University of British Columbia
Department of Mechanical Engineering

MECH 463. Midterm 2, October 29, 2019

Allowed Time: 70 min

Materials admitted: Pen, pencil, eraser, straightedge, simple scientific calculator without programming or communication capabilities, personal hand-written notes within one letter-size sheet of paper (one side).

There are 3 questions in this exam. You are asked to answer all three questions.

The purpose of this test is to evaluate your knowledge of the course material. Orderly presentation demonstrates your knowledge most clearly, while disorganized and unprofessional work creates serious doubt. Marks are assigned accordingly. A bonus of up to 2 marks will be given for exemplary presentation.

Complete the section below **during** the examination time **only.**

NAME:____

SIGNATURE:____

STUDENT NUMBER:____

	Mark Received	**Maximum Mark**
1		6
2		7
3		7
Presentation		2 bonus
Total		20+2

Name: ______________________________

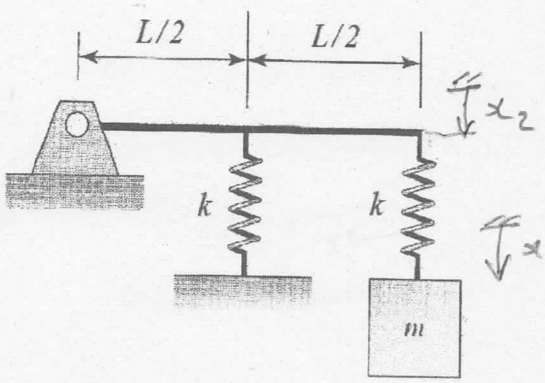
1. A massless rod of length L pinned at its left end and supported by a spring of stiffness k at its midpoint. At its right end, the rod supports a mass m through a spring, also of stiffness k.

   (a) Choose a convenient coordinate system and draw a labeled free-body diagram of the vibrating system.

   (b) Use your free-body diagram to formulate the equation of motion. *(Hint: some careful thought is required here.)*

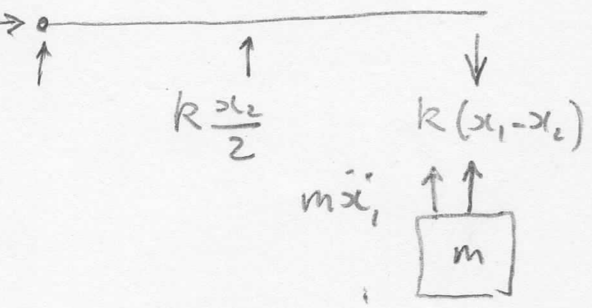
   (c) Solve your equation of motion for natural frequency. Show the needed steps in detail.

   (d) Comment on any notable features of the vibrating system.



(a-b) The system looks a bit unusual because it has two moving parts but only one mass. We will start by using two coordinates and see what happens.

$$\Sigma M_{Left} = k\frac{x_2}{2}\cdot\frac{L}{2} - k(x_1 - x_2)L = 0$$



$$\div kL \rightarrow \frac{x_2}{4} - x_1 + x_2 = 0$$

$$\rightarrow x_2 = \frac{4}{5}x_1$$

$$\Sigma F_{mass} = m\ddot{x}_1 + k(x_1 - x_2) = 0$$

$$= m\ddot{x}_1 + k\left(x_1 - \frac{4}{5}x_1\right) = 0 \quad \rightarrow \quad m\ddot{x}_1 + \frac{k}{5}x_1 = 0$$

(c) Try solution $x_1 = X_1 \cos\omega t$

$$\rightarrow \left(-\omega^2 m + \frac{k}{5}\right)X_1 \cos\omega t = 0$$

For non-trivial solution, true for all t

$$\rightarrow \left(-\omega^2 m + \frac{k}{5}\right) = 0 \quad \rightarrow \quad \omega^2 = \frac{k}{5m}$$

(d) This initially looked like it could possibly be a 2-DOF system. However, the first FBD showed a direct connection between x_2 and x_1. Therefore, it is actually a 1-DOF system with one natural frequency.

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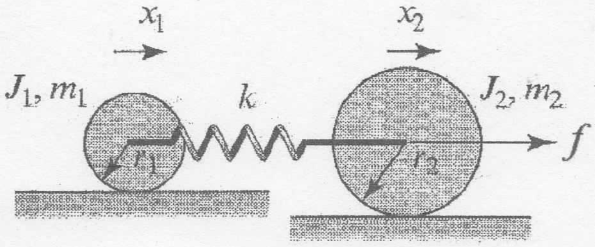
2. Two cylindrical rollers are joined b y a spring of stiffness k. Their radii, mass and moment of inertia respectively are $r_1 = r$, $m_1 = m$, $J_1 = ½\, mr^2$ and $r_2 = 2r$, $m_2 = 2m$, $J_2 = 4mr^2$. An harmonic force $f = F \cos \omega_f t$ acts on the second roller.



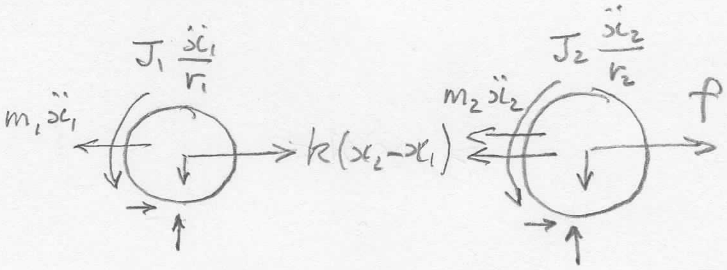
(a) Draw labeled free body diagrams of the rollers.

(b) Formulate a matrix equation of motion for the resulting vibrational displacements of the rollers.

(c) Solve for the steady-state vibration of the first roller.

(d) Sketch the vibration amplitude vs. excitation frequency response of the roller. Comment on and explain any notable features.

(a-b) Take moments about contact points to eliminate contact forces.



$$m_1 \ddot{x}_1 \cdot r_1 + J_1 \frac{\ddot{x}_1}{r_1} - k(x_2 - x_1)\, r_1 = 0$$

$$m_2 \ddot{x}_2 \cdot r_2 + J_2 \frac{\ddot{x}_2}{r_2} + k(x_2 - x_1)\, r_2 = f \cdot r_2$$

Sub. $r_1 = r$, $r_2 = 2r$, $J_1 = \frac{1}{2} mr^2$ $J_2 = 4mr^2$ $m_1 = m$, $m_2 = 2m$

$$m \ddot{x}_1 + \frac{1}{2} m \ddot{x}_1 - k(x_2 - x_1) = 0 \qquad \text{(dividing by } r_1\text{)}$$

$$2m \ddot{x}_2 + m \ddot{x}_2 + k(x_2 - x_1) = f \qquad \text{(dividing by } r_2\text{)}$$

In matrix form:

$$\begin{bmatrix} \frac{3}{2}m & 0 \\ 0 & 3m \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} k & -k \\ -k & k \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ F \end{bmatrix} \cos \omega_f t$$

putting $f = F \cos \omega_f t$

$$\underset{\sim}{M}\, \ddot{\underline{x}} + \underset{\sim}{K}\, \underline{x} = \underline{F} \cos \omega_f t$$

(c) Name: ____________________

Try solution $\underline{x} = \underline{X}\cos\omega_f t \rightarrow \left(-\omega_f^2 \underset{\sim}{M} + \underset{\sim}{K}\right)\underline{X}\cos\omega_f t = \underline{F}\cos\omega_f t$

True for all time $\rightarrow \left(-\omega_f^2 \underset{\sim}{M} + \underset{\sim}{K}\right)\underline{X} = \underline{F}$

$$\begin{bmatrix} k-\frac{3}{2}m\omega_f^2 & -k \\ -k & k-3m\omega_f^2 \end{bmatrix}\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} 0 \\ F \end{bmatrix}$$

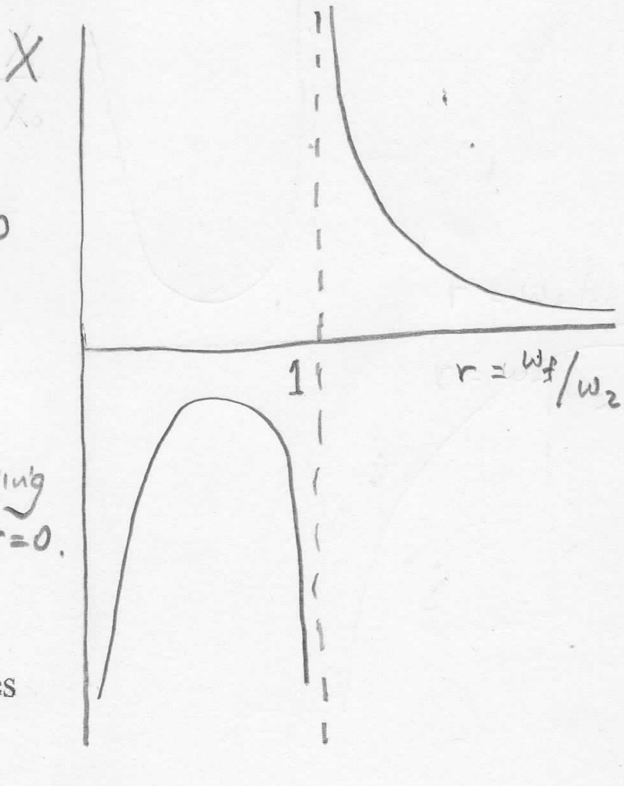
Cramer's Rule:

$$X_1 = \frac{\begin{vmatrix} 0 & -k \\ F & k-3m\omega_f^2 \end{vmatrix}}{\begin{vmatrix} k-\frac{3}{2}m\omega_f^2 & -k \\ -k & k-3m\omega_f^2 \end{vmatrix}} = \frac{Fk}{\left(k-\frac{3}{2}m\omega_f^2\right)\left(k-3m\omega_f^2\right)-k^2}$$

Denominator $= \frac{9}{2}m^2\omega_f^4 - \frac{9}{2}mk\omega_f^2 = \frac{9}{2}m^2\omega_f^2\left(\omega_f^2 - \frac{k}{m}\right)$

From the denominator, we recognize the two natural frequencies as $\omega_1^2 = 0$ and $\omega_2^2 = k/m$. For convenience, we can normalize the response result relative to ω_2

$$X_1 = \frac{F/k}{\frac{9}{2}r^2\left(r^2-1\right)} \quad \text{where } r = \frac{\omega_f}{\omega_2}$$



(d) Resonances at $r=0 \rightarrow \omega_f = 0$ (rigid-body motion) and at $r=1 \rightarrow \omega_f = \sqrt{\frac{k}{m}}$

The response graph looks surprising because it has $-\infty$ response near $r=0$. This graph is the right side of the response graph discussed in class

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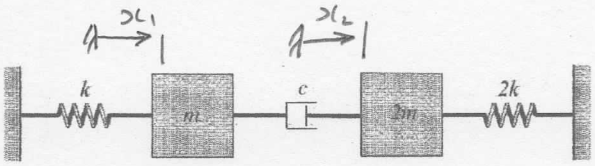
3. Two masses, m and 2m are supported by springs k and 2k, and joined by a damper c, as shown in the diagram.



(a) Draw labeled free-body diagrams of the vibrating system.

(b) Derive the matrix equation of motion from your free-body diagrams.

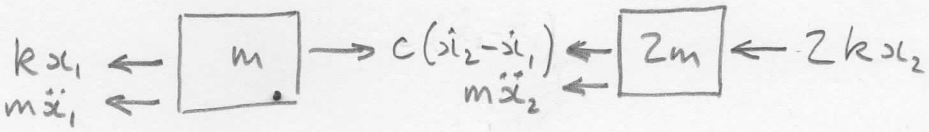
(c) Use the trial solution $\underline{x} = \underline{X}\, e^{\lambda t}$ to find the characteristic solution.

(d) Given that one root of the characteristic equation is $(m\lambda^2 + k)$, find the other root.

(e) Find the undamped and damped natural frequencies and the damping factors of the vibrating system.

(f) Comment on and explain any notable features of your results.

(a)



(b) $$m\ddot{x}_1 - c(\dot{x}_2 - \dot{x}_1) + kx_1 = 0$$

$$2m\ddot{x}_2 + c(\dot{x}_2 - \dot{x}_1) + 2kx_2 = 0$$

In matrix form:

$$\begin{bmatrix} m & 0 \\ 0 & 2m \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} c & -c \\ -c & c \end{bmatrix}\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} + \begin{bmatrix} k & 0 \\ 0 & 2k \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\underset{\sim}{M}\,\ddot{\underline{x}} + \underset{\sim}{C}\,\dot{\underline{x}} + \underset{\sim}{K}\,\underline{x} = \underline{0}$$

(c) Try solution $\underline{x} = \underline{X}\, e^{\lambda t} \rightarrow \left(\lambda^2 \underset{\sim}{M} + \lambda \underset{\sim}{C} + \underset{\sim}{K}\right)\underline{X}\, e^{\lambda t} = \underline{0}$

For non-trivial solution valid for all t, $\det(\lambda^2 \underset{\sim}{M} + \lambda \underset{\sim}{C} + \underset{\sim}{M}) = 0$

$$\rightarrow \begin{vmatrix} m\lambda^2 + c\lambda + k & -c\lambda \\ -c\lambda & 2m\lambda^2 + c\lambda + 2k \end{vmatrix} = 0$$

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$$\rightarrow (m\lambda^2 + c\lambda + k)(2m\lambda^2 + c\lambda + 2k) - c^2\lambda^2 = 0$$

$$= 2m^2\lambda^4 + mc\lambda^3 + 2mk\lambda^2 + 2mc\lambda^3 + c^2\lambda^2 + 2ck\lambda$$
$$+ 2mk\lambda^2 + ck\lambda + 2k^2 - c^2\lambda^2 = 0$$

$$= 2m^2\lambda^4 + 3mc\lambda^3 + 4mk\lambda^2 + 3ck\lambda + 2k^2 = 0$$

(d) Given one root is $m\lambda^2 + k$, divide to find other root.

$m\lambda^2 + k$	$2m^2\lambda^4 + 3mc\lambda^3 + 4mk\lambda^2 + 3ck\lambda + 2k^2$
$\times 2m\lambda^2$	$2m^2\lambda^4 + 2mk\lambda^2$
subtract	$0 \quad 3mc\lambda^3 + 2mk\lambda^2 + 3ck\lambda + 2k^2$
$\times 3c\lambda$	$3mc\lambda^3 + 3ck\lambda$
subtract	$0 + 2mk\lambda^2 + 0 + 2k^2$
$\times 2k$	$2mk\lambda^2 + 2k^2$
subtract	$0 \quad 0$

$\rightarrow$ second root is $\underline{2m\lambda^2 + 3c\lambda + 2k}$

(e)
For standard m-c-k system: $\omega_n^2 = \frac{k}{m}$ $\quad \zeta = \frac{c}{2\sqrt{km}}$ $\quad \omega_d^2 = \omega_n\sqrt{1-\zeta^2}$

For 1st mode: "m" = m, "c" = 0, "k" = k $\rightarrow$ $\underline{\omega_n^2 = \frac{k}{m} \quad \zeta = 0}$ $\quad \omega_d^2 = \omega_n^2$

For 2nd mode "m" = 2m, "c" = 3c, "k" = 2k $\rightarrow$ $\underline{\omega_n^2 = \frac{k}{m}, \quad \zeta = \frac{3c}{4\sqrt{km}}}$

(f) Each mass and spring has same natural frequency $\omega_n^2 = \frac{k}{m}$.
The 1st mode is $u_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and the damper is just translated without length change $\rightarrow$ no damping. The second mode is $u_2 = \begin{bmatrix} 1 \\ -0.5 \end{bmatrix}$ allowing the damper to add damping. There is no stiffness change, so ω_n^2 is the same k/m.