MECH468 : Modern Control Engineering
MECH509 : Controls

L12 : Decomposition for controllability

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Zoom lecture to be recorded and posted on Canvas

Course plan

Topics	CT	DT
Modeling	✓	✓
Stability	✓	✓
→ Controllability/observability		
Realization		
State feedback/observer		
LQR/Kalman filter		

Coordinate transformation (review)

- System $\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) + Du(t) \end{cases}$

- Coordinate transformation

$$z(t) := Tx(t) \quad T\text{: any nonsingular matrix}$$

$$\Rightarrow \begin{cases} \dot{z}(t) = TAT^{-1}z(t) + TBu(t) \\ y(t) = CT^{-1}z(t) + Du(t) \end{cases}$$

  - Does not change transfer function, stability, controllability, observability.
  - Can be used to clarify the structure, and to improve numerical property.

Decomposition for controllability (review)

- If (A,B) is not controllable with $\text{rank}\mathcal{C} = m < n$ then there exists a coordinate transformation (i.e., nonsingular T) that *decomposes* states into controllable part and the uncontrollable part:

$$\left.\begin{array}{l} \dot{x}(t) = Ax(t) + Bu(t) \\ \begin{bmatrix} z_c(t) \\ z_{\bar{c}}(t) \end{bmatrix} := Tx(t) \end{array}\right\} \longrightarrow \begin{bmatrix} \dot{z}_c(t) \\ \dot{z}_{\bar{c}}(t) \end{bmatrix} = \underbrace{\begin{bmatrix} A_c & A_{12} \\ 0 & A_{\bar{c}} \end{bmatrix}}_{TAT^{-1}} \begin{bmatrix} z_c(t) \\ z_{\bar{c}}(t) \end{bmatrix} + \underbrace{\begin{bmatrix} B_c \\ 0 \end{bmatrix}}_{TB} u(t)$$

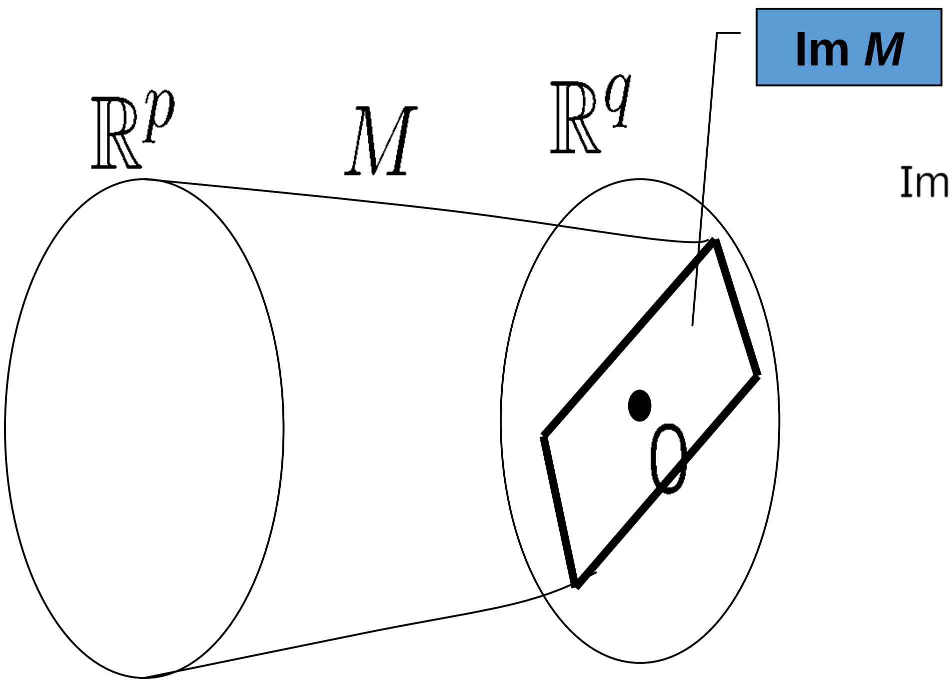
$$\boxed{A_c \in \mathbb{R}^{m \times m}} \qquad \boxed{(A_c, B_c) \text{ is controllable}}$$

Why is decomposition important?

- We can see what is not possible by using control input u. (We cannot affect the uncontrollable part at all.)
- If uncontrollable part is unstable, then we cannot stabilize the system by feedback. (We will learn this in more detail later.)
- This may suggest addition of actuators, or change of plant parameters and actuator locations.
- Next, how to find T?

Image space (space spanned by the column vectors)

For a matrix *M* (*q*-by-*p*): $\mathrm{Im}M := \{y \in \mathbb{R}^q : y = Mx \text{ for some } x \in \mathbb{R}^p\}$



$$M = \begin{bmatrix} 1 & 0 & 2 \\ -2 & 0 & -4 \end{bmatrix}$$

$$\begin{aligned} \mathrm{Im}M &:= \left\{Mx : x \in \mathbb{R}^3\right\} \\ &= \left\{x_1 \begin{bmatrix} 1 \\ -2 \end{bmatrix} + x_3 \begin{bmatrix} 2 \\ -4 \end{bmatrix}, x_1, x_3 \in \mathbb{R}\right\} \\ &= \left\{(x_1 + 2x_3) \begin{bmatrix} 1 \\ -2 \end{bmatrix}, x_1, x_3 \in \mathbb{R}\right\} \\ &= \left\{\alpha \underline{\begin{bmatrix} 1 \\ -2 \end{bmatrix}}, \alpha \in \mathbb{R}\right\} \end{aligned}$$

A basis of Im *M*

How to find *T*?

- We use *image space* of controllability matrix.

$$T^{-1} := [T_c, T_{\bar{c}}] \quad \begin{cases} T_c & : \text{A basis of } \underline{\mathrm{Im}\,\mathcal{C}} \\ T_{\bar{c}} & : \text{ any complement of } T_c \text{ in } \mathbb{R}^n \end{cases}$$

Controllable subspace

Ex. $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$$\mathcal{C} = [B, AB] = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \Rightarrow \text{rank}\,\mathcal{C} = 1 < 2$$

Uncontrollable!

$$\Rightarrow T^{-1} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \Rightarrow TAT^{-1} = \left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & 1 \end{array}\right] \quad TB = \left[\begin{array}{c} 1 \\ \hline 0 \end{array}\right]$$

$$\underbrace{}_{T_c}\ \underbrace{}_{T_{\bar{c}}}$$

Another example

- Ex $A = \begin{bmatrix} 2 & 0 & 0 \\ 2 & 2 & 2 \\ 3 & 0 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$

- Controllability matrix

Uncontrollable!

$$\mathcal{C} = \begin{bmatrix} B, AB, A^2B \end{bmatrix} = \begin{bmatrix} 1 & 2 & 4 \\ -2 & 0 & 8 \\ 1 & 2 & 4 \end{bmatrix} \Rightarrow \text{rank}\mathcal{C} = 2 < 3$$

- Transformation matrix

$$T^{-1} = \Big[\underbrace{\begin{matrix} 1 & 1 \\ -2 & 0 \\ 1 & 1 \end{matrix}}_{T_c} \; \underbrace{\begin{matrix} 1 \\ 0 \\ 0 \end{matrix}}_{T_{\bar{c}}} \Big] \quad \Rightarrow \quad TAT^{-1} = \left[\begin{array}{cc:c} 0 & -2 & -1 \\ 2 & 4 & 4 \\ \hdashline 0 & 0 & -1 \end{array}\right] \quad TB = \left[\begin{array}{c} 1 \\ 0 \\ \hdashline 0 \end{array}\right]$$

Matlab command “ctrbf.m”

```
>> help ctrbf
 CTRBF  Controllability staircase form.

    [ABAR,BBAR,CBAR,T,K] = CTRBF(A,B,C) returns a decomposition
    into the controllable/uncontrollable subspaces.

    If Co=CTRB(A,B) has rank r <= n = SIZE(A,1), then there is a
    similarity transformation T such that

       Abar = T * A * T' ,  Bbar = T * B ,  Cbar = C * T'

    and the transformed system has the form

                | Anc    0 |           | 0 |
       Abar =  ----------  ,  Bbar =  ---  ,  Cbar = [Cnc| Cc].
                | A21   Ac |           |Bc |
                                               -1           -1
    where (Ac,Bc) is controllable, and Cc(sI-Ac)Bc = C(sI-A)B.
```

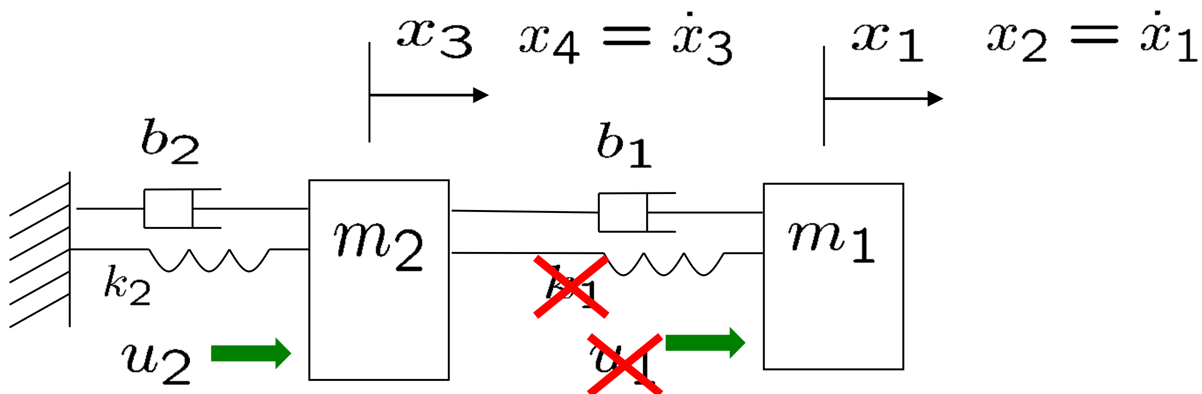
Note the reverse order of the states!

Matlab command for “Another example”

```
>> A=[2 0 0; 2 2 2 ; 3 0 -1];
>> B=[1 -2 1]';
>> C=[1 0 0];
>> [ABAR,BBAR,CBAR,T] = ctrbf(A,B,C)
ABAR =
   -1.0000   -0.0000   -0.0000
   -2.4495    3.3333    0.9428
   -1.7321   -1.8856    0.6667
BBAR =
    0.0000
   -0.0000
   -2.4495
CBAR =
    0.7071   -0.5774   -0.4082
T =
    0.7071         0   -0.7071
   -0.5774   -0.5774   -0.5774
   -0.4082    0.8165   -0.4082
```

Mechanical example

- Mass-spring-damper with $k_1 = u_1 = 0$



$$\dot{x}(t) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \cancel{-k_1/m_1} & -b_1/m_1 & \cancel{k_1/m_1} & b_1/m_1 \\ 0 & 0 & 0 & 1 \\ \cancel{k_1/m_2} & b_1/m_2 & -(\cancel{k_1}+k_2)/m_2 & -(b_1+b_2)/m_2 \end{bmatrix} x(t) + \begin{bmatrix} 0 & 0 \\ 1/m_1 & 0 \\ 0 & 0 \\ 0 & 1/m_2 \end{bmatrix} u(t)$$

Mechanical example (cont'd)

- Controllability matrix $\mathcal{C} = \begin{pmatrix} 0 & 0 & 1 & -3 \\ 0 & 1 & -3 & 7 \\ 0 & 1 & -2 & 4 \\ 1 & -2 & 4 & -9 \end{pmatrix}$

$$\text{rank}\mathcal{C} = 3$$

- Coordinate transformation matrix

$$T^{-1} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & -3 & 1 \\ 0 & 1 & -2 & 0 \\ 1 & -2 & 4 & 0 \end{pmatrix} \quad T = \begin{pmatrix} 0 & 0 & 2 & 1 \\ 2 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & -1 & 0 \end{pmatrix}$$

$$\Rightarrow TAT^{-1} = \left(\begin{array}{ccc|c} 0 & 0 & -1 & 1 \\ 1 & 0 & -2 & 2 \\ 0 & 1 & -3 & 1 \\ \hline 0 & 0 & 0 & 0 \end{array}\right) \quad TB = \left(\begin{array}{c} 1 \\ 0 \\ 0 \\ \hline 0 \end{array}\right)$$

Remarks

- Rank of controllability matrix indicates the number of controllable new states (*z*).
- In the example, z_4 is the uncontrollable state. In fact, z_4 is constant for any input, since $\dot{z}_4(t) = 0$
- This means that, in the original state (*x*)

$$z_4(t) = x_1(t) + x_2(t) - x_3(t) \text{ is constant}$$

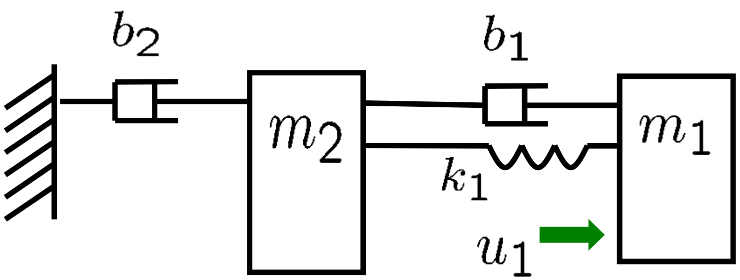
Position of mass 1

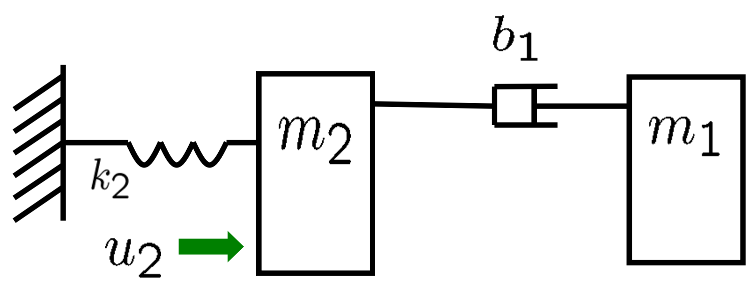
Position of mass 2

Velocity of mass 1

Exercise

- Using Matlab, for the following two systems, compute *T* which decomposes states into controllable and uncontrollable parts, and figure out which state is not controllable (Set all the parameters (*k*,*b*,*m*) to be one.)





Summary

- Decomposition for controllability
  - Image space
  - Controllable subspace
  - Examples
  - Matlab command for decomposition
- Next, observability