a place of mind

MECH468 : Modern Control Engineering
MECH509 : Controls

L25 : Continuous-time finite-horizon LQR (Linear Quadratic Regulator)

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Zoom lecture to be recorded and posted on Canvas

Course plan

Topics	CT	DT
Modeling	✓	✓
Stability	✓	✓
Controllability/observability	✓	✓
Realization	✓	✓
State feedback/observer	✓	✓
LQR/Kalman filter		

6 lectures

Review & topics from now on

- So far, pole placement methods for
    - Control (State feedback)
    - Estimation (Observer)
    - Control + estimation (Observer-based control)
- In the rest of this course, optimal methods for
    - Control (Linear Quadratic Regulator: LQR)
    - Estimation (Kalman filter)
    - Control + estimation (Linear Quadratic Gaussian: LQG)

CT finite-horizon LQR optimal control

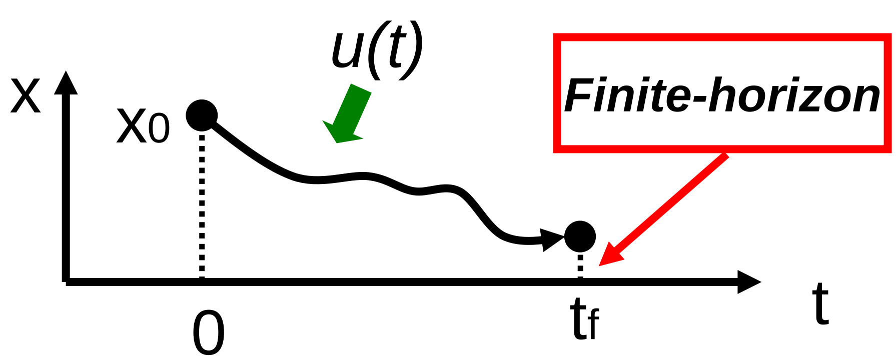
- Problem $\min_{u(\cdot)} J(u(\cdot))$ subj. to $\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ x(0) = x_0 \text{ (given)} \end{cases}$

  - J : $\boxed{\textit{Quadratic}}$ performance index (cost function)

$$J(u(\cdot)) := \int_0^{t_f} \Big[\underbrace{x^T(t)Qx(t)}_{\text{For small state}} + \underbrace{u^T(t)Ru(t)}_{\text{For small input}} \Big] dt + \underbrace{x^T(t_f)Sx(t_f)}_{\text{For small final state}}$$

Design parameters

$$Q \geq 0, \; R > 0, \; S \geq 0$$



Positive (semi-)definite matrix (review) $P = P^T \in \mathbb{R}^{n\times n}$

Terminology	Notation	Definition	Condition
Positive definite	$P > 0$	$x^T P x > 0$ $\forall x (\neq 0) \in \mathbb{R}^n$	All eigenvalues of *P* are positive.
Positive semidefinite	$P \geq 0$	$x^T P x \geq 0$ $\forall x \in \mathbb{R}^n$	All eigenvalues of *P* are nonnegative.

LQR optimal control law

- LQR optimal control is obtained as a state feedback

$$u(t) = -R^{-1}B^T P(t)x(t)$$

Linear

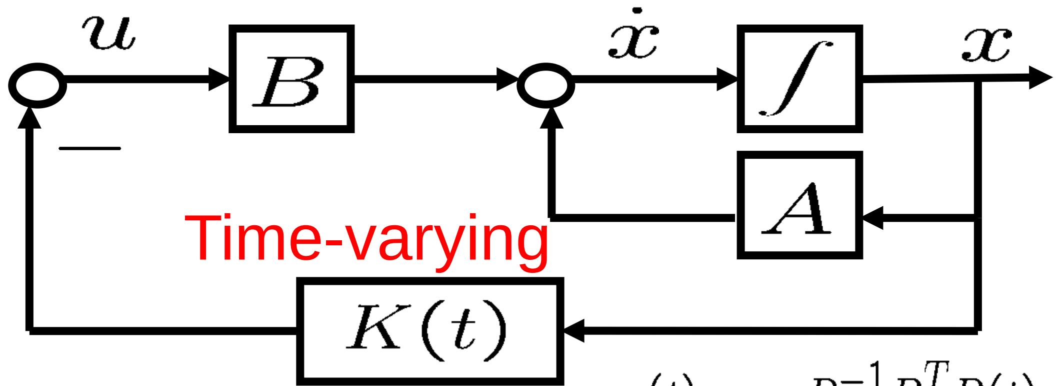
- *P(t)*: unique positive semidefinite and bounded solution to a *matrix Riccati equation*

$$\begin{cases} -\dot{P}(t) &= A^T P(t) + P(t)A - P(t)BR^{-1}B^T P(t) + Q \\ P(t_f) &= S \end{cases}$$

- Optimal performance index $J(u) = x_0^T P(t_0) x_0$

LQR optimal control law (cont'd)

- Block diagram



$$u(t) = -\underbrace{R^{-1}B^TP(t)}_{K(t)}x(t)$$

- Q1: How to solve the matrix Riccati equation?

$$\begin{cases} -\dot{P}(t) &= A^TP(t) + P(t)A - P(t)BR^{-1}B^TP(t) + Q \\ P(t_f) &= S \end{cases}$$

- Q2: How to derive the LQR optimal control law? (Appendix)

How to solve matrix Riccati eq.

- Matrix Riccati equation

$$\begin{cases} -\dot{P}(t) = A^T P(t) + P(t)A - P(t)BR^{-1}B^T P(t) + Q \\ P(t_f) = S \end{cases}$$

- Solution $P(t) = Y(t)X^{-1}(t), \quad t \in [0, t_f]$

  X & Y : solutions to linear matrix differential eq.:

$$\begin{bmatrix} \dot{X}(t) \\ \dot{Y}(t) \end{bmatrix} = \underbrace{\begin{bmatrix} A & -BR^{-1}B^T \\ -Q & -A^T \end{bmatrix}}_{H\text{: Hamiltonian matrix}} \begin{bmatrix} X(t) \\ Y(t) \end{bmatrix}, \quad \begin{bmatrix} X(t_f) \\ Y(t_f) \end{bmatrix} = \begin{bmatrix} I \\ S \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} X(t) \\ Y(t) \end{bmatrix} = e^{H(t-t_f)} \begin{bmatrix} X(t_f) \\ Y(t_f) \end{bmatrix}$$

Proof: Solution of matrix Riccati eq

- Suppose $P(t) = Y(t)X^{-1}(t), \quad t \in [0, t_f]$

$$\begin{bmatrix} \dot{X}(t) \\ \dot{Y}(t) \end{bmatrix} = \begin{bmatrix} A & -BR^{-1}B^T \\ -Q & -A^T \end{bmatrix} \begin{bmatrix} X(t) \\ Y(t) \end{bmatrix}, \quad \begin{bmatrix} X(t_f) \\ Y(t_f) \end{bmatrix} = \begin{bmatrix} I \\ S \end{bmatrix}$$

- Then $P(t_f) = Y(t_f)X^{-1}(t_f) = S$

$$PX = Y \;\Rightarrow\; \dot{P}X + P\dot{X} = \dot{Y}$$

$$\begin{aligned} \Rightarrow\; X^T\dot{P}X &= X^T(\dot{Y} - P\dot{X}) \\ &= X^T(-QX - A^TY - P\left\{AX - BR^{-1}B^TY\right\}) \\ &= X^T\left\{-Q - A^TP - PA + PBR^{-1}B^TP\right\}X \end{aligned}$$

Example

$$\min_{u(\cdot)} \int_0^1 \left(3x(t)^2 + \frac{1}{4}u(t)^2\right) dt + x(1)^2$$

$$\text{subj. to} \begin{cases} \dot{x}(t) & = & 2x(t) + u(t) \\ x(0) & = & x_0 \text{ (given)} \end{cases}$$

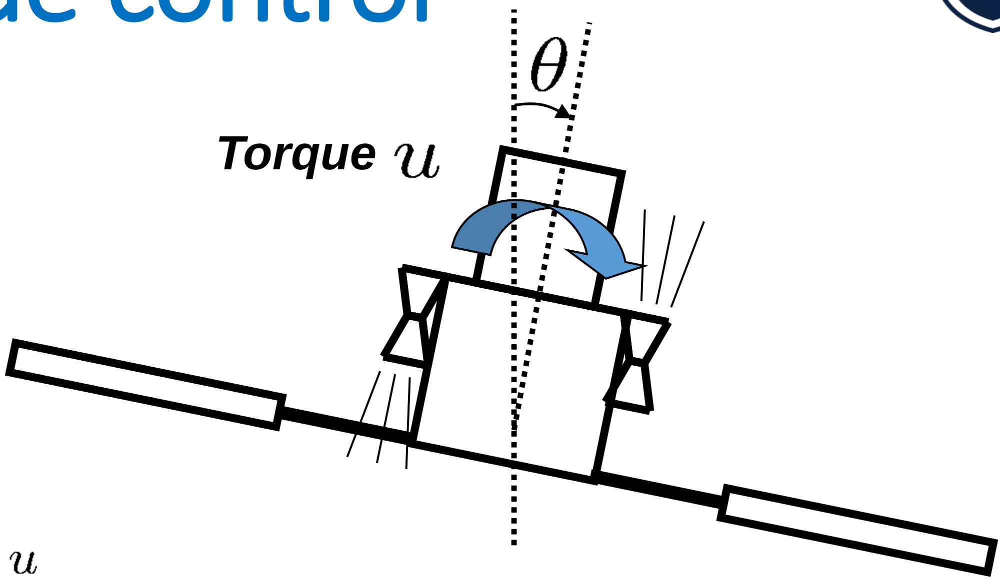
$$\Rightarrow \begin{cases} A = 2, B = 1, Q = 3 \\ R = 1/4, S = 1, t_f = 1 \end{cases}$$

$$\Rightarrow \begin{bmatrix} \dot{X}(t) \\ \dot{Y}(t) \end{bmatrix} = \begin{bmatrix} 2 & -4 \\ -3 & -2 \end{bmatrix} \begin{bmatrix} X(t) \\ Y(t) \end{bmatrix}, \quad \begin{bmatrix} X(1) \\ Y(1) \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} X(t) \\ Y(t) \end{bmatrix} = e^{H(t-1)} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = e^{-4(t-1)} \begin{bmatrix} 3/4 \\ 9/8 \end{bmatrix} + e^{4(t-1)} \begin{bmatrix} 1/4 \\ -1/8 \end{bmatrix}$$

$$\Rightarrow P(t) = Y(t)X^{-1}(t) = \frac{1}{2} \cdot \frac{9e^{-4(t-1)} - e^{4(t-1)}}{3e^{-4(t-1)} + e^{4(t-1)}}$$

Satellite attitude control



- After normalization,

$$\ddot{\theta} = u$$

- SS model $x := [\theta, \dot{\theta}]^T$

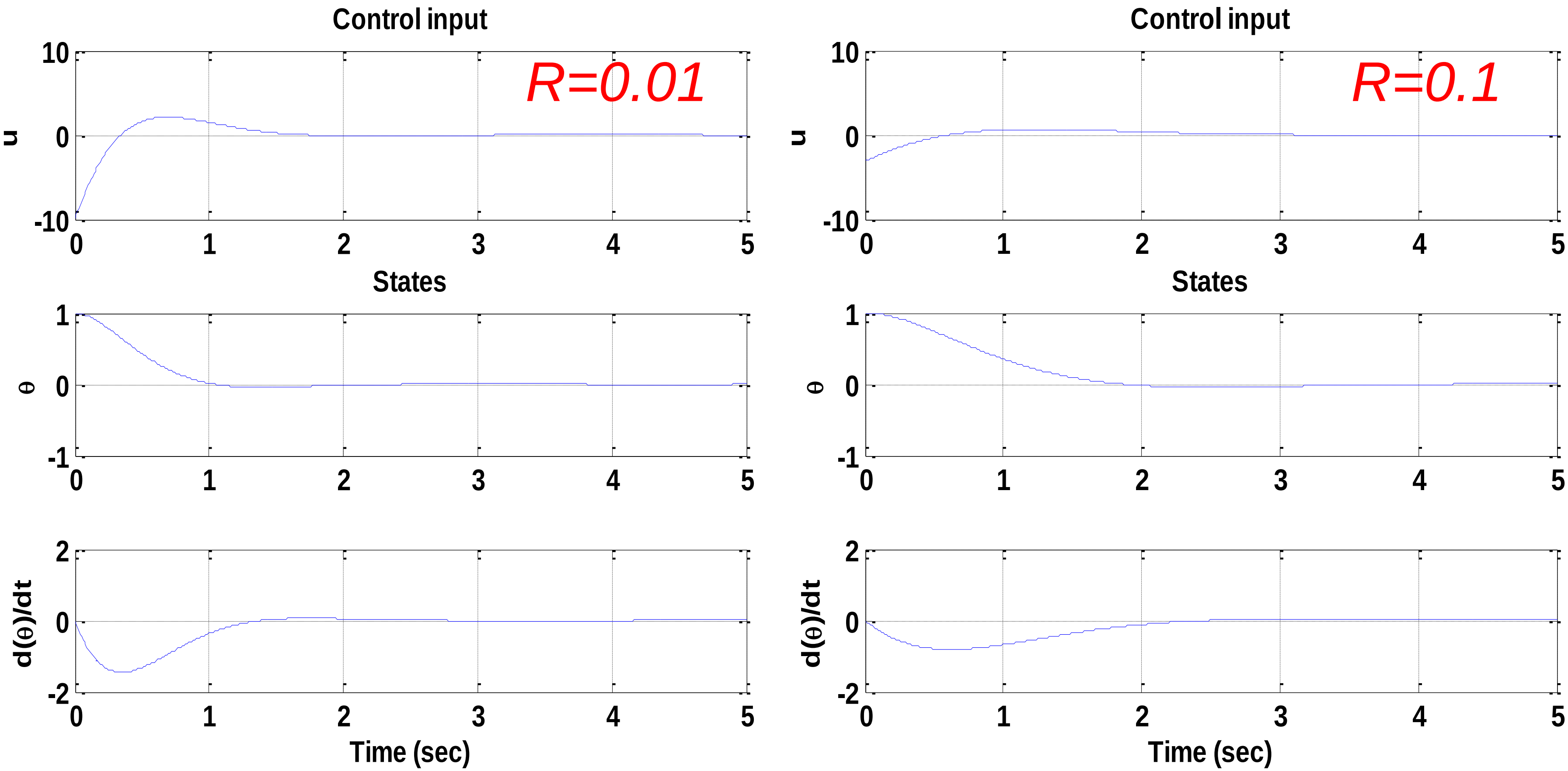
$$\begin{cases} \dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \\ y = \begin{bmatrix} 1 & 0 \end{bmatrix} x \end{cases}$$

- Requirements
  - Small θ
  - Small u

$$\boxed{\begin{aligned} &\min_{u(\cdot)} \int_0^{t_f} \left[x_1^2(t) + Ru^2(t)\right] dt \\ &\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \end{aligned}}$$

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \ B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \ Q = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \ S = 0_2, \ t_f = 5$$

Satellite attitude control (cont'd)



Summary

- CT finite horizon LQR optimal control
  - State feedback with a time-varying feedback gain
  - Matrix Riccati equation
  - Stability is not an issue in finite horizon cases.
  - Extension of finite-horizon LQR optimal control law to time-varying systems is straight-forward.

$$\begin{array}{ccc} A & \longrightarrow & A(t) \\ B & \longrightarrow & B(t) \\ Q & \longrightarrow & Q(t) \\ R & \longrightarrow & R(t) \end{array}$$

- Next, CT infinite-horizon LQR optimal control

Optimality of LQR control law (optional)

1. For any n-by-n symmetric *P(t)* and *x(t)* satisfying

$$\dot{x}(t) = Ax(t) + Bu(t), x(0) = x_0$$

we have $x^T(t_f)P(t_f)x(t_f) - x^T(0)P(0)x(0) = \int_0^{t_f} \frac{d}{dt}\left[x^T(t)P(t)x(t)\right] dt$

$$= \int_0^{t_f} \left[\dot{x}^T(t)P(t)x(t) + x^T(t)\dot{P}(t)x(t) + x^T(t)P(t) \underbrace{\dot{x}(t)}_{Ax(t)+Bu(t)}\right] dt$$

2. Select a *P(t)* s.t. $-\dot{P}(t) = A^T P(t) + P(t)A - P(t)BR^{-1}B^T P(t) + Q,\ P(t_f) = S$

Then $0 = -x^T(t_f)Sx(t_f) + x_0^T P(0)x_0$

$$+ \int_0^{t_f} \left[x^T(t)(P(t)BR^{-1}B^T P(t) - Q)x(t) + u^T(t)B^T Px(t) + x^T(t)PBu(t)\right] dt$$

Optimality of LQR control law (cont'd)

3. By adding the cost function below to both sides

$$J(u(\cdot)) := \int_0^{t_f} \left[x^T(t)Qx(t) + u^T(t)Ru(t)\right] dt + x^T(t_f)Sx(t_f)$$

we have

$$\begin{aligned} J(u(\cdot)) &= x_0^T P(0) x_0 + \int_0^{t_f} \left[x^T(t)P(t)BR^{-1}B^T P(t)x(t) + u^T(t)Ru(t) + u^T(t)B^T Px(t) + x^T(t)PBu(t)\right] dt \\ &= x_0^T P(0) x_0 + \int_0^{t_f} \underbrace{\left[u(t) + R^{-1}B^T P(t)x(t)\right]^T R \left[u(t) + R^{-1}B^T P(t)x(t)\right]}_{\text{Completion of square}} dt \end{aligned}$$

4. Since $R>0$, the function J achieves its minimum when

$$u(t) = -R^{-1}B^T P(t)x(t), t \in \left[0, t_f\right]$$