# MECH468 : Modern Control Engineering
# MECH509 : Controls

## L7 : BIBO stability

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Zoom lecture to be recorded and posted on Canvas

# Course plan

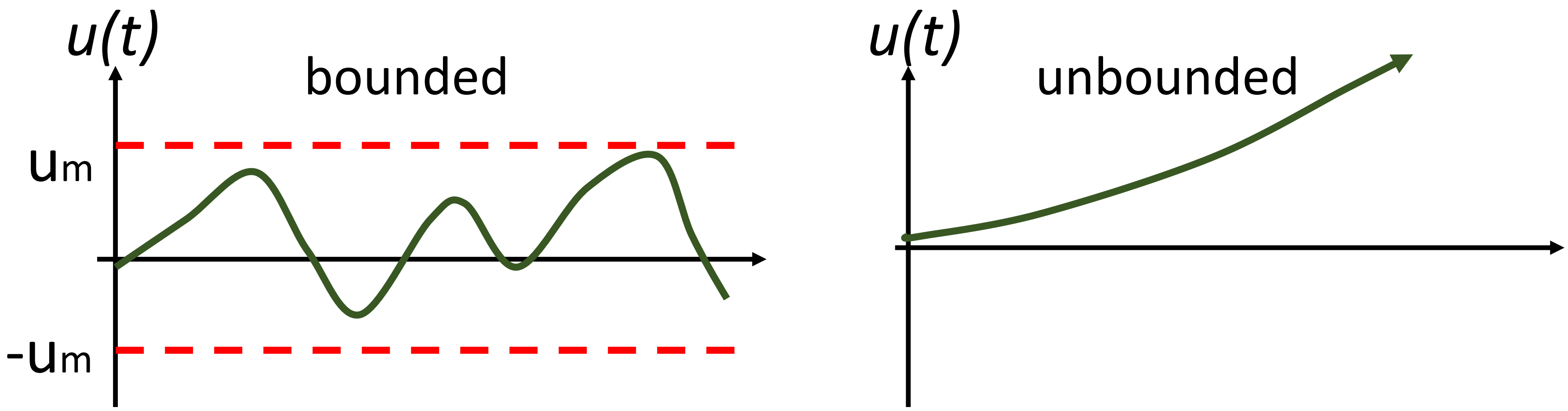
| Topics | CT | DT |
|---|---|---|
| Modeling | ✓ | ✓ |
| Stability | | |
| Controllability/observability | | |
| Realization | | |
| State feedback/observer | | |
| LQR/Kalman filter | | |

# Stability

- Utmost important specification in control design!
  - After stability, performance (tracking, disturbance/noise attenuation, etc.)
- Unstable systems to be stabilized by feedback
- Unstable closed-loop systems are useless.
  - An unstable system may hit mechanical/electrical "stops" (saturation), may break down or burn out.
- In this course, we learn two types of stability:
  - BIBO stability (1 lecture: today)
  - Internal stability (2 lectures)

# Bounded signal

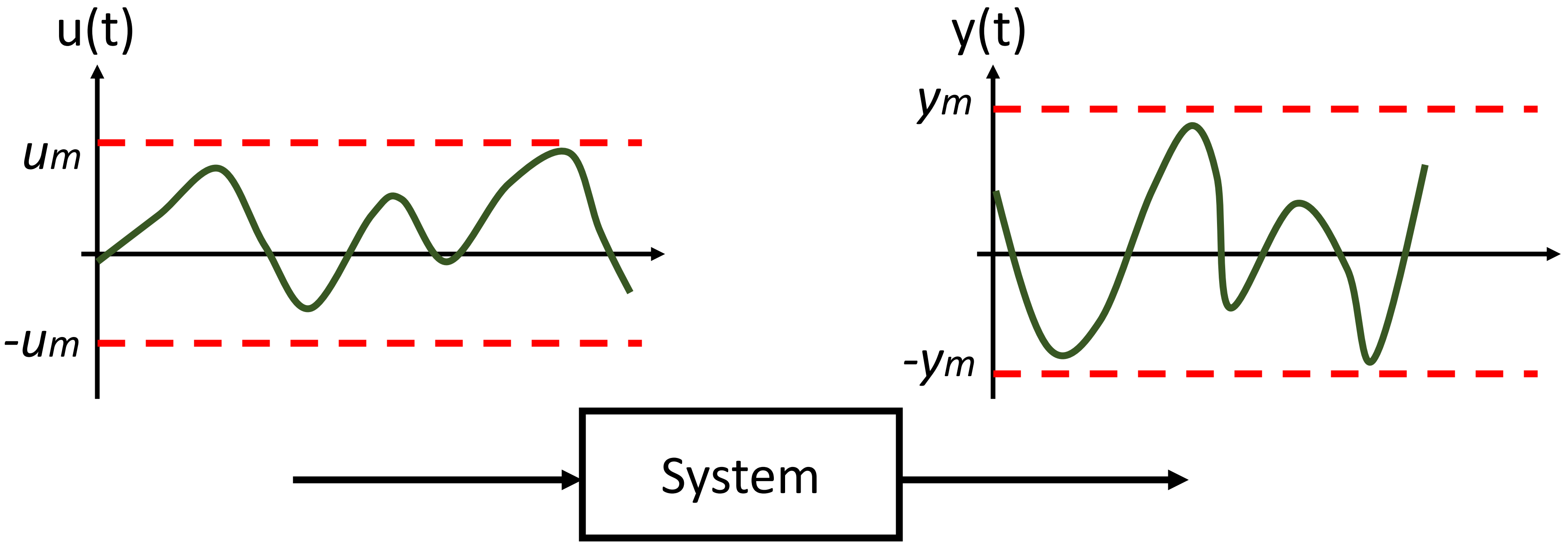
- Definition: A signal *u(t)* is called *bounded* if there exists a positive scalar $u_m$ such that $|u(t)| \leq u_m < \infty, \forall t \geq 0$



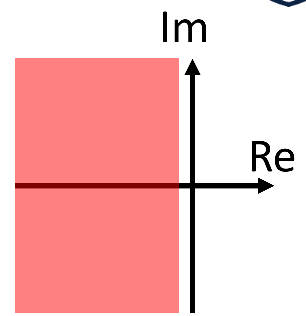

- A vector *u(t)* is bounded if every entry is bounded.

# BIBO stability

- Definition: A system is called *BIBO stable* if every bounded (possibly vector) input *u(t)* excites bounded (possibly vector) output *y(t)*.



# BIBO stability for CT LTI systems




- An CT LTI system *G(s)* is BIBO stable

  *if and only if*

  all the poles of *G(s)* are in the open left half of the complex plane.

Ex. $$G(s) = \frac{s-1}{(s+3)(s+1)}$$ Poles are s=-1,-3

BIBO stable!

Ex. $$G(s) = \frac{s+2}{(s+3)(s-1)}$$ Poles are s=1,-3

Not BIBO stable!

(One can use Routh-Hurwitz criterion.)

# Transfer function (review)

- A transfer function is defined by

$$G(s) := \frac{Y(s)}{R(s)}$$

*Laplace transform of system output*

*Laplace transform of system input*

$$Y(s) = G(s)R(s)$$

$$R(s) \longrightarrow \boxed{G(s)} \longrightarrow$$

- A system is assumed to be at rest. (Zero initial condition)

# From SS to TF (CT case)

- CT LTI SS model $\begin{cases} \dfrac{dx(t)}{dt} &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t) \end{cases}$
- Laplace transform with *x(0)=0*

$$\begin{cases} sX(s) - \cancel{x(0)} &= AX(s) + BU(s) \\ Y(s) &= CX(s) + DU(s) \end{cases}$$

$$\Rightarrow \begin{cases} X(s) &= (sI - A)^{-1}BU(s) \\ Y(s) &= CX(s) + DU(s) \end{cases}$$

*Memorize this!*

$$\Rightarrow Y(s) = \underbrace{\boxed{\left\{C(sI - A)^{-1}B + D\right\}}}_{=:G(s)}U(s)$$

# Simple criteria for BIBO stability

- 1st order polynomial $Q(s) = a_1 s + a_0$

  All roots are in LHP $\Leftrightarrow$ $a_1$ and $a_0$ have the same sign

- 2nd order polynomial $Q(s) = a_2 s^2 + a_1 s + a_0$

  All roots are in LHP $\Leftrightarrow$ $a_2$, $a_1$ and $a_0$ have the same sign

- Higher order polynomial $Q(s) = a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0$

  All roots are in LHP $\Rightarrow$ All $a_k$ have the same sign

# Examples

| Denominator of $G(s)$ | *G(s)* stable? |
|---|---|
| $3s + 5$ | **Yes / No** |
| $-2s^2 - 5s - 100$ | **Yes / No** |
| $523s^2 - 57s + 189$ | **Yes / No** |
| $(s^2 + s - 1)(s^2 + s + 1)$ | **Yes / No** |
| $s^3 + 5s^2 + 10s - 3$ | **Yes / No** |

# BIBO stability for CT LTI SS model

- Example
  - SS model
$$\dot{x} = \begin{bmatrix} -1 & 10 \\ 0 & 1 \end{bmatrix} x + \begin{bmatrix} -2 \\ 0 \end{bmatrix} u$$
$$y = \begin{bmatrix} -2 & 3 \end{bmatrix} x - 2u$$
  - TF
$$\begin{aligned} G(s) &:= C(sI - A)^{-1}B + D \\ &= \begin{bmatrix} -2 & 3 \end{bmatrix} \left( sI - \begin{bmatrix} -1 & 10 \\ 0 & 1 \end{bmatrix} \right)^{-1} \begin{bmatrix} -2 \\ 0 \end{bmatrix} - 2 \\ &= \frac{1}{(s+1)(s-1)} \begin{bmatrix} -2 & 3 \end{bmatrix} \begin{bmatrix} s-1 & 10 \\ 0 & s+1 \end{bmatrix} \begin{bmatrix} -2 \\ 0 \end{bmatrix} - 2 \\ &= \frac{4}{s+1} - 2 \end{aligned}$$

Pole is *s=-1*

(After pole/zero cancellation)

BIBO stable!

# Important remarks

- In this example, CT LTI system is BIBO stable.
- However, from the state equation,

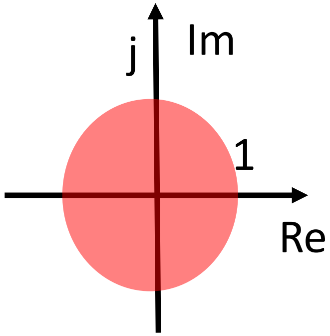
$$\dot{x}_2(t) = x_2(t) \Rightarrow x_2(t) = e^t x_2(0)$$

- For nonzero initial condition of $x_2$, the state $x_2$ goes unbounded, which generates divergent $y$.
- For state-space models, BIBO stability is not good enough to define meaningful stability concept.
- Stability concept that can deal with nonzero initial condition? (Internal stability)

# Summary

- BIBO stability
  - Definition
  - Conditions for both CT and DT LTI systems
  - Examples
- Transform from state-space model to transfer function model
- BIBO stability is not good enough to test the stability of state-space models.
- Next, internal stability

# BIBO stability for DT LTI systems




- An DT LTI system G(z) is BIBO stable

  *if and only if*

  all the poles of *G(z)* are in the open unit disc of the complex plane.

Ex. $$G(z) = \frac{2}{z - 0.5}$$ → Pole is z=0.5

BIBO stable!

Ex. $$G(z) = \frac{1}{z + 1}$$ → Pole is z=-1

Not BIBO stable!

(One can use Jury's test.)

# Z-transform (review)

- Definition: For a sequence *{f[k]: k=0,1,2,....}*

$$\boxed{F(z) = \mathcal{Z}\{f[k]\} := \sum_{k=0}^{\infty} f[k] z^{-k}}$$

- Shift property

$$\begin{aligned}
\mathcal{Z}\{f[k+1]\} &= \sum_{k=0}^{\infty} f[k+1] z^{-k} \\
&= z \sum_{k=0}^{\infty} f[k+1] z^{-(k+1)} \\
&= z(F(z) - f[0])
\end{aligned}$$

# From SS to TF (DT case)

- DT LTI SS model $\begin{cases} x[k+1] &= Ax[k] + Bu[k] \\ y[k] &= Cx[k] + Du[k] \end{cases}$

- Z-transform with x[0]=0

$$\begin{cases} z\{X(z) - \cancel{x[0]}\} &= AX(z) + BU(z) \\ Y(z) &= CX(z) + DU(z) \end{cases}$$

$$\Rightarrow \begin{cases} X(z) &= (zI - A)^{-1}BU(z) \\ Y(z) &= CX(z) + DU(z) \end{cases}$$

*Memorize this!*

$$\Rightarrow Y(z) = \underbrace{\boxed{\left\{C(zI - A)^{-1}B + D\right\}}}_{=:G(z)} U(z)$$