**University of British Columbia**
**Department of Mechanical Engineering**

# MECH468 Modern Control Engineering
# MECH522 Foundations in Control Engineering
# Final exam, Solution Sketch

**Examiner: Dr. Ryozo Nagamune**
**December 9 (Friday), 2016, noon-2:30pm**

Last name, First name

Name: Student #:

Signature:

## Exam policies

- Allowed: Lecture note print-outs, hand-written notes, homework, books.
- Not-allowed: PC, calculators, mobile phones.
- Write all your answers on this booklet. No extra sheet will be provided.
- Motivate your answers properly. (No chance to defend your answers orally.)
- 100 points in total. Mark will be scaled later.

## Before you start ...

- Use washroom before the exam.
- Turn off your mobile phone.
- No eating.
- Questions are NOT allowed.

## If you finish early ...

- If you would like to leave the room **before 2:20pm**, **raise your hand with this booklet**, and **wait at your seat** until the invigilator comes to you and collects your exam booklet.

## To be filled in by the instructor/marker

| Problem # | Mark | Full mark |
|---|---|---|
| 1 | | 40 |
| 2 | | 20 |
| 3 | | 20 |
| 4 | | 20 |
| Total | | 100 |

1. Consider the following continuous-time state-space model:

$$\begin{cases} \dot{x}(t) &= \underbrace{\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}}_{A} x(t) + \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{B} u(t), \\ y(t) &= \underbrace{\begin{bmatrix} 0 & 1 \end{bmatrix}}_{C} x(t). \end{cases} \tag{1}$$

(a) Is this system asymptotically stable, marginally stable, or unstable? You do **not** need to motivate your answer for this question. (5pt)

(b) Linearize the system (1) at equilibrium point $x = [0, 1]^T$ and $u = 0$. (5pt)

(c) From the state-space model above, compute the transfer function $G(s)$ from the input $u$ to the output $y$. (5pt)

(d) Compute the matrix exponential $e^{At}$. (5pt)

(You will find Questions 1-(e) and 1-(f) in the next pages.)

**Write your answer here for Question 1.**

**(a)** unstable

**(b)** With $\delta x := x - x_{eq}$, $\delta u := u - u_{eq}$, and $\delta y := y - y_{eq}$,

$$\begin{cases} \dot{\delta x}(t) &= A\delta x(t) + B\delta u(t), \\ \delta y(t) &= C\delta x(t). \end{cases}$$

**(c)**

$$G(s) = C(sI - A)^{-1}B = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} s & 0 \\ -1 & s \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{s^2}$$

**(d)** Since $A^2 = A^3 = \cdots = 0$, we have

$$e^{At} = I + At + \frac{A^2t^2}{2!} + \cdots = I + At = \begin{bmatrix} 1 & 0 \\ t & 1 \end{bmatrix}.$$

(e) For the state equation in (1), compute the minimum energy control $u(t)$ which transfers the state from $x(0)$ to $x(1)$, where $x(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and $x(1) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. (10pt)

**Write your answer here for Question 1.**

**(e)** The minimum energy control is

$$u(t) = B^T e^{A^T(1-t)} W_c(1)^{-1} \left( x(1) - e^{A \cdot 1} x(0) \right)$$

Compute the controllability Gramian:

$$W_c(1) = \int_0^1 e^{A\tau} B B^T e^{A^T \tau} d\tau = \int_0^1 \begin{bmatrix} 1 \\ \tau \end{bmatrix} \begin{bmatrix} 1 & \tau \end{bmatrix} d\tau = \begin{bmatrix} 1 & 1/2 \\ 1/2 & 1/3 \end{bmatrix}$$

and its inverse:

$$W_c(1)^{-1} = \frac{1}{1/3 - 1/4} \begin{bmatrix} 1/3 & -1/2 \\ -1/2 & 1 \end{bmatrix} = \begin{bmatrix} 4 & -6 \\ -6 & 12 \end{bmatrix}.$$

Thus,

$$\begin{aligned} u(t) &= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1-t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 4 & -6 \\ -6 & 12 \end{bmatrix} \left( \begin{bmatrix} 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) \\ &= -\begin{bmatrix} 1 & 1-t \end{bmatrix} \begin{bmatrix} 4 & -6 \\ -6 & 12 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ &= -\begin{bmatrix} 1 & 1-t \end{bmatrix} \begin{bmatrix} -6 \\ 12 \end{bmatrix} = 12t - 6 \end{aligned}$$

(f) Obtain the continuous-time infinite-horizon LQR optimal control law $u(t)$ which solves the following optimization problem: (10pt)

$$\min_{u(\cdot)} \int_0^\infty \left\{ y^2(t) + u^2(t) \right\} dt, \text{ subject to the state-space model (1).}$$

**Write your answer here for Question 1.**

**(f)** The cost function is rewritten as

$$\min_{u(\cdot)} \int_0^\infty \left\{ x^T(t) C^T C x(t) + u^2(t) \right\} dt.$$

Thus, $Q = C^T C = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ and $R = 1$. The LQR optimal control is

$$u(t) = -R^{-1} B^T P x(t) = -\begin{bmatrix} 1 & 0 \end{bmatrix} P x(t),$$

where $P$ is the unique positive definite solution to the algebraic Riccati equation:

$$\underbrace{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}_{A^T} \underbrace{\begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix}}_{P} + \underbrace{\begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix}}_{P} \underbrace{\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}}_{A} + \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}}_{Q} - \underbrace{\begin{bmatrix} p_1 \\ p_2 \end{bmatrix}}_{PB} \underbrace{1}_{R^{-1}} \underbrace{\begin{bmatrix} p_1 & p_2 \end{bmatrix}}_{B^T P} = 0$$

$$\begin{aligned} (1,1) &: \ 2p_2 - p_1^2 = 0 \\ (1,2) &: \ p_3 - p_1 p_2 = 0 \\ (2,2) &: \ 1 - p_2^2 = 0 \end{aligned}$$

From this, we can obtain the positive definite solution as

$$P = \begin{bmatrix} \sqrt{2} & 1 \\ 1 & \sqrt{2} \end{bmatrix}$$

Thus, the LQR optimal control law is

$$u(t) = -\begin{bmatrix} \sqrt{2} & 1 \end{bmatrix} x(t).$$

2. Consider the transfer matrix

$$G(s) = \begin{bmatrix} \dfrac{1}{s+1} & \dfrac{s+\alpha}{s+1} \\ \dfrac{1}{s+1} & \dfrac{1}{s+1} \end{bmatrix},$$

where $\alpha$ is a positive constant.

(a) Obtain the realization of $G(s)$ in the controllable canonical form. (5pt)

(b) Obtain the realization of $G(s)$ in the observable canonical form. (5pt)

(c) Find $\alpha$ such that the minimal realization of $G(s)$ has only one state (i.e., the size of $A$-matrix becomes 1-by-1). For that $\alpha$, obtain the minimal realization of $G(s)$. (10pt)

**Write your answer here for Question 2.**

**(a)** Controllable canonical realization

$$\begin{cases} \dot{x}(t) &= \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} x(t) + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} u(t) \\ y(t) &= \begin{bmatrix} 1 & \alpha-1 \\ 1 & 1 \end{bmatrix} x(t) + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} u(t) \end{cases}$$

**(b)** Observable canonical realization

$$\begin{cases} \dot{x}(t) &= \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} x(t) + \begin{bmatrix} 1 & \alpha-1 \\ 1 & 1 \end{bmatrix} u(t) \\ y(t) &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x(t) + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} u(t) \end{cases}$$

**(c)** By checking the controllability of the observable canonical realization above, we have the controllability matrix:

$$\mathcal{C} = \begin{bmatrix} 1 & \alpha-1 & -1 & -(\alpha-1) \\ 1 & 1 & -1 & -1 \end{bmatrix}.$$

This matrix will be of rank 1 if and only if $\alpha = 2$, in which case the minimal realization will have only one state.

When $\alpha = 2$, the minimal realization of $G(s)$ is

$$\begin{cases} \dot{x}(t) &= -x(t) + \begin{bmatrix} 1 & 1 \end{bmatrix} u(t) \\ y(t) &= \begin{bmatrix} 1 \\ 1 \end{bmatrix} x(t) + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} u(t) \end{cases}$$

3. Consider the following continuous-time state-space model:

$$\begin{cases} \dot{x}(t) &= \underbrace{\begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix}}_{A} x(t) + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{B} u(t), \\ y(t) &= \underbrace{\begin{bmatrix} 1 & 1 \end{bmatrix}}_{C} x(t) \end{cases}$$

Answer the following questions with proper explanations.

(a) Is this system stabilizable? (5pt)

(b) Is this system detectable? (5pt)

(c) If possible, design a state feedback controller $u(t) = -Kx(t)$ (i.e., obtain a matrix $K$) so that the closed-loop system has an $A$-matrix (i.e., $A-BK$) with eigenvalues at $-1$ and $-2$. If that is not possible, explain the reason why. (5pt)

(d) If possible, design an observer gain $L$ so that the eigenvalues of $A - LC$ are $-1$ and $-2$. If that is not possible, explain the reason why. (5pt)

**Write your answer here for Question 3.**

**(a)** Controllability matrix is

$$\mathcal{C} = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix},$$

which has rank one. Thus, the system is not controllable. A coordinate transformation matrix is

$$T^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = T.$$

The new state space model with this coordinate transformation matrix becomes

$$\begin{cases} \begin{bmatrix} \dot{z}_c(t) \\ \dot{z}_{\bar{c}}(t) \end{bmatrix} &= \underbrace{\begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}}_{TAT^{-1}} \begin{bmatrix} z_c(t) \\ z_{\bar{c}}(t) \end{bmatrix} + \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{TB} u(t), \\ y(t) &= \underbrace{\begin{bmatrix} 1 & 1 \end{bmatrix}}_{CT^{-1}} \begin{bmatrix} z_c(t) \\ z_{\bar{c}}(t) \end{bmatrix}. \end{cases}$$

Since the uncontrollable part of the $A$-matrix is $-1$ which is in the stable region, the system is stabilizable.

**(b)** Observability matrix

$$\mathcal{O} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix},$$

which has full rank. Therefore, it is observable, and thus detectable.

**Write your answer here for Question 3.**

**(c)** Since the specified eigenvalues ($-1$ and $-2$) contain the eigenvalue $-1$ which is not movable by state feedback, it is possible to find such $K$. By direct method,

$$\begin{aligned}
\det\left(sI - \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} k_1 & k_2 \end{bmatrix}\right) &= \det \begin{bmatrix} s+1 & 0 \\ -1+k_1 & s-1+k_2 \end{bmatrix} \\
&= (s+1)(s-1+k_2) = (s+1)(s+2)
\end{aligned}$$

Thus, $k_2 = 3$ and $k_1$ is arbitrary.

**(d)** Since the system is observable, it is possible to assign the eigenvalues of $A - LC$ at any specified locations. By direct method,

$$\begin{aligned}
\det\left(sI - \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} \ell_1 \\ \ell_2 \end{bmatrix} \begin{bmatrix} 1 & 1 \end{bmatrix}\right) &= \det \begin{bmatrix} s+1+\ell_1 & \ell_1 \\ -1+\ell_2 & s-1+\ell_2 \end{bmatrix} \\
&= (s+1+\ell_1)(s-1+\ell_2) - \ell_1(-1+\ell_2) \\
&= s^2 + (\ell_1+\ell_2)s - 1 + \ell_2 \\
&= (s+1)(s+2) = s^2 + 3s + 2
\end{aligned}$$

Thus, $\ell_1 = 0$ and $\ell_2 = 3$.

4. Consider the following discrete-time system:

$$\begin{cases} x[k+1] &= 2x[k] + 2w[k], \\ y[k] &= x[k] + v[k], \end{cases}$$

where $w$ and $v$ are noise terms with:

- expected values $E\{w[k]\} = 0$ and $E\{v[k]\} = 0$ for any $k$, and
- variances $R_w := E\{w^2[k]\} = 1/2$ and $R_v := E\{v^2[k]\} = 1/2$ for any $k$.

(a) Design the (two-step) time-varying Kalman filter. (10pt)

(b) Using the designed time-varying Kalman filter, for initial *a priori* estimate $\hat{x}[0|-1] = 0$ and its error variance $P[0|-1] = 1$, as well as for measurements

$$y[0] = 1/2, \; y[1] = 2/3,$$

compute the state estimates and their error variances, and complete the table below. (10pt)

(**Hint**: Compute all the variances before computing state estimates.)

| $k$ | *a priori* estimate $\hat{x}[k\|k-1]$ | variance $P[k\|k-1]$ | *a posteriori* estimate $\hat{x}[k\|k]$ | variance $P[k\|k]$ |
|---|---|---|---|---|
| 0 | 0 | 1 | 1/3 | 1/3 |
| 1 | 2/3 | 10/3 | 2/3 | 10/23 |

**Write your answer here for Question 4.**

**(a)** $A = 2$, $B_w = 2$, $C = 1$

Measurement update (correction step)

$$\begin{aligned} \hat{x}[k|k] &= \hat{x}[k|k-1] + P[k|k]C^T R_v^{-1}(y[k] - C\hat{x}[k|k-1]) \\ &= \hat{x}[k|k-1] + 2P[k|k](y[k] - \hat{x}[k|k-1]) \\ P[k|k] &= P[k|k-1] - P[k|k-1]C^T(CP[k|k-1]C^T + R_v)^{-1}CP[k|k-1] \\ &= P[k|k-1] - \frac{P[k|k-1]^2}{P[k|k-1] + \frac{1}{2}} = \frac{P[k|k-1]}{2P[k|k-1] + 1} \end{aligned}$$

Time update (prediction step)

$$\begin{aligned} \hat{x}[k+1|k] &= A\hat{x}[k|k] \\ &= 2\hat{x}[k|k] \\ P[k+1|k] &= AP[k|k]A^T + B_w R_w B_w^T \\ &= 4P[k|k] + 2 \end{aligned}$$

**(b)** See the table.