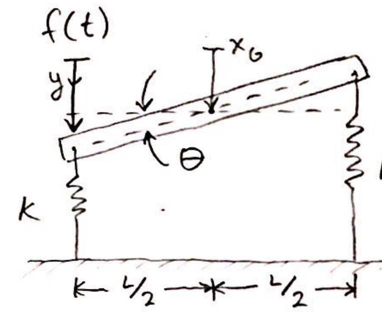
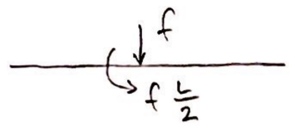
Lecture 14

Generalized Force Example



Equivalent:



$$Q_i = \sum_j F_j \frac{\partial y_j}{\partial q_i}$$

coordinate

or

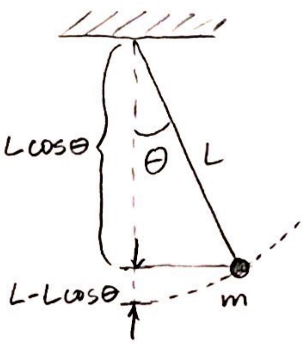
$$Q_i = \sum_j M_j \frac{\partial \phi_j}{\partial q_i}$$

$$y = x_G + \theta \frac{L}{2}$$

For $i=1$: $\quad Q_1 = f \frac{\partial}{\partial x}\left(x_G + \theta \frac{L}{2}\right) = f$

$i=2$: $\quad Q_2 = f \frac{\partial}{\partial \theta}\left(x_G + \theta \frac{L}{2}\right) = \frac{fL}{2}$

Pendulum Example



$$T = \frac{1}{2} m (L\dot{\theta})^2$$

$$V = mgL(1-\cos\theta)$$
$$= mgL\left(1-\left(1-\frac{\theta^2}{2}\right)\right)$$
$$\approx mgL\frac{\theta^2}{2}$$

series expansion: $\cos\theta = 1 - \frac{\theta^2}{2} + \ldots$

Lagrange: $\frac{d}{dt}\left(mL^2\dot{\theta}\right) - 0 + 0 + mgL\theta = 0$

$$\Rightarrow \ddot{\theta} + \frac{g}{L}\theta = 0$$

Matrix Symmetry

Lagrange gives $\frac{\partial V}{\partial q_i}$ → Stiffness matrix

For 1-DOF mass-spring $V = \frac{1}{2}kx^2$

2-DOF mass-spring $V = \frac{1}{2}(k_1+k_2)x_1^2 + \frac{1}{2}(k_2+k_3)x_2^2 + \frac{1}{2}k_2 x_1 x_2$

$$\Leftrightarrow V = \frac{1}{2}k_1 x_1^2 + \frac{1}{2}k_2(x_2-x_1)^2 + \frac{1}{2}k_3 x_2^2$$

In general, for a linear n-DOF system, V is quadratic.

$$V = \Big[\frac{1}{2}\left(k_{11}q_1^2 + k_{22}q_2^2 + \dots k_{NN}q_N^2\right) + \dots$$
$$+ \left(k_{12}q_1q_2 + k_{13}q_1q_3 + \dots + k_{1N}q_1q_N\right) + \dots$$
$$+ \left(k_{23}q_2q_3 + k_{24}q_2q_4 + \dots\right)\Big]$$

We get $[k]$ from $\frac{\partial V}{\partial q_i}$

For $i=1$: $\frac{\partial V}{\partial q_1} = k_{11}q_1 + k_{12}q_2 + \dots k_{1N}q_N$

$i=2$: $\frac{\partial V}{\partial q_2} = k_{12}q_1 + k_{22}q_2 + \dots k_{2N}q_N$

With all values of i:

$$\begin{bmatrix} k_{11} & k_{12} & k_{13} & \dots \\ k_{12} & k_{22} & k_{23} & \dots \\ k_{13} & k_{23} & k_{33} & \dots \\ \vdots & & & \end{bmatrix}\begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ \vdots \end{bmatrix} \Rightarrow \text{symmetrical}$$

k_{12} appears in $k_{12}q_1q_2$, $\frac{\partial V}{\partial q_1} = k_{12}q_2$, $\frac{\partial V}{\partial q_2} = k_{12}q_1$

We typically use normal matrix subscripts and have $k_{ij} = k_{ji}$

$\boxed{\text{Spring Based Coordinates}}$

If each spring has its own coordinate, then:

$$V = \left[\frac{1}{2}\left(k_{11} q_1^2 + k_{22} q_2^2 + k_{NN} q_N^2\right)\right]$$ Only diagonal terms.

For example, $k_{12} q_1 q_2 = \frac{1}{2} k_{12} q_1 q_2 + \frac{1}{2} k_{21} q_2 q_1$

In general, $k_{ij} q_i q_j = \frac{1}{2} k_{ij} q_i q_j + \frac{1}{2} k_{ji} q_j q_i$

$$\Rightarrow [V] = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} k_{ij} q_i q_j$$

$$= \frac{1}{2} \vec{q}^{\,T} [K] \vec{q}$$