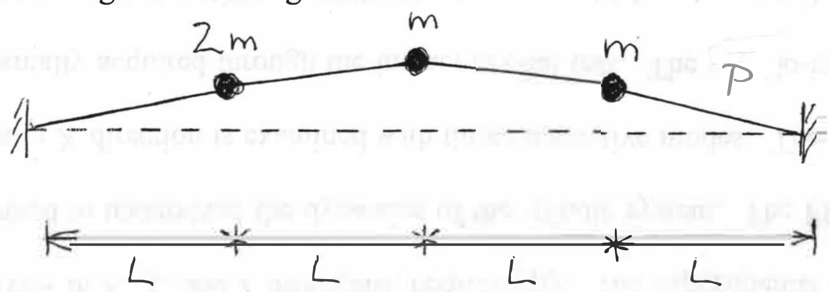
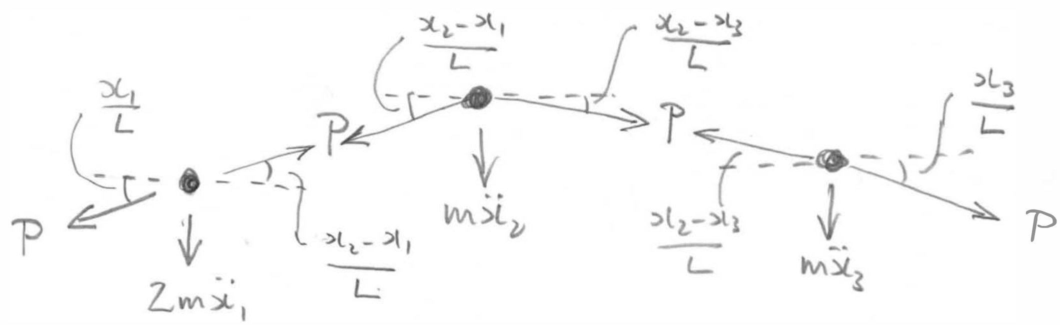
MECH 463 -- Homework 10

1. Three concentrated masses 2m, m, and m are fixed at equal intervals L along the length of a stretched string, of total length 4L, and tension P. The masses can vibrate perpendicular to the length of the string.



(a) Draw free-body diagrams and formulate the equations of motion in matrix format. For convenience, you may write k = P/L.



Use small-angle approximations to define the angles in FBDs.

Vertical equilibrium:

$$-P\frac{x_1}{L} - 2m\ddot{x}_1 + P\left(\frac{x_2-x_1}{L}\right) = 0 \qquad P\left(\frac{x_2-x_3}{L}\right) - m\ddot{x}_3 - P\frac{x_3}{L} = 0$$

$$-P\left(\frac{x_2-x_1}{L}\right) - m\ddot{x}_2 - P\left(\frac{x_2-x_3}{L}\right) = 0$$

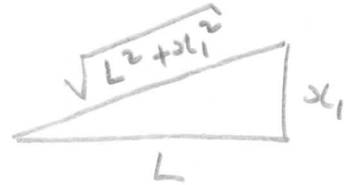
Rearranging and putting into matrix form: $k = \frac{P}{L}$

$$\begin{bmatrix} 2m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{bmatrix} + \begin{bmatrix} 2k & -k & 0 \\ -k & 2k & -k \\ 0 & -k & 2k \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

(b) Use Lagrange's equations to formulate the equations of motion, and confirm that the result is the same as in part (a).

The initial tension P is sufficiently large so that small deflections do not cause a significant tension change. Under these conditions, using the equilibrium state as the zero datum, the potential energy $V = P \times$ string extension.

Consider the leftmost segment:



$$V = P \times \left(\sqrt{L^2 + x_1^2} - L \right)$$

$$= PL \left(\sqrt{1 + \left(\frac{x_1}{L}\right)^2} - 1 \right)$$

$$\simeq PL \left(1 + \frac{x_1^2}{2L^2} + \dots - 1 \right) \quad \text{using binomial series}$$

$$\simeq \frac{P}{2L} x_1^2 = \frac{k}{2} x_1^2 \quad \text{putting } k = \frac{P}{L}$$

Combining similar results for all segments:

$$V = \frac{k}{2} \left(x_1^2 + (x_2 - x_1)^2 + (x_2 - x_3)^2 + x_3^2 \right)$$

Kinetic energy: $T = \frac{m}{2} \left(2\dot{x}_1^2 + \dot{x}_2^2 + \dot{x}_3^2 \right)$

Lagrange Equation: $\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} + \frac{\partial R}{\partial \dot{q}_i} + \frac{\partial V}{\partial q_i} = Q_i$

$i=1 \rightarrow q_i = x_1 \rightarrow \frac{d}{dt}(2m\dot{x}_1) - 0 + 0 + k\left(x_1 - (x_2 - x_1)\right) = 0$

$i=2 \rightarrow q_i = x_2 \rightarrow \frac{d}{dt}(m\dot{x}_2) - 0 + 0 + k\left((x_2 - x_1) + (x_2 - x_3)\right) = 0$

$i=3 \rightarrow q_i = x_3 \rightarrow \frac{d}{dt}(m\dot{x}_3) - 0 + 0 + k\left(-(x_2 - x_3) + x_3\right) = 0$

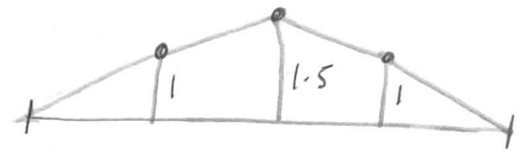
In matrix form:

$$\begin{bmatrix} 2m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{bmatrix} + \begin{bmatrix} 2k & -k & 0 \\ -k & 2k & -k \\ 0 & -k & 2k \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

(c) Write the Rayleigh quotient for the vibrating system. Guess the first mode shape and get an estimate of the corresponding natural frequency.

Guess first mode shape:

$$\underline{v} = \begin{bmatrix} 1 & 1.5 & 1 \end{bmatrix}^T$$



Rayleigh Quotient

$$\omega_R^2 = \frac{\underline{v}^T \underset{\sim}{K} \underline{v}}{\underline{v}^T \underset{\sim}{M} \underline{v}}$$

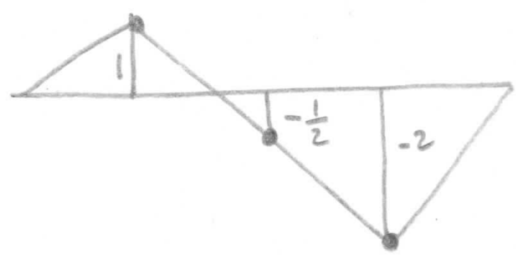
$$\omega_R^2 = \frac{\begin{bmatrix} 1 & 1.5 & 1 \end{bmatrix} \begin{bmatrix} 2k & -k & 0 \\ -k & 2k & -k \\ 0 & -k & 2k \end{bmatrix} \begin{bmatrix} 1 \\ 1.5 \\ 1 \end{bmatrix}}{\begin{bmatrix} 1 & 1.5 & 1 \end{bmatrix} \begin{bmatrix} 2m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m \end{bmatrix} \begin{bmatrix} 1 \\ 1.5 \\ 1 \end{bmatrix}} = \frac{\begin{bmatrix} 1 & 1.5 & 1 \end{bmatrix} k \begin{bmatrix} 0.5 \\ 1 \\ 0.5 \end{bmatrix}}{\begin{bmatrix} 1 & 1.5 & 1 \end{bmatrix} m \begin{bmatrix} 2 \\ 1.5 \\ 1 \end{bmatrix}} = \frac{2.5k}{5.25m}$$

$$\underline{\omega_R^2 = 0.476 \ k/m}$$

(d) Guess the second and third mode shapes and estimate their natural frequencies.

Guess second mode shape:

$$\underline{v} = \begin{bmatrix} 1 & -0.5 & -2 \end{bmatrix}^T$$

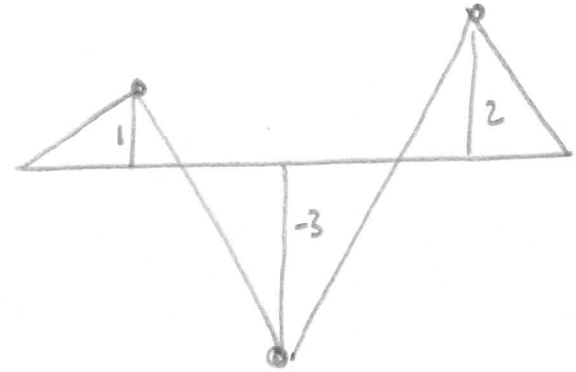


$$\omega_R^2 = \frac{\begin{bmatrix} 1 & -0.5 & -2 \end{bmatrix} \begin{bmatrix} 2k & -k & 0 \\ -k & 2k & -k \\ 0 & -k & 2k \end{bmatrix} \begin{bmatrix} 1 \\ -0.5 \\ -2 \end{bmatrix}}{\begin{bmatrix} 1 & -0.5 & -2 \end{bmatrix} \begin{bmatrix} 2m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m \end{bmatrix} \begin{bmatrix} 1 \\ -0.5 \\ -2 \end{bmatrix}} = \frac{\begin{bmatrix} 1 & -0.5 & -2 \end{bmatrix} k \begin{bmatrix} 2.5 \\ 0 \\ -3.5 \end{bmatrix}}{\begin{bmatrix} 1 & -0.5 & -2 \end{bmatrix} m \begin{bmatrix} 2 \\ -0.5 \\ -2 \end{bmatrix}} = \frac{9.5k}{6.25m}$$

$$\underline{\omega_R^2 = 1.52 \ k/m}$$

Guess third mode shape:

$$\underline{V} = [1 \quad -3 \quad 2]^T$$



$$\omega_R^2 = \frac{[1 \ -3 \ 2]\begin{bmatrix} 2k & -k & 0 \\ -k & 2k & -k \\ 0 & -k & 2k \end{bmatrix}\begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix}}{[1 \ -3 \ 2]\begin{bmatrix} 2m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m \end{bmatrix}\begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix}} = \frac{[1 \ -3 \ 2]\, k \begin{bmatrix} 5 \\ -9 \\ 7 \end{bmatrix}}{[1 \ -3 \ 2]\, m \begin{bmatrix} 2 \\ -3 \\ 2 \end{bmatrix}} = \frac{46k}{15m}$$

$$\underline{\omega_R^2 = 3.067\ k/m}$$

(e) Write the mode shape in general form [1 a b]^T where a and b are factors. Formulate an expression for R(a,b), which is the Rayleigh Quotient for mode shape [1 a b]^T. Use Matlab, Excel or other software to draw a contour plot of this function. Find the natural frequencies and mode shapes from this plot. Compare your results with those from parts (c) and (d). (Hint: plot a in the x direction, b in the y direction, and R(a,b) in the z direction.)

For general mode shape $\underline{V} = [1 \quad a \quad b]^T$

$$\omega_R^2 = \frac{[1 \ a \ b]\begin{bmatrix} 2k & -k & 0 \\ -k & 2k & -k \\ 0 & -k & 2k \end{bmatrix}\begin{bmatrix} 1 \\ a \\ b \end{bmatrix}}{[1 \ a \ b]\begin{bmatrix} 2m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m \end{bmatrix}\begin{bmatrix} 1 \\ a \\ b \end{bmatrix}} = \frac{[1 \ a \ b]\, k \begin{bmatrix} 2-a \\ -1+2a-b \\ -a+2b \end{bmatrix}}{[1 \ a \ b]\, m \begin{bmatrix} 2 \\ a \\ b \end{bmatrix}}$$

$$\omega_R^2 = \frac{2-a-a+2a^2-ab-ab+2b^2}{2+a^2+b^2} \cdot \frac{k}{m}$$

$$\underline{\omega_R^2 = 2 \cdot \frac{1+a^2+b^2-a-ab}{2+a^2+b^2} \cdot \frac{k}{m}}$$

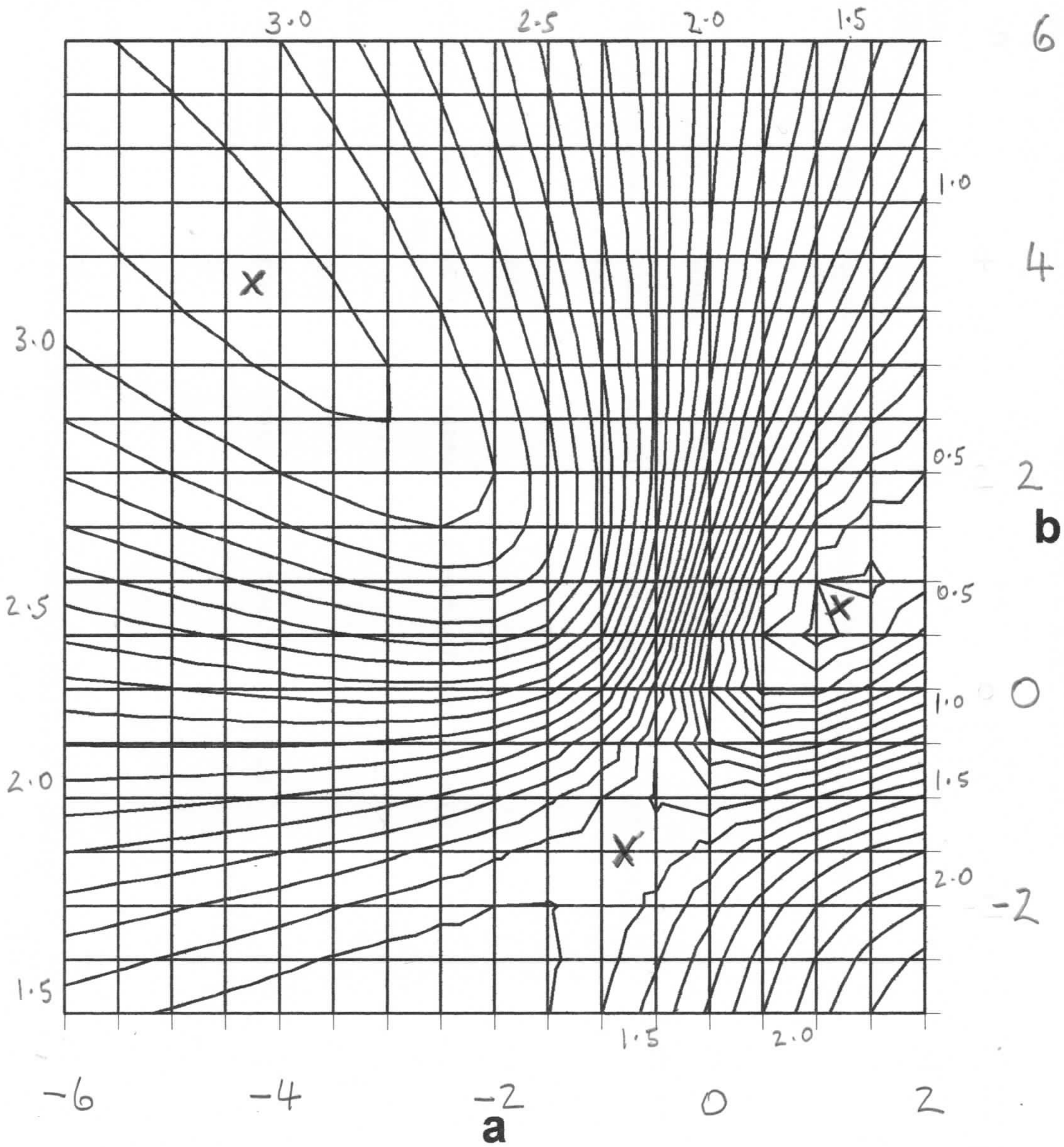
Eigenvalue	=	1.427	Eigenvector	=	1.000	-0.854	-1.489
Eigenvalue	=	0.449	Eigenvector	=	1.000	1.103	0.711
Eigenvalue	=	3.125	Eigenvector	=	1.000	-4.249	3.778

results from solving eigenvalue problem $\underset{\sim}{K}\,\underline{u} = \omega^2 \underset{\sim}{M}\,\underline{u}$

Compare results:

		ω^2			
Mode 1:	guess	0.476	1	1.5	1
	exact	0.449	1	1.103	0.711
Mode 2:	guess	1.520	1	-0.5	-2
	exact	1.427	1	-0.854	-1.489
Mode 3:	guess	3.067	1	-3	2
	exact	3.125	1	-4.249	3.778

We see that the mode shape estimates are crude, but the frequency result are quite good.



Contour plot of R(a,b) from Excel

```
% MECH 463 Homework 10 Q1
%
% Global variables:
% -----------------
% D         eigenvalue matrix
% K         stiffness matrix
% M         mass matrix
% U         mode shape matrix
% V         eigenvector matrix
% wn        natural frequencies

clear all;
close all;

% Assign mass and stiffness matrices
M = [2 0 0; 0 1 0; 0 0 1];
K = [2 -1 0; -1 2 -1; 0 -1 2];

% Solve generalized eigenvalue problem
[V,D] = eig(K,M);

% Extract the natural frequencies from V
wn = sqrt(diag(D));
disp(' ')
disp('Natural Frequencies / sqrt(k/m)')
disp(wn)

% Extract normalized mode shapes from D.
% Eigenvectors V are normalized so that the sum
% of squares of each column = 1.  This is OK,
% but to be consistent with our practice in
% class, we set the first element = 1
U = V ./ V(1,:);
disp('Mode Shapes (in columns)')
disp(U)

% Plot mode shapes
plot(U)
hold on
plot([0 0 0], '--')
xlabel('Mass Index')
ylabel('Displacement')
```

```
>> HW10_Q1

Natural Frequencies / sqrt(k/m)
    0.6698
    1.1945
    1.7676

Mode Shapes (in columns)
    1.0000    1.0000    1.0000
    1.1028   -0.8536   -4.2491
    0.7108   -1.4893    3.7785
```

```matlab
% MECH 463 Homework 10 Q1e
%
% Global variables:
% -----------------
% D         eigenvalue matrix
% J1,J2,J3  moments of inertia
% K         stiffness matrix
% M         mass matrix
% s1,s2,s3  torsional stiffnesses
% U         mode shape matrix
% V         eigenvector matrix
% wn        natural frequencies

clear all;
close all;

% Initialize quantities
a = linspace(-6,2,17);
b = linspace(-3,6,19);

% Compute Rayleigh Quotient
for i = 1:17
    for j = 1:19
        wr(j,i) = 2 * (1+a(i)^2 + b(j)^2 - a(i)
- a(i)*b(j)) ...
                  / (2 + a(i)^2 + b(j)^2);
    end
end

% Plot the Rayleigh Quotient
contour(wr,20)
hold on
xlabel('a  from -6 to 2')
ylabel('b  from -3 to 6')
```