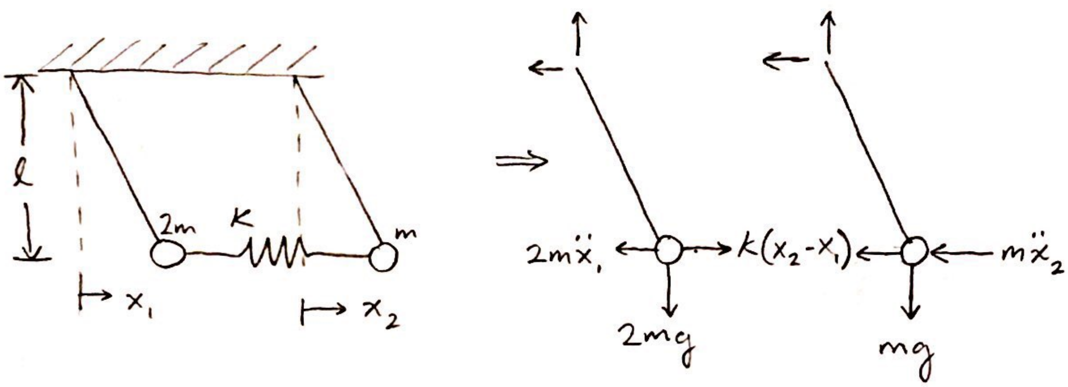
Lecture 8

2 - DOF

Ex: Twin Pendulum:



small θ
$\Rightarrow \sin\theta \simeq \theta$

Moments about tops:

$$2m\ddot{x}_1 l + 2mg x_1 - k(x_2 - x_1) l = 0$$

$$m\ddot{x}_2 l + mg x_2 + k(x_2 - x_1) l = 0$$

Matrix:

$$\begin{bmatrix} 2m & 0 \\ 0 & m \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} \frac{2mg}{l} + k & -k \\ -k & \frac{mg}{l} + k \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow [M]\ddot{\vec{x}} + [k]\vec{x} = \vec{0}$$

Solution: Try $\vec{x} = \vec{X}\cos(\omega t + \phi)$

Same for all problems,

$$\Rightarrow ([k] - \omega^2 [M]) \vec{X} \cos(\omega t + \phi) = \vec{0}$$

For non-trivial: $\det\left[[k] - \omega^2[M]\right] = 0$

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Determinant: $$\det \begin{bmatrix} \frac{2mg}{l} + k - 2m\omega^2 & -k \\ -k & \frac{mg}{l} + km\omega^2 \end{bmatrix} = 0$$

$$\Rightarrow \left(\frac{2mg}{l} + k - 2m\omega^2\right)\left(\frac{mg}{l} + k - m\omega^2\right) - k^2 = 0$$

$$\Rightarrow 2m^2\omega^4 - \left(\frac{4m^2g}{l} + 3mk\right)\omega^2 + \frac{2m^2g^2}{l^2} + \frac{3mgk}{l} = 0$$

$$\Rightarrow \boxed{\omega_1^2 = \frac{g}{l} \quad \text{and} \quad \omega_2^2 = \frac{g}{l} + \frac{3k}{2m}}$$

Mode Shape: Try $\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = C \begin{bmatrix} 1 \\ u \end{bmatrix}$ C is amplitude
u is ratio

Sub: $$\Rightarrow ([K] - \omega^2[M])\vec{X} = \vec{0}$$

$$([K] - \omega^2[M])\, C \begin{bmatrix} 1 \\ u \end{bmatrix} = \vec{0}$$

$$([K] - \omega^2[M]) \begin{bmatrix} 1 \\ u \end{bmatrix} = \vec{0}$$

Expand: $$\begin{bmatrix} \frac{2mg}{l} + k - 2m\omega^2 & -k \\ -k & \frac{mg}{l} + k - m\omega^2 \end{bmatrix} \begin{bmatrix} 1 \\ u \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Top line: $$\frac{2mg}{l} + k - 2m\omega^2 - uk = 0$$

$$\omega_1^2 = \frac{g}{l} \Rightarrow \boxed{u_1 = 1}, \quad \omega_2^2 = \frac{g}{l} + \frac{3k}{2m} \Rightarrow \boxed{u_2 = -2}$$

vibration around mass center

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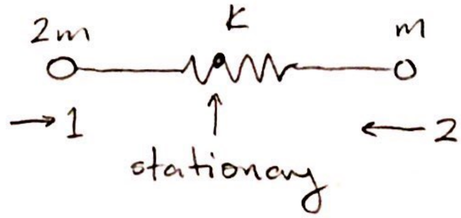
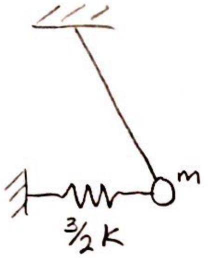
Mode shapes:



Equivalent:



For right-hand mass

Eigenvalue Solutions

$$\left([k] - \omega^2 [M]\right)\vec{X} = \vec{0}$$

$[k]\vec{X} = \omega^2 [M] X$ ⇒ "Generalized Eigenvalue problem"

$\left([M]^{-1}[k]\right)\vec{X} = \omega^2 \vec{X}$ ⇒ Eigenvalue equation

$$\begin{cases} \omega^2 \text{ are the eigenvalues} \rightarrow \text{natural frequencies} \\ \vec{X} \text{ are the eigenvectors} \rightarrow \text{mode shapes} \end{cases}$$

Since $[M]$ and $[k]$ are symmetric, the eigenvalues ω^2 are real.

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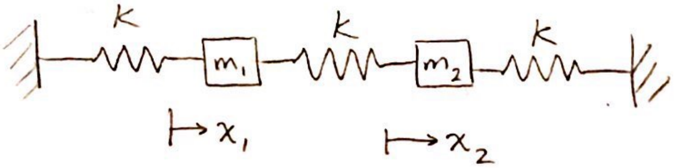
Coordinate Coupling

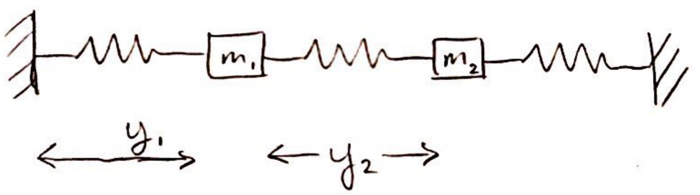


$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\begin{bmatrix} -\ddot{x}_1 \\ -\ddot{x}_2 \end{bmatrix} + \begin{bmatrix} 2k & -k \\ -k & 2k \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

mass-based gives diagonal [M]
$\Rightarrow$ no dynamic coupling

Spring based gives diagonal [K]
$\Rightarrow$ no static coupling

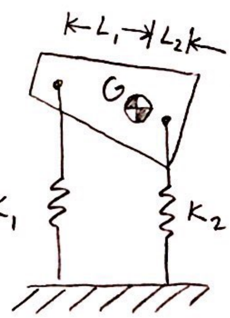
Consider spring based coordinate:



$\Rightarrow$ $x_1 = y_1$

$x_2 = y_1 + y_2$

Ex:



L_1, L_2 are distances to C.O.M.

Total length $L = L_1 + L_2$

Mass m

Moment of Inertia J

No dynamic coupling $\Rightarrow$ height of C.O.M
Rotation

No static coupling $\Rightarrow$ Spring lengths

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