UNIVERSITY OF BRITISH COLUMBIA
DEPARTMENT OF MECHANICAL ENGINEERING
FINAL EXAMINATION, December 2019

MECH 463 – Mechanical Vibrations

Duration: 2.5 hours. Answer all 4 questions.

Materials admitted: Non-communicating, non-programmable calculator, personal handwritten notes within one side of one 8.5x11 sheet.

The purpose of this exam is to evaluate your knowledge of the course material. Orderly presentation demonstrates your knowledge most clearly, while disorganized and unprofessional work creates confusion and doubt (both for the marker and you!)

Write your name on *each* page *during* the examination time.

NAME: ________________________ SIGNATURE: ____________________________________

SECTION: _____________________ STUDENT NUMBER: _____________________________

Student Conduct During Examinations

1. Each examination candidate must be prepared to produce, upon the request of the invigilator or examiner, his or her UBCcard for identification.
2. Examination candidates are not permitted to ask questions of the examiners or invigilators, except in cases of supposed errors or ambiguities in examination questions, illegible or missing material, or the like.
3. No examination candidate shall be permitted to enter the examination room after the expiration of one-half hour from the scheduled starting time, or to leave during the first half hour of the examination. Should the examination run forty-five (45) minutes or less, no examination candidate shall be permitted to enter the examination room once the examination has begun.
4. Examination candidates must conduct themselves honestly and in accordance with established rules for a given examination, which will be articulated by the examiner or invigilator prior to the examination commencing. Should dishonest behaviour be observed by the examiner(s) or invigilator(s), pleas of accident or forgetfulness shall not be received.
5. Examination candidates suspected of any of the following, or any other similar practices, may be immediately dismissed from the examination by the examiner/invigilator, and may be subject to disciplinary action:
6. i. speaking or communicating with other examination candidates, unless otherwise authorized;
   ii. purposely exposing written papers to the view of other examination candidates or imaging devices;
   iii. purposely viewing the written papers of other examination candidates;
   iv. using or having visible at the place of writing any books, papers or other memory aid devices other than those authorized by the examiner(s); and,
   v. using or operating electronic devices including but not limited to telephones, calculators, computers, or similar devices other than those authorized by the examiner(s) — (electronic devices other than those authorized by the examiner(s) must be completely powered down if present at the place of writing).
7. Examination candidates must not destroy or damage any examination material, must hand in all examination papers, and must not take any examination material from the examination room without permission of the examiner or invigilator.
8. Notwithstanding the above, for any mode of examination that does not fall into the traditional, paper-based method, examination candidates shall adhere to any special rules for conduct as established and articulated by the examiner.
9. Examination candidates must follow any additional examination rules or directions communicated by the examiner(s) or invigilator(s).

CANDIDATES MUST IMMEDIATELY STOP WRITING WHEN THE INVIGILATOR ANNOUNCES THE EXAM IS OVER.

#	Mark	Max.
1		10
2		10
3		10
4		10
Total		**40**

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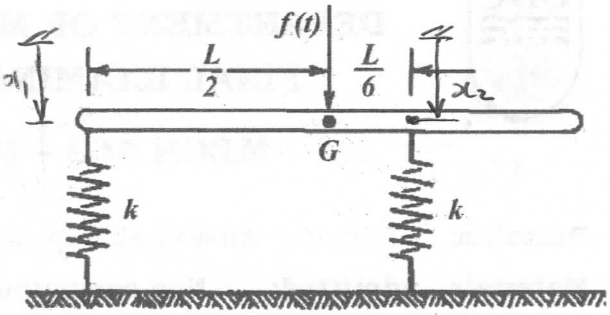
1. A part of a machine contains a uniform slender rod of length L, mass m, and centroidal $J = mL^2/12$. The rod is supported by two springs, each of stiffness k. One spring secures the rod at its left end, and the other spring secures the rod one third the way from the right end. An harmonic vertical force f(t) = F cos ω_f t acts at the centre of mass G.

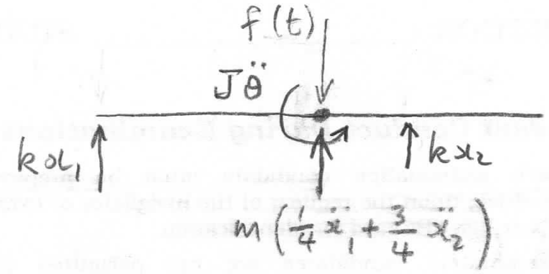


(a) Choose coordinates that will give no static coupling. Formulate the equations of motion in matrix format. Add/subtract rows as needed to make the matrices symmetrical.

(b) Determine the natural frequencies and corresponding mode shapes. Sketch the mode shapes and comment on any interesting features of the system.

(c) Determine and sketch the response of the beam at the excitation point G vs. frequency due the force f(t) = F cos ω_f t.

(a) For no static coupling, choose coordinates x_1 and x_2 based on the springs. By interpolation, the displacement at G is $\frac14 x_1 + \frac34 x_2$



where $J = \frac{1}{12}mL^2$ $\quad \theta \simeq \frac{x_2 - x_1}{2/3\, L}$

From FBD:

$$\Sigma M_2 = \frac{1}{12}mL^2\left(\frac{\ddot x_2-\ddot x_1}{2/3\, L}\right) - \frac{L}{6}m\left(\frac14\ddot x_1+\frac34\ddot x_2\right) - \frac23 Lkx_1 - \frac{L}{6}f(t) = 0$$

$$\Sigma M_1 = \frac{1}{12}mL^2\left(\frac{\ddot x_2-\ddot x_1}{2/3\, L}\right) + \frac{L}{2}m\left(\frac14\ddot x_1+\frac34\ddot x_2\right) + \frac23 Lkx_2 + \frac{L}{2}f(t) = 0$$

$$\Rightarrow \quad \frac18 mL\ddot x_1 - \cancel{\frac18 mL\ddot x_2} + \frac{1}{24}mL\ddot x_1 + \cancel{\frac18 mL\ddot x_2} + \frac23 Lkx_1 - \frac{L}{6}f(t) = 0$$

$$\cancel{\frac18 mL\ddot x_1} - \frac18 mL\ddot x_2 - \cancel{\frac18 mL\ddot x_1} - \frac38 mL\ddot x_2 - \frac23 Lkx_2 + \frac{L}{2}f(t) = 0$$

$$\div \begin{bmatrix} \frac{L}{6} \\ -\frac{L}{6} \end{bmatrix} \rightarrow \begin{bmatrix} m & 0 \\ 0 & 3m \end{bmatrix}\begin{bmatrix} \ddot x_1 \\ \ddot x_2 \end{bmatrix} + \begin{bmatrix} 4k & 0 \\ 0 & 4k \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} f(t) \\ 3f(t) \end{bmatrix}$$

(b)

We've hit the jackpot! Having both $\underset{\sim}{M}$ and $\underset{\sim}{K}$ diagonal means that our spring-based coordinates are also the principal coordinates (not usual). The equations of motion are uncoupled and separate out to:

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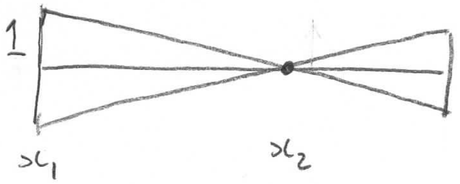
$$m\ddot{x}_1 + 4kx_1 = f(t)$$
$$3m\ddot{x}_2 + 4kx_2 = 3f(t)$$

$\rightarrow \quad \omega_1^2 = 4\frac{k}{m}, \quad \omega_2^2 = \frac{4}{3}\frac{k}{m}$

(The principal coordinates happen not to be in frequency order)

For principal coordinates, the mode shapes are:

$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ 

$\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ 

The nodal points happen to be at the springs.

(c) For first equation, $m\ddot{x}_1 + 4kx_1 = f(t) = F\cos\omega_f t$

try particular solution $x_1 = X_1 \cos\omega_f t$

$$\rightarrow (-m\omega_f^2 + 4k)X_1\cos\omega_f t = F\cos\omega_f t \longrightarrow X_1 = \frac{F}{4k - m\omega_f^2}$$

Similarly, for the second eqn. $\quad 3m\ddot{x}_2 + 4kx_2 = 3F\cos\omega_f t \longrightarrow X_2 = \frac{3F}{4k - 3m\omega_f^2}$

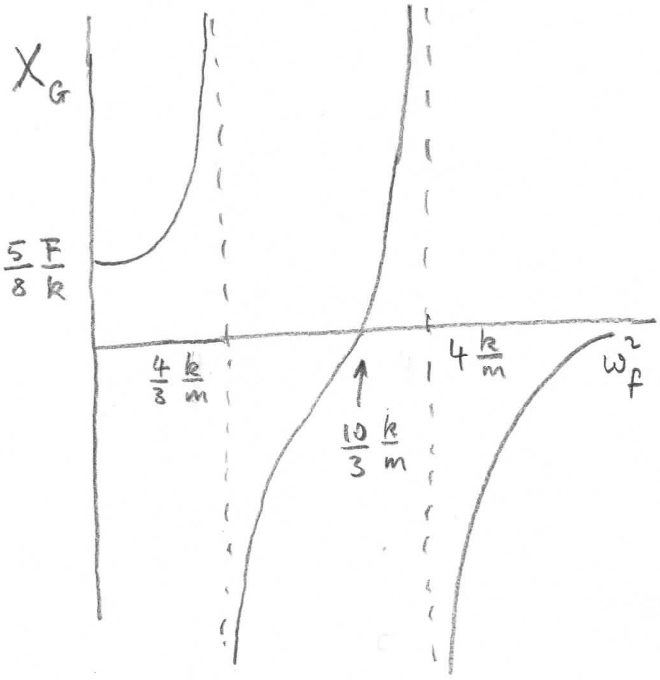
By interpolation:

$$X_G = \frac{1}{4}X_1 + \frac{3}{4}X_2$$

$$X_G = \frac{F}{4}\left(\frac{1}{4k - m\omega_f^2} + \frac{9}{4k - 3m\omega_f^2}\right)$$

$$X_G = \frac{F}{4}\,\frac{(4k - 3m\omega_f^2) + 9(4k - m\omega_f^2)}{(4k - m\omega_f^2)(4k - 3m\omega_f^2)}$$

$$X_G = \frac{(10k - 3m\omega_f^2)F}{(4k - m\omega_f^2)(4k - 3m\omega_f^2)}$$



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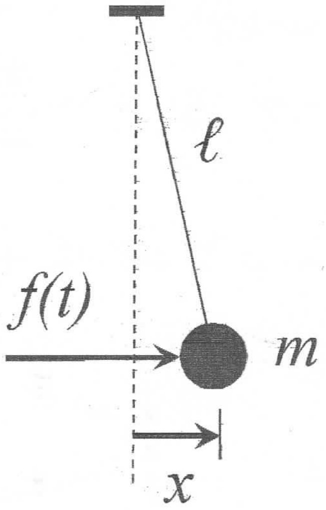
2. A horizontal force f = f(t) acts on a pendulum of mass m and length ℓ.

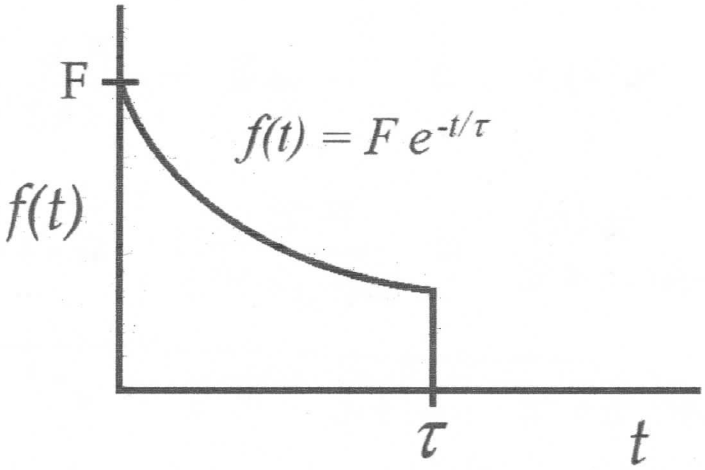
   a. Determine the equivalent spring constant "k" for the pendulum.

   b. Determine the horizontal displacement response x of the system for zero initial conditions, given $f(t) = F\,e^{-t/\tau}$. Assume small vibrations.

   c. Starting from the solution to (a), determine the response of the pendulum to the pulse excitation shown in the diagram for times $t > \tau$.

   d. Is it possible to choose the time constant τ so that the oscillation becomes zero for $t > \tau$? If so, what value of τ should be chosen? If not, why not?



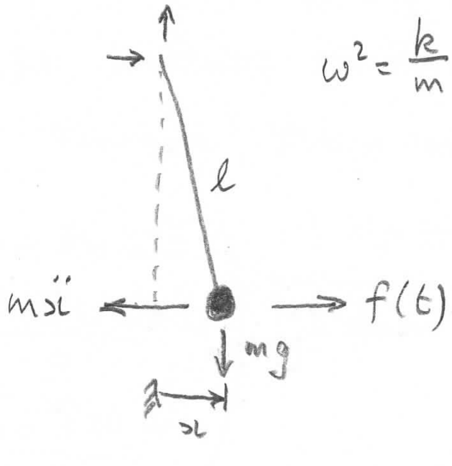


(a) Take moments about top of FBD

$$-m\ddot{x}\,l + f(t)\,l - mg\,x = 0$$

$$\div l \rightarrow m\ddot{x} + \underbrace{\frac{mg}{l}}_{\text{"k"}}x = f(t)$$

The equivalent "k" $= \dfrac{mg}{l}$



$$\omega^2 = \frac{k}{m} = \frac{g}{l}$$

(b) For $f(t) = Fe^{-t/\tau}$ $\rightarrow$ $m\ddot{x} + kx = Fe^{-t/\tau}$

Complementary solution is $x_c = A\cos\omega t - B\sin\omega t$

For particular solution, try $x_p = Xe^{-t/\tau}$

$$\rightarrow \left(\frac{m}{\tau^2} + k\right)Xe^{-t/\tau} = Fe^{-t/\tau} \longrightarrow X = \frac{F}{\frac{m}{\tau^2} + k} = \frac{F\tau^2}{m + k\tau^2}$$

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General solution $x = x_c + x_p$

$$x = A\cos\omega t - B\sin\omega t + \frac{F\tau^2}{m+k\tau^2}e^{-t/\tau}$$

For zero initial conditions:

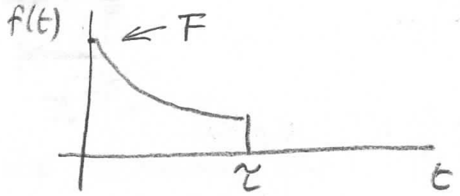
$$x(0) = A - 0 + \frac{F\tau^2}{m+k\tau^2} = 0 \rightarrow A = -\frac{F\tau^2}{m+k\tau^2}$$

$$\dot{x} = -\omega A\sin\omega t - \omega B\cos\omega t - \frac{F\tau}{m+k\tau^2}e^{-t/\tau}$$

$$\dot{x}(0) = 0 - \omega B - \frac{F\tau}{m+k\tau^2} \rightarrow B = -\frac{F\tau/\omega}{m+k\tau^2}$$

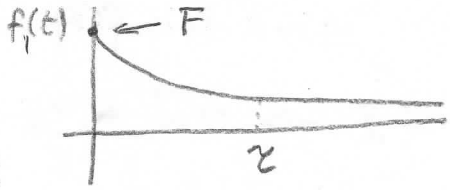
$$\underset{\text{for } t<\tau}{x(t)} = \frac{F\tau}{\omega(m+k\tau^2)}\left(\sin\omega t - \omega\tau\cos\omega t + \omega\tau e^{-t/\tau}\right)$$

(c) At $t=\tau \rightarrow Fe^{-t/\tau} = F\cdot e^{-1}$

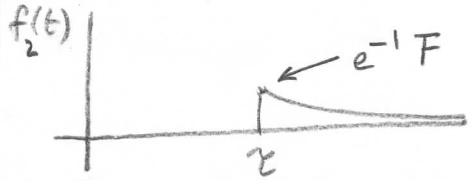


To get specified pulse, need to combine solutions $x(t) - e^{-1}x(t-\tau)$

For $t>\tau$, response is:



$$\frac{F\tau}{\omega(m+k\tau^2)}\left(\begin{array}{l}\sin\omega t - \omega\tau\cos\omega t + \omega\tau e^{-t/\tau} \\ -\dfrac{\sin\omega(t-\tau)}{e} + \dfrac{\omega\tau\cos\omega(t-\tau)}{e} \\ -\dfrac{\omega\tau e^{-(t-\tau)/\tau}}{e}\end{array}\right)$$



We see that $\dfrac{e^{-(t-\tau)/\tau}}{e} = e^{-1}\cdot e^{-\frac{t}{\tau}+1} = e^{-t/\tau}$

$$\underset{\text{for } t>\tau}{x(t)} = \frac{F\tau}{\omega(m+k\tau^2)}\left(\sin\omega t - \frac{\sin\omega(t-\tau)}{e} - \omega\tau\cos\omega t + \frac{\omega\tau\cos\omega(t-\tau)}{e}\right)$$

(d) Even if $\omega\tau = 2\pi$, subtraction does not give zero response because of the $\frac{1}{e}$ term $\rightarrow$ Impossible to zero response.

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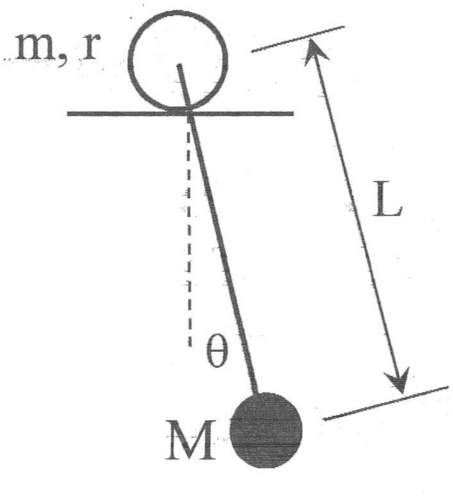
3. The diagram shows a rolling pendulum of length L and mass M, supported by a cylindrical roller of radius r, mass m and centroidal moment of inertia $J = mr^2/2$. The pendulum is rigidly fixed to the roller by a massless rod such that the roller and pendulum move together. The roller rolls without slipping on a horizontal surface.



   i. Describe in words the motion of the pendulum mass and the roller as the pendulum vibrates.

   ii. Use Lagrange's equation to formulate the equation of motion. Assume small vibrations.

   iii. Find a formula for the natural frequency of the rolling pendulum. Check that your result corresponds to a simple pendulum when $r \to 0$.

   iv. For the finite r case, does an increase in M cause an increase or decrease in natural frequency? Comment on your result and give a physical (not just mathematical) explanation.

(i) The pendulum rocks from side to side, with centre of rotation at the contact point. Thus, when mass M moves to the right, mass m moves to the left.

(ii) For small θ, velocity of M $= (L-r)\dot{\theta}$ (for rotation around contact point)

velocity of m $= r\dot{\theta}$

angular velocity of m $= \dot{\theta}$

$\rightarrow$ Kinetic energy $T = \frac{1}{2}M(L-r)^2\dot{\theta}^2 + \frac{1}{2}mr^2\dot{\theta}^2 + \frac{1}{2}J\dot{\theta}^2$

$T = \frac{1}{2}M(L-r)^2\dot{\theta}^2 + \frac{3}{4}mr^2\dot{\theta}^2$

Change in height of M $= L(1-\cos\theta)$ (centre of roller keeps constant height)

$\rightarrow$ Potential energy $V = MgL(1-\cos\theta)$ (no height change for m)

Lagrange's Equation:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}}\right) - \frac{\partial T}{\partial q} + \frac{\partial R}{\partial \dot{q}} + \frac{\partial V}{\partial q} = Q$$

where $q = \theta$, $R = 0$, $Q = 0$

$$\frac{d}{dt}\left(\frac{\partial}{\partial \dot{\theta}}\left(\frac{1}{2}M(L-r)^2\dot{\theta}^2 + \frac{3}{4}mr^2\dot{\theta}^2\right)\right) + \frac{\partial}{\partial \theta}\left(MgL(1-\cos\theta)\right) = 0$$

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$$= \frac{d}{dt}\left(M(L-r)^2\dot{\theta} + \frac{3}{2}mr^2\dot{\theta}\right) + \left(MgL\sin\theta\right) = 0$$

$$= \left(M(L-r)^2 + \frac{3}{2}mr^2\right)\ddot{\theta} + MgL\theta = 0 \qquad \text{for } \theta \to 0$$

(iii) Natural frequency $\omega = \sqrt{\dfrac{MgL}{M(L-r)^2 + \frac{3}{2}mr^2}} = \sqrt{\dfrac{g}{\frac{(L-r)^2}{L} + \frac{3mr^2}{2ML}}}$

When $r \to 0$ $\omega \to \sqrt{\dfrac{g}{L}}$ formula for simple pendulum

(iv) When M gets bigger, the $\frac{m}{M}$ term in the denominator gets smaller and consequently the natural frequency increases. This is contrary to typical experience, where increase in mass decreases natural frequency. (Note mass in the denominator of $\omega = \sqrt{\frac{k}{m}}$).

The natural frequency of a simple pendulum is $\omega = \sqrt{\frac{g}{l}} = \sqrt{\frac{mg/l}{m}}$ where $\frac{mg}{l}$ is the equivalent k.

In this case, both numerator and denominator depend on the same mass, so the effect cancels. For the rolling pendulum, the "stiffness" of the pendulum depends on M, while the "mass" depends on M and m. Thus, increase in M increases the "stiffness" faster than the "mass", so the natural frequency increases.

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4. A graduate student receives a small sample of a metal alloy from a manufacturer. For her research, she needs to know the Young's modulus of the alloy, and she decides to do this by measuring the natural frequencies of the sample. The sample is a plate, 152mm square, 3.07mm thick, and mass 557grams.

   The student drills a small hole in one comer, and suspends the plate by a thin thread. She then taps the plate at each of the four points shown in the diagram, and measures the resulting vibrations using a microphone connected to a spectrum analyzer. The four graphs show the resulting frequency spectra.

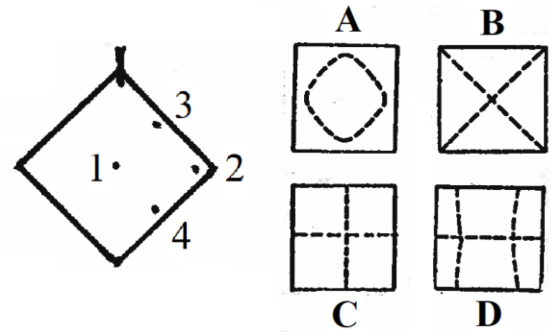
   In the library, the student found a book that shows the first four vibration modes of a square plate. These are reproduced in diagrams A, B, C and D. (They are not in frequency order). The dashed lines represent the nodal lines (zero vibration positions) of the vibration modes. In another book, she found the following formula for the natural frequency f (in Hz) of mode "D" :

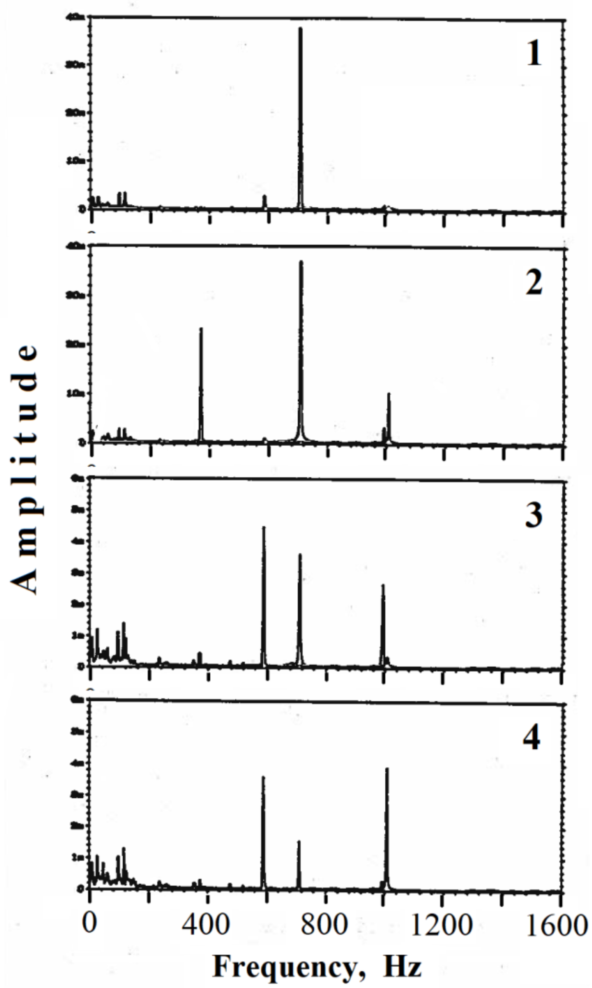
$$f = \frac{35.02}{2\pi\, a^2} \sqrt{\frac{E\, h^2}{12\, \rho\, (1 - \nu^2)}}$$

   where a = side length of the square, h = plate thickness, and ρ = mass density. Identify the frequencies of the four vibration modes from the measured frequency spectra. Explain your identifications. Calculate the Young's modulus of the new alloy (assume ν = 0.3).

   Bonus: Can you identify the direction the plate was rolled during its manufacture?

   (Hint: The ideas useful for this question were also used in Lab 2.)

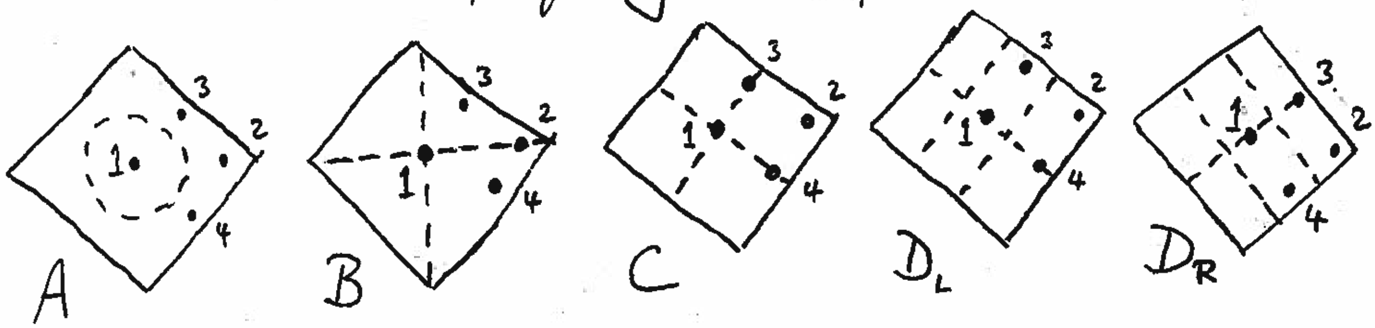




④

The four measured frequency spectra show five prominent natural frequency peaks (and some noise at low frequencies). The frequencies are 376 Hz, 592 Hz, 714 Hz, 994 Hz, 1010 Hz.

We know about various vibration mode shapes, and our task is to determine which vibration mode shape belongs with which natural frequency. The possible mode shapes are:



We notice that there are five possibilities. Mode D has left and right handed versions. The other three modes are symmetric. The key to our mode shape identification is the observation that a mode shape vibration can only be excited by a force applied away from a nodal point or line. If the excitation force is applied at a nodal point or line, that vibration mode will not be excited. From the above diagrams, the following vibration modes are expected to be excited

Measurement 1:	A				
Measurement 2:	A		C	D_L	D_R
Measurement 3:	A	B		D_L	
Measurement 4:	A	B			D_R

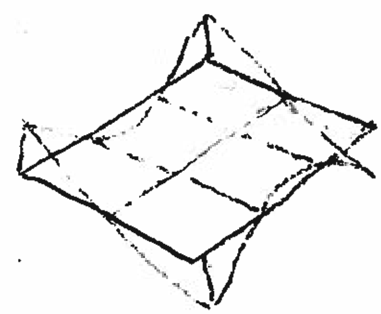
The prominent frequency peaks observed are (in Hz)

Measurement 1:	714					
Measurement 2:	714		396	994	1010	
Measurement 3:	714	592		994		
Measurement 4:	714	592			1010	

Thus, the frequencies of the vibration modes are:

$A = 714$, $B = 592$, $C = 396$, $D_L = 994$, $D_R = 1010$ Hz

For an isotropic material (elastic properties the same in all directions) modes D_L and D_R should have the same natural frequencies. However, the rolling process used to make sheet metals distorts the microscopic structure, and often slightly increases Young's modulus in the rolling direction. We notice that in mode D_R shown in the sketch,



the main bending is in the ⤢ direction.
The lines in the ⤡ direction remain straight, and therefore do not bend significantly. Therefore, the D_R mode is mostly controlled by the Young's modulus in the ⤢ direction.

Rearranging the given formula

$$E = \left(\frac{2\pi a^2 f}{35.02}\right)^2 \cdot \frac{12\rho(1-\nu^2)}{h^2} = \begin{cases} 158\ \text{GPa} & \text{for } f = 994\ \text{Hz} \\ 163\ \text{GPa} & \text{for } f = 1010\ \text{Hz} \end{cases}$$

The rolling direction has the higher E = ⤢