University of British Columbia
Department of Mechanical Engineering

MECH468 Modern Control Engineering
MECH522 Foundations in Control Engineering
Midterm exam

Examiner: Dr. Ryozo Nagamune
February 9 (Friday), 2018, 1-1:50pm

Last name, First name

Name: Student #:

Signature:

Exam policies

- Allowed: One-page letter-size hand-written cheat sheet (both front side and back side)
- Not-allowed: PC, calculators.
- Write all your answers on this booklet. No extra sheet will be provided.
- Motivate your answers properly. (No chance to defend your answers orally.)
- 20 points in total.

Before you start ...

- Use washroom before the exam.
- Turn off your mobile phone.
- No eating.
- Questions are NOT allowed.

If you finish early ...

- Please stay at your seat until the end of exam, i.e., 1:50pm. (You are not allowed to leave the room before the end of exam, except going to washroom.)

To be filled in by the instructor/marker

Problem #	Mark	Full mark
1		12
2		4
3		4
Total		20

1. Consider the following continuous-time system:

$$\begin{cases} \dot{x}(t) &= \begin{bmatrix} -1 & 0 \\ 1 & -1 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 2 \end{bmatrix} u(t) \\ y(t) &= \begin{bmatrix} 1 & 0 \end{bmatrix} x(t) \end{cases}$$

(a) Check the internal stability of the system.

i. Use the eigenvalue criterion. (2pt)

ii. Use the Lyapunov Theorem. (2pt)

(b) Check if the system is controllable. (1pt)

(c) Check if the system is observable. (1pt)

(d) Obtain the Kalman decomposition. Write explicitly which state is controllable/uncontrollable and observable/unobservable. (4pt)

(e) Descretize the system with sampling period T, where T is a positive constant. (Hint: In this question, to obtain the matrix exponential, diagonalization method does not work.) (2pt)
(You can use $\mathcal{L}(\frac{1}{s+a}) = e^{-at}$. $\mathcal{L}(\frac{1}{(s+a)^2}) = te^{-at}$.)

Solution.

(a) Eigenvalues of A-matrix are -1 and -1, both of which are in the open left-half plane. Therefore, the system is asymptotically stable.

Lyapunov equation is

$$\begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} + \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\begin{array}{ll} (1,1) - \text{element} & -p_1 + p_2 - p_1 + p_2 = -1 \\ (1,2) - \text{element} & -p_2 + p_3 - p_2 = 0 \\ (2,2) - \text{element} & -p_3 - p_3 = -1 \end{array}$$

The solution becomes

$$P = \begin{bmatrix} 3/4 & 1/4 \\ 1/4 & 1/2 \end{bmatrix}$$

which is positive definite. Therefore, the system is asymptotically stable.

(b) Controllability matrix is

$$\mathcal{C} = \begin{bmatrix} 0 & 0 \\ 2 & -2 \end{bmatrix}.$$

It does not have full (row) rank. Thus the system is not controllable.

(c) Observability matrix is
$$\mathcal{O} = \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}.$$
It does not have full (column) rank. Thus the system is not observable.

(d)
$$\mathrm{Im}\mathcal{C} = \mathrm{span}\left\{e_2\right\}, \ \mathrm{Ker}\mathcal{O} = \mathrm{span}\left\{e_2\right\}$$
$$T^{-1} = [T_{c\bar{o}}, T_{\bar{c}o}] = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = T$$
By coordinate transformation $z = Tx$, we have the Kalman decomposition
$$\begin{bmatrix} \dot{z}_{c\bar{o}} \\ \dot{z}_{\bar{c}o} \end{bmatrix} = \underbrace{\begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}}_{TAT^{-1}} \begin{bmatrix} z_{c\bar{o}} \\ z_{\bar{c}o} \end{bmatrix} + \underbrace{\begin{bmatrix} 2 \\ 0 \end{bmatrix}}_{TB} u$$
$$y = \underbrace{\begin{bmatrix} 0 & 1 \end{bmatrix}}_{CT^{-1}} \begin{bmatrix} z_{c\bar{o}} \\ z_{\bar{c}o} \end{bmatrix}$$
The state $z_{c\bar{o}}$ is controllable and unobservable, while the state $z_{\bar{c}o}$ is uncontrollable and observable.

(e) Discretized model
$$\begin{cases} x[k+1] & = & A_d x[k] + B_d u[k] \\ y[k] & = & \begin{bmatrix} 1 & 0 \end{bmatrix} x[k] \end{cases}$$

$$\begin{aligned} A_d = e^{AT} &= \mathcal{L}^{-1}\left\{(sI - A)^{-1}\right\} = \mathcal{L}^{-1}\left\{\begin{bmatrix} s+1 & 0 \\ -1 & s+1 \end{bmatrix}^{-1}\right\} \\ &= \mathcal{L}^{-1}\left\{\begin{bmatrix} 1/(s+1) & 0 \\ 1/(s+1)^2 & 1/(s+1) \end{bmatrix}\right\} = \begin{bmatrix} e^{-T} & 0 \\ Te^{-T} & e^{-T} \end{bmatrix} \end{aligned}$$

$$\begin{aligned} B_d = \int_0^T e^{A\tau} d\tau \cdot B &= \int_0^T \begin{bmatrix} e^{-\tau} & 0 \\ \tau e^{-\tau} & e^{-\tau} \end{bmatrix} d\tau \cdot \begin{bmatrix} 0 \\ 2 \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ 2(1 - e^{-T}) \end{bmatrix} \end{aligned}$$

2. For the following controllable discrete-time system:

$$x[k+1] = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} x[k] + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u[k],$$

compute the minimum energy control which transfers the state vector from $x[0] = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ to $x[k_f] = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ for the cases when:

(a) the final time $k_f = 1$. (2pt)

(b) the final time $k_f = 3$. (2pt)

Solution

(a)

$$x[1] - A^1 x[0] = Bu[0]$$

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} u[0]$$

$$\begin{bmatrix} 0 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} u[0]$$

There is no input $u[0]$ satisfying the above equation.

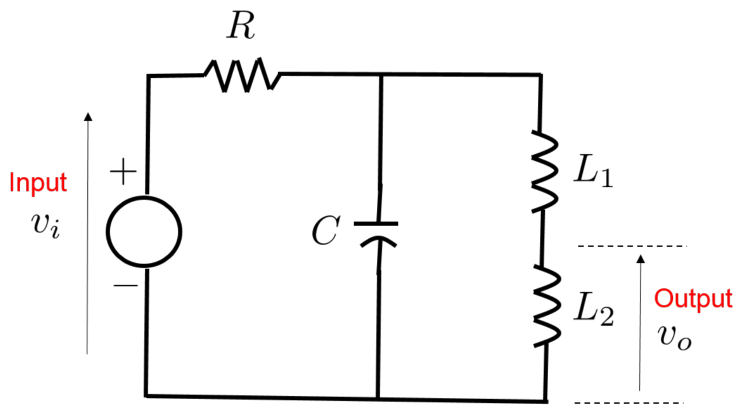
(b)

$$x[3] - A^3 x[0] = \begin{bmatrix} B & AB & A^2B \end{bmatrix} \begin{bmatrix} u[2] \\ u[1] \\ u[0] \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}^3 \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} u[2] \\ u[1] \\ u[0] \end{bmatrix}$$

$$\begin{aligned} \begin{bmatrix} u[2] \\ u[1] \\ u[0] \end{bmatrix} &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \left(\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \right)^{-1} \begin{bmatrix} 0 \\ -2 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1/2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ -2 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \\ 0 \end{bmatrix} \end{aligned}$$

3. Derive a state-space model with two states for the following electric circuit. Here, R is the resistance, C is the capacitance, and L_1 and L_2 are the inductances. The input is the voltage v_i and the output is the voltage v_o (i.e., voltage across L_2). (4pt)



Solution

Kirchhoff voltage law

$$v_i = Ri + \frac{1}{C}\int (i - i_L)$$

$$\frac{1}{C}\int (i - i_L) = (L_1 + L_2)\frac{di_L}{dt}$$

Define states as

$$x_1 := i_L, \; x_2 := \frac{1}{C}\int (i - i_L).$$

Then,

$$\dot{x}_1 = \frac{1}{L_1 + L_2}x_2$$

$$\dot{x}_2 = \frac{1}{C}(i - i_L) = \frac{1}{C}\left(\frac{1}{R}(v_i - x_2) - x_1\right)$$

$$y = L_2\frac{di_L}{dt} = L_2\dot{x}_1 = \frac{L_2}{L_1 + L_2}x_2$$

Thus, the state-space model is

$$\begin{aligned} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} &= \begin{bmatrix} 0 & \frac{1}{L_1+L_2} \\ -\frac{1}{C} & -\frac{1}{CR} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{CR} \end{bmatrix} v_i \\ y &= \begin{bmatrix} 0 & \frac{L_2}{L_1+L_2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \end{aligned}$$

Alternative solution

Define states as

$$x_1 := i_L, \; x_2 := \int (i - i_L).$$

Then,

$$\dot{x}_1 = \frac{1}{C(L_1+L_2)}x_2$$

$$\dot{x}_2 = i - i_L = \frac{1}{R}(v_i - \frac{1}{C}x_2) - x_1$$

$$y = L_2\frac{di_L}{dt} = L_2\dot{x}_1 = \frac{L_2}{C(L_1+L_2)}x_2$$

Thus, the state-space model is

$$\begin{aligned} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} &= \begin{bmatrix} 0 & \frac{1}{C(L_1+L_2)} \\ -1 & -\frac{1}{CR} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{R} \end{bmatrix} v_i \\ y &= \begin{bmatrix} 0 & \frac{L_2}{C(L_1+L_2)} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \end{aligned}$$