MECH468 : Modern Control Engineering
MECH509 : Controls

L8 : Internal stability

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Zoom lecture to be recorded and posted on Canvas

Course plan

Topics	CT	DT
Modeling	✓	✓
→ Stability		
Controllability/observability		
Realization		
State feedback/observer		
LQR/Kalman filter		

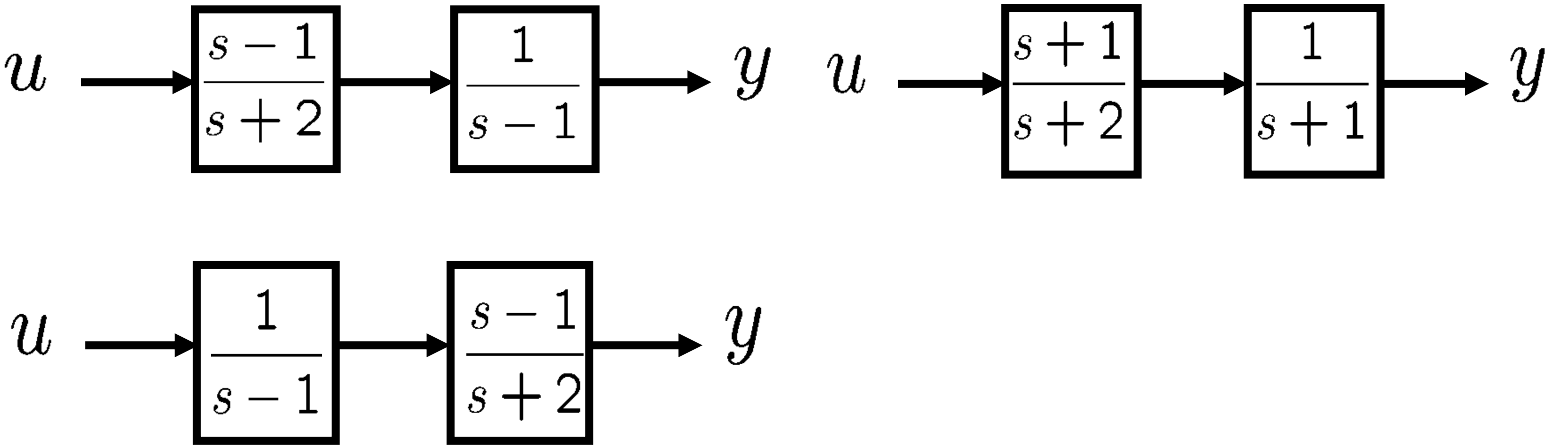
Review & today's topic

- Last lecture was about BIBO stability.
  - BIBO stability cannot deal with stability for state-space models with nonzero initial states.
- Today, we study internal stability for
  - LTI CT : $\dot{x}(t) = Ax(t), \; x(0) = x_0$
  - LTI DT : $x[k+1] = Ax[k], \; x[0] = x_0$

(Note that input is set to be zero.)

Issue in BIBO stability

- Consider three BIBO stable open-loop systems:



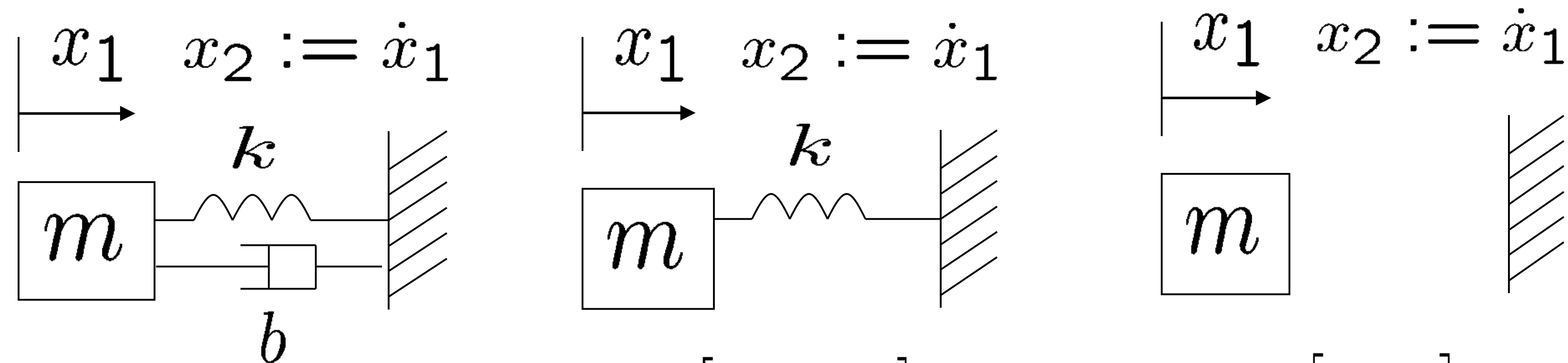
  - Pole/zero cancellation in unstable region by series connection is VERY BAD!
  - *Unstable system MUST be stabilized by feedback!!!*

Issue in BIBO stability (cont'd)

- Why is unstable cancellation by series connection not allowed?
  - Plant model is never exact! (and hence cancellation will not occur in real world.)
  - Neither nonzero initial condition nor input disturbance is allowed. (very fragile)
  - Some internal signal may go unbounded, even if output is bounded.
- BIBO stability cannot detect such cancellation, but internal stability can!

Internal stability: Mechanical example

- Imagine what will happen if we start near *x=0*.



$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -k/m & -b/m \end{bmatrix} x(t)$$

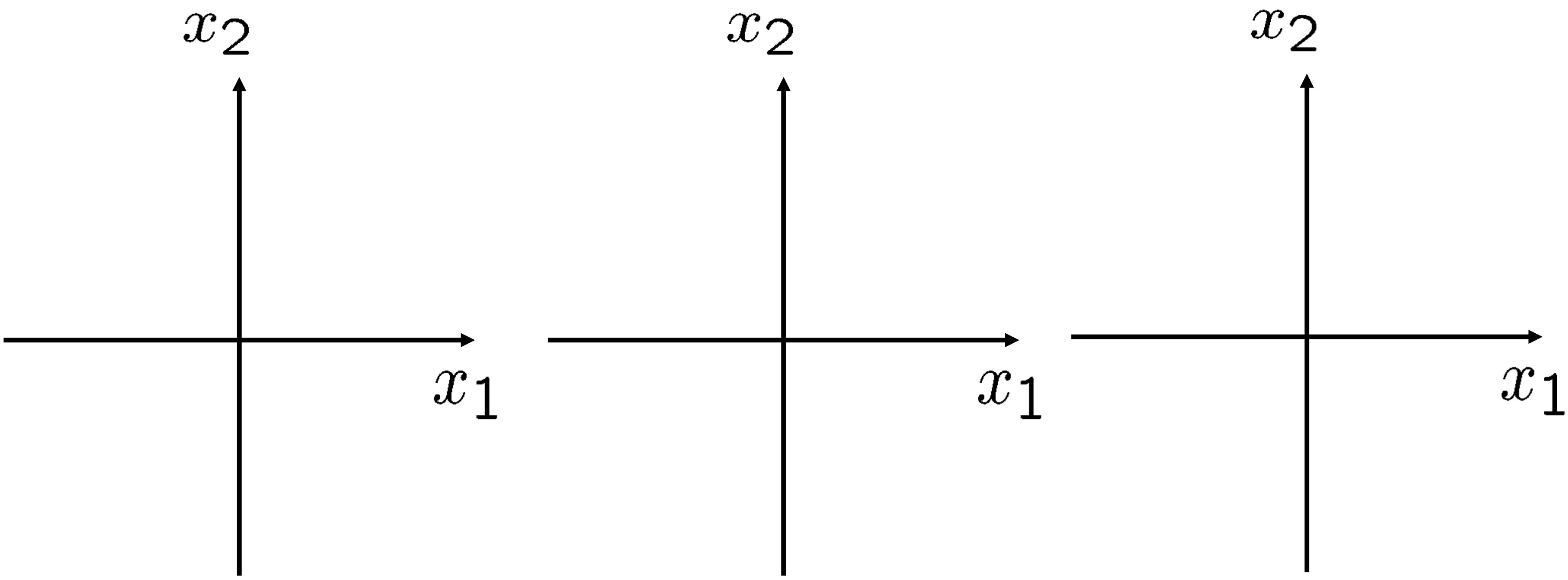
$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -k/m & 0 \end{bmatrix} x(t)$$

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x(t)$$

- Can we explain the difference of the state behavior with the difference of *A*-matrix?

Phase plot

- Plot of state trajectory in state-space



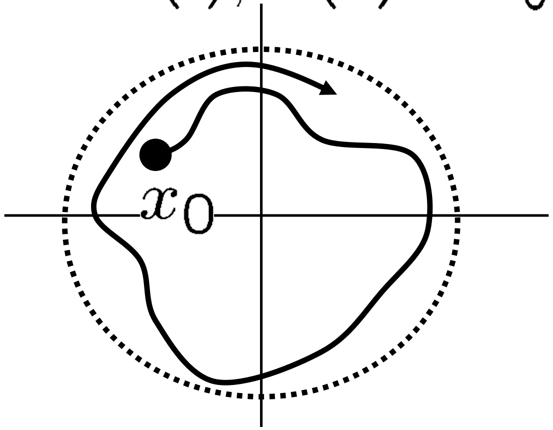
Definition of internal stability

(DT case is analogous.)

- Consider system (no input): $\dot{x}(t) = Ax(t),\ x(0) = x_0$

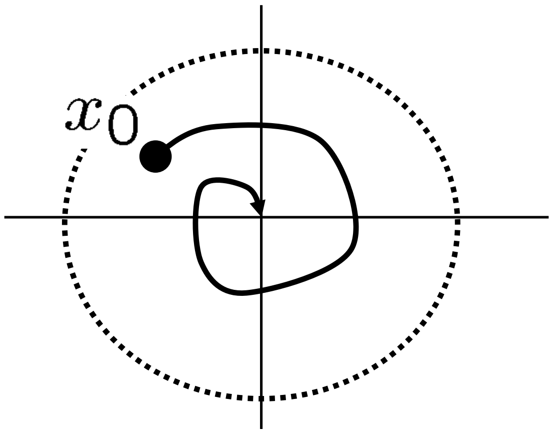
  - The system is marginally stable (stable in the sense of Lyapunov), if, for any *x0*, the following holds for some *M>0*:

$$||x(t)|| \leq M < \infty, \forall t > 0$$



  - The system is asymptotically stable if it is stable and for any *x0*,

$$x(t) \to 0 \text{ as } t \to \infty$$



Eigenvalue criteria

- Eigenvalue of A : λ_i

Stability	CT	DT
Asymptotically stable	$\mathrm{Re}[\lambda_i] < 0, \forall i$	$\lvert\lambda_i\rvert < 1, \forall i$
Marginally stable	$\mathrm{Re}[\lambda_i] \leq 0, \forall i$ For $\mathrm{Re}[\lambda_i] = 0$ $\mathrm{rank}[\lambda_i I - A] = n - m_i$	$\lvert\lambda_i\rvert \leq 1, \forall i$ For $\lvert\lambda_i\rvert = 1$ $\mathrm{rank}[\lambda_i I - A] = n - m_i$

Multiplicity of λ_i

Otherwise, the system is *unstable*, i.e., for some x_0, $x(t)$ is unbounded.

Note: If $m_i=1$, then the rank condition always holds.

Idea of stability condition

- CT

$$\left.\begin{aligned}\dot{x}(t) &= Ax(t)\\ x(0) &= x_0\end{aligned}\right\} \Rightarrow x(t) = e^{At}x_0$$

Examples

$$A = \begin{bmatrix} -2 & 0 \\ 0 & -3 \end{bmatrix} \Rightarrow e^{At} = \begin{bmatrix} e^{-2t} & 0 \\ 0 & e^{-3t} \end{bmatrix} \to 0$$

$$A = \begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix} \Rightarrow e^{At} = \begin{bmatrix} e^{2t} & 0 \\ 0 & e^{-3t} \end{bmatrix} \to \begin{bmatrix} \infty & 0 \\ 0 & 0 \end{bmatrix}$$

- DT

$$\left.\begin{aligned}x[k+1] &= Ax[k]\\ x[0] &= x_0\end{aligned}\right\} \Rightarrow x[k] = A^k x_0$$

Examples

$$A = \begin{bmatrix} 0.1 & 0 \\ 0 & -0.3 \end{bmatrix} \Rightarrow A^k = \begin{bmatrix} (0.1)^k & 0 \\ 0 & (-0.3)^k \end{bmatrix} \to 0$$

$$A = \begin{bmatrix} 1.1 & 0 \\ 0 & -0.3 \end{bmatrix} \Rightarrow A^k = \begin{bmatrix} (1.1)^k & 0 \\ 0 & (-0.3)^k \end{bmatrix} \to \begin{bmatrix} \infty & 0 \\ 0 & 0 \end{bmatrix}$$

For some x_0, $x(t)$ diverges.

Important examples

- System $\dot{x}(t) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} x(t)$ is marginally stable.

  Eigenvalue: 0,0 (m=2) $\underbrace{\text{rank}\,[\lambda I - A]}_{0} = \underbrace{n - m}_{0}$

  In fact, $\left.\begin{array}{l} \dot{x}(t) = 0 \\ x(0) = x_0 \end{array}\right\} \Rightarrow x(t) \equiv x_0$

- System $\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x(t)$ is unstable.

  Eigenvalue: 0,0 (m=2) $\underbrace{\text{rank}\,[\lambda I - A]}_{1} \neq \underbrace{n - m}_{0}$

  In fact, $\left.\begin{array}{l} \dot{x}_1(t) = x_2(t) \\ \dot{x}_2(t) = 0 \\ x(0) = x_0 \end{array}\right\} \Rightarrow x(t) = \begin{bmatrix} x_{20}t + x_{10} \\ x_{20} \end{bmatrix}$ ← *Diverges if x_{20} is nonzero.*

A mechanical example: revisited

- For simplicity, *m=k=b=1*

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} x(t)$$

$$\Downarrow$$

$$\lambda = -\frac{1}{2} \pm \sqrt{3}j$$

$$\Downarrow$$

$$\mathrm{Re}\lambda < 0$$

$$\Downarrow$$

Asymptotically stable

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} x(t)$$

$$\Downarrow$$

$$\lambda = \pm j$$

$$\Downarrow$$

$$\mathrm{Re}\lambda = 0$$

All the eigs are of multiplicity one.

$$\Downarrow$$

Marginally stable

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x(t)$$

$$\Downarrow$$

$$\lambda = 0, 0$$

$$\Downarrow$$

$$\mathrm{Re}\lambda = 0$$

$$\mathrm{rank}(\lambda I - A) = 1$$

$$\Downarrow$$

Unstable

Examples

Are systems with the following A-matrices asymptotically stable, marginally stable, or unstable?

CT DT **CT DT** **CT DT**

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} -1 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Remark on stability for LTV systems

- Example: $\dot{x}(t) = \begin{bmatrix} -1 & e^{2t} \\ 0 & -1 \end{bmatrix} x(t)$
  - Eigenvalues of A-matrix are -1 and -1 for any *t*. However, this LTV system is unstable!!!
  - Why: Solve the differential equation.

$$\begin{cases} x_1(t) &= e^{-t}x_{10} + 0.5(e^t - e^{-t})x_{20} \\ x_2(t) &= e^{-t}x_{20} \end{cases}$$

For an initial state

$$x(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \Rightarrow x(t) = \begin{bmatrix} 0.5(e^t - e^{-t}) \\ e^{-t} \end{bmatrix} \to \begin{bmatrix} \infty \\ 0 \end{bmatrix}$$

Eigenvalue criteria do not work for LTV systems !

Summary

- Internal stability
  - Definition of
    - asymptotic stability
    - marginal stability, and
    - instability.
  - Eigenvalue criteria
  - Examples, phase plot
- Next, Lyapunov Theorem for internal stability

Review of linear algebra

- Vectors $\{v_1, \ldots, v_n\}$ are called linearly independent if the following holds:

$$\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n = 0 \;\Rightarrow\; \alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$$

- Rank of a matrix: the maximum number of linearly independent row (and column) vectors of a matrix
- Examples

$$\text{rank}\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} = 1 \quad \text{rank}\begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} = 2 \quad \text{rank}\begin{bmatrix} 1 & -1 \\ -5 & 5 \end{bmatrix} = 1$$