1. Introduction to Vibrations

MECH 463: Mechanical Vibrations

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Suggested Readings:

1. Topic 1 from notes package.
2. Sections 1.1–1.6 in the course textbook.

Your Notes

T=TEXTBOOK

NP=NOTES PACKAGE

1.1 Learning Objectives

1. Understand the importance of vibrations in mechanical design and learn about force and energy perspectives.
2. Identify degrees of freedom of a mechanical system.
3. Identify different types of vibrations and vibration analysis procedures.
4. Apply principle of superposition (in this and later topics).
5. Develop lumped parameter models (in this and later topics).

• vibrations as a means of energy transfer, or as a derivation of forces [2 approaches]

Your Notes

1.2 Why study vibrations? (T 1.3, NP 1.2)

Fill in the class

• FATIGUE FAILURES
• REASONANCE OF PARTS
• NOISE: VIBRATING OBJECTS ARE SOURCES OF NOISE
• ENERGY LOSSES IN FORM OF UNWANTED VIBRATIONS
• VIBRATIONS CAUSE WEAR: I.E. SQUEAL/ROLLING NOISE IN RAILWAYS
• SUSPENSION DESIGNS: AUTOMOTIVE

Your Notes

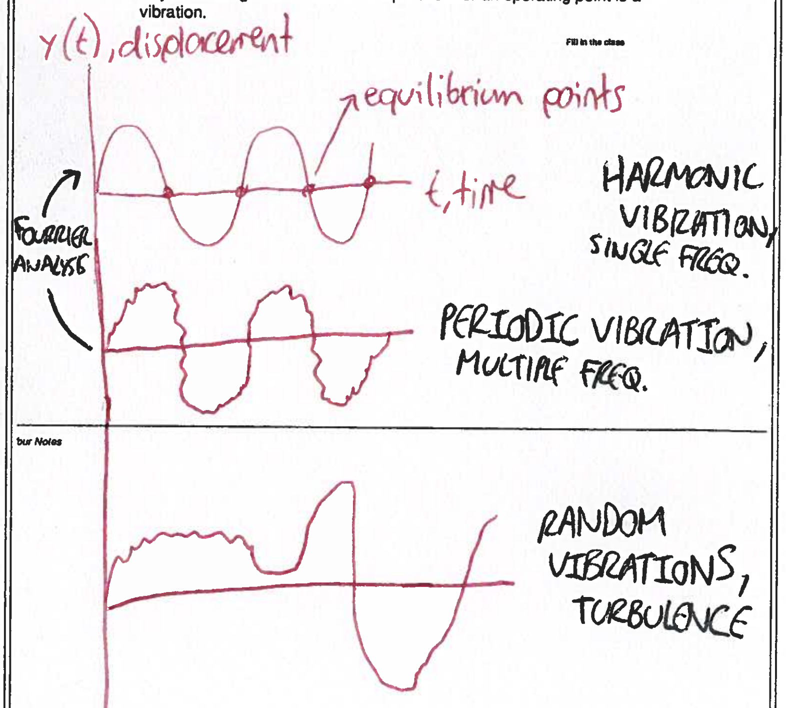
IN MECH 463, WE STUDY DETERMINISTIC VIBRATIONS

• HARMONIC
• PERIODIC

1.3 What is vibration? (T 1.4.1, NP 1.3)

Any fluctuating motion about an equilibrium or an operating point is a vibration.

Fill in the class

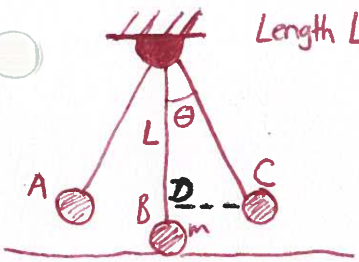


Your Notes

Consider the example of a simple pendulum, comprising a rigid mass suspended by a string or a metal wire, sketched below.

Fill in the class

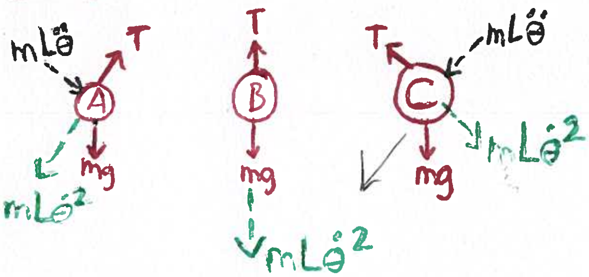
Length L, mass m



DATUM FOR ENERGY ; ~~P.E~~ P.E = V_0

Your Notes

FBD



*T depends on $\dot{\theta}^2$ and $\ddot{\theta}$ accelerations

Q: What brings the pendulum back to equilibrium after initial displacement?

Fill in the class

A: Because there is a restoring moment.

Restoring moment = $mg * \overline{CD}$ = $mgL\sin\theta$ at C ~~B~~ (clockwise)

= $mgL\sin\theta$ at A (c.clockwise)

∴ Tension T does not contribute towards restoring moment

∴ Restoring moment is a function of time

Your Notes

Force Perspective on Vibrations — # 2

Q: Once it is brought to equilibrium why does the pendulum swing through to the other side? Why can't it just creep back to equilibrium and stay there?

Fill in the class

A: Because of momentum, and hence associated inertia of motion.

The mass accelerates towards equilibrium (in the same direction as restoring moment)

* Force View: Interplay between restoring forces and inertial forces

$\boxed{\text{END OF FORCE PERSPECTIVE}}$

Your Notes

*Need a restoring force to have vibrations

- gravity
- elasticity of materials

Energy Perspective on Vibrations — # 1

Fill in the class

let potential energy = V

∴ P.E.$_{DATUM}$ + P.E.$_{MOTION}$ = V

$V_0 + mgL(1-\cos\theta) = V$

let Kinetic energy = $\frac{1}{2}m(L\dot{\theta})^2 = \frac{1}{2}mL^2\dot{\theta}^2 = \frac{1}{2}J_o\dot{\theta}^2$

J_o = mass moment of inertia w.r.t. axis 'o' = $0 + mL^2$ through P.A.T.

$0 \rightarrow J_c$, $mL^2 \rightarrow$ Parallel Axis Theorem

Your Notes

1. Vibration can be viewed as an energy exchange mechanism in which potential energy is continuously transformed into kinetic energy, and vice versa, in a periodic manner.
2. Vibration is an interplay between inertial and restoring forces.
3. All *realistic* vibrations involve dissipative forces as well.
4. Free body diagrams are useful to understand vibration problems.

Your Notes

1.5. Degrees of Freedom (T.1.4.3, NP1.5) — # 1

The number of degrees of freedom (dof) of any mechanical system in motion is defined as the **minimum** number of **independent** co-ordinates required to determine completely the position of all its parts during motion.

Q: What are the dofs a particle and rigid body, if we constrain their motion to a plane?

Fill in the class

particle • motion in a plane

$DOF = 2\ (x, y)$ → constraint: $z = \emptyset$, planar motion

rigid body

$D.O.F = 3\ (x_c, y_c, \theta_c)$

Your Notes

1.5. Degrees of Freedom (T.1.4.3, NP1.5) — # 2

Compound Pendulum:

Fill in the class

*planar motion is assumed

To specify → O: x_o, y_o

→ G: x_G, y_G

constraints: O is fixed

$x_o = 0;\ y_o = 0$

* $\overline{OG}$ is fixed b/c rigid body

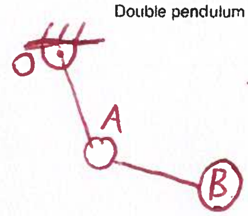
$\boxed{\#D.O.F. = 4 - 3 = 1}$

Your Notes

1.5. Degrees of Freedom (T.1.4.3, NP1.5) — # 3

Double pendulum

Fill in the class



#co-ordinates $= 3 \times 2 = 6$

#points #co-ordinates/point

#constraints

O is fixed = 2 $x_o = 0,\ y_o = 0$

OA is fixed = 1 ∴ OA is rigid

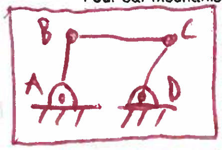
AB is fixed = 1 ∴ AB is rigid

$\boxed{D.O.F = 6 - 4 = 2}$

Your Notes

Four-bar mechanism:

Fill in the class



#CO-ORDINATES $= 4\times2 = 8$

#CONSTRAINTS

A,D FIXED $= 2\times2 = 4$

AB is fixed $= 1$ ∴ rigid

BC is fixed $= 1$ ∴ rigid

CD is fixed $= 1$ ∴ rigid

AD is fixed $= 1$ ∴ rigid

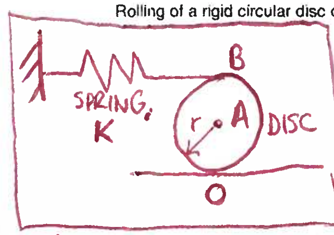
DOUBLE COUNTING

#D.O.F. = #co-ordinates − #constraints $= 8 - 7 = \boxed{1}$ ✓

Your Notes

Rolling of a rigid circular disc on a flat surface:

Fill in the class



#co-ordinates

*rolling without slipping

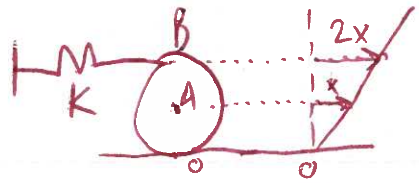
#constraints

OA, OB fixed

O: instantaneous point of contact

Task: Do it yourself

Your Notes



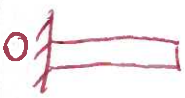
"if A moves x, B moves 2x"

because $OB = 2\times AB$ $(d = 2r)$

1.5. Degrees of Freedom (T.1.4.3, NP1.5) — # 6

Rigid cantilever beam:

Fill in the class



#co-ordinates $2\times2 = 4$

#constraints

OA fixed $= 1$

O, A fixed $= 2$

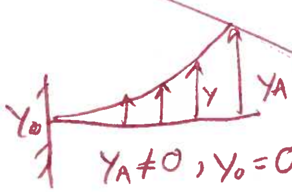
$\Theta = 0$, no rotation $= 1$

∴ D.O.F. $= 4 - 4 = \boxed{0}$

Your Notes

#D.O.F $= \infty$

∵ Need to specify Y-co-ordinate for infinite # of points



$Y_A \neq 0$, $Y_0 = 0$

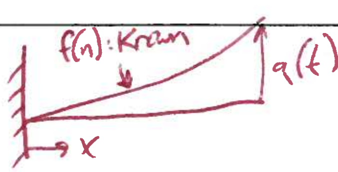
*In a given mode of deformation:

1.5. Degrees of Freedom (T.1.4.3, NP1.5) — # 7

Flexible cantilever beam:

Fill in the class

Your Notes



displacement of beam at any point $= W(x,t) = f(x)\,q(t)$

↑ assumed

∴ #D.OFs $= 1 = q(t)$

1. Kinematic constraints on motion—imposed by supports and other objects in contact— influence the number of degrees of freedom.
2. The choice of co-ordinates is not *unique* and it is *subjective*.
3. Rigid body assumption is essentially a constraint: the distance between two points on the body is always fixed during its motion.
4. Distributed parameter (continuous) systems such as beams and plates require infinite number of dofs to completely specify their motion.
5. Discrete systems require finite number of dofs.

Your Notes

SUMMARY - CH.1

VIBRAT → fluctuating motion about eq. or operating pt.
→ interplay between restoring & inertial forces
→ exchange of energy for Kinematic/potential

D.O.F. → Min # of independent co-ordinates
→ constraints on motion decide # of dof

1. Revise this material.
2. Read the sections 1.6–1.8 from the course package and the corresponding text book sections.
3. Drop me a note on how this session went for you.

Your Notes

In a given/assumed deformation shape with ∞ dof system, can be reduced into a single D.O.F. system

MECH 463 - LECTURE 3

Sept. 12, '13

Principle of superposition

① $\xrightarrow{f_1} \boxed{\text{System}} \xrightarrow{x_1}$ $\quad \xrightarrow{af_1} \boxed{\text{System}} \xrightarrow{ax_2}$ doubling force doubles response (proportionality

② $\xrightarrow{f_2} \boxed{\text{system}} \xrightarrow{x_2}$ $\quad \xrightarrow{bf_2} \boxed{\text{system}} \xrightarrow{bx_2}$

① + ② $\xrightarrow{f_1+f_2} \boxed{\text{system}} \xrightarrow{x_1+x_2}$ $\quad \xrightarrow{af_1+bf_2} \boxed{\text{system}} \xrightarrow{ax_1+bx_2}$

This means we can take a complex input force and break it up into simpler forces. Then, by superposition principle, we can add the response due to eac force acting on its own

Ex: Convolution integral for general forces

$$x_p(t) = \int_{t=0^*}^{t} h(t-z)\, f(z)\, dz$$

steady-state vibr. resp.

Another application of superposition principle

$\xrightarrow{f(t)} \boxed{\text{System}}$

- $f(t)$ is periodic: as in the case of a single cylinder engine $f(t \pm T) = f(t)$, where T is period
- $f(t)$ can be complex (many frequencies)

① split $f(t)$ into simpler forces

$$f(t) = \sum a_n \cdot \cos(n\omega t) + \sum b_n \cdot \sin(n\omega t), \text{ where } \omega = \frac{2\pi}{T}$$

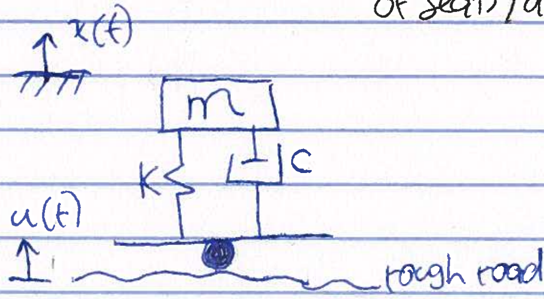
② Find harmonic response due to each frequency: $a_n \cos(n\omega t)$, etc.

③ Add the individual responses due to each force in Fourier Series

Example 3:

Automobile on rough road, vibrations caused by road roughness. Develop 3 m
in increasing order of complexity (a) weight of car (b) elasticity of tires (c) damp
of seats/absorbers/tires

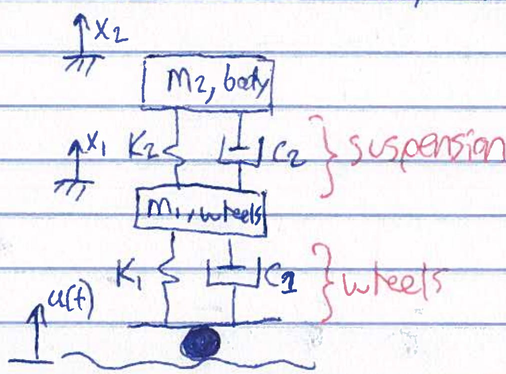
A)



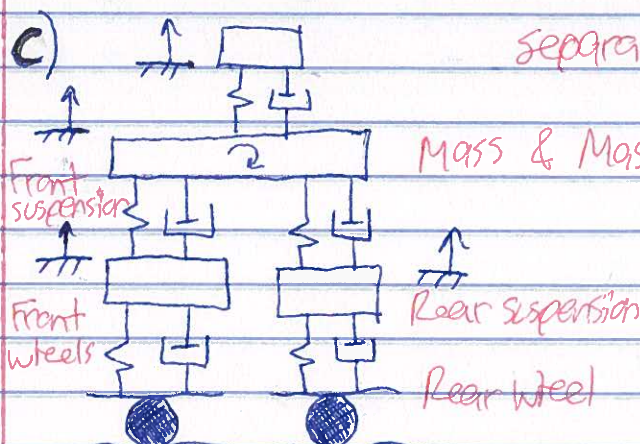
*Base excited vibrations:
Source of vibrations is roughness of ro
Hence, $x(t)$ is random

limitations: can only consider vertical displacements

B)



C)



separating front + rear wheels

Mass & Mass moment of inertia

★ 5 D.O.F.

INDUSTRY USES QUARTER ("B") CAR MODEL

2.1. SDOF– Formulation of Equations of Motion

MECH 463: Mechanical Vibrations

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Suggested Readings:

1. Topic 2.1 from notes package.
2. Sections 2.1–2.2 in the course textbook.

Story So Far ....

Key learning points so far

1. Vibration can be viewed as a fluctuating motion about equilibrium, caused by the interplay between restoring and inertial forces.
2. Steady vibrations can also be viewed as a continuous exchange of energy from one form into other.
3. Harmonic vibration response is fundamental. Why?
4. Single degree of freedom vibration is fundamental. Why?
5. Determining accelerations is essential to understand vibrations. Why?

ning Objectives (NP 2.1)

1. Apply Newton's second law/D'Alembert principle to obtain equations of motion of lumped parameter models.
2. Realize the importance of Kinematics.
3. Learn to construct effective Free Body Diagrams.

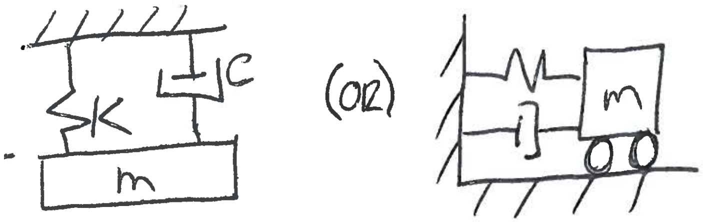
2.1 Introduction–Methods (NP 2.1)

1. Force methods
   1.1 Newton's second law
   1.2 D'Alembert's principle
2. Work-Energy methods
   2.1 Principle of conservation of energy
   2.2 Principle of virtual displacements
   2.3 Lagrange equations

Introduction (NP 2.1) — # 1

Consider the simplest of all sdof systems, a spring-mass model sketched below. p.23 of notes package

Fill in the class



x is an absolute displacement of 'm' measured w.r.t. a fixed observer, taken as positive downwards

NOTE: $\left.\begin{array}{l}\ddot{x} \text{ is also downwards} \\ \dot{x} \text{ is also downwards}\end{array}\right\}$ consistency

2.1. Introduction (NP 2.1) — # 2

Fill in the class

f: external force

x: absolute displacement

m: mass

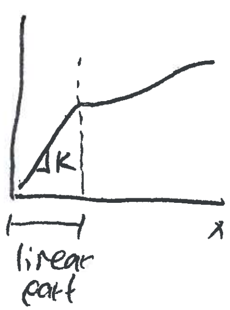
K: stiffness

c: damping

2.2 Elements of Vibration– Spring (NP 2.2) — # 1

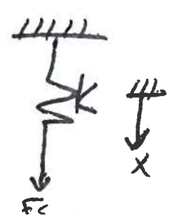
A spring resists relative displacement x between its two ends. How do we find spring constants? Consider a spring with one end fixed and the other end subjected to an external force f_s as sketched below. p.24 of notes package

Fill in the class



∴ for small displacement, the spring is linear:

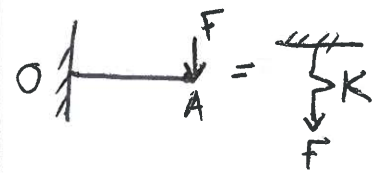
$$\boxed{F_s = K(x-0) = Kx}$$



2.2.2 Elements of Vibration– Spring (NP 2.2) — # 2

Q: Suppose you are given an expression for force f_s as a function of displacement x. How will you determine the spring constant, k using $f_s = kx$?

Fill in the class



K = stiffness at A; changes w/ location

$x = \delta$: δ (from MECH 360) $= \frac{FL^3}{3EI}$; E = Young's, I = 2nd mom. of area

$$\therefore K = \frac{F}{\delta} = \boxed{\frac{3EI}{L^3}}$$

2.2.2 Elements of Vibration– Spring (NP 2.2) — # 3

Q: Suppose an expression for strain energy U as a function of displacement x is given: $U = 10x^2 + 2x^4$. How will you find k ?

Fill in the class

U = elastic strain energy $= 10x^2 + 2x^4$ [given]

$\approx 10x^2$ [linearization; retain powers up to 2]

$\frac{\partial U}{\partial x} = 20x = F =$ force causing x, where x is measured

$$\boxed{K = \frac{F}{x} = 20}$$

2.2.2 Elements of Vibration– Mass (NP 2.2) — # 1

Newton's second law relates *net* force or moment acting on a mass to its *absoute acceleration*: that is, acceleration measured by a fixed observer, via

$$\mathbf{F} = m\ddot{\mathbf{x}}, \quad \ddot{\mathbf{x}} \equiv \frac{d^2\mathbf{x}}{dt^2} \quad \text{(Newton's second law)} \qquad (1a)$$

$$\mathbf{M}_o = J_o\ddot{\boldsymbol{\theta}}, \quad \ddot{\boldsymbol{\theta}} \equiv \frac{d^2\boldsymbol{\theta}}{dt^2} \quad \text{(Newton-Euler's law)} \qquad (1b)$$

Note that bold letters denote vectors. The symbols in the above equations are defined as follows.

Fill in the class

$\vec{F}$: force acting at center of mass

$\vec{x}$: displacement at center of mass

$\ddot{\vec{x}}$: absolute acceleration

$\vec{M}_o$: external moment about O

$\vec{J}_o$: mass moment of inertia about an axis passing through O

$\ddot{\vec{\theta}}_o$: absolute angular acceleration about 'O'

2.2.2 Elements of Vibration– Mass (NP 2.2) — # 2

Another view of the above equation of motion is given by D'Alembert's principle, which converts the above *equations of motion* into *equations of dynamic equilibrium*

negative sign

zero resultants imply equilibrium

$$\mathbf{F} - m\ddot{\mathbf{x}} = \mathbf{0} \qquad (2a)$$

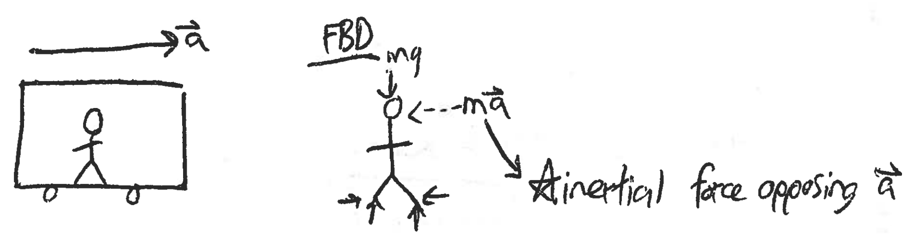
$$\mathbf{M}_o - J_o\ddot{\boldsymbol{\theta}} = \mathbf{0} \qquad (2b)$$

The forces $-m\ddot{\mathbf{x}}$ and $-J_o\ddot{\boldsymbol{\theta}}$ are called *inertial* forces. The negative signs indicate that **inertial forces oppose motion; they act in a direction opposite to the absolute accelerations**. They act at the *centre of mass*.

2.2.2 Elements of Vibration– Mass (NP 2.2) — # 3

Q: You fall backwards when a bus moves forwards from rest; you fall forwards when the bus halts suddenly. Explain this using FBD of a person standing in the bus?

Fill in the class



.3. Force Method (NP 2.3) — # 1

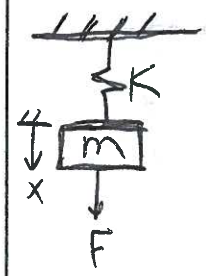
The following steps are to be followed when applying the force method:

1. Isolate the system you want to draw the FBD for.
2. Select an appropriate set of displacement co-ordinates.
3. Draw the FBD of the system for which you seek to determine equations of motion.
4. If you wish to apply Newton's second law, do not indicate inertial forces in the FBD. Use Eq.(1a) or Eq.(1b).
5. Determine absolute accelerations using kinematics
6. Do indicate inertial forces and inertial moments (as required) acting at the centre of mass. Remember that mass moment moments of inertia J_o is about centre of mass in the FBD. Use Eq.(2a) or Eq.(2b).

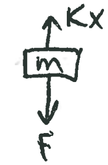
2.3. Force Method (NP 2.3) — # 2

Let us formulate the equation of motion of an undamped spring-mas system using both the approaches. p.30 of notes package

Fill in the class



FBD: Newton w.r.t. fixed frame



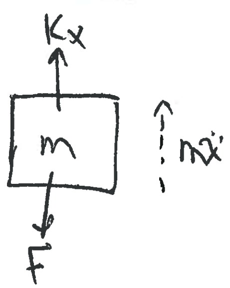
$\therefore \downarrow \sum F_x = m\ddot{x} = \cancel{Kx}\, f - Kx$

$\therefore \boxed{m\ddot{x} + Kx = f}$

3. Force Method (NP 2.3) — # 3

Fill in the class

D'Alembert's principle:



$\downarrow \sum F_x = 0$

$\therefore f - m\ddot{x} - Kx = 0$

$\therefore \boxed{m\ddot{x} + Kx = f}$

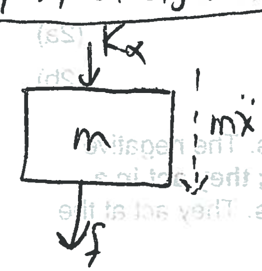
identical as Newton's approach earlier

2.3. Force Method (NP 2.3) — # 4

Q: Can you draw the FBDs when the displacement co-ordinate is chosen as positive in opposite direction to the above?

Fill in the class

Yes, but signs change!



$\sum F_x = -f - Kx - m\ddot{x} = 0$

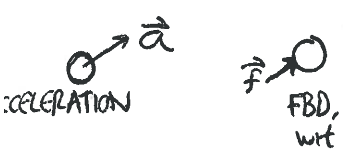
$\boxed{-f = Kx + m\ddot{x}}$

Force Method (NP 2.3) — # 5

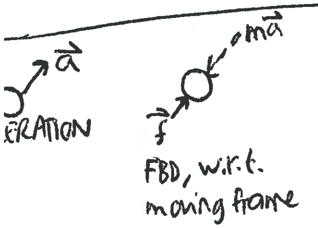
To fix ideas, let us now consider some further simple examples we have already seen in Topic 1.

Point mass, or particle: p.31 of notes package

Fill in the class



$\vec{f} = m\vec{a}$ [NEWTON]



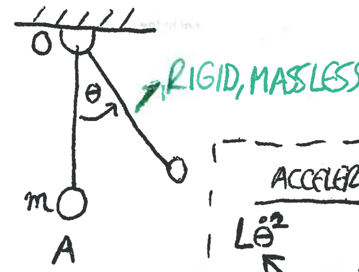
$\vec{f} - m\vec{a} = 0$ [D'Alembert's]

mass: entire mass is at one point; and no mass moment of inertia w.r.t. an axis passing through that point ($J=0$)

2.3. Force Method (NP 2.3) — # 6

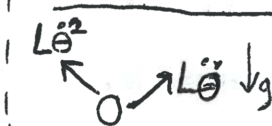
Simple pendulum:

Fill in the class

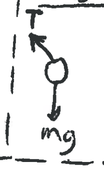


θ +ve ↻ clockwise

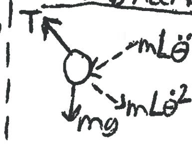
ACCELERATION DIAGRAM



FBD (NEWTON)



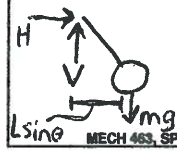
FBD (D'ALEMBERT)



** * SE * SE SLID 21 FOR D'ALE

EQUATIONS OF MOTION

BETTER FOR KEEPING "UNKNOWN FORCES TO A MINIMUM" TO MINIMIZE EFFORT.



$\circlearrowright + \Sigma M_O = J_O\ddot{\theta} - mgL\sin\theta$

$J_O = J_A + mL^2$

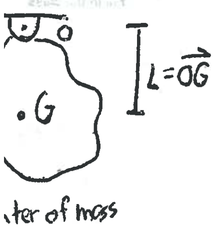
$J_A = 0$, m is point mass

$M_O = \boxed{mL^2\ddot{\theta} - mgL\sin\theta = 0}$

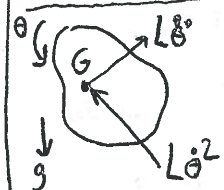
Force Method (NP 2.3) — # 7

Compound pendulum:

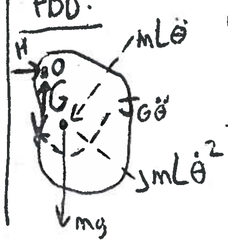
Fill in the class



ACC. DIAGRAM



FBD:



D'ALEMBERT

NEWTON: $\Sigma M_O = J_O\ddot{\theta} \Rightarrow -mgL\sin\theta = J_O\ddot{\theta} \Rightarrow \boxed{J_O\ddot{\theta} + mgL\sin\theta = 0}$

*NOTE: $J_O = J_G + mL^2$ [Parallel Axis Theorem]

D'ALEMBERT $\circlearrowleft \Sigma M_O = 0 = -mgL\sin\theta - mL\ddot{\theta}L - J_O\ddot{\theta} = 0$

$= \boxed{(J_G + mL^2)\ddot{\theta} + mgL\sin\theta = 0}$

Example 3 — # 1

Example 3: Formulate the equations of motion of the following systems using (i) Newton's second law and (ii) D'Alembert's principle.

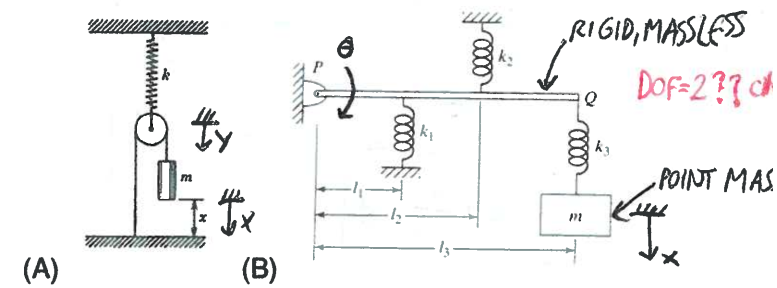


Figure : Figure for example 3.

PART A

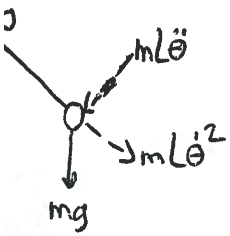
y: absolute displacement of centre of pulley

x: " " centre of mass

GRAVITY can be ignored because 'x' is measured w.r.t. static equilibrium

~~xample 3 — #2~~

'ALEMBERT : SIMPLE PENDULUM [SLIDE 18]



Fill in the class

$\circlearrowleft + \sum M_o = 0 = -mgL\sin\theta - mL^2\ddot{\theta}$

$\therefore \boxed{mL^2\ddot{\theta} + mgL\sin\theta = 0}$

*SAME AS NEWTON, BUT DID NOT REQUIRE PARALLEL AXIS THEOREM; EASIER

Example 3 — # 3

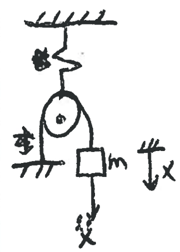
(A) assume rope is massless, and that m is a point mass

Y: ABSOLUTE DISPLACEMENT OF CENTRE OF PULLEY

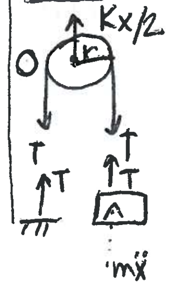
Fill in the class

Use D'Alembert

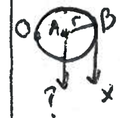
ACCELERATION DIA:



FBD:



ASSUME NO SLIP



O: inst. center of rotati

$x_B = x = 2r\theta$

$x_A = r\theta = \frac{x_B}{2} = \frac{x}{2}$

$\boxed{\text{*if m moves } x\text{, spring moves}}$

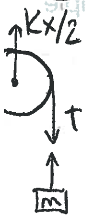
$\therefore \sum M_o = 0 = m\ddot{x}(2r) + \frac{Kx}{2}r$

$\therefore \boxed{m\ddot{x} + \frac{Kx}{4} = 0}$ could verify using Newton's appr

xample 3 — # 4

Now, use Newton's

Fill in the class



$\downarrow \sum F_x = 0 : 2T - \frac{Kx}{2} = 0 \Rightarrow \boxed{T = \frac{Kx}{4}}$ [for pulley]

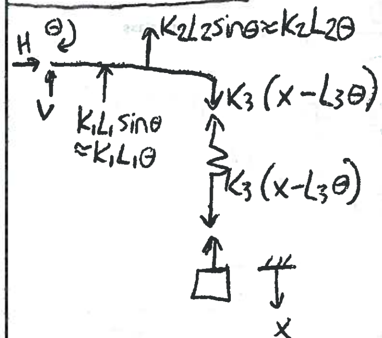
Justified because pulley is massless and has no inertia

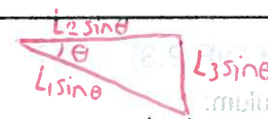
$\sum F_x = m\ddot{x} = -T$

$\therefore \boxed{m\ddot{x} + \frac{Kx}{4} = 0}$

Example 3 — # 5

part B – NEWTON

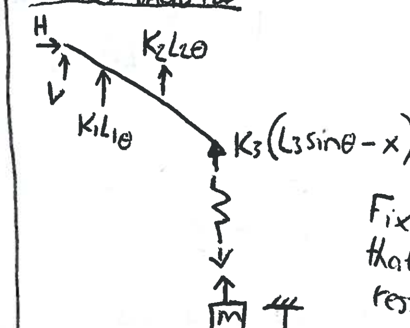




*Are θ & x independent? Is there a constraint?

Fill in the class

EQUALLY VALID FBD



TIPS

• Fix one

• CHECK force is

Fix x=0 and that K_3 force restoring force

mple 3 — # 6

Fill in the class

$\downarrow K_3(x - L_3\sin\theta) \approx K_3(x - L_3\theta) = -K_3L_3\theta$ IF $x = 0$

So, * $\boxed{-K_3L_3\theta}$ is $\uparrow$ direction, hence it is a restoring force

So, * #DOF=?

...an FBD: $\downarrow\Sigma F_x = m\ddot{x} = K_3(L_3\theta - x)$

...ass

$\therefore m\ddot{x} + K_3x = K_3L_3\theta$ (1)

$\hookrightarrow$ x depends on θ

FBD BAR

$\uparrow\Sigma F = m_{bar}\ddot{y}_{CM,BAR}$

$\rightarrow\Sigma F = m_{bar}\ddot{x}_{CM,BAR}$

$\circlearrowleft M_P = J_P\ddot{\theta}$; eliminates H,V

$\circlearrowleft M_P = 0$ $\therefore$ bar has no mass+moment

$(K_1L_1\theta)L_1 + (K_2L_2\theta)L_2 + K_3(L_3\theta - x)L_3 = 0$

$\Rightarrow \boxed{\dfrac{K_1L_1^2\theta + K_2L_2^2\theta + K_3L_3^2\theta}{K_3L_3} = x}$ (2) CONSTRAINT EQ

Example 3 — # 7

- In problem (A) we used the kinematic condition of no slip. Without knowing this kinematic condition, we cannot solve the problem!
- In problem (B) we have two co-ordinates: x and θ. But, we eliminated θ using equilibrium conditions at the point Q where the rod is connected to the spring k_3. We call θ a *passive* co-ordinate since it can be related to the displacement of mass x via equilibrium condition. We will use this idea again when we derive the equivalent spring constant for two springs connected in series.

ning Points

1. A linear elastic spring is governed by Hooke's law $f_s = kx$. For sdof case, the spring constant k can be determined from the energy expression: $k = \frac{\partial^2 U}{\partial x^2}$.
2. A mass is governed by Newton's second law.
3. Newton's second law and D'Alembert's principle are equivalent; both yield identical equations.
4. Inertial forces and moments act at the centre of mass of a rigid body in a direction opposite to the direction of the absolute acceleration.
5. Kinematics is essential to study vibrations.

COMBINE (1) + (2):

$m\ddot{x} + K_3x = K_3L_3\theta = K_3L_3\left[\dfrac{K_3L_3x}{K_1L_1^2 + K_2L_2^2 + K_3L_3^2}\right]$

$\downarrow$ simplify $\downarrow$

$m\ddot{x} + K_3\left[\dfrac{K_1L_1^2 + K_2L_2^2 *}{K_1L_1^2 + K_2L_2^2 + K_3L_3^2}\right]x = 0$

SUMMARY

$\rightarrow$ FBD for mass

$\rightarrow$ FBD for bar

$\rightarrow$ combine to relate, get rid of θ

2.1. SDOF– Formulation of Equations of Motion

MECH 463: Mechanical Vibrations

A. Srikantha Phani
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uggested Readings:

1. Topic 2.1 from notes package.
2. Sections 2.1–2.2 in the course textbook.

2.4. Principle of Conservation of Energy (NP 2.4) — # 1

The principle states that the sum total of potential energy U and kinetic energy T of a *conservative* mechanical system is constant, at every instant in time.

$$\left[T + U = \text{const},\right] \text{or} \left[\frac{d}{dt}[T+U] = 0\right] \qquad (1)$$

It is important to remember that kinetic energy can be due to both translation and rotation. You need to choose a suitable datum to define the potential energy. Let us apply the principle of conservation of energy to the familiar spring mass system to obtain equations of motion.

for simple spring mass:

$$\left.\begin{aligned} K.E. &= T = \tfrac{1}{2}m\dot{x}^2 \\ P.E. &= U = \tfrac{1}{2}Kx^2 \end{aligned}\right\} \frac{d}{dt}\left[T+U\right]=0 \Bigg\} \frac{d}{dt}\left[\frac{1}{2}m\dot{x}^2 + \frac{1}{2}Kx^2\right] = 0$$

Fill in the class

$$\therefore \frac{1}{2}m\frac{d}{dt}(\dot{x}^2) + \frac{1}{2}K\frac{d}{dt}(x^2) = 0$$

Principle of Conservation of Energy (NP 2.4) — # 2

Fill in the class

$$m 2\dot{x}\ddot{x} + \frac{1}{2}K 2x\dot{x} = 0$$

$$\dot{x}\ddot{x} + Kx\dot{x} = 0$$

$$[\ddot{x} + Kx]\dot{x} = 0$$

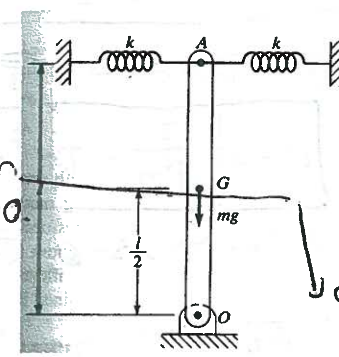
$$x = 0 \qquad \dot{x} = 0 \Rightarrow x = \text{const.} \quad \therefore \text{NO VIBRATIONS}$$

Example 4 — # 1

p.40 in course notes package

Example 4: Write the expressions for potential and kinetic energies of a simple pendulum and apply the principle of conservation of energy to formulate it's equations of motion. Do the same for the inverted pendulum shown below.

Note: G translates and the system rotates about O.



GRAVITY can't be ignored because spring forces do not balance mg in static eq.

chose this point as DATUM

Figure : Figure for example 4.

nple 4 — # 2

$= T = \frac{1}{2} m \left(\frac{L}{2}\dot{\theta}\right)^2 + \frac{1}{2} J_G \dot{\theta}^2 = \frac{1}{2} J_O \dot{\theta}^2$

Recall $J_O = J_G + m\left(\frac{L}{2}\right)^2$

J_G = mass mom. of inert. w.r.t. an axis passing through c.o.m. at point 'G'

$r = V_G - mg\frac{L}{2}(1-\cos\theta) + 2\left(\frac{1}{2}\right) K (L\sin\theta)^2$

V_G = energy associated with datum

$al = r + \upsilon = \text{const.} \rightarrow \frac{dE}{dt} = 0$

$m\left(\frac{L}{2}\dot{\theta}\right)^2 + \frac{1}{2} J_G \dot{\theta}^2 + V_G - mg\frac{L}{2}(1-\cos\theta) + K(L\sin\theta)^2 \Big]$

$^2\left(\frac{L}{2}\dot{\theta}\right)\frac{L}{2}\ddot{\theta} + \frac{1}{2} J_G 2\dot{\theta}\ddot{\theta} + 0 - mg\frac{L}{2}\sin\theta(\dot{\theta}) + 2K(L\sin\theta)L\cos\theta(\dot{\theta}) = 0$

$\left.\frac{mL^2}{4}\right)\ddot{\theta} + 2KL^2\sin\theta\cos\theta - \frac{mg}{2}L\sin\theta\Big]\dot{\theta} = 0$

Example 4 — # 3

Fill in the class

So: $\left[J_G + \frac{mL^2}{4}\right]\ddot{\theta} + 2KL^2\sin\theta\cos\theta - \frac{mgL}{2}\sin\theta = 0$

or $\ddot{\theta} = 0$ for small $\theta \rightarrow \sin\theta \approx \theta$, $\cos\theta \approx 1$

$\therefore \boxed{\left[J_G + \frac{mL^2}{4}\right]\ddot{\theta} + \left[2KL^2 - \frac{mgL}{2}\right]\theta = 0}$

$2KL^2 - \frac{mgL}{2} = 0$ IF $mg = 4KL$ (OR) $L = \boxed{\frac{mg}{4K}}$

→ ZERO STIFFNESS → NO VIBRATION

→ BUCKLING

nple 4 — # 4

Fill in the class

Example 4 — # 5

Fill in the class

mple 4 — # 6

Fill in the class

Example 4 — # 7

Important learning points from this example:

- Gravitational weight tends to destabilise oscillations if the Centre of Mass (C.M.) where *mg* acts, is above the pivot point. It will stabilise the system if the C.M. is below the pivot point. Gravity cannot *always* be ignored! Beware of this!!
- Inertial forces act at C.M. Since the bar is a rigid body and has distributed mass, both inertial force and inertial moment act at C.M. Notice the difference with a simple pendulum on page 32, where there is no inertial moment acting on the pendulum mass: because, it is a point mass having no mass moment of inertia about it's C.M.

mary — # 1

1. We learnt three techniques to arrive at equations of motion: Newton's second law, D'Alembert's principle, and principle of conservation of energy. All give same equations of motion through different routes.
2. A knowledge in Kinematics is essential to determine the absolute velocities and accelerations.
3. D'Alembert's principle gives a complete picture of forces acting in a mechanical system during its motion.
4. Inertial forces and moments act at the centre of mass of a rigid body in a direction opposite to the direction of the absolute acceleration.
5. By adding *inertial forces and moment acting the centre of mass* we can extend the equilibrium concepts of statics to study vibration problems.

Summary — # 2

Q: Compare the three techniques we studied in this topic based on the examples we covered.

Fill in the class

	NEWTON	D'ALEMBERT	ENERGY
KINEMATICS	YES	YES	YES, TO FIND VELOCITIES IN K
FBD	YES	YES	NO
PARALLEL AXIS THEOREM	YES	NO	MAYBE; wrt center of mass
COMPLETE ~~[illegible]~~ USED TO COMPUTE STRESSES	NO	YES	NO FBD
ANALYTICAL, NOT PHYSICALLY INTUITIVE	NO	NO	YES, WE START WITH SCAL FUNCTION AND MINIMIZE I

2.2. SDOF– Equivalent Systems

MECH 463: Mechanical Vibrations

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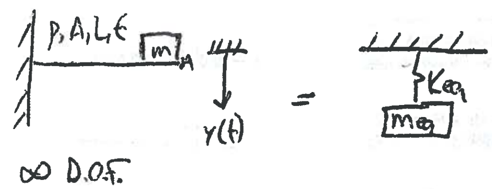
UBC mechanical engineering

Suggested Readings:

1. Topic 2.2 from notes package.
2. Sections 1.7–1.8 in the course textbook.

Learning Objectives (NP 2.5)

1. Determine equivalent spring (translational and rotational) and mass (translational) or mass moment of inertia (rotational) of a multi-component mechanical system at a point of interest.
2. Apply the force and energy methods learned in Topic 2.1.
3. Recognize the advantages and limitations of equivalent systems.



$=$ what are k_{eq}, m_{eq}?

Fill in the class

∞ D.O.F.

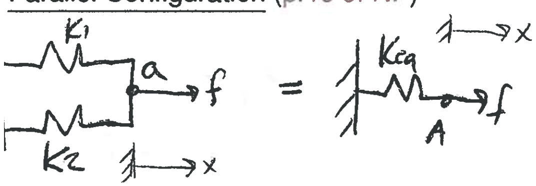
$*$S.D.O.F at (A), only can be applied at point A

Equivalent Spring– Force Method (NP 2.6, T1.7) — # 1

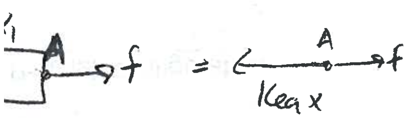
The key idea here is to obtain force-displacement relation at the point of interest.

Parallel Configuration (p.46 of NP)

Fill in the class



at point A

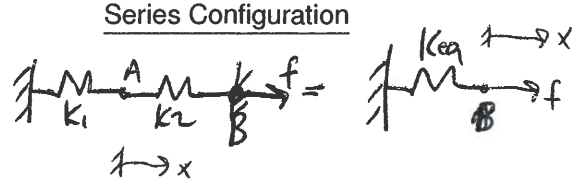


$\Sigma F = 0 \ldots \therefore \boxed{k_{eq} = k_1 + k_2}$

2.5. Equivalent Spring– Force Method (NP 2.6, T1.7) — # 2

Series Configuration

Fill in the class



FBD at point A:

$\therefore \Sigma F = 0 \therefore \boxed{x_2 = \frac{k_1 x_1}{k_1 + k_2}}$ Eqn [1]

internal equilibrium at A eliminates passive co-ordinate

FBD at point B

$f = k_1(x_1 - x_2)$ [2]

combining [1]+[2]:

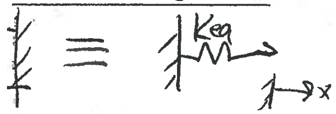
$$\boxed{k_{eq} = \frac{k_1 k_2}{k_1 + k_2}}$$

. Equivalent Spring– Energy Method (NP 2.6, T1.7) — # 1

The key idea here is to obtain potential energy in terms of the displacement co-ordinate at the point of interest.

$PE_{sys} = PE_{in}$ equivalent system

Parallel Configuration



Fill in the class

$, = [U]_{eq}$

$, x^2 + K_2 x^2] = \frac{1}{2} K_{eq} x^2$

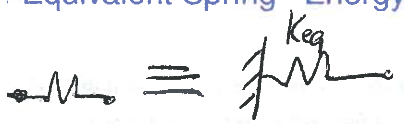
$\therefore \boxed{K_{eq} = K_1 + K_2}$

2.5. Equivalent Spring– Energy Method (NP 2.6, T1.7) — # 2

Series Configuration

Fill in the class

. Equivalent Spring– Energy Method (NP 2.6, T1.7) — # 3



Fill in the class

$s = [U]_{eq}$

$(x_1 - x_2)^2 + \frac{1}{2} K_2 x_2^2 = \frac{1}{2} K_{eq} x_1^2$

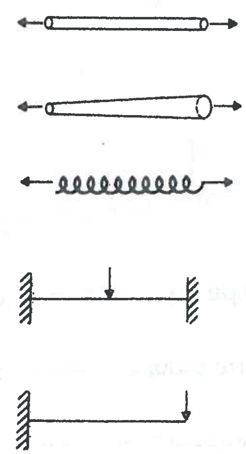
Recall: $x_2 = \frac{K_1 x_1}{K_1 + K_2}$

$x_1 - \frac{K_1 x_1}{K_1 + K_2})^2 + \frac{1}{2} K_2 \left(\frac{K_1 x_1}{K_1 + K_2}\right)^2 = \frac{1}{2} K_{eq} x_1^2$

$\therefore \boxed{K_{eq} = \frac{K_1 K_2}{K_1 + K_2}}$

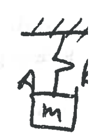
* Do Ex. #3, Part B using this method

2.5. Equivalent Spring– Energy Method (NP 2.6, T1.7) — # 4

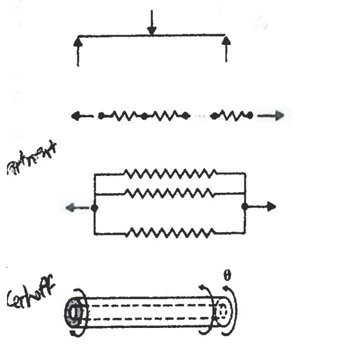


Rod under axial load (l = length, A = cross sectional area)	$k_{eq} = \frac{EA}{l}$
Tapered rod under axial load (D, d = end diameters)	$k_{eq} = \frac{\pi E D d}{4l}$
Helical spring under axial load (d = wire diameter, D = mean coil diameter, n = number of active turns)	$k_{eq} = \frac{G d^4}{8 n D^3}$
Fixed-fixed beam with load at the middle	$k_{eq} = \frac{192 EI}{l^3}$
Cantilever beam with end load	$k_{eq} = \frac{3EI}{l^3}$

* How does equivalent system concept work for: at 'A', when m & K are connected in parallel?



. Equiv...nt Spring– Energy Method (NP 2.6, T1.7) — # 5



Simply supported beam with load at the middle	$k_{eq} = \frac{48EI}{l^3}$
Springs in series	$\frac{1}{k_{eq}} = \frac{1}{k_1} + \frac{1}{k_2} + \cdots + \frac{1}{k_n}$
Springs in parallel	$k_{eq} = k_1 + k_2 + \cdots + k_n$
Hollow shaft under torsion (l = length, D = outer diameter, d = inner diameter)	$k_{eq} = \frac{\pi G}{32l}(D^4 - d^4)$

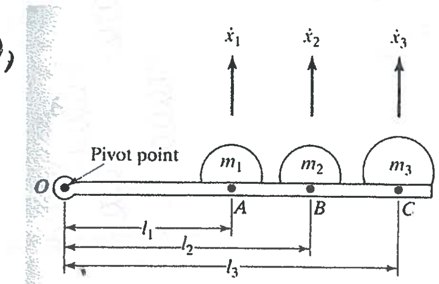
Figure: Equivalent spring constants.

2.5. Equivalent Mass or Equivalent Mass Moment of Inertia (NP 2.7, T1.7) — # 1

The key idea here is to obtain kinetic energy in terms of the velocity co-ordinate at the point of interest.

Translation masses connected by a rigid bar (p.50 of NP)

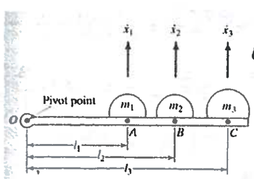
FIND
* Equivalent J at O,
OR
* Equivalent mass at Center of mass



KINEMATICS:
$\dot{x}_1 = l_1\dot{\theta}$
$\dot{x}_2 = l_2\dot{\theta}$
$\dot{x}_3 = l_3\dot{\theta}$

Figure: Translation masses connected by a rigid bar.

. Equivalent Mass or Equivalent Mass Moment of Inertia (NP , T1.7) — # 2



EQ. MASS AT 'A':

$[K.E]_{SYS} = \frac{1}{2} m_{eq}\dot{x}_1^2 = [K.E]_{eq}$

$\therefore [K.E]_{SYS} = \frac{1}{2}J_o\dot{\theta}^2 + \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2 + \frac{1}{2}m_3\dot{x}_3^2$

$\therefore \frac{1}{2}J_{BAR,O}\dot{\theta}^2 = \frac{1}{2}J_{BAR,CM}\dot{\theta}^2 + M_{BAR}\left(\frac{L}{2}\right)^2 \rightarrow$ P.A.T

...M KINEMATICS

$\dot{\theta} = \frac{\dot{x}_1}{L_1}$; $\dot{x}_2 = \frac{L_2}{L_1}\dot{x}_1$; $\dot{x}_3 = \frac{L_3}{L_1}\dot{x}_1$

$\therefore_{SYS} = \frac{1}{2}J_{BAR,O}\left(\frac{\dot{x}_1}{L_1}\right)^2 + \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\left(\frac{L_2}{L_1}\dot{x}_1\right)^2 + \frac{1}{2}m_3\left(\frac{L_3}{L_1}\dot{x}_1\right)^2 = \frac{1}{2}M_{eq}\dot{x}_1^2$

$$\therefore \boxed{M_{eq} = \frac{J_{BAR,O}}{L_1^2} + m_1 + m_2\left(\frac{L_2}{L_1}\right)^2 + m_3\left(\frac{L_3}{L_1}\right)^2}$$

2.5. Equivalent Mass or Equivalent Mass Moment of Inertia (NP 2.7, T1.7) — # 3

Q: Can you see how you can find equivalent mass moment of inertia of the above system? (p.51 of NP)

Fill in the class

Yes. By finding Kinetic in terms of angular velocity $\dot{\theta}$

$K.E. = \frac{1}{2}J_{BAR,CM}\dot{\theta}^2 + \frac{1}{2}M_{BAR}\left(\frac{l_3}{2}\dot{\theta}\right)^2 + \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m \ldots \frac{1}{2}m_3\dot{x}_3^2$

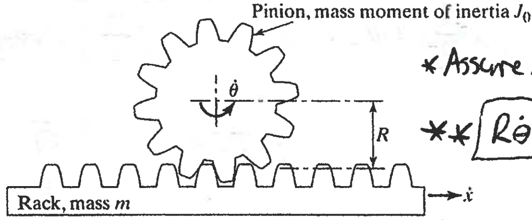
$= \frac{1}{2}\left[J_{BAR,CM} + M_{BAR}\left(\frac{l_3}{2}\right)^2 + m_1 l_1^2 + m_2 l_2^2 + m_3 l_3^2\right]\dot{\theta}$

$= \frac{1}{2}J_{EQ}\dot{\theta}^2$

$$\therefore \boxed{J_{EQ} = J_{BAR,CM} + M_{BAR}\left(\frac{L_3}{2}\right)^2 + m_1 l_1^2 + m_2 l_2^2 + m_3 l_3^2}$$

. Equivalent Mass or Equivalent Mass Moment of Inertia (NP , T1.7) — # 4

Translational and rotational masses coupled



$*$ Assume no slip/backlash:

$** \boxed{R\dot{\theta} = \dot{x}} **$

Figure: Rack and Pinion mechanism.

$= [KE]_{RACK} + [KE]_{PINION}$

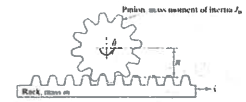
$= \frac{1}{2} m \dot{x}^2 + \frac{1}{2} J_0 \dot{\theta}^2$ → can write in terms of $\dot{\theta}$ to get J_{EQ}, or $\dot{x}$ to get m_{eq}

$= \frac{1}{2} m (R\dot{\theta})^2 + \frac{1}{2} J_0 \dot{\theta}^2 = \frac{1}{2} J_{EQ} \dot{\theta}^2 \therefore \boxed{J_{EQ} = mR^2 + J_0}$

$= \frac{1}{2} m \dot{x}^2 + \frac{1}{2} J_0 \left(\frac{\dot{x}}{R}\right)^2 = \frac{1}{2} m_{eq} \dot{x}^2 \therefore \boxed{m_{eq} = m + \frac{J_0}{R^2}}$

Always check units!

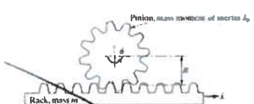
2.5. Equivalent Mass or Equivalent Mass Moment of Inertia (NF 2.7, T1.7) — # 5



→ units of J_0? $[Kgm^2]$

Fill in the class

. Equivalent Mass or Equivalent Mass Moment of Inertia (NP , T1.7) — # 6



Fill in the class

Example 5 — # 1

p.40 in course notes package

Example 5: Determine the equivalent mass and spring constant of the following systems (p.53 of NP)

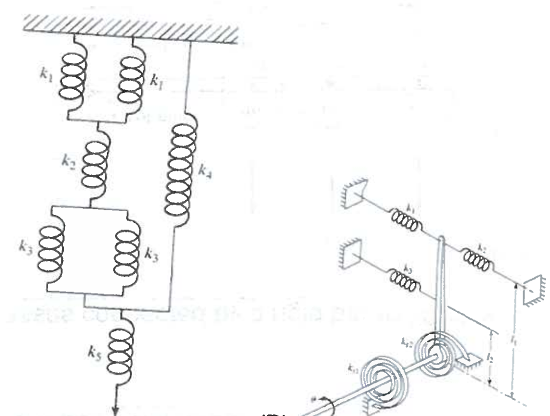
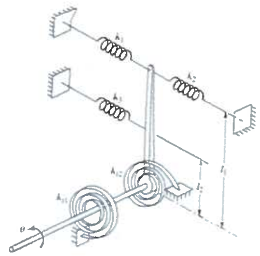


Figure: Figure for example 5.

ımple 5 — # 6

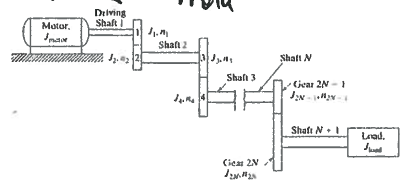


Fill in the class

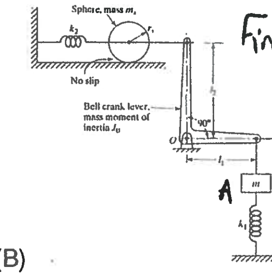
Example 6 — # 1

Example 6: Determine the equivalent mass of the spring-mass system when the spring has a total mass of m_S. Determine the equivalent mass moment of inertia /mass of the systems shown below (p.56 of NP)

Find J_{EQ} of motor



(A)



(B)

Find eq. mass at 'A'

Figure: Figures for example 6.

what if SPRING has a mass, m_s?

m_s is distributed throughout the spring

∞ DOF system

∴ Assume a deformation shape (i.e. method of assumed modes)

ımple 6 — # 2

…aneous length of spring

…te displacement of mass m

$\equiv$ K_{eq}, m_{eq} point of interest: A

…t spring at 'A':

$\ldots_{sys} = U_{EQ}$

$\ldots = \frac{1}{2} K_{eq} x^2$ ∴ $\boxed{K_{eq} = K}$

Fill in the class

Example 6 — # 3

equivalent mass

$KE_{sys} = KE_{EQ}$

$[KE]_m + [KE]_{spring} = \frac{1}{2} M_{eq} \dot{x}^2$

$\therefore \frac{1}{2} m \dot{x}^2 + [KE]_{spring} = \frac{1}{2} M_{eq} \dot{x}^2$

Difficult. Assume deformation shape

$[KE]_{spring} = [KE]_{C.O.M.\ of\ spring} = \frac{1}{2} m_s \left(\frac{\dot{x}}{2}\right)^2 = \frac{1}{8} m_s \dot{x}^2$

$\therefore M_{EQ} = \boxed{m + \frac{m_s}{4}}$

*Center of mass is useful for rigid bodies

NOT CORRECT THIS DOESN'T WORK BEST

Fill in the class

ample 6 — # 4

...CE A NEW CO-ORDINATE 'S' which locates a small element ds

$\cdot$Velocity at 's' of ds = ??

~~Fill in the class~~

...ds

...f 'ds' $= \frac{ms}{L} ds$

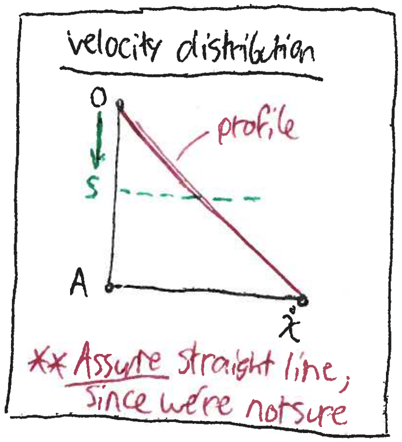
$ds' = \frac{1}{2} \frac{ms}{L} ds \left(V_{ds}\right)^2$

??

at $O = 0$ (fixed)

at $A = \dot{x}$

$ds = \frac{s}{L} \dot{x}$



** Assume straight line, since we're not sure

NOTE:

Correction factor "$\frac{ms}{3}$" is subject to the validity of linear variation of velocities

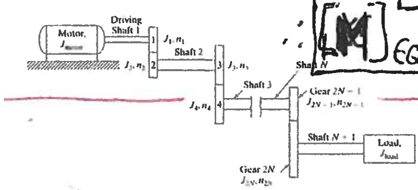
$$\therefore \left[KE_{spring}\right] = \sum \left[KE\right]_{ds} = \sum \frac{1}{2} \frac{ms}{L} ds \left(\frac{s\dot{x}}{L}\right)^2$$

$$= \frac{1}{2} \int_0^L \frac{ms}{L} \frac{s^2}{L^2} \dot{x}^2 ds = \frac{1}{2} \frac{ms}{L_3} \dot{x}^2 \int_0^L s^2 ds$$

$$\therefore \left[KE\right]_{spring} = \frac{1}{2} \left(\frac{ms}{3}\right) \dot{x}^2$$

$$\therefore \boxed{M_{EQ} = m + \frac{ms}{3}}$$

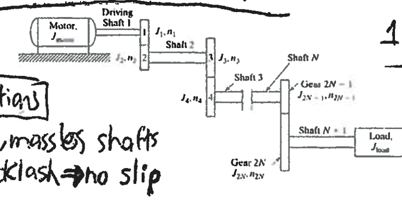
ample 6 — # 6



Fill in the class

Example 6 — # 5

EXAMPLE 6, PART A



Assumptions

- rigid, massless shafts
- no backlash $\Rightarrow$ no slip

1. Find J_{eq} at motor

Fill in the class

$$\left[KE\right]_{sys} = \left[KE\right]_{EQ} = \frac{1}{2} J_{eq} \left(\dot{\theta}_{motor}\right)^2$$

Shaft 1: $\dot{\theta}_{motor}$

J_1: $\dot{\theta}_{motor} = n_1 \dot{\theta}_{motor}$

J_2: $n_2 \dot{\theta}_2$ $\therefore \frac{n_1}{n_2} \dot{\theta}_{motor} = \dot{\theta}_2$ NO SLIP

J_3: $\frac{n_1}{n_2} \dot{\theta}_{motor}$ RIGID SHAFT

J_4: $J_3 = n_4 \dot{\theta}_4 = \frac{n_1 n_3}{n_2} \dot{\theta}_{motor}$ NO SLIP

$\therefore \dot{\theta}_4 = \frac{n_1 n_3}{n_2 n_4} \dot{\theta}_{motor}$

J_5: $\frac{n_3}{n_4} \frac{n_1}{n_2} \dot{\theta}_{motor}$ RIGID SHAFT

AND SO ON

angular $\dot{\theta}$, in terms of $\dot{\theta}_{motor}$

$$SO, \left[KE\right]_{sys} = \frac{1}{2} J_{motor} \left(\dot{\theta}_{motor}\right)^2 + \frac{1}{2} J_1 (\dot{\theta}\ldots$$

$$+ \frac{1}{2} J_2 \left(\frac{n_1}{n_2} \dot{\theta}_{motor}\right)^2 + \frac{1}{2} J_3 (\ldots$$

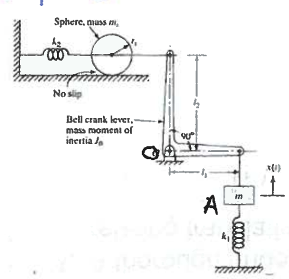
$$+ \frac{1}{2} J_4 \left(\frac{n_1}{n_2} \frac{n_3}{n_4} \dot{\theta}_{motor}\right)^2 +$$

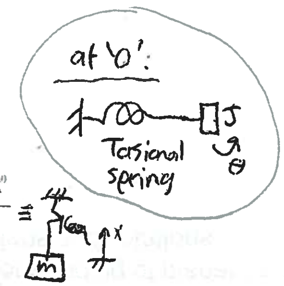
$$\frac{1}{2} J_5 \left(\frac{n_1}{n_2} \frac{n_3}{n_4} \dot{\theta}_{motor}\right)^2 + \ldots$$

$$= \frac{1}{2} J_{eq} \left(\dot{\theta}_{motor}\right)^2$$

$$\boxed{\therefore J_{EQ} = J_{motor} + J_1 + \left(\frac{n_1}{n_2}\right)^2 J_2 + \left(\frac{n_1}{n_2}\right)^2 J_3 + \left(\frac{n_1}{n_2} \frac{n_3}{n_4}\right)^2 J_4 \ldots}$$

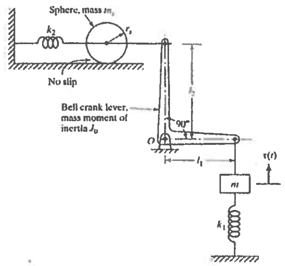
Example 6 — # 7





Fill in the class

ample 6 — # 8



Fill in the class

Summary

1. Equivalent systems simplify the modelling of multi-component systems **at the point of interest**.
2. Equivalent system parameters change with the location of point of interest and hence can't capture entire dynamic response.
3. Equivalent spring is determined from the potential energy and equivalent mass (or mass moment of inertia) is determined from the kinetic energy expression of the system expressed in terms of a **single displacement co-ordinate**.
4. A thorough understanding of planar kinematics is essential in relating translations and rotations.

2.3. Undamped SDOF Response – 1

MECH 463: Mechanical Vibrations

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Suggested Readings:

1. Topic 2.3 from notes package.
2. Sections 1.10, 2.2 and 2.3 from the course textbook.

Learning Objectives

1. Determine undamped free vibration response of a SDOF system.
2. Compute the natural frequencies for free vibrations.
3. Recognize the importance of system parameters and operating conditions on the natural frequencies.
4. Visualize harmonic vibrations in a graphical form.

Far ...

1. We know how to select co-ordinates to describe vibrations.

Fill in the class

2. We know how to compute accelerations and velocities in general planar motion using kinematics.
3. We know how to write equations of motion using force and energy methods.
4. We learned about superposition principle for linear systems.
5. We learned how to deduce equivalent system parameters for continuous ∞ DOF systems.
6. This topic combines all of the above.

Universal SDOF Equations of Motion (NP 2.8)

$$m\ddot{x} + kx = f \quad \text{Translatory vibrations} \tag{1a}$$

$$J_o\ddot{\theta} + k_\theta\theta = M_o \quad \text{Rotatory or torsional vibrations} \tag{1b}$$

Some remarks...

Fill in the class

We focus on harmonic force of the form $f(t) = F_0 \cos \omega t$. Most rotating systems are subjected to forces of this kind. **For general, non harmonic forcing, we make use of superposition principle using Fourier series or in the form of a convolution integral as we shall see in later topics.** See the notes package for detailed mathematical derivations.

. Undamped Vibration Response (NP 2.9)

$$m\ddot{x}_h + kx_h = 0 \quad \text{Homogeneous response/Free vibration} \qquad (2a)$$
$$m\ddot{x}_h + kx_h = f \quad \text{Particular solution/Forced vibration.} \qquad (2b)$$

Adding the above two equations we have the TOTAL response, from the principle of superposition

$$m\ddot{x} + kx = f, \quad x = x_h + x_p \quad \text{TOTAL response} \qquad (3)$$

It is required to specify the initial conditions on the TOTAL response. They can be initial velocity, or initial displacement:

$x(0) = x_0; \quad \dot{x}(0) = \dot{x}_0$ INITIAL conditions **apply on the TOTAL solution**. (4)

Fill in the class

2.10. Free Vibration Response (NP 2.10, T 2.2.4, T 2.2.5+Notes — # 1

$$m\ddot{x}_h + kx_h = 0 \quad \text{Homogeneous response/Free vibration} \qquad (5)$$
$$\therefore x_h(t) = A_1 \cos\omega_n t + A_2 \sin\omega_n t \qquad (6)$$

Notice that we have two unknown constants A_1 and A_2. These can be found using:

1. Initial conditions only when the particular solution is zero, $x_p = 0$.
2. Initial conditions on the TOTAL solution $x = x_h + x_p$, when there is an external force, $x_p \neq 0$.

In free vibration problems, there is no external force $f = 0 \Rightarrow x_p = 0$. Therefore $x = x_h + x_p = x_h$.

). Free Vibration Response (NP 2.10, T 2.2.4, T 2.2.5+Notes) ‡ 2

Imposing initial conditions in Eq.(4) on the total solution, we have

$$x_h(t) = x_0 \Rightarrow A_1 \cos 0 + A_2 \sin 0 = x_0 \Rightarrow A_1 = x_0$$
$$\dot{x}_h(t) = \dot{x}_0 \Rightarrow [-\omega_n A \sin\omega_n t + \omega_n A_2 \cos\omega_n t]_{t=0} = \dot{x}_0$$
$$\omega_n A_2 = \dot{x}_0 \Rightarrow A_2 = \frac{\dot{x}_0}{\omega_n}$$

Thus, the free vibration response of an undamped SDOF system is obtained as given below.

$$\boxed{x = x_h = x_o \cos\omega_n t + \frac{\dot{x}_0}{\omega_n}\sin\omega_n t, \quad \omega_n = \sqrt{\frac{k}{m}} \qquad (7)}$$

In solving problems it is best not to memorize the above formula, but, instead use the form $x_h(t) = A_1 \cos\omega_n t + A_2 \sin\omega_n t$ and determine A_1 and A_2 as appropriate.

2.10. Free Vibration Response (NP 2.10, T 2.2.4, T 2.2.5+Notes — # 3

1. **Free vibration takes place at the system's natural frequency**, irrespective of the initial conditions.
2. Natural frequencies depend only on the stiffness and mass properties of the system. Natural frequency increases with an increase in the stiffness or a *decrease* in the mass.
3. One can estimate natural frequencies from static deflections using the formula $\omega_n = \sqrt{\frac{g}{\delta_{st}}}$. Can you show this?

Fill in the class

4. The S.I. unit of natural frequency is Hertz. One Hz is one cycle per second. In most computer programs, and in mathematics, frequencies are measured in rad/s. In engineering practise one also uses rpm. The following conversion may be useful $1Hz = 2\pi rad/s = \frac{1}{60} rpm$. 1200 rpm is thus 20 Hz or 40π rad/s.

ample 8 — # 1

Example 8: An air-conditioning chiller unit weighs 600 kg is to be supported by four air springs as shown below. Design the air springs such that the natural frequency of vibration of the unit lies between 5 rad/s and 10 rad/s. (p.70 of NP)

this range is decided based on isolations requirements

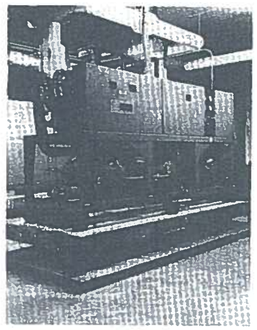


Figure: Figure for example 8.

Example 8 — # 2

Fill in the class

$$W_N = \sqrt{\frac{4K}{m}}$$

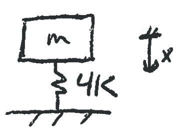
$$5 < \sqrt{\frac{4K}{m}} < 10$$

$$\therefore \boxed{3{,}750\, N/m < K < 15{,}000\, N/m}$$

ample 8 — # 3

Fill in the class

hoosing higher or lower K?



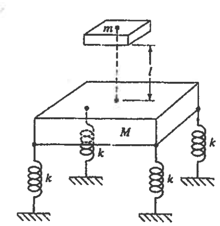
what is the maximum allowed "x"; displacement settlement

olation Requirements will inform the correct choice for 'K'

Topic 2.5

Example 9 — # 1

Example 9: A rigid block of mass M is mounted on four elastic supports, as shown below. A mass m drops from a height l and adheres to the rigid block without rebounding. If the spring constant of each support is k, find the natural frequency of vibration of the system (a) without mass m, and (b) with the mass m. Also find the resulting motion of the system in case (b).(p.72 of NP)



- chose correct/easy co-ord
- impulse-momentum
- choosing correct/easier ref for the co-ordinate

Figure: Figures for example 9.

Fill in the class

ample 9 — # 2

ι – without 'm'

Fill in the class

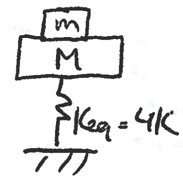
q = 4K , springs in parallel

$\omega_n = \sqrt{\frac{K_{eq}}{m}} = \boxed{\sqrt{\frac{4K}{m}}}$

Example 9 — # 3

case B: Find ω_n with 'm'

Fill in the class



$\omega_n = \sqrt{\frac{K_{eq}}{M+m}} = \boxed{\sqrt{\frac{4K}{M+m}}}$

ample 9 — # 4

after 'm' impacts 'M'

Fill in the class

*what are posible choices for co-ordinates?

1 - $z(t)$: displacement (absolute) of M w.r.t. unstretched springs

2 - $x(t)$: displacement of M w.r.t. static equilibrium of m, M, 4K

3 - $y(t)$: displacement of M w.r.t. static equilibrium of M, K

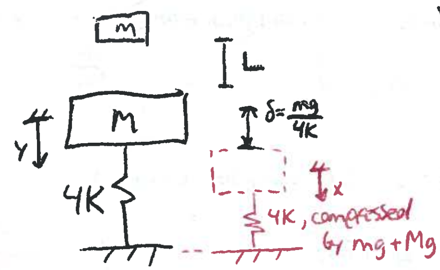
*There is a change in static equilibrium

→ $x(t)$, you can ignore g, gravity

→ $y(t)$ can not ignore m's gravity component; $z(t)$ need gravity

Example 9 — # 5

Fill in the class

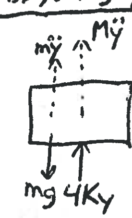


*choice of co-ordinates: x, y

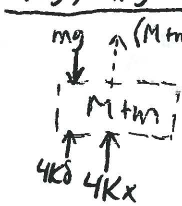
$y(t) = x(t) + \underline{\underline{\delta}}$

$\dot{y}(t) = \dot{x}(t)$

FBD, using $y(t)$



FBD, using $x(t)$



NOTE: $mg = 4K\delta$

$\frac{m\sqrt{2gl}}{\omega_n(M+m)}$ ← $x = x_h + x_p = A_1 \cos\omega_n t + \ldots$

$x(0) = -\delta$

$\dot{x}(0) = m\sqrt{2gl}/(M+m)$

* $mgl = \frac{1}{2}mv^2$ ∴ $v = \sqrt{2gl}$

...ample 9 — # 6

...s of motion, $y(t)$

$\ldots 4Ky - m\ddot{y} - M\ddot{y} = 0$

$\boxed{\ldots)\ddot{y} + 4Ky = mg}$

vibration problem; ...ct of magnitude

Eqns. of motion, $x(t)$

$+\downarrow \sum F_x = 0$

∴ $mg - 4Kx - m\ddot{x} - M\ddot{x} - 4K\delta = 0$

→ Since $mg = 4K\delta$

$\boxed{(M+m)\ddot{x} + 4Kx = 0}$

free vibration problem, force = 0

$\boxed{y(0) = 0}$, rest before impact

Fill in the class

$\dot{y}(0) = ?$

momentum before = mom. after

$m\sqrt{2gl} = (M+m)\dot{y}(0)$

$\boxed{\therefore \dot{y}(0) = \frac{m\sqrt{2gl}}{(M+m)}}$

$x(0) = ?$

$x(0) = y(0) - \delta = -\delta$

$x(0) = \frac{-mg}{4K}$

$\dot{x}(0) = \dot{y}(0) = \frac{m\sqrt{2gl}}{M+m}$

∴ By plugging $y = x + \delta$ into the left equation, we find that both approaches are dependent

*choice of co-ordinate influences the vibration problem you are solving

→ To solve the equations of motion, we must find the initial conditions

→ at $t = 0$ JUST WHEN 'm' IMPACTS 'M'

FINAL ANSWER FOR EX. 7 → $\boxed{x(t) = -\delta\cos(\omega_n t) + \frac{m\sqrt{2gl}}{\omega_n(M+m)}\sin(\omega_n t)}$ $\omega_n = \sqrt{\frac{4K}{M+m}}$

2.11. Different Representations of Response (T 2.2.5, T 1.10+Notes) — # 1

Amplitude-Phase Form

$$\boxed{x = x_h = x_0 \cos\omega_n t + \frac{\dot{x}_0}{\omega_n}\sin\omega_n t, \quad \omega_n = \sqrt{\frac{k}{m}}}$$

negative

Phase-lag form: $x(t) = A\cos(\omega_n t - \phi_0)$, $A = \sqrt{x_0^2 + \left(\frac{\dot{x}_0}{\omega_n}\right)^2}$, $\phi_0 = \tan^{-1}\left(\frac{\dot{x}_0}{x_0\omega\ldots}\right.$ (8a)

positive

Phase-lead form: $x(t) = A\cos(\omega_n t + \phi_0)$, $A = \sqrt{x_0^2 + \left(\frac{\dot{x}_0}{\omega_n}\right)^2}$, $\phi_0 = \tan^{-1}\left(-\frac{\ldots}{x_0\ldots}\right.$ (8b)

Reference: $A\cos(\omega_n t)$

...1. Different Representations of Response (T 2.2.5, T ...0+Notes) — # 2

Complex variable representation

$$x(t) = \underset{\text{Real}}{Re}\left[Ce^{j\omega_n t}\right] \quad (9)$$

→ C is complex, ω_n is real

→ compact representation

$e^{j\omega_n t} = \cos(\omega_n t) + j\sin(\omega_n t)$ [Euler]

Fill in the class

- C contains two real numbers related to A_1, A_2 in $x_h = A_1\cos(\omega_n t) + A_2\sin(\omega_n t)$

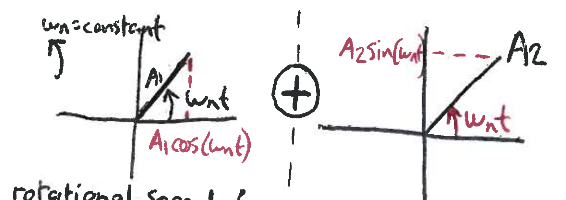
Both the amplitude-phase forms and the complex variable representation of free vibration offers a graphical representation of the vibrations. This graphical method can be used to our great advantage to gain insight into vibration.

Rotating Vector Representation — # 1

The free vibration response can be visualised as a sum of projections of two rotating vectors of amplitudes A_1 and A_2, respectively, as shown below.(p.77 of NP)

$x_h(t) = A_1\cos(\omega_n t) + A_2\sin(\omega_n t)$

Fill in the class



rotational speed of vector of amplitude A_1 = const.

So, free vibrations re... is the sum of horizo... projections of A_1 and... vertical projection of...

$\boxed{\text{This is NOT efficient}}$

* But, there IS a way to make this to make this one vector

tating Vector Representation — # 2

The response forms mentioned in Eq.(8a) and Eq.(8b) are particularly suitable to be visualised as rotating vectors. The response in Eq.(8a) can be visualised as the horizontal projection of a rotating vector of amplitude A and phase lag ϕ_0 as shown below. (p.78 of NP)

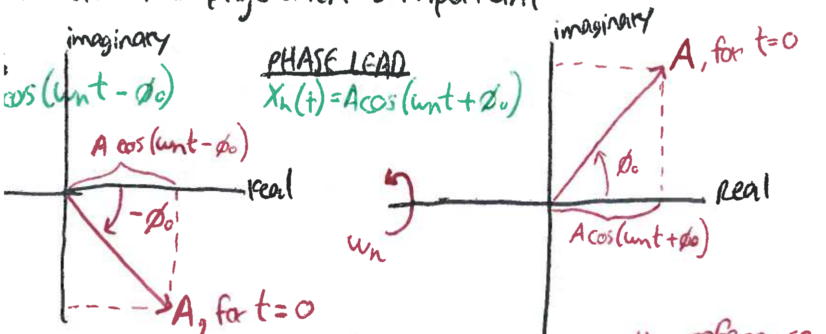
distance $= \omega_n t$ & constant $= \omega_n$

Fill in the class

an write the two vectors as one equation:

$A\cos(\omega_n t - \phi_0)$

the horizontal projection is important



ontal axis corresponds to $\omega_n t = 0$ taken as the reference

Rotating Vector Representation — # 3

Similarly, the response in Eq.(8b) can be visualised as the horizontal projection of a rotating vector of amplitude A and phase lead ϕ_0 as shown above.(p.78 of NP)

[See previous slide]

tating Vector Representation — # 4

Another visualisation is provided by a rotating vector in a complex plane. This rotating vector is called a phasor. Let us represent the displacement, velocity, and acceleration as complex numbers, and then as rotating vectors in the complex plane. We will take the form given Eq.(8a) for the displacement:

$$x(t) = A\cos(\omega_n t - \phi_0) = Re\left[Ae^{j(\omega_n t - \phi_0)}\right]$$

$$\dot{x}(t) = -A\omega_n \sin(\omega_n t - \phi_0) = A\omega_n \cos(\omega_n t - \phi_0 + 90^0)$$

$$= Re\left[A\omega_n e^{j(\omega_n t - \phi_0 + 90^0)}\right] = Re\left[j\omega_n A e^{j(\omega_n t - \phi_0)}\right]$$

$$\ddot{x}(t) = -A\omega_n^2 \cos(\omega_n t - \phi_0) = A\omega_n^2 \cos(\omega_n t - \phi_0 + 180^0)$$

$$= Re\left[A\omega_n^2 e^{j(\omega_n t - \phi_0 + 180^0)}\right] = Re\left[(j\omega_n)^2 A e^{j(\omega_n t - \phi_0)}\right]$$

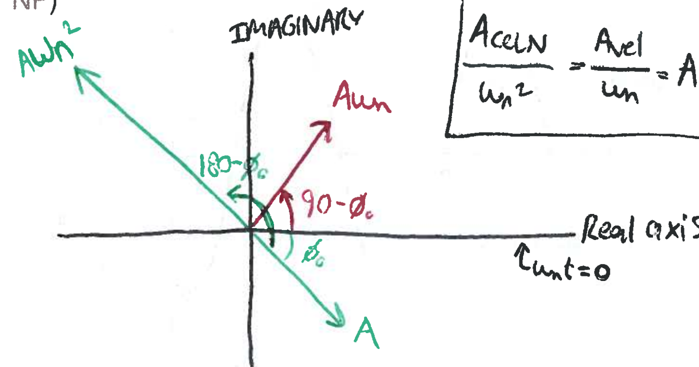
we used $j = e^{j\frac{\pi}{2}}$ and $-1 = j^2 = e^{j\pi}$; Note $\frac{\pi}{2} rad = 90^0, \pi rad = 180^0$.

entiating w.r.t. time scales amplitude by ω_n

ifts phase by 90° in the positive angular velocity direction

Rotating Vector Representation — # 5

We can represent the above in the following graphical form. (p.79 of NP)



$$\boxed{\frac{A_{ccln}}{\omega_n^2} = \frac{A_{vel}}{\omega_n} = A}$$

Fill in the class

Projections along real axis are the only important ones, even for small displacements accelerations that are large due to ω_n^2.

We note that, it is sufficient to know the displacement amplitude to deduce the velocity and acceleration amplitudes for a harmonic motion. **You will find this result useful in the Shaky table experiment.**

$A\omega_n^2$ leads A by 180°, obviously

mmary — # 1

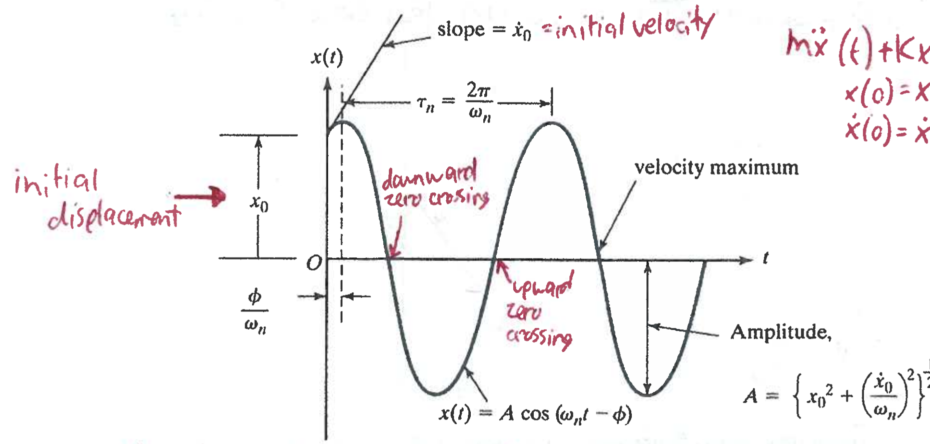
1. Undamped free vibration is specified by the second order, linear, ODE: $m\ddot{x} + kx = 0$ along with the initial conditions: an initial displacement $x(0) = x_0$ and an initial velocity $\dot{x}(0) = \dot{x}_0$.
2. Undamped free vibration response is given by $x = x_h = x_o \cos\omega_n t + \frac{\dot{x}_0}{\omega_n}\sin\omega_n t$, where the *natural frequency*, ω_n, is given by $\omega_n = \sqrt{\frac{k}{m}}$.
3. Undamped free vibration response can also be represented in amplitude-phase form $x(t) = A\cos(\omega_n t - \phi_0)$, which lends itself into a rotating vector representation of harmonic motion.
4. In a harmonic motion at frequency ω rad/s and phase lag ϕ_0, whose displacement is given by $x(t) = A\cos(\omega t - \phi_0)$, the velocity and acceleration amplitudes are related to the displacement amplitude, A, via $A_{velocity} = \omega A$ and $A_{acceleration} = \omega^2 A$. The phase lags are related via $\phi_{0,velocity} = \phi_0 - 90^0$ and $\phi_{0,acceleration} = \phi_0 - 180^0$.
5. The essential features of the free-vibration of a undamped system are shown in the sketch below.

Summary — # 2

Q: Can you explain the meaning of phase-lag from the figure shown? How will the response sketch change if the phase is a lead? (p.81 of NP)

Fill in the class

$m\ddot{x}(t) + Kx = 0$
$x(0) = X_o$
$\dot{x}(0) = \dot{x}_o$



Phase Lag manifests as initial offset in time Δt; $\Delta t = \frac{\phi}{\omega_n}$

mmary — # 3

Fill in the class

$(\omega_n t - \phi)$ lags $A\cos(\omega_n t)$ by a time $\Delta t = \frac{\phi}{\omega_n}$
m positive value for $A\cos(\omega_n t - \phi)$ occurs
conds later than that for $A\cos(\omega_n t)$

(max positive value, max negative value, upward zero crossing, downward zero crossing)
$os(\omega_n t - \phi)$ lags in time by $\frac{\phi}{\omega_n}$ w.r.t. $A\cos(\omega_n t)$

Summary — # 4

Fill in the class

2.3. Undamped SDOF Response – 2

MECH 463: Mechanical Vibrations

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UBC mechanical engineering

Suggested Readings:

1. Topic 2.3 from notes package.
2. Sections 1.10, 2.2 and 2.3 from the course textbook.

Learning Objectives

1. Determine forced vibration response of a SDOF system.
2. Apply the rotating vector technique to identify three regimes of steady forced vibration response.
3. Deduce design guidelines to mitigate forced vibration response.

2 Forced Vibration Response (T 3.3+Notes) — # 1

The particular solution, x_p, when the spring-mass system is subjected to a harmonic force $f(t) = F_0 \cos\omega t$ is governed by

$$\boxed{m\ddot{x}_p + kx_p = F_0 \cos\omega t} \quad \text{CAN EQUALLY USE } F_0 \sin(\omega t) \qquad (1)$$

Assuming

$$\boxed{x_p(t) = X\cos\omega t} \quad \text{can use } X\cos(\omega t - \phi) \qquad (2)$$

in Eq.(1), we find (p.82 of NP) Because the system is linear, it responds only at forcing frequency ll in the class

(2) in (1)

$\omega^2 \cos\omega t) + KX\cos(\omega t) = F_0\cos(\omega t)$

$[-m\omega^2 + K]X\cos(\omega t) = F_0\cos(\omega t)$

$X = \dfrac{F_0\cos(\omega t)}{[K - m\omega^2]\cos(\omega t)} = \dfrac{F_0}{K - m\omega^2}$

$$\therefore \boxed{X_p(t) = \frac{F_0}{K - m\omega^2}\cos(\omega t)}$$

2.12 Forced Vibration Response (T 3.3+Notes) — # 2

$$x_p = \frac{F_0}{k - m\omega^2}\cos\omega t, \quad X = \frac{F_0}{k - m\omega^2} \qquad (3)$$

Therefore, the total response is given by

$$x(t) = x_h + x_p = \overbrace{A_1 \cos\omega_n t + A_2 \sin\omega_n t}^{\text{at } \omega_n} + \overbrace{\frac{F_0}{k - m\omega^2}\cos\omega t}^{\text{at } \omega}, \qquad (4)$$

1. The homogeneous solution is harmonic at *natural* frequency ω_n. The forced vibration or particular solution is harmonic at the *forcing* frequency ω.
2. **The two unknown constants, A_1 and A_2, are to be determined from the initial conditions applied to the total response.**

Substituting the initial conditions $x(0) = x_0$ and $\dot{x}(0) = \dot{x}_0$ in Eq.(4) gives the total response

$x(0) = X_0 \Rightarrow A_1\cos(0) + A_2\sin(0) + \dfrac{F_0}{K - m\omega^2}\cos(0) = X_0$

$A_1 = X_0 - \dfrac{F_0}{K - m\omega^2}$ $\therefore A_1$ is dependant on force

2.12 Forced Vibration Response (T 3.3+Notes) — # 3

(p.84 of NP)

Fill in the class

$= \dot{x}_0 \Rightarrow A_2 \omega_n \cos(0) = V_0 = \dot{x}_0$

$\therefore A_2 = \frac{V_0}{\omega_n}$; A_2 is NOT dependent on force

$A_1 \cos(\omega_n t) + A_2 \sin(\omega_n t) + \frac{F_0}{K - m\omega^2} \cos(\omega t)$

$$\boxed{= \left[x_0 - \frac{F_0}{m\omega^2}\right] \cos(\omega_n t) + \frac{\dot{x}_0}{\omega_n} \sin(\omega_n t) + \frac{F_0}{K - m\omega^2} \cos(\omega t)}$$

2.12 Forced Vibration Response (T 3.3+Notes) — # 4

$$x(t) = \left(x_0 - \frac{F_0}{k - m\omega^2}\right) \cos\omega_n t + \frac{\dot{x}_0}{\omega_n} \sin\omega_n t + \underbrace{\frac{F_0}{k - m\omega^2} \cos\omega t}_{\text{steady-state res}}, \quad (5$$

It is useful to represent $\frac{F_0}{k - m\omega^2}$ in terms of a non-dimensional parameter, called *Dynamic Magnification Factor* (DMF), which is defined as the ratio of the displacement amplitudes in the dynamic and static case as follows

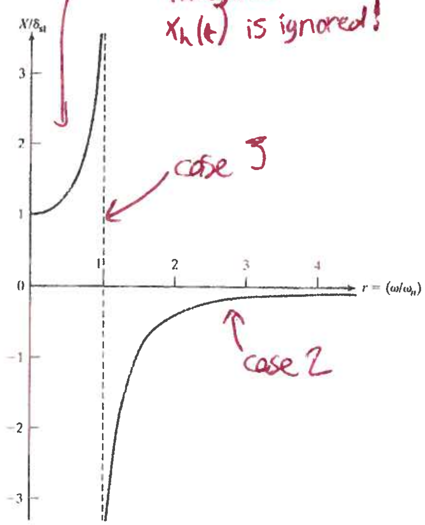
$$\frac{X}{\delta_{st}} = \frac{X}{\frac{F_0}{k}} = \frac{\frac{F_0}{k - m\omega^2}}{\frac{F_0}{k}} = \frac{k}{k - m\omega^2} = \frac{1}{1 - \frac{m\omega^2}{k}} = \frac{1}{1 - \left(\frac{\omega}{\omega_n}\right)^2} \quad (6$$

The DMF is the factor by which the static displacement needs to be multiplied with in order to obtain the dynamic displacement in the steady state, ignoring the homogeneous solution. A plot of the DMF as a function of the non-dimensional frequency ratio $r = \frac{\omega}{\omega_n}$ is shown below:

2.12 Forced Vibration Response (T 3.3+Notes) — # 5

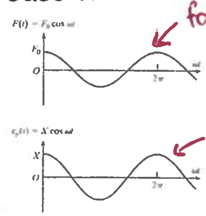
case 1

• Homogeneous Solution $x_h(t)$ is ignored!



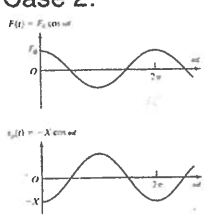
(p.85 of NP)

Case 1:



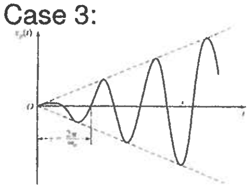
in phase!
$\omega < \omega_n$
$r < 1$

Case 2:



out of phase!
$\omega > \omega_n$
$r > 1$

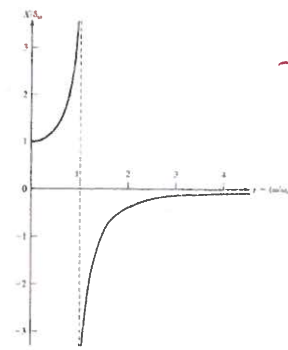
Case 3:



2.12 Forced Vibration Response (T 3.3+Notes) — # 6

`Q: List the important features of the DMF curve?` (p.86 of NP)

Fill in the class



Case 0 : $r = 0$ since $\omega_n \neq 0$

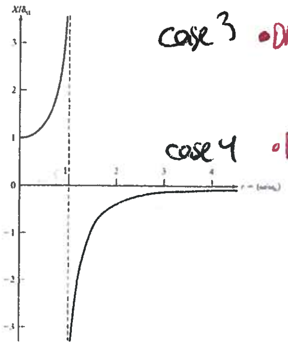
$\frac{X}{\delta_{st}} = \frac{1}{1+2} = 1$

$X = \delta_{st} = \frac{F_0}{K}$ for Statics

$\therefore \boxed{DMF = 1}$

Case 1 : $r < 1$ $\therefore \frac{X}{\delta_{st}} > 1$

2 Forced Vibration Response (T 3.3+Notes) — # 7



case 3 • DMF is unbounded at $r=1$ (case 3)

case 4 • DMF decreases with r for $r>1$

$$\frac{X}{\delta st} = \frac{1}{1-r^2} < 0$$

your own: think of solving Ex #11

Fill in the class

2.12 Forced Vibration Response (T 3.3+Notes) — # 8

Q: Can you explain why X increases with frequency in Case 1, while X decreases with forcing frequency in Case 2? (p.87 of NP)

Fill in the class

• In case 1, $\omega < \omega_n$ $\therefore r = \frac{\omega}{\omega_n} < 1$ hence $X = [DMF]\delta st$

$DMF = \frac{1}{1-r^2} > 1$ and increases until $r = 1$

$\therefore$ if $r\uparrow$, $DMF\uparrow$, $\boxed{X\uparrow}$

• In case 2, $\omega > \omega_n$ $\therefore r = \frac{\omega}{\omega_n} > 1$

$DMF = \frac{1}{1-r^2} < 1$ $\therefore$ if $r\uparrow$, $DMF\downarrow$ $\therefore \boxed{X\downarrow}$

2 Forced Vibration Response (T 3.3+Notes) — # 9

Case 3: $\omega = \omega_n$ [UNDAMPED RESONANT EXCITATION]

The total response in Eq.(5) can be expressed as follows

this term goes to infinity

$\left(\frac{\omega}{\omega_n}\right)^2\Big] = 0$

$$x(t) = x_0 \cos\omega_n t + \frac{\dot{x}_0}{\omega_n}\sin\omega_n t + \delta_{st}\left[\frac{\cos\omega t - \cos\omega_n t}{1-\left(\frac{\omega}{\omega_n}\right)^2}\right] \qquad (7)$$

In the limit $\omega \to \omega_n$ the factor in [] reduces to an indeterminate form $\frac{0}{0}$. Recall from your Calculus that in such cases we use L'Hospital rule. That is, we differentiate the numerator and denominator with respect to ω until such point where the limit is determinate. Let us do this (p.88 of NP)

$$\lim_{\omega\to\omega_n} x_p(t) = ? \Rightarrow \lim_{\omega\to\omega_n} \delta st\left[\frac{\cos\omega t - \cos\omega_n t}{1-\left(\frac{\omega}{\omega_n}\right)^2}\right]$$

Fill in the class

$$\therefore \lim_{\omega\to\omega_n} \frac{f(\omega)}{g(\omega)} \Rightarrow \lim_{\omega\to\omega_n} \frac{df/d\omega}{dg/d\omega}$$

2.12 Forced Vibration Response (T 3.3+Notes) — # 10

$$\therefore \lim_{\omega\to\omega_n} x_p(t) = \lim_{\omega\to\omega_n} \delta_{ST}\left[\frac{d/d\omega\,[\cos\omega t - \cos\omega_n t]}{d/d\omega\,[1-\omega/\omega_n]^2}\right]$$

Fill in the class

$$= \lim_{\omega\to\omega_n} \delta_{ST}\left[\frac{-t\sin(\omega t) - 0}{0 - 2\left(\frac{\omega}{\omega_n}\right)\left(\frac{1}{\omega_n}\right)}\right] = \delta_{ST}\left[\frac{t\sin(\ldots}{-2\frac{\omega}{\omega_n\ldots}}\right.$$

put in $\omega = \omega_n$ $\rightarrow \delta_{ST}\left[\frac{\omega_n t}{2}\sin(\omega_n t)\right]$

So, $x_p(t) = \delta_{st}\left[\frac{\omega_n t}{2}\sin(\omega_n t)\right]$, increases linearly with time

undamped resonant excit

2.12 Forced Vibration Response (T 3.3+Notes) — # 11

Thus the total response when the forcing frequency approaches the natural frequency is given by

$$x(t) = x_0 \cos\omega_n t + \frac{\dot{x}_0}{\omega_n}\sin\omega_n t + \delta_{st}\frac{\omega_n t}{2}\sin\omega_n t \qquad (8)$$

The above result says that the **response grows linearly with time at resonance. In other words, the system becomes unstable!**

`Q: Can you tell the phase relationship between the force and displacement associated with the particular solution at resonance?` (p.89 of NP)

this means lag

~~Fill in~~ the class

$f(t) = F_0 \cos(\omega t)$

$x_p(t) = \delta_{st}\frac{\omega_n t}{2}\sin(\omega_n t) = \frac{\delta_{st}\omega_n t}{2}\cos(\omega_n t - 90^\circ)$

∴ Displacement lags force by 90°

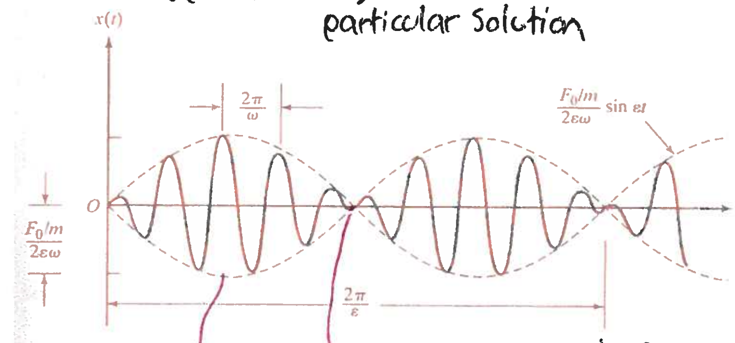
2.12 Forced Vibration Response (T 3.3+Notes) — # 12

Another important phenomenon is observed as the forcing frequency is brought close to resonance leading to *beats*. The response for zero initial velocity and displacement is given by

$$x(t) = \frac{F_0/m}{\omega_n^2 - \omega^2}\left[2\sin\frac{\omega+\omega_n}{2}t\sin\frac{\omega-\omega_n}{2}t\right] = \frac{F_0/m}{2\epsilon\omega}\sin\epsilon t\sin\omega t,$$

$$\epsilon = \frac{\omega_n - \omega}{2}. \qquad (9)$$

this arises due to the interaction between homogeneous and particular solution



Can ... in a... damp...

free & force responses add

free & forced responses cancel

out-of-phase arrang... shaky

Summary of Forced Response

1. The total response of an undamped system subjected to a harmonic force $f(t) = F_0\cos\omega t$ is given by $x(t) = \left(x_0 - \frac{F_0}{k-m\omega^2}\right)\cos\omega_n t + \frac{\dot{x}_0}{\omega_n}\sin\omega_n t + \boxed{\frac{F_0}{k-m\omega^2}\cos\omega t}$.
   ↑ initial condition
2. Free vibration takes place at the natural frequency ω_n while the forced vibration is at ω.
3. With increasing forcing frequency from zero, the response increases reaching an instability at resonance $\omega = \omega_n$ and then decreases for forcing frequencies above resonances. The forced vibration response grows linearly with time at resonance $\omega = \omega_n$.
4. A The response is in-phase with the force for $\omega < \omega_n$; a phase lag of 90^0 at resonance $\omega = \omega_n$; and the response lags behind the force by 180^0 above resonance. The displacement is in exactly the opposite direction to the force. **This is the first counter-intuitive feature we observe in vibration!**
5. The amplitude of the forced vibration can be evaluated from $DMF = \frac{X}{\delta_{st}} = \frac{1}{1-\left(\frac{\omega}{\omega_n}\right)^2}$
6. Beating arises due to the interaction between the free and forced vibration.

Example 10 — # 1

Example 10: A portable shredder used to shred bark, tree branches, and shrub clippings, has a mass of 200 kg resting on tires and support system with an elastic constant of 460 N/mm. The amplitude of the vertical sinusoidal force shown below is 3 kN. Find the maximum vertical displacement, if the shredder operates at 1200 rpm. (p.92 of NP)

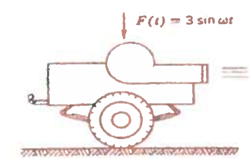


Figure : Figure for example 10.

Fill in the class

ample 10 — # 2

Fill in the class

Example 10 — # 3

Fill in the class

ample 11 — # 1

Example 11: Deduce the expression for forced vibration amplitude X, by using the rotating vector representation. Which forces are dominant below, at, and above the resonant frequency in the vector diagram of forces? (p.94 of NP)

Fill in the class

$m\ddot{x} + Kx = f$

$x = x_p$ only

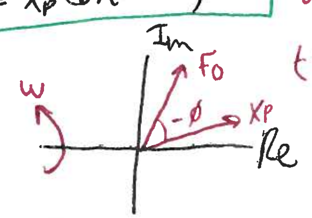
$\rightarrow x_h$ ignored

$\underbrace{-m\ddot{x} - Kx = 0}$

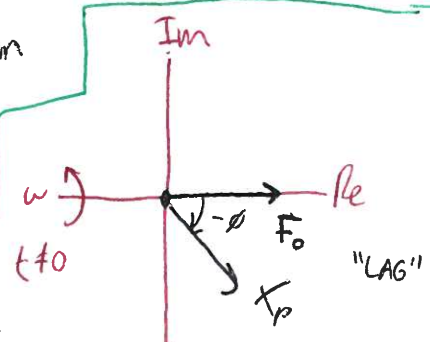
ium of forces = 0 = equilibrium

$f = F_0 \cos\omega t$

$x = X_p \cos(\omega t - \phi)$



$t \neq 0$

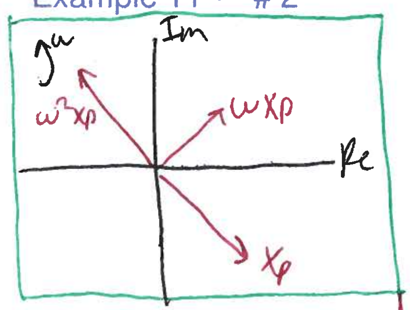


"LAG"

EQUIVALENT

use either

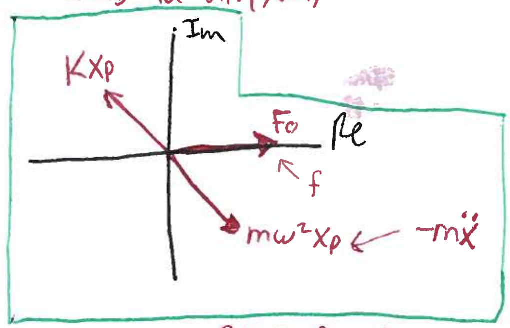
Example 11 — # 2



* Represents $f - m\ddot{x} - Kx = 0$ as Real Vector Diagrams (RVD)

Fill in the class

R.V.D for disp, vel, and acceleration



$\leftarrow -m\ddot{x}$

FORCES

R.V.D for $f, -m\ddot{x}, -Kx$

for equ'brium

$\Sigma F_{RE} = 0$

$\Sigma F_{IM} = 0$

$\therefore \Sigma F = 0 + 0j$

$\boxed{\text{Goal: find } X_p, \phi}$

Example 11 — # 3

Fill in the class

$= 0 \Rightarrow F_0 + \omega^2 m X_p \cos(\phi) - K X_p \sin(\phi) = 0$ (1)

$= 0 \Rightarrow K X_p \cos(90-\phi) - m\omega^2 X_p \sin(\phi) = 0$ (2)

$[K - m\omega^2] X_p \sin\phi = 0$

$\therefore K - m\omega^2 = 0$ or $X_p = 0$ or $\sin\phi = 0$

→ But $K - m\omega^2 \neq 0$ since ω is arbitrary

→ $X_p \neq 0$ otherwise no vibration

→ $\therefore \sin\phi = 0 \rightarrow \boxed{\phi = 0^\circ \text{ or } 180^\circ}$

$\sin(\phi) = 0$, then $\cos(\phi) = 1$ (3)

(3) in (1): $F_0 + m\omega^2 X_p - K X_p = 0 \rightarrow \boxed{X_p = \frac{F_0}{K - m\omega^2}}$

same derivations from R.V.D. and theory

Summary of Topic 2.3 — # 2

5. The total response of an undamped system subjected to a harmonic force $f(t) = F_0 \cos\omega t$ is given by $x(t) = \left(x_0 - \frac{F_0}{k - m\omega^2}\right)\cos\omega_n t + \frac{\dot{x}_0}{\omega_n}\sin\omega_n t + \frac{F_0}{k - m\omega^2}\cos\omega t$. Free vibration takes place at the natural frequency ω_n while the forced vibration is at ω.
6. With increasing forcing frequency from zero, the response increases reaching an instability at resonance $\omega = \omega_n$ and then decreases for forcing frequencies above resonances. The forced vibration response grows linearly with time at resonance $\omega = \omega_n$.
7. The response is in-phase with the force for $\omega < \omega_n$; a phase lag of 90^0 at resonance $\omega = \omega_n$; and the response lags behind the force by 180^0 above resonance. The displacement is in exactly the opposite direction to the force. **This is the first counter-intuitive feature we observe in vibration!**
8. The amplitude of the forced vibration can be evaluated from $DMF = \frac{X}{\delta_{st}} = \frac{1}{1 - \left(\frac{\omega}{\omega_n}\right)^2}$
9. Beating arises due to the interaction between the free and forced vibration. ↘ damping measurement in shaky table lab

Summary of Topic 2.3 — # 1

1. Undamped free vibration is specified by the second order, linea ODE: $m\ddot{x} + kx = 0$ along with the initial conditions: an initial dis placement $x(0) = x_0$ and an initial velocity $\dot{x}(0) = \dot{x}_0$.
2. The undamped free vibration response is given by $x = x_h = x_0 \cos\omega_n t + \frac{\dot{x}_0}{\omega_n}\sin\omega_n t$, where the *natural frequency*, ω_n, is give by $\omega_n = \sqrt{\frac{k}{m}}$.
3. Undamped free vibration response can also be represented i term of the amplitude-form $x(t) = A\cos(\omega_n t - \phi_0)$, which lend itself into a rotating vector representation of harmonic motion.
4. In a harmonic motion at frequency ω rad/s and phase lag ϕ_0 whose displacement is given by $x(t) = A\cos(\omega t - \phi_0)$, the ve locity and acceleration amplitudes are related to the displace ment amplitude, A, via $A_{velocity} = \omega A$ and $A_{acceleration} = \omega^2 A$. Th phase lags are related via $\phi_{0,velocity} = \phi_0 - 90^0$ and $\phi_{0,acceleration} = \phi_0 - 180^0$.

*often we detune the system by slightly altering 'm' or 'K'
↓
shift ω_n away from ω

Case 3: $\omega = \omega_n$

Fo is unbalanced

∴ instabili

*can ad (viscous),

Summary of Topic 2.3 — # 3

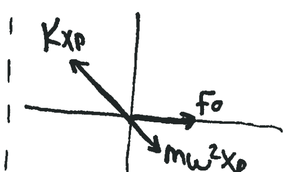
10. Elastic forces dominate below resonance while inertial forces dom inate above the resonance. Thus, low frequency forced vibration can be reduced by stiffening the system, while reducing high fre quency forced vibration requires considerable addition of mass **Adding stiffness has little influence on the DMF well abov resonance!**

Case 0: $\omega = 0$

$\therefore m\omega^2 X_p = 0$

$\boxed{X_p = \frac{F_0}{K}}$

Case 1: $\omega < \omega_n$



$K X_p >> m\omega^2 X_p$

$\therefore \omega < \omega_n$

$\therefore \omega^2 < \omega_n^2$

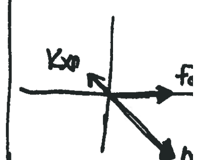
$\therefore \omega^2 < \frac{K}{m}$

$\therefore m\omega^2 < K$

• stiffness force dominates inertial force. Small changes in 'K' will have significant impact on

$X_p \rightarrow$ [illegible]

Case 2: $\omega >$



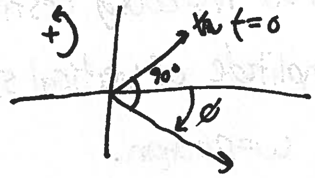
$\omega >> \omega_n$

$\therefore m \uparrow$ means

MECH 463 - LECTURE

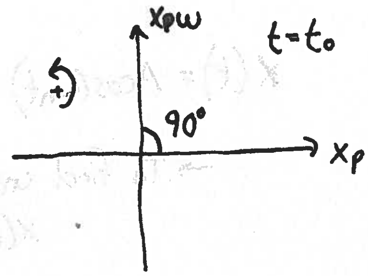
$x_p(t) = X_p\cos(\omega t - \phi)$

X_p lags behind $X_p\cos(\omega t)$ by ϕ



$\dot{x}_p = -X_p\omega\sin(\omega t - \phi)$

$= X_p\omega\cos(\omega t - \phi + 90°) \rightarrow$ TRIG.



** ω = fixed for all rotating vectors

Design Guidelines Summary (Cases 0-3)

Problem: forced steady vibration at 'ω' in a SDOF or equivalent SDOF system.

(1) if $\omega \ll \omega_n$, increase K

(2) if $\omega \gg \omega_n$, increase m

(3) if $\omega \approx \omega_n$, add c (damping) or detune (either <u>m</u> or <u>K</u>)

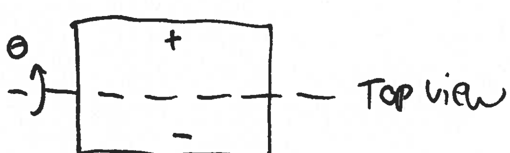
Detuning

· Adding mass is effective at the points where:

- shaky table: in-phase

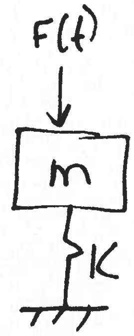
*mass can be added anywhere on the top surface to detune

- out-of-phase



*where to add mass is based on what the system is doing under vibrations

Ex: 10 A portable shredder has $m=200kg$ resting on tires and support system. $K=460N/mm$. Amplitude of vertical sinusoidal force is 3KN. Find max vert. disp. if $\omega=1200rpm$.



$\omega=1200rpm=40\pi\ rad/s$

$$x(t)=A\cos(\omega_n t)+B\sin(\omega_n t)+\frac{F_0}{K-m\omega^2}\sin(\omega t)$$

→ To find unknown constants in $X_h(t)$, assume initial re:

- $x(0)=0$, $\dot{x}(0)=0$

then, solve X_{MAX} after solving A, B

$$A=0 \therefore x(t)=\left[B+\frac{F_0}{K-m\omega^2}\right]\left[\sin(\omega t)\right]$$

2.4. Damped SDOF Response–1

MECH 463: Mechanical Vibrations

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Suggested Readings:

1. Topic 2.4 from notes package for detailed derivations.
2. Sections 2.6 and 3.4 from the course textbook.

Learning Objectives

1. Determine free vibration response of a viscously damped SDOF system.
2. Recognize the trade-offs in transient response design problems.
3. Deduce design guidelines to mitigate vibration response.

Equations of Motion (NP 2.13)

Thus for the spring-mass system with a viscous damper, we can obtain the following equations of motion:

Fill in the class

$$m\ddot{x} + c\dot{x} + kx = f \tag{1}$$

2.14 Damped Vibration Response (NP 2.14) — # 1

$$m\ddot{x}_h + c\dot{x}_h + kx_h = 0 \quad \text{Homogeneous response/Free vibration} \tag{2a}$$
$$m\ddot{x}_p + c\dot{x}_p + kx_p = f \quad \text{Particular solution/Forced vibration.} \tag{2b}$$

Adding the above two equations we have the TOTAL response, from the principle of superposition

$$m\ddot{x} + c\dot{x} + kx = f, \quad x = x_h + x_p \quad \text{TOTAL response} \tag{3}$$

It is required to specify the initial conditions on the TOTAL response. They can be initial velocity, or initial displacement:

$$x(0) = x_0; \quad \dot{x}(0) = \dot{x}_0 \qquad \text{INITIAL conditions } \textbf{apply on the TOTAL solutio} \tag{4}$$

5 Free Vibration Response (NP 2.15, T 2.6) — # 1

$$m\ddot{x}_h + c\dot{x}_h + kx_h = 0 \quad \text{Homogeneous response/Free vibration} \tag{5}$$

To solve the above differential equation we assume a solution of the form $x_h = Xe^{st}$ where X and s are to be determined. Let us insert this *trial* solution into the equation of motion Eq.(5)

$$mXs^2e^{st} + csXe^{st} + KXe^{st} = 0$$
$$\Rightarrow \left[ms^2 + cs + k\right] e^{st} = 0$$
$$\Rightarrow \left[ms^2 + cs + k\right] = 0, \quad \because e^{st} \neq 0$$

to form the *auxiliary* or *characteristic* equation:

$$ms^2 + cs + k = 0 \tag{6}$$

The two roots of of the above *quadratic* equation are

$$s_{1,2} = \frac{-c \pm \sqrt{c^2 - 4km}}{2m} \tag{7}$$

$$x_h(t) = C_1e^{s_1t} + C_2e^{s_2t} \tag{8}$$

2.15 Free Vibration Response (NP 2.15, T 2.6) — # 2

2.15.1 Underdamping $\zeta < 1$ or $c < c_c$ (See the notes for detailed derivations)

$$x_h(t) = e^{-\zeta\omega_n t}\left[A_1 \cos\omega_d t + A_2 \sin\omega_d t\right] = e^{-\zeta\omega_n t} A\cos(\omega_d t - \phi_0) \tag{9}$$

`Q: What can you say about the influence of different parameters on the underdamped free vibration response based on the above?`

Fill in the class

5 Free Vibration Response (NP 2.15, T 2.6) — # 3

2.15 Free Vibration Response (NP 2.15, T 2.6) — # 4

See notes for derivation.

$$\begin{aligned} x = x_h &= e^{-\zeta\omega_n t} A\cos(\omega_d t - \phi_0) \\ \tan\phi_0 &= \frac{\zeta\omega_n x_0 + \dot{x}_0}{\omega_d x_0};\ A = \sqrt{x_0^2 + \left[\frac{\zeta\omega_n x_0 + \dot{x}_0}{\omega_d}\right]^2} \\ x = x_h &= e^{-\zeta\omega_n t}\left[x_0 \cos\omega_d t + \frac{\zeta\omega_n x_0 + \dot{x}_0}{\omega_d}\sin\omega_d t\right] \\ \omega_d &= \omega_n\sqrt{1-\zeta^2} \text{ (Undamped natural frequency)} \end{aligned} \tag{10}$$

The following observations are worth making about the free vibration:

1. **Free vibration takes place at the system's damped natural frequency, slightly below the undamped natural frequency,** irrespective of the initial conditions.

Free Vibration Response (NP 2.15, T 2.6) — # 5

2. Undamped natural frequency depends only on the properties of the system: mass, stiffness, and damping. It increases with an increase in the stiffness or a *decrease* in the mass or damping.

2.15 Free Vibration Response (NP 2.15, T 2.6) — # 6

2.15.2 Critical Damping $\zeta = 1$ or $c = c_c = 2\sqrt{km}$

(See the notes for detailed derivations)

We can let the damping approach the value of 1 in Eq.(10). In this case, $\omega_d = \omega_n\sqrt{1-\zeta^2} \to 0$ and $\cos\omega_d t \to 1$, $\sin\omega_d t \to \omega_d t$. Using these in Eq.(10) gives the free vibration response of a critically damped system as follows.

$$x = x_h = e^{-\omega_n t}\left[x_0 + (\omega_n x_0 + \dot{x}_0)\, t\right] \qquad (11)$$

Free Vibration Response (NP 2.15, T 2.6) — # 7

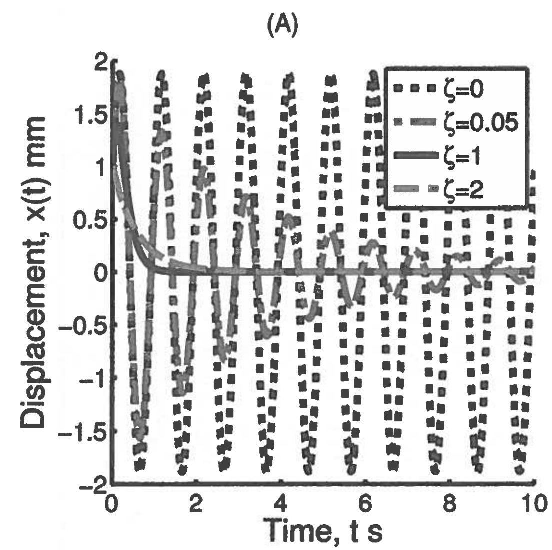
2.15.3 Overdamping $\zeta > 1$ or $c > c_c$

(See the notes for detailed derivations)

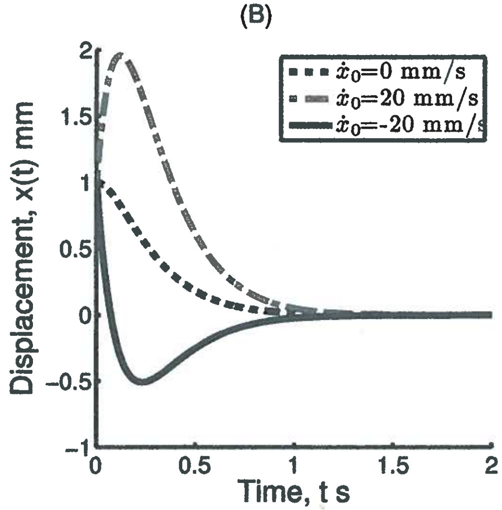
$$x = x_h = e^{-\zeta\omega_n t}\left[\frac{\omega_d x_0 + \zeta\omega_n x_0 + \dot{x}_0}{2\omega_d} e^{\omega_d t} + \frac{\omega_d x_0 - \zeta\omega_n x_0 - \dot{x}_0}{2\omega_d} e^{-\omega_d t}\right]. \qquad (12)$$

2.15 Free Vibration Response (NP 2.15, T 2.6) — # 8

2.15.4 A Comparison of the Three Cases



Free Vibration Response (NP 2.15, T 2.6) — # 9



2.15 Free Vibration Response (NP 2.15, T 2.6) — # 10

Q: *What features do you observe in the responses of the damped systems sketched above?*

Fill in the class

(1) all damped vibration (free) decays to zero with time

(2) for underdamped case, $T_d = \frac{2\pi}{\omega_d} = \frac{2\pi}{\omega_n\sqrt{1-\zeta^2}}$

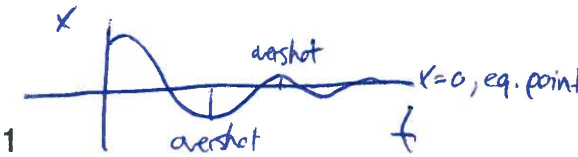
(3) viscous damping leads to exponential decay

(not true for other damping: coulomb gives linear decay)

(4) Maximum displacement: underdamped > critically damped > overdamped

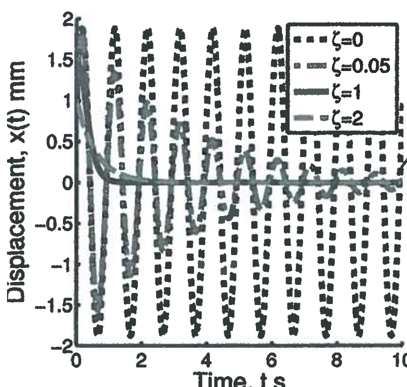
(5) over and critically damped responses are Aperiodic

(6) overshoot is unavoidable in under damping



Free Vibration Response (NP 2.15, T 2.6) — # 11

2.15.5 Logarithmic Decrement



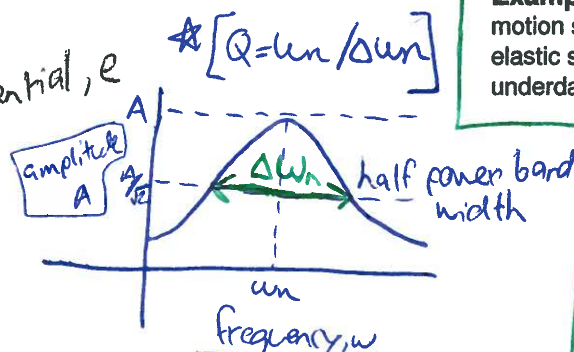
exponential, e

logarithmic decrement

$$\delta = \frac{1}{N}\ln\left(\frac{x_1}{x_{1+N}}\right) = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}} \approx 2\pi\zeta = \frac{2\pi c}{2m\omega_n}. \tag{13}$$

$Q = \frac{1}{2\zeta}$

$\bigstar\ \boxed{Q = \omega_n / \Delta\omega_n}$



Example 15 — # 1

p. 117 in NP

Example 15 : From the consideration of work performed in harmonic motion show that viscous damper dissipates energy over one cycle while an elastic spring does not. Sketch the rotating vector representation of the underdamped free vibrations.

Fill in the class

we Know: force vector → f

displacement vector → x y

$W = f \cdot x \neq 0$ $\quad W = f \cdot y = 0$

- In harmonic vibrations, our intuitive idea does not q

→ choice #1 : $f(t) = F\sin(\omega t)$, $x(t) = X\sin(\omega t - \phi)$ } equivalent

→ choice #2 : $f(t) = F\sin(\omega t + \phi)$, $x(t) = X\sin(\omega t)$

- So, work done by $f(t)$ on $x(t)$: $W(t) = f(t)dx = f($

↳ instantaneous work done

mple 15 — # 2

..work over one period : $\Delta W = \frac{1}{T}\int_0^T W(t)\,dt = \frac{1}{T}\int_0^T f \frac{dx}{dt}\,dt$

$f(t) = F\sin(\omega t+\phi)$; $x(t) = X\sin\omega t$; $T = 2\pi/\omega$

$\frac{1}{T}\int_0^T F\sin(\omega t+\phi)\, X\omega\cos(\omega t)\,dt$ → evaluate by parts

$\boxed{\pi F X\omega\sin\phi}$ phase lag of displacement, ϕ, w.r.t. force dictates average work under one cycle

Case 1: springs f = force on $x(t) = -Kx(t)$

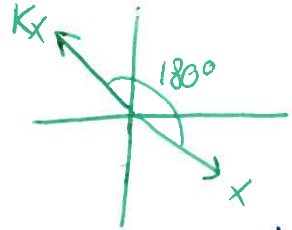
$= -KX\sin(\omega t) = KX\sin(\omega t+180°)$

$\therefore \Delta W = \pi F X\omega\sin\phi = \pi K X\omega\sin(180°)$

$\therefore \boxed{\Delta W = 0}$

Example 15 — # 3

An elastic spring can't remove power from harmonic motion over one



Case 2: viscous damper

f = force exerted by damper on $x(t) = -C\dot{x} = -CX\omega\cos($

$= -CX\omega\sin(\omega t+90°)$

$\therefore \Delta W = \pi F X\sin\phi = \pi(-CX\omega)X\omega\sin(90°) = \boxed{-\pi C X^2\omega}$

→ Negative sign tells us the vibration is losing power to external

mple 15 — # 4

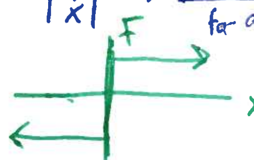
only requirement for a dissipative force is to remove power

the harmonic motion → ϕ can be any non-zero quantity

than π, 2π.

s damping with $\phi = 90°$ is the only one among several forces.

an have $\phi = 30°$, say → $\therefore f(t) = -C\frac{\dot{x}}{|\dot{x}|}$ → $\boxed{\text{coulomb damping}}$ for dry friction



nstable friction-induced oscillations squeal (ex: railway vibrations)

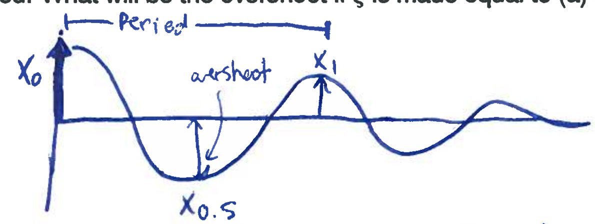
on-viscous damping, we find equivalent viscous damping coefficient

g vector of underdamped free vibration: $x(t) = e^{-\zeta\omega_n t}\cdot\cos(\omega_d t-\phi)$

Example 16 — # 1

p. 119 in NP

Example 16 : A shock absorber is to be designed to limit its overshoot to 15% of its initial displacement when released. Find the damping ratio ζ_0 required. What will be the overshoot if ζ is made equal to (a) $\frac{3}{4}\zeta_0$, and (b) $\frac{5}{4}\zeta_0$.



Fill in the class

Maximum overshoot for viscous damping is $x_{0.5}$

overshoot % = 15% = 0.15 = $\frac{x_{0.5}}{x_0}$

→ Find ζ : $\delta = \frac{1}{N}\ln\left(\frac{x_0}{x_n}\right)$

→ for $N = 0.5$, $\frac{x_0}{x_{0.5}} = \frac{1}{0.15}$

$\therefore \delta = \frac{1}{0.5}\ln(1/0.15) = 3.7942$

$\zeta = 2\pi\zeta /$

mple 16 — # 2

$\delta \cancel{/2\pi}) / \sqrt{1+(\delta/2\pi)^2} = 0.5169 \rightarrow$ for 15% overshoot

$\zeta = \frac{3}{4}(0.5169) = 0.3877 \Rightarrow \delta \Rightarrow 26.6\%$ overshoot

, $\zeta = 1.25(0.5169) \Rightarrow \frac{x_{0.5}}{x_0} = 7\%$ overshoot

Settling time is another consideration

then overshoot ↓ and settling time ↓

*rise time is another consideration

↳ how agile the system is in responding to force

↑ then rise time ↑ not good

Example 16 — # 3

instantly compresses

mple 16 — # 4

Example 17 — # 1

★Do in tutorial

p. 121 in NP

Example 17 : A railroad car of mass 2000 kg travelling at a velocity of $v = 10$ m/s is stopped at the end of the tracks by a spring-damper system, as shown below. If the stiffness of the spring is $\frac{k}{2} = 40$ N/mm and the damping constant is $c = 20$ N-s/mm, determine (a) the maximum displacement of the car after engaging the springs and damper and (b) the time taken to reach the maximum displacement.

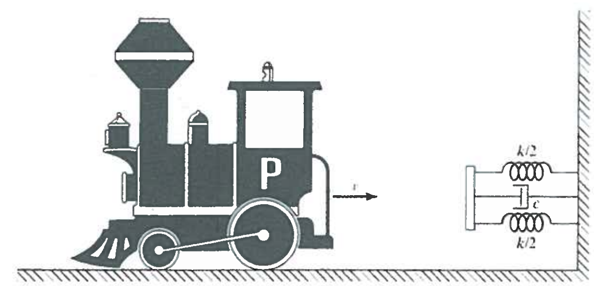


Figure : Figure for example 17.

mary of Free Damped Response

1. There are three categories of damped systems: underdamped, overdamped, and critically damped. Among these three, most of the mechanical systems belong to the underdamped category.
2. Critical damping is engineered when a quick return to the initial configuration in the shortest possible time is desired, such as in recoil mechanisms and shock absorbers.
3. With the passage of time over and critically damped systems gradually turn into underdamped systems.
4. Overdamping does not allow any oscillation but the return to equilibrium is a slow and creeping process.
5. Underdamped response is given by: $x = x_h = e^{-\zeta\omega_n t} A\cos(\omega_d t - \phi_0)$

   $\tan\phi_0 = \frac{\zeta\omega_n x_0 + \dot{x}_0}{\omega_d x_0}$; $A = \sqrt{x_0^2 + \left[\frac{\zeta\omega_n x_0 + \dot{x}_0}{\omega_d}\right]^2}$

   $x = x_h = e^{-\zeta\omega_n t}\left[x_0 \cos\omega_d t + \frac{\zeta\omega_n x_0 + \dot{x}_0}{\omega_d}\sin\omega_d t\right]$

   $\omega_d = \omega_n\sqrt{1-\zeta^2}$ (Undamped natural frequency)
6. Logarithmic decrement is a time-domain measure of damping. It is given by: $\delta = \frac{1}{N}\ln\left(\frac{x_1}{x_{1+N}}\right) = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}} \approx 2\pi\zeta = \frac{2\pi c}{2m\omega_n}$.

Design Guidelines

(1) $\omega_n \downarrow$ for same ~~ζ~~ ζ means settling time $\uparrow$

[low frequency vibrations are hard to suppress]

(2) $\omega_n \uparrow$ for same ζ means settling time $\downarrow$

[high frequency vibrations decay faster]

(3) For a multi-degree of freedom (multimode) system, the vibration persists after initial time is essentially in the fundamental mode

(4) There is a trade-off between overshoot and settling time & ri: time

2.4. Damped SDOF Response–2

MECH 463: Mechanical Vibrations

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Suggested Readings:

1. Topic 2.4 from notes package for detailed derivations.
2. Sections 2.6 and 3.4 from the course textbook.

Learning Objectives

1. Determine forced vibration response of a viscously damped SDOF system.
2. Apply the rotating vector technique to identify three regimes of steady forced vibration response.
3. Deduce design guidelines to mitigate vibration response.

2.16 Harmonically Forced Damped SDOF Response (NP 2.16, T 3.4+Notes)

We restrict this discussion to harmonic forcing of the form $f(t) = f_0 \cos \omega t$. Our interest lies in the particular solution of the second oder ODE:

$$m\ddot{x}_p + c\dot{x}_p + kx_p = f = F_0 \cos \omega t. \tag{1}$$

We can determine the amplitude X and phase lag ϕ of the particular solution, taken to be

$$x_p(t) = X \cos(\omega t - \phi) \tag{2}$$

using the algebraic or graphical methods. Let us solve for the two unknowns using the rotating vector diagram methods.

Example 17 — # 1

p. 124 in NP
Example 17 : Use the rotating vector method to determine X and ϕ of the steady state or particular solution response of a system governed by Eq.(**??**)

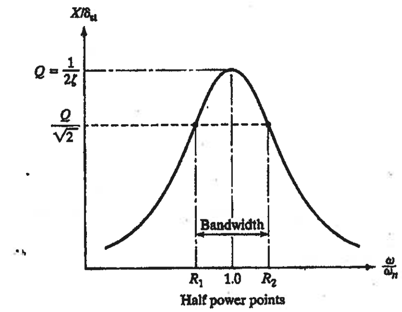
Fill in the class

Forced Damped Response — # 5

Question: How do we measure DMF curve of a practical system such as a machine tool? (p. 129 in NP)

Fill in the class

Forced Damped Response — # 6



Q is defined as follows:

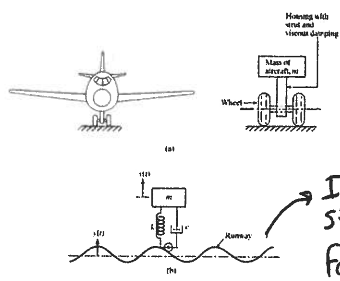
$$Q \approx \frac{1}{2\zeta} \approx \frac{\omega_n}{\Delta\omega}, \quad \Delta\omega = \omega_2 - \omega_1 \text{ (HPBW)}. \qquad (6)$$

Example 18 — # 1

p. 131 in NP

The landing gear of an airplane can be idealised as the spring-mass-damper system shown below. If the runway surface is described by $y(t) = y_0 \cos \omega t$, determine the equations of motion and steady damped vibration response. What design criteria will you use to select the k and c? Open ended problem

k about design criteria need to solve this problem



If this profile is sinusoidal, we use Fourier series to write the profile $y(s) = \sum a_n \cos n\omega_0 s + \sum b_n \sin n\omega_0 s$

Example 18 — # 2

Fill in the class

So, $y(s) = \sum a_n \cos n\omega_0 s + \sum b_n \sin n\omega_0 s$

s = horizontal distance travelled $= vt$

If horizontal velocity 'v' is constant

$y(t) = \sum a_n \cos[n\omega_0 (vt)] + \sum b_n \sin[n\omega_0 (vt)]$

- $y(t)$ is displacement excitation applied at the base. Each a_n/b_n will be a harmonic force at frequency '$n\omega_0$'
- using superposition principle:
  (1) Find response $[X_p]_{n\omega_0}$ for each '$n\omega_0$' forces
  (2) Add all responses due to each harmonic to get total response induced by road with general roughness: $X_p = \sum [X_p]_{n\omega_0}$

Example 18 — # 3

criteria

-tical velocity of the plane as it lands (and horizontal)

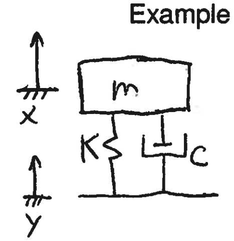
.mass of the plane

-ilure of shock system

-ximum displacement of plane

, of system (plane + shocks)

Example 18 — # 4



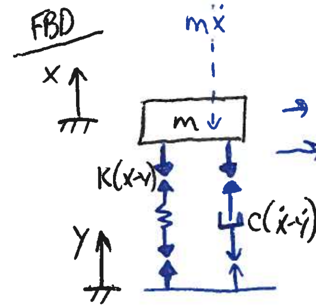
objective : to choose K, C based on design criterio

→ we have a choice for +x or +y for frame

→ we have no choice for positive $\dot{x}, \ddot{x}, \dot{y}, \ddot{y}$ and choose the direction for x and y

- x, y are absolute disp. measured by fixed o w.r.t. static eq.

FBD



→ $x, \dot{x}, \ddot{x}$ in same direction

→ absolute acceleration is $\ddot{x}$, NOT $\ddot{x}-\ddot{y}$

$K(x-y)\downarrow$ with $y=0$ is $Kx\downarrow$ which opposes x

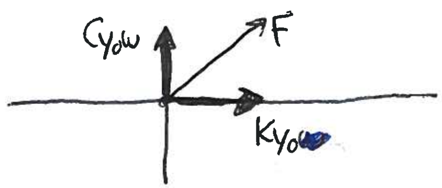
Example 18 — # 5

D:

$=0 \rightarrow -m\ddot{x} - K(x-y) - c(\dot{x}-\dot{y}) = 0$

$m\ddot{x} + c\dot{x} + Kx = Ky + c\dot{y}$

$$\boxed{+Kx + c\dot{x} = KY_0\cos(\omega t) - cY_0\omega\sin(\omega t)}$$



$\tan\theta = \frac{c\omega}{K}$

$F = \sqrt{(KY_0)^2 + (cY_0\omega)^2} = Y_0\sqrt{K^2 + (c\omega)^2}$

Design Guidelines for Forced Response — # 1

→ So, $m\ddot{x} + c\dot{x} + Kx = F\cos(\omega t + \theta)$

$F = Y_0\sqrt{K^2 + (c\omega)^2}$

* Note: $\sin(\omega t) = \cos(\omega t - 90°)$

- We have forced vibration so:

$x_p = X_p\cos(\omega t + \theta - \phi)$

$X_p = \frac{F}{\sqrt{(K - m\omega^2)^2 + (c\omega)^2}}$ $\tan\phi = \frac{c\omega}{K - m\omega^2}$ we know

1, 8, 16, 17, 19, 26, 31, 32

Summary

1. The DMF of a viscously damped system in the steady state is given by:

$$|DMF| = M = \left|\frac{X}{\delta_{st}}\right| = \frac{F_0}{\delta_{st}\sqrt{(k-m\omega^2)^2+(c\omega)^2}} = \frac{1}{\sqrt{(1-r^2)^2+(2\zeta r)^2}}, \quad r = \frac{\omega}{\omega_n}$$

$$\phi = \tan^{-1}\left[\frac{2\zeta r}{1-r^2}\right]$$

2. The maximum magnification occurs slightly below the undamped natural frequency $\omega < \omega_n$, or $r < 1$ at $r = \sqrt{1-2\zeta^2}$ and is given by $M_{max} = \left|\frac{X}{\delta_{st}}\right|_{max} = \frac{1}{2\zeta\sqrt{1-2\zeta^2}}$. At the undamped natural frequency, $\left|\frac{X}{\delta_{st}}\right|_{\omega=\omega_n} = \frac{1}{2\zeta}$
3. The phase lag of the response starts at 0^0 (in-phase) for $r = 0$ and gradually increases to a value of 90^0 at $r = 1$. Eventually, well above resonance $r >> 1$, the phase lag is 180^0 (out-of-phase).
4. Quality factor or Q-factor is a frequency-domain measure of damping, obtained from steady state forced vibration response. It is related to ζ via:$Q \approx \frac{1}{2\zeta} \approx \frac{\omega_n}{\Delta\omega}$, $\Delta\omega = \omega_2 - \omega_1$ (HPBW).

$$\frac{\sqrt{K^2+(c\omega)^2}}{(K-m\omega^2)^2+(c\omega)^2} \cos(\omega t+\theta-\phi) \quad \& \quad \tan\phi = \frac{c\omega}{K-m\omega^2}$$

...eady-state amplitude

$$\frac{\sqrt{K^2+(c\omega)^2}}{(K-m\omega^2)^2+(c\omega)^2}$$ in steady-state

$$= \frac{\text{output disp. amp.}}{\text{input disp. amp.}} = \text{disp. transmissibility} = [TR]_D \equiv \boxed{\frac{\sqrt{K^2+(c\omega)^2}}{\sqrt{(K-m\omega^2)^2+(c\omega)^2}}}$$

We can also show:

$$\boxed{[TR]_D = \frac{\sqrt{1+(2\zeta r)^2}}{\sqrt{(1-r^2)^2+(2\zeta r)^2}}} ; \quad r = \frac{\omega}{\omega_n} ; \quad \zeta = \frac{c}{2m...}$$

*Let's put in some numbers now:

$m = 2000\,kg$

$Y_0 = 0.2\,m$

$\omega = 157.08\,rad/s$

$$\Rightarrow \left|\frac{X}{Y_0}\right| = \frac{\sqrt{K^2+(c\omega)^2}}{\sqrt{(K-m\omega^2)^2+(\omega c)^2}}$$ find ...

$X = 0.1\,m \rightarrow$ limit imposed by problem

$$\therefore \left|\frac{0.1}{0.2}\right| = 0.5 = \cdots \quad \omega^2 = 2.46\times10^4$$

Now, two unknowns:

① chose $K = 5\times10^6\,N/m \longrightarrow c = 158{,}805$

*Need to specify:

① TR_D max

② TR_D at forcing frequency

To reduce forced vibrati... response in steady state (...

(1) what is ω_n? ensure $\omega \neq$ ...

(2) if $\omega = \omega_n$, then change m, K [detuning]

- add m, K at locations with high disp...

(3) If $\omega \neq \omega_n$ then if:
  - $\rightarrow \omega << \omega_n$, change K
  - $\rightarrow \omega >> \omega_n$, change m

- Think about hw problem part ... tomorrow

2.5.Vibration Isolation

MECH 463: Mechanical Vibrations

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Suggested Readings:

1. Topic 2.5 from notes package for detailed derivations.
2. Section 9.10 from the course textbook.

Learning Objectives

1. Know the working principles involved in vibration isolation.
2. Apply SDOF vibration theory to deduce vibration transmissibilities.
3. Design simple SDOF isolators.
4. Appreciate the design trade-offs.

Fill in the class

Design Choices — # 1

1. Appropriate design of restoring (spring) and inertial (mass) elements of a mechanical system. Here, our goal is to avoid resonant frequencies by suitably choosing the spring constants and mass.
2. **We can isolate the source of vibration from the sensitive components we wish to protect.** In the case of a car, vibration from unbalance in engines and from the roughness of the road can be prevented from being transmitted into the passenger cabin. We achieve this using vibration isolators. **We shall study isolation system design in this topic.**
3. Equally applicable is the notion of absorbing vibration energy from a vibrating component. This can be accomplished by channelling away the energy into a secondary device, such as a vibration absorber, effective at selected tuned frequencies. We shall study vibration absorbers later.
4. Somewhat related to the above ideas is the notion of applying an external force (using actuators) to counter vibrations. This method of active vibration control is outside our scope as a detailed knowledge of control theory and actuator and sensor technologies is needed.

ee video on Connect

active control

MECH466...

Design Choices — # 2

Q: Can you list a few situations, other than the car example, where vibration isolation may be useful? p.138 in NP

Fill in the class

1. Installing sensitive optical set-ups
2. Installing sensitive precision machines
3. Installing compressors
4. Installing washing machines
5. Sesmic isolation of building
6. Protecting workers from "white finger" due to vibration exposure in handheld jackhammer

2.17 SDOF Isolation (T9.10+NP) — # 1

think

> The goal of vibration isolation is to reduce the transmitted vibration from a vibrating source to other parts of the system, by **decreasing** ~~the natural frequency of the whole system compared~~ to the disturbing force frequency. This is accomplished by inserting resilient elements (spring mounts, damped spring mounts, pneumatic rubber mounts) between the source of vibration and the object to be protected.

limitations: decreasing natural frequency leads to excessive displacements of the object being studied

2.17 SDOF Isolation (T9.10+NP) — # 2

The effectiveness of an isolation system is quantified using a non-dimensional parameter called Transmissibility. It can be defined as a ratio of transmitted to applied force, or transmitted to applied displacement. Thus we have

$$TR = \frac{F_t}{F} \text{ (force transmissibility)}; \; TR_d = \frac{X}{Y} \text{ (displacement transmissibility)} \quad (1)$$

where F_t, F, X and Y are respectively, transmitted force, applied force, transmitted displacement, and applied displacement.

2.17 SDOF Isolation (T9.10+NP) — # 3

Q: What are the assumptions inherent in the definition of TR or TR_d? Describe situations where one is a better measure than the other. What is the ideal value for a TR?

Fill in the class

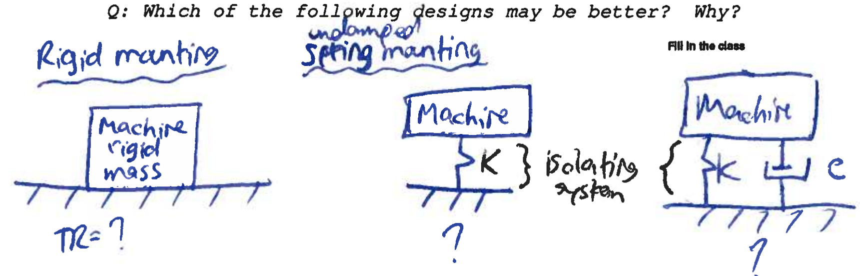
$$[TR]_D = \frac{\sqrt{1+(2\zeta r)^2}}{\sqrt{(1-r^2)^2+(2\zeta r)^2}}$$

1 - No transients
2 - SDOF vibration
3 - TR_D applies at a single 'ω'
4 - Applied force/disp. amplitudes are frequency independent → not true in Shaky Table

2.17 SDOF Isolation (T9.10+NP) — # 4

Q: Which of the following designs may be better? Why?

Fill in the class



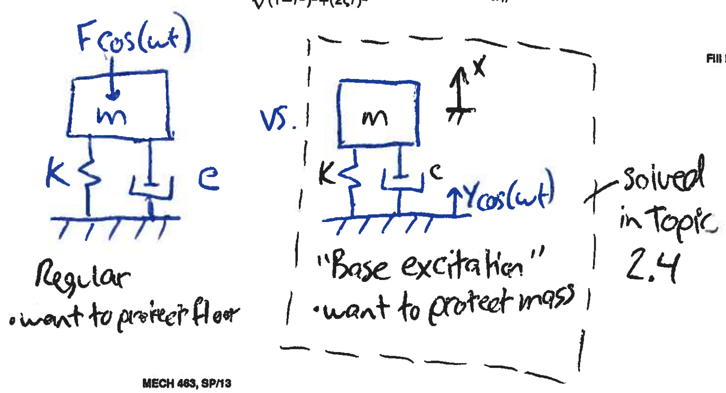
→ Find K, c : isolation system design
→ Machine is source of vibration

Example 20 — # 1

Q: Show that the force and displacement transmissibility of a SDOF system is given by $TR = \frac{F_t}{F} = \frac{\sqrt{k^2+(c\omega)^2}}{\sqrt{(k-m\omega^2)^2+(c\omega)^2}} = \frac{\sqrt{1+(2\zeta r)^2}}{\sqrt{(1-r^2)^2+(2\zeta r)^2}}$ *(force transmissibility) ; and*

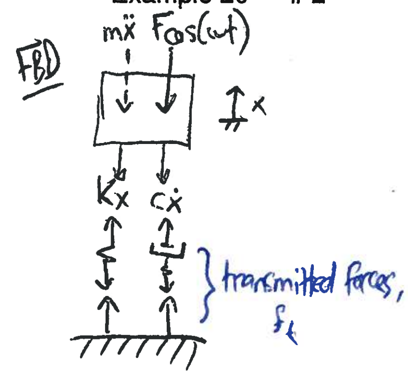
$TR_d = \frac{X}{Y} = \frac{\sqrt{1+(2\zeta r)^2}}{\sqrt{(1-r^2)^2+(2\zeta r)^2}}$, where $r = \frac{\omega}{\omega_n}$. (p.141 in NP)

Fill in the class



Example 20 — # 2

Fill in the class



f_t = transmitted force via K & C to the ground $= c\dot{x} + Kx = F_t \cos($

f = applied force $= F\cos(\omega t)$

$\therefore [TR] = \left|\frac{F_t}{F}\right| = ?$

Example 20 — # 3

Fill in the class

$\boxed{\text{mplitude } F_t \text{ of } \cancel{\text{tr}}\text{transmitted force}}$

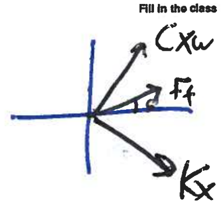
$= c\dot{x} + Kx$

$x = X\cos(\omega t - \phi)$ steady-state forced resp.

$f_t = -cX\omega\sin(\omega t - \phi) + KX\cos(\omega t - \phi)$

$= X\sqrt{K^2 + (c\omega)^2}$

X = steady state forced response



Example 20 — # 4

Fill in the class

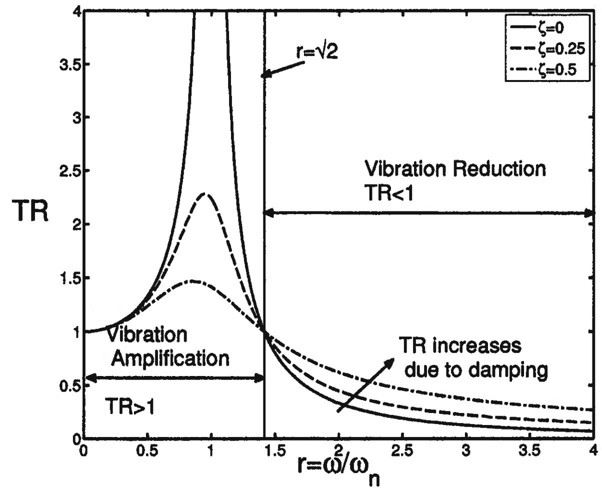
$$F_t = \frac{F}{\sqrt{(K-m\omega^2)^2+(\omega c)^2}}\sqrt{K^2+(\omega c)^2}$$

$$\therefore \left|\frac{F_t}{F}\right| = TR = \frac{\sqrt{K^2+(\omega c)^2}}{\sqrt{(K-m\omega^2)^2+(\omega c)^2}}$$

IS $[TR]_d = \frac{X}{Y}$?

$$[TR]_{ACCLN} = \frac{\omega^2 X}{\omega^2 Y} = \frac{X}{Y} = [TR]_D = [TR]_{velocity}$$

Isolation Design Curve — # 1



Isolation Design Curve — # 2

The following are worth noting

1. For a given operating frequency, say ω, we can choose $r = \frac{\omega}{\omega_n}$ in the above curves by suitably choosing the isolator's spring constant k. If we choose k such that $r > \sqrt{2}$, that is, the natural frequency of the combined system is well below the forcing frequency, then we obtain $TR < 1$ or less force is transmitted. Thus we obtain isolation by making the dynamics of the whole system with the isolator slow compared to the forcing frequency ω.
2. If spring constants are chosen such that $r < \sqrt{2}$, then vibration is amplified and hence more force is transmitted.
3. The above two points suggest that softer springs are better and stiffer springs are undesirable. Thus rigid mounting is the worst case in terms of force transmissibility.
4. Maximum transmissibility occurs around $r \approx 1$. A spring mounted system (or stiffness only isolator) has infinite force transmissibility around $r \approx 1$. Damping helps to reduce the TR dramatically around resonance.
5. There is an increase in the TR value above $r = \sqrt{2}$ due to the damper.

Isolation Design Curve — # 3

Q: Discuss the practical consideration in the choice of k and damping ζ of an isolator? How will you use the TR curve in practice?

Fill in the class

Example 22 — # 1

Q: An electronic instrument panel is to be isolated from a panel that vibrates at frequencies ranging from 25 Hz to 35 Hz. It is desired to have at least 80% vibration isolation in order to prevent damage to the instrument. If the instrument has a weight of 85 N, determine the necessary stiffness of a spring mounting. What is the limitation of this design? (p.148 in NP)

Fill in the class

Basic Rotordynamics

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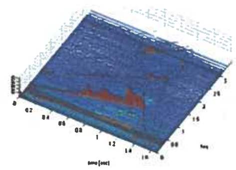
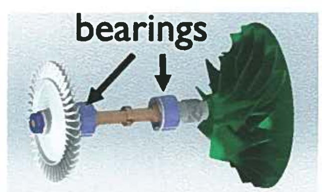


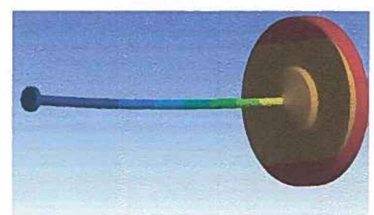
topics

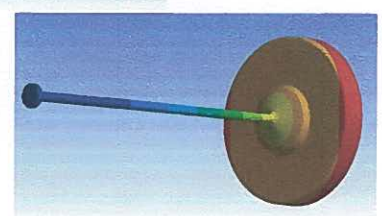
- Introduction & motivation
- Modeling rotating system dynamics
  - Jeffcott Rotor model (1 disk + shaft)
  - Whirling at constant speed
  - Self centering
  - Effect of damping & bearing forces
  - Anisotropic bearings and elliptical whirling

Real rotating machines

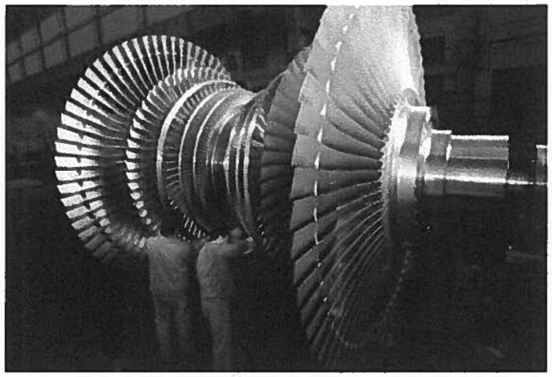


Vibrating vs. rotating

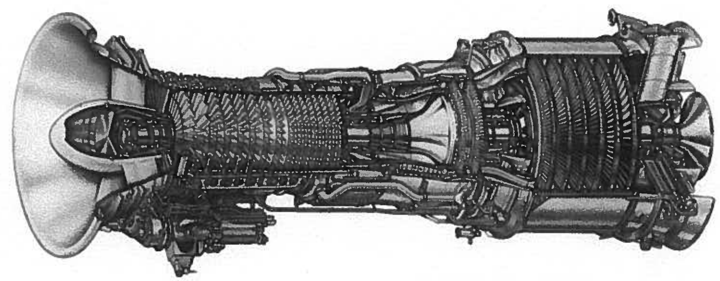




Typical machine with rotating elements

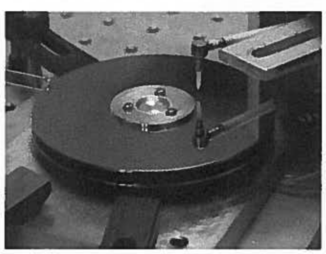


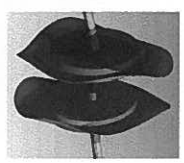
Gas turbine



General Electric LM2500 Gas Turbine

Hard disk vibration





Why teach dynamics of rotating structures?

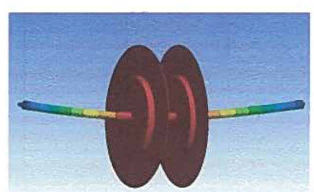
- Many machines contain rotating elements
- Rotating structures contain considerable energy
- Rotating machine operate at a range of speeds changing their behavior
- Theory of rotating machines >100 years old

Without words

ROTATING MACHINE VIBRATION & SIMPLE MODELS

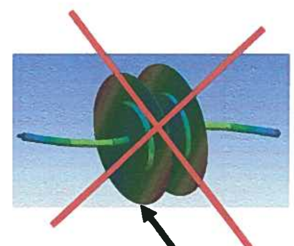
Basic assumptions

- Jeffcott rotor model
- No Gyroscopic effect
- Single mode dynamics / massless shaft
- No torsion (stiff)
- Only bending

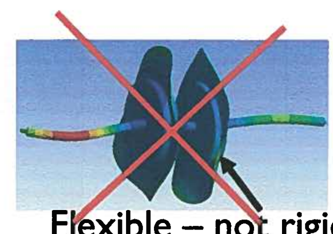


Basic model assumptions

- No consideration of higher frequencies
- No consideration of disk flexibility



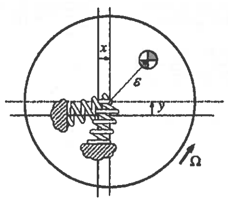
Gyroscopic effect

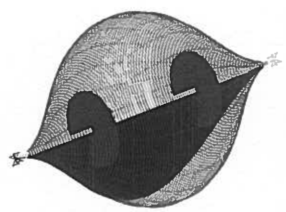


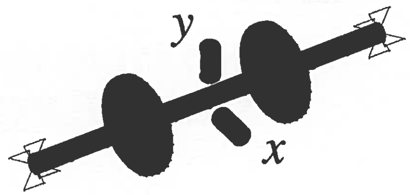
Flexible – not rigid

One degree of freedom – Jeffcott rotor

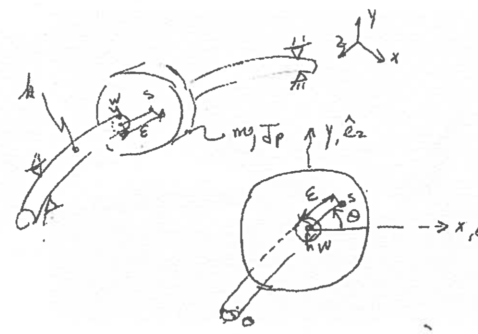
- One mode
- Response to unbalance
- Isotropic supports



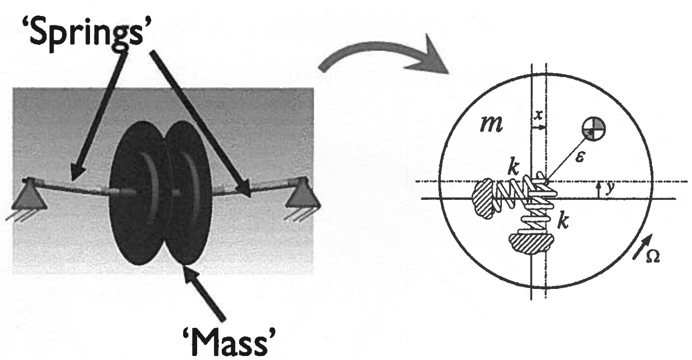




model

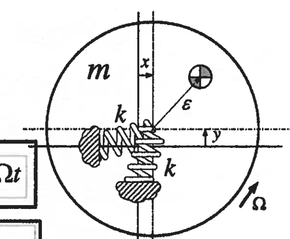


Jeffcott Rotor model



Jeffcott rotor

- constant speed $\dot{\Omega} = 0$
- Unbalance



$$x_{cg} = x + \varepsilon \cos \Omega t \qquad y_{cg} = y + \varepsilon \sin \Omega t$$

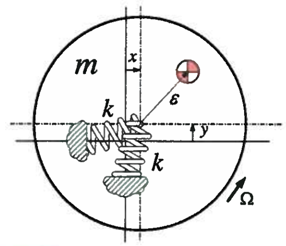
$$\frac{d}{dt}\left(m \frac{d}{dt} x_{cg}\right) = -kx \qquad \frac{d}{dt}\left(m \frac{d}{dt} y_{cg}\right) = -ky$$

$$\begin{aligned} m\ddot{x} + kx &= m\varepsilon\Omega^2 \cos \Omega t \\ m\ddot{y} + ky &= m\varepsilon\Omega^2 \sin \Omega t \end{aligned}$$

$$r \triangleq x + iy$$

$$m\ddot{r} + kr = m\varepsilon\Omega^2 e^{i\Omega t}$$

Jeffcott rotor – steady state response



$$m\ddot{r} + kr = m\varepsilon\Omega^2 e^{i\Omega t}$$

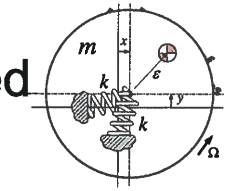
Put in Eq. $r = Ae^{i\Omega t} + Be^{-i\Omega t}$

$$r = \frac{\varepsilon\Omega^2}{\omega_n^2 - \Omega^2} e^{i\Omega t} \qquad \omega_n^2 = \frac{k}{m}$$

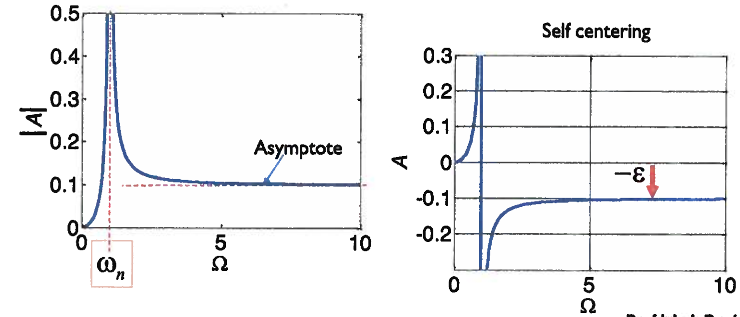
Isotropic! supports

orbit

Response vs speed



$$r = \frac{\varepsilon\Omega^2}{\omega_n^2 - \Omega^2} e^{i\Omega t} = Ae^{i\Omega t} \qquad \omega_n^2 = \frac{k}{m}$$

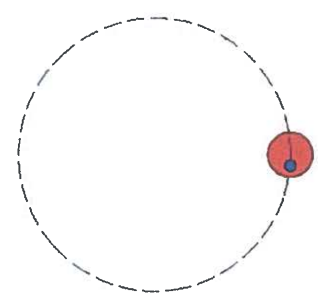


Animate whirl

$\omega_n > \Omega$ $A > 0$ | $\omega_n < \Omega$ $A < 0$ | $\omega_n \ll \Omega$ $A = -\varepsilon < 0$

Animate whirl

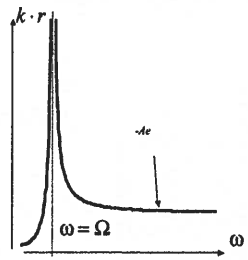
$\omega_n \approx \Omega$ $A \to \infty$



$phase\ lag\ \frac{\pi}{2}$

forces @ bearings

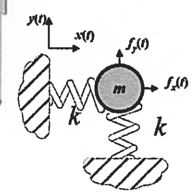
- Reaction force



$$f\_bearing = kr(t) = kA\exp(i\Omega t) = k\frac{\frac{\Omega^2}{\omega_n^2}e}{\left(1-\frac{\Omega^2}{\omega_n^2}\right)}\exp(i\Omega t)$$

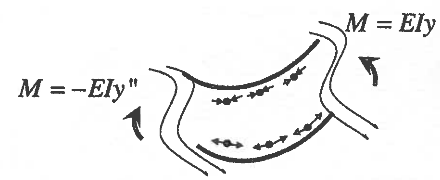
$$f_x(t) = k\frac{\frac{\Omega^2}{\omega_n^2}e}{\left(1-\frac{\Omega^2}{\omega_n^2}\right)}\cos\Omega t$$

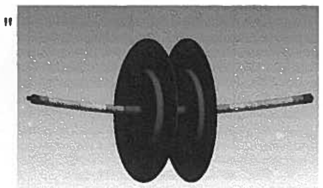
$$f_y(t) = k\frac{\frac{\Omega^2}{\omega_n^2}e}{\left(1-\frac{\Omega^2}{\omega_n^2}\right)}\sin\Omega t$$



Shaft stress

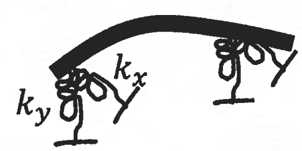
- Is shaft in tension or compression?
- Is stress alternating (fatigue), at what rate?

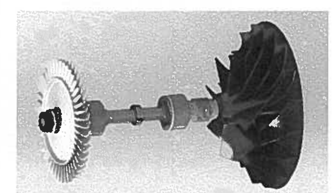




Rotors in anisotropic bearings

unequal stiffness





Total stiffness (springs in series)

Shaft + bearing + foundation

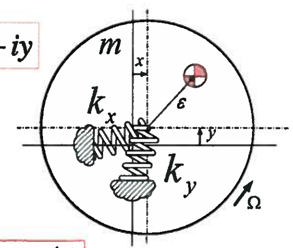
Usually $k_x \neq k_y$ $\boxed{\omega_x^2 = \frac{k_x}{m}}$ $\boxed{\omega_y^2 = \frac{k_y}{m}}$

Anisotropic bearings

$$\ddot{x} + \omega_x^2 x = \varepsilon\Omega^2 \cos\Omega t$$

$$\ddot{y} + \omega_y^2 y = \varepsilon\Omega^2 \sin\Omega t$$

$$r \triangleq x + iy$$



$$\omega_x^2 \triangleq \omega_n^2 + \omega_\Delta^2, \quad \omega_y^2 \triangleq \omega_n^2 - \omega_\Delta^2,$$

$$\ddot{r} + \omega_n^2 r + \omega_\Delta^2 \bar{r} = \Omega^2 \varepsilon e^{i\Omega t}$$

$$\Rightarrow \quad r = r_+ e^{i\Omega t} + r_- e^{-i\Omega t}$$

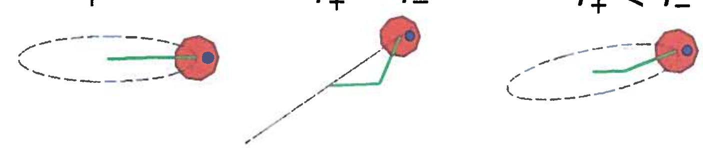
$$r_+ = \frac{\Omega^2 \varepsilon}{2} \frac{(\omega_x^2 + \omega_y^2 - 2\Omega^2)}{(\omega_x^2 - \Omega^2)(\omega_y^2 - \Omega^2)} \qquad r_- = \frac{\Omega^2 \varepsilon}{2} \frac{(\omega_y^2 - \omega_x^2)}{(\omega_x^2 - \Omega^2)(\omega_y^2 - \Omega^2)}$$

Forward & backward whirl

- Forward whirl takes place when $r_+ > r_-$
- Backward whirl $r_+ < r_-$

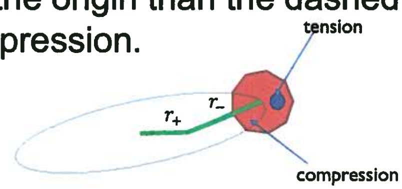
$$r = r_+ e^{i\Omega t} + r_- e^{-i\Omega t}$$

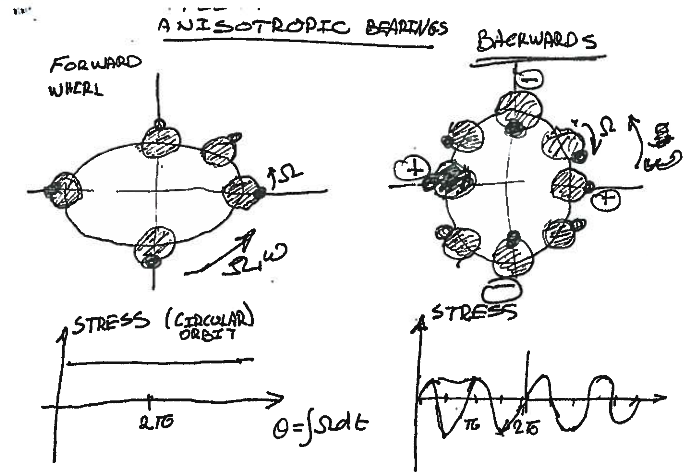
$r_+ > r_-$ $\qquad$ $r_+ = r_-$ $\qquad$ $r_+ < r_-$



Stress & anisotropic bearings

- Consider the blue dot (cg) as a material fiber.
- As long as it is further than the dashed line, it is in tension
- If it is closer to the origin than the dashed line, it is in compression.



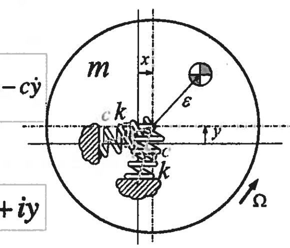


Damped *Rotors*

Damping + unbalance

- Unbalance

$$\frac{d}{dt}\left(m\frac{d}{dt}x_{cg}\right)=-kx-c\dot{x} \qquad \frac{d}{dt}\left(m\frac{d}{dt}y_{cg}\right)=-ky-c\dot{y}$$



$$\ddot{x}+2\zeta\omega_n\dot{x}+\omega_n^2x=\varepsilon\Omega^2\cos\Omega t$$

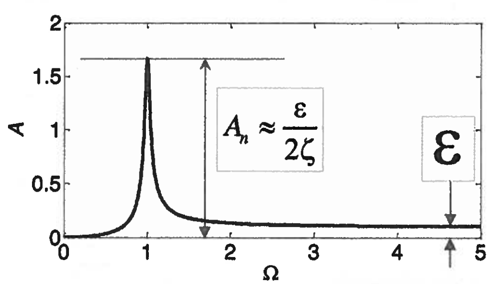
$$\ddot{y}+2\zeta\omega_n\dot{y}+\omega_n^2y=\varepsilon\Omega^2\sin\Omega t$$

$$r\triangleq x+iy$$

$$\ddot{r}+2\zeta\omega_n\dot{r}+\omega_n^2r=\varepsilon\Omega^2e^{i\Omega t}$$

$$\Rightarrow \quad r=\frac{\varepsilon\Omega^2}{\omega_n^2-\Omega^2+i2\zeta\omega_n\Omega}e^{i\Omega t}$$

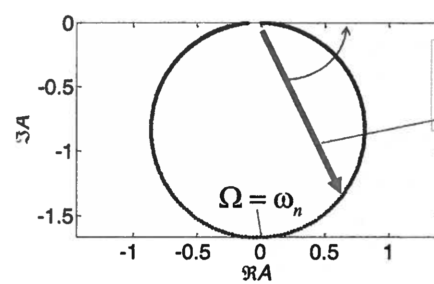
Amplitude vs speed



- Does damping affect self centering?

$$r=\frac{\varepsilon\Omega^2}{\omega_n^2-\Omega^2+i2\zeta\omega_n\Omega}e^{i\Omega t} \xrightarrow{\Omega\gg\omega_n} r=\frac{\varepsilon\Omega^2}{-\Omega^2}e^{i\Omega t}=-\varepsilon e^{i\Omega t}$$

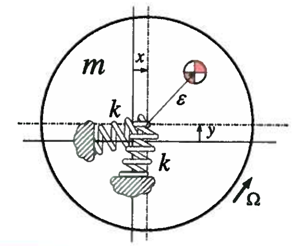
Polar plot



$$A=\frac{\varepsilon\Omega^2}{\omega_n^2-\Omega^2+i2\zeta\omega_n\Omega}$$

Rotors in body-fixed coordinates

Jeffcott rotor – steady state response



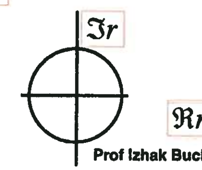
$$m\ddot{r} + kr = m\varepsilon\Omega^2 e^{i\Omega t}$$

Put in Eq.

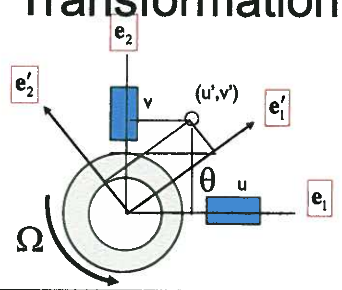
$$r = Ae^{i\Omega t} + BAe^{-i\Omega t}$$

$$r = \frac{\varepsilon\Omega^2}{\omega_n^2 - \Omega^2} e^{i\Omega t} \qquad \omega_n^2 = \frac{k}{m}$$

orbit



Transformation of axes

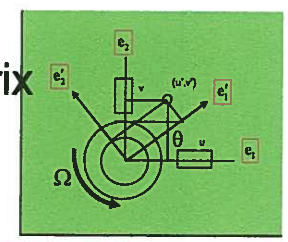


$$\theta = \int_0^t \Omega(\tau)\, d\tau$$

vector $\mathbf{r} = u\mathbf{e}_1 + v\mathbf{e}_2 = u'\mathbf{e}_1' + v'\mathbf{e}_2'$

Complex representation $r = u + \mathrm{i}\, v$ $\quad \rho = u' + \mathrm{i}\, v'$

Transformation matrix (proof)



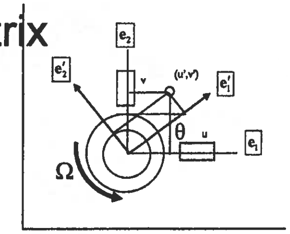
$$\mathbf{r}\cdot\mathbf{e}_1' = u\mathbf{e}_1\cdot\mathbf{e}_1' + v\mathbf{e}_2\cdot\mathbf{e}_1' = u'\mathbf{e}_1'\cdot\mathbf{e}_1' + v'\mathbf{e}_2'\cdot\mathbf{e}_1'$$

$$\mathbf{r}\cdot\mathbf{e}_1' = u\cos\theta + v\,s\,in\theta = u'$$

$$\mathbf{r}\cdot\mathbf{e}_2' = u\mathbf{e}_1\cdot\mathbf{e}_2' + v\mathbf{e}_2\cdot\mathbf{e}_2' = u'\mathbf{e}_1'\cdot\mathbf{e}_2' + v'\mathbf{e}_2'\cdot\mathbf{e}_2'$$

$$\mathbf{r}\cdot\mathbf{e}_2' = -u\sin\theta + v\cos\theta = v'$$

Transformation matrix (proof)



$$\begin{bmatrix}\cos\theta & \sin\theta \\ -\sin\theta & \cos\theta\end{bmatrix}\begin{pmatrix}u\\v\end{pmatrix}=\begin{pmatrix}u'\\v'\end{pmatrix}$$

$$\begin{pmatrix}u\\v\end{pmatrix}=\begin{bmatrix}\cos\theta & -\sin\theta \\ \sin\theta & \cos\theta\end{bmatrix}\begin{pmatrix}u'\\v'\end{pmatrix}$$

← Matrix transformation of cords

$$u'\cos\theta - v'\sin\theta + i\left(u'\sin\theta + v'\cos\theta\right) = u + i\,v$$

$\boxed{\rho e^{i\theta} = r}$ ← Complex transformation of cords

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EQ of motion in body cords

Dynamics of Rotating structures

- Stationary cords & transformation

$$\ddot{r} + \omega_n^2 r = \varepsilon\Omega^2 e^{i\Omega t} \qquad \rho e^{i\theta} = \rho e^{i\Omega t} = r$$

$$\left(\ddot{\rho} + 2i\,\Omega\dot{\rho} - \rho\Omega^2\right)e^{i\Omega t} + \omega_n^2\rho e^{i\Omega t} = \varepsilon\Omega^2 e^{i\Omega t}$$

$$\left(\ddot{\rho} + 2i\,\Omega\dot{\rho} - \rho\Omega^2\right) + \omega_n^2\rho = \varepsilon\Omega^2$$

A strain gauge would measure that

$$\rho = \overbrace{\frac{\varepsilon\Omega^2}{\omega_n^2 - \Omega^2}}^{\text{steady}} + \overbrace{\rho_1 e^{i\lambda_1 t} + \rho_2 e^{i\lambda_2 t}}^{\text{transient}}$$

Prof Izhak Bucher

transient in body cords

Dynamics of Rotating structures

$$\left(\ddot{\rho} + 2i\,\Omega\dot{\rho} - \rho\Omega^2\right) + \omega_n^2\rho = \cancel{\varepsilon\Omega^2}\ \boxed{=0}$$

Propose a solution $\rho = \rho_0 e^{\lambda t}$

$$\left(\lambda^2 + 2i\lambda\Omega + \omega_n^2 - \Omega^2\right)\rho_0 e^{\lambda t} = 0 \Rightarrow \lambda_{1,2} = -i\left(\omega_n \pm \Omega\right)$$

A strain gauge would measure that

$$\rho = \overbrace{\frac{\varepsilon\Omega^2}{\omega_n^2 - \Omega^2}}^{\text{steady}} + \overbrace{\rho_1 e^{-i(\omega_n+\Omega)t} + \rho_2 e^{i(\omega_n-\Omega)t}}^{\text{transient}}$$

When a disturbance occurs, the apparent frequency is shifted

Prof Izhak Bucher

List of reference books

Dynamics of Rotating structures

Prof Izhak Bucher

Most relevant books

- **Dynamics of rotating systems,** 2005
  - Giancarlo Genta
- **Rotordynamik (German Edition),** 2007
  - Robert Gasch, Rainer Nordmann, Herbert Pfützner

Specific books (rotating machines)

- Ehrich, Fredric F. editor, Handbook of rotordynamics
- Childs, Dara, Turbomachinery rotordynamics : 1993.
- Lalanne, Michel, Rotordynamics prediction in engineering
- Vance, John M. Machinery Vibration and Rotordynamics
- Adams, Maurice L. Rotating machinery vibration : 2001.
- Wowk, Victor, Machinery vibration : balancing 1995.
- Kramer, Erwin, Dynamics of rotors and foundations, 1993.

arbitrary, mathematically inexpressible

2.6. General Excitation

MECH 463: Mechanical Vibrations

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UBC mechanical engineering

Suggested Readings:

1. Topic 2.6 from notes package for detailed derivations.
2. Sections 4.2–4.5 from the course textbook.

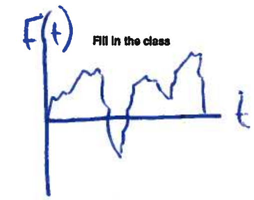
Learning Objectives

1. Determine the response to general forcing function: step, impulse, shocks.
2. Recognize the importance of harmonic response.
3. Appreciate interrelations among different forms of response.
4. Apply Fourier series.

We only consider $x_p(t)$!

→ Find response for a general $f(t)$ in $m\ddot{x} + c\dot{x} + kx = f(t)$

Fill in the class



Introduction

Q: Which aspects of the solution (response) will be different for a non-harmonic force? p.154 in NP

Fill in the class

- x_h does not change, so:

$$x_h(t) = e^{-\zeta\omega_n t}\left[C_1\cos(\omega_d t) + C_2\sin(\omega_d t)\right]$$

- $x_p(t)$ changes depending on $f(t)$

Available methods — # 1

Approach 1: Use ODE Theory (Look up table)

$m\ddot{x} + c\dot{x} + kx = f(t)$

Fill in the class

Particular Integrals

For linear differential equations with constant coefficients:

$f(t)$ ← Right-hand side	Trial P.I. → $x_p(t)$
constant	a
x^n (n integer)	$a x^n + b x^{n-1} + \dots$
ex: $f(t) = e^{kt}$ ← e^{kx}	$a e^{kx}$ → use: $x_p(t) = a e^{kt}$
$x e^{kx}$	$(a x + b) e^{kx}$
$x^n e^{kx}$	$(a x^n + b x^{n-1} + \dots) e^{kx}$
$\sin px$, $\cos px$	$a \sin px + b \cos px$
$e^{kx} \sin px$, $e^{kx} \cos px$	$e^{kx}(a \sin px + b \cos px)$

A dictionary of Particular Solutions for linear ODEs. Note that we replace x with t and k with ω for vibration problems, since time t is the independent variable.

This approach does not always work!

Available methods — # 2

$\ddot{x}+c\dot{x}+Kx=Fe^{j\omega t}$ $j=\sqrt{-1}$

ess $x_p(t)=Xe^{j\omega t}$; using the table last page

es: $\left[m(j\omega)^2+j\omega c+K\right]Xe^{j\omega t}=Fe^{j\omega t}$

$\therefore \boxed{X_p(t)=\dfrac{F}{K-m\omega^2+j\omega c}}$

good way to solve practical problems

Fill in the class

Available methods — # 3

Approach 2: Use Fourier Series → If $f(t)$ is periodic (or periodisab

In Fourier analysis we express the force in the following form

given $f(t)$

$f(t)=\frac{a_0}{2}+\sum_{n=1}^{n=\infty}[\underbrace{a_n\cos n\omega t}_{\text{force}}+\underbrace{b_n\sin n\omega t}_{\text{force}}]$, $\omega=\frac{2\pi}{T}$, where T is the period of the force.

This method works for periodic forces and periodisable forces. We treat each term in the Fourier series as a force and find the associated response. **Using the principle of superposition:**

$x_p(t)=x_{a_0}+\sum_{n=1}^{n=\infty}[\underbrace{x_{a_n}}_{a_n}+\underbrace{x_{b_n}}_{b_n}]$.

due to "$\frac{a_0}{2}$"

"$f(t)$ is the sum of harmonic forces"

$\tan\phi_n=\frac{Cn\omega}{K-m(n\omega$

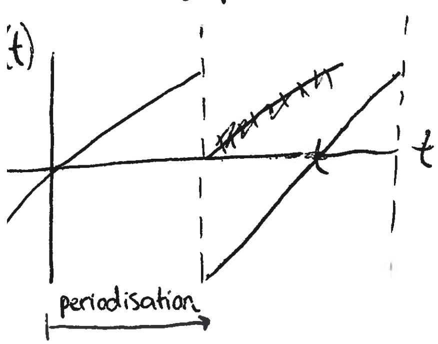
Fill in the class

$X_{a_n}=\frac{a_n}{\sqrt{[K-m(n\omega)^2]^2+(}}$

*Finding a_n and b_n is a math problem

Available methods — # 4

f function is not periodic?



*The Fourier Series for $f(t)$ does not converge to true value at $t=t_0$

*Fourier Series converges to zero (average value)

*Think about alternate periodisations

Available methods — # 5

Approach 3: Superposition principle with impulse response

This can deal with any forcing. Here, we break up any time series representing the force (periodic or not) in to a sum of shifted impulse functions (or Dirac delta functions).

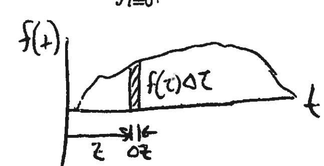
The response of the system due to the original force is given by the sum of the responses of the system, calculated for each shifted impulse.

If $h(t)$ is the impulse response then the response for any force $f(t)$ is:

$x_p(t)=\int_{t=0^+}^{t}h(t-\tau)f(\tau)d\tau$

$\boxed{\int f(t)\,dt = \text{impulse}\ [\text{N·s units}]}$

Fill in the class



$f(\tau)\Delta\tau$ is applied at

$f(t)=\sum \underbrace{f(\tau)\Delta(\tau)\delta(t-\tau)}_{\text{shifted impulse}}$

Available methods — # 6

$$\int_{\tau=0}^{\tau=t} h(t-\tau) f(\tau) d\tau$$

→ integral for x_p requires a computer.

→ trapezoidal rule

Fill in the class

Approach 2, see Topic 2.6, part 2

Approach 3, see Assignment #5

Available methods — # 7

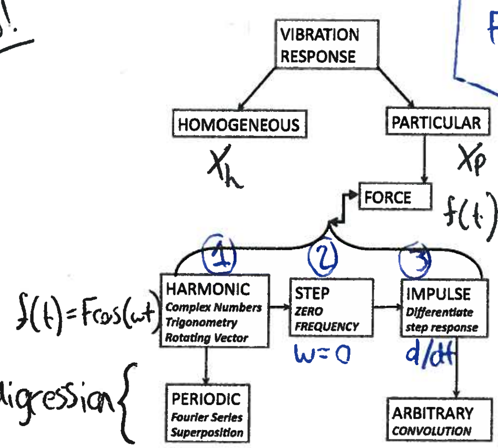
Q: List the advantages and drawbacks of each of the above methods? p.157 in NP

	Approach 1	Approach 2	App. 3
Principle	Look-up table	Fourier Series, Superposition	Impulse response, Superposition
Methodology	Analytical	Analytical, numerical	numerical, computer-oriented
Rough road? Gust excitation	X	X	✓
Periodic Force	✓	✓	✓
Aperiodic Force	Table, check	X	✓
Easiness?	Easy	Moderate	Difficult

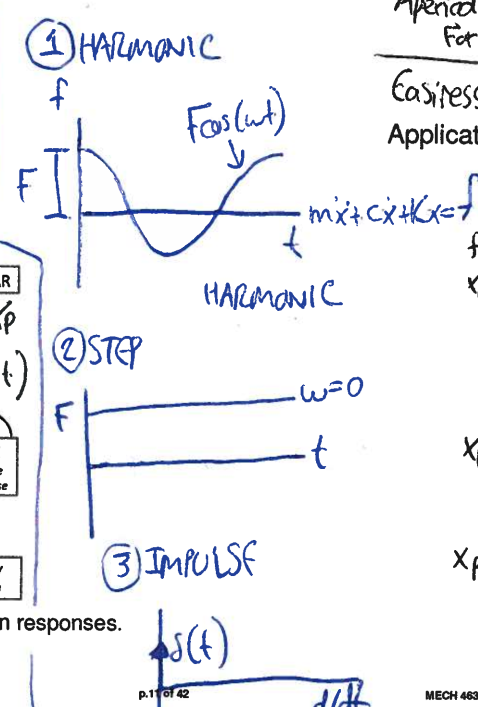
Fill in the class

Interrelations

Important Map!

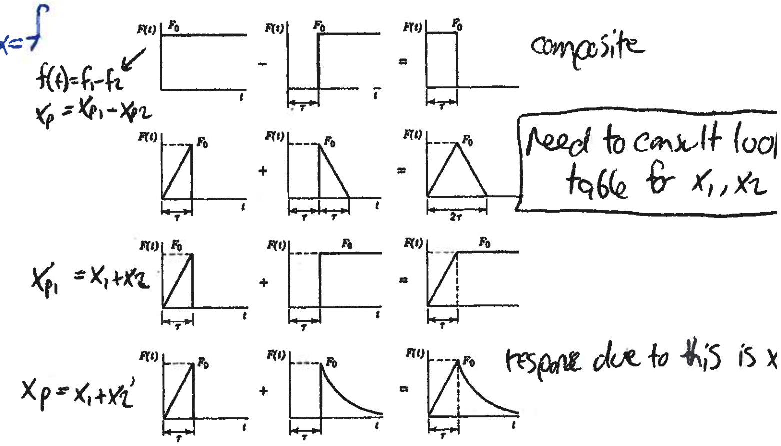


Relationship among different forced vibration responses.



So, $\frac{d}{dt}$(step response) = impulse response $= \int_{\tau=0}^{\tau=t} h(t-\tau) f(\tau) d\tau$

Application of Superposition Principle — # 1



Superposition is a powerful principle for linear systems.

Application of Superposition Principle — # 2

Fill in the class

2.15. Step Response — # 1

Following the map presented earlier in Fig.(11), we can evaluate the step response from the harmonic response by setting $\omega = 0$:

$$m\ddot{x} + c\dot{x} + kx = f = F_0 = [F_0 \cos(\omega t)]_{\omega=0}$$

$$x(t) = e^{-\zeta\omega_n t}\left[C_1 \cos\omega_d t + C_2 \sin\omega_d t\right] + \frac{F_0}{k} \tag{1}$$

$$C_1 = x_0 - \frac{F_0}{k}; \quad C_2 = \frac{\dot{x}_0 + \zeta\omega_n\left[x_0 - \frac{F_0}{k}\right]}{\omega_d}$$

$X(0) = X_0 \Rightarrow C_1 + \frac{F_0}{K} = X_0$

$C_2 = X_0 - \frac{F_0}{K}$

$\dot{X}(0) = \dot{X}_0 \Rightarrow$ gives C2

$\boxed{\text{C1, C2 depend on } x_p(t) \text{ through initial conditions}}$

Fill in the class

2.15. Step Response — # 2

Note that C_1 and C_2 were found from the initial conditions imposed on the TOTAL response. This must always be followed.

The unit step response $F_0 = 1$ tells us three things:

1. Rise time: Time taken from 10% to 90% of final value (0.1 to 0.9)
2. Settling time: Response reaches within 1% or 5% of final value (0.99 or 0.95).
3. Overshoot: % of final value by which the response rises initially.

2.15. Step Response — # 3

If $\zeta \uparrow$ damping

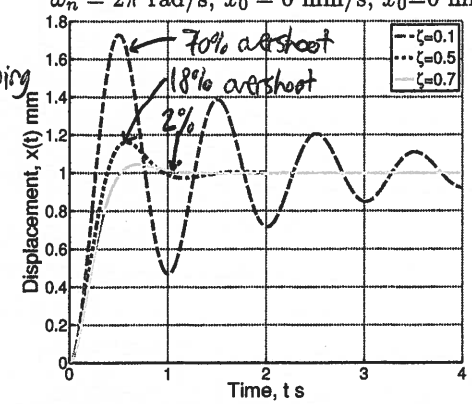
then, overshoot ↓

settling time ↓

rise time ↑

*The system will not respond to applied input instantly

$\omega_n = 2\pi$ rad/s; $\dot{x}_0 = 0$ mm/s; $x_0 = 0$ mm



Design trade-o

Unit step response of a SDOF system, initially at rest, for different damping ratios. Notice that higher is the damping ratio faster is the settling time, lower overshoot, but increased rise time!

Summary — # 1

1. Harmonic response is the fundamental response. Setting $\omega = 0$ in the harmonic response gives the step response. Differentiating step response gives impulse response. Convolution gives reponse for any force.
2. The step response is given by

$$m\ddot{x} + c\dot{x} + kx = f = F_0 = [F_0 \cos(\omega t)]_{\omega=0}$$

$$x(t) = e^{-\zeta\omega_n t}\left[C_1 \cos\omega_d t + C_2 \sin\omega_d t\right] + \frac{F_0}{k} \tag{3}$$

$$C_1 = x_0 - \frac{F_0}{k}; \quad C_2 = \frac{\dot{x}_0 + \zeta\omega_n\left[x_0 - \frac{F_0}{k}\right]}{\omega_d}.$$

Higher damping ratios ζ lead to lower overshoot, faster settling time, but result in slow rise time.

3. The impulse response of a SDOF system is given by $h(t) = \frac{1}{m\omega_d} e^{-\zeta\omega_n t} \sin\omega_d t$.
4. The response of a SDOF system to any arbitrary force is given by the convolution integral: $x_p(t) = \int_{t=0^+}^{t} h(t-\tau)f(\tau)d\tau$. This integral is usually evaluated numerically using a computer. Hand calculations are possible only for the simpler cases.

Summary — # 2

5. The homogenous response must be added to the above in order to determine the two unknown constants. Thus, we have for the most general case, the total response: $x(t) = e^{-\zeta\omega_n t}\left[C_1 \cos\omega_d t + C_2 \sin\omega_d t\right] + \int_{t=0^+}^{t} h(t-\tau)f(\tau)d\tau$. C_1 and C_2 must be found by imposing initial conditions on the total response $x(t)$.