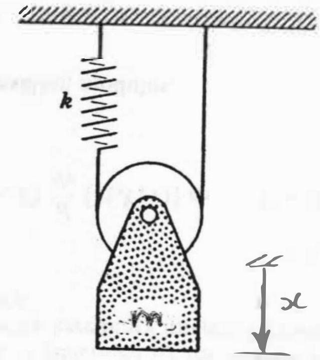
MECH 463 -- Tutorial 1

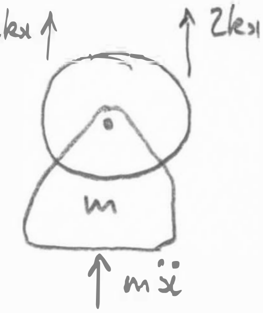
1. A mass m hanging from a pulley that is supported on a cable and a spring of stiffness k.



Let x = displacement of m

$\rightarrow$ spring extension $= 2x$

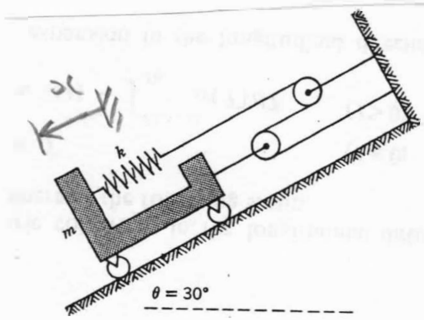
$\rightarrow$ cable tension $= 2kx$



$\Sigma F = m\ddot{x} + 4kx = 0 \rightarrow \omega = \sqrt{\frac{4k}{m}}$

NOTE: x = displ. from equilibrium position.

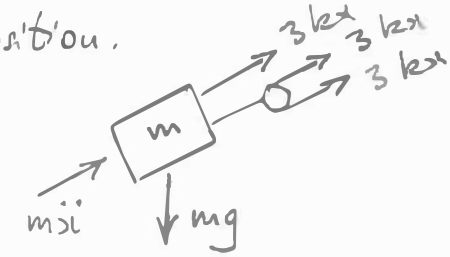
2. A wheeled cart of mass m on a plane inclined at an angle of 30°, supported by a pulley system and a spring of stiffness k. (You may ignore the mass of the pulleys).



Likely, it would be OK to ignore gravity and to consider displacements from the equilibrium position. Just to make sure, let's include gravity and define x as the displacement from the unstretched spring position.

$\rightarrow$ spring extension $= 3x$

$\rightarrow$ cable tension $= 3kx$



$$\Sigma F = m\ddot{x} + 9kx - mg\sin\theta = 0$$

$$\rightarrow m\ddot{x} + 9kx = mg\sin\theta = k\delta$$

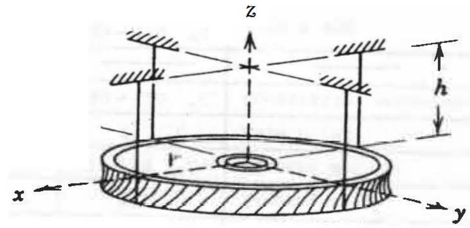
where
δ = static deflection
$\delta = mg\sin\theta / k$

If alternatively we had defined $y = x - \delta$ = displacement from the equilibrium position, we would have got

$m\ddot{y} + 9ky = 0$. In either case: $\omega = \sqrt{\frac{9k}{m}}$

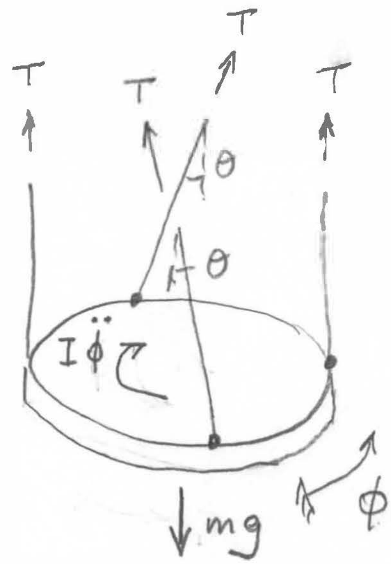
Changing θ only changes the static deflection. It does not change the natural frequency.

3. A circular gear of mass m, moment of inertia I, radius r, and supported on four strings of length h. (The gear vibrates by rotation in the x-y plane).



Let θ = string rotation angle

ϕ = gear rotation angle



Arc length $= r\phi \simeq h\theta$

$\rightarrow \theta = \frac{r}{h}\phi$ (for small angles)

Let T = string tension

Vertical $\sum F = 4T\cos\theta - mg = 0$

$\rightarrow T = \frac{mg}{4\cos\theta} \simeq \frac{mg}{4}$ for small angles

$$\sum M = I\ddot{\phi} + 4rT\sin\theta = 0$$

$$= I\ddot{\phi} + 4r\frac{mg}{4}\frac{r}{h}\phi = 0 \quad \rightarrow \quad I\ddot{\phi} + \frac{mgr^2}{h}\phi = 0$$

$$\rightarrow \omega = \sqrt{\frac{mgr^2}{hI}} = \sqrt{\frac{2g}{h}} \text{ for } I = \tfrac{1}{2}mr^2$$

independent of r

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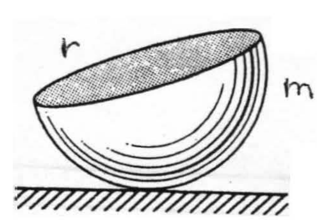
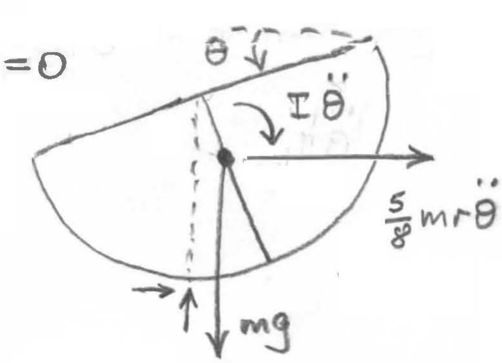
4. A solid hemisphere of radius r and mass m, rolling on a horizontal plane without slipping.



The centroid is $3/8\,r$ from upper surface.

$\rightarrow$ $5/8\,r$ from contact point.

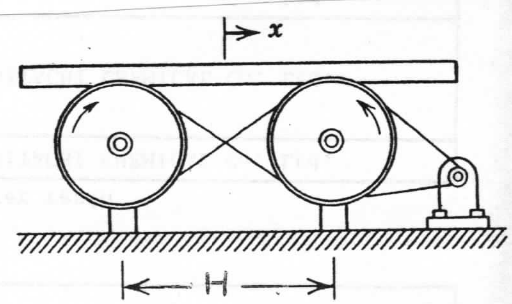


Contact point $\sum M = I\ddot{\theta} + \frac{5}{8}r\cdot\frac{5}{8}mr\ddot{\theta} + \frac{3}{8}mgr\sin\theta = 0$

$\simeq \left(\frac{83}{320} + \frac{25}{64}\right)mr^2\ddot{\theta} + \frac{3}{8}mgr\theta = 0$ for small angles

$= \frac{13}{20}mr^2\ddot{\theta} + \frac{3}{8}mgr\theta = 0 \rightarrow \omega = \sqrt{\frac{15g}{26r}}$

5. A testing machine for measuring dynamic friction coefficient μ. A rod of length L, mass m, rests on two counter-rotating pulleys with centres H apart.



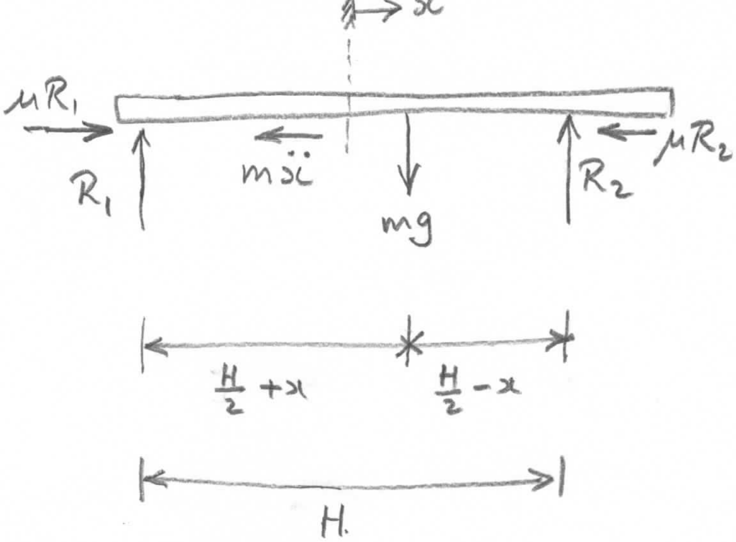
Let x = displacement from centre.

R_1 and R_2 = vertical reaction forces

$$\Sigma M_2 = R_1 H - mg\left(\frac{H}{2} - x\right) = 0$$

$$\rightarrow R_1 = mg\left(\frac{1}{2} - \frac{x}{H}\right)$$

$$\rightarrow R_2 = mg\left(\frac{1}{2} + \frac{x}{H}\right)$$



Let μ = friction coefficient

$$\text{Horiz. } \Sigma F = m\ddot{x} + \mu R_2 - \mu R_1 = 0$$

$$= m\ddot{x} + \mu mg\left(\frac{1}{2} + \frac{x}{H}\right) - \mu mg\left(\frac{1}{2} - \frac{x}{H}\right) = 0$$

$$= m\ddot{x} + 2\mu mg\frac{x}{H} = 0 \quad \longrightarrow \quad \omega = \sqrt{\frac{2\mu g}{H}}$$

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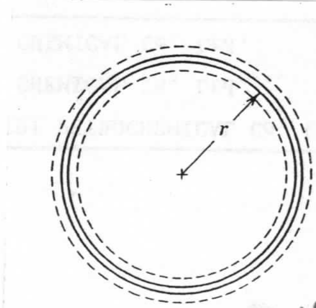
6. A thin circular ring of radius r, mass density ρ, Young's modulus E, and cross section area A, vibrates radially.



Let x = radial displacement

original length of segment $= r\,d\theta$

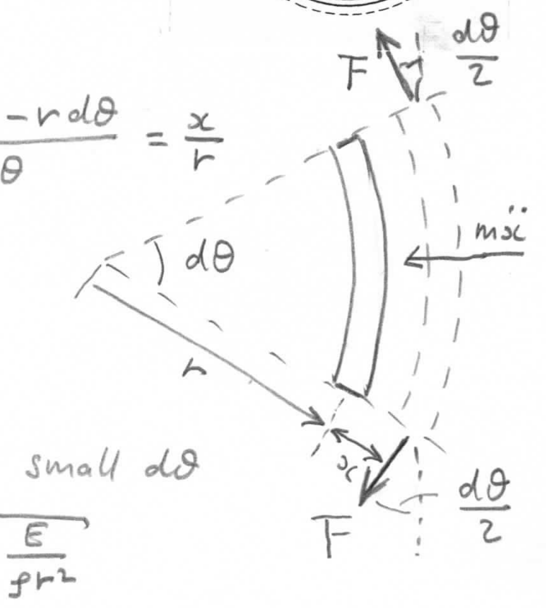
Mass of segment, $m = \rho A r\,d\theta$

Displaced length of segment $= (r+x)\,d\theta$

$$\text{Circumferential strain, } \varepsilon = \frac{\text{change in length}}{\text{original length}} = \frac{(r+x)\,d\theta - r\,d\theta}{r\,d\theta} = \frac{x}{r}$$

Hooke's Law: stress $\sigma = E\varepsilon = \frac{Ex}{r}$

Circ. force $F = \sigma A = \frac{EAx}{r}$



$$\text{Horiz. } \Sigma F = m\ddot{x} + 2F\sin\frac{d\theta}{2} = 0$$

$$= \rho A r\,d\theta\,\ddot{x} + \frac{2EAx}{r}\frac{d\theta}{2} = 0 \quad \text{for small } d\theta$$

$$\longrightarrow \rho\ddot{x} + \frac{E}{r^2}x = 0 \quad \longrightarrow \quad \omega = \sqrt{\frac{E}{\rho r^2}}$$