a place of mind

MECH468 : Modern Control Engineering
MECH509 : Controls

L15 : Controllability and observability for discrete-time systems

Dr. Ryozo Nagamune
Department of Mechanical Engineering
University of British Columbia

Zoom lecture to be recorded and posted on Canvas

Course plan

Topics	CT	DT
Modeling	✓	✓
Stability	✓	✓
Controllability/observability	✓	→
Realization		
State feedback/observer		
LQR/Kalman filter		

Why controllability & observability?

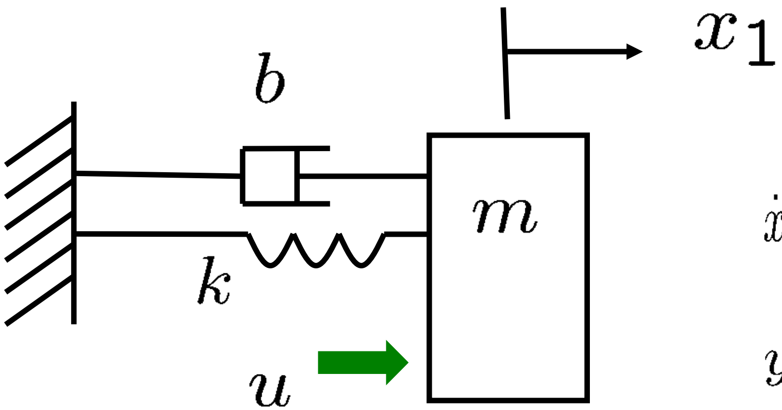
- Definitions of controllability (open-loop) and observability (estimation of past) are not so practical.
- They are important properties in many feedback controller and state estimator design techniques.
- The lack of these properties limits achievable performance. (later in this course)
- Note that stability, controllability, and observability are independent properties.

Review & today's topic

- So far, for CT systems, we learned controllability and observability:
  - Definitions
  - Conditions
  - Duality
  - Kalman decomposition
- Today, we study discrete-time counterpart (which is almost the same as CT results; actually easier to understand/prove results than CT cases!).

An example

- Mass-spring-damper



$$x_2 = \dot{x}_1$$

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -k/m & -b/m \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1/m \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(t)$$

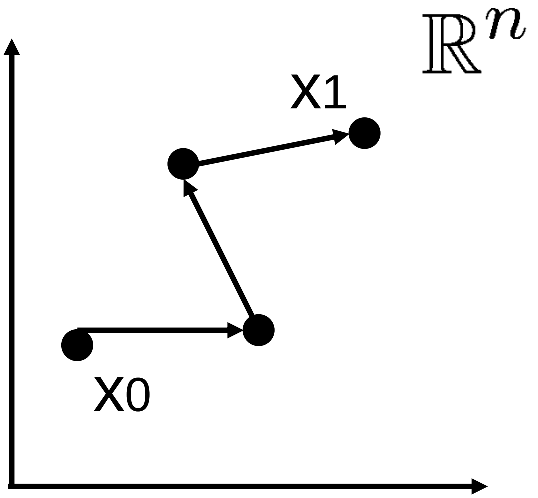
  - Controllable and observable CT system
  - If we discretize it by ZOH with period *T*, is it still "controllable" and/or "observable"?
  - Definitions of controllability & observability for DT systems?

Controllability of DT LTI system

- Consider a state equation

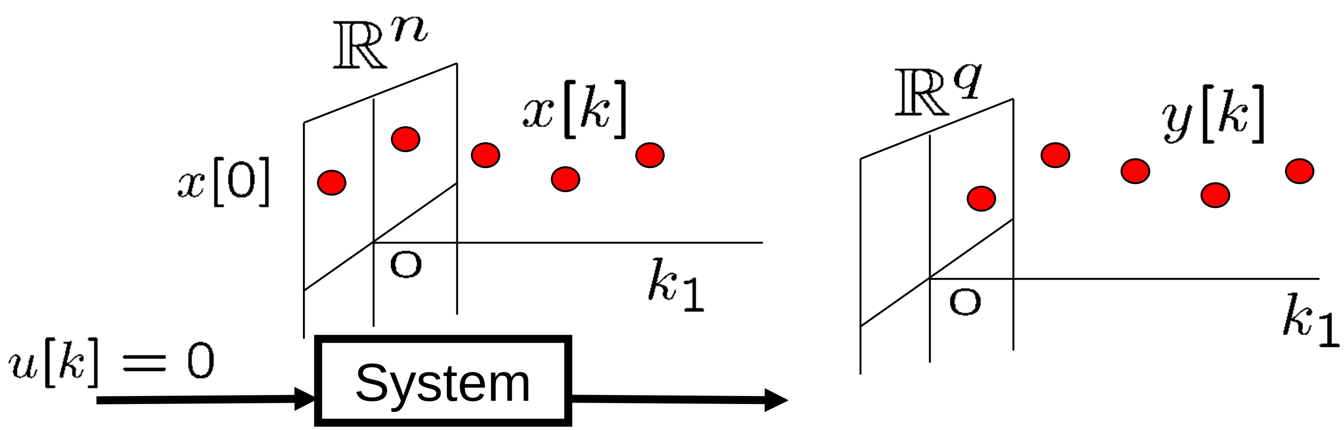
$$x[k+1] = Ax[k] + Bu[k], \quad A \in \mathbb{R}^{n\times n},\ B \in \mathbb{R}^{n\times p}$$

- *Definition*: The system above, or (*A*,*B*), is called *controllable* if, for *any* initial state x_0 and *any* final state x_1, there is an input sequence *u[0],u[1],…* of finite length which transfers from x_0 to x_1.



Observability of DT LTI system

- System equations (no input) $\begin{cases} x[k+1] = Ax[k], & A \in \mathbb{R}^{n\times n} \\ y[k] = Cx[k], & C \in \mathbb{R}^{q\times n} \end{cases}$
- *Assumptions*: *y[k]*: measurable, *x[0]*: unknown.
- *Definition*: The system above, or (*A*,*C*), is called *observable* if, there is a finite $k_1>0$ such that *y* over time interval $[0,k_1]$ determines *uniquely* *x[0]*.



Conditions

$$\begin{cases} x[k+1] = Ax[k] + Bu[k], & A \in \mathbb{R}^{n \times n},\ B \in \mathbb{R}^{n \times p} \\ y[k] = Cx[k] + Du[k], & C \in \mathbb{R}^{q \times n},\ D \in \mathbb{R}^{q \times p} \end{cases}$$

- Controllable *if and only if* the *controllability matrix* C_d has full row rank.

$$\mathcal{C}_d := \left[B, AB, \cdots, A^{n-1}B \right]$$

- Observable *if and only if* the *observability matrix* O_d has full column rank.

$$\mathcal{O}_d := \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

Derivation for controllability condition

- Solve recursively $x[k+1] = Ax[k] + Bu[k]$

$$
\begin{aligned}
x[1] &= Ax[0] + Bu[0] \\
x[2] &= Ax[1] + Bu[1] = A^2x[0] + \begin{bmatrix} B & AB \end{bmatrix} \begin{bmatrix} u[1] \\ u[0] \end{bmatrix} \\
x[3] &= Ax[2] + Bu[2] = A^3x[0] + \begin{bmatrix} B & AB & A^2B \end{bmatrix} \begin{bmatrix} u[2] \\ u[1] \\ u[0] \end{bmatrix} \\
&\vdots
\end{aligned}
$$

$$
x[n] - A^n x[0] = \begin{bmatrix} B & \cdots & A^{n-1}B \end{bmatrix} \begin{bmatrix} u[n-1] \\ \vdots \\ u[0] \end{bmatrix}
$$

- For any *x[n]* and *x[0]*, this has a solution *u[0],…,u[n-1] if and only if* $\operatorname{rank} \begin{bmatrix} B & \cdots & A^{n-1}B \end{bmatrix} = n$

Simple examples

$$x[0] = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \; x[2] = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

- Uncontrollable case

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \; B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\Rightarrow \; x[2] - A^2 x[0] = \begin{bmatrix} B & AB \end{bmatrix} \begin{bmatrix} u[1] \\ u[0] \end{bmatrix}$$

$$\Rightarrow \; \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u[1] \\ u[0] \end{bmatrix}$$

No solution.

- Controllable case

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \; B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\Rightarrow \; x[2] - A^2 x[0] = \begin{bmatrix} B & AB \end{bmatrix} \begin{bmatrix} u[1] \\ u[0] \end{bmatrix}$$

$$\Rightarrow \; \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} u[1] \\ u[0] \end{bmatrix}$$

There is a solution.

Derivation for observability condition

- Solve recursively $x[k+1] = Ax[k]$ ➡ $x[k] = A^k x[0]$

- Substitute this into $y[k] = Cx[k]$ ➡ $y[k] = CA^k x[0]$

➡ $$\begin{bmatrix} y[0] \\ \vdots \\ y[n-1] \end{bmatrix} = \underbrace{\begin{bmatrix} C \\ \vdots \\ CA^{n-1} \end{bmatrix}}_{\mathcal{O}_d} x[0]$$

Given (measured)

➡ There is a unique *x[0] if and only if* $\text{rank}\,\mathcal{O}_d = n$

Simple examples

$$x[0] = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$ *(Unknown)*

- Unobservable case

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \; C = \begin{bmatrix} 0 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} y[0] \\ y[1] \end{bmatrix} = \begin{bmatrix} C \\ CA \end{bmatrix} x[0] = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x[0] = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow x[0] = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \cdots$$

Non-unique solutions.

- Observable case

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \; C = \begin{bmatrix} 1 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} y[0] \\ y[1] \end{bmatrix} = \begin{bmatrix} C \\ CA \end{bmatrix} x[0] = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x[0] = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \Rightarrow x[0] = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Unique solution.

Minimum energy control

- If controllable, find the input with minimum energy (least-squares sum), i.e., solve

$$\min_{u[\cdot]} \sum_{k=0}^{k_f-1} u^T[k]u[k] \quad \text{subj. to} \quad \begin{cases} x[k+1] = Ax[k] + Bu[k] \\ x[0] = x_0, \;\; x[k_f] = x_f \end{cases}$$

$$x[k_f] - A^{k_f}x[0] = \underbrace{\begin{bmatrix} B & \cdots & A^{k_f-1}B \end{bmatrix}}_{=:\mathcal{C}_d[k_f]} \begin{bmatrix} u[k_f-1] \\ \vdots \\ u[0] \end{bmatrix}$$

LS solution

$$\begin{bmatrix} u[k_f-1] \\ \vdots \\ u[0] \end{bmatrix} = \mathcal{C}_d[k_f]^T (\mathcal{C}_d[k_f] \cdot \mathcal{C}_d[k_f]^T)^{-1} (x[k_f] - A^{k_f}x[0])$$

Remark

- Recall that, for CT systems, if (*A*,*B*) is controllable, then any state transfer is possible in any (epsilon) time.
- However, this is NOT the case for DT systems.
- Ex. $A = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}, \; B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ ➡ $\mathcal{C}_d = \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -3 \end{bmatrix}$
    - 1-step state transfer is possible *only when* $x[1] - Ax[0] \in \mathrm{Im}B$
      $(x[1] - Ax[0] = Bu[0])$
    - 2-step state transfer is always (i.e., for any *x[2] & x[0]*) possible in this example. $x[2] - A^2 x[0] = \begin{bmatrix} B & AB \end{bmatrix} \begin{bmatrix} u[1] \\ u[0] \end{bmatrix}$

An example: revisited

- Set *m=1, k=4, b=0.*

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -k/m & -b/m \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1/m \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(t)$$

- Discretize the system with sampling period *T*.

$$A_d = e^{AT} = \mathcal{L}^{-1}\left\{(sI - A)^{-1}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s^2+4}\begin{bmatrix} s & 1 \\ -4 & s \end{bmatrix}\right\} = \begin{bmatrix} \cos 2T & \frac{1}{2}\sin 2T \\ -2\sin 2T & \cos 2T \end{bmatrix}$$

$$B_d = \int_0^T e^{A\tau} d\tau \cdot B = \begin{bmatrix} -\frac{1}{4}\cos 2T + \frac{1}{4} \\ \frac{1}{2}\sin 2T \end{bmatrix}$$

An example (cont'd)

- Is the discretized system controllable?

$$C_d = \begin{bmatrix} -\frac{1}{4}\cos 2T + \frac{1}{4} & -\frac{1}{4}(\cos^2 2T - \sin^2 2T - \cos 2T) \\ \frac{1}{2}\sin 2T & \sin 2T \cos 2T - \frac{1}{2}\sin 2T \end{bmatrix}$$

$$\Rightarrow \quad -\det C_d = \frac{1}{4}(\cos 2T - 1)(\underbrace{\sin 2T \cos 2T}_{\frac{1}{2}\sin 4T} - \frac{1}{2}\sin 2T) - \frac{1}{4}(\underbrace{\cos^2 2T - \sin^2 2T}_{\cos 4T} - \cos 2T)\frac{1}{2}\sin 2T$$

$\Rightarrow$ $\det C_d = 0$ if $T = \frac{\pi}{2}, \pi, \frac{3\pi}{2}, \cdots$ $\Rightarrow$ *Uncontrollable for some T!*

Summary

- DT controllability and observability (very similar to CT results.).
- Minimum energy control
- Duality & Kalman decomposition apply to DT systems, too, in an exactly same way as CT cases.
- Next, realization theory