**Problem 1**

**(a)** What performance specifications are useful in selecting a sensor for a specific application? (10 points)

**(b)** An accelerometer is used to measure the displacement of a mechanical system, as follows. The accelerometer signal is sampled at the rate of 1 kHz. The sampled signal is Fourier transformed and the resulting frequency spectrum (complex) is divided by $-\omega^2$ where $\omega$ is the frequency with respect to which the spectrum is expressed.

**1.** What is the purpose of dividing the spectrum by $-\omega^2$? Comment on this approach of measuring displacement. (10 points)

**2.** Explaining and fully justifying your answer, estimate a frequency value up to which the displacement measured by this method would be valid. (20 points)

**Problem 2**

**(a)** Explain the terms "mechanical loading" and "electrical loading" with regard to control system instrumentation.

(10 points)

**(b)** Consider a mechanical system modeled as a mass-spring system of mass $M$ and stiffness $K$. A force $f(t)$ is applied to the system and the displacement $y$ is to be measured. A displacement sensor of mass $m$ is rigidly mounted on the system to measure the displacement, as shown in Figure 1.

**1.** Obtain an expression in the frequency domain for the loading error (as a fraction with respect to the true value) due to the mass of the sensor.

Express your answer in terms of the following parameters:

$p = \dfrac{m}{M} =$ mass ratio

$r = \dfrac{\omega}{\omega_n} =$ nondimensional frequency

where $\omega$ is the frequency variable (in the frequency domain) and $\omega_n = \sqrt{\dfrac{K}{M}}$ is the natural frequency of the system.

(35 points)

**2.** If $p = 0.01$ in what frequency range of operation (measurement) will the loading error be $\leq 1\%$? (15 points)

*Note*: Neglect gravity.

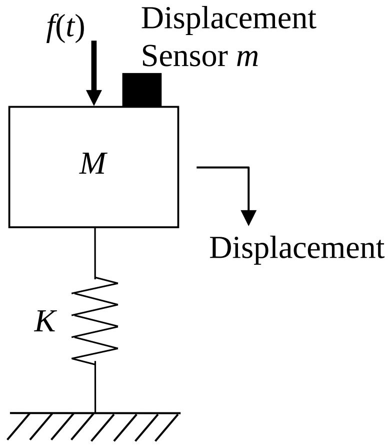


**Figure 1: System with the displacement sensor.**

## Solutions

**Problem 1**

(a)

Useful performance specifications for sensors:

- Sensitivity
- Dynamic range
- Resolution
- Linearity
- Drift (zero, full-scale)
- Useful frequency range of operation
- Bandwidth (and speed of response)
- Impedances (input, output)

(b)

1. Dividing the frequency spectrum by $j\omega$ is equivalent to integrating the corresponding time signal. Accordingly, division by $(j\omega)^2 = -\omega^2$ is equivalent to double-integration of the time signal. The acceleration spectrum is converted into the displacement spectrum by this method.

   This approach has several advantages. Numerical integration (in the time domain) is not used. Hence, the associated numerical errors and reduction of the signal

bandwidth will not enter into the computation. Also, due to the division by-$\omega^2$, high-frequency noise will be filtered out to a great degree.
The main disadvantage of the method is the fact that a sufficient number of data samples will be needed to perform the digital Fourier transform (by FFT) and thereby to compute the frequency spectrum of the measured signal. Hence, the computation is not real-time.

2. Sampling rate = 1000 Hz
   Nyquist frequency = 500 Hz
   In the ideal case, the displacement measurement would be valid up to 500 Hz. But, even if a good anti-aliasing filter is used, some effects of aliasing will be present. As a result, typically only the first 80% of the spectrum is used. Hence, the measurement will be practically valid up to about 400 Hz.

## Problem 2

(a)

**Mechanical Loading:** When an instrument (such as a sensor) is mounted on an object, the mechanical signals (motion, force, etc.) of the object will be changed. This is termed mechanical loading. In the case of a sensor to measure a mechanical signal of the object, then, what is measured is not the true original signal but rather a modified signal. The resulting error is called loading error. Typically, mechanical loading can be reduced by decreasing the inertia and increasing the stiffness of the mounted object.

**Electrical Loading:** When an electrical device (circuit) is connected to an electrical system, the electrical signals (voltage, current, etc.) of the original system will be changed. This is termed electrical loading. In the case of a sensor to measure an electrical signal of the original system, then, what is measured is not the true original signal but rather a modified signal. The resulting error is called loading error. Typically, electrical loading can be reduced by making the input impedance of the connected circuit much larger than the output impedance of the original system.

(b)

1. Equation of motion without the sensor:

$$M\ddot{y} + Ky = f(t)$$

Equation of motion with the sensor:

$$(M+m)\ddot{\tilde{y}} + K\tilde{y} = f(t)$$

In the Laplace domain:

$$y = \frac{f}{Ms^2 + K}$$

$$\tilde{y} = \frac{f}{(M+m)s^2 + K}$$

Hence,

$$\frac{\tilde{y}}{y} = \frac{Ms^2 + K}{(M+m)s^2 + K} \qquad \text{(i)}$$

In the Laplace domain, fractional loading error

$$e = \frac{\tilde{y} - y}{y} = \frac{\tilde{y}}{y} - 1 \qquad \text{(ii)}$$

Substitute (i) in (ii):

$$e = \frac{Ms^2 + K}{(M+m)s^2 + K} - 1 = -\frac{ms^2}{(M+m)s^2 + K}$$

$$= -\frac{m/Ms^2}{(1+m/M)s^2 + K/M} = -\frac{ps^2}{(1+p)s^2 + \omega_n{}^2}$$

In the frequency domain ($s = j\omega$) we have

$$e = \frac{p\omega^2}{-(1+p)\omega^2 + \omega_n{}^2} = \frac{p(\omega/\omega_n)^2}{-(1+p)(\omega/\omega_n)^2 + 1}$$

Or,

$$e = \frac{pr^2}{1-(1+p)r^2} \qquad \text{(iii)}$$

2. Note from (iii) that the error $e$ will be +ve when $1-(1+p)r^2 > 0$, and –ve when $1-(1+p)r^2 < 0$. We must consider both cases because it is the magnitude of the error, not the sign, that matters.
   With $p = 0.01$, for the loading error $e$ not to exceed 0.01 (i.e., 1%), we must have,

   $$\frac{0.01r^2}{1-(1+0.01)r^2} < 0.01 \qquad \text{(iv)}$$

   and

   $$\frac{0.01r^2}{(1+0.01)r^2 - 1} < 0.01 \qquad \text{(v)}$$

   (iv) gives: $r^2 < \frac{1}{2.01}$

   (v) gives: $r^2 > 100$

   Hence, the suitable frequency range of operation is:

   $$\omega < 0.705\omega_n \text{ and } \omega > 10\omega_n$$

***Note*****:** If the system had damping, then the phase difference between $y$ and $\tilde{y}$ should to be taken into account when computing the error. Specifically, the magnitude of the corresponding complex numbers should be used. Except for this, the approach would be the same.