# MECH468 : Modern Control Engineering
# MECH509 : Controls

## L26 : Continuous-time infinite-horizon LQR

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Zoom lecture to be recorded and posted on Canvas

# Course plan

| Topics | CT | DT |
|---|---|---|
| Modeling | ✓ | ✓ |
| Stability | ✓ | ✓ |
| Controllability/observability | ✓ | ✓ |
| Realization | ✓ | ✓ |
| State feedback/observer | ✓ | ✓ |
| LQR/Kalman filter | → | |

# Review & today's topic

- Last lecture
  - CT finite-horizon LQR optimal control
    - State feedback with a time-varying feedback gain
    - Matrix Riccati equation
- Today
  - CT infinite-horizon LQR optimal control
    - State feedback with a constant feedback gain
    - Algebraic Riccati equation

# CT infinite-horizon LQR optimal control

- Problem $$\min_{u(\cdot)} J(u(\cdot)) \quad \text{subj. to} \quad \begin{cases} \dot{x}(t) &= Ax(t) + Bu(t) \\ x(0) &= x_0 \text{ (given)} \end{cases}$$

- *J* : $\boxed{\textit{Quadratic}}$ performance index (cost function)

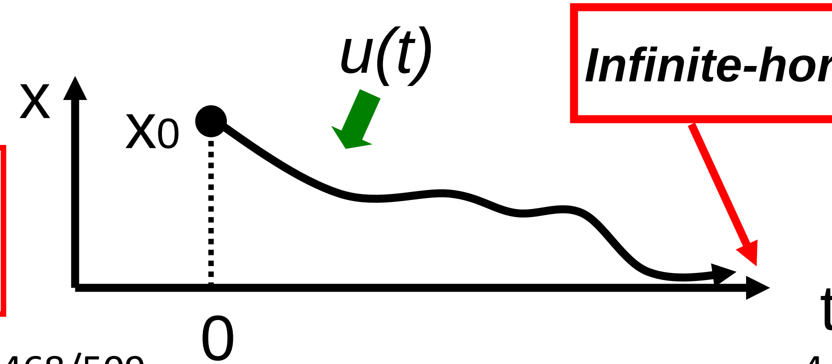
$$J(u(\cdot)) := \int_0^\infty \Big[ \underbrace{x^T(t)Qx(t)}_{\text{For small state}} + \underbrace{u^T(t)Ru(t)}_{\text{For small input}} \Big] \, dt$$

Design parameters

$$Q \geq 0, \;\; R > 0$$

**Assumptions: (*A*,*B*) controllable & (*A*,*Q*) observable**

***Infinite-horizon***



# LQR optimal control law

- LQR optimal control is obtained as a state feedback

$$u(t) = -\underbrace{R^{-1}B^TP}_{K}x(t) \quad \boxed{\textit{Linear}}$$

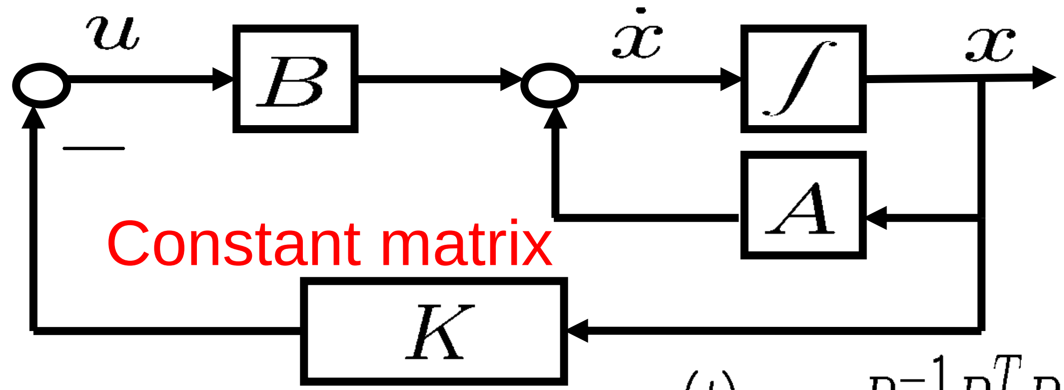
  - *P*: unique positive definite solution to an *algebraic Riccati equation (ARE)*

$$A^TP + PA - PBR^{-1}B^TP + Q = 0$$

  - CL system is stable, i.e., Re[eig(*A*-*BK*)]<0
  - Optimal performance index $J(u) = x_0^T P x_0$

# LQR optimal control law (cont'd)

- Block diagram




$$u(t) = -\underbrace{R^{-1}B^T P}_{K}\,x(t)$$

- ARE is obtained by setting *P(t)*=constant in matrix Riccati equation for finite-horizon LQR problem.

$$-\cancelto{0}{\dot{P}(t)} = A^T P(\cancel{t}) + P(\cancel{t})A - P(\cancel{t})BR^{-1}B^T P(\cancel{t}) + Q$$

# How to solve ARE

$$A^T P + PA - PBR^{-1}B^T P + Q = 0$$

1. Numerically in Matlab.

$$P = \text{are}(A, BR^{-1}B^T, Q)$$

$$[P, \Lambda_-, K] = \text{care}(A, B, Q, R)$$

2. Direct method (Next slide)
3. Method with Hamiltonian matrix (not covered)

Note: The uniqueness of the positive definite solution to ARE is guaranteed by the assumptions "(*A*,*B*) is controllable and (*A*,*Q*) is observable."

# How to solve ARE: Direct method

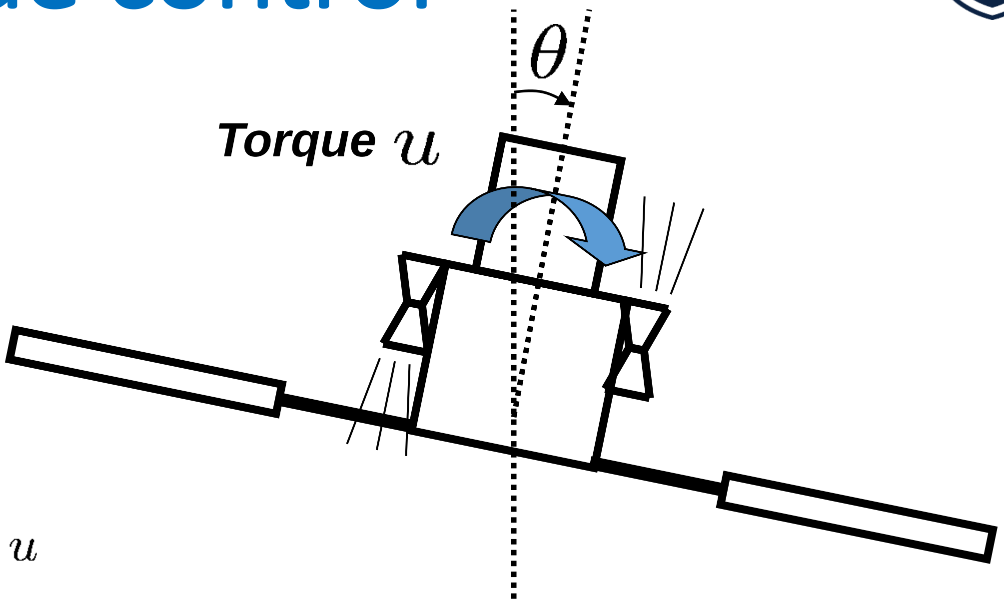
- Example $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \; B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, R = 1, \; Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$\underbrace{\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}}_{A^T} \underbrace{\begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix}}_{P} + \underbrace{\begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix}}_{P} \underbrace{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}_{A} - \underbrace{\begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix}}_{P} \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{B} \underbrace{1}_{R^{-1}} \underbrace{\begin{bmatrix} 0 & 1 \end{bmatrix}}_{B^T} \underbrace{\begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix}}_{P} + \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{Q} = 0$$

$$\left.\begin{aligned} (1,1) &: 1 - p_2^2 = 0 \\ (1,2) &: p_1 - p_2 p_3 = 0 \\ (2,2) &: 1 + 2p_2 - p_3^2 = 0 \end{aligned}\right\} \quad P = \begin{bmatrix} \sqrt{3} & 1 \\ 1 & \sqrt{3} \end{bmatrix}$$

*P must be positive definite!*

# Satellite attitude control




- After normalization,

$$\ddot{\theta} = u$$

- SS model $x := [\theta, \dot{\theta}]^T$

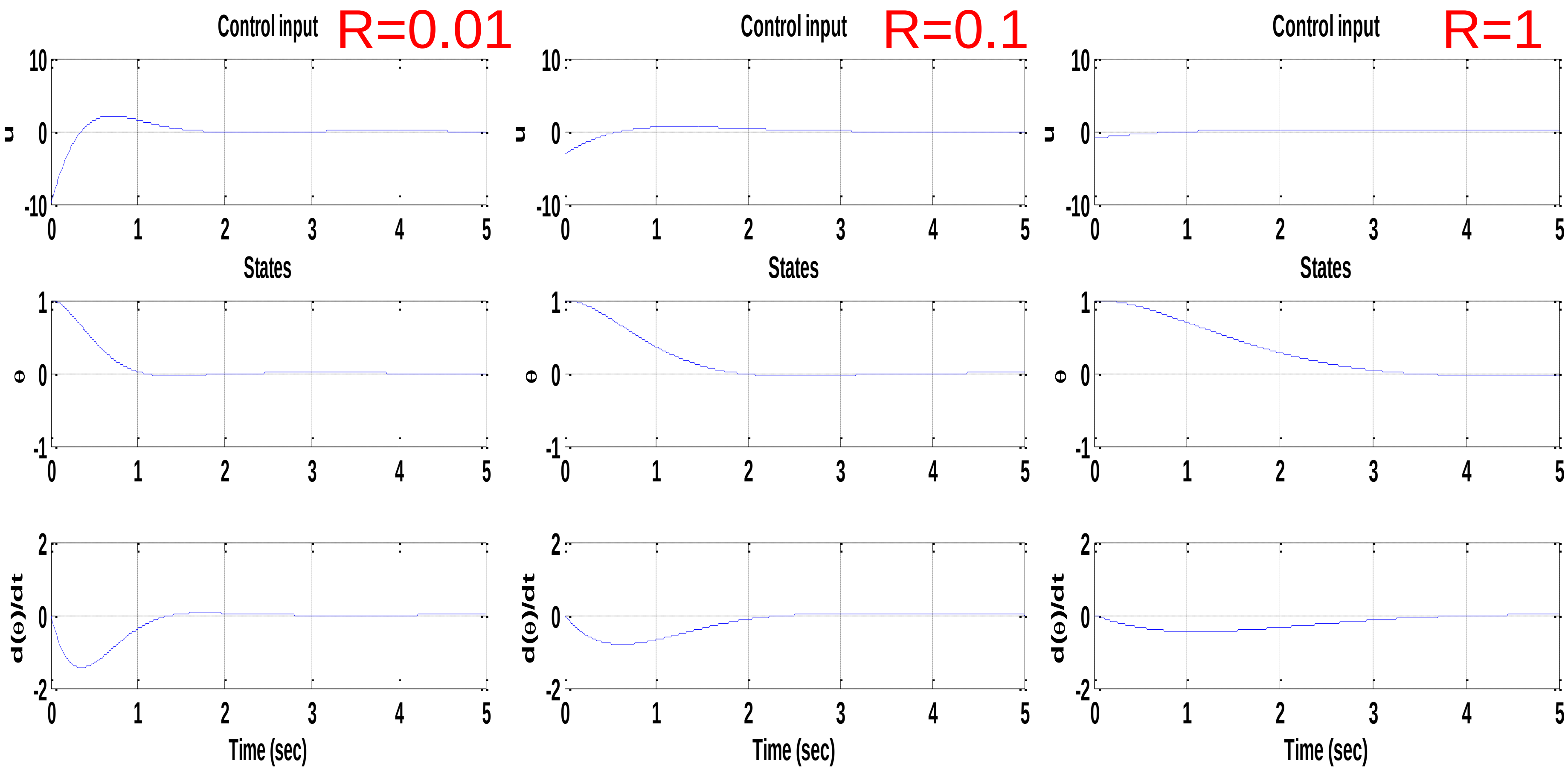
$$\begin{cases} \dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \\ y = \begin{bmatrix} 1 & 0 \end{bmatrix} x \end{cases}$$

- Requirements
  - Small $\theta$
  - Small $u$

$$\min_{u(\cdot)} \int_0^{\infty} \left[ x_1^2(t) + R u^2(t) \right] dt$$

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

# Satellite attitude control (cont'd)



# Satellite attitude control: Analysis

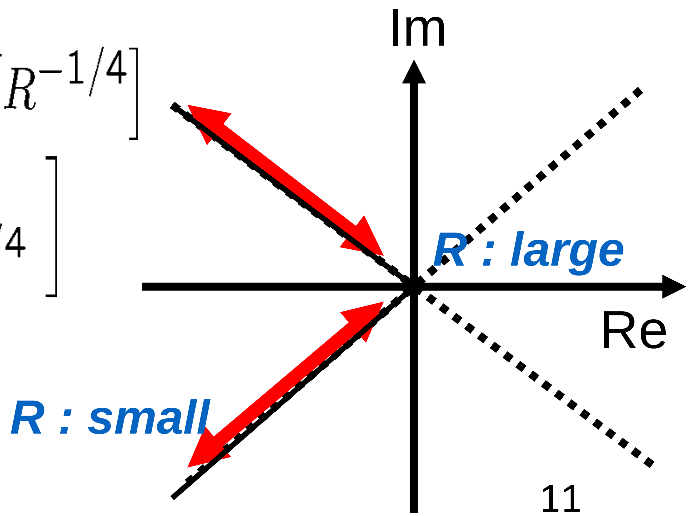
- Problem

$$\min_{u(\cdot)} \int_0^\infty x_1(t)^2 + Ru(t)^2 \quad \text{subj. to} \quad \begin{cases} \dot{x}(t) & = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \\ x(0) & = x_0 \end{cases}$$

- ARE $A^T P + PA - PBR^{-1}B^T P + Q = 0$ ➡ $P = \begin{bmatrix} \sqrt{2}R^{1/4} & R^{1/2} \\ R^{1/2} & \sqrt{2}R^{3/4} \end{bmatrix}$

- Control gain $K = R^{-1}B^T P = \left[ R^{-1/2}, \sqrt{2}R^{-1/4} \right]$
- CL *A*-matrix $A_{cl} = \begin{bmatrix} 0 & 1 \\ -R^{-1/2} & -\sqrt{2}R^{-1/4} \end{bmatrix}$

$$\text{eig}(A_{cl}) = -\frac{1}{\sqrt{2}R^{1/4}} \pm \frac{1}{\sqrt{2}R^{1/4}} j$$



# LQR with output cost

- Optimization problem

$$\min_{u(\cdot)} J(u(\cdot)) \quad \text{subj. to} \begin{cases} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t), \;\; x(0) = x_0 \end{cases}$$

  - $J$ : Quadratic performance index (cost function)

$$J(u(\cdot)) := \int_0^\infty \left[y^T(t)Qy(t) + u^T(t)Ru(t)\right] dt, \quad Q \geq 0, \; R > 0$$

- This problem is reduced to standard LQR problem

$$J(u(\cdot)) := \int_0^\infty \left[x^T(t)\overbrace{C^T Q C}^{Q_{new}}x(t) + u^T(t)Ru(t)\right] dt$$

with $\underbrace{Cx(t)}_{y(t)}$

# Matlab commands for LQR

- "lqr.m": solver for $$\boxed{\begin{aligned}&\min \int_0^\infty \left[x^T(t)Qx(t)+u^T(t)Ru(t)+2x^T(t)Nu(t)\right]dt\\&\text{subj. to } \dot{x}(t)=Ax(t)+Bu(t)\end{aligned}}$$

$$[K,P,E]=\mathrm{lqr}(\mathrm{sys},Q,R,N)$$

$K$ feedback gain $R^{-1}B^TP$
$P$ solution to the ARE
$E$ closed-loop eigenvalues $\mathrm{eig}(A-BK)$

- "lqry.m": solver for $$\boxed{\begin{aligned}&\min \int_0^\infty \left[y^T(t)Qy(t)+u^T(t)Ru(t)+2y^T(t)Nu(t)\right]dt\\&\text{subj. to } \dot{x}(t)=Ax(t)+Bu(t),\ y(t)=Cx(t)+Du(t)\end{aligned}}$$

$$[K,P,E]=\mathrm{lqry}(\mathrm{sys},Q,R,N)$$

# Why LQR?

- Solvable!
- Practical! (Linear state feedback control; not open-loop)
- Intuitive tuning of design parameters ($Q$, $R$, $S$)
- Good robustness (Not covered in this course)
  - Gain margin: Infinity (CT case)
  - Phase margin: 60 degree (CT case)
- Various generalizations
  - LQR with output cost
  - LQR with an integrator (Appendix)

# Summary

- CT infinite-horizon LQR optimal control
  - State feedback with a constant feedback gain
  - Algebraic Riccati Equation (ARE)
  - Various extensions
    - LQR with output cost
    - LQR with an integrator (Appendix)
- Next, discrete-time LQR optimal control

# LQR with an integrator

- Problem $\min_{u(\cdot)} J(u(\cdot))$ subj. to $\begin{cases} \dot{x}(t) = Ax(t) + Bu(t), \; x(0) = 0 \\ y(t) = Cx(t) \end{cases}$

  - $J$ : Quadratic performance index (cost function)

$$J(u(\cdot)) := \int_0^\infty \Big[ \underbrace{(r - y(t))^T Q (r - y(t))}_{\text{For small deviation from reference r}} + \underbrace{\dot{u}^T(t) R \dot{u}(t)}_{\text{For small input rate of change}} \Big] \, dt$$

- Design parameters $Q \geq 0, \; R(t) > 0$

# Reduction to standard LQR

- Consider an auxiliary state vector: $\tilde{x} = \begin{bmatrix} \dot{x} \\ e \end{bmatrix}, \quad e := r - y$
- Then, $$\frac{d}{dt}\tilde{x}(t) = \begin{bmatrix} \ddot{x} \\ \dot{e} \end{bmatrix} = \underbrace{\begin{bmatrix} A & 0 \\ -C & 0 \end{bmatrix}}_{\tilde{A}} \tilde{x} + \underbrace{\begin{bmatrix} B \\ 0 \end{bmatrix}}_{\tilde{B}} \dot{u}$$

$$J(u(\cdot)) = \int_0^\infty \left[ \tilde{x}^T(t) \underbrace{\begin{bmatrix} 0 \\ I \end{bmatrix} Q \begin{bmatrix} 0 & I \end{bmatrix}}_{\tilde{Q}} \tilde{x}(t) + \dot{u}^T(t) R \dot{u}(t) \right] dt$$

- Problem is reduced to a standard LQR problem:

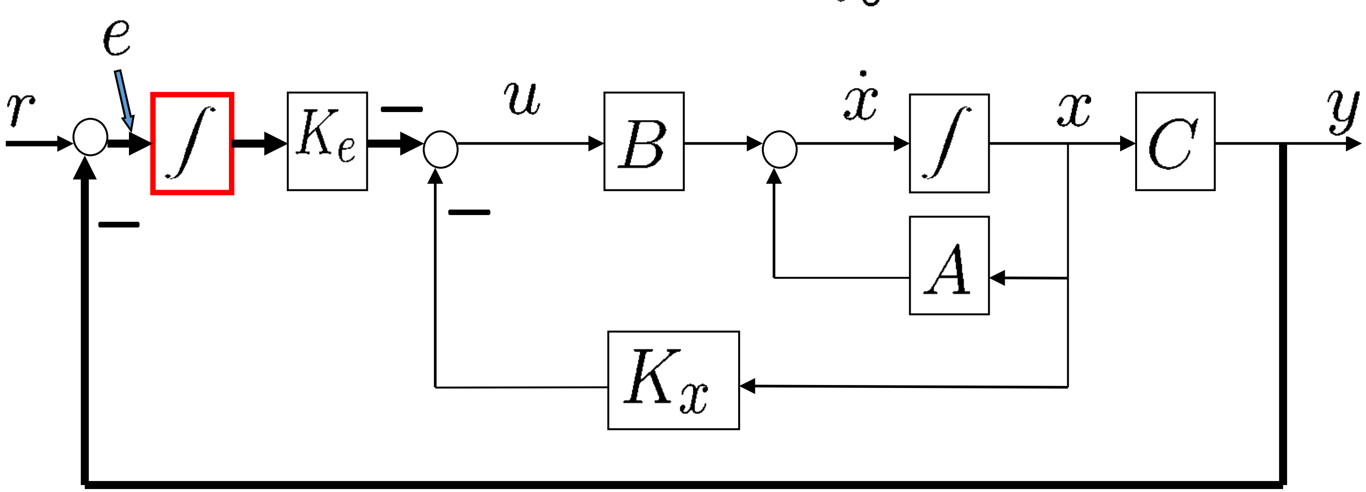
$$\min_{\dot{u}(\cdot)} \int_0^\infty \left[ \tilde{x}^T(t) \tilde{Q} \tilde{x}(t) + \dot{u}^T(t) R \dot{u}(t) \right] dt \quad \text{subj. to} \quad d\tilde{x}(t)/dt = \tilde{A}\tilde{x}(t) + \tilde{B}\dot{u}(t)$$

# Reduction to LQR (cont'd)

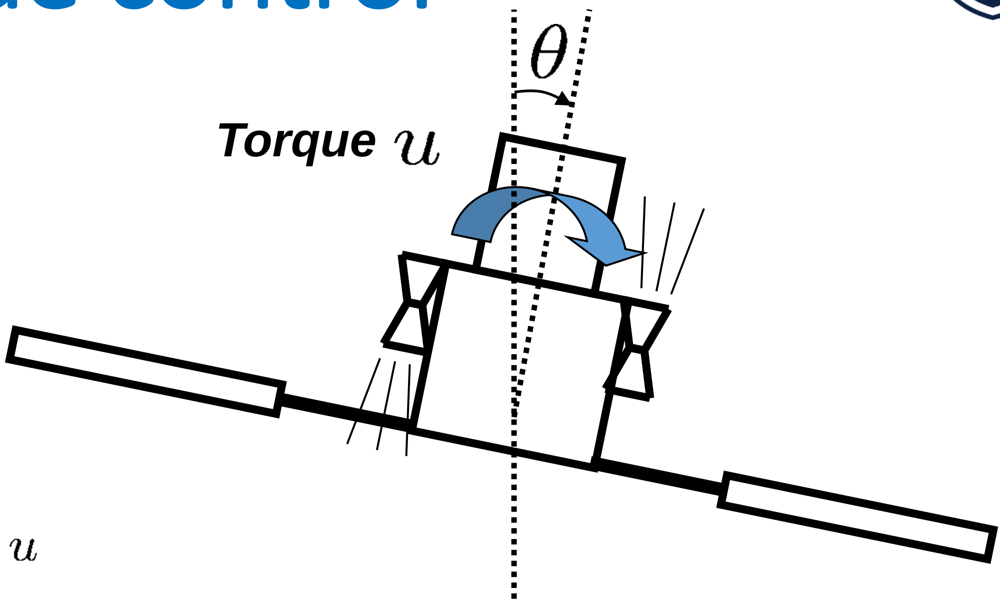
- LQR optimal control is

$$\dot{u}(t) = -R^{-1}\tilde{B}^T\tilde{P}\tilde{x}(t) = -K_x\dot{x}(t) - K_e e(t)$$

$$\Rightarrow \quad u(t) = -K_x x(t) - K_e \int_0^t e(\tau)d\tau$$



# Satellite attitude control




- After normalization,

$$\ddot{\theta} = u$$

- SS model $x := [\theta, \dot{\theta}]^T$

$$\begin{cases} \dot{x} &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \\ y &= \begin{bmatrix} 1 & 0 \end{bmatrix} x \end{cases}$$

- Requirements
  - Step change of $\theta$
  - Small *u*

# Satellite attitude control (cont'd)

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \; B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \; Q = 1$$