LTI Systems Summary

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1 Signals and Systems

Signals

Functions of time, e.g., $v(t)$, $i(t)$.

Systems

- "Constraints" among signals, e.g., circuit diagram.
- "Mapping" from input signals to output signals, e.g., block diagram.

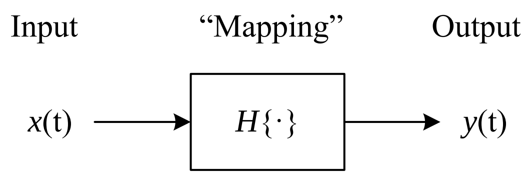

Figure 1: System viewed as an operator that maps $x(t)$ to $y(t)$.

2 System Properties

Memoryless

A system is memoryless, or static, if the output at a certain time only depends on the input at the same time, e.g., $y(t) = x(t)^2, y(t) = ax(t)$.

Linear

A system is linear if the response to a linear combination of inputs is the same linear combination of the individual responses, i.e., superposition holds.

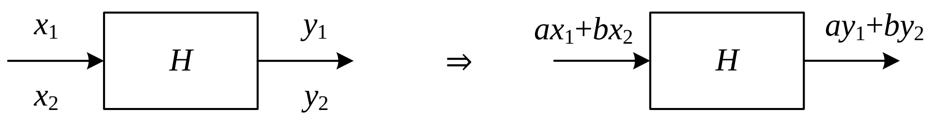

Figure 2: Linear system.

Time-invariant

A system is time-invariant if the response to a time-shifted input is the same response shifted by the same time.

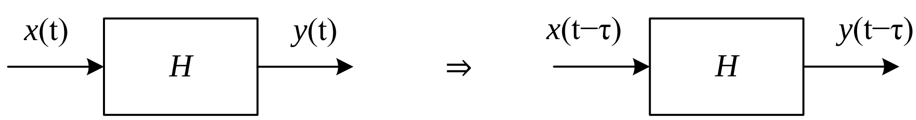

Figure 3: Time-invariant system.

Causal

A system is causal if the output at a certain time does not depend on the future input. That is, the output at time $t = t_o$ only depends on the input during $t \leq t_0$. A linear and time-invariant (LTI) system is causal if and only if the impulse response is

$$h(t) = 0 \quad \text{for} \quad t < 0.$$

Bounded-input Bounded-output (BIBO) Stable

A system is BIBO stable if the response to a bounded input is always bounded.

3 Linear Time-invariant (LTI) System

Delta Function

Delta function $\delta(t)$ has the following properties.

Infinite amplitude:	$\delta(0) \to \infty$
Unity area:	$\int_{-\infty}^{\infty} \delta(t)\,\mathrm{d}x = 1$
Sampling:	$x(t)\delta(t-\tau) = x(\tau)\delta(t-\tau)$

An arbitrary signal $x(t)$ can be expressed as a superposition of delta functions.

$$
\begin{aligned}
x(t) &= x(t) \underbrace{\int_{-\infty}^{\infty} \delta(t-\tau)\,\mathrm{d}\tau}_{1} \\
&= \int_{-\infty}^{\infty} x(t)\delta(t-\tau)\,\mathrm{d}\tau \\
&= \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)\,\mathrm{d}\tau
\end{aligned}
$$

Impulse Response

LTI system can be characterized by its impulse response $h(t)$.

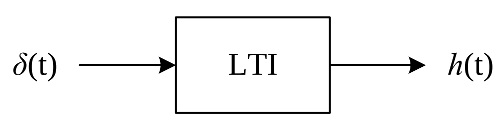

Figure 4: Impulse response.

- Causal system: $h(t) = 0$ for $t < 0$
- BIBO stable system: $\int_{-\infty}^{\infty} |h(t)|\,\mathrm{d}t < \infty$
- Finite impulse response (FIR) system: duration of $h(t)$ is finite
- Infinite impulse response (IIR) system: duration of $h(t)$ is infinite

Convolution

The response $y(t)$ to an arbitrary input $x(t)$ can be obtained via convolution integral.

$$
\begin{aligned}
x(t) &= \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)\,\mathrm{d}\tau \implies \\
y(t) &= \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\,\mathrm{d}\tau
\end{aligned}
$$

Convolution operator ($*$) is commutative.

$$\int_{-\infty}^{\infty} x(\tau)h(t-\tau)\,\mathrm{d}\tau = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)\,\mathrm{d}\tau \quad \Longrightarrow \quad h(t) * x(t) = x(t) * h(t)$$

The dummy variable τ is referred to as a *lag variable*.

Graphical Interpretation for Convolution

Given an input signal $x(t)$ and an impulse response $h(t)$

1) Draw $x(\tau)$ and $h(\tau)$ on the τ-domain.

2) Flip $x(\tau)$ about the origin to draw $x(-\tau)$.

3) Shift $x(-\tau)$ by t along the τ-axis to draw $x(-(\tau - t)) = x(t-\tau)$.

4) Sweep $x(t-\tau)$ over $h(\tau)$ and carry out the integral.

Step Response

The step response $s(t)$, i.e., the response to the unit step function $u(t)$, is equivalent to the integral of the impulse response $h(t)$

$$s(t) = \int_{-\infty}^{\infty} h(\tau)u(\tau - t)\,\mathrm{d}\tau = \int_{-\infty}^{t} h(\tau)\,\mathrm{d}\tau\,.$$

Auto-correlation and Cross-correlation*

(Deterministic) auto-correlation and cross-correlation, often used in signal processing, are related to convolution integral as follows.

$$R_{xx}(\tau) \equiv \int_{-\infty}^{\infty} x(t)x(t-\tau)\,\mathrm{d}t = x(\tau) * x(-\tau)$$
$$R_{xy}(\tau) \equiv \int_{-\infty}^{\infty} x(t)y(t-\tau)\,\mathrm{d}t = x(\tau) * y(-\tau)$$

Note that $x(-\tau)$ is the time-reversed copy of $x(\tau)$.

4 Transfer Function

Eigenvalue and Eigenfunction

- Complex exponential $e^{s_o t}$ is the eigenfunction of LTI systems.
- Drawn in a 3D space consisting of real- imaginary- and time-axes, a complex exponential looks like a spiral along the time axis.
- The response of an LTI system to the input $e^{s_o t}$ is the same complex exponential scaled by a complex number $H(s_o)$. In other words, the complex exponential 'picks up' or 'samples' the gain at the particular frequency s_o.
- This scaling factor $H(s)$ viewed as a function of complex frequency $s \in \mathbb{C}$ is referred to as the *transfer function*.

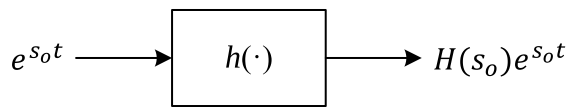

Figure 5: LTI system response to a complex exponential.

Relation to the Laplace Transform

$$\text{for} \quad x(t) = e^{st} \quad \rightarrow \quad y(t) = \int_{-\infty}^{\infty} h(\tau) e^{s(t-\tau)} \, \mathrm{d}\tau$$
$$= e^{st} \underbrace{\int_{-\infty}^{\infty} h(\tau) e^{-s\tau} \, \mathrm{d}\tau}_{H(s)}$$

Transfer function $H(s)$ can be derived from the Laplace transform of $h(t)$, i.e.,

$$H(s) = \int_{-\infty}^{\infty} h(t) e^{-st} \, \mathrm{d}t$$

when the integral converges.

If the system is causal,

$$\int_{-\infty}^{\infty} h(t) e^{-st} \, \mathrm{d}t = \int_{0^-}^{\infty} h(t) e^{-st} \, \mathrm{d}t \, .$$

Laplace Transform Properties

- Laplace transform converts *convolution* to *multiplication*. This allows us to handle a cascade of LTI systems as a product of transfer functions.
- Laplace transform converts *differential equations* to *algebraic equations*. This allows us to use a rational transfer function (e.g., $a(s)/b(s)$) to describe a system.

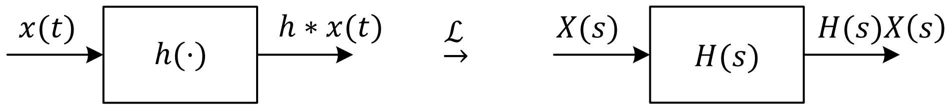

Figure 6: Laplace transform converts convolution to multiplication.

Rational Transfer Function

A rational transfer function $H(s)$ has polynomials for both the numerator and denominator. A counterexample is

$$H(s) = \frac{1}{s+1} e^{-sT}.$$

When a transfer function is non-rational, as above, it cannot be fully characterized with the poles and zeros.

A rational transfer function is *proper* when the order of the denominator is equal to or larger than the order of the numerator. For example,

$$H(s) = \frac{s^2+1}{s^2+s+1}.$$

The frequency response of a proper transfer function is bounded at high frequency. In the above example, $H(s) \to 1$ as $s \to \infty$.

A rational transfer function is *strictly proper* when the order of the denominator is larger than the order of the numerator. For example,

$$H(s) = \frac{s+1}{s^2+s+1}$$

The frequency response of a strictly proper transfer function approaches zero at high frequencies. In the above example, $H(s) \to \frac{1}{s} \to 0$ as $s \to \infty$. This is a common characteristic of physical systems.

Region of Convergence (ROC)*

The set of values $s \in \mathbb{C}$ for which the integral $\int_{-\infty}^{\infty} h(t)e^{-st}\,\mathrm{d}t$ converges.

Inverse Laplace transform*

$$h(t) = \frac{1}{2\pi j} \int_{\sigma - j\infty}^{\sigma + j\infty} H(s) e^{st}\,\mathrm{d}s$$

Here, the integral should be carried along a line that is in parallel with the $j\omega$ axis inside the ROC. In this course, we never carry out the inverse Laplace transform formula. Instead, we use partial fraction expansion and mapping methods to find the corresponding time-domain signal $x(t)$.

5 Frequency Response

Eigenvalue and Eigenfunction

- Complex sinusoid $e^{j\omega_o t}$ is an eigenfunction of LTI systems.
- It is a special case of complex exponential with $s_o = j\omega_o$.
- The response of an LTI system to $e^{j\omega_o t}$ is the same complex sinusoid scaled by a complex number $H(j\omega_o)$. In other words, the complex exponential 'picks up' or 'samples' the gain at the particular frequency ω_o.
- This scaling factor $H(j\omega)$ viewed as a function of frequency $\omega \in \mathbb{R}$ is referred to as *frequency response.*

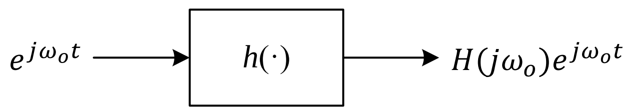

Figure 7: LTI system response to a complex sinusoid.

Relation to the Fourier Transform

Let us find the response to a complex sinusoid via convolution integral:

$$\begin{aligned} \text{for} \quad x(t) = e^{j\omega t} \quad \rightarrow \quad y(t) &= \int_{-\infty}^{\infty} h(\tau) e^{j\omega(t-\tau)}\,\mathrm{d}\tau \\ &= e^{j\omega t} \underbrace{\int_{-\infty}^{\infty} h(\tau) e^{-j\omega\tau}\,\mathrm{d}\tau}_{H(j\omega)} \end{aligned}$$

It shows that frequency response $H(j\omega)$ can be derived from the Fourier transform of $h(t)$

$$H(j\omega) = \int_{-\infty}^{\infty} h(t) e^{-j\omega t}\,\mathrm{d}t\,,$$

when the integral converges. The integral converges if $\int_{-\infty}^{\infty} |h(t)| < \infty$.

The inverse Fourier transform is defined as follows.

$$h(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} H(j\omega) e^{j\omega t}\,\mathrm{d}\omega$$

Note the duality between the Fourier transform and the inverse Fourier transform.

Gain and Phase

Frequency response $H(j\omega)$ in a polar form is

$$H(j\omega) = |H(j\omega)|e^{j\angle H(j\omega)}$$

- $M \equiv |H(j\omega)|$ is the *gain*.
- $\phi \equiv \angle H(j\omega)$ is the *phase*.

In terms of the gain and magnitude, the response of $H(s)$ to an input complex sinusoid is

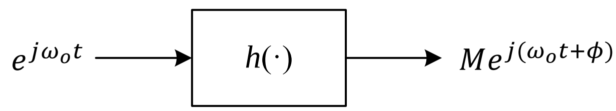

Figure 8: LTI system response to a complex sinusoid $e^{j\omega_o t}$.

The response of $H(s)$ to an input real sinusoid can be derived by taking real parts of the input and output – Euler's formula $e^{j\omega_o t} = \cos(\omega_o t) + j\sin(\omega_o t)$.

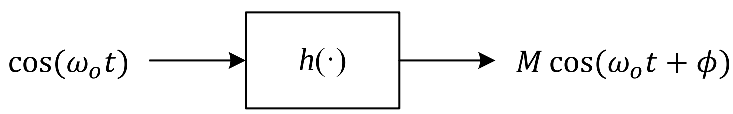

Figure 9: LTI system response to a real sinusoid $\cos(\omega_o t)$.

Bode's Gain-phase Relation for Minimum Phase Systems

- For a given gain curve, there exist multiple systems that differ in phase curves.
- Among those, the one that has the smallest phase lag is called *minimum phase system*.
- For minimum phase systems, there exists a unique relation between the gain curve and phase curve (i.e., $\phi \approx n \times 90°$, where n is the slope of the Bode gain curve).
- For non-minimum phase systems, you need to be careful in drawing the Bode plot.

Spectrum and Frequency Response

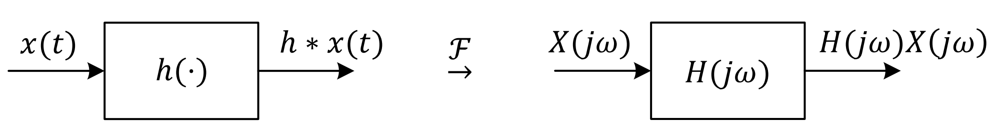

Figure 10: Fourier transform converts convolution to multiplication.

- $X(j\omega)$ is referred to as the spectrum of the signal.
- $H(j\omega)$ is referred to as the frequency response of the system.

Fourier Transform Properties

- Converts *convolution* in time domain to *multiplication* in frequency domain.
  This allows us to handle a cascade of LTI systems as a product of frequency responses.

- Converts *multiplication* in time domain to *convolution* in frequency domain.
  This allows us to visualize signal processing techniques, such as modulation and sampling, in frequency domain.

6 Why Frequency-domain Approaches?

It simplifies the math

- Convolution $\rightarrow$ Multiplication
- Differential equation $\rightarrow$ Algebraic equation

There exist nice graphical representations

- Pole-zero map for $H(s)$ (only for rational $H(s)$)
- Bode plot for $H(j\omega)$ (for a wider class of systems)

There exist nice design tools based on the graphical representations

- Root locus with pole-zero map (difficult when a system has too many poles and zeros)
- Loop shaping with Bode plot (for a wider class of systems)
- We can measure $H(j\omega)$ and use it for loop shaping design (even for unstable systems).

References

[1] A. V. Oppenheim and A. S. Willsky, *Signals & Systems*, 2nd ed. Upper Saddle River, NJ: Prentice Hall, Aug. 1996.

Table 1: Fourier Transform – Selected Properties [1, p.328]

Property	Signal	Fourier Transform
	$x(t)$	$X(j\omega)$
	$y(t)$	$Y(j\omega)$
Linearity	$ax(t) + by(t)$	$aX(j\omega) + bY(j\omega)$
Scaling	$x(at)$	$\frac{1}{\lvert a\rvert} X\left(\frac{j\omega}{a}\right)$
Time shifting	$x(t - t_0)$	$e^{-j\omega t_0} X(j\omega)$
Frequency shifting	$e^{j\omega_0 t} x(t)$	$X(j(\omega - \omega_0))$
Convolution	$\int_{-\infty}^{\infty} x(\tau) y(t - \tau)\,\mathrm{d}\tau$	$X(j\omega) Y(j\omega)$
Multiplication	$x(t) y(t)$	$\frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\theta) Y(j(\omega - \theta))\,\mathrm{d}\theta$
Differentiation in time	$\frac{\mathrm{d}}{\mathrm{d}t} x(t)$	$j\omega X(j\omega)$
Differentiation in frequency	$tx(t)$	$j \frac{\mathrm{d}}{\mathrm{d}\omega} X(j\omega)$
Parseval's theorem	$\int_{-\infty}^{\infty} \lvert x(t)\rvert^2\,\mathrm{d}t = \frac{1}{2\pi} \int_{-\infty}^{\infty} \lvert X(j\omega)\rvert^2\,\mathrm{d}\omega$	

Table 2: Laplace Transform – Selected Properties [1, p.691]

Property	Signal	Laplace Transform	ROC
	$x(t)$	$X(s)$	R
	$x_1(t)$	$X_1(s)$	R_1
	$x_2(t)$	$X_2(s)$	R_2
Linearity	$ax_1(t) + bx_2(t)$	$aX_1(s) + bX_2(s)$	At least $R_1 \cap R_2$
Time shifting	$x(t - t_0)$	$e^{-st_0} X(s)$	R
Frequency shifting	$e^{s_0 t} x(t)$	$X(s - s_0)$	Shifted version of R
Time scaling	$x(at)$	$\frac{1}{\lvert a\rvert} X\left(\frac{s}{a}\right)$	Scaled version of R
Convolution	$x_1(t) * x_2(t)$	$X_1(s) X_2(s)$	At least $R_1 \cap R_2$
Differentiation in time	$\frac{\mathrm{d}}{\mathrm{d}t} x(t)$	$sX(s)$	At least R
Differentiation in frequency	$-tx(t)$	$\frac{\mathrm{d}}{\mathrm{d}s} X(s)$	R
Integration in time	$\int_{-\infty}^{t} x(\tau)\,\mathrm{d}\tau$	$\frac{1}{s} X(s)$	At least $R \cap \{\Re\{s\}\}$
Initial-value theorem	If $x(t)$ is causal and does not have singularities at $t = 0$ $x(0^+) = \lim_{s \to \infty} sX(s)$		
Final-value theorem	If $x(t)$ is causal and has finite limit as $t \to \infty$ $\lim_{t \to \infty} x(t) = \lim_{s \to \infty} sX(s)$		