MECH468 : Modern Control Engineering
MECH509 : Controls

L21 : State feedback
Stabilizability

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Zoom lecture to be recorded and posted on Canvas

Course plan

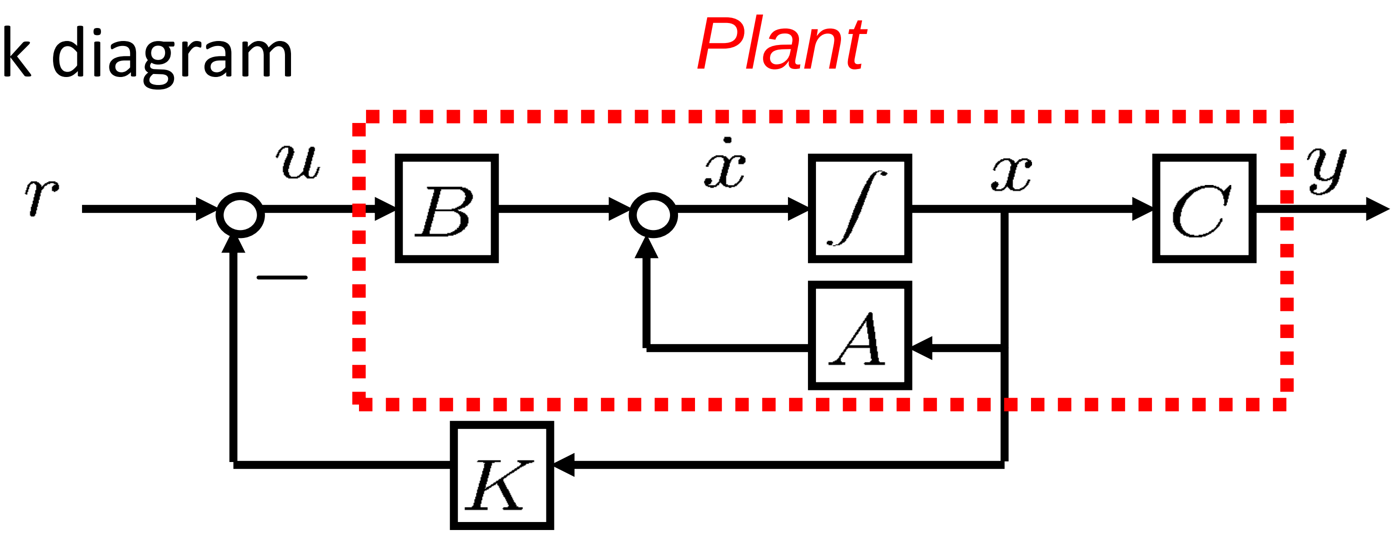
Topics	CT	DT
Modeling	✓	✓
Stability	✓	✓
Controllability/observability	✓	✓
Realization	✓	✓
→ State feedback/observer		
LQR/Kalman filter		

Review & today's topic

- In the last two lectures
  - State feedback
  - Pole placement theorem: *Arbitrary pole placement is possible by a state feedback u=-Kx if and only if (A,B) is controllable*
  - Direct method & canonical form method to compute the feedback gain *K* ("place.m")
- Today's topics
  - Stabilizability
  - Where to place closed-loop poles
  - Appendix: Lyapunov equation method (not covered)

State feedback (review)

- Block diagram



- Open-loop and closed-loop systems

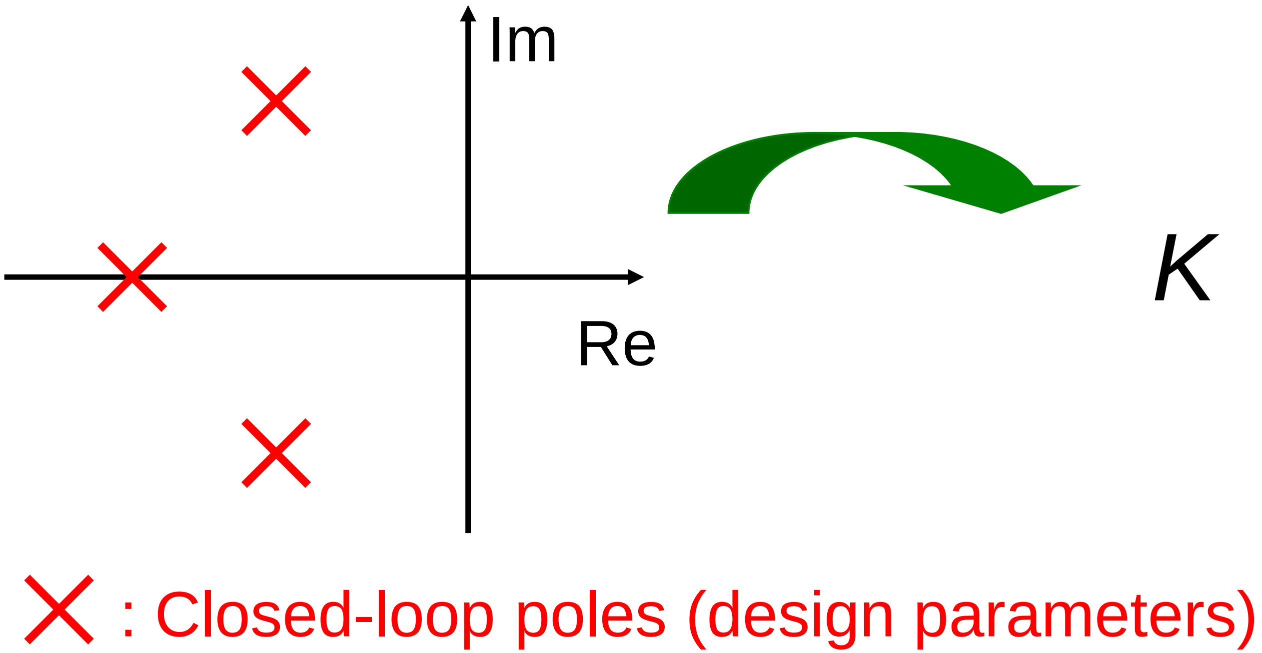
$$\begin{cases} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) \end{cases}$$

$$u(t) = -Kx(t) + r(t)$$

$$\Rightarrow \begin{cases} \dot{x}(t) &= (A - BK)x(t) + Br(t) \\ y(t) &= Cx(t) \end{cases}$$

Pole placement theorem (review)

- *If (A,B) is controllable, the eigenvalues of (A-BK) can be placed arbitrarily (provided that they are symmetric with respect to the real axis).*



Stabilizability

- Suppose that (*A*,*B*) is NOT controllable.
- If the "uncontrollable part" of *A*-matrix is stable, then the system is called *stabilizable*.

$$\dot{x}(t) = Ax(t) + Bu(t)$$

Controllable pair

$$z(t) := Tx(t) \quad \Rightarrow \quad \begin{bmatrix} \dot{z}_c(t) \\ \dot{z}_{\bar{c}}(t) \end{bmatrix} = \begin{bmatrix} A_c & A_{12} \\ 0 & A_{\bar{c}} \end{bmatrix} \begin{bmatrix} z_c(t) \\ z_{\bar{c}}(t) \end{bmatrix} + \begin{bmatrix} B_c \\ 0 \end{bmatrix} u(t)$$

Eigenvalues of this cannot be changed by state feedback.

Stabilizability (cont'd)

$$
\begin{aligned}
A - BK &= T^{-1}\begin{bmatrix} A_c & A_{12} \\ 0 & A_{\bar{c}} \end{bmatrix} T - T^{-1}\begin{bmatrix} B_c \\ 0 \end{bmatrix} K \\
&= T^{-1}\left\{\begin{bmatrix} A_c & A_{12} \\ 0 & A_{\bar{c}} \end{bmatrix} - \begin{bmatrix} B_c \\ 0 \end{bmatrix} KT^{-1}\right\} T \\
&\quad (KT^{-1} =: \begin{bmatrix} K_c & K_{\bar{c}} \end{bmatrix}) \\
&= T^{-1}\begin{bmatrix} A_c - B_c K_c & A_{12} - B_c K_{\bar{c}} \\ 0 & A_{\bar{c}} \end{bmatrix} T
\end{aligned}
$$

$$
\Rightarrow \quad \mathrm{eig}(A - BK) = \underbrace{\mathrm{eig}(A_c - B_c K_c)}_{\text{Arbitrarily assignable}} \cup \underbrace{\mathrm{eig}(A_{\bar{c}})}_{\text{Not movable!}}
$$

Remarks on stabilizability

- If a system is controllable, it is stabilizable.
- Stabilizability is necessary (but may not be sufficient) for successful feedback control.
- The real plant needs to be modified (e.g. by adding actuators, changing the location of actuators, or changing the types of actuators) if
  - the system is not stabilizable, or
  - the uncontrollable part is stable, but limits the closed-loop performance (e.g., the un-movable eigenvalues are too close to the imaginary axis).

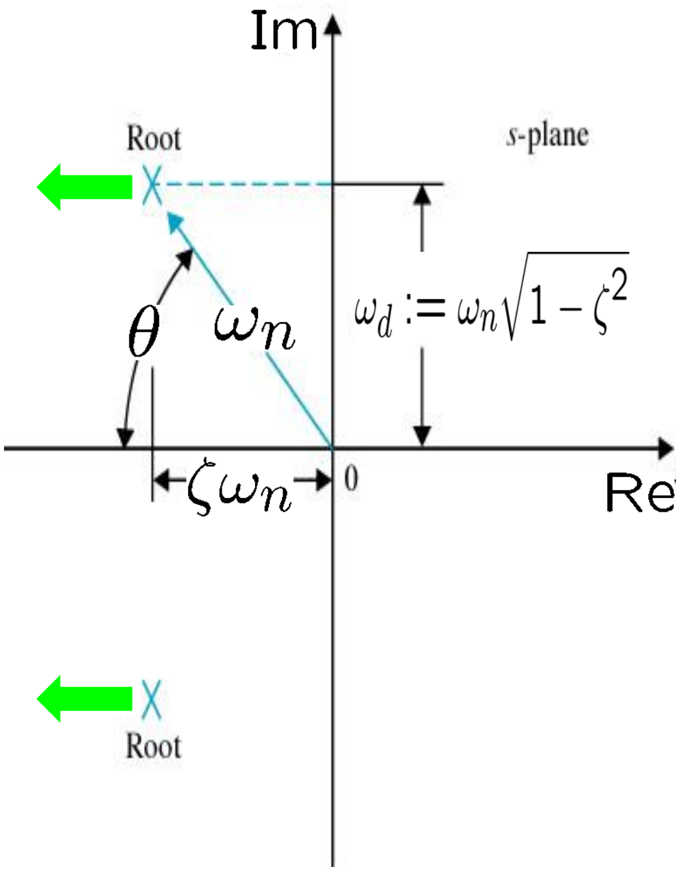
Where to place closed-loop poles Rules of thumbs

- Move the poles for improvement of stability and performance.
- Do not try to move poles farther than necessary.
  - Control input should not be too large.

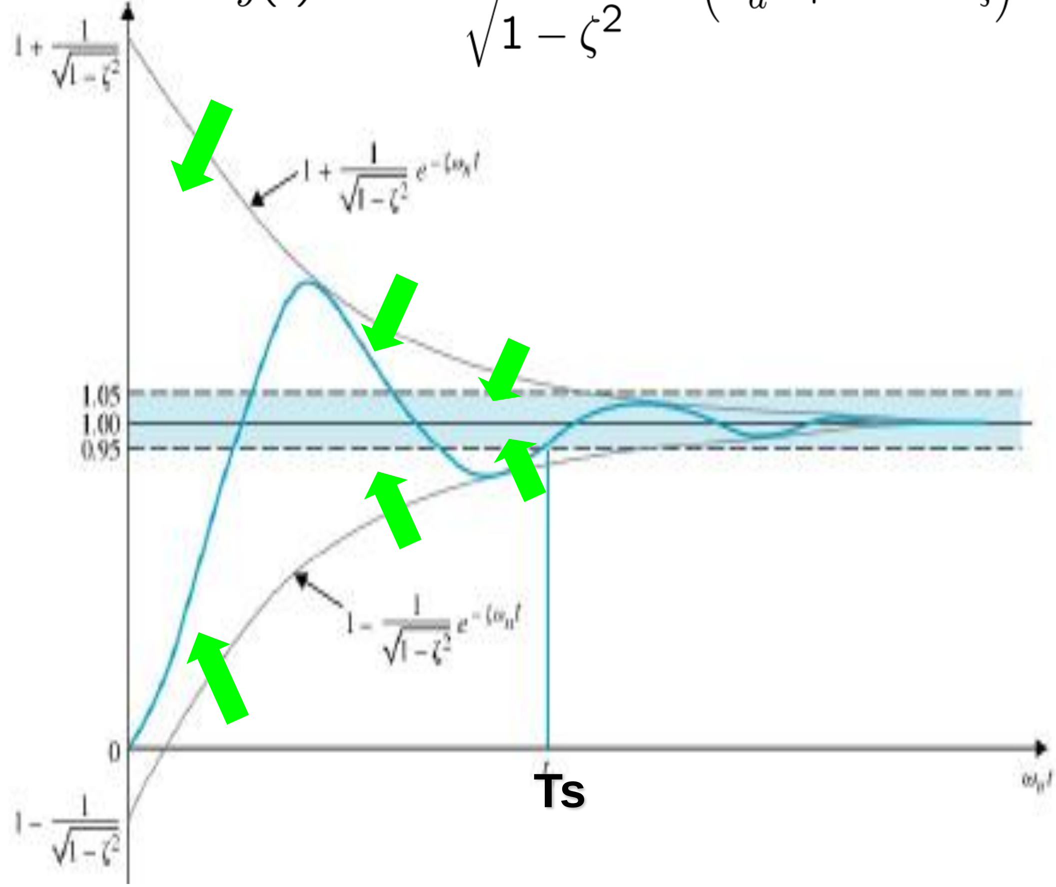
$$K := \underbrace{[\alpha_n - a_n, \cdots, \alpha_1 - a_1]}_{\text{(desired CL) – (OL)}} T$$

- *Place the poles in similar distances from the origin*, to make the control effort efficient.
  - Control effort depends on the farthest poles, while speed depends on the closest poles to the origin.

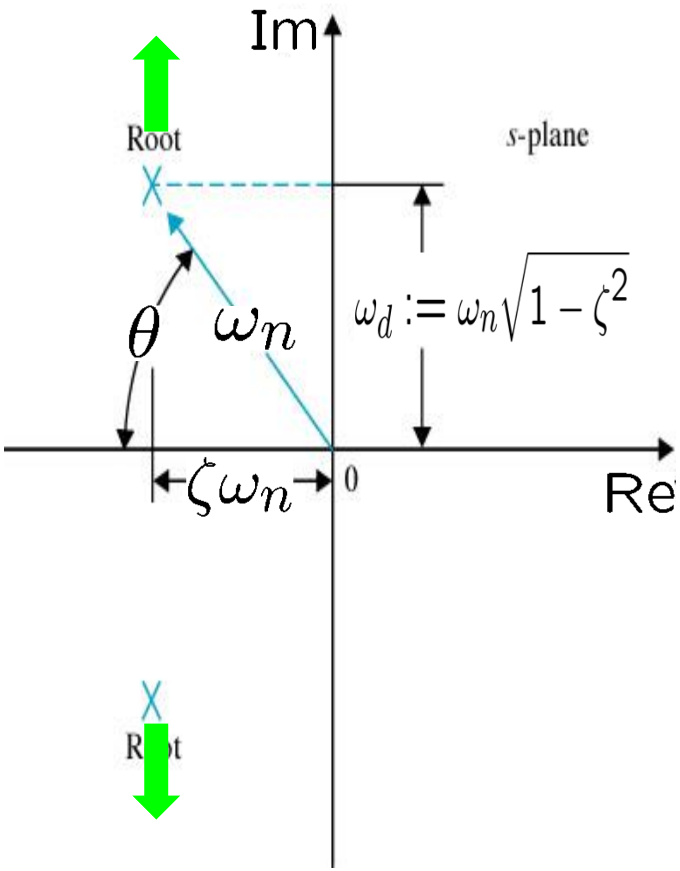
Influence of real part of poles (How to decrease settling time)



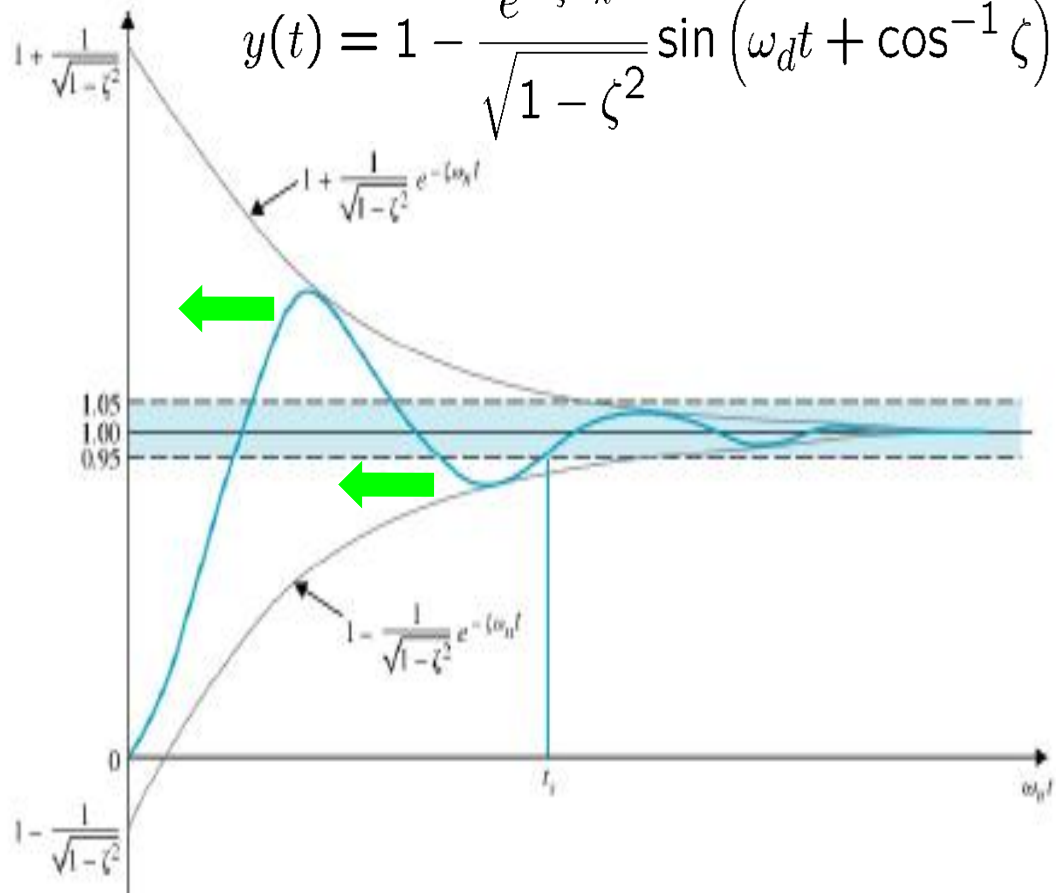
$$y(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\sin\left(\omega_d t + \cos^{-1}\zeta\right)$$



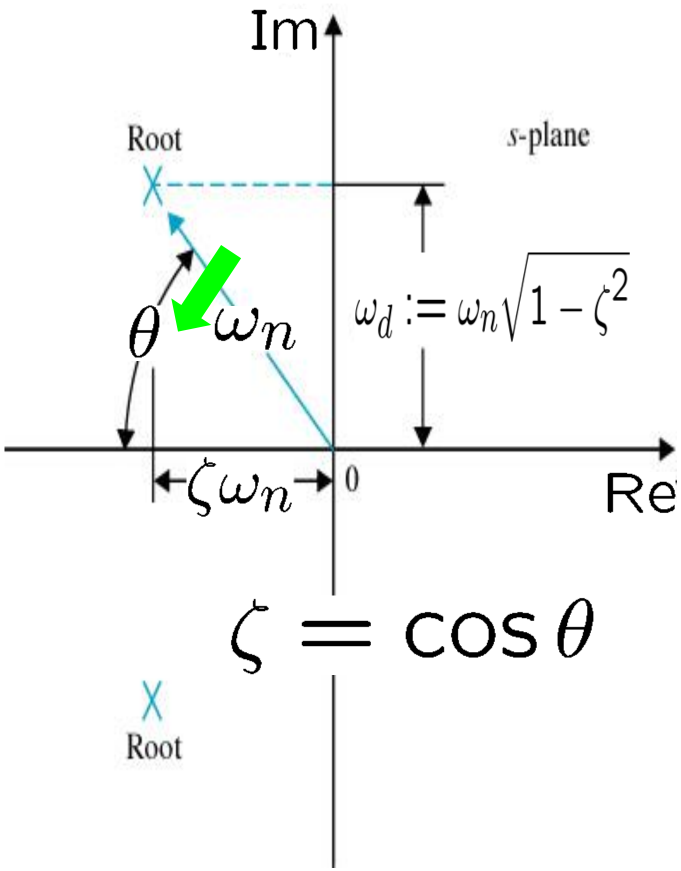
Influence of imag. part of poles (How to increase oscillation freq.)



$$y(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\sin\left(\omega_d t + \cos^{-1}\zeta\right)$$

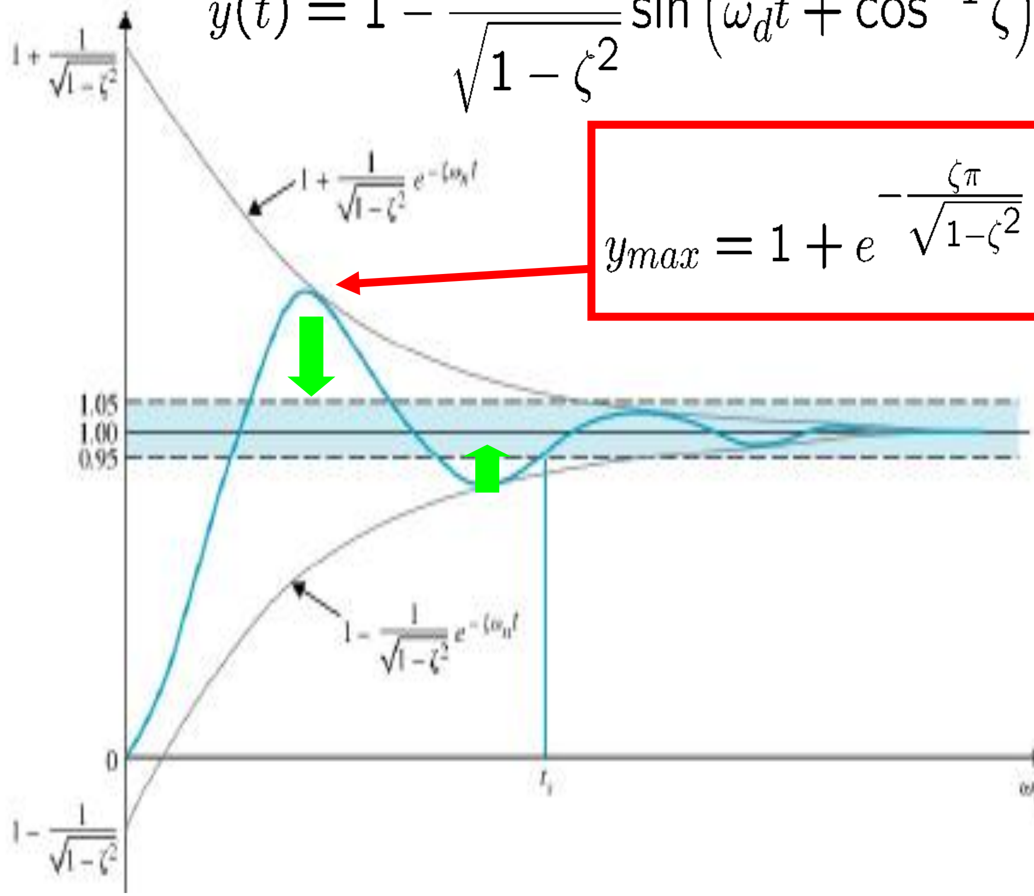


Influence of angle of poles (How to decrease overshoot)



$$\zeta = \cos\theta$$

$$y(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\sin\left(\omega_d t + \cos^{-1}\zeta\right)$$



$$\boxed{y_{max} = 1 + e^{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}}}$$

Example: DC motor position control

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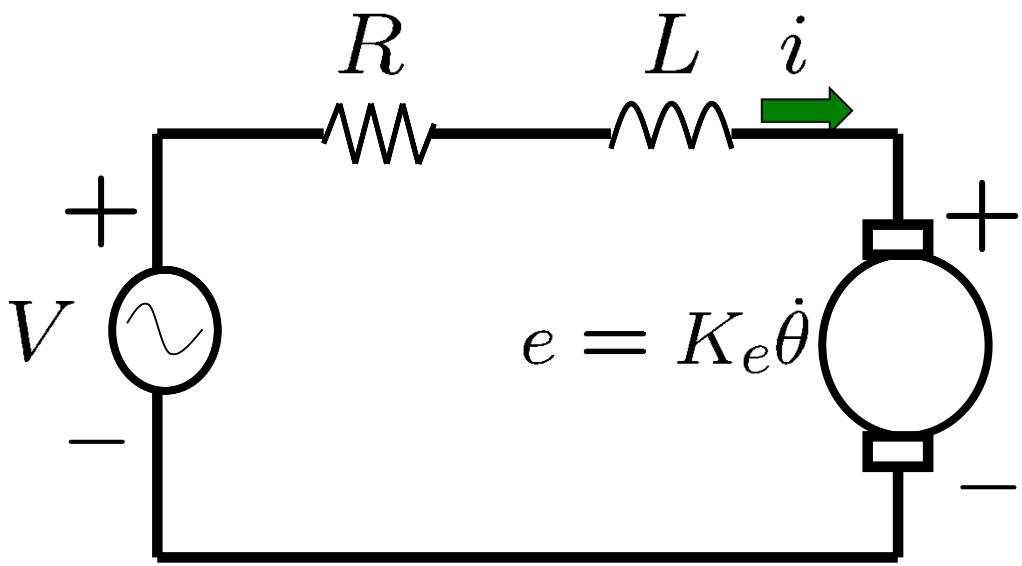
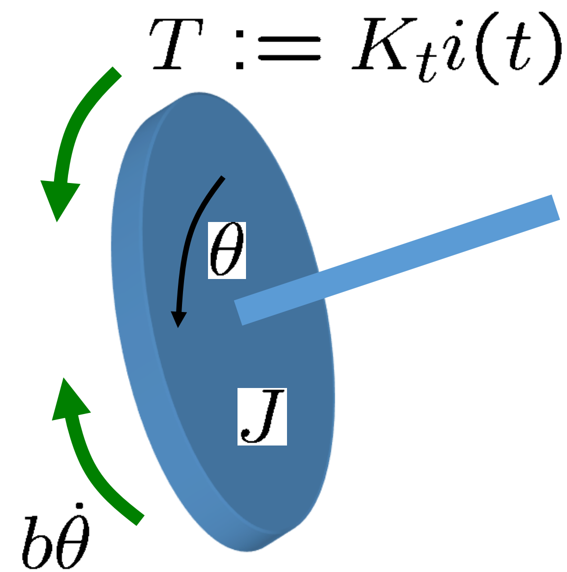




$$
\begin{aligned}
J\ddot{\theta}(t) &= K_t i(t) - b\dot{\theta}(t) \\
V(t) &= Ri(t) + L\tfrac{d}{dt}i(t) + K_e\dot{\theta}(t)
\end{aligned}
$$

DC motor position control (cont'd)

- State-space model

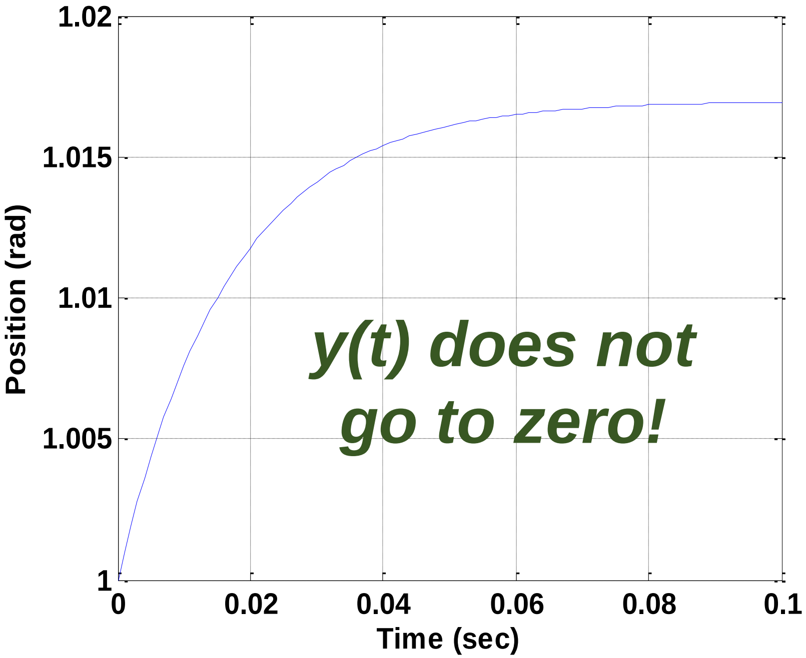
$$\begin{cases} \frac{d}{dt}\begin{bmatrix}\theta\\ \dot{\theta}\\ i\end{bmatrix} = \begin{bmatrix}0 & 1 & 0\\ 0 & -b/J & K_t/J\\ 0 & -K_e/L & -R/L\end{bmatrix}\begin{bmatrix}\theta\\ \dot{\theta}\\ i\end{bmatrix} + \begin{bmatrix}0\\ 0\\ 1/L\end{bmatrix}V(t)\\ \theta(t) = \begin{bmatrix}1 & 0 & 0\end{bmatrix}\begin{bmatrix}\theta\\ \dot{\theta}\\ i\end{bmatrix}\end{cases}$$

J	moment of inertia	$3.2284 \cdot 10^{-6}$	$\mathrm{kg} \cdot \mathrm{m}^2$
b	damping coefficient	$3.0577 \cdot 10^{-6}$	$\mathrm{kg} \cdot \mathrm{m}^2/\mathrm{s}$
$K_t = K_e$	emf constant	$2.74 \cdot 10^{-2}$	$\mathrm{N} \cdot \mathrm{m}/\mathrm{Amp}$

$$R = 4\Omega \quad L = 2.75 \cdot 10^{-6} H$$

DC motor position control (cont'd)

- Specifications: For $\begin{bmatrix} \theta(0) & \dot{\theta}(0) & i(0) \end{bmatrix}^T = \begin{bmatrix} 1 & 1 & 10 \end{bmatrix}^T$
  - *r(t)=0*
  - Settling time < 40 ms
  - Overshoot < 16%
  - Zero steady state error
- Open-loop system
  - Poles = 0, -59.226
    -1.4545E+6
  - Not asymptotically stable!

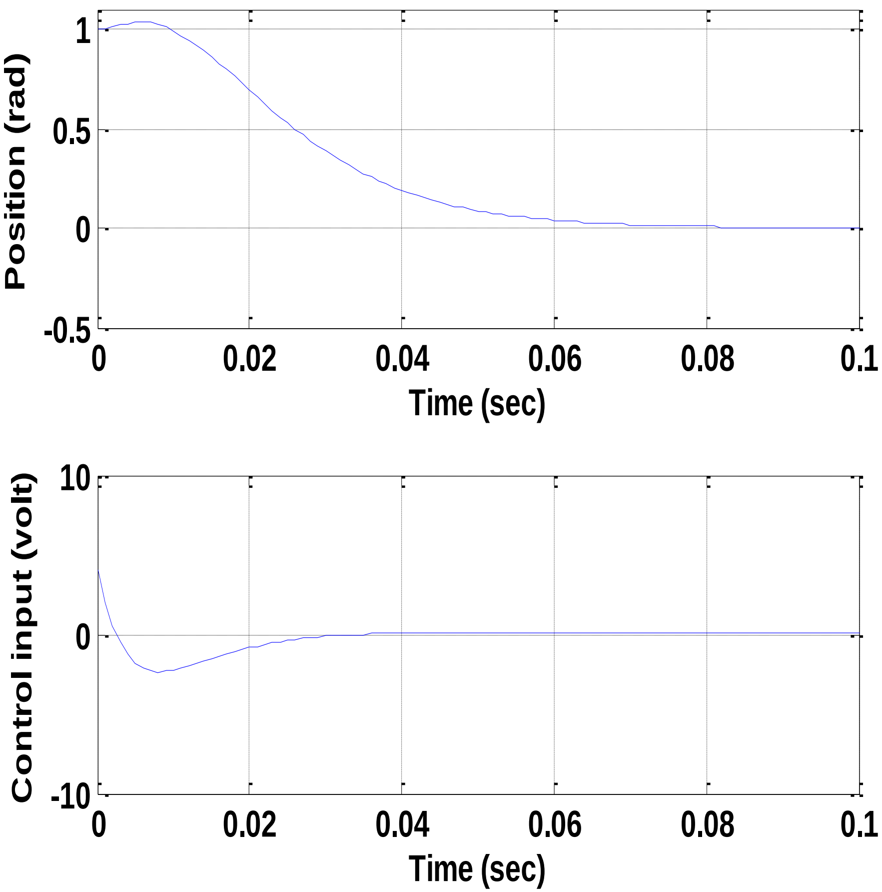


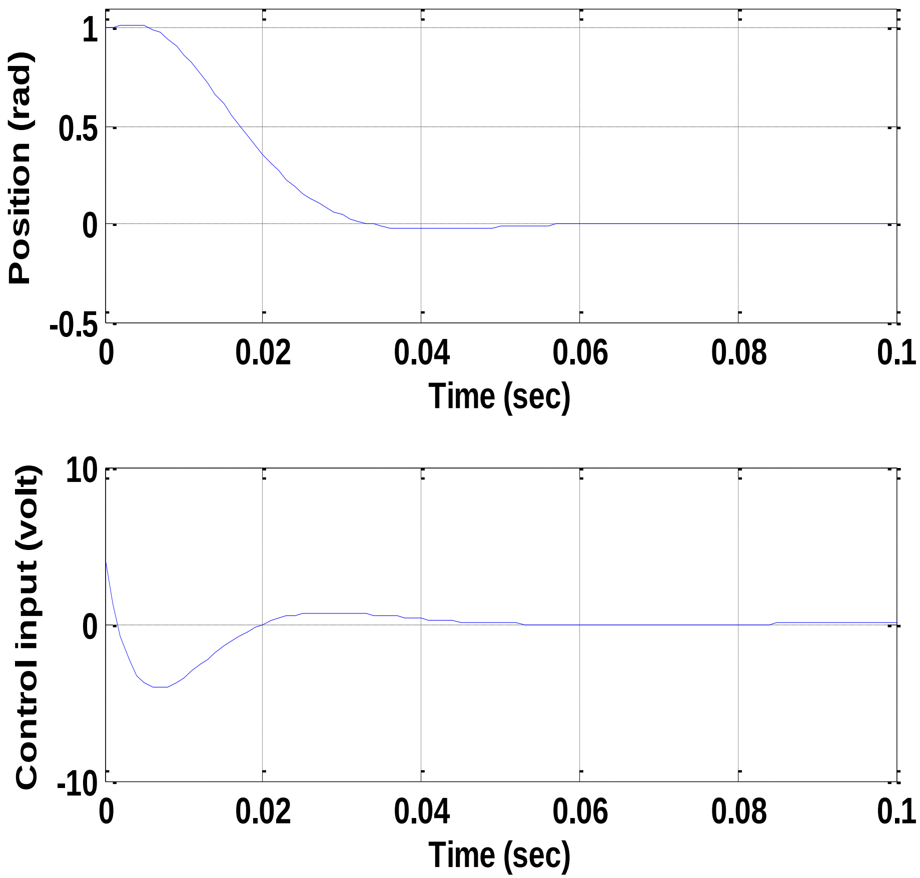
Feedback control for stability & performance!

DC motor position control (cont'd)

poles: $-200, \ -100 \pm 10j$

poles: $-200, \ -100 \pm 100j$





Exercise

- Try the simulation by yourselves! (Matlab code “motorposition.m” is posted on Canvas.) Change the pole locations, and get a feeling how responses are affected by the pole location.

Summary

- State feedback
  - Stabilizability
  - Where to place closed-loop poles (one always needs trial-and-error)
    - Rules of thumbs
    - Optimal method (We will learn more later in LQR.)
  - Lyapunov equation method to design state feedback gain (Appendix, not covered in the exam)
- Next,
  - Tracking (servo) control

Lyapunov equation method

Step 0: Check whether (A,B) is controllable. If it is, go to Step 1.

Step 1: Select a matrix F with desired eigenvalues s.t.
$\sigma(F) \cap \sigma(A) = \emptyset$ (σ : set of eigenvalues)

Step 2: Select a matrix $\bar{K} \in \mathbb{R}^{p \times n}$: $(F, \bar{K})$ is observable

Step 3: Solve the Lyapunov equation w.r.t. T

$$AT - TF = B\bar{K}$$

Step 4: If T is singular, redo from Step 3 with different $\bar{K}$

Step 5: If T is nonsingular, $K = \bar{K}T^{-1}$

Remarks on Lyapunov eq. method

- How to construct F
  - Suppose that the desired poles are $\lambda_1, \alpha_1 \pm j\beta_1, \alpha_2 \pm j\beta_2$
    Then, F can be chosen as

$$F = \begin{bmatrix} \lambda_1 & 0 & 0 & 0 & 0 \\ 0 & \alpha_1 & \beta_1 & 0 & 0 \\ 0 & -\beta_1 & \alpha_1 & 0 & 0 \\ 0 & 0 & 0 & \alpha_2 & \beta_2 \\ 0 & 0 & 0 & -\beta_2 & \alpha_2 \end{bmatrix}$$

- How to construct $\bar{K}$: Select randomly!
- Condition for the unique solution of the Lyapunov equation:

$$AX - XB = C, A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{m \times m} \qquad \boxed{\sigma(A) \cap \sigma(B) = \emptyset}$$

Idea of Lyapunov eq. method

- From Steps 3 & 5, $K = \bar{K}T^{-1}$

$$AT - TF = B\bar{K} \iff (A - BK)T = TF$$

$$\iff A - BK = TFT^{-1}$$

- If T is nonsingular

$$\{\text{eigenvalues of } A - BK\} = \{\text{eigenvalues of } F\}$$

- Conditions $\begin{cases} (A, B) : \text{controllable} \\ (F, \bar{K}) : \text{observable} \end{cases}$

  are necessary for the Lyapunov equation to have a nonsingular solution. (Proof omitted.)

An example

- SS model $\dot{x}(t) = \underbrace{\left[\begin{array}{cc} 1 & 3 \\ 3 & 1 \end{array}\right]}_{A} x(t) + \underbrace{\left[\begin{array}{c} 1 \\ 0 \end{array}\right]}_{B} u(t)$
- CL poles $-1 \pm 2j$
  - Step 1: $F = \left[\begin{array}{cc} -1 & 2 \\ -2 & -1 \end{array}\right]$
  - Step 2: $\bar{K} = \left[\begin{array}{cc} 1 & 0 \end{array}\right]$
  - Step 3: $AT - TF = B\bar{K}$ $\quad T = \text{lyap}(A, -F, -B\bar{K})$
  - Step 5: $K = \bar{K}T^{-1} = \left[\begin{array}{cc} 4 & \frac{17}{3} \end{array}\right]$