MECH468 Modern Control Engineering
MECH509 Controls

Homework 3. Due: March 8 (Monday), 11:59 pm, 2021.

Solutions

1 Theoretical (hand-calculation) questions

1. Obtain controllable canonical form realization for the following transfer matrices **by hand-calculations**.

(a) $G(s) = \begin{bmatrix} \dfrac{1}{s^2+s} & \dfrac{1}{s^2} \end{bmatrix} = \dfrac{1}{s^3+s^2}\left\{ \begin{bmatrix} 1 & 1 \end{bmatrix} s + \begin{bmatrix} 0 & 1 \end{bmatrix} \right\}$

$$\text{CCF:} \left\{ \begin{array}{rcl} \dot{x} & = & \begin{bmatrix} 0_2 & I_2 & 0_2 \\ 0_2 & 0_2 & I_2 \\ 0_2 & 0_2 & -I_2 \end{bmatrix} x + \begin{bmatrix} 0_2 \\ 0_2 \\ I_2 \end{bmatrix} u \\ y & = & \begin{bmatrix} 0 & 1 & 1 & 1 & 0 & 0 \end{bmatrix} x \end{array} \right.$$

$$\text{OCF:} \left\{ \begin{array}{rcl} \dot{x} & = & \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & -1 \end{bmatrix} x + \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 0 & 0 \end{bmatrix} u \\ y & = & \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} x \end{array} \right.$$

OCF is minimal because it is controllable (or because it is MISO system).

(b) $G(s) = \begin{bmatrix} \dfrac{1}{s^2+s} \\ \dfrac{1}{s^2} \end{bmatrix} = \dfrac{1}{s^3+s^2}\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} s + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$

$$\text{CCF:} \left\{ \begin{array}{rcl} \dot{x} & = & \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u \\ y & = & \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} x \end{array} \right.$$

$$
\text{OCF:}\left\{\begin{array}{rcl}
\dot{x} & = & \begin{bmatrix} 0_2 & 0_2 & 0_2 \\ I_2 & 0_2 & 0_2 \\ 0_2 & I_2 & -I_2 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} u \\
y & = & \begin{bmatrix} 0_2 & 0_2 & I_2 \end{bmatrix} x
\end{array}\right.
$$

CCF is minimal because it is observable (or because it is SIMO system).

(c) $$G(s) = \begin{bmatrix} \dfrac{1}{s+1} & \dfrac{1}{s(s+1)} \\ \dfrac{1}{s(s+1)} & \dfrac{1}{s^2} \end{bmatrix} = \frac{1}{s^2(s+1)} \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} s^2 + \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} s + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

$$
\text{CCF:}\left\{\begin{array}{rcl}
\dot{x} & = & \begin{bmatrix} 0_2 & I_2 & 0_2 \\ 0_2 & 0_2 & I_2 \\ 0_2 & 0_2 & -I_2 \end{bmatrix} x + \begin{bmatrix} 0_2 \\ 0_2 \\ I_2 \end{bmatrix} u \\
y & = & \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \end{bmatrix} x
\end{array}\right.
$$

$$
\text{OCF:}\left\{\begin{array}{rcl}
\dot{x} & = & \begin{bmatrix} 0_2 & 0_2 & 0_2 \\ I_2 & 0_2 & 0_2 \\ 0_2 & I_2 & -I_2 \end{bmatrix} x + \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 1 \\ 1 & 1 \\ 1 & 0 \\ 0 & 0 \end{bmatrix} u \\
y & = & \begin{bmatrix} 0_2 & 0_2 & I_2 \end{bmatrix} x
\end{array}\right.
$$

Both CCF and OCF are non-minimal. Using CCF, obtain the kernel space of the observability matrix:

$$\ker\mathcal{O} = \text{span}\,\{e_1, e_2 - e_3\}$$

So, the coordinate transformation matrix for Kalman decomposition is

$$T^{-1} = \Big[\underbrace{e_3, e_4, e_5, e_6}_{T_o}, \underbrace{e_1, e_2 - e_3}_{T_{\bar{o}}} \Big]$$

Then,

$$TAT^{-1} = \left[\begin{array}{cccc|cc} 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{array}\right], \; TB = \left[\begin{array}{cc} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ \hline 0 & 0 \\ 0 & 0 \end{array}\right]$$

$$CT^{-1} = \left[\begin{array}{cccc|cc} 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \end{array}\right]$$

By eliminating the last two states (which are unobservable states), we can obtain the minimal realization.

2. Obtain observable canonical form realization for the transfer matrices above **by hand calculations**.

3. Obtain minimal realization for the transfer matrices above **by hand calculations**.

   In finding the minimal realization of (c), after obtaining a non-minimal realization **by hand-calculation**, you **can use Matlab** to compute Im$\mathcal{C}$ or ker$\mathcal{O}$, a coordinate transformation matrix T^{-1}, and T, TAT^{-1}, TB and CT^{-1}. Do NOT use Matlab command `minreal.m`.

2 Matlab question

In HW1 and HW2, you got state-space models for the pendulum system and the inverted pendulum system, respectively. For each model, check the minimality of the state-space models.

Solution: Verify controllability and observability for each model.