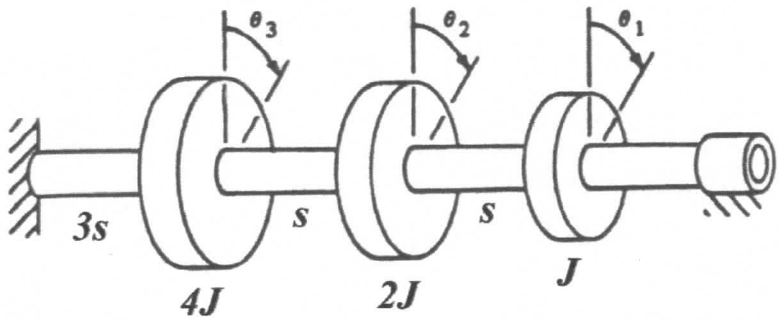
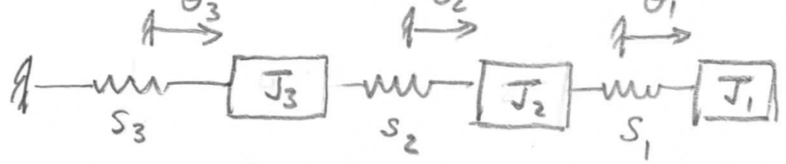
MECH 463 -- Homework 8

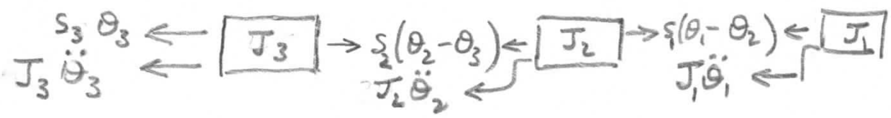
1. Three gear wheels are mounted on a flexible shaft. The left end of the shaft is fixed and the right end rotates freely in a journal bearing. The moments of inertia of the three gear wheels are respectively J, 2J and 4J. The torsional stiffnesses of the parts of the shaft between the gear wheels are respectively s, s and 3s. (Why are we not interested in the stiffness in the part of the shaft furthest to the right ?) Formulate the equations of motion of the system and determine the mass and stiffness matrices. Use Matlab routine `[V,D] = eig(A,B)` to evaluate the natural frequencies. Plot the mode shapes. Do they have the expected shapes ?



This is a thinly disguised version of our old friend:



From FBD:



$$J_1\ddot{\theta}_1 + s_1(\theta_1-\theta_2) = 0$$

$$J_2\ddot{\theta}_2 + s_2(\theta_2-\theta_3) - s_1(\theta_1-\theta_2) = 0$$

$$J_3\ddot{\theta}_3 + s_3\theta_3 - s_2(\theta_2-\theta_3) = 0$$

where $J_1 = J \quad J_2 = 2J \quad J_3 = 4J$

$s_1 = s \quad s_2 = s \quad s_3 = 3s$

$$\begin{bmatrix} J & 0 & 0 \\ 0 & 2J & 0 \\ 0 & 0 & 4J \end{bmatrix}\begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \\ \ddot{\theta}_3 \end{bmatrix} + \begin{bmatrix} s_1 & -s_1 & 0 \\ -s_1 & s_1+s_2 & -s_2 \\ 0 & -s_2 & s_2+s_3 \end{bmatrix}\begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \underset{\sim}{M}\ddot{\underline{\theta}} + \underset{\sim}{K}\ddot{\underline{\theta}} = \underline{0}$$

Use Matlab eig(K, M) to solve $\underset{\sim}{K}\underline{u} = \omega_n^2 M\underline{u}$

Eigenvalues are ω_n^2 and eigenvectors are mode shapes.

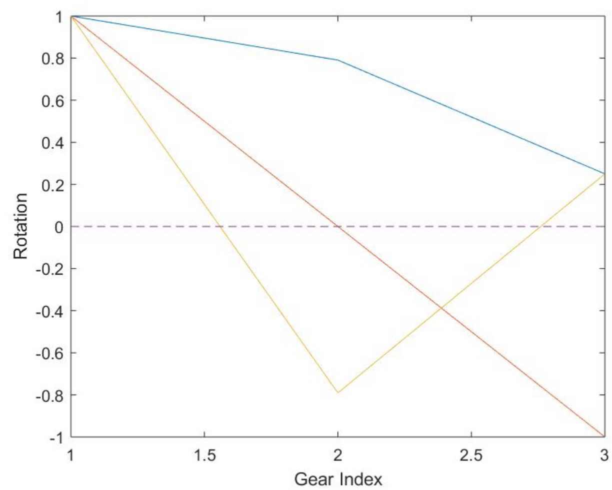
For solution $\underline{\theta} = C\underline{u}\cos\omega_n t \rightarrow \ddot{\underline{\theta}} = -\omega_n^2 C\underline{u}\cos_n t$

$$\rightarrow (-\omega_n^2\underset{\sim}{M} + \underset{\sim}{K})\underline{u}C\cos\omega_n t = \underline{0}$$

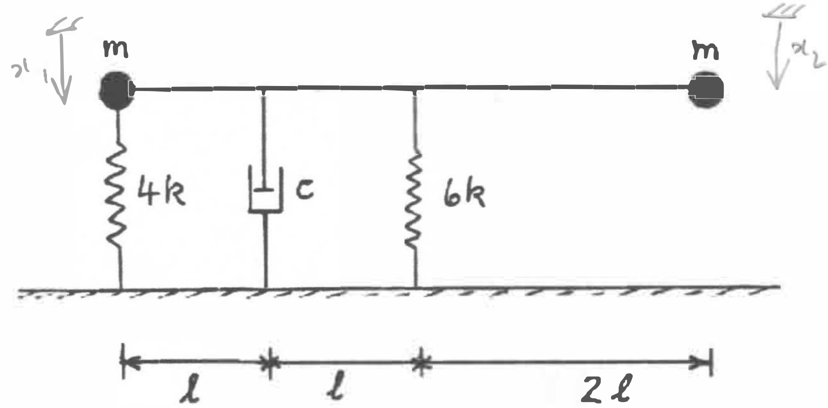
$$\rightarrow \underset{\sim}{K}\underline{u} = \omega_n^2\underset{\sim}{M}\underline{u} \leftarrow \text{use eig}(K, M) \text{ to solve}$$

```
% MECH 463 Homework 08 Q1
%
% Global variables:
% -----------------
% D         eigenvalue matrix
% J1,J2,J3  moments of inertia
% K         stiffness matrix
% M         mass matrix
% s1,s2,s3  torsional stiffnesses
% U         mode shape matrix
% V         eigenvector matrix
% wn        natural frequencies

clear all;
close all;

% Initialize quantities
J1 = 1;
J2 = 2;
J3 = 4;
s1 = 1;
s2 = 1;
s3 = 3;

% Assign mass and stiffness matrices
M = [J1 0 0; 0 J2 0; 0 0 J3];
K = [s1 -s1 0; -s1 s1+s2 -s2; 0 -s2 s2+s3];

% Solve generalized eigenvalue problem
[V,D] = eig(K,M);

% Extract the natural frequencies from V
wn = sqrt(diag(D));
disp(' ')
disp('Natural Frequencies')
disp(wn)

% Extract normalized mode shapes from D.
% Eigenvectors V are normalized so that the sum
% of squares of each column = 1.  This is OK,
% but to be consistent with our practice in
% class, we set the first element = 1
U = V ./ V(1,:);
disp('Mode Shapes (in columns)')
disp(U)

% Plot mode shapes
plot(U)
hold on
plot([0 0 0], '--')
xlabel('Gear Index')
ylabel('Rotation')
```

```
>> HW08_Q1

Natural Frequencies
    0.4576
    1.0000
    1.3381

Mode Shapes (in columns)
    1.0000    1.0000    1.0000
    0.7906    0.0000   -0.7906
    0.2500   -1.0000    0.2500
```

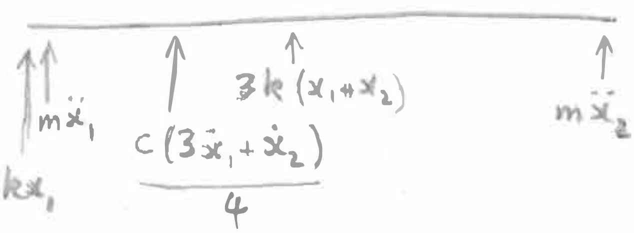


2. The diagram shows an idealized damped vibrating system. The rod supporting the two masses may be assumed to be rigid and to have negligible mass. Find the natural frequencies, damping factors and mode shapes of the system.



Rearrange the matrix equation of motion into double size matrix form and use Matlab routine `[V,D] = eig(A,B)` to evaluate the natural frequencies and damping factors. Confirm that the results are the same as found in Tutorial 8.

Take moments about the two masses:



$$\ell\left(4m\ddot{x}_1 + 16x_1 + \frac{3c(3\dot{x}_1+\dot{x}_2)}{4} + 6k(x_1+x_2)\right) = 0$$

$$\ell\left(4m\ddot{x}_1 + 6k(x_1+x_2) + \frac{c(3\dot{x}_1+\dot{x}_2)}{4}\right) = 0$$

In matrix form:

$$\begin{bmatrix} 4m & 0 \\ 0 & 4m \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} \frac{9}{4}c & \frac{3}{4}c \\ \frac{3}{4}c & \frac{1}{4}c \end{bmatrix}\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} + \begin{bmatrix} 22k & 6k \\ 6k & 6k \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Proceeding algebraically, the solution is:

$$\omega_{n1} = \sqrt{\frac{k}{m}} \qquad \zeta_1 = 0 \qquad \omega_{d1} = \sqrt{\frac{k}{m}} \qquad u_1 = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$$

$$\omega_{n2} = \sqrt{\frac{6k}{m}} = \underline{2.4495\sqrt{\frac{k}{m}}} \qquad \zeta_2 = \frac{5}{16}\cdot\frac{c}{\sqrt{6km}} = \underline{0.1276}$$

$$\omega_{d2} = \omega_{n2}\sqrt{1-\zeta^2} = \underline{2.4295}\sqrt{\frac{k}{m}} \qquad u_2 = \begin{bmatrix} 1 \\ \frac{1}{3} \end{bmatrix}$$

$$\left(\lambda \underbrace{\begin{bmatrix} \underset{\sim}{C} & \underset{\sim}{M} \\ \underset{\sim}{M} & \underset{\sim}{0} \end{bmatrix}}_{\underset{\sim}{B}} + \underbrace{\begin{bmatrix} \underset{\sim}{K} & \underset{\sim}{0} \\ \underset{\sim}{0} & -\underset{\sim}{M} \end{bmatrix}}_{\underset{\sim}{A}}\right)\underbrace{\begin{bmatrix} \underline{u} \\ \lambda\underline{u} \end{bmatrix}}_{\underline{v}} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \underline{\underset{\sim}{A}\underline{v} = -\lambda\underset{\sim}{B}\underline{v}}$$

```
% MECH 463 Homework 08 Q2
%
% Global variables:
% -----------------
% A         double-size stiffness matrix
% B         double-size mass matrix
% C         damping matrix
% D         eigenvalue matrix
% DD        diagonal of eigenvalue matrix
% I         ascending sequence of wd
% K         stiffness matrix
% M         mass matrix
% U         mode shape matrix
% V         double-size eigenvector matrix
% wd        damped natural frequencies
% wn        undamped natural frequencies

clear all;
close all;

% Assign matrices
M = [4 0; 0 4];
C = [2.25 0.75; 0.75 0.25];
K = [22 6; 6 6];

% Find undamped natural frequencies
% and arrange in order of increasing wn
% (necessary to correlate with wd)
[V,D] = eig(K,M);
wn = sqrt(diag(D));
wn = sort(wn);
disp(' ')
disp('Undamped Natural Frequencies')
disp(wn)

% Form double-size matrices
A(1:2,1:2) = K;
A(1:2,3:4) = zeros(2);
A(3:4,1:2) = zeros(2);
A(3:4,3:4) = -M;
B(1:2,1:2) = C;
B(1:2,3:4) = M;
B(3:4,1:2) = M;
B(3:4,3:4) = zeros(2);

% Solve generalized eigenvalue problem
[V,D] = eig(A,B);

% Eigenvalues come in complex conjugate pairs.
% Imaginary parts equal the damped nat. freq.
% Real parts equal damping ratio x undamped
nat. freq.

% Extract the damped natural freqs from D and
% arrange solution in order of increasing wd

disp('Complex Eigenvalues')
DD = -diag(D);
[DD,I] = sort(DD);
V = V(:,I);
disp(DD)
wd = [imag(DD(2)) imag(DD(4))]';
wd = sort(abs(wd));
disp('Damped Natural Frequencies')
disp(wd)

% Extract the damping factors from D
z = [real(DD(2)) real(DD(4))]' ./ wn;
disp('Damping factors')
disp(z)

% Extract normalized mode shapes from D.
% Eigenvectors V are normalized so that the sum
% of squares of each column = 1.  This is OK,
% but to be consistent with our practice in
% class, we set the first element = 1
V = V ./ V(1,:);
U = [V(1:2,2) V(1:2,4)];
disp('Mode Shapes (in columns)')
disp(U)
```

```
>> HW08_Q2

Undamped Natural Frequencies
    1.0000
    2.4495

Complex Eigenvalues
  -0.0000 - 1.0000i
  -0.0000 + 1.0000i
  -0.3125 - 2.4295i
  -0.3125 + 2.4295i

Damped Natural Frequencies
    1.0000
    2.4295

Damping factors
    0.0000
    0.1276

Mode Shapes (in columns)
   1.0000 + 0.0000i   1.0000 + 0.0000i
  -3.0000 + 0.0000i   0.3333 - 0.0000i
```