MECH468 : Modern Control Engineering
MECH509 : Controls

L3 : State-space models

Dr. Ryozo Nagamune
Department of Mechanical Engineering
University of British Columbia

Zoom lecture to be recorded and posted on Canvas

Review and today's topic

- Last lecture was about model classifications.
  - Continuous-time and discrete-time
  - Memoryless, causal and noncausal
  - Lumped and distributed
  - Time-invariant and time-varying
  - Linear and nonlinear
- Today, we introduce linear state-space models to describe causal lumped linear systems.

Linear state-space models

Continuous-time (CT)

$$\begin{cases} \dfrac{dx(t)}{dt} & = & A(t)x(t) + B(t)u(t) \\ y(t) & = & C(t)x(t) + D(t)u(t) \end{cases}$$

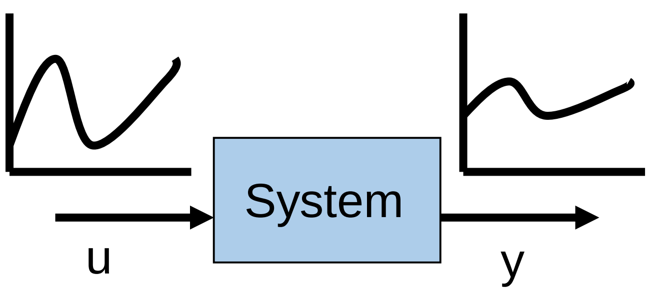
$t \in \mathbb{R}$ (Real numbers)

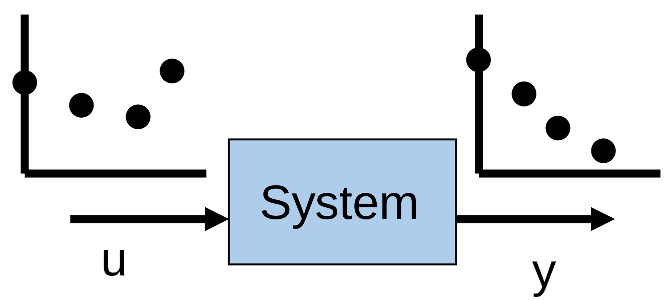
Discrete-time (DT)

$$\begin{cases} x[k+1] & = & A[k]x[k] + B[k]u[k] \\ y[k] & = & C[k]x[k] + D[k]u[k] \end{cases}$$

$k \in \mathbb{Z}$ (Integers)

x : state vector
u : input vector
y : output vector





Remarks

- The first equation, called *state equation*, is
  - a first-order ordinary differential (CT case) equation.
  - a first-order difference (DT case) equation.
- The second equation, called *output equation*, is an algebraic equation.
- Two equations are called *state-space model*.
- If a system is time-invariant, the matrices A, B, C, D are constant (independent of time).
- Pay attention to sizes of matrices and vectors. They must be always compatible!

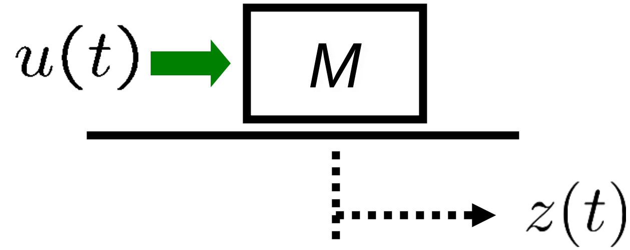
Examples

- Mass with a driving force
- Mass-spring-damper system
- RLC circuit
- DC motor

Mass with a driving force

- By Newton's law, we have

$$M\ddot{z}(t) = u(t)$$



where *u*: input force, *y*: output position

- Define state variables $x_1 := z,\ \ x_2 := \dot{z}$
- Then,

$$\begin{cases} \dot{x}_1 = \dot{z} = x_2 \\ \dot{x}_2 = \ddot{z} = \frac{1}{M}u \\ y = z \end{cases} \Rightarrow \begin{cases} \frac{d}{dt}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1/M \end{bmatrix} u(t) \\ y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \end{cases}$$

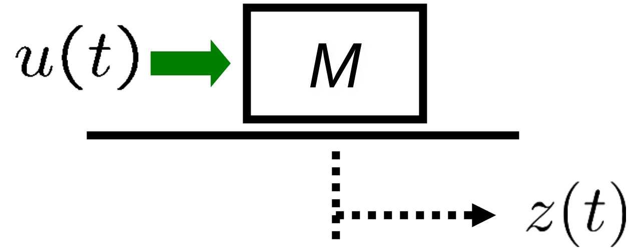
Mass with a driving force
Another SS model

- Define state variables $x_1 := \dot{z},\ x_2 := z$
- Then, $$\begin{cases} \frac{d}{dt}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 1/M \\ 0 \end{bmatrix} u(t) \\ y(t) = \begin{bmatrix} 0 & 1 \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \end{cases}$$
  - Derivation 1: Directly calculate.
  - Derivation 2: Using the SS model in the previous slide, exchange rows/columns of *A*, *B*, *C* matrices.
- *SS model is not unique; it depends on the selection of states.*

Mass with a driving force

- By Newton's law, we have

$$M\ddot{z}(t) = u(t)$$



where *u*: input force, *y*: output velocity

State-space model 1

$$x_1 := z, \;\; x_2 := \dot{z}$$

$$\begin{cases} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1/M \end{bmatrix} u \\ y = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \end{cases}$$

State-space model 2

$$x := \dot{z}$$

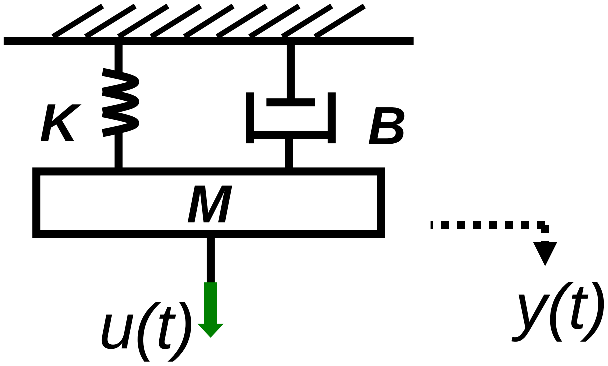
$$\begin{cases} \dot{x} = (1/M)u \\ y = x \end{cases}$$

Q: *Both models are correct.*
But which is better?
(Minimal realization in Lec. 18)

Mass-spring-damper system

- By Newton's law,

$$M\ddot{y}(t) = u(t) - Ky(t) - B\dot{y}(t)$$

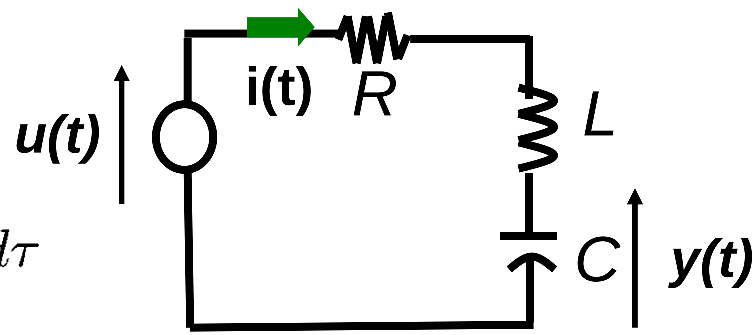


- Define state variables: $x_1(t) := y(t),\ x_2(t) := \dot{y}(t)$

$$\Rightarrow \begin{cases} \dfrac{d}{dt}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -K/M & -B/M \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1/M \end{bmatrix} u(t) \\ y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \end{cases}$$

RLC circuit

- By Kirchhoff's voltage law



$$u(t) = Ri(t) + L\frac{di(t)}{dt} + \frac{1}{C}\int i(\tau)d\tau$$

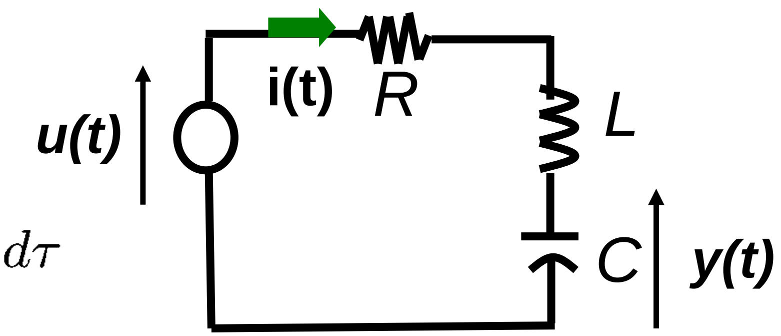
- Define state variables:
  - Current for inductor
  - Voltage for capacitor

$$x_1(t) := i(t), \ x_2(t) := \frac{1}{C}\int i(\tau)d\tau$$

$$\begin{cases} \frac{d}{dt}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} -R/L & -1/L \\ 1/C & 0 \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 1/L \\ 0 \end{bmatrix} u(t) \\ y(t) = \begin{bmatrix} 0 & 1 \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \end{cases}$$

RLC circuit: another SS model

- By Kirchhoff's voltage law



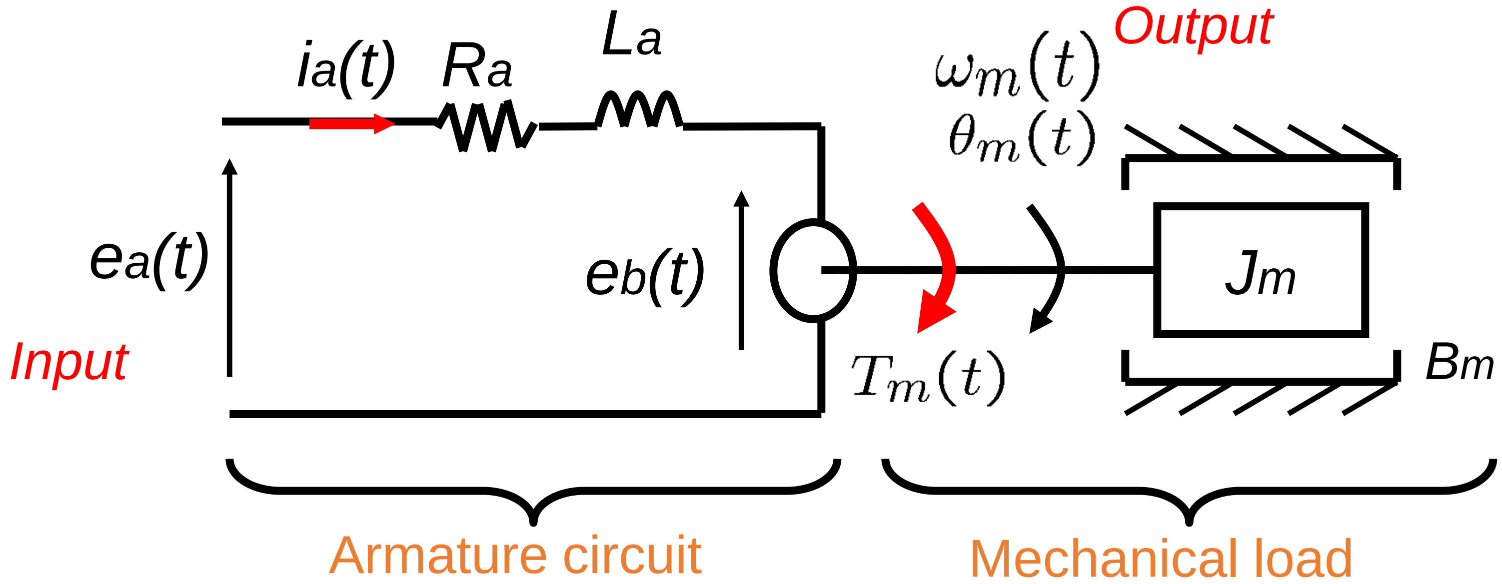
$$u(t) = Ri(t) + L\frac{di(t)}{dt} + \frac{1}{C}\int i(\tau)d\tau$$

- Define state variables:
  - Current for inductor
  - C*(Voltage for capacitor)

$$x_1(t) := i(t),\ x_2(t) := \int i(\tau)d\tau$$

$$\Rightarrow \begin{cases} \frac{d}{dt}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} -R/L & -1/LC \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 1/L \\ 0 \end{bmatrix} u(t) \\ y(t) = \begin{bmatrix} 0 & 1/C \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \end{cases}$$

Model of DC motor



"a" :armature

e_a :applied voltage

i_a :armature current

"b" :back EMF

"m" :mechanical

θ_m :angular position

ω_m :angular velocity

J_m : rotor inertia

B_m : viscous friction

Modeling of DC motor

- Armature circuit $$e_a(t) = R_a i_a(t) + L_a \frac{di_a(t)}{dt} + e_b(t)$$
- Connection between mechanical/electrical parts
  - Motor torque $$T_m(t) = K_i i_a(t)$$
  - Back EMF $$e_b(t) = K_b \omega_m(t)$$
- Mechanical load $$J_m \dot{\omega}_m(t) = T_m(t) - B_m \omega_m(t)$$
- Angular position $$\omega_m(t) = \dot{\theta}_m(t)$$

DC motor: output = speed

- By substitution, $$e_a(t) = R_a i_a(t) + L_a \frac{di_a(t)}{dt} + K_b \omega_m(t)$$ $$J_m \dot{\omega}_m(t) = K_m i_a(t) - B_m \omega_m(t)$$
- Define state variables $x_1(t) := i_a(t),\ x_2(t) := \omega_m(t)$

$$\Rightarrow \begin{cases} \frac{d}{dt}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} -R_a/L_a & -K_b/L_a \\ K_m/J_m & -B_m/J_m \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 1/L_a \\ 0 \end{bmatrix} e_a(t) \\ y(t) = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \end{cases}$$

DC motor: output = position

- By substitution,

$$e_a(t) = R_a i_a(t) + L_a \frac{di_a(t)}{dt} + K_b \omega_m(t)$$

$$J_m \dot{\omega}_m(t) = K_m i_a(t) - B_m \omega_m(t)$$

$$\omega_m(t) = \dot{\theta}_m(t)$$

- Define state variables $x_1(t) := i_a(t),\ x_2(t) := \theta_m(t),\ x_3(t) := \dot{\theta}_m(t)$

$$\Rightarrow \begin{cases} \frac{d}{dt}\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = \begin{bmatrix} -R_a/L_a & 0 & -K_b/L_a \\ 0 & 0 & 1 \\ K_m/J_m & 0 & -B_m/J_m \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} 1/L_a \\ 0 \\ 0 \end{bmatrix} e_a(t) \\ y(t) = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} \end{cases}$$

DC motor: two outputs

- Output vector $y(t) := \begin{bmatrix} \theta(t) \\ \omega(t) \end{bmatrix}$

- Define state variables $x_1(t) := i_a(t),\ x_2(t) := \theta_m(t),\ x_3(t) := \dot{\theta}_m(t)$

$$\Rightarrow \left\{ \begin{array}{lcl} \frac{d}{dt}\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} & = & \begin{bmatrix} -R_a/L_a & 0 & -K_b/L_a \\ 0 & 0 & 1 \\ K_m/J_m & 0 & -B_m/J_m \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} 1/L_a \\ 0 \\ 0 \end{bmatrix} e_a(t) \\ y(t) & = & \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} \end{array} \right.$$

Summary

- Linear state-space model
- Examples for continuous-time systems
  - Mechanical systems
  - Electrical systems
  - DC motor
- Discrete-time state-space models obtained by discretization of continuous-time systems will be explained later (in Lecture 6).
- Next, linearization

Review of linear algebra

- Product of a matrix and a vector

$$
\begin{aligned}
Ax &= \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \\
&= \begin{bmatrix} a_{11}x_1 + \cdots + a_{1n}x_n \\ \vdots \\ a_{n1}x_1 + \cdots + a_{nn}x_n \end{bmatrix} \\
&= \underbrace{x_1 \begin{bmatrix} a_{11} \\ \vdots \\ a_{n1} \end{bmatrix}}_{1^{st} \text{ column of } A} + \cdots + \underbrace{x_n \begin{bmatrix} a_{1n} \\ \vdots \\ a_{nn} \end{bmatrix}}_{n^{th} \text{ column of } A}
\end{aligned}
$$