1. Answer the following true-or-false questions. Write (T) (meaning *true*) or (F) (meaning *false*). **No need to motivate your answers.** (2pt each)

Below, consider the continuous-time linear time-invariant system

$$\begin{cases} \dot{x}(t) &= Ax(t) + Bu(t), \\ y(t) &= Cx(t) + Du(t), \end{cases} \tag{1}$$

where x, u and y denote respectively state, input and output vectors. By applying the state coordinate transformation $z = Tx$ with a nonsingular matrix T, we can obtain a system:

$$\begin{cases} \dot{z}(t) &= TAT^{-1}z(t) + TBu(t), \\ y(t) &= CT^{-1}z(t) + Du(t). \end{cases} \tag{2}$$

(a) If the system (1) is asymptotically stable, then it is always observable.

(b) If the system(1) is observable, then it is always asymptotically stable.

(c) If the system (1) is observable, then it is always detectable.

(d) If the system (1) is detectable, then it is always observable.

(e) If the system (1) is detectable, then it is always asymptotically stable.

(f) If the system (1) is asymptotically stable, then it is always detectable.

(g) If the system (1) is stabilizable, then it is always detectable.

(h) If the system (1) is detectable, then it is always stabilizable.

(i) If the system (1) is observable, then the system (2) is always observable.

(j) If the system (1) is detectable, then the system (2) is always detectable.

Question	**Write your answer here**
(a)	F
(b)	F
(c)	T
(d)	F
(e)	F
(f)	T
(g)	F
(h)	F
(i)	T
(j)	T

2. Select **only one** correct statement, **by circling one of the numbers i, ii, iii or iv**, for the following sentences. **No need to motivate your answers.**

(3pt each)

(a) If we linearize the state equation $\dot{x}(t) = -\sin x(t) + \cos u(t)$ around an input $u_0 = \frac{\pi}{2}$, then the corresponding equilibrium input x_0 and the linearized state equation will be ($\delta x(t) := x(t) - x_0$, $\delta u(t) := u(t) - u_0$):

i. $x_0 = \frac{\pi}{2}$ and $\delta\dot{x}(t) = -\delta x(t) + \delta u(t)$.

ii. $x_0 = -\frac{\pi}{2}$ and $\delta\dot{x}(t) = -\delta x(t) + \delta u(t)$.

iii. $x_0 = \frac{\pi}{2}$ and $\delta\dot{x}(t) = \delta x(t) - \delta u(t)$.

iv. None of i, ii, iii is correct.

(b) If we discretize (with the zero-order-hold method) a continuous-time linear time-invariant system which is controllable, observable and asymptotically stable, then the discretized system for any sampling time is:

i. always controllable, observable and asymptotically stable.

ii. always controllable and observable, but not necessarily asymptotically stable.

iii. always observable and asymptotically stable, but not necessarily controllable.

iv. None of i, ii, iii is correct.

(c) For a state equation $x[k+1] = -x[k] + 2w[k]$ where the expected value and variance of w and given by $E\{w\} = 1$ and $R_w = 1$, respectively, the prediction step of the Kalman filter will be:

i. $\hat{x}[k+1|k] = -\hat{x}[k|k] + 2$ and $P[k+1|k] = P[k|k] + 2$.

ii. $\hat{x}[k+1|k] = -\hat{x}[k|k] + 2$ and $P[k+1|k] = P[k|k] + 4$.

iii. $\hat{x}[k+1|k] = -\hat{x}[k|k]$ and $P[k+1|k] = P[k|k] + 2$.

iv. $\hat{x}[k+1|k] = -\hat{x}[k|k]$ and $P[k+1|k] = P[k|k] + 4$.

(d) For an output equation $y[k] = x[k] + v[k]$ where the expected value and variance of v and given by $E\{v\} = 1$ and $R_v = 1$, respectively, the correction step of the Kalman filter will be:

i. $\begin{cases} \hat{x}[k|k] = \hat{x}[k|k-1] + P[k|k](y[k] - \hat{x}[k|k-1] - 1), \\ P[k|k] = \frac{P[k|k-1]+1}{P[k|k-1]}. \end{cases}$

ii. $\begin{cases} \hat{x}[k|k] = \hat{x}[k|k-1] + P[k|k](y[k] - \hat{x}[k|k-1] - 1), \\ P[k|k] = \frac{P[k|k-1]}{P[k|k-1]+1}. \end{cases}$

iii. $\begin{cases} \hat{x}[k|k] = \hat{x}[k|k-1] + P[k|k](y[k] - \hat{x}[k|k-1]), \\ P[k|k] = \frac{P[k|k-1]+1}{P[k|k-1]}. \end{cases}$

iv. $\begin{cases} \hat{x}[k|k] = \hat{x}[k|k-1] + P[k|k](y[k] - \hat{x}[k|k-1]), \\ P[k|k] = \frac{P[k|k-1]}{P[k|k-1]+1}. \end{cases}$

(e) By infinite-horizon LQR optimal control with weighting matrices $Q \geq 0$ and $R > 0$ and controllable (A, B) and observable (A, Q), the closed-loop system becomes:

i. always asymptotically stable.
ii. always marginally stable.
iii. always unstable.
iv. None of i, ii, iii is correct.

(f) For a system $\dot{x}(t) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$, by selecting an appropriate control input $u(t)$, it is possible to transfer state:

i. from $x(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}^{\cancel{T}}$ to $x(1) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}^{\cancel{T}}$.

ii. from $x(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}^{\cancel{T}}$ to $x(1) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}^{\cancel{T}}$.

iii. from $x(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}^{\cancel{T}}$ to $x(1) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}^{\cancel{T}}$.

iv. All of i, ii, iii are correct.

(g) In the infinite-horizon LQR problem with a cost function

$$\min_{u(\cdot)} \int_0^\infty \left(Qx(t)^2 + Ru(t)^2\right) dt, \quad Q > 0,\ R > 0,$$

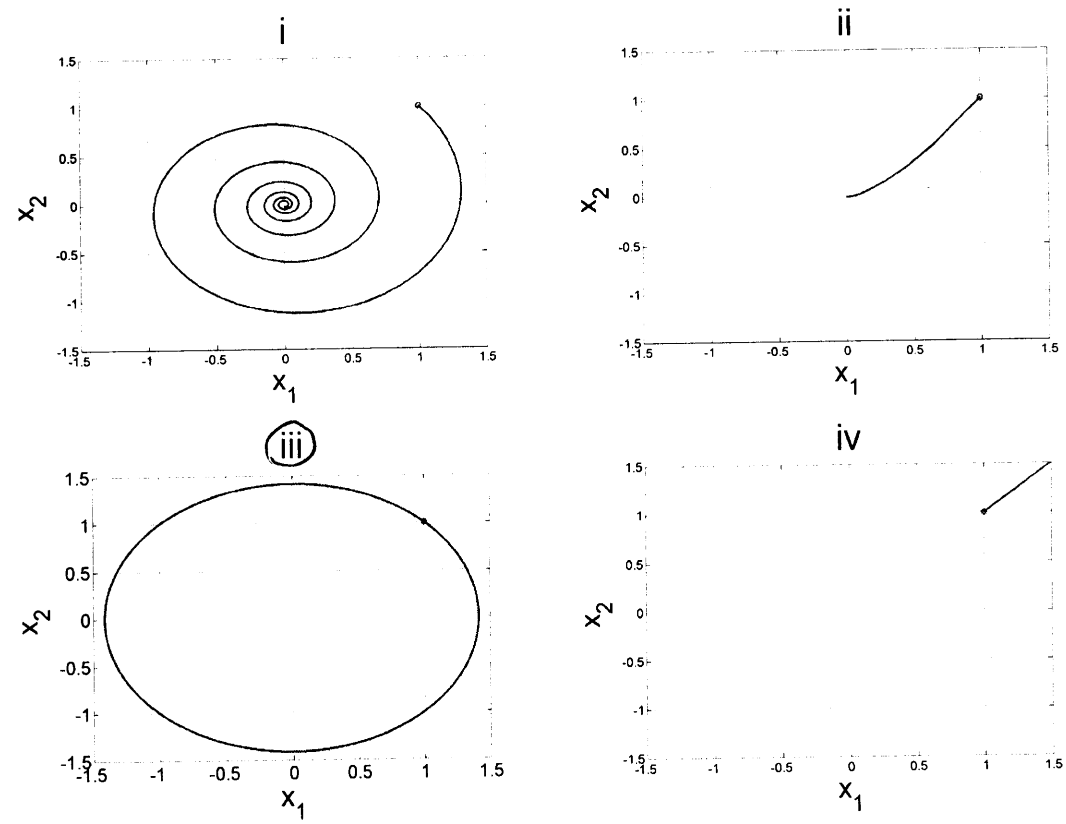
with a state equation (for example, $\dot{x}(t) = -x(t) + u(t)$), during the design iteration while searching for appropriate Q and R, if we would like to reduce the input amplitude, then we should:

i. Increase Q without changing R.
ii. Increase R without changing Q.
iii. Increase Q and R by the same multiple (for example, $2Q$ and $2R$).
iv. None of i, ii, iii is correct.

(h) In the observer-based state-feedback controller design using pole-placement technique, there are two types of poles, that is, the eigenvalues of $A-BK$, denoted by $\sigma(A-BK)$ and the eigenvalues of $A-LC$, denoted by $\sigma(A-LC)$. As a rule of thumb, we should place the poles so that:

i. $\sigma(A-BK)$ and $\sigma(A-LC)$ are located in similar distances from the origin.
ii. $\sigma(A-BK)$ are located far left, compared to $\sigma(A-LC)$.
iii. $\sigma(A-LC)$ are located far left, compared to $\sigma(A-BK)$.
iv. None of i, ii, iii is correct.

(i) A state equation $\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ with an initial condition $x(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ has the following phase plot (small 'o'-mark at $(x_1, x_2) = (1, 1)$ indicates the initial condition):



(j) A continuous-time linear state-space model

$$\begin{cases} \dot{x}(t) &= \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t), \\ y(t) &= \begin{bmatrix} 1 & 0 \end{bmatrix} x(t), \end{cases}$$

is:

i. stabilizable and detectable.
ii. stabilizable but not detectable.
iii. detectable but not stabilizable.
iv. neither stabilizable nor detectable.

Write your answer here for Problem 3.

(a) $G(s) = \dfrac{2}{s+2\alpha}$ $\quad \therefore$ CCF $\begin{cases} \dot{x} = -2\alpha x + u \\ y = 2x \end{cases}$

(b) $\alpha = 1$

$$A_{aug} = \begin{bmatrix} -2 & 0 \\ -2 & 0 \end{bmatrix}, \quad B_{aug} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad K_{aug} = \begin{bmatrix} k & k_a \end{bmatrix}$$

$$\det\left[\lambda I - (A_{aug} - B_{aug}K_{aug})\right] = \det\begin{bmatrix} \lambda+2+k & +k_a \\ 2 & \lambda \end{bmatrix}$$

$$= \lambda(\lambda+2+k) \overset{-}{\cancel{+}} 2k_a$$

$$= \lambda^2 + (2+k)\lambda \overset{-}{\cancel{+}} 2k_a = \lambda^2 + 3\lambda + 2$$

$$\Rightarrow k = +1, \quad k_a = -1$$

(c) $\alpha \neq 1$ $\quad \det\left[\lambda I - (A_{aug} - B_{aug}K_{aug})\right]$

$$= \lambda^2 + \underbrace{(2\alpha + k)}_{>0}\lambda \overset{-}{\cancel{+}} 2k_a$$

$$\alpha > \frac{-k}{2} = \frac{-1}{2}$$

Write your answer here for Problem 4.

(a) $\mathcal{C} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$ rank $\mathcal{C} = 2$ $\therefore$ controllable

(b) $\mathcal{O} = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}$ rank $\mathcal{O} = 2$ $\therefore$ observable

$2y^2 = \underbrace{x^T C^T 2 C x}_{Q}$

(c)
$$\underbrace{\begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}}_{A^T}\underbrace{\begin{bmatrix} P_1 & P_2 \\ P_2 & P_3 \end{bmatrix}}_{P} + \underbrace{\begin{bmatrix} P_1 & P_2 \\ P_2 & P_3 \end{bmatrix}}_{P}\underbrace{\begin{bmatrix} -1 & 0 \\ 1 & -1 \end{bmatrix}}_{A} + \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}}_{Q}$$
$$- \underbrace{\begin{bmatrix} P_1 & P_2 \\ P_2 & P_3 \end{bmatrix}}_{P}\underbrace{\begin{bmatrix} 1 \\ -1 \end{bmatrix}}_{B}\underbrace{1^{-1}}_{R^{-1}}\underbrace{\begin{bmatrix} 1 & -1 \end{bmatrix}}_{B^T}\underbrace{\begin{bmatrix} P_1 & P_2 \\ P_2 & P_3 \end{bmatrix}}_{P} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

(d) (1,1) $2(-P_1+P_2) - (P_1-P_2)^2 = 0 \Rightarrow (P_1-P_2)(P_1-P_2+2) = 0$

(1,2) $-P_2+P_3-P_2 - (P_1-P_2)(P_2-P_3) = 0$

(2,2) $2(-P_3) + 2 - (P_2-P_3)^2 = 0$

If $P_1-P_2+2=0$, then (1,2) $\Rightarrow -2P_2+P_3+2(P_2-P_3)=0 \Rightarrow P_3 = 0$ ✗

(diagonal entry >0)

If $P_1-P_2=0$, then (1,2) $\Rightarrow P_3 = 2P_2$

(2,2) $\Rightarrow -4P_2+2-P_2^2 = 0$

$P_2^2+4P_2-2=0 \Rightarrow P_2 = -2\pm\sqrt{6}$

$P_3 = 2P_2 > 0 \Rightarrow P_2 = -2+\sqrt{6}$

$P = \begin{bmatrix} P_2 & P_2 \\ P_2 & 2P_2 \end{bmatrix}$, $P_2 = -2+\sqrt{6}$

(P is positive definite.)

(e) $u = -[1\ \ -1]Px = -[0\ \ 2-\sqrt{6}]x$
$= P_2 x_2$

(f) $A-Bk = \begin{bmatrix} -1 & 0 \\ 1 & -1 \end{bmatrix} - \begin{bmatrix} 1 \\ -1 \end{bmatrix}[0\ \ 2-\sqrt{6}] = \begin{bmatrix} -1 & -2+\sqrt{6} \\ 1 & 1-\sqrt{6} \end{bmatrix}$

Ch. eq. $(\lambda+1)(\lambda-1+\sqrt{6}) - (-2+\sqrt{6}) = 0$

$\lambda^2 + \underbrace{\sqrt{6}}_{>0}\lambda + \underbrace{1}_{>0} = 0$ asym. stable!