MECH 421: Mechatronics System Instrumentation
2020/21 Winter Session – Term 2

Homework 2 – Solution

Assigned: Jan 22, 2021
Due: Jan 29, 2021

Let us consider a non-inverting amplifier shown in Fig. 1. We assume that the op-amp input impedance R_i is **infinite**, the output impedance R_o is **non-zero**, and the open-loop gain A is a **finite constant**. There is a disturbance current I_o injected into the amplifier output terminal.

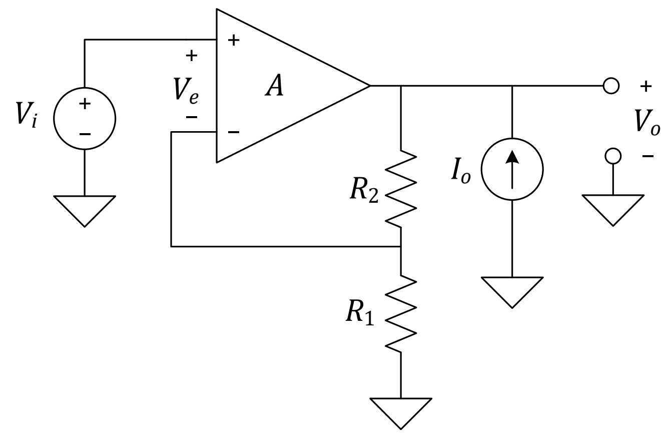


Figure 1: Schematic of a non-inverting amplifier.

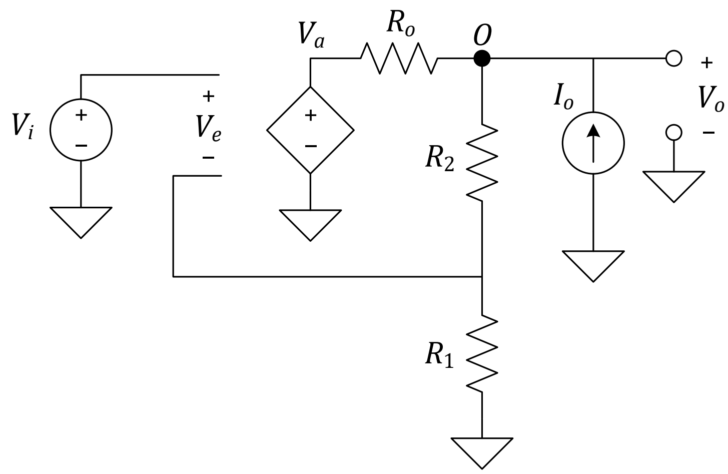


Figure 2: Equivalent circuit model.

Fig. 2 shows the equivalent circuit model. Here, the output stage of the op-amp is modeled as a dependent voltage source with a series output impedance R_o. The output from the dependent voltage source is $V_a = AV_e$, where V_e is the differential voltage applied to the op-amp input signal port. Note that the amplifier output voltage V_o is not equal to V_a because $R_o \neq 0$.

Problem 1

(a) Find an expression for V_o in terms of V_a and I_o as the input variables.

Answer: Apply KCL at the node marked with O in Fig. 1. Since the voltage at the node O is V_o, KCL states that

$$\frac{V_o - V_a}{R_o} + \frac{V_o}{R_1 + R_2} = I_o.$$

Re-arranging the above equation for V_o leads to

$$\boxed{V_o = \frac{R_1 + R_2}{R_o + R_1 + R_2} V_a + \frac{R_o + R_1 + R_2}{R_o(R_1 + R_2)} I_o}$$

Alternatively, one can derive the same equation using the superposition method, i.e, find the portions of the voltage due to V_a and I_o separately and sum them up.

(b) Find an expression for V_e in terms of V_i and V_o as the input variables.

Answer: The feedback voltage V_f, i.e., the voltage at the node between R_1 and R_2, can be obtained by applying the voltage divider rule to V_o.

$$V_f = \frac{R_1}{R_1 + R_2} V_o.$$

Therefore, the differential voltage V_e becomes

$$\boxed{V_e = V_i - \frac{R_1}{R_1 + R_2} V_o.}$$

(c) Complete the block diagram in Fig. 3.

Answer:

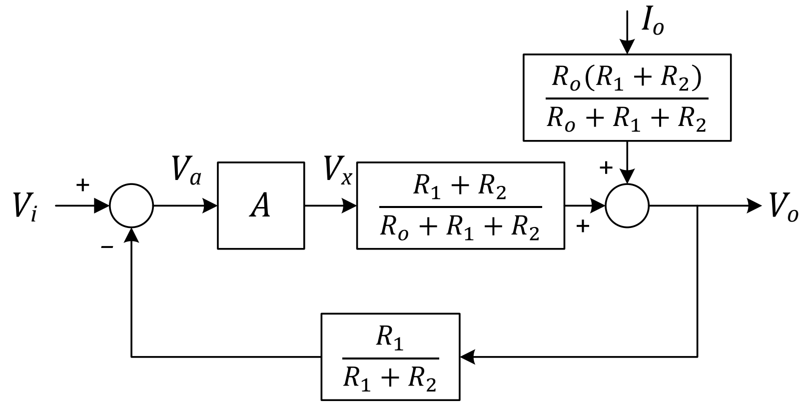


Figure 3: Block diagram.

Problem 2

For $R_o = 50\,\Omega$, $R_1 = 1\,\mathrm{k}\Omega$, and $R_2 = 9\,\mathrm{k}\Omega$, find the amplifier gain V_o/V_i when

(a) $A = 10^5$

Answer: Break the loop right before the summing junction and identify the loop transmission

$$L.T. = -A\left(\frac{R_1+R_2}{R_o+R_1+R_2}\right)\left(\frac{R_1}{R_1+R_2}\right).$$

Apply the Black's formula from V_i to V_o:

$$\frac{V_o}{V_i} = \frac{\text{Forward gain}}{1-L.T.} = \frac{A\left(\frac{R_1+R_2}{R_o+R_1+R_2}\right)}{1+A\left(\frac{R_1+R_2}{R_o+R_1+R_2}\right)\left(\frac{R_1}{R_1+R_2}\right)}.$$

Substituting $R_o = 50\,\Omega$, $R_1 = 1\,\mathrm{k}\Omega$, $R_2 = 9\,\mathrm{k}\Omega$, and $A = 10^5$ into the above leads to

$$\boxed{\frac{V_o}{V_i} = 9.999}$$

(b) $A \to \infty$

Answer: When $A \to \infty$, the above analytic expression for V_o/V_i simplifies to

$$\frac{V_o}{V_i} = \frac{R_1+R_2}{R_1}$$

Substituting $R_1 = 1\,\mathrm{k}\Omega$ and $R_2 = 9\,\mathrm{k}\Omega$ into the above leads to

$$\boxed{\frac{V_o}{V_i} = 10}$$

Note that R_o does not affect the amplifier gain when $A \to \infty$.

Problem 3

For $R_o = 50\,\Omega$, $R_1 = 1\,\mathrm{k}\Omega$, and $R_2 = 9\,\mathrm{k}\Omega$, find the amplifier output impedance V_o/I_o when

(a) $A = 10^5$

Answer: Apply the Black's formula from I_o to V_o

$$\frac{V_o}{I_o} = \frac{\text{Forward gain}}{1 - L.T.} = \frac{\frac{R_o(R_1+R_2)}{R_o+R_1+R_2}}{1 + A\left(\frac{R_1+R_2}{R_o+R_1+R_2}\right)\left(\frac{R_1}{R_1+R_2}\right)}$$

Substituting $R_o = 50\,\Omega$, $R_1 = 1\,\mathrm{k}\Omega$, $R_2 = 9\,\mathrm{k}\Omega$, and $A = 10^5$ into the above leads to

$$\boxed{\frac{V_o}{I_o} = 0.005\,\Omega}$$

(b) $A \to \infty$

Answer: When $A \to \infty$, the above analytic expression converges to zero. Therefore,

$$\boxed{\frac{V_o}{I_o} = 0\,\Omega}$$

Problem 4

Let us consider an op-amp circuit shown in Figure 4. We assume that the op-amp is **ideal**, i.e., the input impedance R_i is infinite, the output impedance R_o is zero, and the open-loop gain A is infinite.

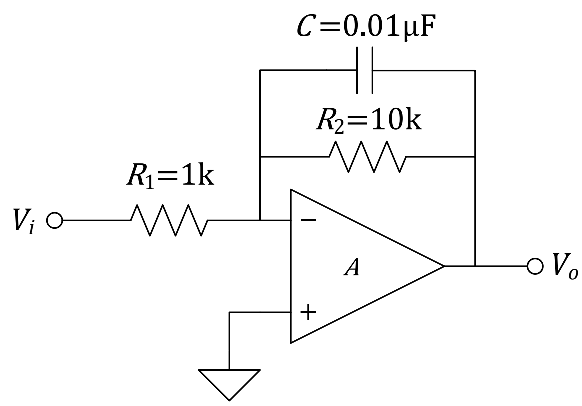


Figure 4: Schematic of a non-inverting amplifier.

(a) Derive the transfer function $V_o(s)/V_i(s)$

Answer:

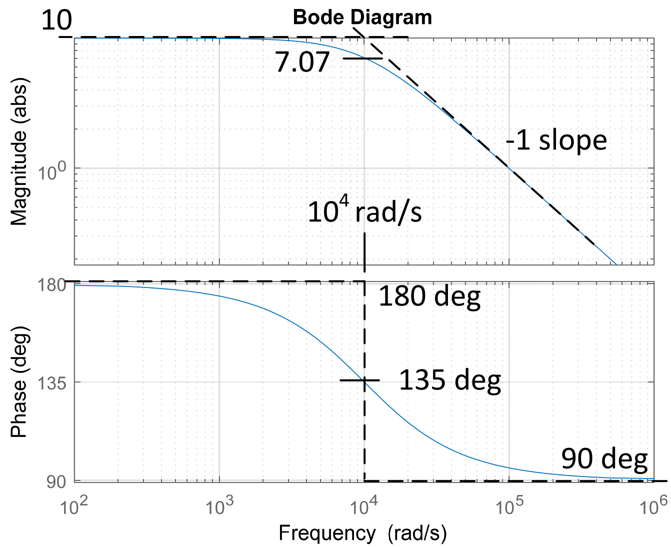
$$\frac{V_o}{V_i} = -\frac{Z_2}{Z_1} = -\frac{1}{R_1}\left(\frac{1}{Cs} \parallel R_2\right) = -\frac{R_2}{R_1}\left(\frac{1}{R_2Cs+1}\right)$$

Therefore,

$$\boxed{\frac{V_o}{V_i} = -10\left(\frac{1}{0.0001s+1}\right)}$$

(b) Plot the Bode plot and the step response.

Answer:



Note that the op-amp circuit behaves as a constant gain at low frequencies and as an integrator at high frequencies. The transition occurs at a frequency (break frequency) determined by C and R_2. Below this frequency the parallel branch consisting of C and R_2 looks like a resistor, whereas above this frequency the parallel branch looks like a capacitor.