University of British Columbia
Department of Mechanical Engineering

MECH468 Modern Control Engineering
MECH522 Foundations in Control Engineering
Final exam

Examiner: Dr. Ryozo Nagamune
April 11 (Wednesday), 2018, noon-2:30pm

Last name, First name

Name: Student #:

Signature:

Exam policies

- Allowed: Lecture note print-outs, hand-written notes, homework, books.
- Not-allowed: PC, calculators, mobile phones.
- Write all your answers on the provided exam booklet.
- Motivate your answers properly. (No chance to defend your answers orally.)
- 100 points in total. Mark will be scaled later.

Before you start ...

- Use washroom before the exam.
- Turn off your mobile phone.
- No eating.
- Questions are NOT allowed.

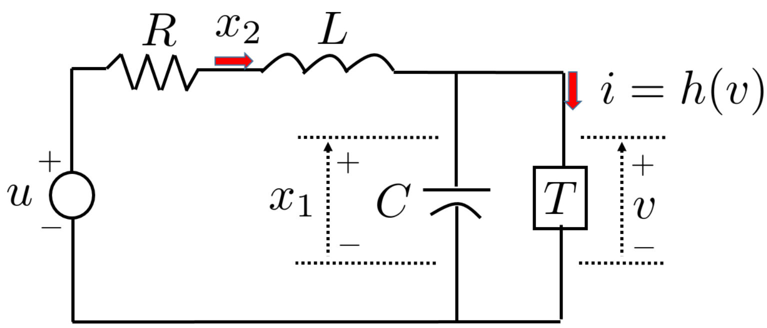
If you finish early ...

- If you would like to leave the room **before 2:20pm**, **raise your hand with this booklet**, and **wait at your seat** until the invigilator comes to you and collects your exam booklet.

To be filled in by the instructor/marker

Problem #	Mark	Full mark
1		20
2		20
3		20
4		20
5		20
Total		100

1. Consider the electric circuit depicted below. Here, the notations R, L and C respectively denote the resistance, inductance and capacitance, and u is the voltage source. An electrical element T has the characteristic $i = h(v)$, where i is the current through T and v is the voltage across T, and h is a nonlinear function which is differentiable with respect to v (i.e., $h'(v)$ exists).



(a) Let x_1 be the voltage across the capacitance, and x_2 be the current through the inductance. Prove that the state equation for this system is described as follows. (10pt)

$$\begin{aligned}\dot{x}_1(t) &= -\frac{1}{C}h(x_1(t)) + \frac{1}{C}x_2(t)\\ \dot{x}_2(t) &= -\frac{1}{L}x_1(t) - \frac{R}{L}x_2(t) + \frac{1}{L}u(t)\end{aligned}$$

(b) Linearize the state equation above around the operating point $(x_1, x_2, u) = (x_{10}, x_{20}, u_0)$. (6pt)

(c) Express x_{20} and u_0 as functions of x_{10}. (4pt)

<u>**Solution**</u>

(a) By Kirchhoff's current and voltage laws, we have

$$\begin{aligned}x_2 &= C\dot{x}_1 + h(x_1)\\ u &= Rx_2 + L\dot{x}_2 + x_1\end{aligned}$$

By manipulating these equations, we can reach the state equation.

(b) By introducing the deviation variables

$$\delta x_1 := x_1 - x_{10},\ \delta x_2 := x_2 - x_{20},\ u := u - u_0,$$

the linearized state equation can be written as

$$\begin{aligned}\dot{\delta x}_1(t) &= -\frac{1}{C}h'(x_{10})\delta x_1(t) + \frac{1}{C}\delta x_2(t)\\ \dot{\delta x}_2(t) &= -\frac{1}{L}\delta x_1(t) - \frac{R}{L}\delta x_2(t) + \frac{1}{L}\delta u(t).\end{aligned}$$

It can also be written in a matrix form as

$$\frac{d}{dt}\begin{bmatrix}\delta x_1\\ \delta x_2\end{bmatrix} = \begin{bmatrix}-\frac{h'(x_{10})}{C} & \frac{1}{C}\\ -\frac{1}{L} & -\frac{R}{L}\end{bmatrix}\begin{bmatrix}\delta x_1\\ \delta x_2\end{bmatrix} + \begin{bmatrix}0\\ \frac{1}{L}\end{bmatrix}u$$

(c) In the nonlinear state equation, we set the derivative terms to be zero. Then, we have

$$x_{20} = h(x_{10}), \quad u_0 = x_{10} + Rx_{20} = x_{10} + Rh(x_{10}).$$

2. Obtain minimal realizations of the following transfer functions. After obtaining minimal realizations, check if the realization is indeed minimal. (10pt-each)

(a) $G_1(s) = \begin{bmatrix} \dfrac{s^2+3s+2}{s^2+2s+1} \\ \dfrac{1}{s+2} \end{bmatrix}$

(b) $G_2(s) = \begin{bmatrix} \frac{1}{s} & \frac{4}{s} \\ \frac{2}{s} & \frac{8}{s} \end{bmatrix}$

<u>**Solution**</u>

(a)

$$G_1(s) = \begin{bmatrix} \dfrac{s^2+3s+2}{s^2+2s+1} \\ \dfrac{1}{s+2} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} \dfrac{s+1}{(s+1)^2} \\ \dfrac{1}{s+2} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \frac{1}{s^2+3s+2} \begin{bmatrix} s+2 \\ s+1 \end{bmatrix}$$

Using the controllable canonical realization, we have

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \left[\begin{array}{cc|c} 0 & 1 & 0 \\ -2 & -3 & 1 \\ \hline 2 & 1 & 1 \\ 1 & 1 & 0 \end{array}\right]$$

This realization is observable because the matrix C is full column rank, and thus the observability matrix $\mathcal{O}$ is also full column rank. Therefore, this realization is indeed minimal.

(b)

$$G_2(s) = \begin{bmatrix} \frac{1}{s} & \frac{4}{s} \\ \frac{2}{s} & \frac{8}{s} \end{bmatrix} = \frac{1}{s} \begin{bmatrix} 1 & 4 \\ 2 & 8 \end{bmatrix}$$

Using the observable canonical realization, we have

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \left[\begin{array}{cc|cc} 0 & 0 & 1 & 4 \\ 0 & 0 & 2 & 8 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right]$$

This realization is not controllable because the controllability matrix

$$\mathcal{C} = \begin{bmatrix} 1 & 4 & 0 & 0 \\ 2 & 8 & 0 & 0 \end{bmatrix}$$

is of rank 1, i.e., not full row rank. To obtain the minimal realization, we take the decomposition for controllability.

$$T^{-1} := \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \Rightarrow T = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$$

$$TAT^{-1} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \; TB = \begin{bmatrix} 1 & 4 \\ 0 & 0 \end{bmatrix}, \; CT^{-1} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$$

By eliminating the uncontrollable part, we can obtain the minimal realization as

$$\begin{bmatrix} A_{co} & B_{co} \\ C_{co} & D \end{bmatrix} = \left[\begin{array}{c|cc} 0 & 1 & 4 \\ \hline 1 & 0 & 0 \\ 2 & 0 & 0 \end{array}\right]$$

One can verify that this realization is indeed minimal, because it is controllable and observable.

3. For the following state-space model, design an observer-based state-feedback controller. For the controller design, select the pole locations so that (20pt)

- state estimation error converges to zero in about 0.4 second, and
- (2%) settling time for initial condition excitation becomes about 1 second.

$$\begin{aligned} \dot{x}(t) &= \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \\ y(t) &= \begin{bmatrix} 0 & 1 \end{bmatrix} x(t) \end{aligned}$$

Solution: To satisfy the requirements, we specify the pole locations as

$$\frac{4}{|\mathrm{Re}|} = 1 \Rightarrow \mathrm{eig}(A - BK) = \{-4, -4\}$$

$$\frac{4}{|\mathrm{Re}|} = 0.4 \Rightarrow \mathrm{eig}(A - LC) = \{-10, -10\}$$

Using the direct method, we can obtain matrices K and L as follows.

$$\begin{aligned} \det(sI - (A - BK)) &= \det \begin{bmatrix} s & -1 \\ -(1 - k_1) & s - (-1 - k_2) \end{bmatrix} \\ &= s^2 + (1 + k_2)s + k_1 - 1 \\ &= s^2 + 8s + 16 \\ K &= \begin{bmatrix} 17 & 7 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} \det(sI - (A - LC)) &= \det \begin{bmatrix} s & -(1 - \ell_1) \\ -1 & s - (-1 - \ell_2) \end{bmatrix} \\ &= s^2 + (1 + \ell_2)s + \ell_1 - 1 \\ &= s^2 + 20s + 100 \\ L &= \begin{bmatrix} 101 \\ 19 \end{bmatrix} \end{aligned}$$

The observer equation is

$$\dot{\hat{x}} = A\hat{x} + Bu + L(y - C\hat{x}),$$

and the state (estimate) feedback controller equation is

$$u = -K\hat{x}.$$

(The structure of the observer-based state-feedback controller is given in the lecture slide, and thus omitted here.)

4. Determine whether each statement is True or False.

   - If your answer is 'True', provide an explanation to support your answer.
   - If your answer is 'False', provide a counter-example with two states, with an explanation, to support your answer. In counter-examples, use non-zero B-matrix and non-zero C-matrix.

   One example is given below. (10pt-each)

   Example If a linear time-invariant system is stable, then it is controllable.
   Answer False
   Counter-example

$$\dot{x} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u$$

   Explanation This system is stable because the eigenvalues of A-matrix are -1 and -1, both of which are in the open left-half plane. However, it is not controllable because the rank of the controllability matrix $\mathcal{C} = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$ is one, i.e., $\mathcal{C}$ is not full rank.

   (a) If a linear time-invariant system is unstable, it is not controllable.
   Solution
   Answer False
   Counter-example

$$\dot{x} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$

   Explanation This system is unstable because the eigenvalues of A-matrix are 1 and 1, which are in the open right-half plane. However, it is controllable because the controllability matrix $\mathcal{C} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ has full rank.

   (b) If a linear time-invariant system is detectable, it is observable.
   Solution

Answer False
Counter-example

$$\dot{x} = \begin{bmatrix} -1 & 0 \\ 1 & -1 \end{bmatrix} x, \; y = \begin{bmatrix} 1 & 0 \end{bmatrix} x$$

Explanation This system is detectable because it is stable. However, it is not observable because the observability matrix $\mathcal{O} = \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}$ does not have full rank.

5. Consider the following continuous-time infinite-horizon linear quadratic regulator (LQR) problem, where α is a positive constant.

$$\min_{u(\cdot)} \int_0^\infty \left\{ \alpha x_2(t)^2 + u_1(t)^2 + u_2(t)^2 \right\} dt$$

$$\text{subject to} \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}$$

(a) Design the LQR control law. (10pt)

(b) Prove that the designed LQR control law stabilizes closed-loop system for any $\alpha > 0$. (5pt)

(c) For the closed-loop system with the designed LQR control law and $\alpha = 3$, draw the state trajectory in (x_1, x_2)-plane when the initial state is $x(0) = (1, 1)$. (**Hint:** The state trajectory must converge to $(0, 0)$ (i.e., origin of the (x_1, x_2)-plane because the closed-loop system is stable.) (5pt)

<u>**Solution**</u>

(a) By noting that

$$Q = \begin{bmatrix} 0 & 0 \\ 0 & \alpha \end{bmatrix}, \; R = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

Algebraic Riccati Equation becomes

$$\underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}_{A^T} \underbrace{\begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix}}_{P} + \underbrace{\begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix}}_{P} \underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}_{A} + \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & \alpha \end{bmatrix}}_{Q} - \underbrace{\begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix}}_{P} \underbrace{I_2}_{BR^{-1}B^T} \underbrace{\begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix}}_{P} = 0$$

We get the following equations.

$$\begin{aligned}
&(1,1) \qquad p_2 + p_2 - (p_1^2 + p_2^2) = 0 \\
&(1,2) \qquad p_3 + p_1 - p_2(p_1 + p_3) = 0 \Rightarrow (1 - p_2)(p_1 + p_3) = 0 \\
&(2,2) \qquad p_2 + p_2 + \alpha - (p_2^2 + p_3^2) = 0
\end{aligned}$$

Since P is positive definite, it must be $p_1 > 0$ and $p_3 > 0$, i.e., $p_1 + p_3 > 0$. Thus, from the second equation, we have $p_2 = 1$. Then, from the first and third equations, we have $p_1 = 1$ and $p_3 = \sqrt{1 + \alpha}$.

The LQR optimal control law is

$$u = -R^{-1}B^TPx = -\left[\begin{array}{cc} 1 & 1 \\ 1 & \sqrt{1+\alpha} \end{array}\right] x$$

(b) The closed-loop A-matrix is

$$A - R^{-1}B^TP = \left[\begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array}\right] - \left[\begin{array}{cc} 1 & 1 \\ 1 & \sqrt{1+\alpha} \end{array}\right] = \left[\begin{array}{cc} -1 & 0 \\ 0 & -\sqrt{1+\alpha} \end{array}\right].$$

This matrix has negative eigenvalues. Therefore, the closed-loop system is stable for any $\alpha > 0$.

(c) When $\alpha = 3$, the closed-loop system equation is

$$\dot{x} = \left[\begin{array}{cc} -1 & 0 \\ 0 & -2 \end{array}\right] x \Rightarrow \left\{\begin{array}{l} x_1(t) = e^{-t}x_{10} \\ x_2(t) = e^{-2t}x_{20} \end{array}\right.$$

Thus,

$$x_2(t) = \left(\underbrace{\frac{x_1(t)}{x_{10}}}_{e^{-t}}\right)^2 x_{20}$$

When $(x_{10}, x_{20}) = (1, 1)$, the state trajectory will satisfy the equation

$$x_2(t) = x_1(t)^2.$$

Just draw a parabolic curve $x_2 = x_1^2$ from $(1, 1)$ to $(0, 0)$, with an arrow converging to the origin.

— — — — [END OF FINAL EXAM] — — — —