MECH468 Modern Control Engineering
MECH509 Controls

Homework 2. Due: February 15 (Monday), 11:59 pm, 2021.

1 Theoretical (hand-calculation) questions

Let us consider the following continuous-time system

$$\begin{aligned} \dot{x} &= \begin{bmatrix} 0 & 1 & 0 \\ -1 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} u, \\ y &= \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} x. \end{aligned}$$

1. Check if the system is BIBO stable.

   Hint: For a block-diagonal matrix $M = \begin{bmatrix} M_1 & 0 \\ 0 & M_2 \end{bmatrix}$, $M^{-1} = \begin{bmatrix} M_1^{-1} & 0 \\ 0 & M_2^{-1} \end{bmatrix}$.

   Solution: The transfer function from u to y is obtained by

$$\begin{aligned} C(sI-A)^{-1}B &= \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \left(sI - \begin{bmatrix} 0 & 1 & 0 \\ -1 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right)^{-1} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} s & -1 & 0 \\ 1 & s+2 & 0 \\ 0 & 0 & s \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} \frac{s+2}{(s+1)^2} & \frac{1}{(s+1)^2} & 0 \\ -\frac{1}{(s+1)^2} & \frac{s}{(s+1)^2} & 0 \\ 0 & 0 & 1/s \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{(s+1)^2} \\ \frac{s}{(s+1)^2} \\ 0 \end{bmatrix} = \frac{1}{s+1}. \end{aligned}$$

   Since the pole location is $s=-1$ which is in the open left-half plane, this system is BIBO stable.

2. Check if the system is asymptotically stable, marginally stable, or unstable.

Hint: For a block-diagonal matrix $M = \begin{bmatrix} M_1 & 0 \\ 0 & M_2 \end{bmatrix}$, the set of eigenvalues of M consists of the set of eigenvalues of M_1 and the set of eigenvalues of M_2.

Solution: The eigenvalues of the A-matrix are -1, -1 and 0. Therefore, the system is marginally stable.

3. Check the controllability.

Solution: The controllability matrix is

$$\mathcal{C} = \begin{bmatrix} 0 & 1 & -2 \\ 1 & -2 & 3 \\ 0 & 0 & 0 \end{bmatrix}.$$

Thus, rank$\mathcal{C} = 2$ which is less than 3, and the system is not controllable.

4. Check the observability.

Solution: The observability matrix is

$$\mathcal{O} = \begin{bmatrix} 1 & 1 & 1 \\ -1 & -1 & 0 \\ 1 & 1 & 0 \end{bmatrix}.$$

Thus, rank$\mathcal{O} = 2$ which is less than 3, and the system is not observable.

5. Obtain the Kalman decomposition.

Solution: The controllable subspace and the unobservable subspace are respectively

$$\mathrm{Im}\mathcal{C} = \mathrm{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

$$\mathrm{Ker}\mathcal{O} = \mathrm{span}\left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \right\}.$$

Thus, we can obtain the coordinate transformation matrix T^{-1} as

$$T^{-1} = \begin{bmatrix} T_{co} & T_{c\bar{o}} & T_{\bar{c}o} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

From this T^{-1}, we can get T and coordinate transformations as follows.

$$T = \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$TAT^{-1} = \begin{bmatrix} -1 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \; TB = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \; CT^{-1} = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix}.$$

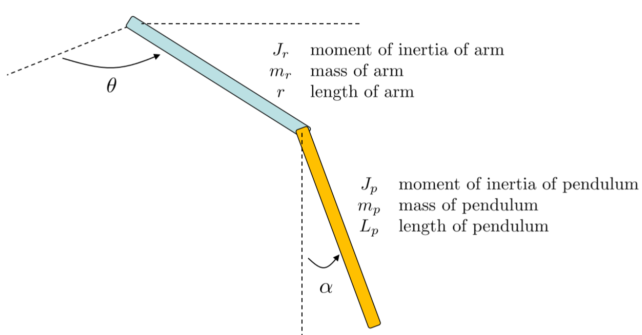
Thus, the Kalman decomposition of the original system becomes

$$\begin{aligned} \frac{d}{dt}\begin{bmatrix} z_{co} \\ z_{c\bar{o}} \\ z_{\bar{c}o} \end{bmatrix} &= \begin{bmatrix} -1 & \color{red}{0} & 0 \\ 1 & -1 & 0 \\ \color{red}{0} & \color{red}{0} & 0 \end{bmatrix}\begin{bmatrix} z_{co} \\ z_{c\bar{o}} \\ z_{\bar{c}o} \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \\ \color{red}{0} \end{bmatrix} u \\ y &= \begin{bmatrix} 1 & \color{red}{0} & 1 \end{bmatrix}\begin{bmatrix} z_{co} \\ z_{c\bar{o}} \\ z_{\bar{c}o} \end{bmatrix} \end{aligned}$$

(Note: The red-colored 0's are due to the structure of Kalman decomposition. Although the selection of T is not unique, those 0's must be 0 regardless of your selection of T.)

2 Matlab question

Consider a rotary pendulum shown below. This system has been taken from https://www.quanser.com/products/qube-servo-2/. All the equations and parameter values were given in HW1.



In HW1, we derived the linearized model for the pendulum system, i.e., around

$$\theta = 0,\ \dot{\theta} = 0,\ \alpha = 0,\ \dot{\alpha} = 0.$$

Task:

1. By hand-calculation, derive the linearized model for the inverted pendulum system, i.e., around

$$\theta = 0,\ \dot{\theta} = 0,\ \alpha = \pi,\ \dot{\alpha} = 0.$$

Hint: See HW1 for the derivation of the linearized model for the pendulum system, and think how to modify it.

Solution: The equations of motion can be written (no derivation is required here) as

$$(J_r + J_p \sin^2 \alpha)\ddot{\theta} + m_p r\ell \cos\alpha\ddot{\alpha} + 2J_p \sin\alpha\cos\alpha\dot{\theta}\dot{\alpha} - m_p r\ell \sin\alpha\dot{\alpha}^2 = \tau - b_r\dot{\theta},$$

$$J_p\ddot{\alpha} + m_p r\ell\cos\alpha\ddot{\theta} - J_p\sin\alpha\cos\alpha\dot{\theta}^2 + m_p g\ell\sin\alpha = -b_p\dot{\alpha},$$

where the notations are indicated in the figure, and $\ell := L_p/2$.

If we approximate the system around $\theta = 0$ and $\alpha = \pi$, using the Taylor series expansion:

$$\begin{aligned}
\sin\alpha &\approx \sin\pi + \cos\pi \times (\alpha - \pi) = -(\alpha - \pi), \\
\cos\alpha &\approx \cos\pi - \sin\pi \times (\alpha - \pi) = -1, \\
\sin^2\alpha &\approx \sin^2\pi + 2\sin\pi\cos\pi \times (\alpha - \pi) = 0,
\end{aligned}$$

we can simplify these equations as

$$J_r\ddot{\theta} - m_p r\ell\ddot{\alpha} = \tau - b_r\dot{\theta},$$

$$-m_p r\ell\ddot{\theta} + J_p\ddot{\alpha} = -b_p\dot{\alpha} + m_p g\ell(\alpha - \pi).$$

From these two equations, we can derive

$$\ddot{\theta} = \frac{1}{J_t}\left\{J_p(\tau - b_r\dot{\theta}) - m_p r\ell(b_p\dot{\alpha} - m_p g\ell(\alpha - \pi))\right\}$$

$$\ddot{\alpha} = \frac{1}{J_t}\left\{-J_r(b_p\dot{\alpha} - m_p g\ell(\alpha - \pi)) + m_p r\ell(\tau - b_r\dot{\theta})\right\}$$

where

$$J_t := J_r J_p - (m_p r \ell)^2.$$

By introducing the state variables as

$$x_1 := \theta,\ x_2 := \dot{\theta},\ x_3 := \alpha - \pi,\ x_4 := \dot{\alpha},$$

and the input and outputs as

$$u := \tau,\ y_1 := \theta,\ y_2 = \alpha,$$

we can get the state-space model as

$$\begin{aligned} \dot{x} &= Ax + Bu, \\ y &= Cx, \end{aligned}$$

where

$$A := \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -J_p b_r/J_t & (m_p \ell)^2 r g/J_t & -m_p r \ell b_p/J_t \\ 0 & 0 & 0 & 1 \\ 0 & -m_p r \ell b_r/J_t & J_r m_p g \ell/J_t & -J_r b_p/J_t \end{bmatrix}, \quad B := \frac{1}{J_t}\begin{bmatrix} 0 \\ J_p \\ 0 \\ m_p r \ell \end{bmatrix}$$

$$C := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

2. Using Simulink and the linearized model, simulate for the case when all the initial states are zero except $\alpha(0) = \pi + 0.1$ [rad], and with no input. Plot the outputs $\theta(t)$ and $\alpha(t)$.

3. For both (pendulum and inverted pendulum) linearized systems, compute the eigenvalues of A-matrices and determine the internal stability.

Attach your Matlab code(s) (m-file and Simulink block) in your report.

<u>Solution:</u>

- Pendulum's linearized model (HW1) is marginally stable, because:

```
>> eig(A)
ans =
   0.0000 + 0.0000i
  -4.8479 + 0.0000i
  -3.0749 +15.1276i
  -3.0749 -15.1276i
```

- Inverted pendulum's linearized model is unstable, because:

  ```
  >> eig(A)
  ans =
           0
    -20.8385
     -4.0042
     13.8450
  ```

Note: The requirement to attach Matlab codes to your homework assignments is:

- for making sure that each student did the homework independently, and
- for pointing out possible errors if the marker feels something is wrong.

We will not aim at checking your Matlab codes in detail.