Vectors, Frames & Coordinates

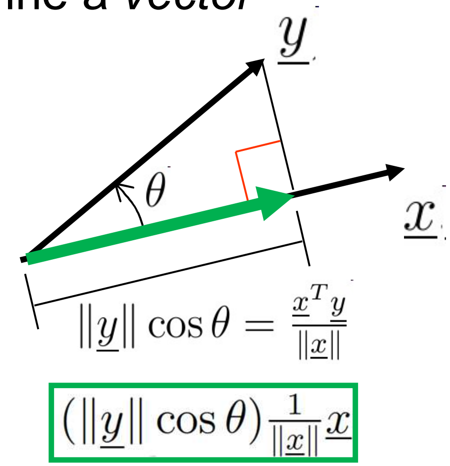
$$\underline{x}, \quad \underline{\mathbf{x}}, \quad \vec{x}, \quad \vec{\mathbf{x}}, \quad \mathbf{x}$$

Vector $\underline{x}$: A physical displacement in space or other physical vector (velocity, force, etc). Vectors can be added and scaled. The properties of vector addition and multiplication with a scalar define a *vector space.*

Scalar or dot product between two vectors:

$$\underline{x} \cdot \underline{y} = <\underline{x}, \underline{y}> = \underline{x}^T \underline{y} = \|\underline{x}\| \|\underline{y}\| \cos \angle(\underline{x}, \underline{y})$$

$$\|\underline{x}\| = \sqrt{\underline{x}^T \underline{x}}$$

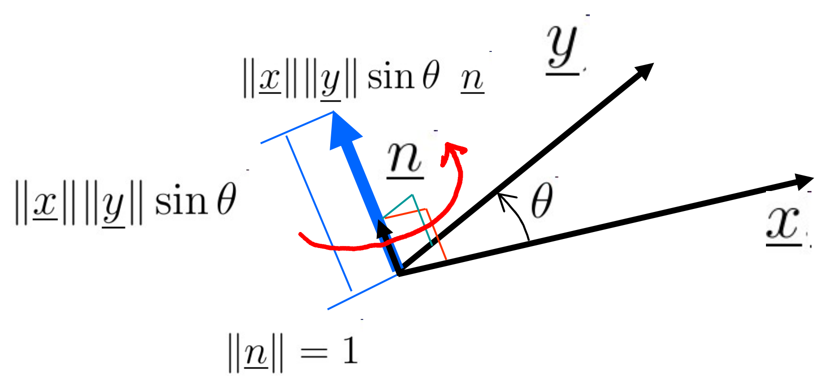


Vector product between two vectors:

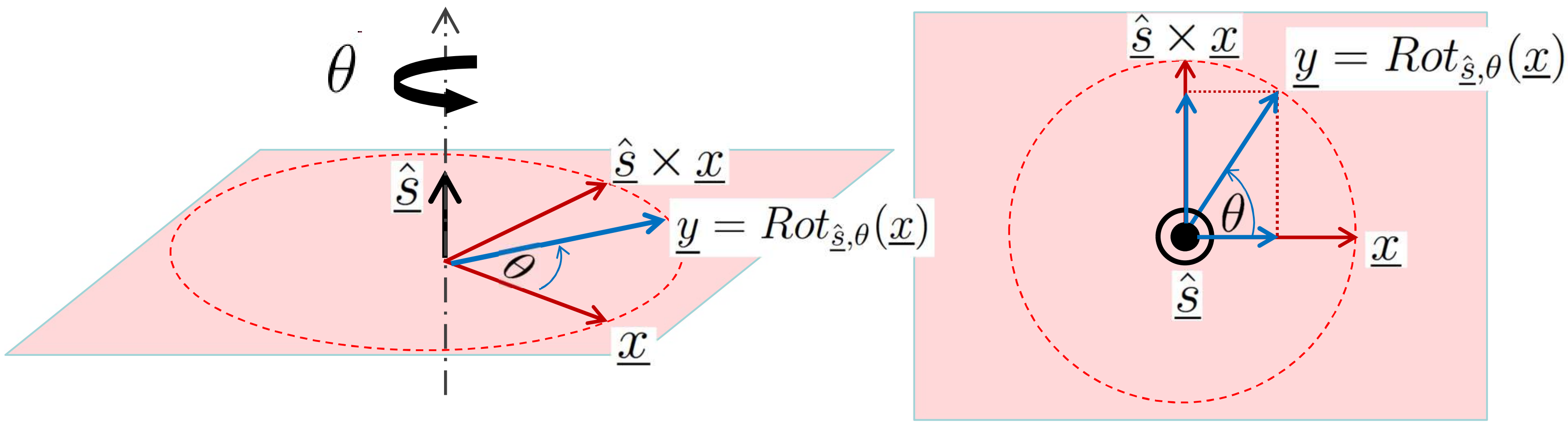
$$\underline{x} \times \underline{y} = \|\underline{x}\| \|\underline{y}\| \sin \angle(\underline{x}, \underline{y}) \;\; \underline{n}$$

$$\underline{y} \times \underline{x} = -\underline{x} \times \underline{y}$$

$$\underline{x} \times \underline{x} = \underline{0}$$



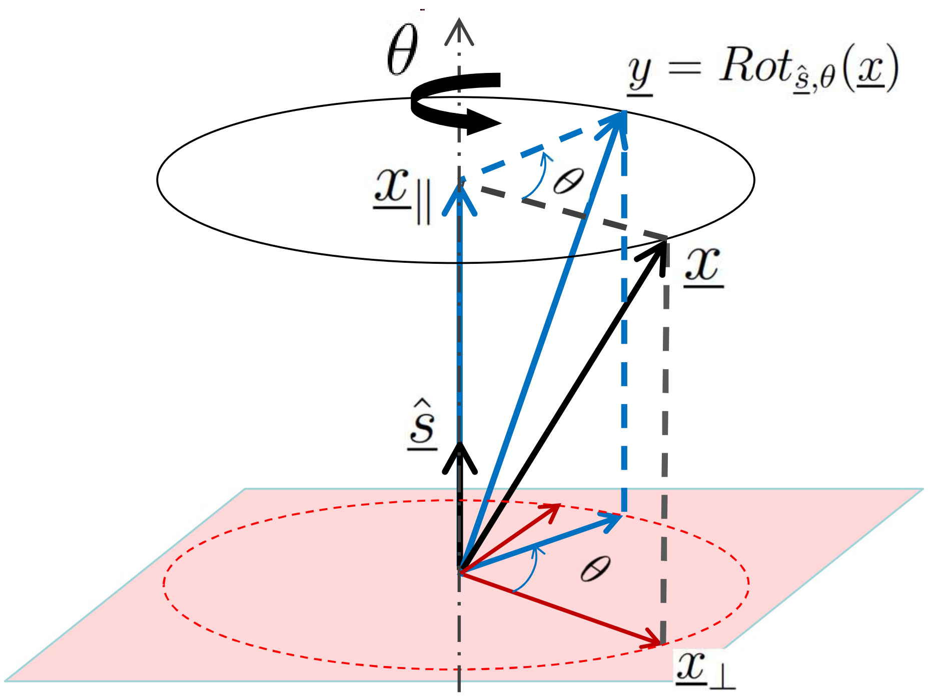
Rotation of a vector – first in a plane

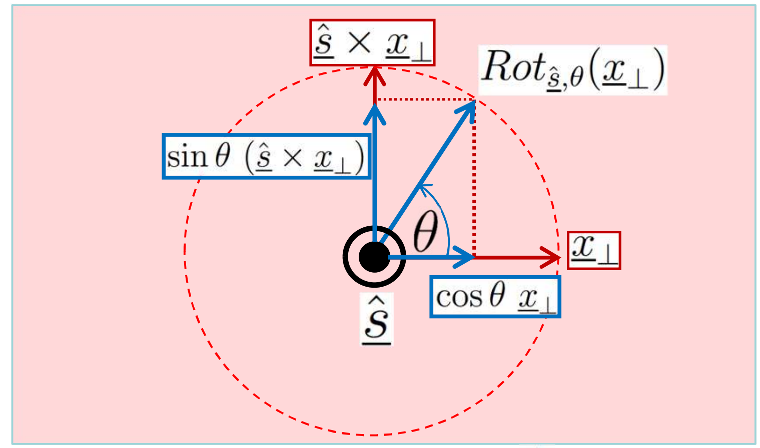


$$\|\underline{\hat{s}} \times \underline{x}\| = \|\underline{x}\|$$

$$\underline{y} = \underline{x}\cos\theta + (\underline{\hat{s}} \times \underline{x})\sin\theta$$

Rotation in 3D – vector rotation formula & Rodrigues rotation formula





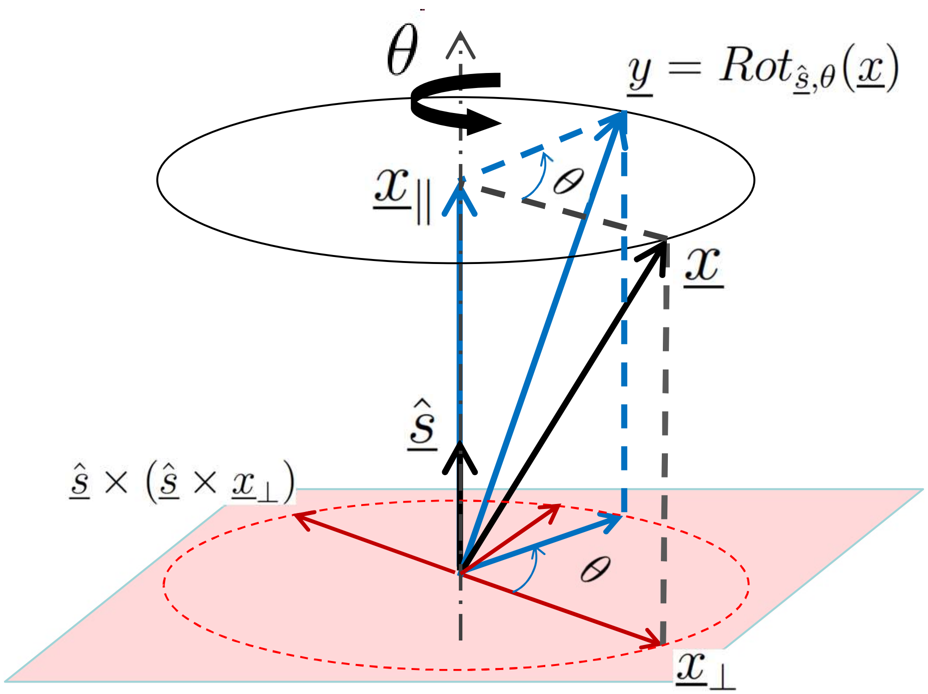
View from the tip of $\hat{\underline{s}}$

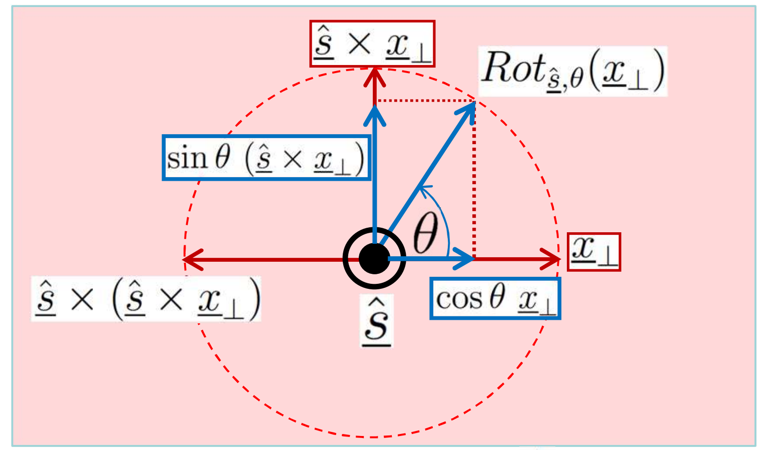
$$\underline{y} = Rot_{\hat{\underline{s}},\theta}(\underline{x}) = Rot_{\hat{\underline{s}},\theta}(\underline{x}_{||} + \underline{x}_\perp)$$
$$= Rot_{\hat{\underline{s}},\theta}(\underline{x}_{||}) + Rot_{\hat{\underline{s}},\theta}(\underline{x}_\perp)$$

$$\boxed{Rot_{\hat{\underline{s}},\theta}(\underline{x}_{||} + \underline{x}_\perp) = \underline{x}_{||} + \cos\theta\ \underline{x}_\perp + \sin\theta\ (\hat{\underline{s}} \times \underline{x}_\perp)}$$

vector rotation formula

Rotation in 3D – Rodrigues Rotation Formula





View from the tip of $\underline{\hat{s}}$

$$\underline{y} = Rot_{\underline{\hat{s}},\theta}(\underline{x}) = \underline{x}_{||} + \cos\theta\ \underline{x}_{\perp} + \sin\theta\ (\underline{\hat{s}} \times \underline{x}_{\perp})$$

$$\underline{x}_{||} = \underline{x} - \underline{x}_{\perp}$$

$$= \underline{x} + (\cos\theta - 1)\underline{x}_{\perp} + \sin\theta\ (\underline{\hat{s}} \times \underline{x})$$

$$\underline{\hat{s}} \times (\underline{\hat{s}} \times \underline{x}_{\perp}) = -\underline{x}_{\perp}$$

$$\underline{\hat{s}} \times \underline{x}_{\perp} = \underline{\hat{s}} \times (\underline{x}_{||} + \underline{x}_{\perp})$$

$$= \underline{\hat{s}} \times \underline{x}$$

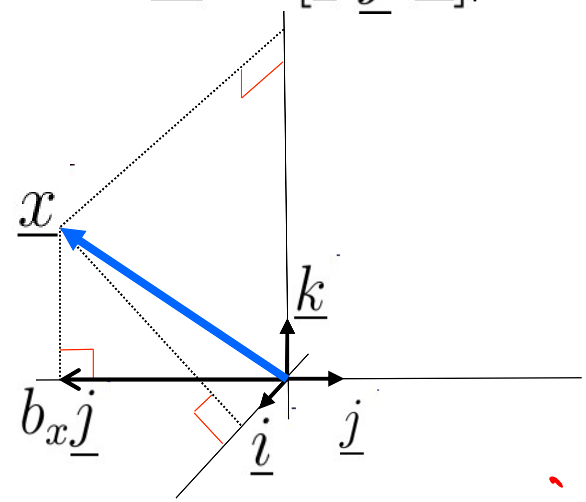
$$\boxed{Rot_{\underline{\hat{s}},\theta}(\underline{x}) = \underline{x} + \sin\theta\ \underline{\hat{s}} \times \underline{x} + (1 - \cos\theta)\ \underline{\hat{s}} \times (\underline{\hat{s}} \times \underline{x})}$$

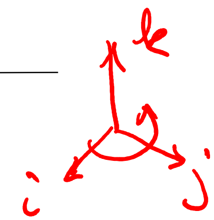
Rodrigues rotation formula

Vectors in 3D are usually specified in terms of their projections onto a right-handed orthonormal *basis* or *frame*

$$\underline{C} \triangleq [\underline{i}\ \underline{j}\ \underline{k}]$$

$$\underline{x} = a_x \underline{i} + b_x \underline{j} + c_x \underline{k}$$





where *right-handed orthonormal* means that the following hold:

$$\begin{bmatrix} \underline{i}^T \underline{i} & \underline{i}^T \underline{j} & \underline{i}^T \underline{k} \\ \underline{j}^T \underline{i} & \underline{j}^T \underline{j} & \underline{j}^T \underline{k} \\ \underline{k}^T \underline{i} & \underline{k}^T \underline{j} & \underline{k}^T \underline{k} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\underline{i} \times \underline{j} = \underline{k} \quad \underline{k} \times \underline{i} = \underline{j} \quad \underline{j} \times \underline{k} = \underline{i}$$

$a_x,\ b_x,\ c_x$ are the *coordinates* of $\underline{x}$ in frame $\underline{C}$ and we write:

$$\underline{x} = a_x \underline{i} + b_x \underline{j} + c_x \underline{k} = \begin{bmatrix} \underline{i} & \underline{j} & \underline{k} \end{bmatrix} \begin{bmatrix} a_x \\ b_x \\ c_x \end{bmatrix} = \underline{C} x$$

For the scalar and vector product this means:

$$\underline{C} \triangleq [\underline{i}\ \underline{j}\ \underline{k}] \qquad \underline{x} = a_x\underline{i} + b_x\underline{j} + c_x\underline{k} \qquad \underline{y} = a_y\underline{i} + b_y\underline{j} + c_y\underline{k}$$

$$(\underline{C}x)^T(\underline{C}y) = x^T \underline{C}^T\underline{C}\, y \quad ; \quad \underline{C}^T\underline{C} = \text{identity}$$

$$\underline{x}^T\underline{y} = (\underline{C}x)^T(\underline{C}y) \triangleq x^T y \triangleq a_x a_y + b_x b_y + c_x c_y$$

$$\underline{x} = (a_x\underline{i} + b_x\underline{j} + c_x\underline{k}) \times (a_y\underline{i} + b_y\underline{j} + c_y\underline{k})$$

$$\underline{i} \times \underline{j} = \underline{k} \quad ; \quad \underline{k} \times \underline{i} = \underline{j} \quad ; \quad \underline{j} \times \underline{k} = \underline{i}$$

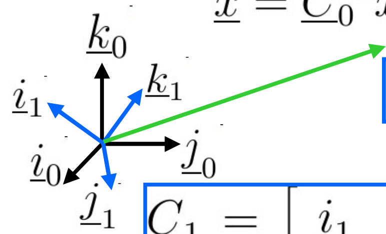
$$\underline{i} \times \underline{i} = \underline{j} \times \underline{j} = \underline{k} \times \underline{k} = \underline{0}$$

$$\underline{x} \times \underline{y} = (b_x c_y - c_x b_y)\underline{i} + (c_x a_y - a_x c_y)\underline{j} + (a_x b_y - b_x a_y)\underline{k}$$

$$= \underline{C}\begin{bmatrix} b_x c_y - c_x b_y \\ c_x a_y - a_x c_y \\ a_x b_y - b_x a_y \end{bmatrix} \left(= \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ a_x & b_x & c_x \\ a_y & b_y & c_y \end{vmatrix} \right) = \underline{C}\begin{bmatrix} 0 & -c_x & b_x \\ c_x & 0 & -a_x \\ -b_x & a_x & 0 \end{bmatrix}\begin{bmatrix} a_y \\ b_y \\ c_y \end{bmatrix}$$

Changing coordinates:

$$\underline{x} = \underline{C}_0\,{}^0x$$

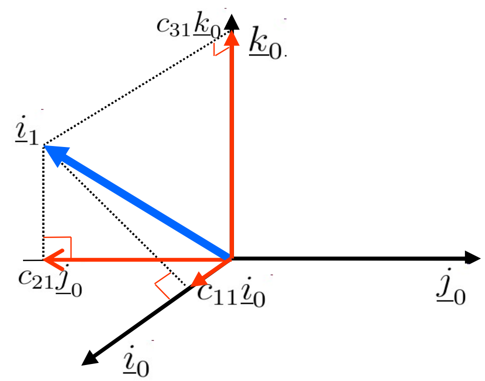


$$\underline{x} = \underline{C}_0\,{}^0x$$

$$\boxed{\underline{x} = \underline{C}_1\,{}^1x}$$

$$\underline{C}_0 = \begin{bmatrix} \underline{i}_0 & \underline{j}_0 & \underline{k}_0 \end{bmatrix}$$

$$\boxed{\underline{C}_1 = \begin{bmatrix} \underline{i}_1 & \underline{j}_1 & \underline{k}_1 \end{bmatrix}}$$



$$\underline{i}_1 = \underline{i}_0 c_{11} + \underline{j}_0 c_{21} + \underline{k}_0 c_{31} = [\underline{i}_0\ \underline{j}_0\ \underline{k}_0] \begin{bmatrix} c_{11} \\ c_{21} \\ c_{31} \end{bmatrix}$$

$$\underline{j}_1 = \underline{i}_0 c_{12} + \underline{j}_0 c_{22} + \underline{k}_0 c_{32}$$

$$\underline{k}_1 = \underline{i}_0 c_{13} + \underline{j}_0 c_{23} + \underline{k}_0 c_{33}$$

$$\underline{C}_1 = \begin{bmatrix} \underline{i}_1 & \underline{j}_1 & \underline{k}_1 \end{bmatrix} = \begin{bmatrix} \underline{i}_0 & \underline{j}_0 & \underline{k}_0 \end{bmatrix} \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix} \triangleq \underline{C}_0\,{}^0C_1$$

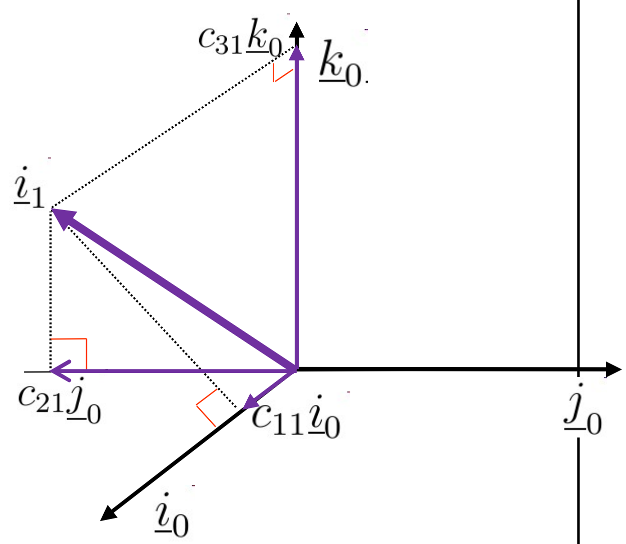
The columns of ${}^0C_1$ are the coordinates of the old frame $\underline{C}_1$ with respect to the new one $\underline{C}_0$

$${}^0C_1 = \begin{bmatrix} \underline{i}_0^T \\ \underline{j}_0^T \\ \underline{k}_0^T \end{bmatrix} \begin{bmatrix} \underline{i}_1 & \underline{j}_1 & \underline{k}_1 \end{bmatrix} = \begin{bmatrix} \underline{i}_0^T\underline{i}_1 & \underline{i}_0^T\underline{j}_1 & \underline{i}_0^T\underline{k}_1 \\ \underline{j}_0^T\underline{i}_1 & \underline{j}_0^T\underline{j}_1 & \underline{j}_0^T\underline{k}_1 \\ \underline{k}_0^T\underline{i}_1 & \underline{k}_0^T\underline{j}_1 & \underline{k}_0^T\underline{k}_1 \end{bmatrix}$$

$$\underline{C}_0\,{}^0x = \underline{x} = \underline{C}_1\,{}^1x = \underline{C}_0\,{}^0C_1\,{}^1x$$

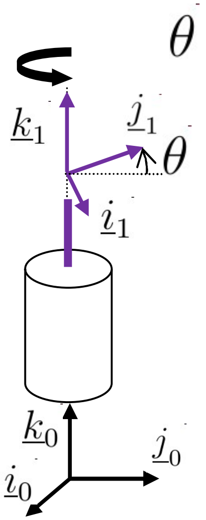
$$\boxed{{}^0x = {}^0C_1\,{}^1x}$$

Changing coordinates – rotation about $\underline{k}_0$:

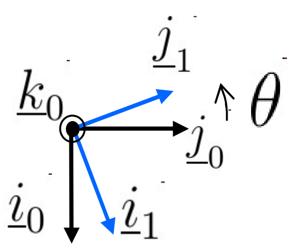


$$\underline{C}_0 = \begin{bmatrix} \underline{i}_0 & \underline{j}_0 & \underline{k}_0 \end{bmatrix} \qquad \underline{C}_1 = \begin{bmatrix} \underline{i}_1 & \underline{j}_1 & \underline{k}_1 \end{bmatrix}$$

$$\begin{aligned} \underline{i}_1 &= \underline{i}_0 c_{11} + \underline{j}_0 c_{21} + \underline{k}_0 c_{31} \\ \underline{j}_1 &= \underline{i}_0 c_{12} + \underline{j}_0 c_{22} + \underline{k}_0 c_{32} \\ \underline{k}_1 &= \underline{i}_0 c_{13} + \underline{j}_0 c_{23} + \underline{k}_0 c_{33} \end{aligned}$$



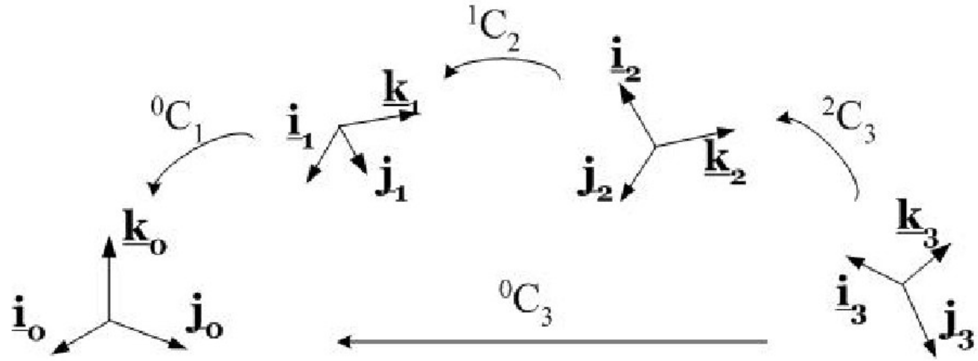
$$\begin{aligned} \underline{i}_1 &= \underline{i}_0 \cos\theta + \underline{j}_0 \sin\theta + \underline{k}_0 0 \\ \underline{j}_1 &= \underline{i}_0 (-\sin\theta) + \underline{j}_0 \cos\theta + \underline{k}_0 0 \\ \underline{k}_1 &= \underline{i}_0 0 + \underline{j}_0 0 + \underline{k}_0 1 \end{aligned}$$



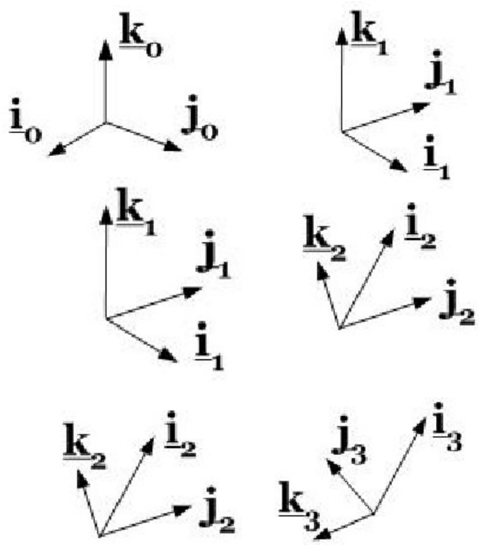
$${}^0C_1 = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \triangleq R(k, \theta)$$

Sequence of Frames

$$
\begin{aligned}
\underline{C}_3 &= \underline{C}_2\,{}^2C_3 \\
&= \underline{C}_1\,{}^1C_2\,{}^2C_3 = \\
&= \underline{C}_0\,{}^0C_1\,{}^1C_2\,{}^2C_3 \\
&= \underline{C}_0\,{}^0C_3
\end{aligned}
$$



Euler Angles: A method of describing the relative orientation of one frame w.r.t. another as a sequence of 3 rotations about the principal axes of the **moving frame**, e.g., Z-Y-X Euler Angles:

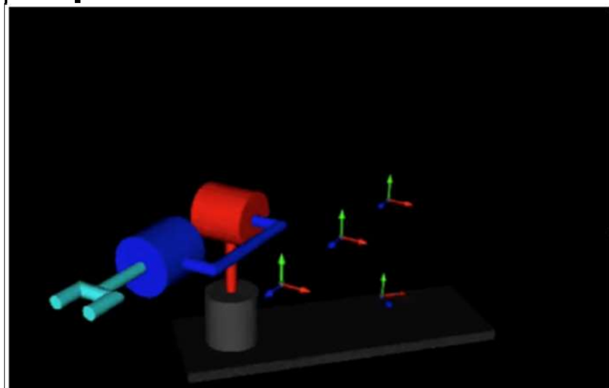


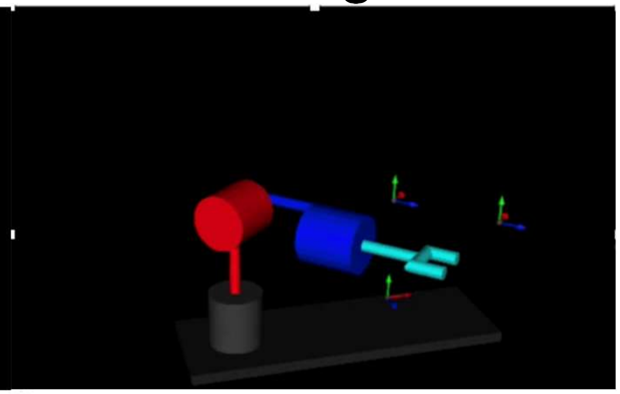
$$
{}^0C_1 \;=\; R(k,\theta_1) = \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 & 0 \\ \sin\theta_1 & \cos\theta_1 & 0 \\ 0 & 0 & 1 \end{bmatrix}
$$

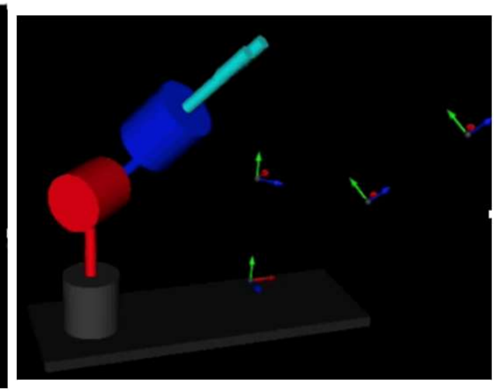
$$
{}^1C_2 \;=\; R(j,\theta_2) = \begin{bmatrix} \cos\theta_2 & 0 & \sin\theta_2 \\ 0 & 1 & 0 \\ -\sin\theta_2 & 0 & \cos\theta_2 \end{bmatrix}
$$

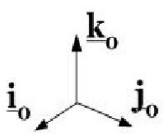
$$
{}^2C_3 \;=\; R(i,\theta_3) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_3 & -\sin\theta_3 \\ 0 & \sin\theta_3 & \cos\theta_3 \end{bmatrix}
$$

Spherical wrist – Z-Y-X Euler angles

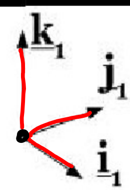




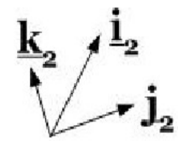




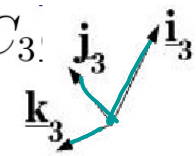
$$\underline{C}_1 = \underline{C}_0 \, {}^0C_1$$

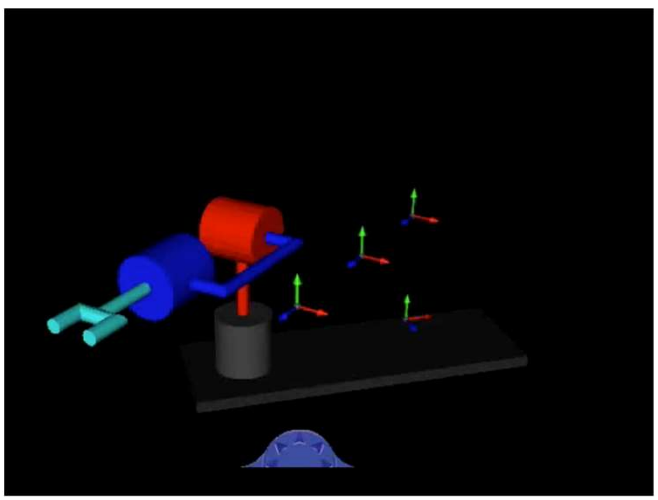


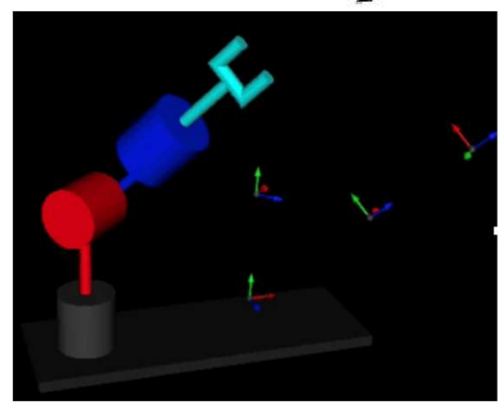
$$\underline{C}_2 = \underline{C}_1 \, {}^1C_2$$



$$\underline{C}_3 = \underline{C}_2 \, {}^2C_3$$







$$\begin{aligned} \underline{C}_3 &= \underline{C}_2 \, {}^2C_3 \\ &= \underline{C}_1 \, {}^1C_2 \, {}^2C_3 \\ &= \underline{C}_0 \, {}^0C_1 \, {}^1C_2 \, {}^2C_3 \end{aligned}$$

Coordinate invariance of scalar product and vector product

It is important to distinguish between physical *vectors* and physical transformations and their *coordinate representations.*

Coordinates are vectors themselves (the set of 3-tuples with component-wise addition and scaling), but there are many coordinates that can represent the same physical vector. Physical transformations of physical vectors should be independent of coordinate choice!

In particular, the scalar product is coordinate invariant:

$$ {}^0x^T\,{}^0y = [\,{}^0C_1\,{}^1x]^T[\,{}^0C_1\,{}^1y] = {}^1x^T\,{}^0C_1^T\,{}^0C_1\,{}^1y = {}^1x^T\,{}^1y $$

$$ \underline{x}^T\underline{y} = {}^0x^T\,{}^0y = {}^1x^T\,{}^1y $$

and the vector product is coordinate invariant:

$$ {}^0x \times {}^0y = [\,{}^0C_1\,{}^1x] \times [\,{}^0C_1\,{}^1y] = {}^0C_1[\,{}^1x \times {}^1y] $$

$$ \underline{x} \times \underline{y} = \underline{C}_0(\,{}^0x \times {}^0y) = \underline{C}_1(\,{}^1x \times {}^1y\,) $$

Matrix representations of linear functions

$$\underline{y} = \underline{\underline{f}}(\underline{x}) = \underline{\underline{A}}\ \underline{x}$$

Consider a linear function acting on 3D vectors, e.g. vector product, rotation, reflection:

$$\underline{y} = \underline{\underline{f}}(\underline{x}) = \underline{\underline{A}}\ \underline{x}$$

$$\underline{x} = \underline{C}x \qquad\qquad \underline{y} = \underline{C}y$$

$$\begin{tikzcd} \{\text{Vectors}\} \arrow[r, "\underline{\underline{A}}"] & \{\text{Vectors}\} \\ \{\text{Coordinates}\} \arrow[u, "\underline{C}"] \arrow[r, "A"'] & \{\text{Coordinates}\} \arrow[u, "\underline{C}"'] \end{tikzcd}$$

$$\underline{C} = [\underline{i}\ \ \underline{j}\ \ \underline{k}]$$

$$x \qquad\qquad y = Ax$$

Then $\underline{\underline{A}}$ has a matrix representation A:

$$\underline{y} = \underline{\underline{f}}(a_x\underline{i} + b_x\underline{j} + c_x\underline{k}) = a_x\underline{\underline{f}}(\underline{i}) + b_x\underline{\underline{f}}(\underline{j}) + c_x\underline{\underline{f}}(\underline{k}) = [\underline{\underline{f}}(\underline{i})\ \ \underline{\underline{f}}(\underline{j})\ \ \underline{\underline{f}}(\underline{k})]\begin{bmatrix} a_x \\ b_x \\ c_x \end{bmatrix}$$

$$\underline{\underline{f}}(\underline{i}) = a_{11}\underline{i} + a_{21}\underline{j} + a_{31}\underline{k} = [\underline{i}\ \ \underline{j}\ \ \underline{k}]\begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \end{bmatrix}$$

$$\underline{\underline{f}}(\underline{j}) = a_{12}\underline{i} + a_{22}\underline{j} + a_{32}\underline{k} = [\underline{i}\ \ \underline{j}\ \ \underline{k}]\begin{bmatrix} a_{12} \\ a_{22} \\ a_{32} \end{bmatrix}$$

$$\underline{\underline{f}}(\underline{k}) = a_{13}\underline{i} + a_{23}\underline{j} + a_{33}\underline{k} = [\underline{i}\ \ \underline{j}\ \ \underline{k}]\begin{bmatrix} a_{13} \\ a_{23} \\ a_{33} \end{bmatrix}$$

$$\Rightarrow [\underline{\underline{f}}(\underline{i})\ \ \underline{\underline{f}}(\underline{j})\ \ \underline{\underline{f}}(\underline{k})] = \underbrace{[\underline{i}\ \ \underline{j}\ \ \underline{k}]}_{\underline{C}}\ \underbrace{\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}}_{\triangleq A}$$

$$\underline{y} = \underline{C}\ y = \underline{C}\ Ax \quad \Rightarrow \quad \boxed{y = Ax}$$

$\boxed{\text{Columns of } A = \text{coordinates of image of } \underline{C},\ [\underline{\underline{f}}(\underline{i})\ \ \underline{\underline{f}}(\underline{j})\ \ \underline{\underline{f}}(\underline{k})] \text{, in } \underline{C}}$

What is the coordinate representation of the vector product function

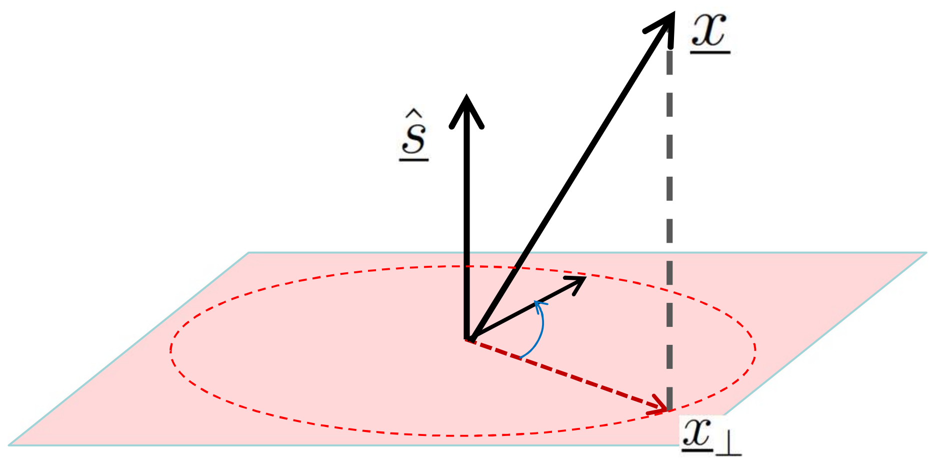
$$\underline{y} = \underline{f}(\underline{x}) = \underline{\underline{S}}\,\underline{x} = \underline{k} \times \underline{x}$$

in frame $\underline{C} = [\underline{i} \;\; \underline{j} \;\; \underline{k}]$ **?**

$$\begin{tikzcd}
\{\text{Vectors}\} \arrow[r, "\underline{\underline{S}}"] & \{\text{Vectors}\} \\
\{\text{Coordinates}\} \arrow[u, "\underline{C}"] \arrow[r, "S"] & \{\text{Coordinates}\} \arrow[u, "\underline{C}"']
\end{tikzcd}$$

Screw operator $\underline{y} = \underline{f}(\underline{x}) = \underline{\underline{A}}_s \underline{x} = \underline{s} \times \underline{x}$

can be interpreted as projection on orthogonal plane to s, followed by a rotation of ninety degrees about s and scaling by $||s||$



$$s \times \triangleq \begin{bmatrix} 0 & -c_s & b_s \\ c_s & 0 & -a_s \\ -b_s & a_s & 0 \end{bmatrix}$$

What is the coordinate representation of

$$\underline{y} = \underline{f}(\underline{x}) = \underline{\underline{A}}_s \underline{x} = \underline{s} \times \underline{x}$$

$$\underline{s} = a_s \underline{i} + b_s \underline{j} + c_s \underline{k}$$

in frame $\underline{C} = [\underline{i} \;\; \underline{j} \;\; \underline{k}]$ **?**

$$\begin{tikzcd}
\{\text{Vectors}\} \arrow[r, "\underline{\underline{A}}_s"] & \{\text{Vectors}\} \\
\{\text{Coordinates}\} \arrow[u, "\underline{C}"] \arrow[r, "A_s"'] & \{\text{Coordinates}\} \arrow[u, "\underline{C}"']
\end{tikzcd}$$

Coordinate invariance leads to the following formula regarding the skew operator, for any right-handed orthonormal Q:

$$\boxed{(Qs)\times = Q(s\times)Q^T}$$

$$\begin{tikzcd}
\{{}^0\text{Coordinates}\} \arrow[r, "({}^0s\times)"] & \{{}^0\text{Coordinates}\} \\
\{{}^1\text{Coordinates}\} \arrow[u, "{}^0C_1"] \arrow[r, "({}^1s\times)"'] & \{{}^1\text{Coordinates}\} \arrow[u, "{}^0C_1"']
\end{tikzcd}$$

$$({}^0s\times) = ({}^0C_1\,{}^1s)\times = {}^0C_1({}^1s\times)\,{}^0C_1^T$$

Geometric rotations are linear functions with matrix representations

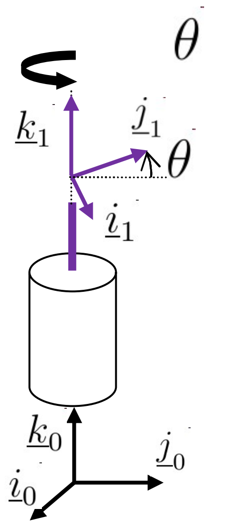
$$\underline{C}_0 = \begin{bmatrix} \underline{i}_0 & \underline{j}_0 & \underline{k}_0 \end{bmatrix} \qquad \underline{C}_1 = \begin{bmatrix} \underline{i}_1 & \underline{j}_1 & \underline{k}_1 \end{bmatrix} \qquad \underline{C}_1 = \underline{\underline{Rot}}(\underline{k}_0, \theta)\underline{C}_0$$

$$\underline{\underline{Rot}}(\underline{k}_0, \theta)\underline{C}_0 = \underline{C}_0 R(k, \theta)$$

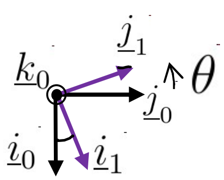
$$\underline{x} = \underline{C}_0 {}^0x \qquad \underline{y} = \underline{C}_0 {}^0y$$

$$\begin{tikzcd}
\{\text{Vectors}\} \arrow[r, "\underline{\underline{Rot}}(\underline{k}_0, \theta)"] & \{\text{Vectors}\} \\
\{\text{Coordinates}\} \arrow[u, "\underline{C}_0"] \arrow[r, "R(k, \theta)"'] & \{\text{Coordinates}\} \arrow[u, "\underline{C}_0"']
\end{tikzcd}$$

$${}^0y = R(k, \theta)\, {}^0x$$



$$\begin{aligned}
\underline{i}_1 &= \underline{i}_0 \cos\theta + \underline{j}_0 \sin\theta + \underline{k}_0 0 \\
\underline{j}_1 &= \underline{i}_0 (-\sin\theta) + \underline{j}_0 \cos\theta + \underline{k}_0 0 \\
\underline{k}_1 &= \underline{i}_0 0 + \underline{j}_0 0 + \underline{k}_0 1
\end{aligned}$$



$${}^0C_1 = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \triangleq R(k, \theta) \qquad {}^0y = {}^0C_1\, {}^0x$$

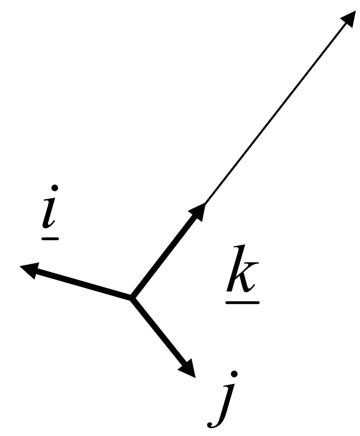
Structure of Rotations $\underline{\underline{Q}}$

- How do we choose a frame for the coordinate representation of $\underline{\underline{Q}}$?

- Rotation axis $\underline{s}$ is not changed

- Choose $\underline{i}, \underline{j}$ such that

$$\underline{i}, \underline{j} \perp \underline{k}, \quad \underline{i} \perp \underline{j}, \quad \underline{i} \times \underline{j} = \underline{k}$$



$$\begin{tikzcd} \{\text{Vectors}\} \arrow[r, "\underline{\underline{Q}}"] & \{\text{Vectors}\} \\ \{\text{Coordinates}\} \arrow[u, "\underline{C}"] \arrow[r, "Q"] & \{\text{Coordinates}\} \arrow[u, "\underline{C}"'] \end{tikzcd}$$

$$\underline{\underline{Q}}\ \underline{C} = \underline{C}\ Q$$

The rotation matrix $R(k,\theta)$ **has the following exponential form:**

$$R(k,\theta) = e^{\theta k\times}$$

$$k = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad (k\times) = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

e^{At} **defined as solution to** $\dot{X} = AX,\ X(0) = I$

Solve by taking Laplace transforms: $\hat{X}(s) = \mathcal{L}\left\{X(t)\right\}$

$$s\hat{X}(s) - X(0) = A\hat{X}(s)$$

$$s\hat{X}(s) - I = A\hat{X}(s)$$

$$(sI - A)\hat{X}(s) = I$$

$$X(t) = \mathcal{L}^{-1}\left\{\hat{X}(s)\right\} \qquad \boxed{X(t) = \mathcal{L}^{-1}\left\{(sI - A)^{-1}\right\}}$$

$$A = (k\times) \qquad (k\times) = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$sI - A = sI - (k\times) = \begin{bmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & s \end{bmatrix} + \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} s & 1 & 0 \\ -1 & s & 0 \\ 0 & 0 & s \end{bmatrix}$$

$$(sI - A)^{-1} = \begin{bmatrix} \frac{s}{s^2+1} & -\frac{1}{s^2+1} & 0 \\ \frac{1}{s^2+1} & \frac{s}{s^2+1} & 0 \\ 0 & 0 & \frac{1}{s} \end{bmatrix}$$

$$e^{(k\times)t} = \begin{bmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$e^{(k\times)\theta} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

General exponential form of a rotation matrix:

$$
\begin{array}{ccc}
\{{}^0\text{Coordinates}\} & \xrightarrow{\boxed{{}^0R = ?}} & \{{}^0\text{Coordinates}\} \\
{}^0C_1 \uparrow & & \uparrow {}^0C_1 \\
\{{}^1\text{Coordinates}\} & \xrightarrow[\boxed{{}^1R = e^{\theta(k\times)}}]{} & \{{}^1\text{Coordinates}\}
\end{array}
$$

$$Q \triangleq {}^0C_1$$

$${}^1R = e^{\theta({}^1s\times)} = e^{\theta(k\times)}$$

$${}^0R = Q\,{}^1RQ^T$$

$$
\begin{aligned}
{}^0R &= Qe^{\theta k\times}Q^T \\
&= Q[I + \theta(k\times) + \tfrac{1}{2!}\theta^2(k\times)^2 + \tfrac{1}{3!}\theta^3(k\times)^3 + \cdots]Q^T \\
&= QQ^T + \theta Q(k\times)Q^T + \tfrac{1}{2!}\theta^2 Q(k\times)QQ^T(k\times)Q^T + \cdots \\
&= I + \theta(Qk\times) + \tfrac{1}{2!}\theta^2(Qk\times)^2 + \cdots \\
&= e^{\theta(Qk)\times} = e^{\theta({}^0s\times)}
\end{aligned}
$$

$${}^1s = k$$

$${}^0s = {}^0C_1\,{}^1s = {}^0C_1 k = Qk$$

The coordinate representation of a rotation matrix is:

$$\boxed{R = e^{\theta(s\times)}}$$

θ **= rotation angle**

s **= coords of rotation axis (normalized to unit length)**

General exponential form of a rotation matrix: $\underline{\underline{Q}}$

$$
\begin{tikzcd}
\{\text{Vectors}\} \arrow[r, "\underline{\underline{Q}}"] & \{\text{Vectors}\} \\
\{\text{Coordinates}\} \arrow[u, "\underline{C}"] \arrow[r, "Q"] & \{\text{Coordinates}\} \arrow[u, "\underline{C}"']
\end{tikzcd}
\qquad
\underline{\underline{Q}}\,\underline{C} = \underline{C}\,\underline{Q}
$$

$$
\boxed{\underline{\underline{Q}} = e^{\theta\, \underline{s}\times} = e^{\theta\, (\underline{C}s)\times} = \underline{C}\; e^{\theta\, s\times}\; \underline{C}^{T}}
$$

$$
\boxed{Rot_{\underline{\hat{s}},\theta}(\underline{x}) = \underline{x} + \sin\theta\ \underline{\hat{s}} \times \underline{x} + (1 - \cos\theta)\, \underline{\hat{s}} \times (\underline{\hat{s}} \times \underline{x})}
$$

Rodrigues rotation formula

Properties of rotation matrices

$$Q^T Q = Q Q^T = I \qquad (Qs)\times = Q(s\times)Q^T$$

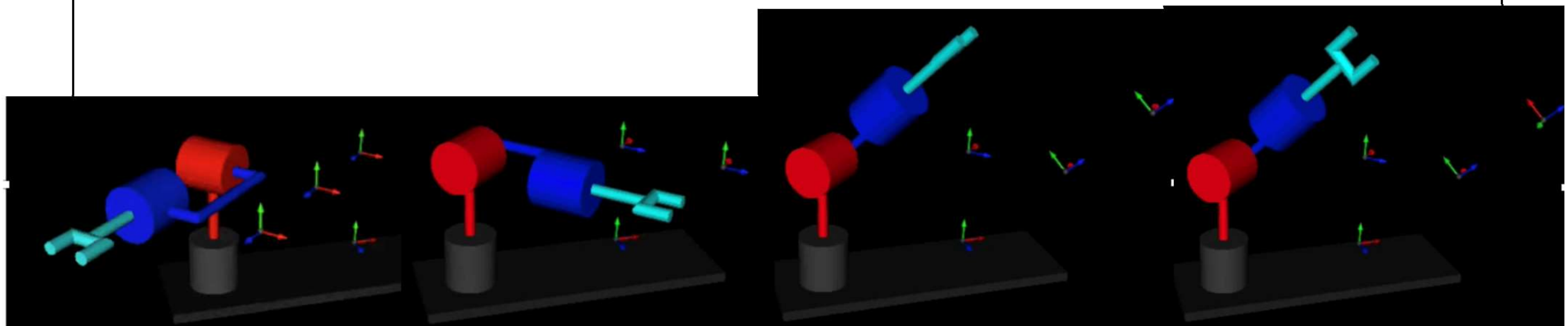
$$Qe = \lambda e \qquad |\lambda| = 1 \qquad \lambda \in \{e^{-j\theta}, e^{j\theta}, 1\} \qquad Qe = e$$

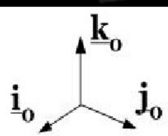
$$Q = e^{\theta(s\times)}$$

$$Q = I + \sin\theta(s\times) + (1 - \cos\theta)(s\times)^2$$

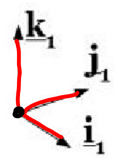
Rodrigues rotation formula

Spherical wrist – Z-Y-X Euler angles

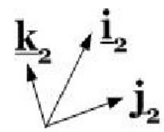




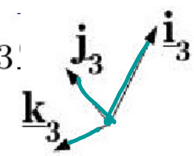
$$\underline{C}_1 = \underline{C}_0\,{}^0C_1$$

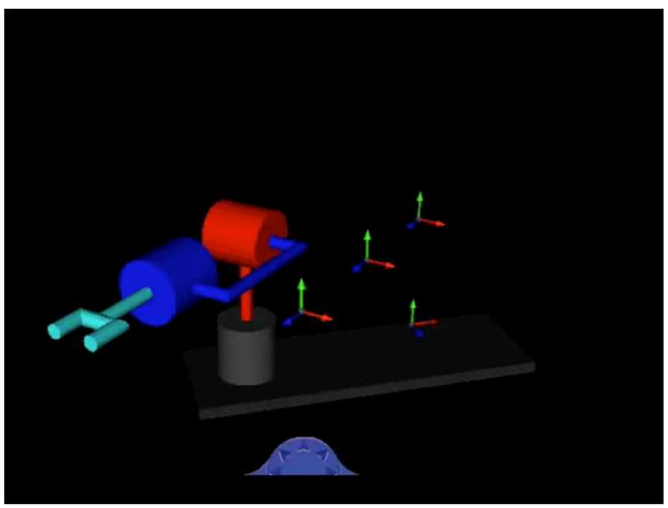


$$\underline{C}_2 = \underline{C}_1\,{}^1C_2$$



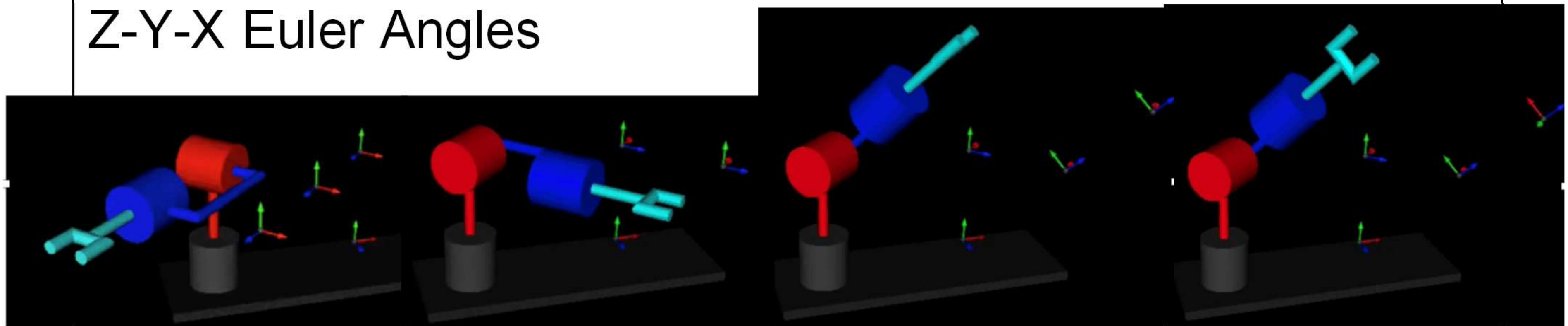
$$\underline{C}_3 = \underline{C}_2\,{}^2C_3$$

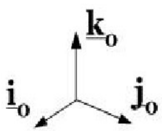




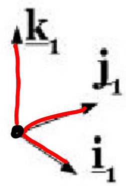
$$\begin{aligned}\underline{C}_3 &= \underline{C}_2\,{}^2C_3\\ &= \underline{C}_1\,{}^1C_2\,{}^2C_3\\ &= \underline{C}_0\,{}^0C_1\,{}^1C_2\,{}^2C_3\end{aligned}$$

Z-Y-X Euler Angles

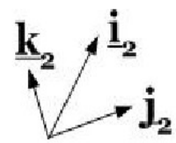




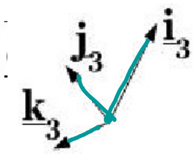
$$\underline{C}_1 = \underline{C}_0\, {}^0C_1$$



$$\underline{C}_2 = \underline{C}_1\, {}^1C_2$$

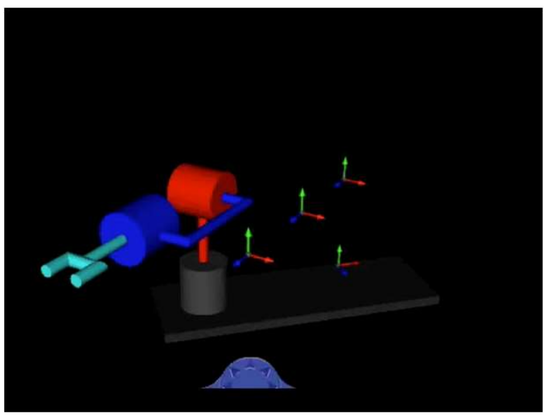


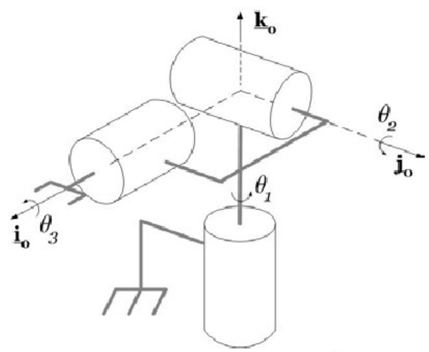
$$\underline{C}_3 = \underline{C}_2\, {}^2C_3$$



$$\underline{C}_1 = e^{\theta_1 \underline{k}_0 \times}\underline{C}_0 = \underline{C}_0 e^{\theta_1 k \times}$$

$$\underline{C}_2 = e^{\theta_2 \underline{j}_1 \times}\underline{C}_1 = \underline{C}_1 e^{\theta_2 j \times}$$





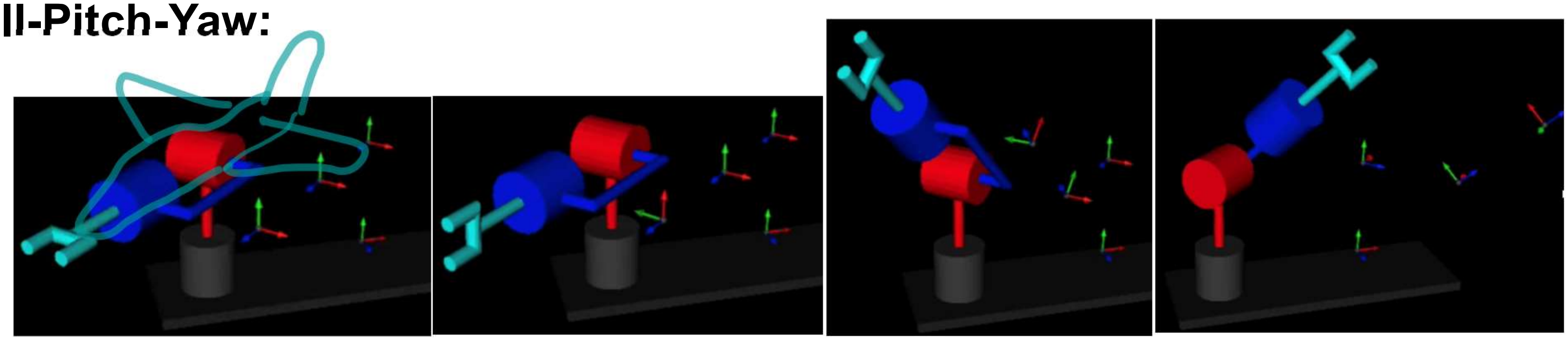
$$\underline{C}_3 = e^{\theta_3 \underline{i}_2 \times}\underline{C}_2 = \underline{C}_2 e^{\theta_3 i \times}$$

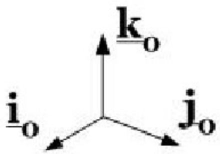
$$= \underline{C}_1 e^{\theta_2 j \times} e^{\theta_3 i \times}$$

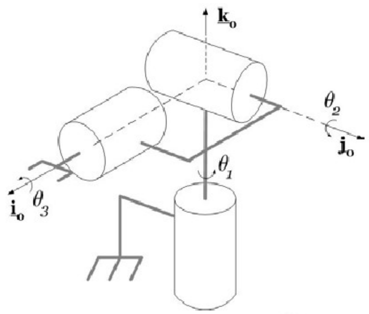
$$= \underline{C}_0 e^{\theta_1 k \times} e^{\theta_2 j \times} e^{\theta_3 i \times}$$

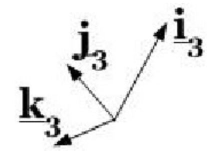
Current frame sequence of rotations written from left to right!

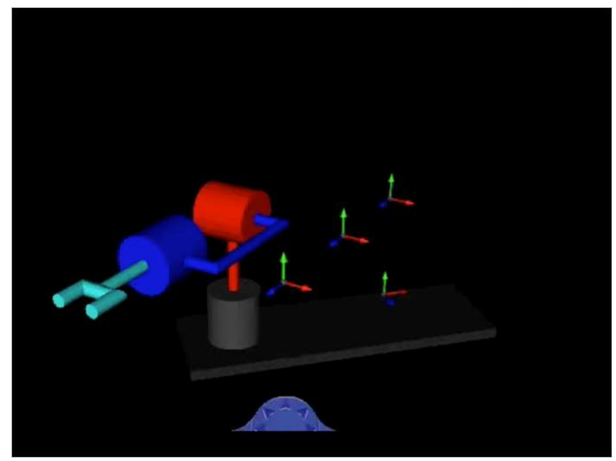
Roll-Pitch-Yaw:









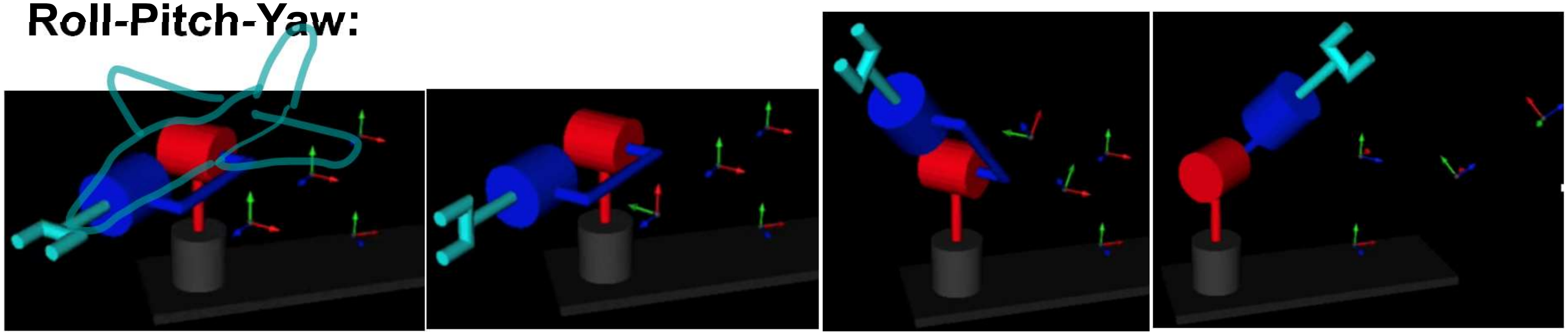


$$
\begin{aligned}
\underline{C}_3 &= e^{\theta_1 \underline{k}_0 \times} e^{\theta_2 \underline{j}_0 \times} e^{\theta_3 \underline{i}_0 \times} \underline{C}_0 \\
&= e^{\theta_1 \underline{k}_0 \times} e^{\theta_2 \underline{j}_0 \times} \underline{C}_0 e^{\theta_3 i \times} \\
&= e^{\theta_1 \underline{k}_0 \times} \underline{C}_0 e^{\theta_2 j \times} e^{\theta_3 i \times} \\
&= \underline{C}_0 e^{\theta_1 k \times} e^{\theta_2 j \times} e^{\theta_3 i \times}
\end{aligned}
$$

$$\underline{C}_3 = \underline{C}_0 e^{\theta_1 k \times} e^{\theta_2 j \times} e^{\theta_3 i \times}$$

Base frame sequence of rotations written from right to left!

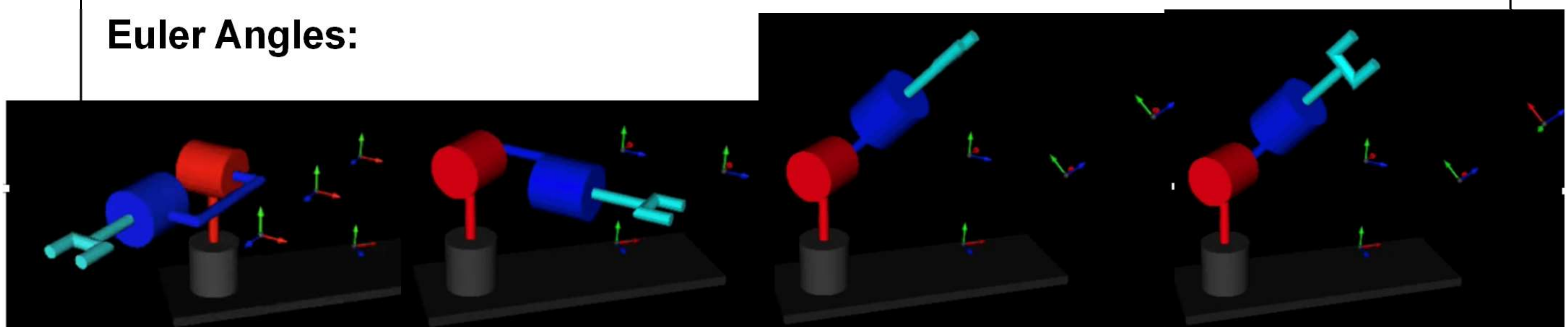
Roll-Pitch-Yaw:



$$\underline{C}_3 = e^{\theta_1 \underline{k}_0 \times}(e^{\theta_2 \underline{j}_0 \times}(e^{\theta_3 \underline{i}_0 \times}\underline{C}_0))$$

$$\underline{C}_3 = \underline{C}_0 e^{\theta_1 k \times} e^{\theta_2 j \times} e^{\theta_3 i \times}$$

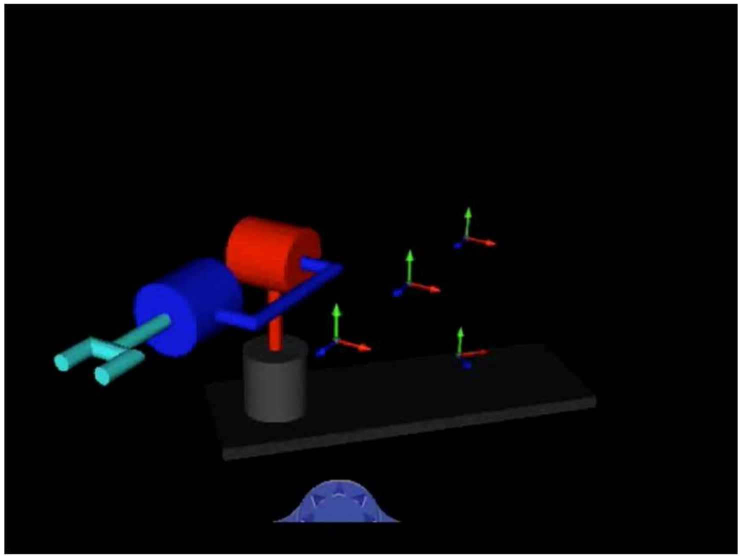
Euler Angles:



$$\underline{C}_3 = e^{\theta_3 \underline{i}_2 \times}(e^{\theta_2 \underline{j}_1 \times}(e^{\theta_1 \underline{k}_0 \times}\underline{C}_0))$$

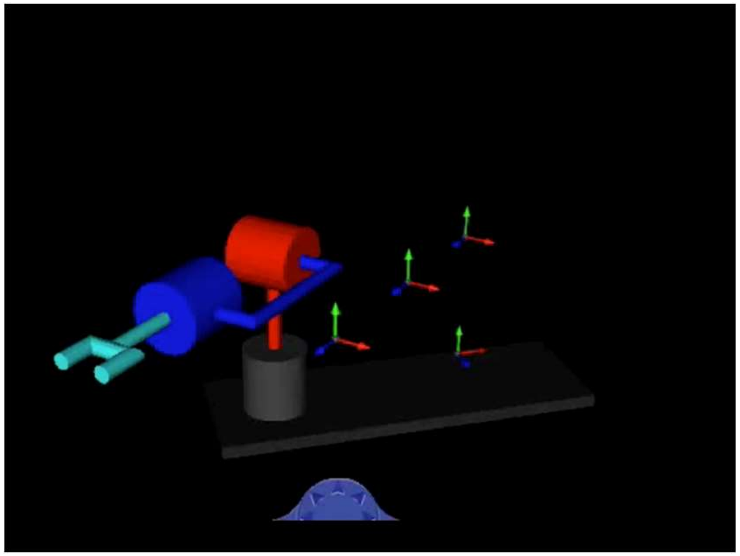
$$\underline{C}_3 = \underline{C}_0 e^{\theta_1 k \times} e^{\theta_2 j \times} e^{\theta_3 i \times}$$

Roll-Pitch-Yaw:



$$\underline{C}_3 = e^{\theta_1 \underline{k}_0 \times}(e^{\theta_2 \underline{j}_0 \times}(e^{\theta_3 \underline{i}_0 \times}\underline{C}_0))$$
$$\underline{C}_3 = \underline{C}_0 e^{\theta_1 k \times} e^{\theta_2 j \times} e^{\theta_3 i \times}$$

Euler Angles:

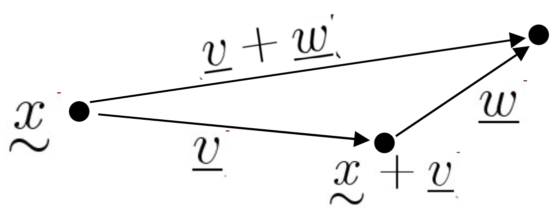


$$\underline{C}_3 = e^{\theta_3 \underline{i}_2 \times}(e^{\theta_2 \underline{j}_1 \times}(e^{\theta_1 \underline{k}_0 \times}\underline{C}_0))$$
$$\underline{C}_3 = \underline{C}_0 e^{\theta_1 k \times} e^{\theta_2 j \times} e^{\theta_3 i \times}$$

Points, Coordinate Systems and Coordinates

<u>Affine Space:</u> A set, $\mathcal{A}$, of elements having the following properties:

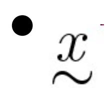
1. Elements of the set are **points** which can be displaced by vectors (a succession of displacements is achieved by addition of vectors).



$$(\underset{\sim}{x} + \underline{v}) + \underline{w} = \underset{\sim}{x} + (\underline{v} + \underline{w})$$

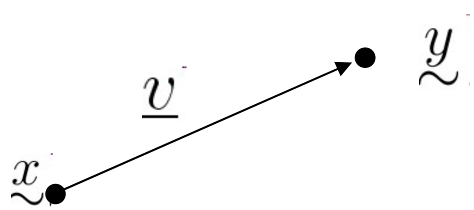
$$(\underset{\sim}{x} + \underline{v}) + \underline{w} = \underset{\sim}{x} + (\underline{v} + \underline{w})$$

2. The unique zero vector, $\underline{\mathbf{0}}$, does not displace elements of $\mathcal{A}$.



$$\underset{\sim}{x} + \underline{0} = \underset{\sim}{x}$$

3. Any two (ordered) points uniquely determines a displacement.

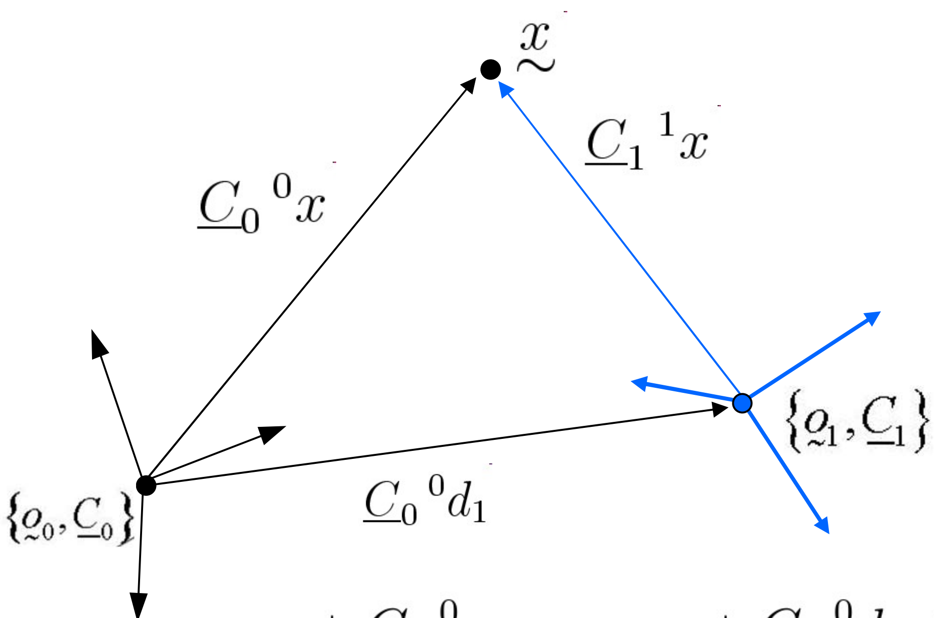


$$\underset{\sim}{y} = \underset{\sim}{x} + \underline{v}$$

Origin: A reference point from which other points may be specified in terms of vectors. Denote by $\underset{\sim}{o}$

Coordinate System: A pair consisting of an *origin* and a *frame* which permit any point in the affine space to be specified. Denote by $\{\ \underset{\sim}{o},\ \underline{C}\ \}$

Coordinate systems change:



$$\underset{\sim}{x} = \underset{\sim}{o}_0 + \underline{C}_0\,{}^0x$$
$$= \underset{\sim}{o}_1 + \underline{C}_1\,{}^1x$$
$$\underline{C}_1 = \underline{C}_0\,{}^0C_1$$
$$\underset{\sim}{o}_1 = \underset{\sim}{o}_0 + \underline{C}_0\,{}^0d_1$$

$$\underset{\sim}{o}_0 + \underline{C}_0\,{}^0x = \underset{\sim}{o}_0 + \underline{C}_0\,{}^0d_1 + \underline{C}_0\,{}^0C_1\,{}^1x$$

$$\underline{C}_0\,{}^0x = \underline{C}_0\,{}^0d_1 + \underline{C}_0\,{}^0C_1\,{}^1x \qquad {}^0x = {}^0d_1 + {}^0C_1\,{}^1x$$

Relating different homogenous coordinate representations

Frame Relationship: $\underline{C}_1 \;=\; \underline{C}_0\,{}^0C_1$

Origin Relationship: $\underset{\sim}{o}_1 \;=\; \underset{\sim}{o}_0 + \underline{C}_0\,{}^0d_1$

Homogenous Coordinate Relationship: ${}^0x = {}^0d_1 + {}^0C_1\,{}^1x$

$$\begin{bmatrix} {}^0x \\ 1 \end{bmatrix} = \begin{bmatrix} {}^0C_1 & {}^0d_1 \\ 0^T & 1 \end{bmatrix} \begin{bmatrix} {}^1x \\ 1 \end{bmatrix} \triangleq {}^0T_1 \begin{bmatrix} {}^1x \\ 1 \end{bmatrix}$$

Homogenous (Affine) Transformation Matrix: Often convenient to relate representations using a single matrix:

$$\begin{bmatrix} {}^0x \\ 1 \end{bmatrix} = \begin{bmatrix} {}^0C_1 & {}^0d_1 \\ 0^T & 1 \end{bmatrix} \begin{bmatrix} {}^1x \\ 1 \end{bmatrix} \triangleq {}^0T_1 \begin{bmatrix} {}^1x \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} \underline{C}_1 & \underset{\sim}{o}_1 \\ 0^T & 1 \end{bmatrix} = \begin{bmatrix} \underline{C}_0 & \underset{\sim}{o}_0 \\ 0^T & 1 \end{bmatrix} \underbrace{\begin{bmatrix} {}^0C_1 & {}^0d_1 \\ 0^T & 1 \end{bmatrix}}_{{}^0T_1}.$$

Homogenous transformation matrices can be used to relate representations of both *points* (by augmenting with “1”) and *vectors* (by augmenting with “0”):

$$\underset{\sim}{x} = \begin{bmatrix} x \\ 1 \end{bmatrix} \qquad \underline{v} = \begin{bmatrix} v \\ 0 \end{bmatrix}$$

$$\underset{\sim}{x} + \underline{v} = \begin{bmatrix} x \\ 1 \end{bmatrix} + \begin{bmatrix} v \\ 0 \end{bmatrix} = \begin{bmatrix} x+v \\ 1 \end{bmatrix}$$

$$\underset{\sim}{x}' - \underset{\sim}{x} = \begin{bmatrix} x' \\ 1 \end{bmatrix} - \begin{bmatrix} x \\ 1 \end{bmatrix} = \begin{bmatrix} x'-x \\ 0 \end{bmatrix}$$

Geometric Object / Transformation		**Coordinate Representation**
Vector	$\underline{x} = \underline{C}x$	x in frame $\underline{C}$
Point	$\underset{\sim}{x} = \underset{\sim}{o} + \underline{C}x$	x in coord. system $\{\underset{\sim}{o}, \underline{C}\}$
Scalar Product	$\underline{y} = \underline{s}^T \underline{x}$	$y = s^T x$
Vector Product	$\underline{y} = \underline{s} \times \underline{x}$	$y = \underline{C}^T[(\underline{C}s)\times]\underline{C}x = s \times x$ in frame $\underline{C}$
Rotation	$\underline{y} = \underline{\underline{Q}}\underline{x} = e^{\theta \underline{s}\times}\underline{x}$	$y = \underline{C}^T \underline{\underline{Q}}\underline{C}\, x = Qx = e^{\theta s \times}\, x$ in frame $\underline{C}$

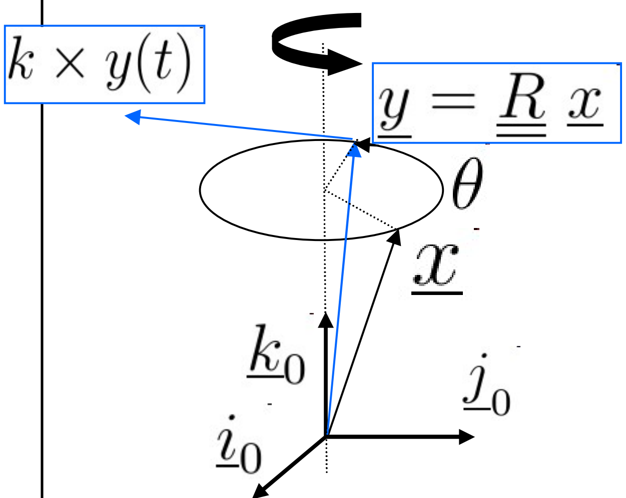
Rigid Motion
$\underset{\sim}{y} = \underset{\sim}{o} + \underline{d} + \underline{\underline{Q}}(\underset{\sim}{x} - \underset{\sim}{o})$
(first rotation then translation)

$y = Qx + d$ in
coordinate system $\{\underset{\sim}{o}, \underline{C}\}$

Consider the motion of a point with respect to $\underline{C}_0 = \begin{bmatrix} \underline{i}_0 & \underline{j}_0 & \underline{k}_0 \end{bmatrix}$

$$y(t) = R(t)x = e^{(k\times)t}x$$

$$(k\times) = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad e^{(k\times)t} = \begin{bmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



$$\dot{y}(t) = \dot{R}(t)x = \frac{d}{dt}e^{(k\times)t}x$$

$$= \frac{d}{dt}\begin{bmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & 1 \end{bmatrix} x$$

$$= \begin{bmatrix} -\sin t & -\cos t & 0 \\ \cos t & -\sin t & 0 \\ 0 & 0 & 0 \end{bmatrix} x$$

$$= (k\times)R(t)x = (k\times)y(t) = k \times y(t)$$

$$\boxed{\dot{R}(t) = (k\times)R(t)}$$

Angular Velocity

What is the angular velocity of frame $\underline{C}_1(t)$ with respect to $\underline{C}_0(t)$

$$\underline{C}_1(t) = \underline{C}_0(t)\,{}^0C_1(t) = \underline{C}_0(t)Q(t) = \underline{C}_0(t)Q$$

$$\frac{d}{dt}Q = \dot{Q} = \Omega Q$$

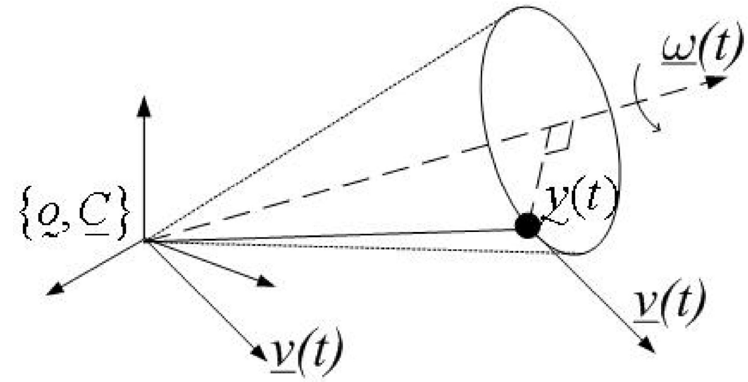
$$\Omega^T = -\Omega$$

$${}^0\omega_{1,0}\times \; = \; \Omega = \dot{Q}Q^T = {}^0\dot{C}_1(t)\,{}^0C^T{}_1(t)$$

$$\underline{\omega}_{1,0} = \underline{C}_0\,{}^0\omega_{1,0}$$

Physical Interpretation:

$$\underset{\sim}{y}(t) = \underset{\sim}{o} + \underline{\underline{Q}}(t)(\underset{\sim}{x} - \underset{\sim}{o})$$
$$= \underset{\sim}{o} + \underline{C}Q(t)x$$



$$\underline{v}(t) = \dot{\underset{\sim}{y}}(t) = \underline{C}\dot{Q}(t)x = \underline{C}v(t)$$
$$= (\underline{C}\omega) \times \underline{C}Q(t)x$$
$$= \underline{C}(\omega\times)\underline{C}^T\underline{C}Q(t)x$$
$$= \underline{C}(\omega\times)Q(t)x$$
$$= \underline{\omega}(t) \times (\underset{\sim}{y}(t) - \underset{\sim}{o})$$

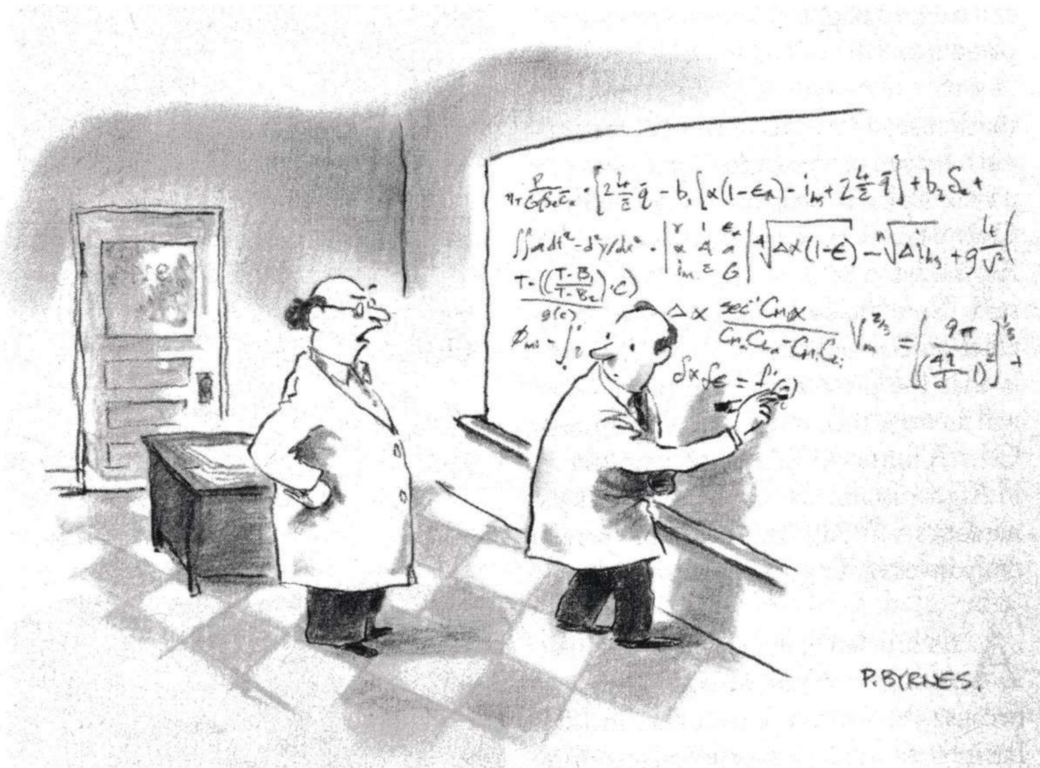
Addition of Angular Velocities

Consider a sequence of 3 time-varying frames: $\underline{C}_{i-1}(t)$, $\underline{C}_i(t)$ and $\underline{C}_{i+1}(t)$

Can show (see eqns (87)-(95)):

$$^{i-1}\omega_{i+1,i-1}\times = \left({}^{i-1}\omega_{i,i-1} + {}^{i-1}\omega_{i+1,i} \right)\times$$

What does this mean physically?



"Duh."