a place of mind

MECH468 : Modern Control Engineering
MECH509 : Controls

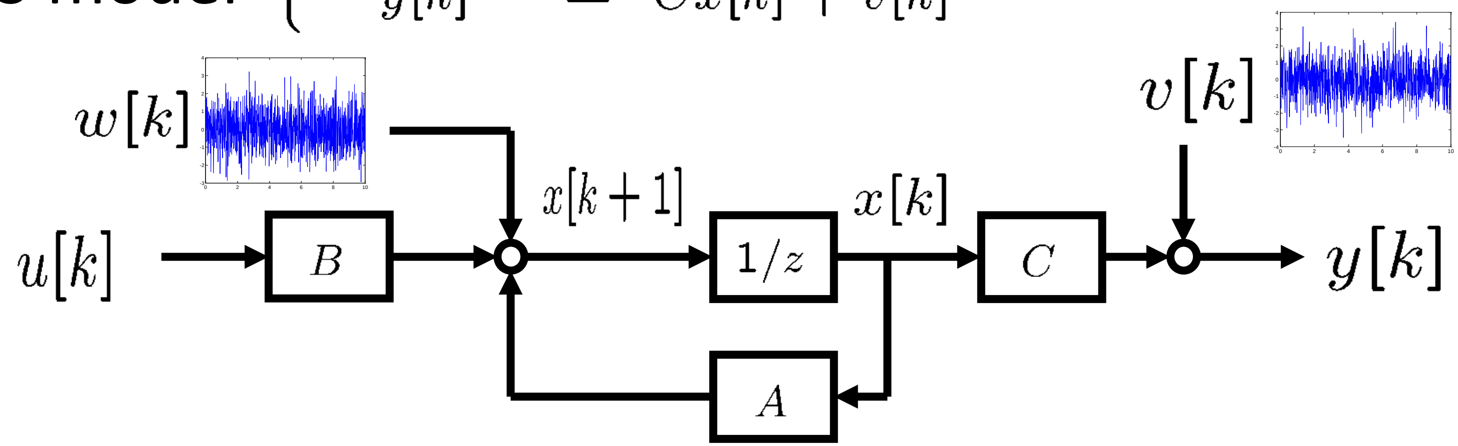
L29 : Discrete-time Kalman filter

Dr. Ryozo Nagamune
Department of Mechanical Engineering
University of British Columbia

Zoom lecture to be recorded and posted on Canvas

DT linear system with noise

- SS model $\left\{ \begin{array}{rcl} x[k+1] & = & Ax[k] + Bu[k] + w[k] \\ y[k] & = & Cx[k] + v[k] \end{array} \right.$



- Assumptions
  - *w[k]*: $E\{w[k]\} = 0,\ E\left\{w[k]w[k]^T\right\} = R_w,\ E\left\{w[i]w[j]^T\right\} = 0, \forall i \neq j$
  - *v[k]*: $E\{v[k]\} = 0,\ E\left\{v[k]v[k]^T\right\} = R_v,\ \underbrace{E\left\{v[i]v[j]^T\right\} = 0, \forall i \neq j}_{\textit{white noise}}$
  - (*v*,*w*) are uncorrelated $E\left\{w[i]v[j]^T\right\} = 0, \forall i, j$

Some terminologies & notation

- A priori (i.e. before taking measurement) estimate of *x[k]* and error covariance

$\hat{x}[k|k-1]$: estimate of *x[k]* from measurement up to time *k-1*

$P[k|k-1]$: error covariance $E\left\{(\hat{x}[k|k-1]-x[k])(\hat{x}[k|k-1]-x[k])^T\right\}$

- A posteriori (i.e. after taking measurement) estimate of *x[k]* and error covariance

$\hat{x}[k|k]$: estimate of *x[k]* from measurement up to time *k*

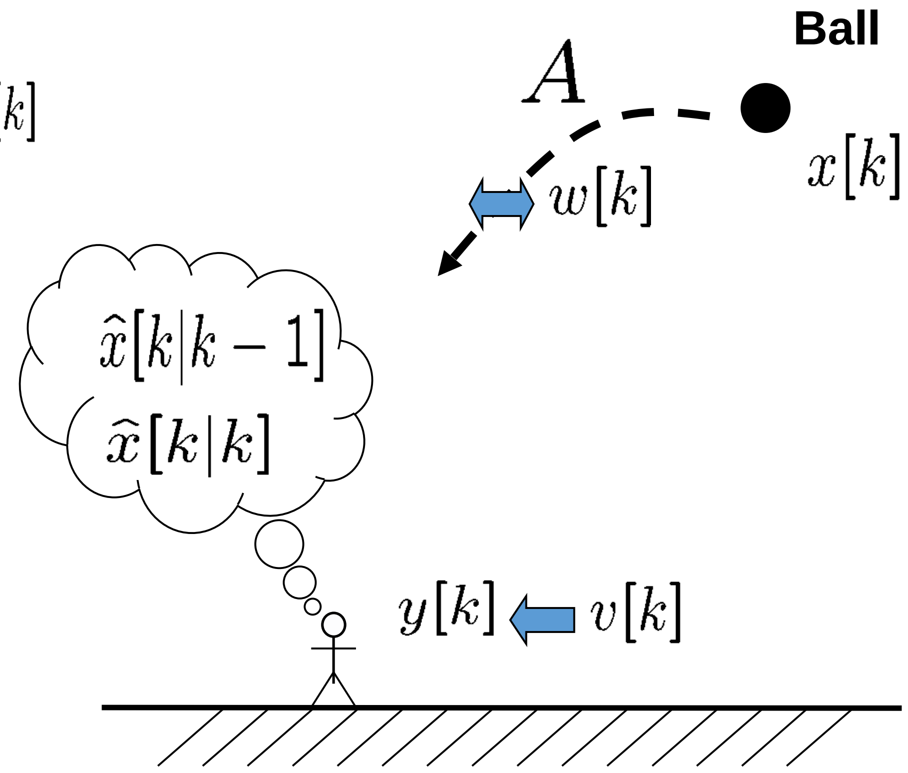
$P[k|k]$: error covariance $E\left\{(\hat{x}[k|k]-x[k])(\hat{x}[k|k]-x[k])^T\right\}$

An intuitive example

- Catching a ball

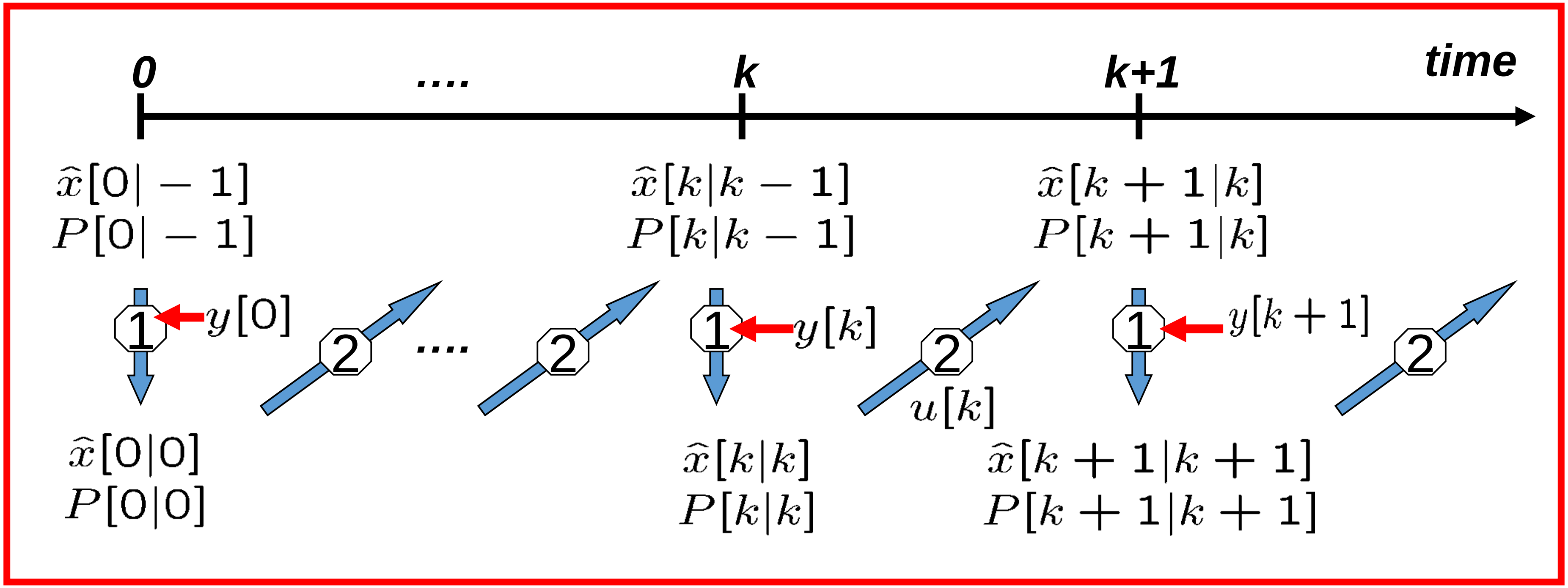
$$\begin{cases} x[k+1] &= Ax[k] + Bu[k] + w[k] \\ y[k] &= Cx[k] + v[k] \end{cases}$$

  - *x* : position & velocity of a ball
  - *u*=0
  - *w* : wind
  - *y* : position measurement by eyes
  - *v* : noise because of bad eye-sight



Idea of Kalman filter

$$\begin{cases} x[k+1] &= Ax[k] + Bu[k] + w[k] \text{ .... } ② \\ y[k] &= Cx[k] + v[k] \text{ .... } ① \end{cases}$$



Discrete-time Kalman filter

Initial conditions

- Estimate $\hat{x}[0|-1]$
- Error cov. $P[0|-1]$

Remark: Error cov. can be computed offline.

① Measurement update (correction)

- Estimate $\hat{x}[k|k] = \hat{x}[k|k-1] + P[k|k]C^T R_v^{-1}(y[k] - C\hat{x}[k|k-1])$
- Error cov. $P[k|k] = P[k|k-1] - P[k|k-1]C^T \left(CP[k|k-1]C^T + R_v\right)^{-1} CP[k|k-1]$

② Time update (prediction)

- Estimate $\hat{x}[k+1|k] = A\hat{x}[k|k] + Bu[k]$
- Error cov. $P[k+1|k] = AP[k|k]A^T + R_w$

Measurement update: Derivation

$$① \quad y[k] = Cx[k] + v[k]$$

- Suppose that we have estimate & covariance:

$$\hat{x}[k|k-1] \text{ and } P[k|k-1]$$

- We update this “old” estimate & cov. based on the “new” data *y[k]*. Then, the new estimate and error covariance are given by using recursive least-squares as:

$$\hat{x}[k|k] = \hat{x}[k|k-1] + P[k|k]C^T R_v^{-1}(y[k] - C\hat{x}[k|k-1])$$

$$\begin{aligned} P[k|k] &:= E\left\{ (\hat{x}[k|k] - x[k]) (\hat{x}[k|k] - x[k])^T \right\} \\ &= \left(P[k|k-1]^{-1} + C^T R_v^{-1} C \right)^{-1} \\ &= P[k|k-1] - P[k|k-1]C^T \left(CP[k|k-1]C^T + R_v \right)^{-1} CP[k|k-1] \end{aligned}$$

Time update: Derivation

$$② \quad x[k+1] = Ax[k] + Bu[k] + w[k]$$

- Suppose that we have estimate & covariance:

$$\hat{x}[k|k] \text{ and } P[k|k]$$

- From the state equation above, we can predict the estimate of *x[k+1]* from the measurement up to time *k* as

$$\begin{aligned} \hat{x}[k+1|k] &= E\{A\hat{x}[k|k] + Bu[k] + w[k]\} \\ &= A\hat{x}[k|k] + Bu[k] + \underbrace{E\{w[k]\}}_{\rightarrow\, 0} \end{aligned}$$

$$\begin{aligned} P[k+1|k] &:= E\left\{(\hat{x}[k+1|k] - x[k+1])(\hat{x}[k+1|k] - x[k+1])^T\right\} \\ &= E\left\{(A(\hat{x}[k|k] - x[k]) - w[k])(A(\hat{x}[k|k] - x[k]) - w[k])^T\right\} \end{aligned}$$

Time update: Derivation (cont'd)

$$P[k+1|k] = A\underbrace{E\left\{(\hat{x}[k|k]-x[k])(\hat{x}[k|k]-x[k])^T\right\}}_{P[k|k]}A^T + \underbrace{E\left\{w[k]w[k]^T\right\}}_{R_w}$$
$$-\underline{AE\left\{(\hat{x}[k|k]-x[k])w[k]^T\right\}} - E\left\{w[k](\hat{x}[k|k]-x[k])^T\right\}A^T$$

Prove that this term is zero.

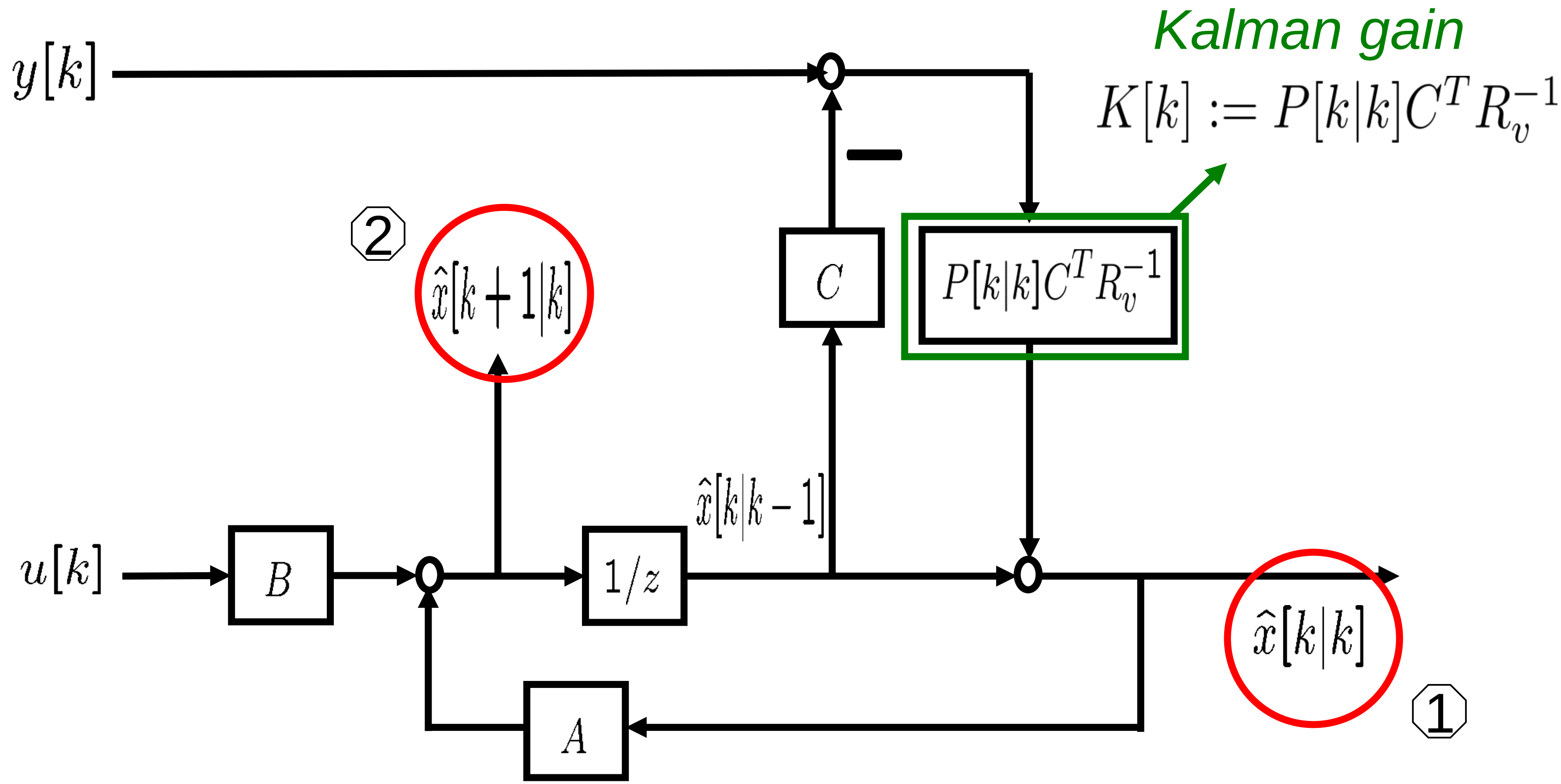
- One can prove (with a lengthy calculation) that

$$\hat{x}[k|k]-x[k]$$
$$= P[k|k]P[k|k-1]^{-1}A(\hat{x}[k-1|k-1]-x[k-1]) - P[k|k]P[k|k-1]^{-1}w[k-1] + P[k|k]C^T R_v^{-1} v[k]$$

- By induction, the left-hand side can be written as a linear combination of *w[k-1],w[k-2],…,v[k],v[k-1],…* Thus,

$$E\left\{(\hat{x}[k|k]-x[k])w[k]^T\right\} = 0$$

DT Kalman filter: Block diagram



One-step Kalman filter
Combination of two steps ①+②

- Estimate

$$\begin{cases} \underline{\hat{x}[k|k]} = \hat{x}[k|k-1] + P[k|k]C^T R_v^{-1}(y[k] - C\hat{x}[k|k-1]) \\ \hat{x}[k+1|k] = A\underline{\hat{x}[k|k]} + Bu[k] \end{cases}$$

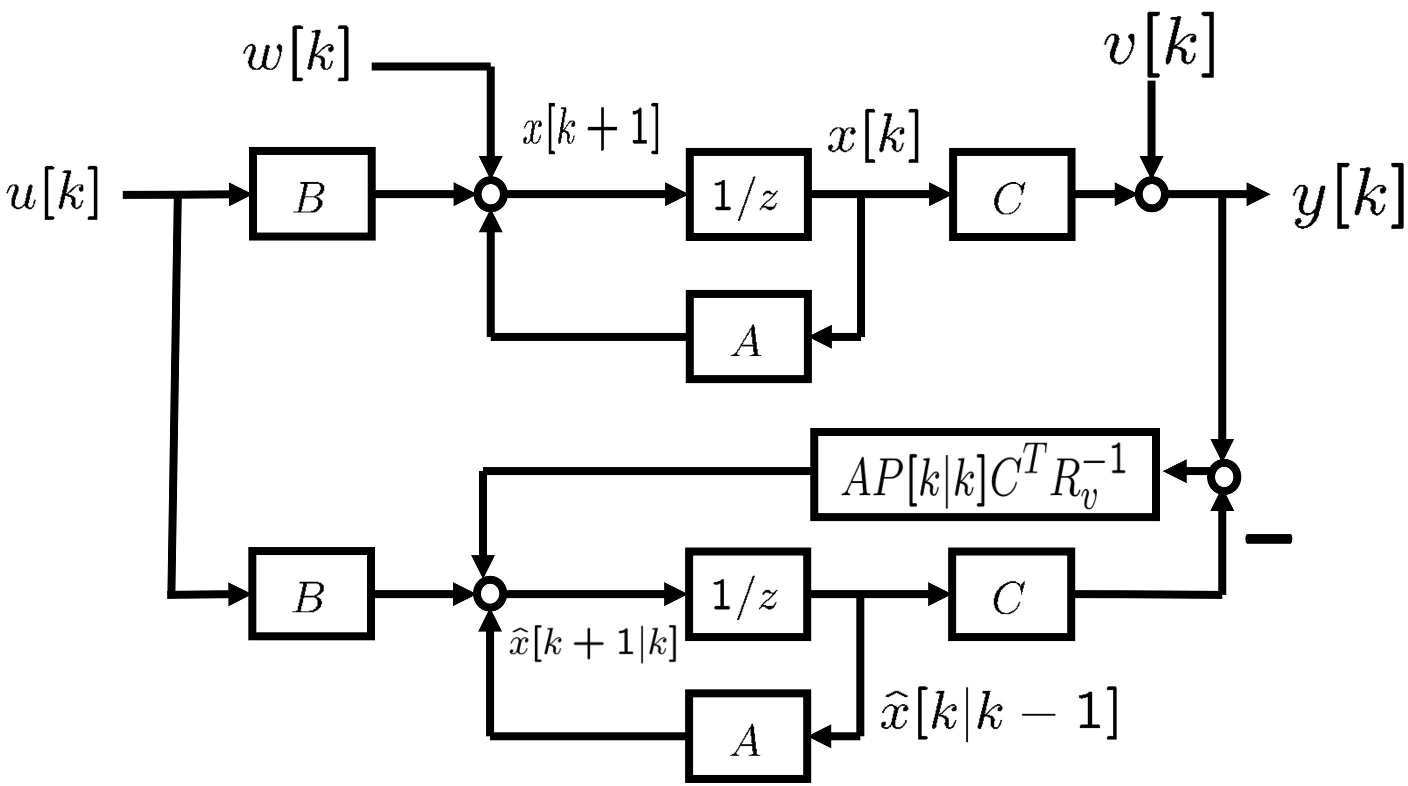
$$\Rightarrow \quad \hat{x}[k+1|k] = A\hat{x}[k|k-1] + Bu[k] + AP[k|k]C^T R_v^{-1}(y[k] - C\hat{x}[k|k-1])$$

- Error covariance

$$\begin{cases} \underline{P[k|k]} = P[k|k-1] - P[k|k-1]C^T\left(CP[k|k-1]C^T + R_v\right)^{-1} CP[k|k-1] \\ P[k+1|k] = A\underline{P[k|k]}A^T + R_w \end{cases}$$

$$\Rightarrow \quad P[k+1|k] = A\left\{P[k|k-1] - P[k|k-1]C^T\left(CP[k|k-1]C^T + R_v\right)^{-1} CP[k|k-1]\right\}A^T + R_w$$

One-step Kalman filter
Block diagram



Observer structure with a time-varying gain!

Another one-step Kalman filter Combination of two steps ②+①

- Estimate

$$\begin{cases} \hat{x}[k+1|k+1] = \underline{\hat{x}[k+1|k]} + P[k+1|k+1]C^T R_v^{-1}(y[k+1] - C\underline{\hat{x}[k+1|k]}) \\ \underline{\hat{x}[k+1|k]} = A\hat{x}[k|k] + Bu[k] \end{cases}$$

$$\Rightarrow \quad \hat{x}[k+1|k+1] = A\hat{x}[k|k] + Bu[k] + P[k+1|k+1]C^T R_v^{-1}(y[k+1] - C(A\hat{x}[k|k] + Bu[k]))$$

- Error covariance

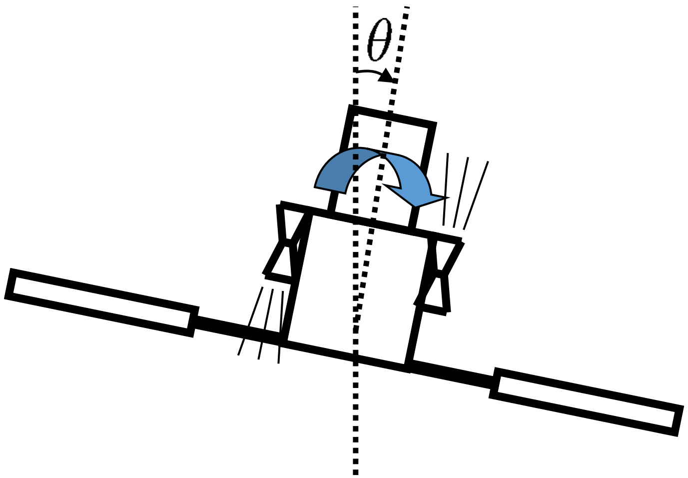
$$\begin{cases} P[k+1|k+1] = \left(\underline{P[k+1|k]}^{-1} + C^T R_v C\right)^{-1} \\ \underline{P[k+1|k]} = AP[k|k]A^T + R_w \end{cases}$$

$$\Rightarrow \quad P[k+1|k+1] = \left((AP[k|k]A^T + R_w)^{-1} + C^T R_v C\right)^{-1}$$

Satellite attitude estimation

- CT model with $x := [\theta, \dot{\theta}]^T$

$$\begin{cases} \dot{x} &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} w \\ y &= \begin{bmatrix} 1 & 0 \end{bmatrix} x \end{cases}$$



- Discretization with period T

$$\begin{cases} x[k+1] &= \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix} x[k] + \overbrace{\begin{bmatrix} \frac{T^2}{2} \\ T \end{bmatrix}}^{B_w} w[k] \\ y[k] &= \begin{bmatrix} 1 & 0 \end{bmatrix} x[k] + v[k] \end{cases}$$

Replace Rw with BwRwBw' in previous discussions.

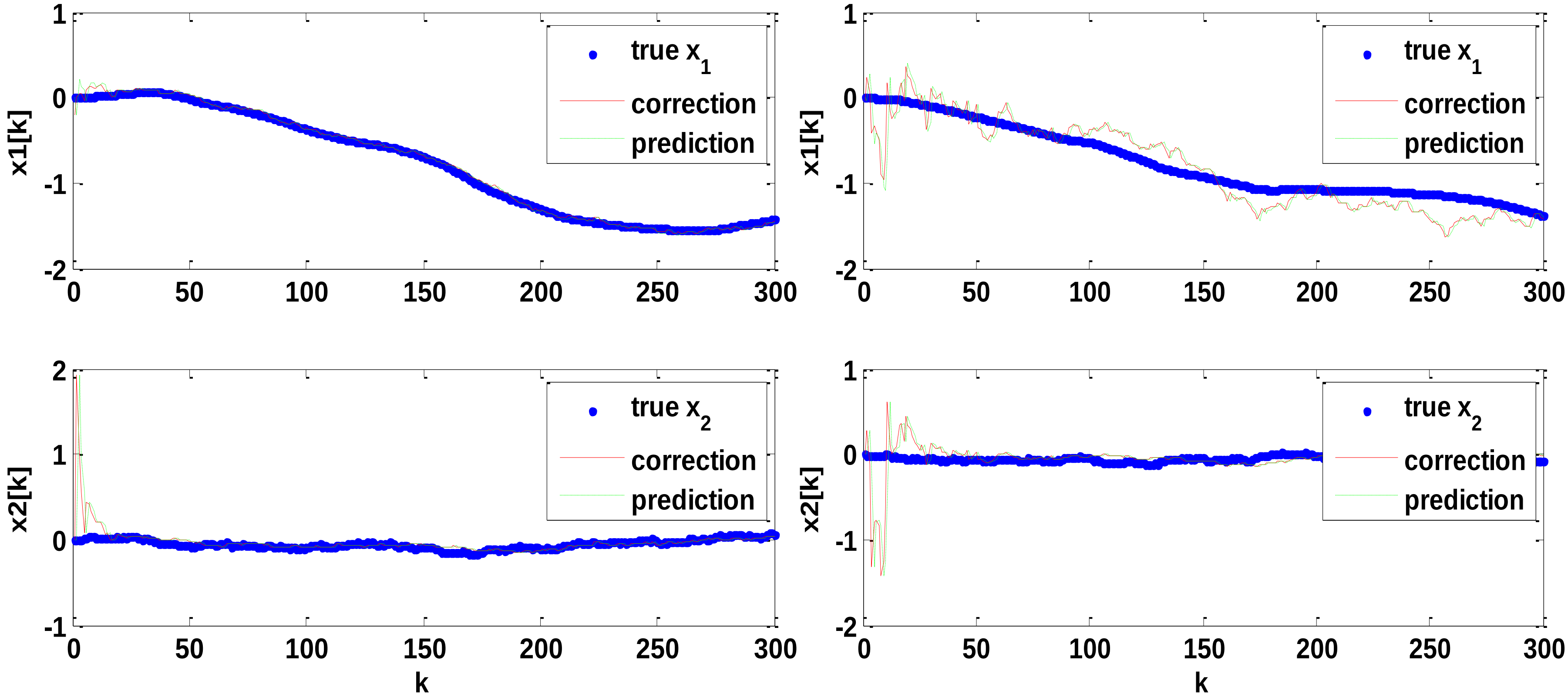
$T = 0.1$ (sec)

We assume some measurement noise.

Satellite attitude estimation

$$R_w = 0.1^2 \quad \hat{x}[0|-1] = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad P[0|-1] = \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix}$$

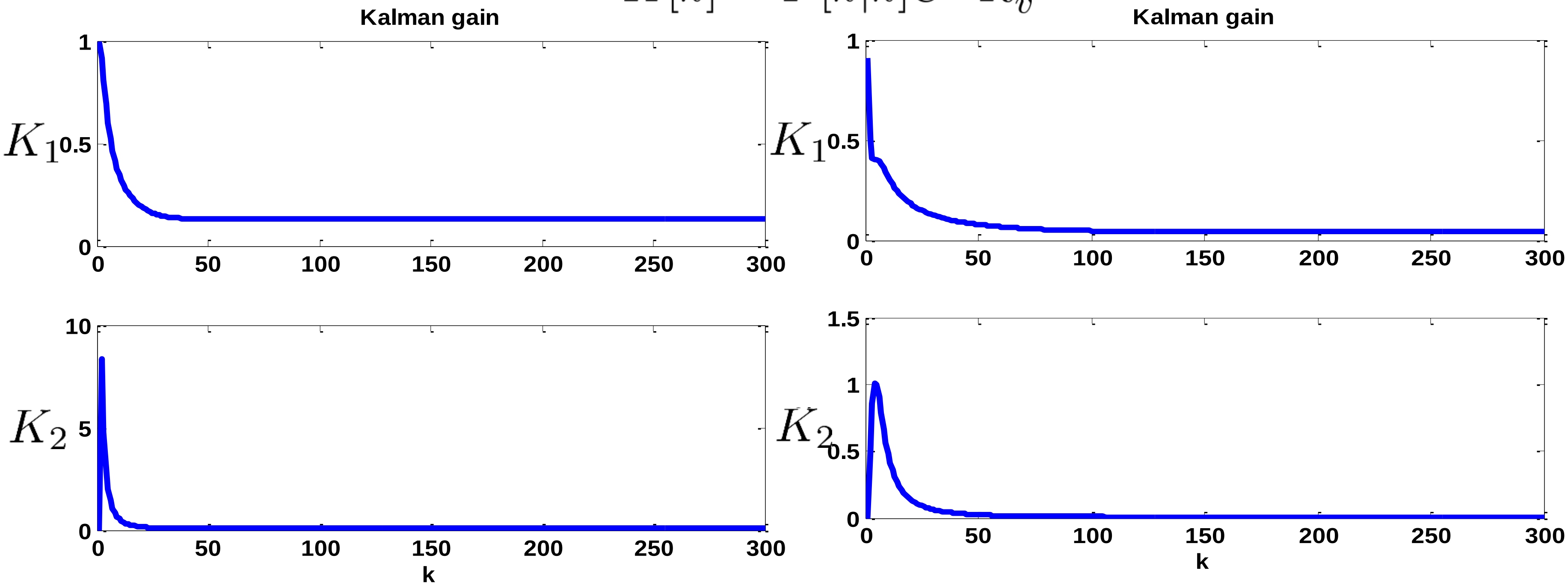
$R_v = 0.1^2$ (deg) $\qquad\qquad$ $R_v = 1$ (deg)



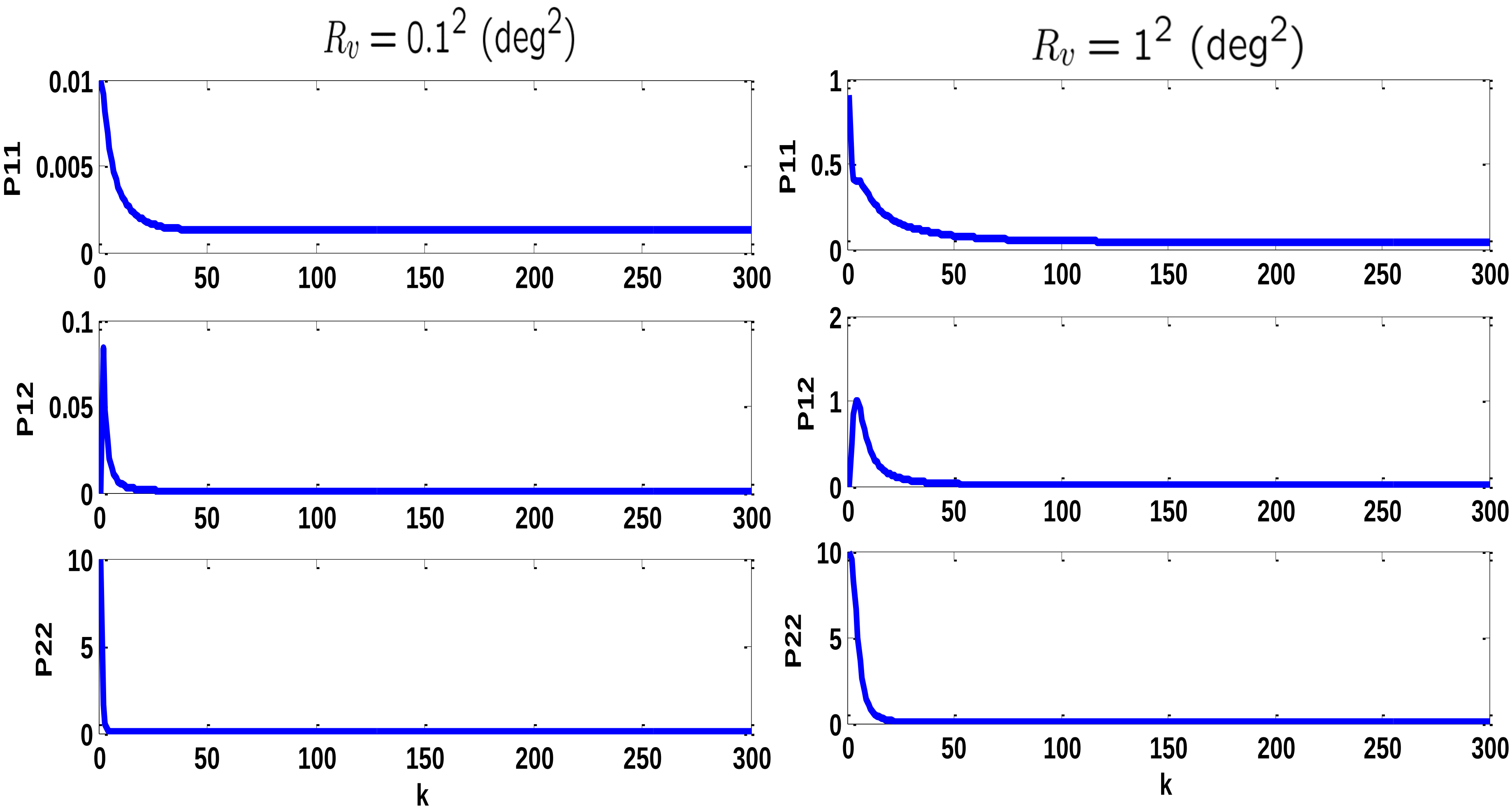
Satellite attitude estimation

- If *Rv* is small, the measurement is more accurate. In such case, Kalman gain is generally large.
- Kalman gain converges fairly quickly.

$R_v = 0.1^2\ (\mathrm{deg}^2)$ $\quad K[k] = P[k|k]C^T R_v^{-1}$ $\quad R_v = 1\ (\mathrm{deg}^2)$



Satellite attitude estimation
Plot of *P[k|k]*



Remarks on Kalman filter

- Error covariance and Kalman gain do not depend on measurements *y[k]*. Thus, these can be pre-computed off-line, and stored in a computer for implementation.
- Error covariance and Kalman gain are almost constants. (Steady-state Kalman filter is presented in next lecture.)
- Kalman filter minimizes $P[k|k]$
  - the optimal state estimator (among both linear and nonlinear filters) if *w* and *v* are Gaussian.
  - the optimal linear state estimator for non-Gaussian *w* and *v*.

Remarks (cont'd)

- A variety of extensions
  - Continuous-time case (Kalman-Bucy filter)
  - Colored (non-white) disturbance and noise
  - Correlated disturbance and noise
  - Nonlinear Kalman filter
    - Extended Kalman filter
    - Unscented Kalman filter
    - Particle filter
  - Adaptive Kalman filter
  - Robust Kalman filter

Summary

- Discrete-time Kalman filter
  - Prediction step (estimate propagation based on state equation)
  - Correction step (estimate update based on least-squares with a new measurement)
- One-step Kalman filter has an observer structure with a time-varying gain.
- Next,
  - Steady-state Kalman filter
  - Duality between LQR and Kalman filter
  - LQG