MECH 463 -- 2018 Final Exam

1. (a) Explain in words what is meant by a mode shape. Describe two basic feature of a mode shape that are assumed when choosing a suitable trial function for the matrix solution of a vibrating multi-DOF system.

A mode shape describes the relative vibration amplitudes and phase of the individual degrees of freedom of a system vibrating at one of its natural frequencies. For a n-DOF system, there are n distinct mode shapes corresponding to the n natural frequencies.

The two basic features assumed when choosing a suitable trial function for the vibration solution of an undamped multi-DOF system are:

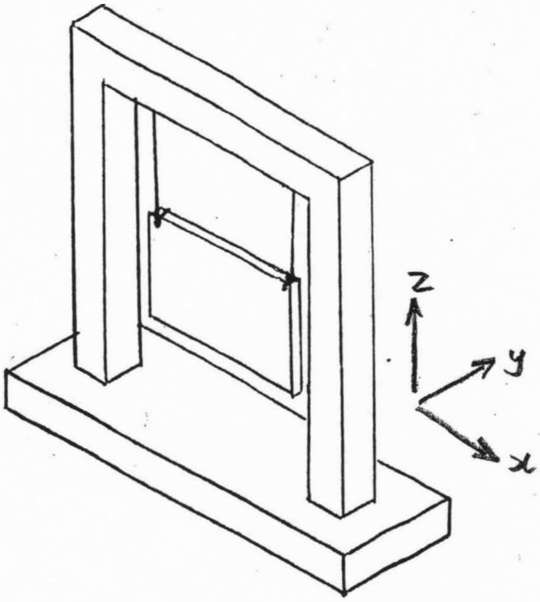
1) when vibrating at a given natural frequency, all parts of the system vibrate in phase or exactly out of phase.
2) at each natural frequency, there is a definite ratio of the vibration amplitudes of each part of the system (negative ratio for out-of-phase parts). This is the mode shape.

With these features in mind, a suitable trial function is

$$\underline{x} = \underline{X} \cos(\omega t + \phi)$$

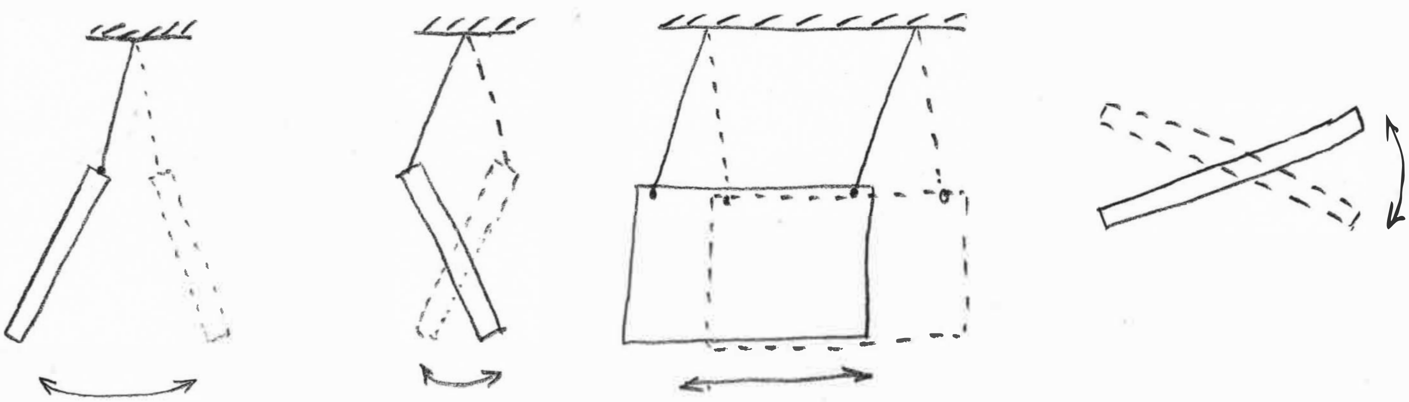
where $\underline{X}$ is a vector whose elements describe the mode shape. The constancy of the phase is indicated by the common term $\cos(\omega t + \phi)$

1. (b) An ornamental sign board hangs at the entrance of a garden. It is rectangular in shape and hangs from two non-stretching cables. Identify the number of degrees of freedom, and explain why. Use your experience to suggest realistic mode shapes.



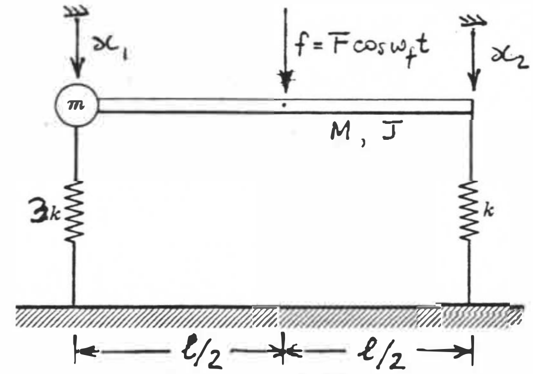
A completely free plate would have six degrees of freedom (3 translations and 3 rotations).
The two non-stretching strings provide constraints that prevent translation in the z direction and rotation around the y axis.
After deducting the two constraints, there remain <u>4 DOF</u>

For this symmetrical case, the mode shapes are:



view in x direction | view in x direction | view in y direction | view in z direction

2. A uniform rod of length ℓ is supported by two springs, one of stiffness k and one of stiffness 3k. The rod has mass M = 3m, and polar moment of inertia about its centre $J = M\ell^2/12$. An additional mass m is attached at the end of the rod that is supported by the spring of stiffness 3k. An oscillating vertical force $f = F \cos \omega_f t$ is applied at the centre of the rod.



(a) Formulate the equations of motion of the system using Lagrange's Equations.

(b) Determine the response amplitudes due to the oscillating excitation force f.

(c) Comment on and explain your results found in (b).

(a)

At the centre of mass of the rod, displacement $= \dfrac{x_1 + x_2}{2}$

rotation $= \dfrac{x_2 - x_1}{\ell}$

$$\rightarrow T = \frac{1}{2} m \dot{x}_1^2 + \frac{1}{2} M \left(\frac{\dot{x}_1 + \dot{x}_2}{2}\right)^2 + \frac{1}{2} J \left(\frac{\dot{x}_2 - \dot{x}_1}{\ell}\right)^2$$

$$V = \frac{1}{2} 3k x_1^2 + \frac{1}{2} k x_2^2$$

$$R = 0$$

Let y = displacement at force application point

$\rightarrow y = \dfrac{x_1 + x_2}{2}$

$$Q_1 = f \frac{\partial y}{\partial x_1} = f \frac{\partial}{\partial x_1}\left(\frac{x_1 + x_2}{2}\right) = \frac{f}{2}$$

$$Q_2 = f \frac{\partial y}{\partial x_2} = f \frac{\partial}{\partial x_2}\left(\frac{x_1 + x_2}{2}\right) = \frac{f}{2}$$

Recall Lagranges Equations:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} + \frac{\partial V}{\partial q_i} + \frac{\partial R}{\partial q_i} = Q_i$$

For $i = 1$ $\longrightarrow$ $m\ddot{x}_1 + \dfrac{M}{4}(\ddot{x}_1 + \ddot{x}_2) + \dfrac{J}{\ell^2}(\ddot{x}_1 - \ddot{x}_2) + 3kx_1 = \dfrac{f}{2}$

For $i = 2$ $\longrightarrow$ $\dfrac{M}{4}(\ddot{x}_1 + \ddot{x}_2) - \dfrac{J}{\ell^2}(\ddot{x}_1 - \ddot{x}_2) + kx_2 = \dfrac{f}{2}$

In matrix form:

$$\begin{bmatrix} m+\frac{M}{4}+\frac{J}{l^2} & \frac{M}{4}-\frac{J}{l^2} \\ \frac{M}{4}-\frac{J}{l^2} & \frac{M}{4}+\frac{J}{l^2} \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} 3k & 0 \\ 0 & k \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{f}{2} \\ \frac{f}{2} \end{bmatrix}$$

Substitute $M = 3m$. $J = \frac{1}{12}Ml^2 = \frac{1}{4}ml^2$

$$\begin{bmatrix} 2m & \frac{m}{2} \\ \frac{m}{2} & m \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} 3k & 0 \\ 0 & k \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{f}{2} \\ \frac{f}{2} \end{bmatrix}$$

(b) Try solution $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}\cos\omega_f t$

$$\rightarrow \begin{bmatrix} 3k-2m\omega_f^2 & -\frac{m}{2}\omega_f^2 \\ -\frac{m}{2}\omega_f^2 & k-m\omega_f^2 \end{bmatrix}\begin{bmatrix} X_1 \\ X_2 \end{bmatrix}\cos\omega_f t = \begin{bmatrix} F/2 \\ F/2 \end{bmatrix}\cos\omega_f t$$

This is true for all t $\longrightarrow$ $\cos\omega_f t \neq 0$

$$\begin{bmatrix} 3k-2m\omega_f^2 & -\frac{m}{2}\omega_f^2 \\ -\frac{m}{2}\omega_f^2 & k-m\omega_f^2 \end{bmatrix}\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} F/2 \\ F/2 \end{bmatrix}$$

Solving by Cramer's rule:

$$X_1 = \frac{F/2\left(k-m\omega_f^2+\frac{m}{2}\omega_f^2\right)}{(3k-2m\omega_f^2)(k-m\omega_f^2)-\left(\frac{m}{2}\omega_f^2\right)^2}$$

$$= \frac{F/2\left(k-\frac{m}{2}\omega_f^2\right)}{3k^2-5mk\omega_f^2+7/4\,m^2\omega_f^4}$$

$$= \frac{F/2\left(k-\frac{m}{2}\omega_f^2\right)}{\left(k-\frac{m}{2}\omega_f^2\right)\left(3k-\frac{7}{2}m\omega_f^2\right)} = \frac{F}{(6k-7m\omega_f^2)}$$

Similarly,

$$X_2 = \frac{F/2\left(3k - 2m\omega_f^2 + \frac{m}{2}\omega_f^2\right)}{(3k-2m\omega_f^2)(k-m\omega_f^2) - \left(\frac{m}{2}\omega_f^2\right)^2}$$

$$= \frac{3F/2\left(k - \frac{m}{2}\omega_f^2\right)}{\left(k-\frac{m}{2}\omega_f^2\right)\left(3k-\frac{7}{2}m\omega_f^2\right)} = \frac{3F}{(6k-7m\omega_f^2)} = 3X_1$$

(c) The denominator equals zero and hence the responses X_1 and X_2 become unbounded when $\omega_f^2 = \frac{6}{7} k/m$. This value of ω_f^2 corresponds to one of the two natural frequencies of the system. There is no unbounded response at the second natural frequency, $\omega_f^2 = 2\, k/m$ because the excitation force f is applied at the nodal point of this mode. The second mode can therefore not be excited.

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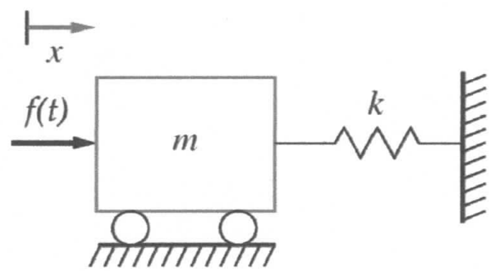
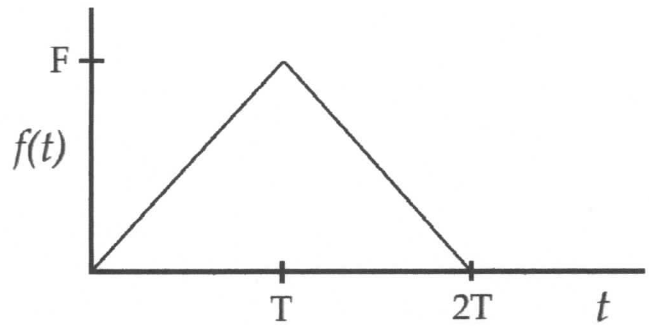
3. (a) A force f = f(t) acts on a simple mass-spring system. Determine the response of the system for zero initial conditions, given f(t) = at.

(b) Starting from the solution to (a), determine the response of the system to the ramp-pulse excitation shown in the diagram for times t > 2T.





$m\ddot{x} + kx = f(t)$

(a) For $f(t) = at$ $\rightarrow$ $m\ddot{x} + kx = at$

Complementary solution: $x = A\cos\omega t - B\sin\omega t$

For particular solution, try $x = Ct$ (same form as RHS)

Substituting: $0 + kCt = at$ $\rightarrow$ $C = \frac{at}{k}$

$\rightarrow$ General solution is $x = A\cos\omega t - B\sin\omega t + \frac{at}{k}$

Initial conditions. $x(0) = A - 0 + 0 = 0$ $\rightarrow$ $A = 0$

$\dot{x}(t) = -\omega A\sin\omega t - \omega B\cos\omega t + \frac{a}{k}$

$\dot{x}(0) = 0 - \omega B + \frac{a}{k} = 0$ $\rightarrow$ $B = \frac{a}{\omega k}$

$\rightarrow$ $x = \frac{a}{\omega k}(\omega t - \sin\omega t)$

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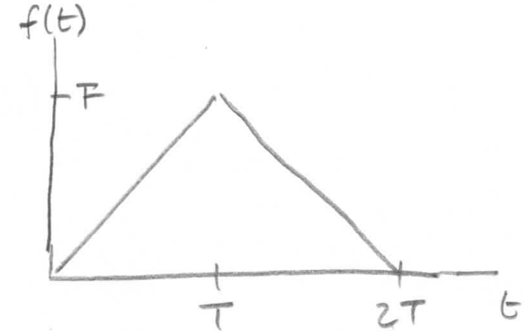
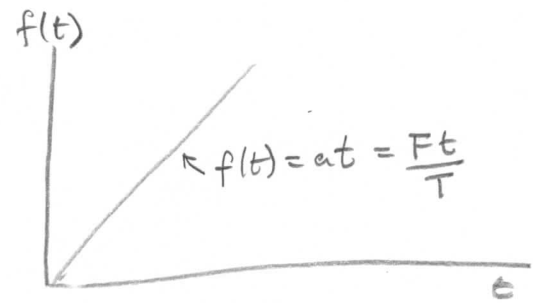
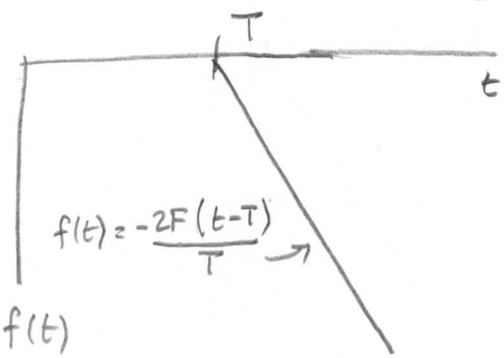
(b) Using the principle of superposition, the triangular pulse can be considered as the sum of three ramp functions, as shown.



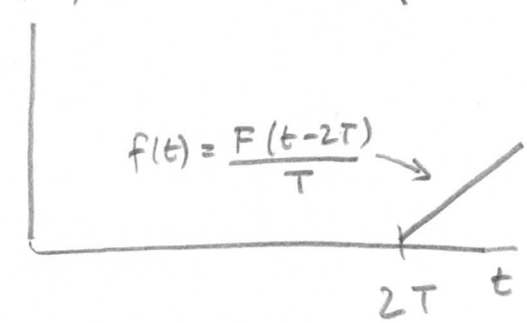
For $t > 2T$ (where $a = \frac{F}{T}$)



$$x = \frac{F}{\omega kT}\Big[\omega t - \sin\omega t - 2\omega(t-T) + 2\sin\omega(t-T) + \omega(t-2T) - \sin\omega(t-2T)\Big]$$



$$x = \frac{F}{\omega kT}\Big[\omega t - 2\omega t + 2\omega T + \omega t - 2\omega T - \sin\omega t + 2\sin\omega t\cos\omega T - 2\cos\omega t\sin\omega T - \sin\omega t\cos 2\omega T + \cos\omega t\sin 2\omega T\Big]$$



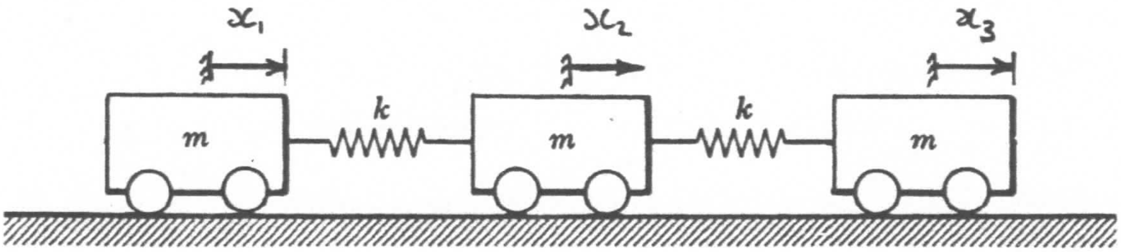
$$x = \frac{F}{\omega kT}\Big[\sin\omega t\left(2\cos\omega T - 1 - \cos 2\omega T\right) + \cos\omega t\left(\sin 2\omega T - 2\sin\omega T\right)\Big]$$

$$x = \frac{F}{\omega kT}\left[\begin{array}{l}\sin\omega t\left(2\cos\omega T - 2\cos^2\omega T\right) \\ + \cos\omega t\left(2\sin\omega T\cos\omega t - 2\sin\omega T\right)\end{array}\right]$$

$$x = \frac{F}{\omega kT}\left[\begin{array}{l}2\sin\omega t\cos\omega T\left(1 - \cos\omega T\right) \\ -2\cos\omega t\sin\omega T\left(1 - \cos\omega T\right)\end{array}\right]$$

$$\underline{x = \frac{2F}{\omega kT}\left(1 - \cos\omega T\right)\sin\omega(t-T)}$$

4. Three train cars are on a level track. The cars each have mass m and are connected together by couplings of stiffness k, as shown in the diagram.



(a) Formulate the equations of motion and identify the mass and stiffness matrices.

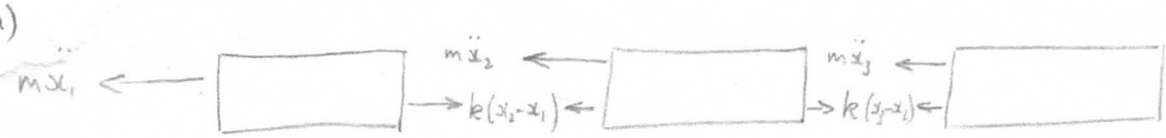
(b) Based on physical features of the arrangement of train cars, identify two of the three vibration mode shapes of the system by inspection. Explain the reasoning for your identification.

(c) Use mode shape orthogonality to identify the third mode shape.

(d) Use the Rayleigh method to evaluate the natural frequencies corresponding to the three vibration mode shapes identified above.

(e) Give simple physical explanations for the three natural frequencies found.

(a)



From free body diagrams

$$m\ddot{x}_1 - k(x_2-x_1) = 0$$
$$m\ddot{x}_2 + k(x_2-x_1) - k(x_3-x_2) = 0$$
$$m\ddot{x}_3 + k(x_3-x_2) = 0$$

In matrix form:

$$\underbrace{\begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m \end{bmatrix}}_{\underset{\sim}{M}} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{bmatrix} + \underbrace{\begin{bmatrix} k & -k & 0 \\ -k & 2k & -k \\ 0 & -k & k \end{bmatrix}}_{\underset{\sim}{K}} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

(b) System is semi-definite, so one vibration mode corresponds to a rigid body motion $\longrightarrow \underline{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

System is symmetrical, so a second vibration mode involves opposing motions of the outer two masses, with the centre one remaining still $\longrightarrow \underline{u}_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$

(c) The third mode is orthogonal to the other two.

Say $\underline{u}_3 = \begin{bmatrix} 1 \\ \alpha \\ \beta \end{bmatrix}$ where α and β are to be determined.

$$\longrightarrow \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} m & & \\ & m & \\ & & m \end{bmatrix} \begin{bmatrix} 1 \\ \alpha \\ \beta \end{bmatrix} = 0 \longrightarrow m(1+\alpha+\beta) = 0$$

$$\text{and } \begin{bmatrix} 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} m & & \\ & m & \\ & & m \end{bmatrix} \begin{bmatrix} 1 \\ \alpha \\ \beta \end{bmatrix} = 0 \longrightarrow m(1 \quad -\beta) = 0$$

$$\longrightarrow \alpha = -2,\ \beta = 1 \longrightarrow \underline{u}_3 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

(d) Rayleigh Quotient

$$\omega_R^2 = \frac{\underline{v}^T \underset{\sim}{K} \underline{v}}{\underline{v}^T \underset{\sim}{M} \underline{v}} = \frac{\begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} k & -k & 0 \\ -k & 2k & -k \\ 0 & -k & k \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}{\begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} m & & \\ & m & \\ & & m \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}} = \frac{\begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}}{\begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} m \\ m \\ m \end{bmatrix}} = \underline{0}$$

as expected for a rigid body motion

For second mode

$$\omega_R^2 = \frac{\begin{bmatrix} 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} k & -k & 0 \\ -k & 2k & -k \\ 0 & -k & k \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}}{\begin{bmatrix} 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} m & & \\ & m & \\ & & m \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}} = \frac{\begin{bmatrix} 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} k \\ 0 \\ -k \end{bmatrix}}{\begin{bmatrix} 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} m \\ 0 \\ -m \end{bmatrix}} = \underline{k/m}$$

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For third mode

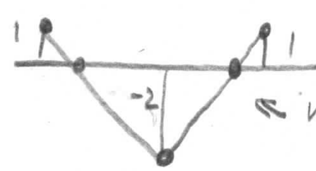
$$\omega_R^2 = \frac{\begin{bmatrix}1 & -2 & 1\end{bmatrix}\begin{bmatrix}k & -k & 0\\ -k & 2k & -k\\ 0 & -k & k\end{bmatrix}\begin{bmatrix}1\\-2\\1\end{bmatrix}}{\begin{bmatrix}1 & -2 & 1\end{bmatrix}\begin{bmatrix}m & & \\ & m & \\ & & m\end{bmatrix}\begin{bmatrix}1\\-2\\1\end{bmatrix}} = \frac{\begin{bmatrix}1 & -2 & 1\end{bmatrix}\begin{bmatrix}3k\\-6k\\3k\end{bmatrix}}{\begin{bmatrix}1 & -2 & 1\end{bmatrix}\begin{bmatrix}m\\-2m\\m\end{bmatrix}} = \underline{\frac{3k}{m}}$$

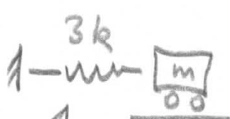
(e)

The first mode shape [1 1 1] corresponds to a rigid-body motion of a neutrally-stable system. The natural frequency is therefore zero.

The second mode shape [1 0 -1] corresponds to a symmetrical vibration of the two outer cars with the centre one stationery. In this case, each outer car behaves as a 1-DOF mass-spring system  for which $\omega^2 = k/m$

The third mode shape [1 -2 1] has nodal points 1/3 from the centre car



Each end car behaves as a 1-DOF mass-spring system  for which $\omega^2 = \frac{3k}{m}$

spring stiffness = 3k because $\frac{1}{3}$ of the spring is active

(stiffness $\propto \frac{1}{\text{length}}$)