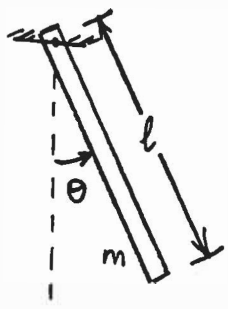
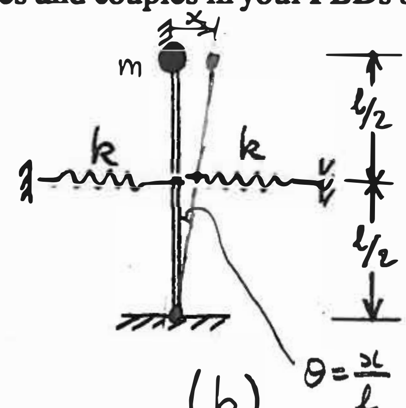
MECH463 -- Tutorial 3

1. Draw free body diagrams for each of the three vibrating systems shown in the diagrams. Formulate the equations of motion and identify the natural frequencies.

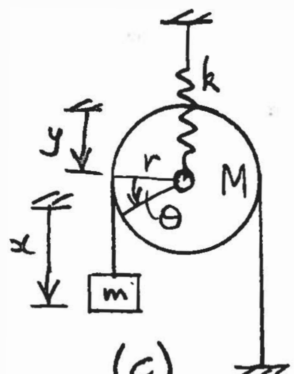
   Component (a) is a compound pendulum made of a uniform bar of length ℓ, mass M, and centroidal polar moment of inertia $J = m\ell^2/12$, that pivots around its upper end. Component (b) is an inverted pendulum with a mass m attached at the upper end of a light, stiff rod. The rod, which has length ℓ and pivots at the bottom, is supported by horizontal springs of stiffness k at its midpoint. Component (c) is a circular pulley of mass M, radius r, $J = \frac{1}{2}Mr^2$, supported by a spring of stiffness k at its centre. A light, stiff string passes around the pulley and secures a mass m. (Hint: Remember to include the inertia forces and couples in your FBDs based on the centres of mass.)



(a)



(b)

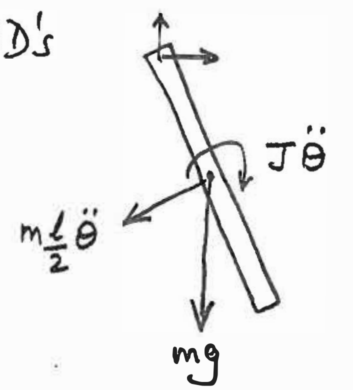


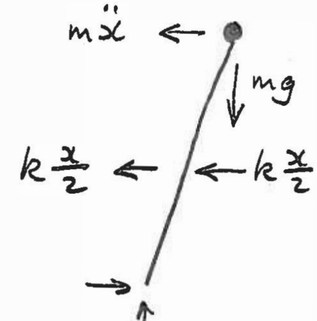
(c)

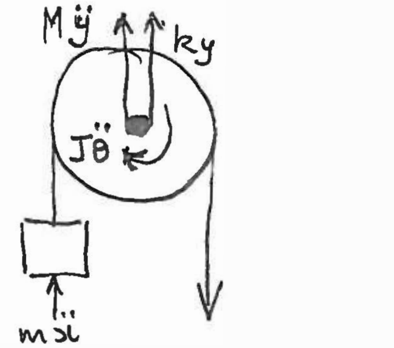
$$y = \frac{x}{2}$$

$$\theta = \frac{x-y}{r} = \frac{x}{2r}$$

FBD's







(a)

Take moments about top (to avoid unknown reactions)

$$m\frac{\ell}{2}\ddot{\theta}\cdot\frac{\ell}{2} + J\ddot{\theta} + mg\frac{\ell}{2}\sin\theta = 0$$

For small angles, $\sin\theta \simeq \theta$

$$\rightarrow \left(\frac{m\ell^2}{4} + \frac{m\ell^2}{12}\right)\ddot{\theta} + \frac{mg\ell}{2}\theta = 0$$

$$\rightarrow \frac{m\ell^2}{3}\ddot{\theta} + \frac{mg\ell}{2}\theta = 0$$

$$\omega^2 = \frac{3g}{2\ell}$$

(b)

Assume small angles.

Take moments about ~~bottom~~

$$m\ell\ddot{x} + 2\left(k\frac{x}{2}\right)\cdot\frac{\ell}{2} - mgx = 0$$

$$m\ell\ddot{x} + \left(k\frac{\ell}{2} - mg\right)x = 0$$

$$\omega^2 = \frac{k}{2m} - \frac{g}{\ell}$$

Stable only if $k > \frac{2mg}{\ell}$

(c)

Take moments about right side (to avoid unknown string tension)

$$2r\cdot m\ddot{x} + rM\ddot{y} + rky + J\ddot{\theta} = 0$$

$$2mr\ddot{x} + \frac{1}{2}Mr\ddot{x} + \frac{1}{2}krx + \frac{1}{2}Mr^2\cdot\frac{\ddot{x}}{2r} = 0$$

$$\left(\frac{3}{4}M + 2m\right)\ddot{x} + \frac{k}{2}x = 0$$

$$\omega^2 = \frac{2k}{3M + 8m}$$

2. A shock absorber is required that will have an overshoot of not more than 15% of its initial displacement when released. Determine the needed damping factor.

From classnotes, $x = e^{-\zeta\omega_n t}\left(A\cos\omega_d t - B\sin\omega_d t\right)$ where $\omega_d = \omega_n\sqrt{1-\zeta^2}$

$\rightarrow$ velocity, $\dot{x} = e^{-\zeta\omega_n t}\left(-\zeta\omega_n(A\cos\omega_d t - B\sin\omega_d t) - \omega_d(A\sin\omega_d t + B\cos\omega_d t)\right)$

$= e^{-\zeta\omega_n t}\left((A\zeta\omega_n + B\omega_d)\cos\omega_d t + (B\omega_d - A\zeta\omega_n)\sin\omega_d t\right)$

Given $\dot{x} = 0$ when $t = 0$ $\rightarrow$ $A\zeta\omega_n + B\omega_d = 0$

$\dot{x} = e^{-\zeta\omega_n t}\left(B\omega_d - A\zeta\omega_n\right)\sin\omega_d t$

The maximum point of the first overshoot, t_1 occurs when $\dot{x}$ next reaches 0 $\rightarrow \sin\omega_d t = 0 \rightarrow \omega_d t_1 = \pi \rightarrow t_1 = \frac{\pi}{\omega_d}$

At $t = t_0 = 0$ $\quad x_0 = e^{-\zeta\omega_n t_0}\left(A\cos\omega_d t_0 - B\sin\omega_d t_0\right) = A$

At $t = t_1 = \frac{\pi}{\omega_d}$ $\quad x_1 = e^{-\zeta\omega_n t_1}\left(A\cos\omega_d t_1 - B\sin\omega_d t_1\right) = -Ae^{-\zeta\pi\omega_n/\omega_d}$

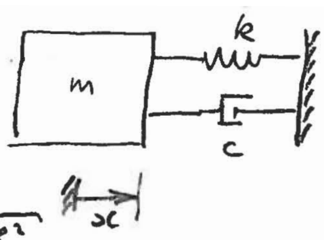
$= -Ae^{-\zeta\pi/\sqrt{1-\zeta^2}}$

Overshoot ratio $= \frac{-x_1}{x_0} = e^{-\zeta\pi/\sqrt{1-\zeta^2}} = 0.15$

$\rightarrow \frac{-\zeta\pi}{\sqrt{1-\zeta^2}} = \ln(0.15) = -1.90 \rightarrow \frac{\zeta}{\sqrt{1-\zeta^2}} = 0.60 \rightarrow \underline{\zeta = 0.5}$

3. A spring-damper system consisting of a spring k = 40N/mm in parallel with a damper c = 10N.s/mm is installed at the end of a railway siding. A freight car of mass 2000kg rolls along the siding and hits the spring-damper at speed 10 m/s. Determine (a) the maximum displacement of the spring-damper, and (b) the time taken to reach the maximum displacement.

After impact, the freight car and the spring-damper combine to form a 1-DOF system.



Given $m = 2000$ kg, $c = 10\times10^3$ N.s/m, $k = 40\times10^3$ N/m

$\omega_n = \sqrt{\frac{k}{m}} = 4.47$ rad/s. $\quad \zeta = \frac{c}{2\sqrt{km}} = 0.56 \quad \omega_d = \omega_n\sqrt{1-\zeta^2} = 3.70$ rad/s.

From classnotes, $x = e^{-\zeta\omega_n t}\left(A\cos\omega_d t - B\sin\omega_d t\right)$

Given $x = 0$ when $t = 0$ $\rightarrow A = 0 \rightarrow x = -Be^{-\zeta\omega_n t}\sin\omega_d t$

Velocity $\quad \dot{x} = Be^{-\zeta\omega_n t}\left(\zeta\omega_n\sin\omega_d t - \omega_d\cos\omega_d t\right)$

Given $\dot{x} = 10$ when $t = 0$ $\longrightarrow$ $10 = -B\omega_d \rightarrow B = -2.70$

Maximum displacement occurs at $t = t_1$ when $\dot{x} = 0$

$$\rightarrow \quad 0 = Be^{-\zeta\omega_n t_1}\left(\zeta\omega_n \sin\omega_d t_1 - \omega_d \cos\omega_d t_1\right)$$

$$\rightarrow \quad \zeta\omega_n \sin\omega_d t_1 - \omega_d\cos\omega_d t_1 = 0 \quad \rightarrow \tan\omega_d t_1 = \frac{\sin\omega_d t_1}{\cos\omega_d t_1} = \frac{\omega_d}{\zeta\omega_n}$$

$$\rightarrow \quad \tan\omega_d t_1 = \frac{\sqrt{1-\zeta^2}}{\zeta} = 1.48 \quad \rightarrow \omega_d t_1 = 0.98 \quad \rightarrow \underline{t_1 = 0.265 \text{ sec}}$$

At $t = t_1 \quad \rightarrow \quad x = -Be^{-\zeta\omega_n t_1} \sin\omega_d t_1$

$$x_1 = 2.70\, e^{-0.56\times 4.47\times 0.265} \sin(0.98) = \underline{1.15\text{ m}}$$

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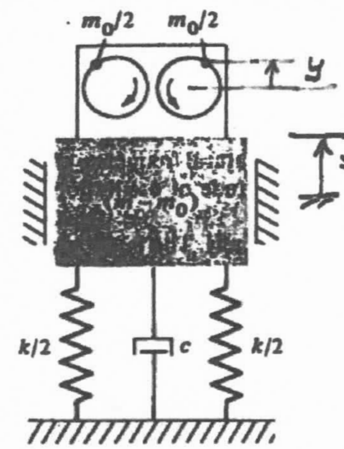
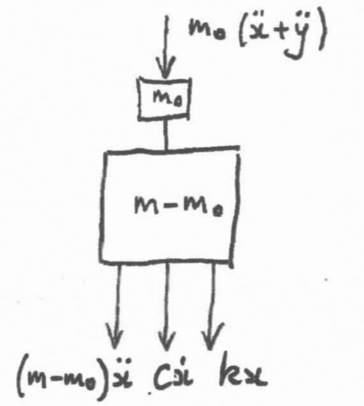
4. A machine with a total mass m = 50 kg contains a shaft mechanism of effective mass m_0 that rotates at 1800 rpm. The machine rests on springs of combined stiffness k = 200 kN/m, but the damping constant c is unknown. Due to an imbalance in the shaft, there is a steady-state vibration with amplitude 1 mm. The amplitude of the vibration force transmitted to the floor is 278N. Determine (a) the damping constant c, and (b) the unbalanced moment $m_0\varepsilon$.



Natural frequency, $\omega = \sqrt{k/m} = \sqrt{\dfrac{200000}{50}} = 63.2 \text{ rad/s}$

Rotation frequency, $\omega_f = \dfrac{1800\times 2\pi}{60} = 188.5 \text{ rad/s}$

Frequency ratio $r = \dfrac{\omega_f}{\omega} = \dfrac{188.5}{63.2} = 3.0$



Let x = displacement of the machine

Let y = displacement of eccentric mass <u>relative</u> to the machine

$\rightarrow \quad x+y$ = displacement of eccentric mass

$$y = \varepsilon\cos\omega_f t \quad \rightarrow \quad \ddot{y} = -\omega_f^2\varepsilon\cos\omega_f t$$

From FBD $\rightarrow \quad m_0(\ddot{x}+\ddot{y}) + (m-m_0)\ddot{x} + c\dot{x} + kx = 0$

$$\rightarrow \quad m\ddot{x} + c\dot{x} + kx = -m_0\ddot{y} = \omega_f^2 m_0\varepsilon\cos\omega_f t$$

$$= \omega_f^2 m_0 \varepsilon\, \mathrm{Re}\left[e^{i\omega_f t}\right]$$

Try solution $x = \mathrm{Re}\left[De^{i\omega_f t}\right]$ where D is complex, and substitute the entire trial solution into the equation of motion.

$$\rightarrow (-m\omega_f^2 + i\omega_f c + k) D e^{i\omega_f t} = \omega_f^2 m_o \varepsilon e^{i\omega_f t}$$

For a solution valid for all $t \rightarrow e^{i\omega_f t} \neq 0$

$$\rightarrow D = \frac{\omega_f^2 m_o \varepsilon}{-m\omega_f^2 + i\omega_f c + k} = \frac{\omega_f^2 m_o \varepsilon / k}{1 - \frac{m}{k}\omega_f^2 + i\frac{\omega_f c}{k}} = \frac{r^2 \frac{m_o}{m}\varepsilon}{(1-r^2) + i 2\zeta r}$$

where $\omega^2 = k/m$ $\quad r = \frac{\omega_f}{\omega}$ $\quad \zeta = \frac{c}{2\sqrt{km}}$

Amplitude, $\quad |D| = \dfrac{r^2 \frac{m_o}{m}\varepsilon}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}}$

Transmitted Force, $\quad F = kx + c\dot{x}$

$$= Re\left[(k + i\omega_f c) D e^{i\omega_f t}\right] \quad \text{substituting } x = Re\left[De^{i\omega_f t}\right]$$

$$= Re\left[(1 + i2\zeta r) kDe^{i\omega_f t}\right]$$

$$= Re\left[\frac{k r^2 \frac{m_o}{m}\varepsilon (1 + i2\zeta r)}{(1-r^2) + i2\zeta r} e^{i\omega_f t}\right] \quad \text{substituting } D$$

Force amplitude - $|F|$ - $\dfrac{k r^2 \frac{m_o}{m}\varepsilon \sqrt{1 + (2\zeta r)^2}}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}}$

Dynamic stiffness $= \dfrac{|F|}{|D|} = \dfrac{278}{0.001} = k\sqrt{1+(2\zeta r)^2} = 278000$

$\rightarrow 2\zeta r = 0.96$ $\qquad \zeta = 0.16$ $\qquad c = 2\zeta\sqrt{mk} = \underline{1020 \text{ N.s/m}}$

From amplitude equation:

$$m_o\varepsilon = \frac{|D| m \sqrt{(1-r^2)^2 + (2\zeta r)^2}}{r^2} = \frac{0.001 \times 50\sqrt{(1-3.0^2)^2 + (0.96)^2}}{3.0^2}$$

$$= \underline{0.045 \text{ kg.m}}$$