MECH 463
Oct 22

Lecture 13

2^{nd} midterm Oct 29^{th} - Up to this lecture

Lab teams doing labs this week, have 1 week report extension.

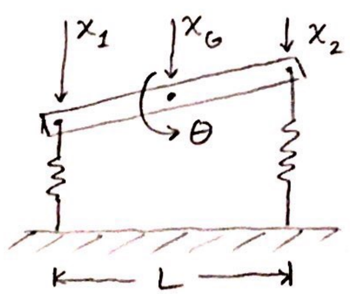
Multi-DOF systems

Generalized Coordinates - Independent set of parameters that uniquely define the position of the system

A coordinate is independent when it cannot be expressed as a combination as a combination of other coordinates

Conversely, if it can, there exists a constraint.

Ex:



$$x_G = \frac{1}{2}(x_1 + x_2)$$

$$\Theta = \frac{1}{L}(x_1 - x_2)$$

$$\begin{pmatrix}\text{\# Generalized} \\ \text{coordinates}\end{pmatrix} = \begin{pmatrix}\text{\# All} \\ \text{coordinates}\end{pmatrix} - \begin{pmatrix}\text{\# constraints}\end{pmatrix}$$

Let $\vec{q}$ = generalized coordinates $(q_1, q_2, q_3, \ldots)$

$\boxed{\text{Lagrange's Equations}}$ - Minimized Energy Approach

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} + \frac{\partial R}{\partial \dot{q}_i} + \frac{\partial V}{\partial q_i} = Q_i$$

Note $\dot{q}$ in denominators

T - kinetic energy (e.g. $\frac{1}{2} m \dot{x}^2$)

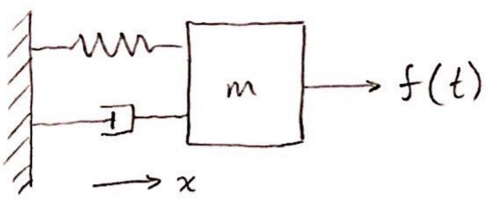
V - potential energy (e.g. $\frac{1}{2} k x^2$)

R - dissipation function (e.g. $\frac{1}{2} c \dot{x}^2$, not an energy)

Q_i - generalized "force" → force @ generalized coordinate q_i (or moment)

For numerical stability, do not mix angular and displacement coordinates.

$\boxed{\text{1-DOF example}}$



$$T = \frac{1}{2} m \dot{x}^2$$

$$V = \frac{1}{2} k x^2$$

$$R = \frac{1}{2} c \dot{x}^2$$

$$Q = f(t)$$

$x \leftrightarrow q$ and $i = 1$

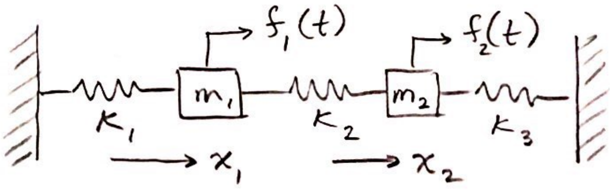
Lagrange: $$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}}\right) - \frac{\partial T}{\partial x} + \frac{\partial R}{\partial \dot{x}} + \frac{\partial V}{\partial x} = f(t)$$

$$\frac{d}{dt}\left(\frac{\partial}{\partial \dot{x}}\left(\frac{1}{2} m \dot{x}^2\right)\right) - \frac{\partial}{\partial x}\left(\frac{1}{2} m \dot{x}\right) + \frac{\partial}{\partial \dot{x}}\left(\frac{1}{2} c \dot{x}^2\right) + \frac{\partial}{\partial x}\left(\frac{1}{2} k x^2\right) = f(t)$$

$$\frac{d}{dt}\left(m\dot{x}\right) - 0 + c\dot{x} + kx = f(t)$$

$$\Rightarrow m\ddot{x} + c\dot{x} + kx = f(t) \quad \text{as expected}$$

2-DOF example



$$T = \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 \dot{x}_2^2$$

$$V = \frac{1}{2} k_1 x_1^2 + \frac{1}{2} k_2 (x_2 - x_1)^2 + \frac{1}{2} k_3 x_2^2$$

$$R = 0$$

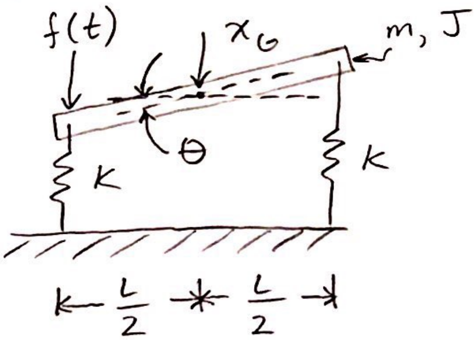
$Q_1 = f_1(t)$ and $Q_2 = f_2(t)$

For x_1: $\frac{d}{dt}(m_1 \dot{x}_1) - 0 + k_1 x_1 - k_2 (x_2 - x_1) = f_1(t)$

For x_2: $\frac{d}{dt}(m_2 \dot{x}_2) - 0 + k_2 (x_2 - x_1) + k_3 x_2 = f_2(t)$

$$\Rightarrow \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k & k_2 + k_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} f_1(t) \\ f_2(t) \end{bmatrix}$$

Mixed Coordinate example



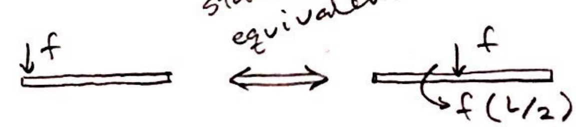
$$J = \frac{1}{12} m L^2$$

$$T = \frac{1}{2} m_G \dot{x}_G^2 + \frac{1}{2} J \dot{\theta}^2$$

$$V = \frac{1}{2} k \left(x_G + \frac{L}{2}\theta\right)^2 + \frac{1}{2} k \left(x_G - \frac{L}{2}\theta\right)^2$$

$$R = 0$$

For Q:



$$\Rightarrow Q_1 = f(t), \quad Q_2 = f(t) \cdot (L/2)$$

For x_G: $\frac{d}{dt}(m \dot{x}_G) - 0 + k\left(x_G + \frac{L}{2}\right) + k\left(x_G - \frac{L}{2}\right) + 0 = f(t)$

For θ: $\frac{d}{dt}(J\dot{\theta}) - 0 + k\left(x_G + \frac{L}{2}\theta\right)\left(\frac{L}{2}\right) - k\left(x_G - \frac{L}{2}\theta\right)\left(\frac{L}{2}\right) + 0 = f(t)\left(\frac{L}{2}\right)$

$$\Rightarrow \frac{1}{12} m L^2 \ddot{\theta} + \frac{1}{2} k L^2 \theta = f(t)\left(\frac{L}{2}\right)$$