University of British Columbia
Department of Mechanical Engineering

MECH468 Modern Control Engineering
MECH522 Foundations in Control Engineering
Midterm exam

Examiner: Dr. Ryozo Nagamune
February 8 (Friday), 2019, 1-1:50pm

Last name, First name

Name: Student #:

Signature:

Exam policies

- Allowed: One-page letter-size hand-written cheat sheet (both front side and back side)
- Not-allowed: PC, calculators.
- Write all your answers on this booklet. No extra sheet will be provided.
- Motivate your answers properly. (No chance to defend your answers orally.)
- 20 points in total.

Before you start ...

- Use washroom before the exam.
- Turn off your mobile phone.
- No eating.
- Questions are NOT allowed.

If you finish early ...

- Please stay at your seat until the end of exam, i.e., 1:50pm. (You are not allowed to leave the room before the end of exam, except going to washroom.)

To be filled in by the instructor/marker

Problem #	Mark	Full mark
1		5
2		10
3		5
Total		20

1. Answer the following true-or-false questions. Write (True) or (False). **No need to motivate your answers.** (0.5pt each)

   (a) Linear state-space control theory was established around 2010 (i.e., about 10 years ago), and that is why it is called "modern" control theory.

   (b) When a linear time-invariant system is not observable, it is possible to recover the observability by adding actuators.

   (c) Any asymptotically stable linear time-invariant system is controllable.

   (d) A symmetric matrix which is not positive definite is negative definite.

   (e) All eigenvalues of a real symmetric matrix are real numbers.

   (f) If we discretize a continuous-time controllable model with zero-order hold, the discretized model is guaranteed to be controllable.

   (g) Minimum energy control is useful in controlling unstable systems.

   (h) If a system is not controllable and not observable, by the Kalman decomposition, there are always some states $z_{\bar{c}\bar{o}}$ which are uncontrollable and unobservable.

   (i) A continuous-time system $\dot{x} = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} x$ is marginally stable.

   (j) If a linear time-invariant system ($\dot{x} = Ax + Bu$, $y = Cx$) is BIBO stable, then it is always asymptotically stable.

Question	**Write (True) or (False)**
(a)	**False**
(b)	**False**
(c)	**False**
(d)	**False**
(e)	**True**
(f)	**False**
(g)	**False**
(h)	**False**
(i)	**True**
(j)	**False**

2. Consider the following continuous-time state space model.

$$\begin{cases} \dot{x}(t) &= \begin{bmatrix} 0 & 1 \\ -3 & -4 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t), \\ y(t) &= \begin{bmatrix} 1 & 1 \end{bmatrix} x(t). \end{cases}$$

(a) Check the stability of the system by using the Lyapunov method. (2pt)

(b) Check the controllability and the observability. (2pt)

(c) Find the Kalman decomposition. Write explicitly which state is controllable/uncontrollable and observable/unobservable. (2pt)

(d) Obtain the transfer function from the input u to the output y. (2pt)

(e) Compute the matrix exponential e^{At}. (2pt)

Solution

(a) Lyapunov equation

$$\begin{bmatrix} 0 & -3 \\ 1 & -4 \end{bmatrix} \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} + \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -3 & -4 \end{bmatrix} = - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

This can be written element-wise as

$$\begin{array}{ll} (1,1) & -3p_2 - 3p_2 = -1 \\ (1,2) & -3p_3 + p_1 - 4p_2 = 0 \\ (2,2) & p_2 - 4p_3 + p_2 - 4p_3 = -1 \end{array}$$

These equations are solved as

$$P := \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} = \begin{bmatrix} 7/6 & 1/6 \\ 1/6 & 1/6 \end{bmatrix} = \frac{1}{6} \begin{bmatrix} 7 & 1 \\ 1 & 1 \end{bmatrix}$$

Due to the Sylvester criterion, we can prove that the matrix P is positive definite. Therefore, the system is asymptotically stable.

(b) Controllability

$$\text{rank}\mathcal{C} = \text{rank} \begin{bmatrix} 0 & 1 \\ 1 & -4 \end{bmatrix} = 2 \ \Rightarrow \ \text{Controllable}$$

Observability

$$\text{rank}\mathcal{O} = \text{rank} \begin{bmatrix} 1 & 1 \\ -3 & -3 \end{bmatrix} = 1 \ \Rightarrow \ \text{Not observable}$$

(c) Since the system is controllable, we decompose the states into observable state and unobservable state.

$$\ker\mathcal{O} = \text{span}\left\{\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right\}$$

Thus, we can take T^{-1} as

$$T^{-1} = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}.$$

In this case, $T = T^{-1}$. Thus, the Kalman decomposition becomes

$$\begin{aligned}
\begin{bmatrix} z_{co} \\ z_{c\bar{o}} \end{bmatrix} &= \underbrace{\begin{bmatrix} -3 & 0 \\ 3 & -1 \end{bmatrix}}_{TAT^{-1}} \begin{bmatrix} z_{co} \\ z_{c\bar{o}} \end{bmatrix} + \underbrace{\begin{bmatrix} 1 \\ -1 \end{bmatrix}}_{TB} u \\
y &= \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{CT^{-1}} \begin{bmatrix} z_{co} \\ z_{c\bar{o}} \end{bmatrix}
\end{aligned}$$

(d) Since only the controllable and observable part affects the transfer function, we can obtain the transfer function from u to y as

$$C_{co}(sI - A_{co})^{-1}B_{co} = 1 \cdot (s+3)^{-1} \cdot 1 = \frac{1}{s+3}$$

(e) Using the Laplace transform method,

$$\begin{aligned}
e^{At} &= \mathcal{L}^{-1}\left\{(sI - A)^{-1}\right\} \\
&= \mathcal{L}^{-1}\left\{\begin{bmatrix} s & -1 \\ 3 & s+4 \end{bmatrix}^{-1}\right\} \\
&= \mathcal{L}^{-1}\left\{\frac{1}{(s+1)(s+3)} \begin{bmatrix} s+4 & 1 \\ -3 & s \end{bmatrix}\right\} \\
&= \mathcal{L}^{-1}\left\{\frac{1}{s+1} \cdot \frac{1}{2} \begin{bmatrix} 3 & 1 \\ -3 & -1 \end{bmatrix} + \frac{1}{s+3} \cdot \left(-\frac{1}{2}\right) \begin{bmatrix} 1 & 1 \\ -3 & -3 \end{bmatrix}\right\} \\
&= \frac{1}{2}\left\{e^{-t} \begin{bmatrix} 3 & 1 \\ -3 & -1 \end{bmatrix} + e^{-3t} \begin{bmatrix} -1 & -1 \\ 3 & 3 \end{bmatrix}\right\}
\end{aligned}$$

3. Consider the following system of nonlinear equations:

$$\begin{aligned} &\ddot{p}(t) + \sin(\theta(t)) - (\dot{\theta}(t))^2 = 0, \\ &\ddot{\theta}(t) + p(t)\dot{p}(t)\dot{\theta}(t) + p(t)\cos\theta(t) = \tau(t). \end{aligned}$$

(This model is a simplified one for a ball and beam system, but you don't need to know this fact to solve the following questions.)

(a) By considering the input u and the output y respectively as

$$u(t) := \tau(t), \quad y(t) := p(t),$$

and by introducing the state variables as

$$x_1(t) := p(t), \quad x_2(t) := \dot{p}(t), \quad x_3(t) := \theta(t), \quad x_4(t) := \dot{\theta}(t),$$

derive a nonlinear state-space model. (2pt)

(b) Prove that a point $(x_0, u_0, y_0) = (0, 0, 0)$ is an equilibrium point. (1pt)

(c) Around the equilibrium point $(x_0, u_0, y_0) = (0, 0, 0)$, linearize the nonlinear state-space model obtained in (a). Do NOT use the small angle approximations ($\sin\theta \approx \theta$, $\cos\theta \approx 1$ for small $|\theta|$.) (2pt)

——————— (END OF MIDTERM EXAM) ———————

Solution

(a) A nonlinear state-space model:

$$\begin{aligned} \dot{x}(t) &= \begin{bmatrix} x_2(t) \\ x_4^2(t) - \sin x_3(t) \\ x_4(t) \\ -x_1(t)x_2(t)x_4(t) - x_1(t)\cos x_3(t) + u(t) \end{bmatrix}, \\ y(t) &= x_1(t). \end{aligned}$$

(b) By substituting $(x_0, u_0, y_0) = (0, 0, 0)$ into the state equation and the output equation, both left-hand sides and right-hand sides become zeros. Thus, the point $(x_0, u_0, y_0) = (0, 0, 0)$ is an equilibrium point.

(c) Linearization of the nonlinear system around the equilibrium point is obtained by the following.

$$A = \left.\frac{\partial f}{\partial x}\right|_{x=0} = \left.\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -\cos x_3 & 2x_4 \\ 0 & 0 & 0 & 1 \\ -x_2 x_4 - \cos x_3 & -x_1 x_4 & x_1 \sin x_3 & -x_1 x_2 \end{bmatrix}\right|_{x=0,u=0}$$

$$= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \end{bmatrix}$$

$$B = \left.\frac{\partial f}{\partial u}\right|_{x=0,u=0} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}$$