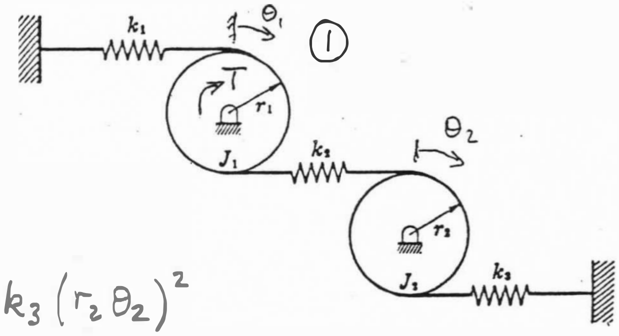
MECH 463 – Homework 9

This homework gives some practice at using Lagrange's Equations. All the questions here are reproduced from Homework 3. For each system, draw the displaced shape, and mark the displacements and rotations at key points in terms of the specified coordinate system (which is chosen for practice, not necessarily for convenience.) In terms of these coordinates, write the kinetic and potential energies of the system. Take the partial derivatives and substitute into Lagrange's equations to get the equations of motion. You need not solve the equations of motion for the natural frequencies and mode shapes unless you want the practice.

1. A wire with three springs k_1, k_2 and k_3 passes over two pulleys of radius r_1 and r_2 and polar moment of inertia J_1 and J_2. A torque T(t) is applied to the first pulley.



Kinetic energy:

$$T = \frac{1}{2} J_1 \dot{\theta}_1^2 + \frac{1}{2} J_2 \dot{\theta}_2^2$$

Potential energy:

$$V = \frac{1}{2} k_1 (r_1 \theta_1)^2 + \frac{1}{2} k_2 (r_1\theta_1 + r_2\theta_2)^2 + \frac{1}{2} k_3 (r_2\theta_2)^2$$

Generalized forces, $Q_1 = T$, $Q_2 = 0$. Also $R = 0$

Recall Lagrange's Equations

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} + \frac{\partial V}{\partial q_i} + \frac{\partial R}{\partial \dot{q}_i} = Q_i$$

For $i = 1$, $q_i = \theta_1$, $\dot{q}_i = \dot{\theta}_1$, $Q_i = T$

$$\frac{d}{dt}(J_1 \dot{\theta}_1) - 0 + k_1 r_1^2 \theta_1 + k_2 r_1 (r_1\theta_1 + r_2\theta_2) + 0 = T$$

For $i = 2$, $q_i = \theta_2$, $\dot{q}_i = \dot{\theta}_2$, $Q_i = 0$

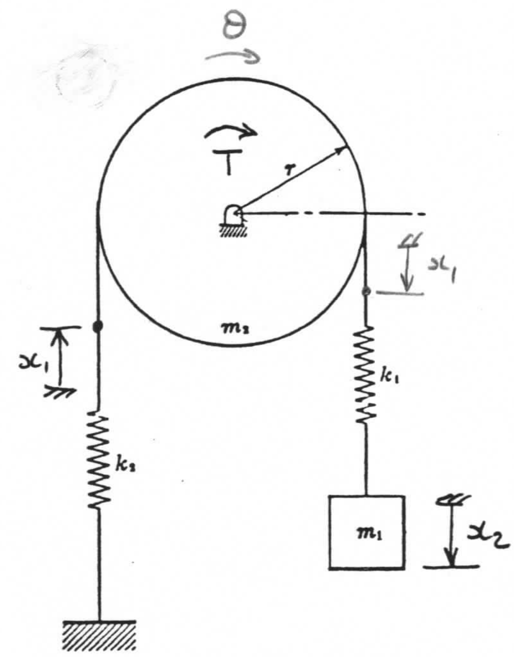
$$\frac{d}{dt}(J_2 \dot{\theta}_2) - 0 + k_2 r_2 (r_1\theta_1 + r_2\theta_2) + k_3 r_2^2 \theta_2 + 0 = 0$$

In matrix notation:

$$\begin{bmatrix} J_1 & 0 \\ 0 & J_2 \end{bmatrix} \begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix} + \begin{bmatrix} (k_1+k_2) r_1^2 & k_2 r_1 r_2 \\ k_2 r_1 r_2 & (k_2+k_3) r_2^2 \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} T \\ 0 \end{bmatrix}$$

2. A wire with springs k_1 and k_2 passes over a pulley, radius r, mass m_2 and moment of inertia $J = \frac{1}{2} m_2 r^2$, and supports a second mass m_1. A torque T(t) acts on the pulley.

Rotation of pulley $= \theta = \dfrac{x_1}{r}$



$$T = \tfrac{1}{2} J \dot{\theta}^2 + \tfrac{1}{2} m_1 \dot{x}_2^2$$

$$= \tfrac{1}{4} m_2 \dot{x}_1^2 + \tfrac{1}{2} m_1 \dot{x}_2^2 \qquad J = \tfrac{1}{2} m_2 r^2$$

$$V = \tfrac{1}{2} k_2 x_1^2 + \tfrac{1}{2} k_1 (x_2 - x_1)^2$$

Note: no need to include gravity because displacements are measured from the equilibrium position.

$$Q_1 = \frac{\partial \theta}{\partial x_1} \cdot T = \frac{T}{r}$$

$$Q_2 = \frac{\partial \theta}{\partial x_2} \cdot T = 0$$

Recall Lagrange's equations

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} + \frac{\partial V}{\partial q_i} + \frac{\partial R}{\partial \dot{q}_i} = Q_i$$

For $i = 1$ $\quad q_i = x_1 \quad \dot{q}_i = \dot{x}_1$

$$\frac{d}{dt}\left(\tfrac{1}{2} m_2 \dot{x}_1\right) - 0 + k_2 x_1 + k_1 (x_2 - x_1)(-1) + 0 = \frac{T}{r}$$

For $i = 2$ $\quad q_i = x_2 \quad \dot{q}_i = \dot{x}_2$

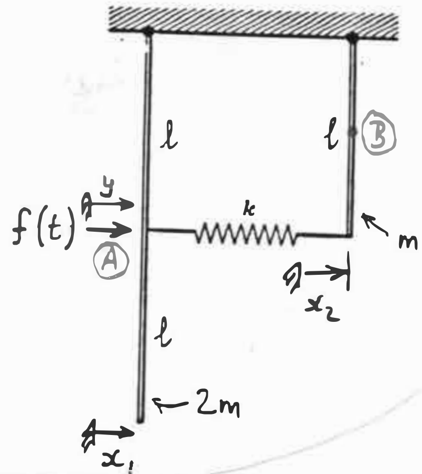
$$\frac{d}{dt}\left(m_1 \dot{x}_2\right) - 0 + k_1 (x_2 - x_1) + 0 = 0$$

In matrix form:

$$\begin{bmatrix} \frac{1}{2} m_2 & 0 \\ 0 & m_1 \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} k_1 + k_2 & -k_1 \\ -k_1 & k_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} T/r \\ 0 \end{bmatrix}$$

.3. Two uniform rods are pinned at their upper ends and connected together by a spring of stiffness k, a distance ℓ down. The long rod has mass 2m and length 2ℓ. The short rod has mass m and length ℓ. A force f(t) is applied at the midpoint of the long rod.

Ⓐ and Ⓑ are the centres of mass of the two rods.



At Ⓐ displacement $= \frac{x_1}{2} = y$

rotation $= \frac{x_1}{2\ell}$

At Ⓑ displacement $= \frac{x_2}{2}$

rotation $= \frac{x_2}{\ell}$

Kinetic energy, $T = \frac{1}{2}(2m)\left(\frac{\dot{x}_1}{2}\right)^2 + \frac{1}{2}\left(\frac{1}{12}\cdot 2m\cdot(2\ell)^2\right)\left(\frac{\dot{x}_1}{2\ell}\right)^2$

$+ \frac{1}{2} m \left(\frac{\dot{x}_2}{2}\right)^2 + \frac{1}{2}\left(\frac{1}{12} m \ell^2\right)\left(\frac{\dot{x}_2}{\ell}\right)^2$

$= \frac{1}{3} m \dot{x}_1^2 + \frac{1}{6} m \dot{x}_2^2$

Potential energy, $V = \frac{1}{2} k \left(x_2 - \frac{x_1}{2}\right)^2 + 2mg\left(\ell - \sqrt{\ell^2 - \left(\frac{x_1}{2}\right)^2}\right)$

$+ mg\left(\frac{\ell}{2} - \sqrt{\left(\frac{\ell}{2}\right)^2 - \left(\frac{x_2}{2}\right)^2}\right)$

$= \frac{1}{2} k \left(x_2 - \frac{x_1}{2}\right)^2 + 2mg\ell\left(1 - \left(1 - \left(\frac{x_1}{2\ell}\right)^2\right)^{1/2}\right)$

$+ \frac{mg\ell}{2}\left(1 - \left(1 - \left(\frac{x_2}{\ell}\right)^2\right)^{1/2}\right)$

$\simeq \frac{1}{2} k \left(x_2 - \frac{x_1}{2}\right)^2 + \frac{mg x_1^2}{4\ell} + \frac{mg x_2^2}{4\ell}$

Generalized force $Q_1 = \frac{\partial y}{\partial x_1} f(t)$ where y is the displacement associated with $f(t) = \frac{x_1}{2}$

$= \frac{1}{2} f(t)$

$Q_2 = \frac{\partial y}{\partial x_2} f(t) = 0$

Recall Lagrange's equations

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} + \frac{\partial V}{\partial q_i} + \frac{\partial R}{\partial \dot{q}_i} = Q_i$$

For $i=1$ $\quad q_i = x_1 \quad \dot{q}_i = \dot{x}_1$

$$\frac{d}{dt}\left(\frac{2}{3} m \dot{x}_1\right) - 0 + k\left(x_2 - \frac{x_1}{2}\right)\cdot\left(-\frac{1}{2}\right) + \frac{mg x_1}{2l} + 0 = \frac{1}{2} f(t)$$

For $i=2$ $\quad q_i = x_2 \quad \dot{q}_i = \dot{x}_2$

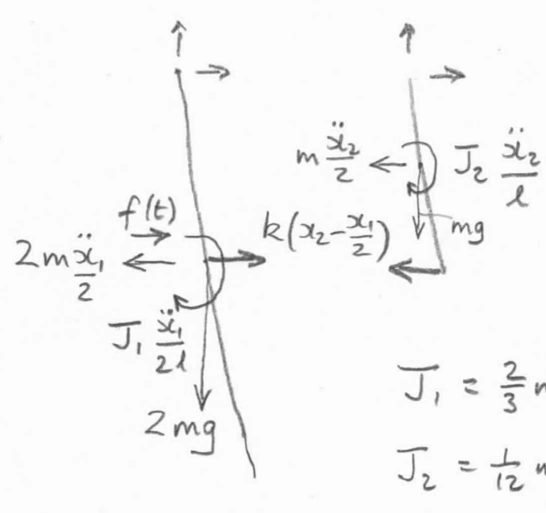
$$\frac{d}{dt}\left(\frac{1}{3} m \dot{x}_2\right) - 0 + k\left(x_2 - \frac{x_1}{2}\right) + \frac{mg x_2}{2l} + 0 = 0$$

In matrix notation:

$$\begin{bmatrix} \frac{2}{3} m & 0 \\ 0 & \frac{1}{3} m \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} \frac{k}{4} + \frac{mg}{2l} & -\frac{k}{2} \\ -\frac{k}{2} & k + \frac{mg}{2l} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} f(t) \\ 0 \end{bmatrix}$$

Alternatively, use free body diagram.

Take moments about tops of pendulums



$$2m\frac{\ddot{x}_1}{2}\cdot l + J_1 \frac{\ddot{x}_1}{2l} + 2mg\cdot\frac{x_1}{2} - kl\left(x_2 - \frac{x_1}{2}\right) - f(t)\, l = 0$$

$$m\frac{\ddot{x}_2}{2}\cdot\frac{l}{2} + J_2 \frac{\ddot{x}_2}{l} + mg\cdot\frac{x_2}{2} + kl\left(x_2 - \frac{x_1}{2}\right) = 0$$

$J_1 = \frac{2}{3} m l^2$

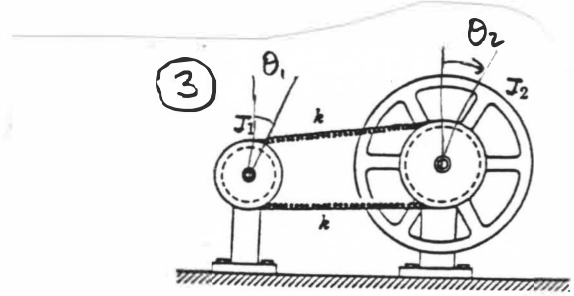
$J_2 = \frac{1}{12} m l^2$

$$\rightarrow \frac{4}{3} ml\, \ddot{x}_1 + \left(\frac{kl}{2} + mg\right) x_1 - kl x_2 = f(t)$$

$$\frac{1}{3} ml\, \ddot{x}_2 - \frac{kl}{2} x_1 + \left(kl + \frac{mg}{2}\right) x_2 = 0$$

Divide first equation by $2l$ and second by l to get our previous results.

4. A movie projector reel drive consists of a tightly wound helical spring, stretched around two grooved pulleys. The pulleys have radii r_1 and r_2 and centroidal moments of inertia J_1 and J_2. The stiffness of each exposed length of spring is k.



$$T = \tfrac{1}{2} J_1 \dot{\theta}_1^2 + \tfrac{1}{2} J_2 \dot{\theta}_2^2$$

$$V = \tfrac{1}{2} k \left(r_2\theta_2 - r_1\theta_1\right)^2 + \tfrac{1}{2} k \left(r_1\theta_1 - r_2\theta_2\right)^2$$

$$Q_1 = Q_2 = 0$$

Recall Lagrange's equations

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} + \frac{\partial V}{\partial q_i} + \frac{\partial R}{\partial \dot{q}_i} = Q_i$$

For $i = 1$ $\quad q_i = x_1 \quad \dot{q}_i = \dot{x}_1$

$$\frac{d}{dt}\left(J_1\dot{\theta}_1\right) - 0 + k\left(r_2\theta_2 - r_1\theta_1\right)(-r_1) + k\left(r_1\theta_1 - r_2\theta_2\right)(r_1) + 0 = 0$$

For $i = 2$ $\quad q_i = x_2 \quad \dot{q}_i = \dot{x}_2$

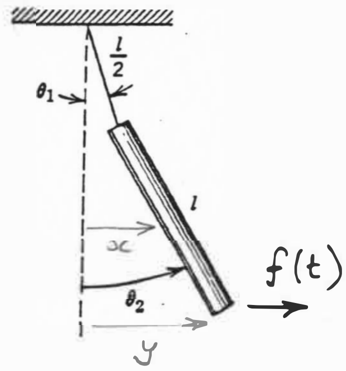
$$\frac{d}{dt}\left(J_2\dot{\theta}_2\right) - 0 + k\left(r_2\theta_2 - r_1\theta_1\right)(r_2) + k\left(r_1\theta_1 - r_2\theta_2\right)(-r_2) + 0 = 0$$

In matrix form:

$$\begin{bmatrix} J_1 & 0 \\ 0 & J_2 \end{bmatrix}\begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix} + \begin{bmatrix} 2kr_1^2 & -2kr_1r_2 \\ -2kr_1r_2 & 2kr_2^2 \end{bmatrix}\begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

5, A uniform slender rod of length ℓ is suspended at one end as a pendulum by a light string of length $\frac{1}{2}\ell$. A force f(t) acts at the lower end of the rod.

Assume small displacements.



Let x = lateral displacement of centre of rod

y = lateral displacement of lower end of rod

$$\rightarrow \quad x = \tfrac{l}{2}\theta_1 + \tfrac{l}{2}\theta_2$$

$$y = \tfrac{l}{2}\theta_1 + l\,\theta_2$$

$$T = \tfrac{1}{2} m\dot{x}^2 + \tfrac{1}{2} J\dot{\theta}_2^2$$

$$= \tfrac{1}{2} m\left(\tfrac{l}{2}\dot{\theta}_1 + \tfrac{l}{2}\dot{\theta}_2\right)^2 + \tfrac{1}{2}\left(\tfrac{1}{12} m l^2\right)\dot{\theta}_2^2$$

$$\tfrac{1}{8} m l^2\left(\dot{\theta}_1 + \dot{\theta}_2\right)^2 + \tfrac{1}{24} m l^2 \dot{\theta}_2^2$$

$$V = mg\left(\tfrac{l}{2}(1-\cos\theta_1) + \tfrac{l}{2}(1-\cos\theta_2)\right)$$

$$\simeq \frac{mgl}{4}\left(\theta_1^2 + \theta_2^2\right) \qquad \text{using} \quad \cos\theta \simeq 1 - \frac{\theta^2}{2}$$

$$Q_1 = \frac{\partial y}{\partial \theta_1} f(t) = \tfrac{l}{2} f(t) \qquad Q_2 = \frac{\partial y}{\partial \theta_2} f(t) = l\, f(t)$$

Recall Lagranges equations

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} + \frac{\partial V}{\partial q_i} + \frac{\partial R}{\partial \dot{q}_i} = Q_i$$

For $i=1$ $\quad q_i = \theta_1 \quad \dot{q}_i = \dot{\theta}_1$

$$\frac{d}{dt}\left(\tfrac{1}{4} m l^2(\dot{\theta}_1 + \dot{\theta}_2)\right) - 0 + \frac{mgl}{2}\theta_1 + 0 = \tfrac{l}{2} f(t)$$

For $i=2$ $\quad q_i = \theta_2 \quad \dot{q}_i = \dot{\theta}_2$

$$\frac{d}{dt}\left(\tfrac{1}{4} m l^2(\dot{\theta}_1 + \dot{\theta}_2) + \tfrac{1}{12} m l^2 \dot{\theta}_2\right) - 0 + \frac{mgl}{2}\theta_2 + 0 = l\, f(t)$$

Divide by l and put in matrix form $\rightarrow$

$$\begin{bmatrix} ml/4 & ml/4 \\ ml/4 & ml/3 \end{bmatrix}\begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix} + \begin{bmatrix} \frac{mg}{2} & 0 \\ 0 & \frac{mg}{2} \end{bmatrix}\begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} f(t) \\ f(t) \end{bmatrix}$$