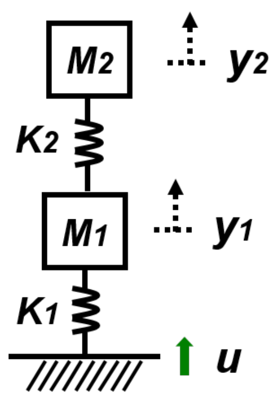
MECH468 Modern Control Engineering
MECH509 Controls

Homework 1. Due: January 29 (Friday), 11:59 pm, 2021.

1 Theoretical questions

Q1. Derive the state-space model for the mass-spring system, where the input u is the displacement of the ground (assume that the ground goes up and down, like a bumpy road), and the outputs are the displacement y_1 and y_2 indicated in the figure.



Q2. Linearize the following nonlinear state-space model around the equilibrium point $x_{1o} = 1$, $x_{2o} = 0$, $u_{1o} = 1$, $u_{2o} = 0$.

$$\begin{aligned} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} &= \begin{bmatrix} -2x_1x_2^2 + x_2u_1 \\ -e^{-x_1u_1}x_2 + x_1u_2 \end{bmatrix} \\ y &= x_1u_1u_2. \end{aligned}$$

Q3. Calculate the matrix exponential e^{At} for the following A matrix.

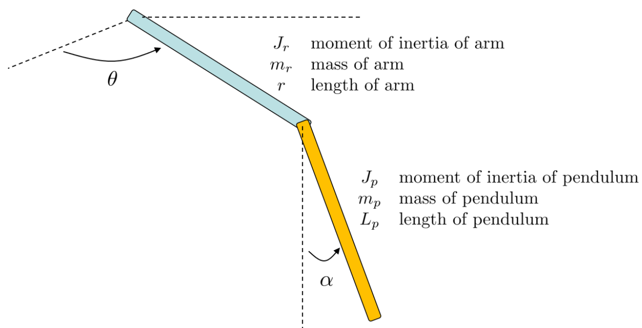
$$A = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

Q4. Discretize the following continuous-time state equation with zero-order-hold, with a sampling time $T > 0$.

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u.$$

2 Matlab question

Consider a rotary pendulum shown below. This system has been taken from https://www.quanser.com/products/qube-servo-2/.



The equations of motion can be written (no derivation is required here) as

$$(J_r + J_p \sin^2 \alpha)\ddot{\theta} + m_p r \ell \cos\alpha \ddot{\alpha} + 2J_p \sin\alpha\cos\alpha\dot{\theta}\dot{\alpha} - m_p r\ell \sin\alpha\dot{\alpha}^2 = \tau - b_r\dot{\theta},$$

$$J_p\ddot{\alpha} + m_p r\ell\cos\alpha\ddot{\theta} - J_p\sin\alpha\cos\alpha\dot{\theta}^2 + m_p g\ell\sin\alpha = -b_p\dot{\alpha},$$

where the notations are indicated in the figure, and $\ell := L_p/2$.

If we approximate the system around $\theta = 0$ and $\alpha = 0$, using $\sin\alpha \approx \alpha$, $\cos\alpha \approx 1$ and $\sin^2\alpha \approx 0$, we can simplify these equations as

$$J_r\ddot{\theta} + m_p r\ell\ddot{\alpha} = \tau - b_r\dot{\theta},$$

$$m_p r\ell\ddot{\theta} + J_p\ddot{\alpha} = -b_p\dot{\alpha} - m_p g\ell\alpha.$$

From these two equations, we can derive

$$\ddot{\theta} = \frac{1}{J_t}\Big\{J_p(\tau - b_r\dot{\theta}) + m_p r\ell(b_p\dot{\alpha} + m_p g\ell\alpha)\Big\}$$

$$\ddot{\alpha} = \frac{1}{J_t}\left\{-J_r(b_p\dot{\alpha} + m_p g\ell\alpha) - m_p r\ell(\tau - b_r\dot{\theta})\right\}$$

where

$$J_t := J_r J_p - (m_p r\ell)^2.$$

By introducing the state variables as

$$x_1 := \theta,\ x_2 := \dot{\theta},\ x_3 := \alpha,\ x_4 := \dot{\alpha},$$

and the input and outputs as

$$u := \tau,\ y_1 := \theta,\ y_2 = \alpha,$$

we can get the state-space model as

$$\begin{aligned} \dot{x} &= Ax + Bu, \\ y &= Cx, \end{aligned}$$

where

$$A := \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -J_p b_r/J_t & (m_p\ell)^2 rg/J_t & m_p r\ell b_p/J_t \\ 0 & 0 & 0 & 1 \\ 0 & m_p r\ell b_r/J_t & -J_r m_p g\ell/J_t & -J_r b_p/J_t \end{bmatrix}, \quad B := \frac{1}{J_t}\begin{bmatrix} 0 \\ J_p \\ 0 \\ -m_p r\ell \end{bmatrix}$$

$$C := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

The parameter values are given in the table below.

$$J_r := \frac{1}{3}m_r r^2, \quad J_p = \frac{1}{3}m_p L_p^2, \quad \ell = \frac{L_p}{2}.$$

Notation	Meaning	Value and unit
m_r	rotary arm mass	0.095 kg
r	rotary arm length	0.085 m
b_r	viscous friction coefficient	0.001 Nms/rad
m_p	pendulum mass	0.024 kg
L_p	pendulum length	0.129 m
b_p	viscous friction coefficient	5×10^{-5} Nms/rad
g	gravitational acceleration	9.81 m/s^2

Task: Using Simulink, simulate for the case when all the initial states are zero except $\alpha(0) = 0.1$ [rad], and with no input. Plot the outputs $\theta(t)$ and $\alpha(t)$. Add your Matlab code(s) (m-file and Simulink block) in your report.