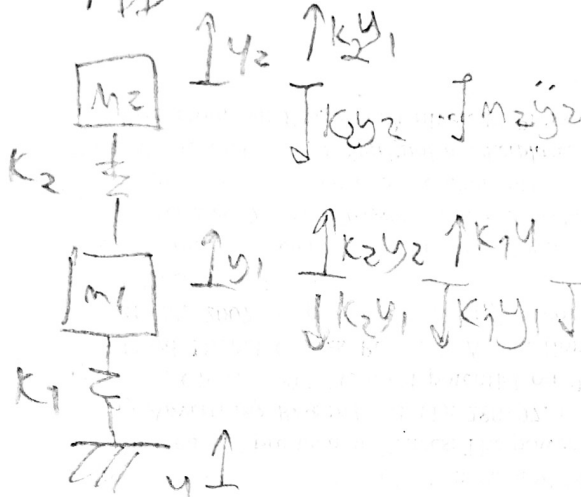


Raghamnon Prakitpong #63205165

Q1

FBD



force balance eq'n

$$\Rightarrow 0 = m_2 \ddot{y}_2 + k_2 y_2 - k_2 y_1$$

$$\Rightarrow 0 = m_1 \ddot{y}_1 + k_1 y_1 + k_2 y_1 - k_2 y_2 - k_1 y_1$$

State Space

$$[x] = \begin{bmatrix} y_1 \\ y_2 \\ \dot{y}_1 \\ \dot{y}_2 \end{bmatrix} \quad [\dot{x}] = \begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \\ \ddot{y}_1 \\ \ddot{y}_2 \end{bmatrix} \quad [u] = [u]$$

$$[y] = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$[\dot{x}] = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ k_2 & -k_2 & 0 & 0 \\ -(k_2 + k_1) & k_2 & 0 & 0 \end{bmatrix} [x] + \begin{bmatrix} 0 \\ 0 \\ 0 \\ k_1 \end{bmatrix} [u]$$

$$[y] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} [x]$$

Q2  $f = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix}$   $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$   $u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$   $x_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$   $u_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\frac{\partial f}{\partial x} = \begin{bmatrix} -2x_2 \\ (-u_1)(-e^{-x_1 u_1})x_2 + u_2 \end{bmatrix}$   $y = [y]$

$\left. \frac{\partial f}{\partial x} \right|_{x=x_0, u=u_0} = \begin{bmatrix} -2(0)^2 & -1(1)(-e^{-1(0)})(0) + 1 \\ -1(e^{-1})0 + 0 & -e^{-1(0)} \end{bmatrix}$

$= \begin{bmatrix} 0 & 1 \\ 0 & -\frac{1}{e} \end{bmatrix}$

$\frac{\partial f}{\partial u} = \begin{bmatrix} x_2 & 0 \\ -x_1(-e^{-x_1 u_1})x_2 & x_1 \end{bmatrix}$

$\left. \frac{\partial f}{\partial u} \right|_{x=x_0, u=u_0} = \begin{bmatrix} 0 & 0 \\ -(1)(-e^{-1(0)})0 & 1 \end{bmatrix}$

$= \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

$[\dot{x}] = \begin{bmatrix} 0 & 1 \\ 0 & -1/e \end{bmatrix} [x] + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} [u]$

$\frac{\partial y}{\partial x} = \begin{bmatrix} u_1 u_2 & 0 \end{bmatrix}$

$\left. \frac{\partial y}{\partial x} \right|_{x=x_0, u=u_0} = \begin{bmatrix} 1(0) & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \end{bmatrix}$

$$\frac{\partial y}{\partial u} = \begin{bmatrix} x_1 & u_1 \end{bmatrix}$$

$$\frac{\partial y}{\partial u} \bigg|_{x=x_0, u=u_0} = \begin{bmatrix} 1(0) & 1(1) \end{bmatrix} = \begin{bmatrix} 0 & 1 \end{bmatrix}$$

$$[y] = \begin{bmatrix} 0 & 1 \end{bmatrix} [u]$$

Q3.  $A = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$

$A^2 = \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix}$

$A^3 = \begin{bmatrix} -1 & 3 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -8 \end{bmatrix}$

$A^k = \begin{bmatrix} (-1)^k & k(-1)^{k-1} & 0 \\ 0 & (-1)^k & 0 \\ 0 & 0 & (-2)^k \end{bmatrix}$

$$e^{At} = I + At + \frac{(At)^2}{2!} + \dots = \begin{bmatrix} e^{-t} & te^{-t} & 0 \\ 0 & e^{-t} & 0 \\ 0 & 0 & e^{-2t} \end{bmatrix}$$

$e^{-t} = 1 - t + \frac{t^2}{2!} - \frac{t^3}{3!} + \dots$

$e^{-2t} = 1 - 2t + \frac{4t^2}{2!} - \dots$

$te^{-t} = 0 + t - \frac{2t^2}{2!} + \frac{3t^3}{3!} - \frac{4t^4}{4!} + \dots = t - \frac{t^2}{1!} + \frac{t^3}{2!} - \frac{t^4}{3!} + \dots$

$= \sum_{h=0}^{\infty} \frac{t^{h+1}}{(h+1)!} = te^{-t}$

Q4

$$\dot{x}(t) \begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix} x(t) \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \rightarrow x[k+1] = A_d x[k] + B_d u[k]$$

$\underbrace{\quad}_A \quad \underbrace{\quad}_B \quad \downarrow A_d = e^{AT} \quad \downarrow B_d = \left( \int_0^T e^{A(T-\tau)} d\tau \right) B$

$$A_d = e^{AT} = e^{AT} \text{ where } t = T \text{ (sampling time)}$$

$$\text{find eig of } A \rightarrow 0 = \det(1I - A) = \det \begin{pmatrix} \lambda & -1 \\ 6 & \lambda+5 \end{pmatrix}$$

$$0 = \lambda(\lambda+5) + 6$$

$$= \lambda^2 + 5\lambda + 6$$

$$= (\lambda+3)(\lambda+2)$$

$$\lambda = -2, -3$$

distinct ✓

$$\text{eigvals } \lambda_1 = -2 \rightarrow \begin{pmatrix} -2 & -1 \\ 6 & 3 \end{pmatrix} x_1 = 0 \rightarrow x_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

$$\lambda_2 = -3 \rightarrow \begin{pmatrix} -3 & -1 \\ 6 & 2 \end{pmatrix} x_2 = 0 \rightarrow x_2 = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$$

$T = \begin{bmatrix} 1 & 1 \\ -2 & -3 \end{bmatrix}$

$$A_d = e^{AT} = T e^{Pt} T^{-1} = \begin{bmatrix} 1 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} e^{-2t} & 0 \\ 0 & e^{-3t} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -2 & -3 \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} e^{-2t} & e^{-3t} \\ -2e^{-2t} & -3e^{-3t} \end{bmatrix} \begin{bmatrix} -3 & -1 \\ 2 & 1 \end{bmatrix} \frac{1}{-3+2}$$

$$= \begin{bmatrix} -3e^{-2t} + 2e^{-3t} & -e^{-2t} + e^{-3t} \\ 6e^{-2t} - 6e^{-3t} & 2e^{-2t} - 3e^{-3t} \end{bmatrix} (-1) \rightarrow \text{substitute } t \text{ for } T \text{ (sampling time)}$$

$$= e^{-2T} \begin{bmatrix} 3 & 1 \\ -6 & -2 \end{bmatrix} + e^{-3T} \begin{bmatrix} -2 & -1 \\ 6 & 3 \end{bmatrix}$$

$$Bd = \left( \int_0^T e^{A\tau} d\tau \right) \cdot B$$

$$= \left( \int_0^T e^{-2\tau} \begin{bmatrix} 3 & 1 \\ -6 & -2 \end{bmatrix} + e^{-3\tau} \begin{bmatrix} -2 & -1 \\ 6 & 3 \end{bmatrix} d\tau \right) \cdot B$$

$$= \left( \frac{e^{-2\tau}}{-2} \begin{bmatrix} 3 & 1 \\ -6 & -2 \end{bmatrix} + \frac{e^{-3\tau}}{-3} \begin{bmatrix} -2 & -1 \\ 6 & 3 \end{bmatrix} \right) \Bigg|_0^T \cdot B$$

$$= \left( \frac{e^{-2T} - 1}{-2} \begin{bmatrix} 3 & 1 \\ -6 & -2 \end{bmatrix} + \frac{e^{-3T} - 1}{-3} \begin{bmatrix} -2 & -1 \\ 6 & 3 \end{bmatrix} \right) \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= \frac{e^{-2T} - 1}{-2} \begin{bmatrix} 1 \\ -2 \end{bmatrix} + \frac{e^{-3T} - 1}{-3} \begin{bmatrix} -1 \\ 3 \end{bmatrix}$$

MATLAB section:

I wrote the variables in Matlab then load it into Simulink.

```
q2_vars.m
1 - mr = 0.095;
2 - r = 0.085;
3 - br = 0.001;
4 - mp = 0.024;
5 - Lp = .129;
6 - bp = .00005;
7 - g = 9.81;
8 - Jr = mr*r*r/3;
9 - Jp = mp*Lp*Lp/3;
10 - l = Lp/2;
11 - Jt = Jr*Jp - (mp*r*l)^2;
```

Fig 1: Matlab code

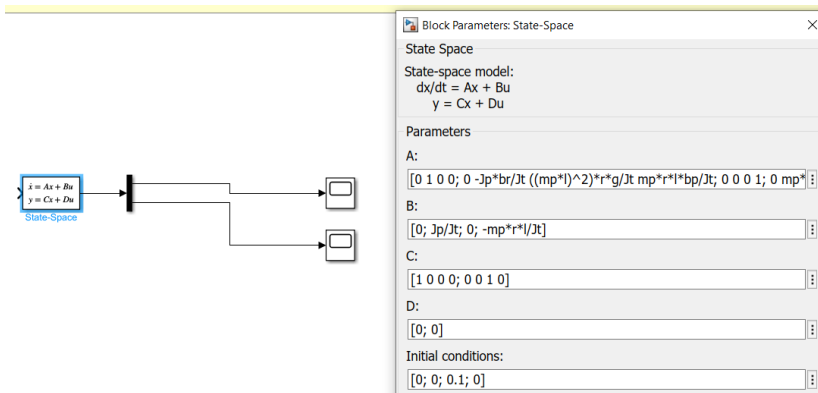


Fig 2: Simulink model

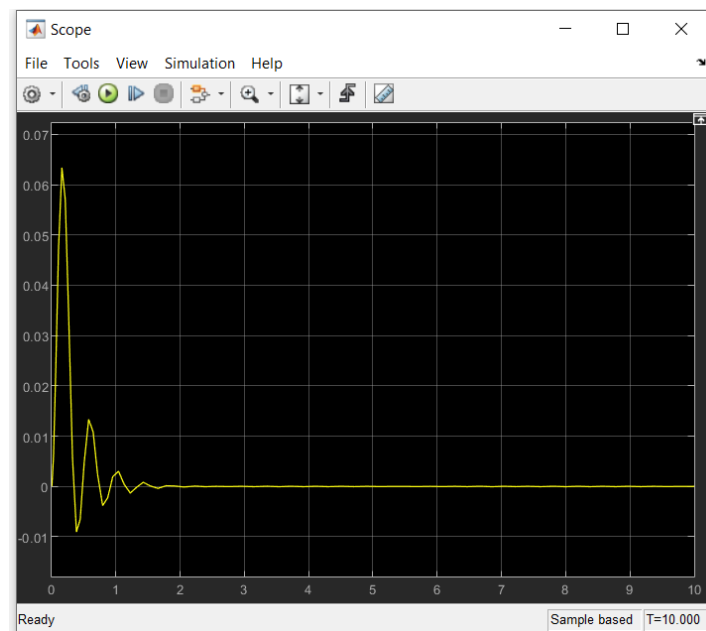


Fig 3: Angle over time

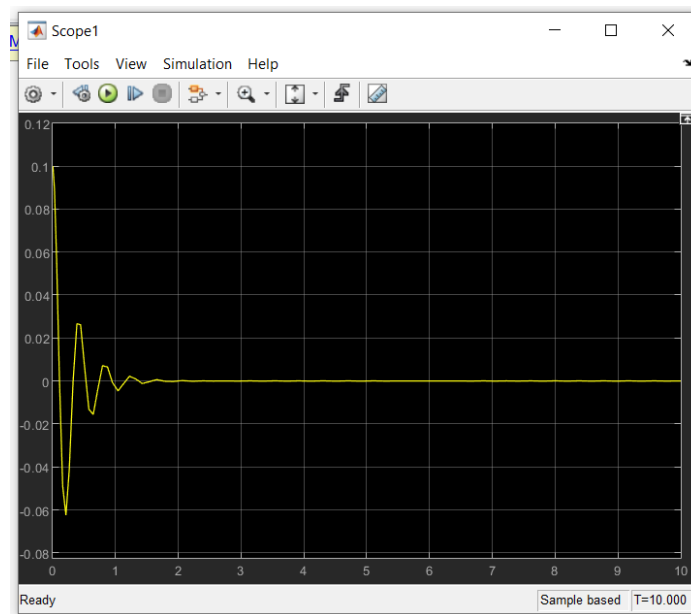


Fig 4: Speed over time