mass 20,000 kg, and the upper of mass 10,000 kg, both supported by steel columns at the corners. After construction, the stiffness of the building is tested by applying horizontal loads on each of the two concrete slabs. When a horizontal force of 1000 N is applied to the lower slab, the displacement of that slab is 1mm. When a horizontal force of 1000 N is applied to the upper slab, the displacement of that slab is 3mm. Formulate the equations of motion in terms of the Principal Coordinates. Identify the natural frequencies and mode shapes.



This is a thinly disguised version of:

For a force F_1 applied to m_1, $k_1 = \frac{F_1}{\delta_1} = \frac{1000}{0.001} = 1\ MN/m$

For a force F_2 applied to m_2, $k^* = \frac{F_2}{\delta_2} = \frac{1000}{0.003} = 0.33\ MN/m$

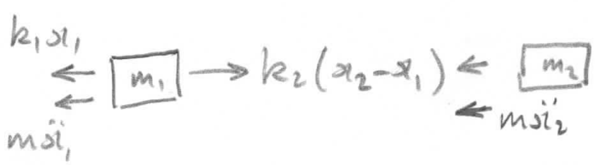
For springs in series $\frac{1}{k^*} = \frac{1}{k_1} + \frac{1}{k_2} \rightarrow \frac{1}{k_2} = \frac{1}{k^*} - \frac{1}{k_1} \rightarrow k_2 = \frac{k^* k_2}{k_1 - k^*}$

$\rightarrow k_2 = \frac{0.33 \times 1}{1 - 0.33} = 0.5\ MN/m$

From FBD:

$m_1\ddot{x}_1 - k_1 x_1 - k_2(x_2 - x_1) = 0$

$m_2\ddot{x}_2 + k_2(x_2 - x_1) = 0$



In matrix format:

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightarrow \underset{\sim}{M}\,\underline{\ddot{x}} + \underset{\sim}{K}\,\underline{x} = \underline{0}$$

Try solution $\underline{x} = \underline{X}\cos(\omega t + \phi) \rightarrow (-\omega^2 \underset{\sim}{M} + \underset{\sim}{K})\,\underline{X}\cos(\omega t + \phi) = \underline{0}$

For a non-trivial solution valid for all $t \rightarrow \det(-\omega^2 \underset{\sim}{M} + \underset{\sim}{K}) = 0$

$$\rightarrow \begin{vmatrix} k_1 + k_2 - m_1\omega^2 & -k_2 \\ -k_2 & k_2 - m_2\omega^2 \end{vmatrix} = 0 \rightarrow (k_1 + k_2 - m_1\omega^2)(k_2 - m_2\omega^2) - k^2 = 0$$

$$\rightarrow m_1 m_2 \omega^4 - (m_1 k_2 + m_2 k_1 + m_2 k_2)\omega^2 + k_1 k_2 = 0$$

In a "real" system where the numbers have decimal values, we would typically continue numerically. Here we can continue algebraically.

Here $k_1 = 2k_2 \quad m_1 = 2m_2$

$$\rightarrow 2m_2^2 \omega^4 - 5m_2 k_2 \omega^4 + 2k_2^2 = 0$$

$$\omega^2 = \frac{5m_2k_2 \pm \sqrt{25m_2^2k_2^2 - 16m_2^2k_2^2}}{4m_2^2} = \frac{m_2}{2k_2} \text{ or } \frac{2m_2}{k_2}$$

For mode shapes, put $\underline{x} = \begin{bmatrix} 1 \\ u \end{bmatrix} C\cos(\omega t + \phi)$

$$\rightarrow \left[-\omega^2 \underset{\sim}{M} + \underset{\sim}{K}\right]\begin{bmatrix} 1 \\ u \end{bmatrix} C\cos(\omega t + \phi) = \underline{0}$$

For non-trivial solution valid for all t

$$\begin{bmatrix} 3k_2 - 2m_2\omega^2 & -k_2 \\ -k & k_2 \end{bmatrix}\begin{bmatrix} 1 \\ u \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

From first line: $3k_2 - 2m_2\omega^2 - k_2 u = 0$

$$\rightarrow u = 3 - \frac{2m_2}{k_2}\omega^2 \qquad \rightarrow u_1 = 3 - 1 = 2 \qquad u_2 = 3 - 4 = -1$$

$\rightarrow$ Mode shapes are $\underline{\begin{bmatrix} 1 \\ 2 \end{bmatrix} \text{ for } \omega_1^2 = \frac{m_2}{2k_2}}$ and $\underline{\begin{bmatrix} 1 \\ -1 \end{bmatrix} \text{ for } \omega_2^2 = \frac{2m_2}{k_2}}$

$$\omega_1^2 = \frac{k_2}{2m_2} = \frac{0.5 \times 10^6}{2 \times 10\,000} = 250 \rightarrow f_1 = \frac{\sqrt{250}}{2\pi} = \underline{2.5 \text{ Hz}}$$

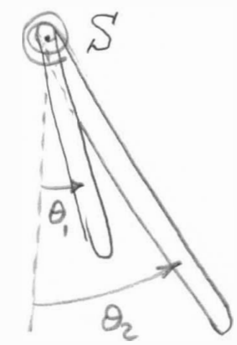
$$\omega_2^2 = \frac{2k_2}{m_2} = \frac{2 \times 0.5 \times 10^6}{10000} = 1000 \rightarrow f_2 = \frac{\sqrt{1000}}{2\pi} = \underline{5.0 \text{ Hz}}$$

Q2.

Choose angular coordinates θ_1 and θ_2

Masses of rods, $m_1 = \rho l$, $m_2 = \rho L$

MI's of rods, $J_1 = \frac{1}{12}(\rho l) l^2$, $J_2 = \frac{1}{12}(\rho L) L^2$

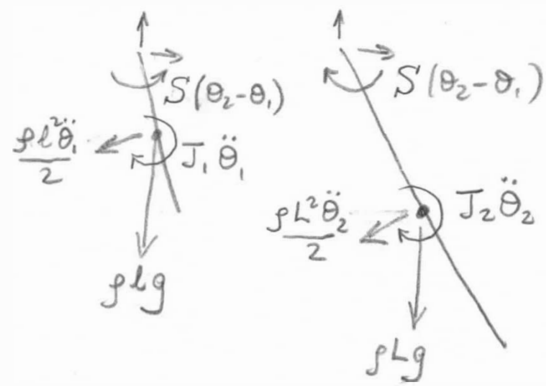


System

Moment balance about upper ends of the rods: (for small oscillations)

$$\rho \frac{l^2 \ddot{\theta}_1}{2} \cdot \frac{l}{2} + \frac{1}{12}\rho l^3 \ddot{\theta}_1 + \rho l g \cdot \tfrac{l}{2}\theta_1 - S(\theta_2 - \theta_1) = 0$$

$$\rho \frac{L^2 \ddot{\theta}_2}{2} \cdot \frac{L}{2} + \frac{1}{12}\rho L^3 \ddot{\theta}_2 + \rho L g \cdot \frac{L}{2}\theta_2 + S(\theta_2 - \theta_1) = 0$$



Free body diagrams

In matrix form:

$$\begin{bmatrix} \frac{\rho l^3}{3} & 0 \\ 0 & \frac{\rho L^3}{3} \end{bmatrix} \begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix} + \begin{bmatrix} \frac{\rho l^2}{2} g + S & -S \\ -S & \frac{\rho L^2}{2} g + S \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Divide both equations by $\rho g l^2$:

$$\begin{bmatrix} \frac{1}{3}\frac{l}{g} & 0 \\ 0 & \frac{1}{3}\frac{l}{g}\cdot\left(\frac{L}{l}\right)^3 \end{bmatrix} \begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix} + \begin{bmatrix} \frac{1}{2} + \frac{S}{\rho g l^2} & -\frac{S}{\rho g l^2} \\ -\frac{S}{\rho g l^2} & \frac{1}{2}\left(\frac{L}{l}\right)^2 + \frac{S}{\rho g l^2} \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Substitute $\mu = l/L$ and $\beta = \frac{S}{\rho g l^2}$

$$\underbrace{\begin{bmatrix} \frac{1}{3}\frac{l}{g} & 0 \\ 0 & \frac{1}{3}\frac{l}{g}\cdot\frac{1}{\mu^3} \end{bmatrix}}_{\underset{\sim}{M}} \begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix} + \underbrace{\begin{bmatrix} \frac{1}{2} + \beta & -\beta \\ -\beta & \frac{1}{2}\cdot\frac{1}{\mu^2} + \beta \end{bmatrix}}_{\underset{\sim}{K}} \underbrace{\begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix}}_{\underset{\sim}{\theta}} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\rightarrow \underset{\sim}{M}\,\underline{\ddot{\theta}} + \underset{\sim}{K}\,\underline{\theta} = \underline{0}$$

Try solution $\underline{\theta} = \underline{\Theta} \cos(\omega t - \phi)$

$$\longrightarrow \left(\underset{\sim}{K} - \omega^2 \underset{\sim}{M} \right) \underline{\Theta} \cos(\omega t - \phi) = 0$$

For a non-trivial solution, $\left| \underset{\sim}{K} - \omega^2 \underset{\sim}{M} \right| = 0$

$$\longrightarrow \begin{vmatrix} \tfrac{1}{2} + \beta - \tfrac{1}{3}\Omega^2 & -\beta \\ -\beta & \tfrac{1}{2} \cdot \frac{1}{\mu^2} + \beta - \tfrac{1}{3} \cdot \frac{1}{\mu^3} \cdot \Omega^2 \end{vmatrix} = 0$$

putting $\Omega^2 = \omega^2 l / g$

$$\rightarrow \left(\tfrac{1}{2} + \beta - \tfrac{1}{3}\Omega^2 \right)\left(\tfrac{1}{2} \cdot \frac{1}{\mu^2} + \beta - \tfrac{1}{3} \cdot \frac{1}{\mu^3}\Omega^2 \right) - \beta^2 = 0$$

$\leftarrow$ Characteristic equation

Multiply by $36\mu^3$ to clear the fractions

$$\left(3 + 6\beta - 2\Omega^2\right)\left(3\mu + 6\beta\mu^3 - 2\Omega^2\right) - 36\beta^2\mu^3 = 0$$

$$= 9\mu + 18\beta\mu^3 - 6\Omega^2 + 18\beta\mu + \cancel{36\beta^2\mu^3} - 12\beta\Omega^2 - 6\mu\Omega^2 - 12\beta\mu^3\Omega^2 + 4\Omega^4 - \cancel{36\beta^2\mu^3} = 0$$

$$= 4\Omega^4 - \left(6 + 12\beta + 6\mu + 12\beta\mu^3\right)\Omega^2 + 9\mu + 18\beta\mu^3 + 18\beta\mu = 0$$

$$= a\Omega^4 + b\Omega^2 + c = 0$$

where

$$\boxed{\begin{aligned} a &= 4 \\ b &= -6\left(1 + 2\beta + \mu + 2\beta\mu^3\right) \\ c &= 9\left(\mu + 2\beta\mu^3 + 2\beta\mu\right) \end{aligned}}$$

$$\rightarrow \boxed{\Omega^2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}}$$

Coupled Pendulum

This question illustrates the effect of mode shape coupling in a vibrating system. In the present context, the word "coupling" has a different meaning to the coordinate coupling discussed in class. Here coupling refers to a connection between two or more **mode shapes** of a vibrating system.

The vibrating system considered in this project consists of two compound pendulums connected together by a torsional spring of stiffness S. If we consider the extreme case where the torsional stiffness of the spring S is reduced to zero, the spring disappears and we are left with two entirely separate 1-dof pendulums. There is no connection of any kind between them and they are totally "uncoupled". When $S > 0$, the spring physically couples the two separate 1-dof sub-systems into the combined 2-dof system. In particular, the spring couples the **mode shapes** of the two pendulums. Such coupling can also occur in less physically obvious ways, but the effect of coupling together different mode shapes is the same.

This homework illustrates a characteristic of mode shape coupling called "avoided crossing" or "curve veering". In this example, we see the phenomenon when we look at graphs of the natural frequency parameter Ω^2 vs. the length ratio $\mu = \ell/L$. Figure 1 shows that the frequency curves for the uncoupled case, $\beta = S/(\rho g \ell^2) = 0$, consists of two intersecting straight lines. Figure 2, for the coupled case, $S > 0$, $\beta > 0$, shows the curves approaching one other and then turning away to follow each others original path. Such avoided crossings of the frequency curves are quite common in vibrating systems, but are often not recognized.

Figure 1 shows the dimensionless frequency curves for the no coupling case, $\beta = 0$. Here, the torsional spring is effectively absent and the two pendulums vibrate independently at their own natural frequencies. The frequency of the first pendulum is fixed, and is $\omega^2 = 3g/2\ell$, $\Omega^2 = \omega^2\ell/g = 1.5$. This result corresponds to the horizontal line in Figure 1. The frequency of the second pendulum depends on L, and is $\omega^2 = 3g/2L$, $\Omega^2 = \omega^2\ell/g = 1.5\ell/L = 1.5\mu$. This result corresponds to the diagonal line in Figure 1.

Figure 2 shows the dimensionless frequency curves for the finite coupling case, $\beta = 0.05$. The curves no longer intersect one another, but show the avoided crossing phenomenon. Figure 3 shows how the avoided crossings become more prominent as β increases. However, even though the frequency curves do not cross for $\beta > 0$, the mode shapes effectively do. Figure 4 shows the percent θ_1 component in the mode shapes for $\beta = 0.05$. The interchange of mode shapes becomes steeper for smaller β values, theoretically becoming a step function for $\beta = 0$. Such a step change in mode shape corresponds to moving between the diagonal and horizontal lines at $\mu = 1$ in Figure 1.

In Figure 3, the point $\Omega^2 = 1.5$, $\mu = 1$, is seen to be a special point because all the lower frequency curves pass through it. This point corresponds to the particular case where both pendulums have equal lengths and are vibrating in phase with the same amplitude. Under these conditions, the spring is not deformed, and does not contribute in any way to the vibration. Thus, the stiffness of this non-acting spring (described by β) has no effect on the vibration frequency.

Figure 3 illustrates two features that are common to all vibrating systems, independent of any considerations of mode shape coupling. Firstly, any addition of stiffness (here described by increase in β) **increases** all the natural frequencies. The only exception is when the stiffness is added at a non-acting point of a given vibration mode (corresponding here to $\Omega^2 = 1.5$, $\mu = 1$). Then, the associated natural frequency remains unchanged. Secondly, any addition of mass (here described by decrease in μ) **decreases** all the natural frequencies. The only exception is when the mass is added at a non-acting point. Again, the associated natural frequency remains unchanged. The second exception is not illustrated in Figure 3, and occurs only when mass is added at a nodal point.

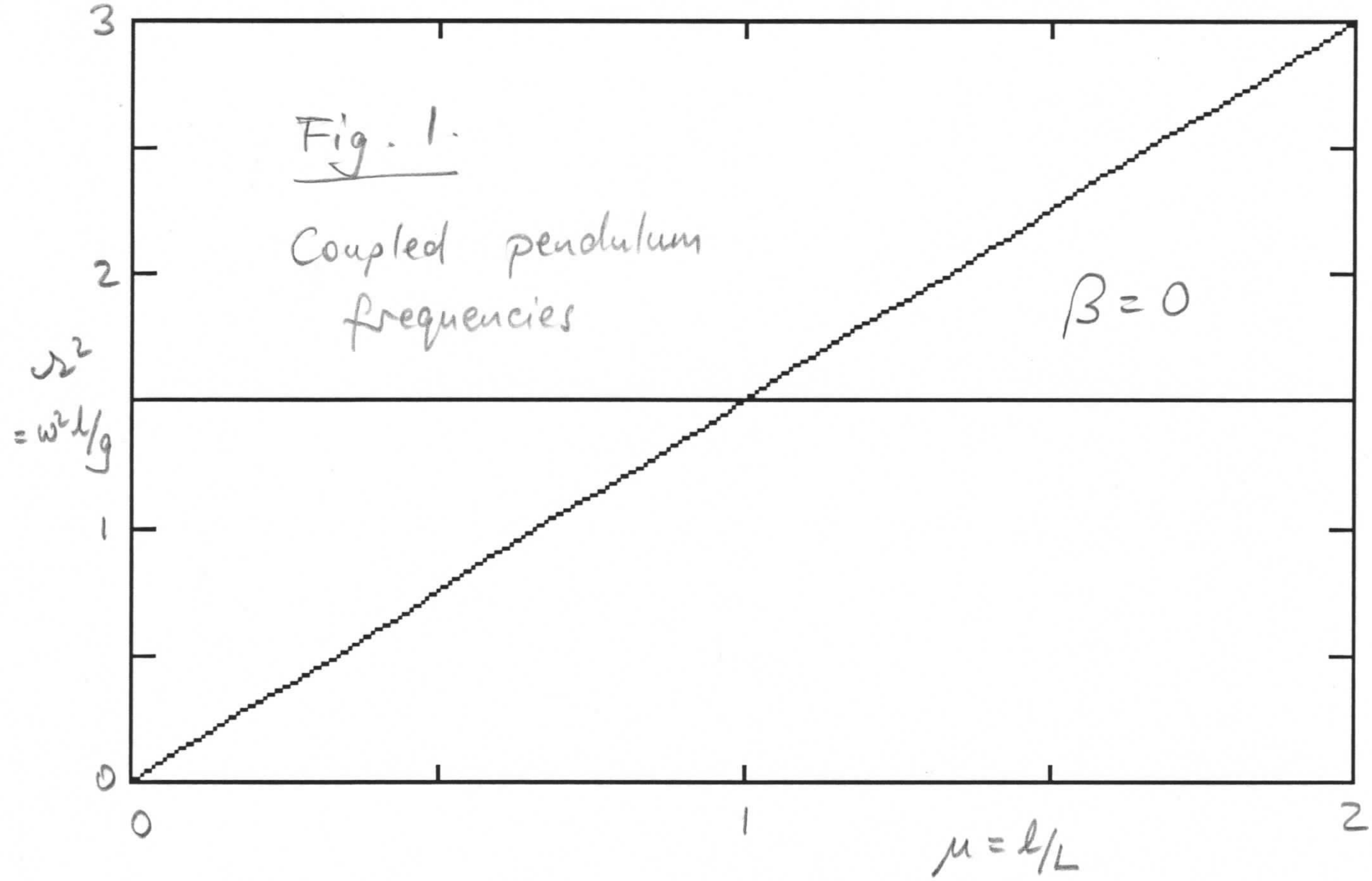


Fig. 1.
Coupled pendulum frequencies
$\beta = 0$
$\Omega^2 = \omega^2 l/g$
$\mu = l/L$
0
1
2
3

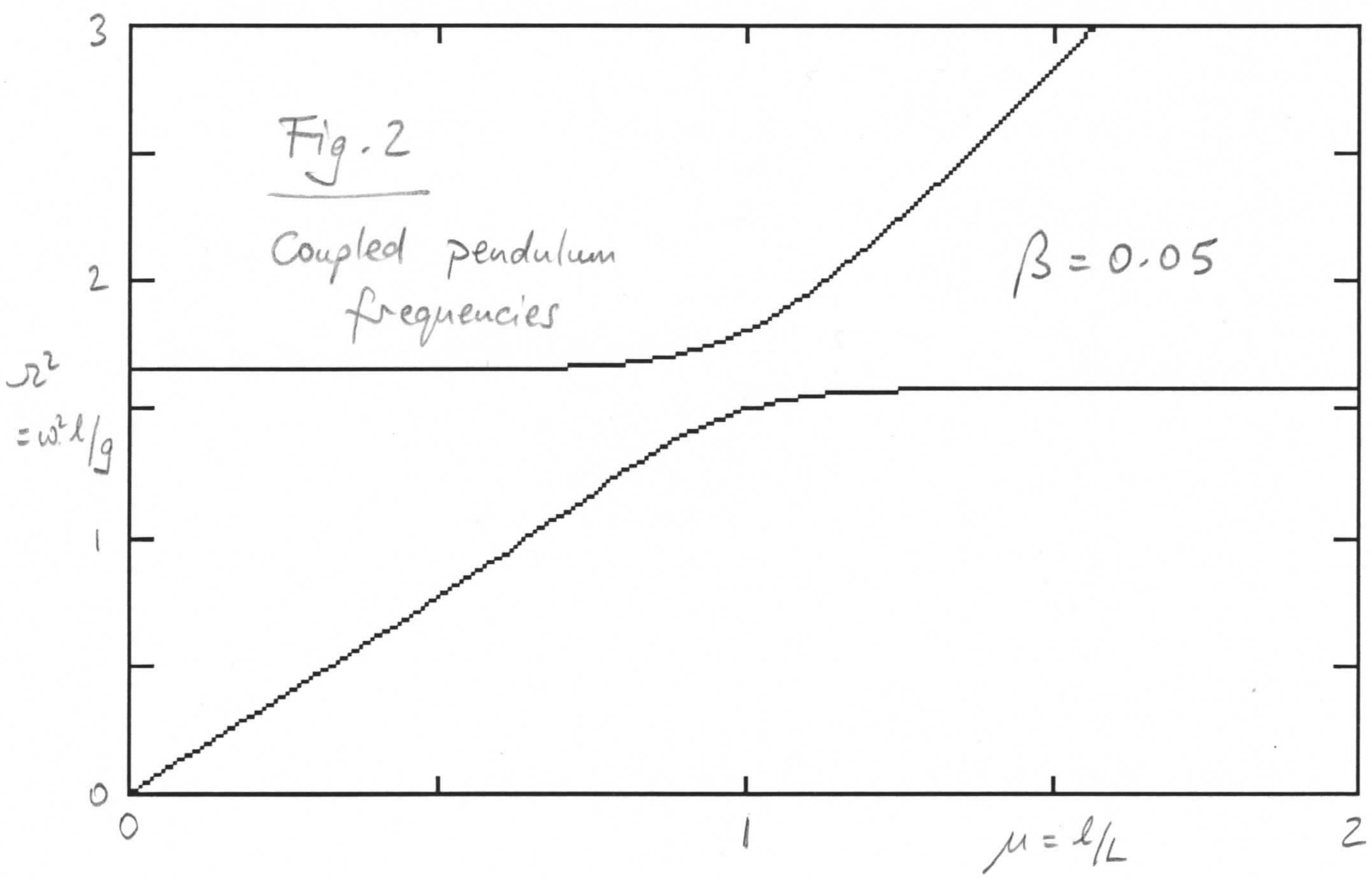


Fig. 2
Coupled pendulum frequencies
$\beta = 0.05$
$\Omega^2 = \omega^2 l/g$
$\mu = l/L$
0
1
2
3

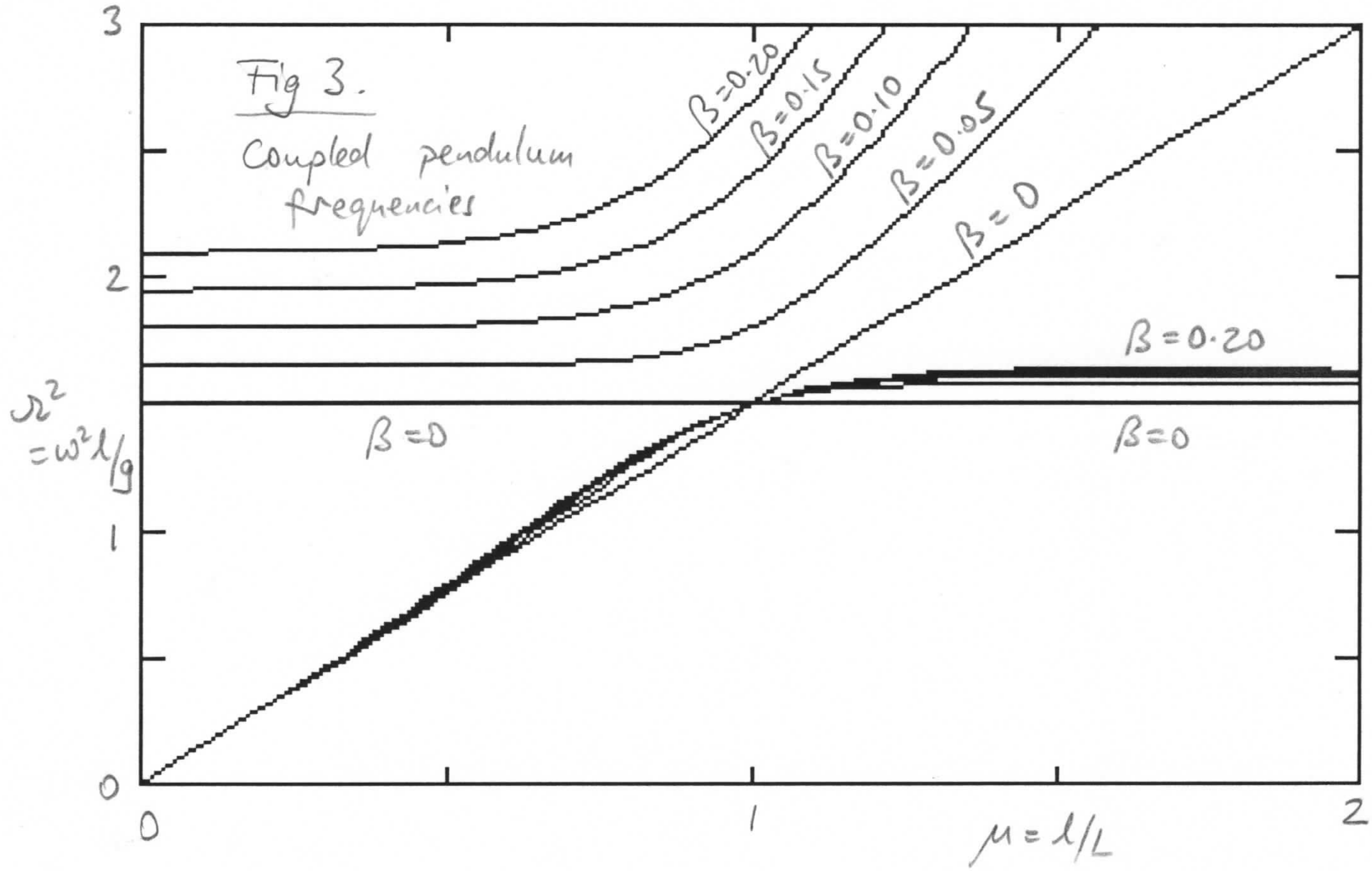


Fig 3.
Coupled pendulum frequencies
β=0.20
β=0.15
β=0.10
β=0.05
β=0
β=0.20
β=0
β=0
$\Omega^2 = \omega^2 l/g$
$\mu = l/L$
0
1
2
3
0
1
2

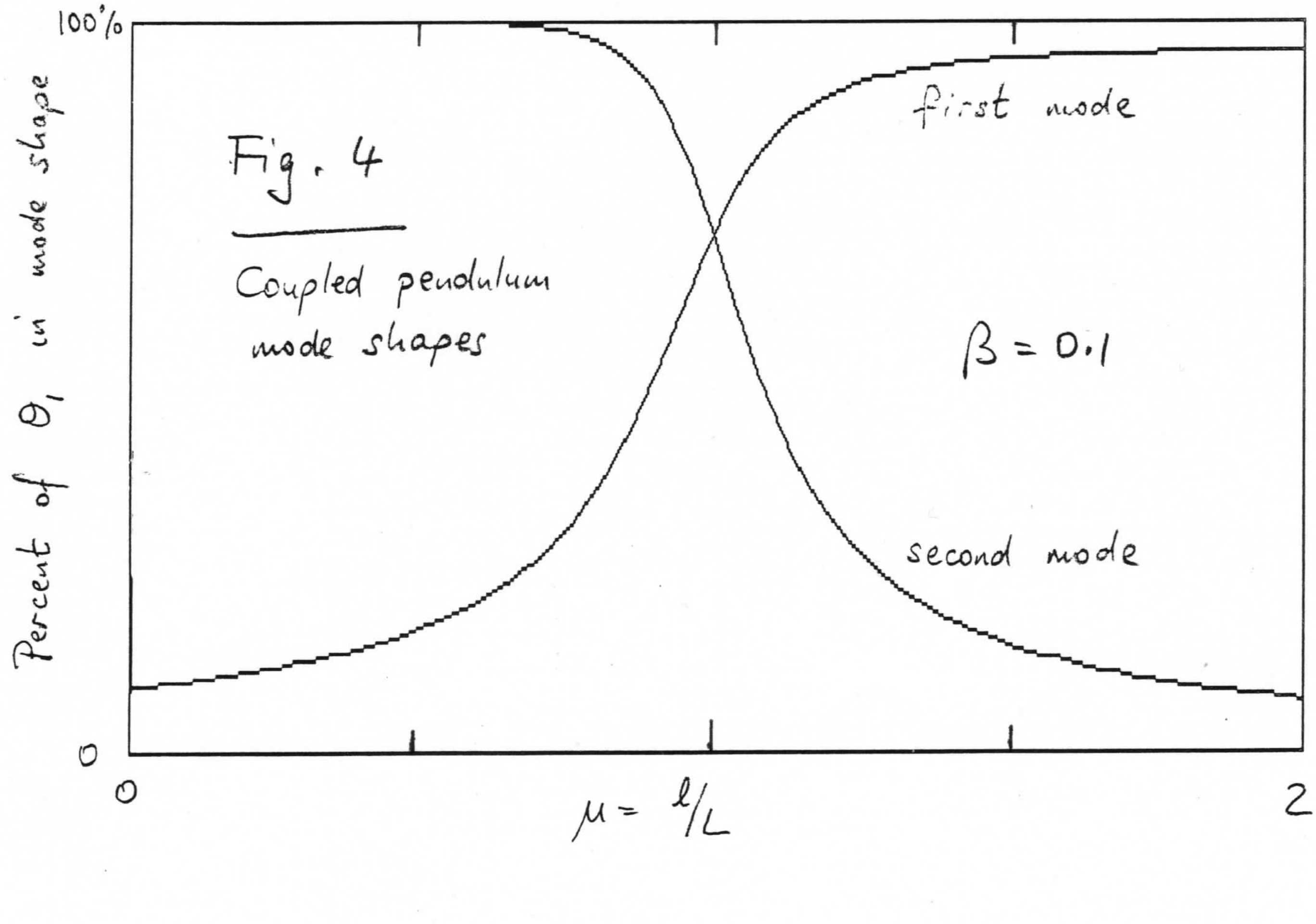


Fig. 4
Coupled pendulum mode shapes
first mode
second mode
β = 0.1
Percent of θ_1 in mode shape
100%
0
0
2
$\mu = l/L$