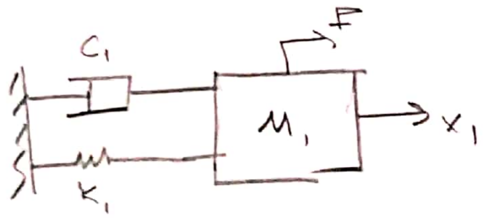
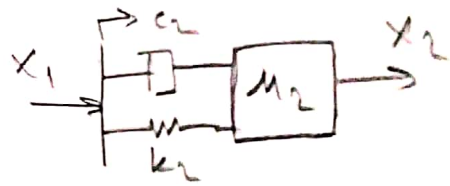
Tutorial 1

Why using Laplace!!

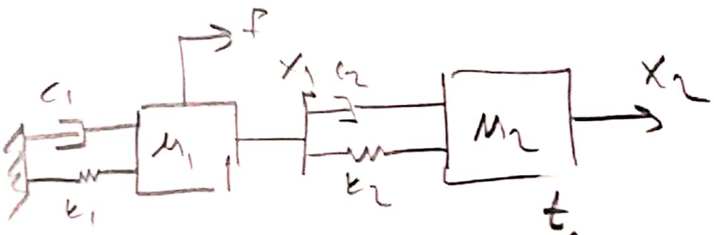
- Transforming differential equations to algebric operations.
- Describing the interaction of dynamic systems as multiplication instead of convolution integral.





Sys ① $\dfrac{x_1}{F(t)} = G_1(t)$ Sys ② $\dfrac{x_2(t)}{x_1(t)} = G_2(t)$

Sys ① + ②



$$\frac{x_2}{F(t)} = G_1(t) * G_2(t) = \int_0^t G_1(\tau)\, G_2(t-\tau)\, d\tau$$

whereas in Laplace Domain

$$\frac{X_2(s)}{F(s)} = G_1(s)\, G_2(s)$$

Laplace Transform

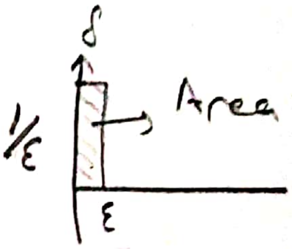
$$\mathcal{L}[f_{(t)}] = F_{(s)} = \int_0^{+\infty} f_{(t)} e^{-st} dt \qquad s: \text{complex variable}$$

$$\mathcal{L}[f_{(t)} * g_{(t)}] = \mathcal{L}\left[\int_0^{t} f(\tau) \cdot g(t-\tau) \cdot d\tau\right] = F_{(s)} \cdot G_{(s)} \qquad (*)$$

Laplace Transform of Impulse function

Definition 1) $\delta(t) = \begin{cases} +\infty & t = 0 \\ 0 & t \neq 0 \end{cases}$, $\int_{-\infty}^{\infty} \delta(t)\, dt = 1$

$$\mathcal{L}[\delta(t)] = \int_0^{+\infty} \delta(t) \cdot e^{-st} dt = \int_{0^-}^{0^+} \delta(t) e^{-st} dt$$



$0 \leftrightarrow +\infty \Rightarrow 0^- < t < 0^+$ (e^{-st} is continuous in this interval)

$$\mathcal{L}[\delta(t)] = \int_{0^-}^{0^+} \delta(t)\, dt = 1 \qquad \left(e^{-st}\Big|_{t=0} = 1 \right)$$

Definition 2 $\quad \delta(t) = \lim_{\varepsilon \to 0} r_\varepsilon(t)$

where $r_\varepsilon(t)$ is unit pulse function: $r_\varepsilon(t) = \frac{1}{\varepsilon}[1(t) - 1(t-\varepsilon)]$

$$= \begin{cases} 1/\varepsilon & 0 \leq t \leq \varepsilon \\ 0 & \text{otherwise} \end{cases}$$

and $1(t)$ is unit step function $\quad 1(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases}$

Laplace Transformation of Step function

$$f_{(t)} = \begin{cases} u & t \geq 0 \\ 0 & t < 0 \end{cases}$$

$$F_{(s)} = \int_0^{+\infty} u \cdot e^{-st}\, dt = u \int_0^{+\infty} e^{-st}\, dt = u\left(-\frac{1}{s} e^{-st} \Big|_0^{+\infty}\right)$$

$$= u\left(-\frac{1}{s} \underbrace{\cancel{e^{-s(\infty)}}}_{0} + \frac{1}{s} \underbrace{e^{-s(0)}}_{1}\right) = \frac{u}{s}$$

Laplace Transformation of Exponential Function

$$f_{(t)} = A e^{-\alpha t} \qquad t \geq 0$$

$$\mathcal{L}\left[A e^{-\alpha t}\right] = \int_0^{+\infty} A \cdot e^{-\alpha t} \cdot e^{-st} = A \int_0^{\infty} e^{-(\alpha+s)t}\, dt =$$

$$= A\left(-\frac{1}{\alpha+s} e^{-(\alpha+s)t}\right)\Big|_0^{+\infty}$$

$$= A\left[\left(-\frac{\overset{0}{\cancel{e^{-(\alpha+s)t}}}}{s+\alpha}\right)\Big|_{t=\infty} - \left(\frac{-\overset{1}{\cancel{e^{-(\alpha+s)t}}}}{s+\alpha}\right)\Big|_{t=0}\right] = \frac{A}{s+\alpha}$$

Laplace Transformation of Ramp Function

$$f_{(t)} = \begin{cases} At & t \geq 0 \\ 0 & t < 0 \end{cases}$$

$$\mathcal{L}[f_{(t)}] = \int_0^\infty \underbrace{t}_{u}\, \underbrace{A e^{-st}\, dt}_{dv} = A\left\{ \underbrace{t\frac{e^{-st}}{-s}}_{uv}\Big|_0^\infty - \int_0^\infty \underbrace{\frac{e^{-st}}{-s}\, dt}_{v.du} \right\}$$

$$= A\left\{ \underbrace{\lim_{t\to\infty}\left(\frac{t e^{-st}}{-s}\right)}_{1} - \underbrace{\lim_{t\to 0}\left(\overset{0}{\frac{t e^{-st}}{-s}}\right)}_{2} + \underbrace{\int_0^\infty \left(\frac{e^{-st}}{s}\right) dt}_{3} \right\}$$

① $\lim\limits_{t\to\infty} \frac{t e^{-st}}{-s} = -\frac{1}{s} \lim\limits_{t\to\infty} \left(\frac{t}{e^{st}}\right)^{\frac{\infty}{\infty}} = -\frac{1}{s} \lim\limits_{t\to\infty}\left(\frac{1}{s e^{st}}\right) = 0$

L'hopital's rule

② 0

③ $\rightarrow \int_0^\infty \left(\frac{e^{-st}}{s}\right) dt = \left(-\frac{1}{s^2} e^{-st}\right)\Big|_0^\infty = 0 - \left(-\frac{1}{s^2}\right) = \frac{1}{s^2}$

$$\mathcal{L}[At] = \frac{A}{s^2}$$

L Hospital $\lim\limits_{x\to c} \frac{f(x)}{g(x)} = \lim\limits_{x\to c} \frac{f'(x)}{g'(x)}$ / $\frac{0}{0}, \frac{\infty}{\infty}$

Integration By Part: $d(uv) = du.v + u\,dv$

$\rightarrow \int u\,dv = uv - \int v\,du$

Laplace Transform of Sin (wt)?

$$A = \int_0^\infty \underbrace{\sin(wt)}_{u}\,\underbrace{e^{-st}}_{dv}\,dt \qquad s>0$$

$u = \sin(wt) \qquad v = -\frac{1}{s}e^{-st}$

$$A = u.v\Big|_0^\infty - \int_0^\infty v\,du = \underbrace{\sin(wt)\left(-\frac{1}{s}e^{-st}\right)\Big|_0^\infty}_{1} - \underbrace{\int_0^\infty\left(-\frac{1}{s}e^{-st}\right)(w\cos(wt))dt}_{2}$$

1 $\quad e^{-st}\Big|_{t=\infty} = 0$ and $\sin(wt)\Big|_{t=0} = 0 \qquad$ 1 $= 0$

1, 2 $\Rightarrow \quad A = \frac{w}{s}\int \underbrace{\cos(wt)}_{u}\,\underbrace{e^{-st}}_{dv}\,dt$

$u = \cos wt$

$v = -\frac{1}{s}e^{-st}$

$$A = \frac{w}{s}\left[uv\Big|_0^\infty - \int_0^\infty v\,du\right]$$

$$= \frac{w}{s}\left[\underbrace{\cos(wt)\left(-\frac{1}{s}e^{-st}\right)\Big|_0^\infty}_{3} - \underbrace{\int_0^\infty\left(-\frac{1}{s}e^{-st}\right)(-w\sin(wt))dt}_{4}\right]$$

3 $\rightarrow e^{-st}\Big|_{t=\infty} = 0$ and $\cos(wt)\left(-\frac{1}{s}e^{-st}\right)\Big|_{t=0} = -\frac{1}{s}$

3 $= 0 - \left(-\frac{1}{s}\right) = \frac{1}{s}$

4 $\quad \int_0^\infty\left(-\frac{1}{s}e^{-st}\right)(-w\sin(wt))\,dt = \frac{w}{s}\int_0^\infty \sin(wt)e^{-st}\,dt = \frac{w}{s}A$

3, 4 $\rightarrow A = \frac{w}{s}\left[\frac{1}{s} - \frac{w}{s}A\right] \rightarrow A\left(1+\frac{w^2}{s^2}\right) \rightarrow \boxed{A = \frac{w}{s^2+w^2}}$

HW: Laplace Transform of $\cos(wt)$??