< Duality between the Step Response & Bode plot >

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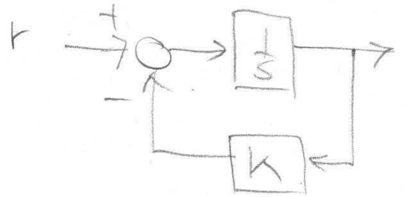
- Single-integrator feedback

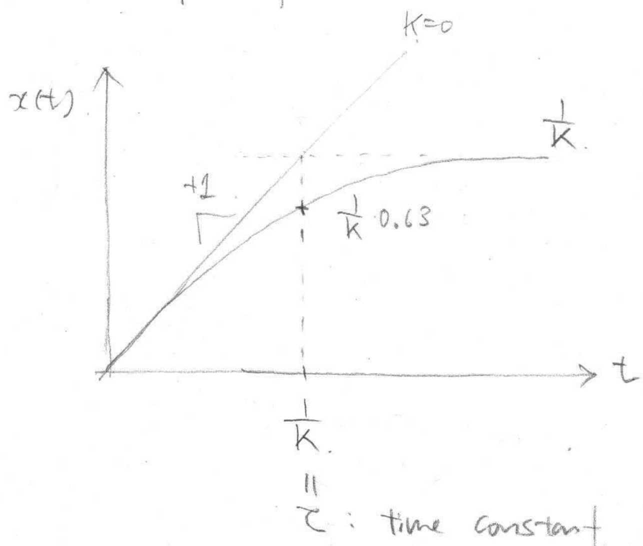


$L.T. = -\frac{K}{s}$ "Loop transmission"

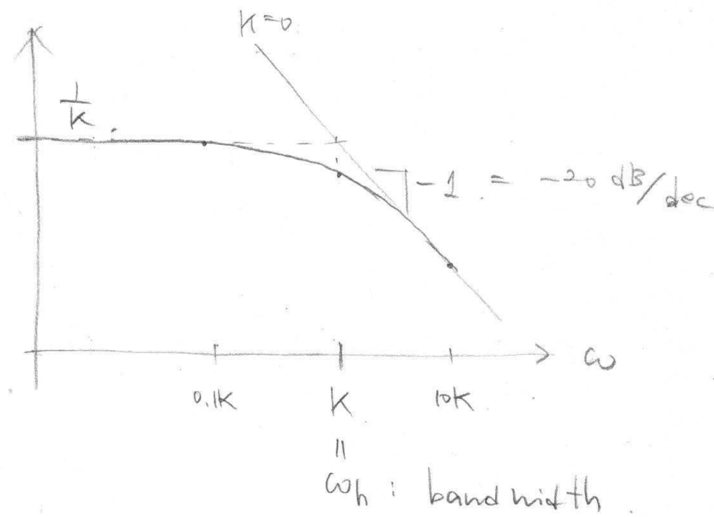
$L(s) = \frac{K}{s}$ "Loop Return Ratio"

$$G(s) = \frac{X}{R} = \frac{\frac{1}{s}}{1+L(s)} = \frac{\frac{1}{s}}{1+\frac{K}{s}} = \frac{1}{s+K}.$$

- Step Resp.



Bode plot



- Note the "duality" between the step resp. & freq. resp.

Time constant : $\frac{1}{K}$	Band width : K
Initial resp. : t	High-freq. resp : $\frac{1}{s}$
Final value : $\frac{1}{K}$	DC gain : $\frac{1}{K}$
No overshoot	No frequency peak

① We can "infer" the key features of step response (time-domain representation) from the Bode plot (freq-domain representation).

② When feedback is non-effective ($\omega \gg K$) : $G(s) \simeq \frac{1}{s}$

effective ($\omega \ll K$) : $G(s) \simeq \frac{1}{K}$.

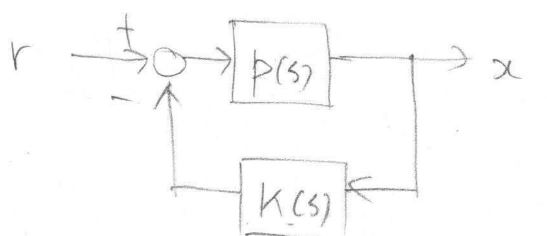
< Generalization to monotionic $L(j\omega)$ >

- In terms of $L(j\omega) = \frac{K}{j\omega}$, feedback is effective when

$$\left|\frac{K}{j\omega}\right| > 1 \iff |L(j\omega)| > 1.$$

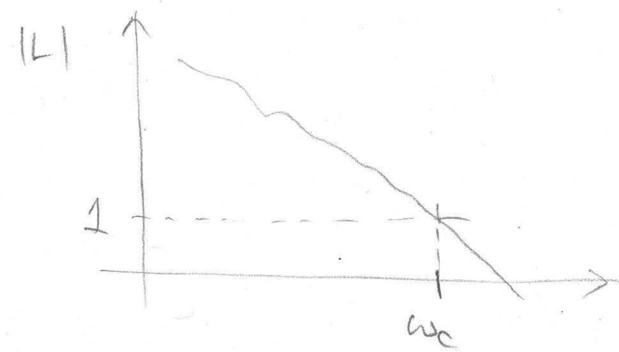
- In other words, feedback is effective in frequencies where the loop return ratio magnitude (Loop Gain) is higher than unity

- Consider a general feedback system.



$$L(s) = P(s)\,K(s).$$

and supose $|L(j\omega)|$ monotonically decreases with frequency.



ω_c : (unity-gain) cross-over freq.

$$|L(j\omega_c)| = 1.$$

By generalizing the single-integrator example, we can say

- Feedback is effective below ω_c. $\iff$ $L(j\omega) > 1$.

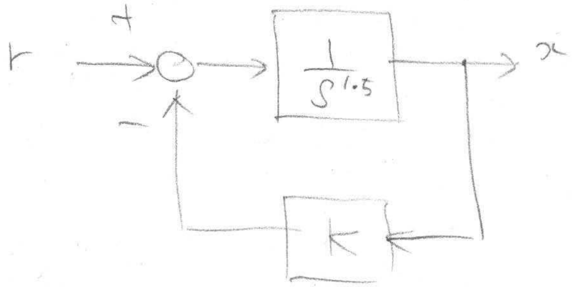
$$G(s) = \frac{P(s)}{1+L(s)} = \begin{cases} P(s) & \text{when } |L(j\omega)| \ll 1 \\ \frac{P(s)}{L(s)} = \frac{1}{K(s)} & \text{when } |L(j\omega)| \gg 1 \end{cases}$$

< Fractional-order Integrator >

· Let's check it with an exotic system. $P(s) = \frac{1}{s^{1.5}}$!?

Q. Does such a system exist? Yes. PA13 has $A(s) \simeq \frac{1}{s^{1.2}}$

Q. How can we handle such a system? → Frequency Resp



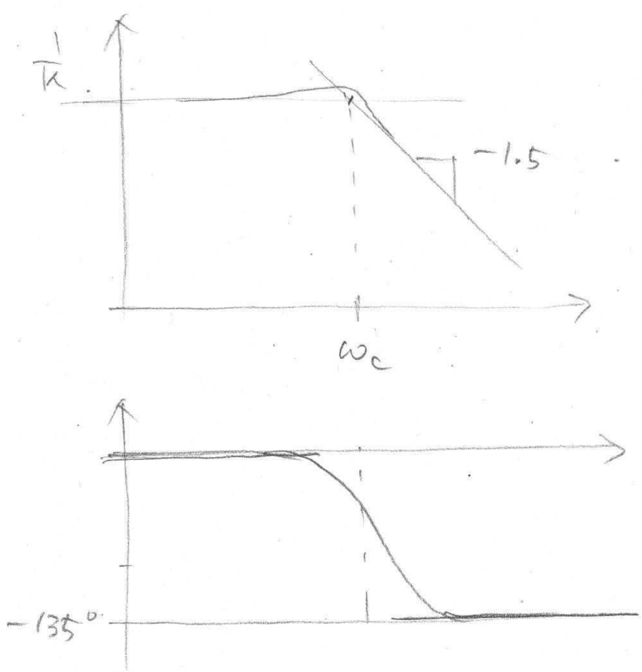
$$L(s) = \frac{K}{s^{1.5}}$$

$$L(j\omega) = \frac{K}{(j\omega)^{1.5}} = \frac{K}{\omega^{1.5}} \cdot \frac{1}{\left(e^{j\frac{\pi}{2}}\right)^{1.5}}$$

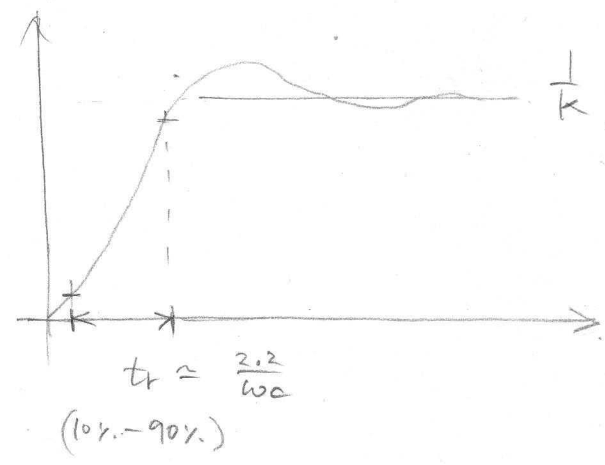
$$= \frac{K}{\omega^{1.5}} e^{-j\underbrace{\left(\frac{\pi}{2} \times 1.5\right)}_{135^\circ}}$$

$$|L(j\omega)| = \frac{K}{\omega^{1.5}} \Rightarrow \omega_c = K^{\frac{1}{1.5}}$$

Bode plot

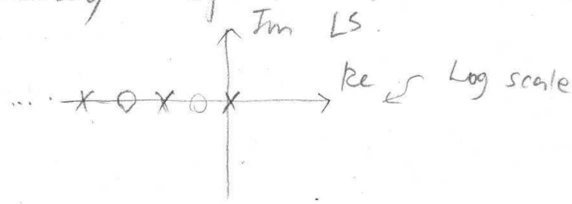


Step Resp.? $\Longrightarrow$



· We can "infer" not all but key features of step responses from the frequency response

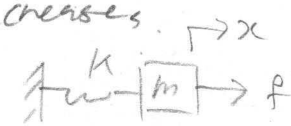
· We can approximate $\frac{1}{\sqrt{s}}$ with ...

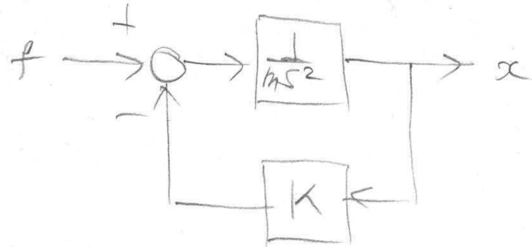


· MATLAB Demo.

< Double-integrator Feedback >

Let's see what happens if the integrator order increases.





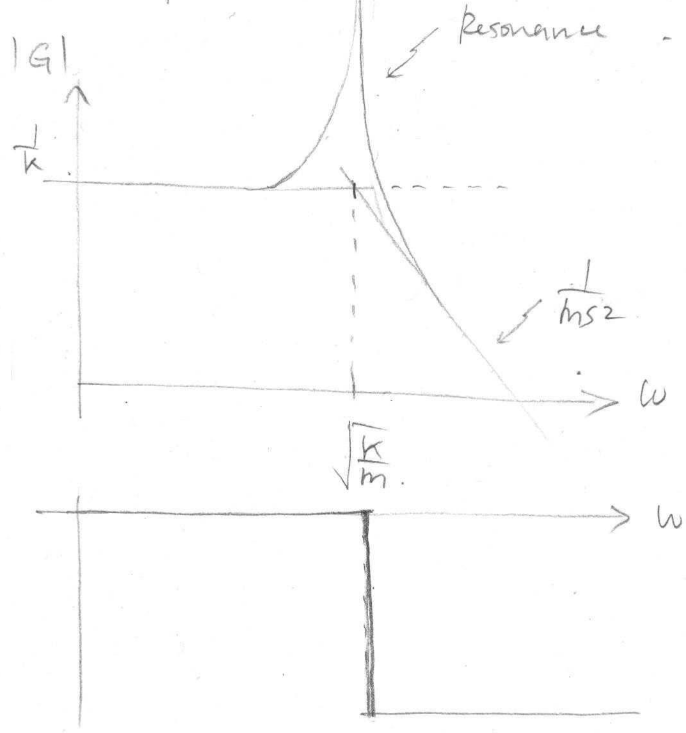
$\frac{1}{m}t$

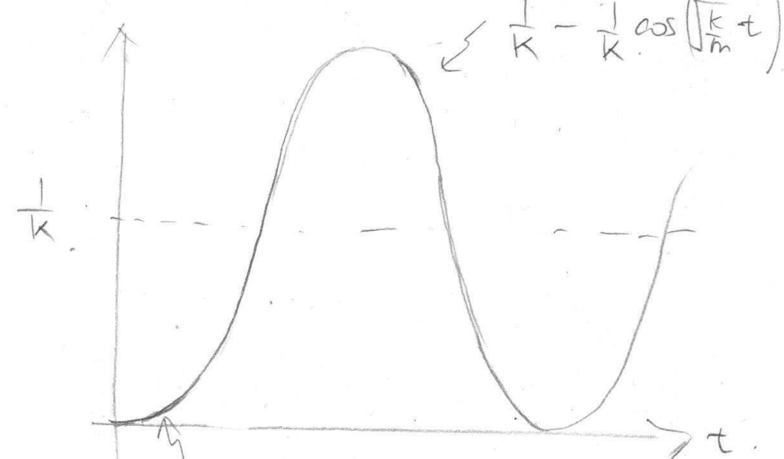
$$L.T. = -\frac{K}{ms^2}$$

$$L(s) = \frac{K}{ms^2}$$

$$G(s) = \frac{\frac{1}{ms^2}}{1+\frac{K}{ms^2}} = \frac{1}{ms^2+K}$$

Bode plot





Initial Resp $\simeq \frac{1}{m}\frac{t^2}{2}$

* We can find this waveform by solving $m\ddot{x} + kx = u(t)$

$u(t)$: unit step.

$x = x_p + x_h$

particular, Homogeneous

$x_p = \frac{1}{k}$

$x_h = c_1 \cos\left(\sqrt{\frac{k}{m}}\,t\right) + c_2 \sin\left(\sqrt{\frac{k}{m}}\,t\right)$

Initial Conditions : $x(0) = 0$, $\dot{x}(0) = 0$

$x = \frac{1}{k} + c_1 \cos(\omega_r t) + c_2 \sin(\omega_r t)$

$c_1 = -\frac{1}{k}, \quad c_2 = 0$

$\Rightarrow \quad x = \frac{1}{k} - \frac{1}{k}\cos(\omega_r t)$

$|L(j\omega_c)| = 1 \Rightarrow \frac{k}{m\omega^2} = 1 \quad \therefore \omega_c = \sqrt{\frac{k}{m}}$

For $\omega \gg \omega_c$, $G(s) \simeq \frac{1}{ms^2}$

For $\omega \ll \omega_c$, $G(s) \simeq \frac{1}{k}$

For $\omega \simeq \omega_c$, it depends on the "phase margin"

$$\boxed{\phi_m \triangleq \angle L(j\omega_c) + 180°}$$

But don't need it for loop shaping