University of British Columbia
Department of Mechanical Engineering

MECH468 Modern Control Engineering, Final exam
MECH522 Foundations in Control Engineering, Final exam

Examiner: Dr. Ryozo Nagamune

April 21 (Tuesday), 2020, 8:30am-11am
(Upload your answer sheets on Canvas "Assignments" by 11:30am)

Exam policies

- Allowed: Open-book. Any distributed material and any textbook.
- Not-allowed: Calculators. Matlab. Web-browsing.
- Write all your answers on **your own sheets**.
- Motivate your answers properly. (No chance to defend your answers orally.)
- **No questions are allowed.**
- 100 points in total. Mark may be scaled later.

After you finish the exam ...

- Scan, or take a photo of, your answer sheets.
- Upload the pdf-files of your answer sheets on Canvas "Assignments".
- **Do not send an inquiry email to the instructor** even if you are not sure whether your uploading was successful. You will be contacted by the instructor if he cannot find it on Canvas.

Marking scheme

Question #	Expected duration	Full mark
Q1	about 30 min	30 %
Q2	about 30 min	20 %
Q3	about 45 min	30 %
Q4	about 45 min	20 %
Total	about 150 min	100 %

Q1. Select **only one** correct statement for each the following sentences. **There is no need to motivate your answers. Your mark will solely depend on your selected statement.** (3pt each)

(a) State-space models can be used to represent:

i. only linear time-invariant systems.
ii. not only linear but also nonlinear time-invariant systems.
iii. not only linear time-invariant but also linear time-varying systems.
iv. nonlinear time-varying systems.

(b) If we linearize the nonlinear state equation $\dot{x} = -x^3 + x + u$ around the operating point $(x_0, u_0) = (-1, 0)$, the linearized model is:

i. $\delta\dot{x} = \delta x + \delta u$.
ii. $\delta\dot{x} = -\delta x + \delta u$.
iii. $\delta\dot{x} = -2\delta x + \delta u$.
iv. None of i, ii, iii is correct.

(c) Consider two matrices $A_1 = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}$ and $A_2 = \begin{bmatrix} 0 & 1 \\ -100 & -20 \end{bmatrix}$.

The continuous-time system $\dot{x} = Ax$ is:

i. asymptotically stable for both $A = A_1$ and $A = A_2$.
ii. marginally stable for $A = A_1$ but asymptotically stable for $A = A_2$.
iii. asymptotically stable for $A = A_1$ but marginally stable for $A = A_2$.
iv. None of i, ii, iii is correct.

(d) For the nonlinear system $\dot{x} = -x^3 - x$, the following can be the Lyapunov function:

i. $V(x) = x$.
ii. $V(x) = x^2$.
iii. $V(x) = x^3$.
iv. None of i, ii, iii is correct.

(e) Consider two symmetric matrices $M_1 = \begin{bmatrix} 1 & -2 \\ -2 & 1 \end{bmatrix}$ and $M_2 = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$.

Then,

i. both M_1 and M_2 are positive definite.
ii. M_1 is positive definite but M_2 is not positive definite.
iii. M_1 is not positive definite but M_2 is positive definite.
iv. None of i, ii, iii is correct.

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(f) A linear time-invariant system

$$\begin{cases} \dot{x} &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \\ y &= \begin{bmatrix} 1 & 0 \end{bmatrix} x \end{cases}$$

is:

i. both stabilizable and detectable.
ii. stabilizable but not detectable.
iii. not stabilizable but detectable.
iv. neither stabilizable nor detectable.

(g) For the 1-by-2 transfer matrix $G(s) = \begin{bmatrix} \frac{1}{s^2} & \frac{1}{s} \end{bmatrix}$, the McMillan degree is:

i. 2.
ii. 3.
iii. 4.
iv. None of i, ii, iii is correct.

(h) If a linear time-invariant system $\dot{x} = Ax + Bu$, $y = Cx$ is asymptotically stable, then the system is:

i. always both stabilizable and detectable.
ii. always stabilizable but not always detectable.
iii. not always stabilizable but always detectable.
iv. None of i, ii, iii is correct.

(i) The solution to the ordinary differential equation $\dot{x} = -x$ with the boundary condition $x(1) = 2$ is

i. $x(t) = 2$.
ii. $x(t) = 2e^{-t+1}$.
iii. $x(t) = 2e^{-t-1}$.
iv. None of i, ii, iii is correct.

(j) Discrete-time finite-horizon LQR requires offline computation of the controller gain $K[k]$, while one-step Kalman filter requires offline computation of the error covariance matrix $P[k|k-1]$.

i. Both $K[k]$ and $P[k|k-1]$ are computed forward in time k.
ii. Both $K[k]$ and $P[k|k-1]$ are computed backward in time k.
iii. $K[k]$ is computed forward in time k, while $P[k|k-1]$ is computed backward in time k.
iv. $K[k]$ is computed backward in time k, while $P[k|k-1]$ is computed forward in time k.

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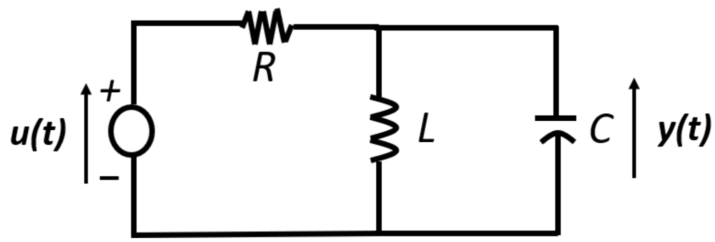
Q2. Derive the state-space model of the following systems. Your answers should be in a matrix-vector form: (10pt each)

$$\begin{cases} \dot{x} &= Ax + Bu, \\ y &= Cx + Du. \end{cases}$$

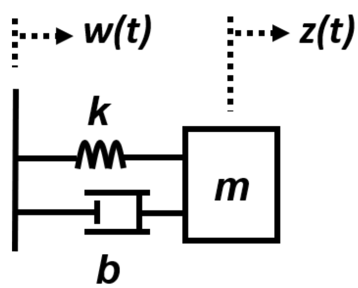
(a) An electrical circuit in the figure below, where

- the input voltage is $u(t)$,
- the output voltage is $y(t)$, and
- R, L, and C are resistance, inductance, and capacitance, respectively.



(b) A mass-spring damper system in the figure below, where

- the input is the **velocity** $\dot{w}(t)$ (where w is the displacement of the massless plate at the left-side of the figure),
- the three outputs are position $z(t)$, velocity $\dot{z}(t)$, and acceleration $\ddot{z}(t)$ of the mass m, and
- m, b, and k are mass, damping constant, and spring constant, respectively.



Hint: Take the displacement w as one of the states.

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Q3. For the following continuous-time state-space equation, answer the following questions.

$$\begin{cases} \dot{x} = \underbrace{\begin{bmatrix} 2 & 1 \\ 0 & -2 \end{bmatrix}}_{A} x + \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{B} u \\ y = \underbrace{\begin{bmatrix} 0 & 1 \end{bmatrix}}_{C} x \end{cases}$$

(a) Compute the matrix exponential e^{At}. (5pt)

(b) Check the controllability of the system. (5pt)

(c) Check the stabilizability of the system. (5pt)

(d) Select appropriate closed-loop poles, and design a stabilizing state-feedback controller $u = -Kx$. (5pt)

(e) Obtain the infinite-horizon LQR controller $u = -K_{LQR}x$ which solves the following optimization problem: (10pt)

$$\min_{u(\cdot)} \int_0^\infty \left[3y^2(t) + u^2(t)\right] dt.$$

In this question, you do not need to check the solvability condition, but make sure:

- to verify that your obtained solution to the algebraic Riccati equation is positive definite,
- to present your obtained controller gain K_{LQR}, and
- to verify the closed-loop stability with the designed controller.

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Q4. Let us consider to estimate the unknown state $x[k]$ from noisy measurements. The discrete-time state-space model can be written as

$$\begin{cases} x[k+1] = x[k] + w[k] \\ y[k] = x[k] + v[k] \end{cases}$$

where k is the time-index, $y[k]$ is the measurement at time k, and $w[k]$ and $v[k]$ are the process noise and the measurement noise at time k, respectively. Assume that both $v[k]$ and $w[k]$ are Gaussian white noise with the mean values and the variances given as

$$E\{w[k]\} = 0, \quad E\{w[k]^2\} = 1, \quad k = 0, 1, 2, \ldots,$$
$$E\{v[k]\} = 0, \quad E\{v[k]^2\} = 2 \quad k = 0, 1, 2, \ldots,$$

$$E\{w[j]v[k]\} = 0, \; j = 0, 1, 2, \ldots, \; k = 0, 1, 2, \ldots,$$

where $E\{\cdot\}$ denotes the expected value. We will use the following standard notations:

$\hat{x}[k|k-1]$ and $P[k|k-1]$: A priori estimate of $x[k]$ and its error variance

$\hat{x}[k|k]$ and $P[k|k]$: A posteriori estimate of $x[k]$ and its error variance

(a) Write down the recursive equations for the state estimates and their error variances in the time-varying Kalman filter. (10pt)

$$\begin{aligned} P[k+1|k] &= \cdots \\ P[k|k] &= \cdots \\ \hat{x}[k+1|k] &= \cdots \\ \hat{x}[k|k] &= \cdots \end{aligned}$$

(b) Now, assume that we get the measurements $y[1]$, $y[2]$. By using time-varying Kalman filter, fill out the following table. Initial estimate $\hat{x}[0|0]$ and its error variance $P[0|0]$ are given in the table. (10pt)

You are NOT allowed to use any calculator!

k	$\hat{x}[k\|k-1]$	$P[k\|k-1]$	$\hat{x}[k\|k]$	$P[k\|k]$
0	N/A	N/A	0	0
1				
2				

——— (END OF FINAL EXAM) ———