1 Mathematical Preliminaries

1.1 Vectors, frames and coordinates

Throughout this course, vectors will be defined as displacements (*not points*) in three-dimensional space, with vector addition and multiplication by a scalar defined in the usual way.

Commonly used geometric manipulations of vectors are the dot product between two vectors, the vector product between two vectors, and the rotation of a vector about another vector.

The dot product, inner product or scalar product between $\underline{x}$ and $\underline{y}$ is a scalar

$$\underline{x} \cdot \underline{y} = < \underline{x}, \underline{y} > = \underline{x}^T \underline{y} = \|\underline{x}\| \|\underline{y}\| \cos \angle(\underline{x}, \underline{y}) \quad , \tag{1}$$

where θ is the angle between $\underline{x}$ and $\underline{y}$ and $\| \ \|$ denotes the Euclidean length of a vector. We note that the projection of $\underline{y}$ onto a unit vector $\hat{\underline{x}}$ is $(\underline{y} \cdot \hat{\underline{x}})\, \hat{\underline{x}}$.

The vector product between two vectors

$$\underline{x} \times \underline{y} = \|\underline{x}\| \|\underline{y}\| \sin \angle(\underline{x}, \underline{y}) \ \hat{\underline{n}} \tag{2}$$

is perpendicular to both $\underline{x}$ and $\underline{y}$, has length equal to the product of the two lengths times $|\sin \angle(\underline{x}, \underline{y})|$, and its direction is defined by the right-hand rule. From the definition of the vector product $\underline{x} \times \underline{x} = 0$ for any vector $\underline{x}$, $\underline{x} \times \underline{y} = -\underline{y} \times \underline{x}$.

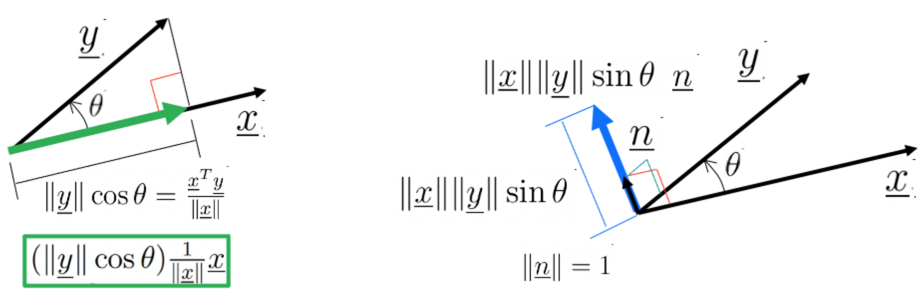

Figure 1: Scalar and vector products between vectors.

A vector $\underline{x}$ can be rotated by an angle θ about an axis of rotation $\hat{\underline{s}}$ to yield a vector $\underline{y} = Rot_{\hat{\underline{s}},\theta}(\underline{x})$. Without loss of generality, the axis of rotation is considered to be unit length. Consider first rotation in the plane $\hat{\underline{s}}_{\perp}$, orthogonal to $\hat{\underline{s}}$, and let $\underline{x}$ be a vector in this plane.

With reference to Figure 2, we note that $\hat{\underline{s}} \times \underline{x}$ also lies in the plane $\hat{\underline{s}}_{\perp}$, and it is perpendicular to $\underline{x}$. Now $\|\hat{\underline{s}} \times \underline{x}\| = \|\underline{x}\|$ and therefore $\underline{y} = \underline{x} \cos \theta + (\hat{\underline{s}} \times \underline{x}) \sin \theta$.

With reference to Figure 3, $\underline{x}$ can be projected onto the axis of rotation and onto the plane orthogonal to it:

$$\underline{x} = \underline{x}_{\|} + \underline{x}_{\perp} \tag{3}$$

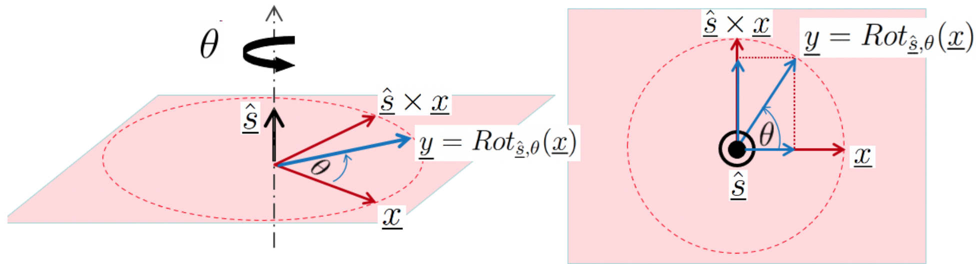

Figure 2: Geometry of 2D rotation: $\underline{x}$ is rotated about the unit axis $\underline{\hat{s}}$ into vector $\underline{y}$.

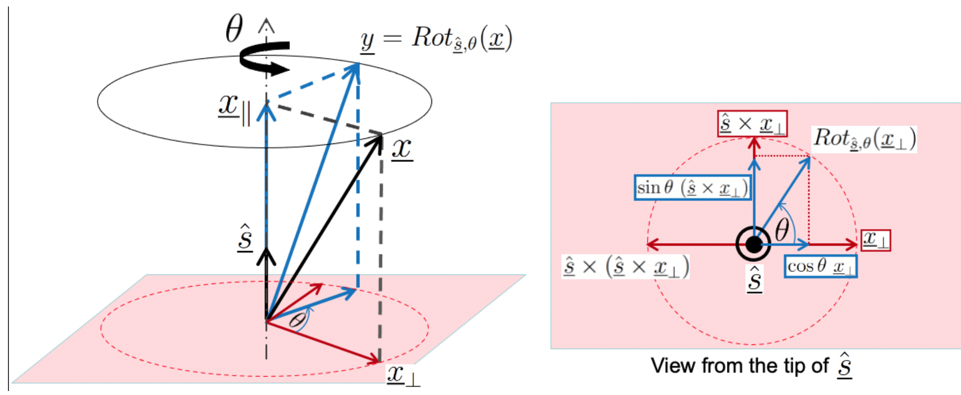

Figure 3: Geometry of 3D rotation: $\underline{x}$ is rotated about the unit axis $\underline{\hat{s}}$ into vector $\underline{y}$. The vector $\underline{x}$ is projected onto the rotation axis ($\underline{x}_{\|}$) and onto the plane perpendicular to the axis ($\underline{x}_{\perp}$). Only $\underline{x}_{\perp}$ is rotated as shown on the right.

A rotation does not change a vector that coincides with the axis or rotation, while the rotation in the plane is given by

$$Rot_{\underline{\hat{s}},\theta}(\underline{x}_{\perp}) = \cos\theta\ \underline{x}_{\perp} + \sin\theta\ (\underline{\hat{s}} \times \underline{x}_{\perp})\ . \tag{4}$$

Combining these, we obtain *the vector rotation formula*:

$$\begin{aligned} Rot_{\underline{\hat{s}},\theta}(\underline{x}) &= Rot_{\underline{\hat{s}},\theta}(\underline{x}_{\|} + \underline{x}_{\perp}) = Rot_{\underline{\hat{s}},\theta}(\underline{x}_{\|}) + Rot_{\underline{\hat{s}},\theta}(\underline{x}_{\perp}) && (5)\\ &= \underline{x}_{\|} + \cos\theta\ \underline{x}_{\perp} + \sin\theta\ (\underline{\hat{s}} \times \underline{x}_{\perp})\ . && (6) \end{aligned}$$

We can substitute $\underline{x}_{\|} = \underline{x} - \underline{x}_{\perp}$ and $\underline{\hat{s}} \times \underline{x}_{\perp} = \underline{\hat{s}} \times (\underline{x}_{\|} + \underline{x}_{\perp}) = \underline{\hat{s}} \times \underline{x}$ to obtain:

$$\begin{aligned} Rot_{\underline{\hat{s}},\theta}(\underline{x}) &= \underline{x} - \underline{x}_{\perp} + \cos\theta\ \underline{x}_{\perp} + \sin\theta\ (\underline{\hat{s}} \times \underline{x}) && (7)\\ &= \underline{x} + (\cos\theta - 1)\underline{x}_{\perp} + \sin\theta\ (\underline{\hat{s}} \times \underline{x}) && (8) \end{aligned}$$

From Figure 3, we note that $\underline{x}_{\perp} = -\underline{\hat{s}} \times (\underline{\hat{s}} \times \underline{x})$, from which we determine the *Rodrigues Rotation Formula*:

$$\underline{y} = Rot_{\underline{\hat{s}},\theta}(\underline{x}) = \underline{x} + \sin\theta\ \underline{\hat{s}} \times \underline{x} + (1 - \cos\theta)\ \underline{\hat{s}} \times (\underline{\hat{s}} \times \underline{x}) \tag{9}$$

expressing the action of a general rotation on an input vector $\underline{x}$ to generate an output vector $\underline{y}$ in terms of vector products only.

In most cases, vectors are expressed in terms of other vectors. Any vector $\underline{x}$ can be expressed as a *linear combination* of three linearly independent vectors that form a *basis* or *frame*.

For example, in the right-handed orthonormal frame $\underline{C} \triangleq [\underline{i}\ \underline{j}\ \underline{k}] =$ we have that

$$\underline{x} = a_x\underline{i} + b_x\underline{j} + c_x\underline{k} = \begin{bmatrix} \underline{i} & \underline{j} & \underline{k} \end{bmatrix} \begin{bmatrix} a_x \\ b_x \\ c_x \end{bmatrix} = \underline{C}x \ . \tag{10}$$

We say that a_x, b_x, c_x are the *coordinates* of $\underline{x}$ *in base or frame* $\underline{C}$ and that $x \triangleq [a_x\ \ b_x\ \ c_x]^T$ is the *coordinate representation* of $\underline{x}$ in frame $\underline{C}$.

The *inner product or scalar product* of any two vectors $\underline{x} = a_x\underline{i} + b_x\underline{j} + c_x\underline{k}$ and $\underline{y} = a_y\underline{i} + b_y\underline{j} + c_y\underline{k}$ is *a scalar* given by

$$\underline{x}^T\underline{y} = (\underline{C}x)^T(\underline{C}y) \triangleq x^Ty \triangleq a_xa_y + b_xb_y + c_xc_y \tag{11}$$

while the length (or norm) of a vector $\underline{x}$ is given by

$$\|\underline{x}\| \triangleq \sqrt{x^Tx} \ . \tag{12}$$

The *vector product* of any two vectors $\underline{x} = a_x\underline{i} + b_x\underline{j} + c_x\underline{k}$ and $\underline{y} = a_y\underline{i} + b_y\underline{j} + c_y\underline{k}$ is *a vector* given by

$$\underline{x} \times \underline{y} = (\underline{C}x) \times (\underline{C}y) \triangleq \underline{C}(x \times y) \triangleq \underline{C} \begin{bmatrix} b_xc_y - c_xb_y \\ c_xa_y - a_xc_y \\ a_xb_y - b_xa_y \end{bmatrix} \left(= \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ a_x & b_x & c_x \\ a_y & b_y & c_y \end{vmatrix}\right) \ . \tag{13}$$

Note that the coordinates of the vector product can be written in matrix form as

$$x \times y = (x\times)y = \begin{bmatrix} 0 & -c_x & b_x \\ c_x & 0 & -a_x \\ -b_x & a_x & 0 \end{bmatrix} y = -(y\times)x = -\begin{bmatrix} 0 & -c_y & b_y \\ c_y & 0 & -a_y \\ -b_y & a_y & 0 \end{bmatrix} x \ , \tag{14}$$

where the *skew* operator $s\times$ is defined by

$$s\times \triangleq \begin{bmatrix} 0 & -c_s & b_s \\ c_s & 0 & -a_s \\ -b_s & a_s & 0 \end{bmatrix} \ . \tag{15}$$

A frame $\underline{C}_0 = [\underline{i}_0\ \underline{j}_0\ \underline{k}_0]$ is *orthonormal* if its vectors are of unit length ($\underline{i}_0^T\underline{i}_0 = 1,\ \underline{j}_0^T\underline{j}_0 = 1,\ \underline{k}_0^T\underline{k}_0 = 1$) and mutually orthogonal ($\underline{i}_0^T\underline{j}_0 = 0,\ \underline{j}_0^T\underline{k}_0 = 0,\ \underline{k}_0^T\underline{i}_0 = 0$). Thus $\underline{C}_0$ is orthonormal if $\underline{C}_0^T\underline{C}_0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \triangleq I$. A frame $\underline{C}_0$ is *right-handed* if $\underline{i}_0 \times \underline{j}_0 = \underline{k}_0$.
Only right-handed orthonormal frames will be used throughout this course.

Since one can choose infinitely many frames, there are infinitely many ways to represent the same vector $\underline{x}$:

$$\underline{x} = \underline{C}x = \underline{C}_0\,{}^0x = \underline{C}_1\,{}^1x = \cdots \tag{16}$$

How can we relate different coordinate representations?
Consider two frames $\underline{C}_0 = \begin{bmatrix} \underline{i}_0 & \underline{j}_0 & \underline{k}_0 \end{bmatrix}$, $\underline{C}_1 = \begin{bmatrix} \underline{i}_1 & \underline{j}_1 & \underline{k}_1 \end{bmatrix}$. Each vector of one frame can be expressed in terms of the vectors of the other. In particular, we can write

$$\begin{aligned} \underline{i}_1 &= \underline{i}_0 c_{11} + \underline{j}_0 c_{21} + \underline{k}_0 c_{31} \\ \underline{j}_1 &= \underline{i}_0 c_{12} + \underline{j}_0 c_{22} + \underline{k}_0 c_{32} \\ \underline{k}_1 &= \underline{i}_0 c_{13} + \underline{j}_0 c_{23} + \underline{k}_0 c_{33} \end{aligned}$$

or

$$\underline{C}_1 = \begin{bmatrix} \underline{i}_1 & \underline{j}_1 & \underline{k}_1 \end{bmatrix} = \begin{bmatrix} \underline{i}_0 & \underline{j}_0 & \underline{k}_0 \end{bmatrix} \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix} \triangleq \underline{C}_0\,{}^0C_1\ , \tag{17}$$

or

$${}^0C_1 = \begin{bmatrix} \underline{i}_0^T \\ \underline{j}_0^T \\ \underline{k}_0^T \end{bmatrix} \begin{bmatrix} \underline{i}_1 & \underline{j}_1 & \underline{k}_1 \end{bmatrix} = \begin{bmatrix} \underline{i}_0^T\underline{i}_1 & \underline{i}_0^T\underline{j}_1 & \underline{i}_0^T\underline{k}_1 \\ \underline{j}_0^T\underline{i}_1 & \underline{j}_0^T\underline{j}_1 & \underline{j}_0^T\underline{k}_1 \\ \underline{k}_0^T\underline{i}_1 & \underline{k}_0^T\underline{j}_1 & \underline{k}_0^T\underline{k}_1 \end{bmatrix}\ . \tag{18}$$

Therefore, since

$$\underline{C}_0\,{}^0x = \underline{x} \ = \ \underline{C}_1\,{}^1x = \underline{C}_0\,{}^0C_1\,{}^1x \tag{19}$$

we have that

$$ {}^0x = {}^0C_1 \, {}^1x \ , \tag{20} $$

where *the columns of* ${}^0C_1$ *are the coordinates of the* $\underline{C}_1$ *frame vectors in frame* $\underline{C}_0$ (direction cosines of $\underline{C}_1$ frame vectors relative to $\underline{C}_0$ frame).

It is important to note that the inner product (11) and the vector product (13) have geometric meaning (see ((1), (2)) and are *independent* of the choice of frame $\underline{C}$. Equation (11) yields the same result for any orthonormal $\underline{C}$, *i.e.* $\underline{x}^T\underline{y} = x^Ty = {}^0x^T\,{}^0y = {}^1x^T\,{}^1y = \cdots$, as long as the frames $\underline{C}_0$, $\underline{C}_1$, ... in which the the coordinates of $\underline{x}$, $\underline{y}$ are available are orthonormal.
Equation (13) yields the same result for any orthonormal, right-handed $\underline{C}$, *i.e.*, $\underline{x} \times \underline{y} = \underline{C}(x \times y) = \underline{C}_0({}^0x \times {}^0y) = \underline{C}_1({}^1x \times {}^1y) = \cdots$, as long as $\underline{C}_0$, $\underline{C}_1$, ... are orthonormal and right-handed. Indeed, for any rotation matrix Q and any $s, t \in I\!R^3$, $(Qs) \times (Qt) = Q(s \times t)$. To see this, note that for all a, s, t in $I\!R^3$,

$$ a^T(s \times t) = \det[a\ s\ t] = \det Q \det[a\ s\ t] = \det[Qa\ Qs\ Qt] \tag{21} $$

$$ = (Qa)^T((Qs) \times (Qt)) = a^TQ^T((Qs) \times (Qt)) \ . \tag{22} $$

In terms of the skew-matrix notation (15), this is the same as stating that $(\underline{Q}s)\times = \underline{Q}(s\times)\underline{Q}^T$.

Example 1.1 Consider two orthonormal frames $\underline{C}_0$ and $\underline{C}_1$, attached to a fixed base and the link of a revolute joint, respectively, as shown in Figure 4. If the link rotates about the common $\underline{k}$-vectors of the frames, we have

$$ \begin{array}{rcrcrcr} \underline{i}_1 = & & \underline{i}_0 \cos\theta & + & \underline{j}_0 \sin\theta & + & \underline{k}_0 0 \\ \underline{j}_1 = & & \underline{i}_0(-\sin\theta) & + & \underline{j}_0 \cos\theta & + & \underline{k}_0 0 \\ \underline{k}_1 = & & \underline{i}_0 0 & + & \underline{j}_0 0 & + & \underline{k}_0 1 \end{array} \tag{23} $$

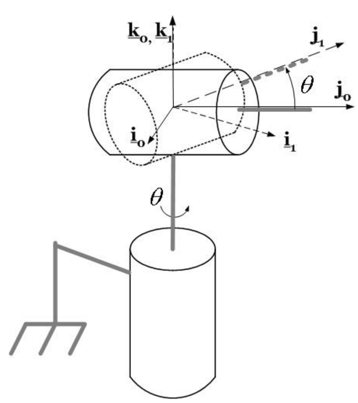

Figure 4: Rotation about $\underline{k}$ axis.

and therefore

$$ {}^0C_1 = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \stackrel{\Delta}{=} R(k,\theta) \ . \tag{24} $$

Thus, the coordinates ${}^0x$ and ${}^1x$ of a vector $\underline{x}$ in frame $\underline{C}_0$ and $\underline{C}_1$, respectively, are related by ${}^0x = R(k,\theta)\,{}^1x$.

Typically, we have a *sequence* of frames, as shown in Figure 5.

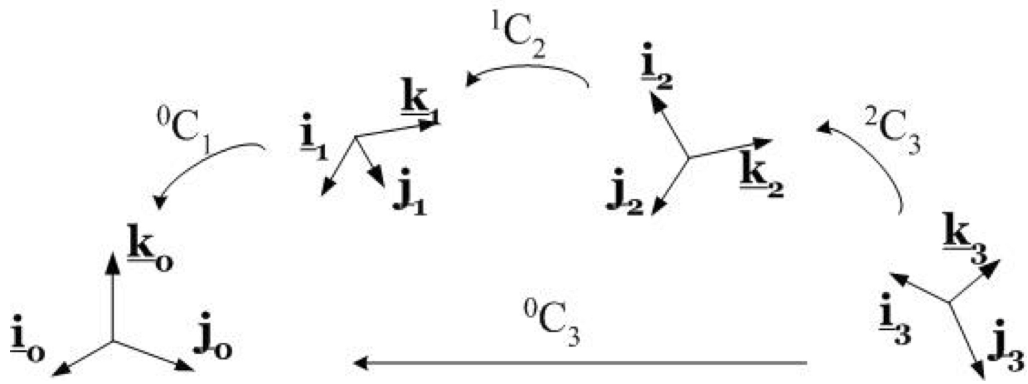

Figure 5: A sequence of frames.

Since

$$
\begin{aligned}
\underline{C}_3 &= \underline{C}_2\,{}^2C_3 && (25)\\
&= \underline{C}_1\,{}^1C_2\,{}^2C_3 && (26)\\
&= \underline{C}_0\,{}^0C_1\,{}^1C_2\,{}^2C_3 && (27)\\
&\stackrel{\Delta}{=} \underline{C}_0\,{}^0C_3 && (28)
\end{aligned}
$$

we have that

$$ {}^0x = {}^0C_3\,{}^3x = {}^0C_1\,{}^1C_2\,{}^2C_3\,{}^3x \ . \tag{29} $$

Example 1.2 *Z-Y-X Euler or Spherical Wrist.*
A spherical wrist has a sequence of orthogonal, intersecting axes, as shown in Figure 6. The matrices ${}^0C_1$, ${}^1C_2$, ${}^2C_3$, are the so-called *basic rotation matrices*:

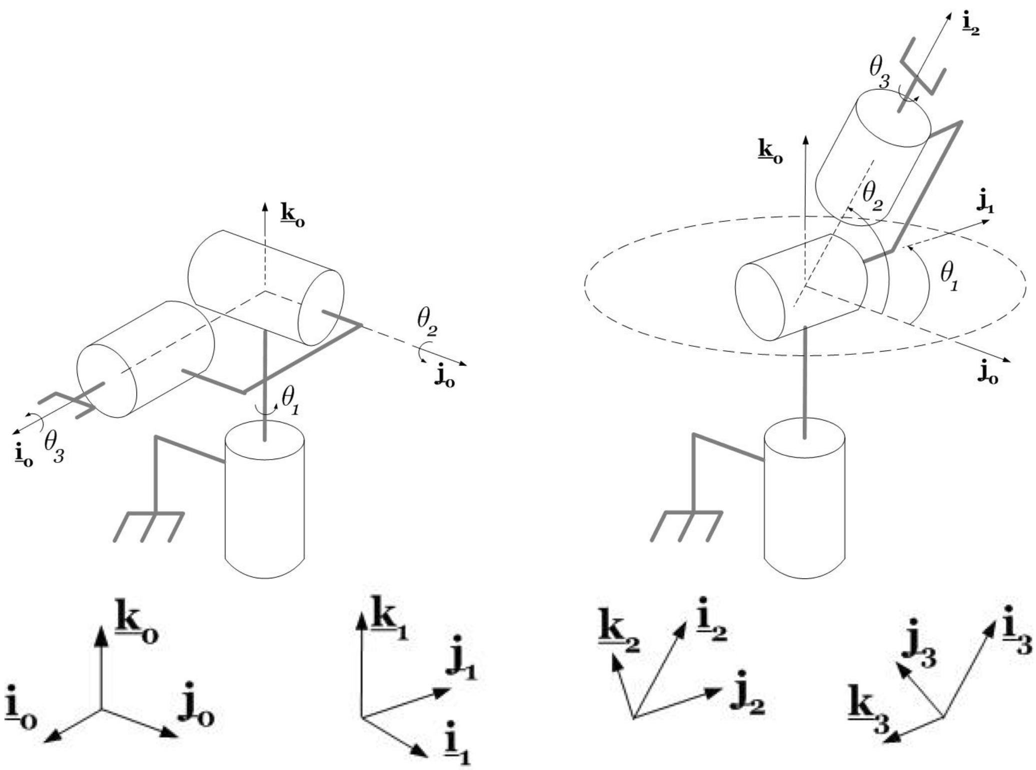

Figure 6: Z-Y-X Euler wrist

$$
\begin{aligned}
{}^0C_1 &= R(k,\theta_1) = \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 & 0 \\ \sin\theta_1 & \cos\theta_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} && (30) \\
{}^1C_2 &= R(j,\theta_2) = \begin{bmatrix} \cos\theta_2 & 0 & \sin\theta_2 \\ 0 & 1 & 0 \\ -\sin\theta_2 & 0 & \cos\theta_2 \end{bmatrix} && (31) \\
{}^2C_3 &= R(i,\theta_3) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_3 & -\sin\theta_3 \\ 0 & \sin\theta_3 & \cos\theta_3 \end{bmatrix} && (32)
\end{aligned}
$$

and ${}^0C_3 = R(k,\theta_1)R(j,\theta_2)R(i,\theta_3)$.

1.2 Points, coordinate systems, and coordinates

Vectors are added, multiplied by a scalar; there is a *single* zero vector, etc.. Points in physical space cannot be appropriately represented by vectors (what is the sum of two points? what is the zero point?).

Instead, points should be represented as the elements of a set with the following properties:

1. Points are displaced by vectors and a succession of displacements is achieved by the addition of vectors.
2. The zero vector does not displace a point.
3. Any two points uniquely determine a displacement.

Formally, a set of elements having the above properties is called an *affine space* A modelled on a vector space V. Given a point $\underset{\sim}{x} \in A$ and a vector $\underline{v} \in V$, we denote by $\underset{\sim}{x} + \underline{v}$ the point to which $\underset{\sim}{x}$ is displaced by $\underline{v}$. Then, the properties listed above can be formally repeated as

1. For all $\underset{\sim}{x} \in A$, $\underline{v}$, $\underline{w} \in V$,

$$(\underset{\sim}{x} + \underline{v}) + \underline{w} = \underset{\sim}{x} + (\underline{v} + \underline{w}) \ . \tag{33}$$

2. For all $\underset{\sim}{x} \in A$,

$$\underset{\sim}{x} + \underline{0} = \underset{\sim}{x} \ . \tag{34}$$

3. For all pairs $\underset{\sim}{x}, \underset{\sim}{x\prime} \in A$, there exists a unique $\underline{v} \in V$, denoted by $\underline{v} = \underset{\sim}{x\prime} - \underset{\sim}{x}$, such that $\underset{\sim}{x\prime} = \underset{\sim}{x} + \underline{v}$.

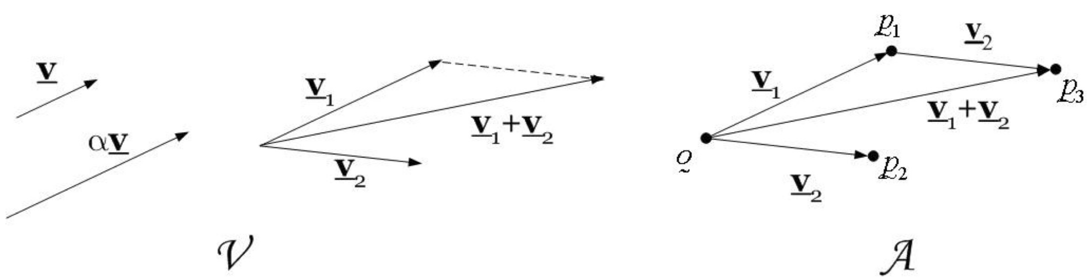

Figure 7: Vector spaces *vs* affine spaces.

Points are specified in terms of displacements from a *fixed point* $\underset{\sim}{o}$ called the origin. The displacements (which are just vectors) are usually specified by their coordinates in a given frame, *e.g.*, the canonical frame $\underline{C} = [\underline{i}\ \underline{j}\ \underline{k}]$.

The pair $\{\ \underset{\sim}{o}, \underline{C}\ \}$ is called a *coordinate system* and we say that a point $\underset{\sim}{x} = \underset{\sim}{o} + \underline{C}x$ has *coordinates* x in the *coordinate system* $\{\ \underset{\sim}{o}, \underline{C}\ \}$. The entries of x are sometimes called affine or homogeneous coordinates.

As with vectors, there are infinitely many coordinate systems (origins *and* frames). Consider two coordinate systems representations of the same point $\underset{\sim}{x}$:

$$\underset{\sim}{x} = \underset{\sim}{o}_0 + \underline{C}_0\, {}^0x = \underset{\sim}{o}_1 + \underline{C}_1\, {}^1x \ . \tag{35}$$

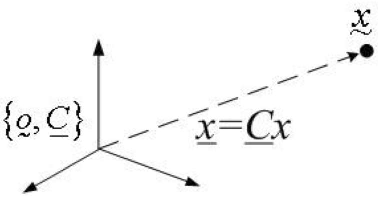

Figure 8: Coordinate system for affine spaces (points).

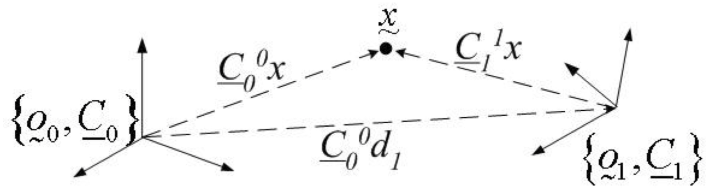

Figure 9: Affine coordinate system changes.

As before, the vectors of one frame can be written in terms of the other. In addition, the origin of one coordinate system can be written with respect to the other, leading to

$$\begin{aligned} \underline{C}_1 &= \underline{C}_0\, {}^0C_1 \\ \underset{\sim}{o}_1 &= \underset{\sim}{o}_0 + \underline{C}_0\, {}^0d_1\ . \end{aligned} \tag{36}$$

Then

$$\underline{C}_0\, {}^0x = (\underset{\sim}{o}_1 - \underset{\sim}{o}_0) + \underline{C}_1\, {}^1x \tag{37}$$

$$= \underline{C}_0\, {}^0d_1 + \underline{C}_0\, {}^0C_1\, {}^1x \tag{38}$$

and therefore

$${}^0x = {}^0d_1 + {}^0C_1\, {}^1x\ . \tag{39}$$

The columns of ${}^0C_1$ are the coordinates of the frame vectors $\underline{C}_1$ in frame $\underline{C}_0$ and the entries of ${}^0d_1$ are the coordinates of the vector $\underset{\sim}{o}_1 - \underset{\sim}{o}_0$, in frame $\underline{C}_0$. Thus, unlike coordinate frame changes for vectors, which require specifying three column vectors (the columns of ${}^0C_1$), *coordinate system* changes for points require specifying *four* column vectors (the columns of ${}^0C_1$ and the coordinates of $\underset{\sim}{o}_1 - \underset{\sim}{o}_0$ in frame $\underline{C}_0$).

In order to specify coordinate system changes by conventional matrix multiplications, augment coordinates for points by a 1:

$$\begin{bmatrix} {}^0x \\ 1 \end{bmatrix} = \begin{bmatrix} {}^0C_1 & {}^0d_1 \\ 0^T & 1 \end{bmatrix} \begin{bmatrix} {}^1x \\ 1 \end{bmatrix} \triangleq {}^0T_1 \begin{bmatrix} {}^1x \\ 1 \end{bmatrix}\ . \tag{40}$$

This is called an *affine* or *homogeneous* transformation and completely specifies an *affine* coordinate system change. The relationship between the coordinate systems can also be written formally in matrix form:

$$\begin{bmatrix} \underline{C}_1 & \underset{\sim}{o}_1 \\ 0^T & 1 \end{bmatrix} = \begin{bmatrix} \underline{C}_0 & \underset{\sim}{o}_0 \\ 0^T & 1 \end{bmatrix} \underbrace{\begin{bmatrix} {}^0C_1 & {}^0d_1 \\ 0^T & 1 \end{bmatrix}}_{{}^0T_1} . \tag{41}$$

Note: Formally, with $\underset{\sim}{x} = \begin{bmatrix} x \\ 1 \end{bmatrix}$ denoting points, $\underline{v} = \begin{bmatrix} v \\ 0 \end{bmatrix}$ vectors, we have that

$\frac{d}{dt}\underset{\sim}{x} = \begin{bmatrix} \dot{x} \\ 0 \end{bmatrix}$ *i.e.*, the time derivative of a particle position is a particle velocity, that

$\underset{\sim}{x} + \underline{v} = \begin{bmatrix} x \\ 1 \end{bmatrix} + \begin{bmatrix} v \\ 0 \end{bmatrix} = \begin{bmatrix} x+v \\ 1 \end{bmatrix}$ *i.e.*, the "sum" of a point with a vector is a point, and

$\underset{\sim}{x'} - \underset{\sim}{x} = \begin{bmatrix} x' \\ 1 \end{bmatrix} - \begin{bmatrix} x \\ 1 \end{bmatrix} = \begin{bmatrix} x'-x \\ 0 \end{bmatrix}$ *i.e.*, the "difference" of two points is a vector.

1.3 Linear Transformations and Rotations

In examples 1.1, 1.2, frames $\underline{C}_i$ were obtained from frames $\underline{C}_{i-1}$ by single-axis rotations, and the coordinates of $\underline{C}_i$ in terms of $\underline{C}_{i-1}$ where readily available. In general, any *linear transformation* $\underline{x} \to \underline{\underline{A}}\,\underline{x}$ can be specified by its *coordinate or matrix representation* A with respect to any frame $\underline{C}$.

If

$$\underline{y} = \underline{\underline{A}}\,\underline{x} \tag{42}$$

$$\underline{C}y = \underline{\underline{A}}\,\underline{C}\,x \tag{43}$$

$$\underline{C}y \triangleq \underline{C}Ax \tag{44}$$

and therefore

$$y = Ax \ , \tag{45}$$

with the columns of A being the coordinates of the image vectors $\underline{\underline{A}}\,\underline{C}$ in base frame $\underline{C}$.

Example 1.3 *Vector product function.* For a given vector $\underline{s}$, define the linear transformation $\underline{\underline{A}}_{\underline{s}}$ by $\underline{\underline{A}}_{\underline{s}}\underline{x} = \underline{s} \times \underline{x}$. Let us determine the matrix representation $A_{\underline{s}}$ of $\underline{\underline{A}}_{\underline{s}}$ for $\underline{C} = [\underline{i}\ \underline{j}\ \underline{k}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $\underline{s} = a_s\underline{i} + b_s\underline{j} + c_s\underline{k}$.

Because

$$\begin{array}{lll} \underline{i} \times \underline{i} = \underline{0} \ ; & \underline{j} \times \underline{j} = \underline{0} \ ; & \underline{k} \times \underline{k} = \underline{0} \\ \underline{i} \times \underline{j} = \underline{k} \ ; & \underline{j} \times \underline{k} = \underline{i} \ ; & \underline{k} \times \underline{i} = \underline{j} \end{array} \tag{46}$$

we have that

$$\begin{array}{rcccccc} \underline{\underline{A}}_{\underline{s}} \underline{i} = & \underline{s} \times \underline{i} = & & + & \underline{j} c_s & + & \underline{k}(-b_s) \\ \underline{\underline{A}}_{\underline{s}} \underline{j} = & \underline{s} \times \underline{j} = & \underline{i}(-c_s) & + & & + & \underline{k}(a_s) \\ \underline{\underline{A}}_{\underline{s}} \underline{k} = & \underline{s} \times \underline{k} = & \underline{i}(b_s) & + & \underline{j}(-a_s) & & , \end{array} \tag{47}$$

or

$$\underline{\underline{A}}_{\underline{s}} [\underline{i} \ \underline{j} \ \underline{k}] = [\underline{s} \times \underline{i} \quad \underline{s} \times \underline{j} \quad \underline{s} \times \underline{k}] \ = [\underline{i} \ \underline{j} \ \underline{k}] \begin{bmatrix} 0 & -c_s & b_s \\ c_s & 0 & -a_s \\ -b_s & a_s & 0 \end{bmatrix} \ . \tag{48}$$

Therefore

$$\underline{\underline{A}}_{\underline{s}} \underline{C} = \underline{C} A_{\underline{s}} = \underline{C}(s\times) \tag{49}$$

where the $\times$ or skew operator was defined before and $s\times$ is the representation of $\underline{\underline{A}}_{\underline{s}}$ in frame $\underline{C}$.

Consider now a linear function $\underline{\underline{Q}}$: $\mathbb{R}^3 \to \mathbb{R}^3$ that preserves the lengths of vectors, *i.e.*, $\|\underline{\underline{Q}}(\underline{x})\| = \|\underline{x}\|$, for all $\underline{x} \in \mathbb{R}^3$. Such a norm-preserving linear transformation is called either a *rotation* or a *reflection*.
To see why, let us examine the eigenvalues and eigenvectors of $\underline{\underline{Q}}$. An eigenvalue-eigenvector pair of $\underline{\underline{Q}}$ is defined as $\{\lambda, \underline{e}\}$, with non-zero $\underline{e}$, such that $\underline{\underline{Q}}\underline{e} = \lambda \underline{e}$. Because $\underline{\underline{Q}}$ is norm-preserving, all eigenvalues satisfy $|\lambda| = 1$. Since the eigenvalues are the roots of a third order polynomial with real coefficients, the set of eigenvalues of $\underline{\underline{Q}}$ is given by either $\{e^{-j\theta}, e^{j\theta}, 1\}$, in which case $\underline{\underline{Q}}$ is a "rotation", or by $\{e^{-j\theta}, e^{j\theta}, -1\}$, in which case $\underline{\underline{Q}}$ is called a "reflection". The angle θ is the *angle of rotation/reflection*.

It can be shown that the eigenvectors $\underline{u}$, $\underline{v}$ and $\underline{w}$ of $\underline{\underline{Q}}$, corresponding to $e^{-j\theta}$, $e^{j\theta}$ and ± 1, respectively, are mutually orthogonal, and so are the vectors $\underline{i}_1 = \frac{Re\underline{u}}{\|Re\underline{u}\|}$, $\underline{j}_1 = -\frac{Im\underline{v}}{\|Im\underline{v}\|}$, $\underline{w}$. The vector $\underline{w}$ is the *axis of rotation/reflection*. Choose $\underline{k}_1$ to be equal to either $\underline{w}$ or $-\underline{w}$, whichever makes $[\underline{i}_1 \ \underline{j}_1 \ \underline{k}_1] \stackrel{\Delta}{=} \underline{C}_1$, into a right-handed orthonormal frame.

If $\underline{\underline{Q}}$ is a rotation,

$$\left.\begin{array}{lcl} \underline{\underline{Q}} \ \underline{u} & = & e^{-j\theta}\underline{u} \\ \underline{\underline{Q}} \ \underline{v} & = & e^{j\theta}\underline{v} \\ \underline{\underline{Q}} \ \underline{w} & = & \underline{w} \end{array}\right\} \Longrightarrow \left\{\begin{array}{lcccc} \underline{\underline{Q}} \ \underline{i}_1 & = & \underline{i}_1 \cos\theta & + & \underline{j}_1 \sin\theta \\ \underline{\underline{Q}} \ \underline{j}_1 & = & \underline{i}_1(-\sin\theta) & + & \underline{j}_1 \cos\theta \\ \underline{\underline{Q}} \ \underline{k}_1 & = & & & \underline{k}_1 \end{array}\right. \tag{50}$$

and therefore

$$\underline{\underline{Q}}\ [\underline{i}_1\ \underline{j}_1\ \underline{k}_1] = [\underline{i}_1\ \underline{j}_1\ \underline{k}_1] \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \triangleq [\underline{i}_1\ \underline{j}_1\ \underline{k}_1]\ {}^1Q\ . \tag{51}$$

Thus, in frame $\underline{C}_1 = [\underline{i}_1\ \underline{j}_1\ \underline{k}_1]$, the matrix representation ${}^1Q$ of $\underline{\underline{Q}}$ is the familiar basic rotation matrix $R(k, \theta)$.

It can be shown that

$$\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} = e^{\theta \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \times} \triangleq e^{\theta\ k\times}\ . \tag{52}$$

Note that $k = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ are the coordinates of $\underline{k}_1$ in $\underline{C}_1$. Let $\underline{C}_0$ be any other right-handed orthonormal frame. With $\underline{C}_1 = \underline{C}_0{}^0C_1$, the coordinates of the eigenvector $\underline{k}_1$ in $\underline{C}_0$ are then given by ${}^0k_1 = {}^0C_1 k$. Since

$$\begin{aligned}
\underline{\underline{Q}}\underline{C}_0 &= \underline{\underline{Q}}\underline{C}_1\ {}^1C_0 && (53)\\
&= \underline{C}_1\ \ {}^1Q\ \ {}^1C_0 && (54)\\
&= \underline{C}_0\ \ {}^0C_1\ \ {}^1Q\ \ {}^1C_0 && (55)\\
&= \underline{C}_0\ \ {}^0C_1\ \ (e^{\theta\ k\times})\ \ {}^1C_0 && (56)\\
&= \underline{C}_0\ \ {}^0C_1\ \ (e^{\theta\ k\times})\ \ {}^0C_1^T && (57)\\
&= \underline{C}_0\ e^{\theta({}^0C_1\ k)\times} && (58)\\
&= \underline{C}_0\ e^{\theta({}^0k_1)\times}\ . && (59)
\end{aligned}$$

Therefore, regardless of the frame $\underline{C}$ relative to which $\underline{\underline{Q}}$ is expressed, we have that

$$\underline{\underline{Q}}\underline{C} = \underline{C}e^{\theta\ s\times}\ , \tag{60}$$

where $\underline{s} = \underline{C}s$ is the axis of rotation and θ is the angle of rotation. Alternatively, we shall use the following "coordinate-free" notation:

$$\underline{\underline{Q}} = e^{\theta\underline{s}\times}\ . \tag{61}$$

Note: The above matrix exponential representation of rotation is known as *Euler's Rotation Theorem*, which states that given any two frames $\underline{C}_0$, $\underline{C}_1$, $\underline{C}_0$ can be aligned to $\underline{C}_1$ by a rotation about a fixed axis $\underline{s}$.

Let $\underline{C}_1 = \underline{C}_0\, {}^0C_1 = \underline{C}_0 e^{\theta({}^0s\times)}$ where $\underline{s} = \underline{C}_0\, {}^0s$ is the axis of rotation. Consider the matrix differential equation

$$\frac{d}{d\phi}Q(\phi) = ({}^0s\times)Q(\phi) \tag{62}$$

with initial condition $Q(0) = I$. Then $Q(\phi) = e^{\phi({}^0s\times)}$, $Q(\phi)$ is a rotation about a *fixed* axis ${}^0s$ and $\underline{C}_1 = \underline{C}_0 Q(\theta)$.

Example 1.4 *Roll-Pitch-Yaw.*

Let us re-visit the spherical wrist example 1.2. The frame $\underline{C}_3$ attached to the last link of the wrist was obtained by a sequence of three rotations: the first one about the $\underline{k}_0$ axis of the base frame, ($\underline{C}_1 = \underline{C}_0 e^{\theta_1 k\times}$), the second one about the $\underline{j}_1$ axis of the *rotated* frame $\underline{C}_1$, ($\underline{C}_2 = \underline{C}_1 e^{\theta_2 j\times}$), and the third one about the $\underline{i}_2$ axis of the *twice rotated* frame $\underline{C}_2$, ($\underline{C}_3 = \underline{C}_2 e^{\theta_3 i\times}$).

The frame $\underline{C}_3$ can also be obtained from a sequence of three fixed rotations about axes defined in the base frame $\underline{C}_0$. These are the *roll* (about $\underline{i}_0$) of θ_3, *pitch* (about $\underline{j}_0$) of θ_2, and *yaw* (about $\underline{k}_0$) of θ_1:

$$\begin{aligned}
\underline{C}_3 &= e^{\theta_1 \underline{k}_0\times}[e^{\theta_2 \underline{j}_0\times}(e^{\theta_3 \underline{i}_0\times}\underline{C}_0)] && (63)\\
&= e^{\theta_1(\underline{C}_0 k)\times} e^{\theta_2(\underline{C}_0 j)\times} e^{\theta_3(\underline{C}_0 i)\times}\underline{C}_0 && (64)\\
&= \underline{C}_0 e^{\theta_1 k\times} e^{\theta_2 j\times} e^{\theta_3 i\times} \quad . && (65)
\end{aligned}$$

The two sequences of rotations give the same result, as expected.
In finding out the orientation of the distal link of an open kinematic linkage, the current frame approach would be equivalent to rotating each link, in sequence, starting from the proximal link (connected to mechanical ground) and ending with the distal link. The base frame approach would be equivalent to rotating each link, in sequence, starting with the distal link and ending with the proximal link.

1.4 Rigid Motions

We define a *rigid motion* of points $\underset{\sim}{x} \to \underset{\sim}{f}(\underset{\sim}{x})$ to be a function from 3-D space (viewed as an affine space) into itself such that distances between points are preserved. Consider a coordinate system $\{\underset{\sim}{o}, \underline{C}\}$, in which $\underset{\sim}{x} = \underset{\sim}{o} + \underline{C}x$, and define $f : I\!R^3 \to I\!R^3$ as the transformation of coordinates of $\underset{\sim}{x}$ under $\underset{\sim}{f}$, *i.e.* $\underset{\sim}{f}(\underset{\sim}{x}) - \underset{\sim}{f}(\underset{\sim}{o}) = \underline{C}f(x)$.

Since distances between points are preserved by $\underset{\sim}{f}$, we have that for any two points $\underset{\sim}{x}_1, \underset{\sim}{x}_2$,

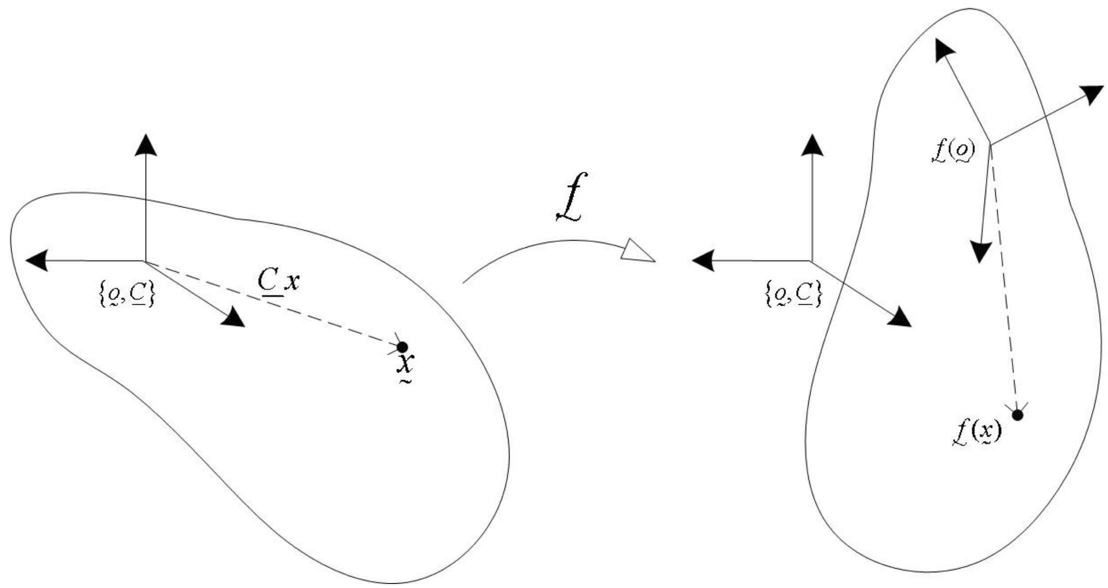

Figure 10: A mapping function of a rigid motion.

$$\|\underset{\sim}{f}(\underset{\sim}{x}_1) - \underset{\sim}{f}(\underset{\sim}{x}_2)\| \;=\; \|\underset{\sim}{x}_1 - \underset{\sim}{x}_2\| \tag{66}$$

or

$$\|\underset{\sim}{f}(\underset{\sim}{o}) + \underline{C}f(x_1) - \underset{\sim}{f}(\underset{\sim}{o}) - \underline{C}f(x_2)\| \;=\; \|\underset{\sim}{o} + \underline{C}x_1 - \underset{\sim}{o} - \underline{C}x_2\| \tag{67}$$

or

$$\|\underline{C}(f(x_1) - f(x_2))\| \;=\; \|\underline{C}(x_1 - x_2)\| \;\; . \tag{68}$$

Because $f(0) = 0$ and $\|f(x_1) - f(x_2)\| = \|x_1 - x_2\|$, f is linear, $f(x) = Qx$ with Q an orthonormal matrix. Indeed, to show linearity, note that for all $x_1, x_2 \in I\!R^3$,

$$\begin{aligned} x_1^T x_2 &= \tfrac{1}{2}[\|x_1\|^2 + \|x_2\|^2 - \|x_1 - x_2\|^2] \qquad \text{(parallelogram law)} \\ &= \tfrac{1}{2}[\|f(x_1)\|^2 + \|f(x_2)\|^2 - \|f(x_1) - f(x_2)\|^2] \\ &= f(x_1)^T f(x_2) \end{aligned}$$

and therefore, for any α, $\beta \in I\!R$,

$$\begin{aligned} & \|f(\alpha x_1 + \beta x_2) - \alpha f(x_1) - \beta f(x_2)\|^2 \\ = \;& \|f(\alpha x_1 + \beta x_2)\|^2 + \alpha^2\|f(x_1)\|^2 + \beta^2\|f(x_2)\|^2 - 2\alpha f(\alpha x_1 + \beta x_2)^T f(x_1) \\ & -2\beta f(\alpha x_1 + \beta x_2)^T f(x_2) + 2\alpha\beta f(x_1)^T f(x_2) \\ = \;& \|(\alpha x_1 + \beta x_2) - \alpha x_1 - \beta x_2\|^2 = 0 \;\; . \end{aligned}$$

From $x_1^T Q^T Q x_2 = x_1^T x_2$, for all x_1, $x_2 \in I\!R^3$ it follows that $Q^T Q = Q Q^T = I$, so Q is orthonormal (just pick $x_1 = [1\ 0\ 0]^T$, $x_2 = [1\ 0\ 0]^T$, ($x_1 = [1\ 0\ 0]^T$, $x_2 = [0\ 1\ 0]^T$, etc.).

Now, with $f(x) = Qx$, Q orthonormal, it follows that

$$
\begin{aligned}
\underset{\sim}{f}(\underset{\sim}{x}) &= \underset{\sim}{f}(\underset{\sim}{o}) + \underline{C}Qx && (69)\\
&= \underset{\sim}{f}(\underset{\sim}{o}) + \underline{\underline{Q}}(\underset{\sim}{x} - \underset{\sim}{o}) && (70)\\
&= \underset{\sim}{o} + [\underset{\sim}{f}(\underset{\sim}{o}) - \underset{\sim}{o}] + \underline{\underline{Q}}(\underset{\sim}{x} - \underset{\sim}{o}) && (71)\\
&= \underset{\sim}{o} + \underbrace{\underline{d}}_{TRANSLATION} + \underbrace{\underline{\underline{Q}}(\underset{\sim}{x} - \underset{\sim}{o})}_{ROTATION/REFLECTION} && (72)\\
&= \underset{\sim}{o} + \underline{C}d + \underline{C}Qx \ . && (73)
\end{aligned}
$$

Thus a rigid motion of points can always be interpreted as a rotation/reflection (rotation if $\det Q = 1$ in the above), followed by a translation. After the motion, the coordinates x of a point $\underset{\sim}{x}$ become $d + Qx$.

1.5 Angular Velocity

Consider the time-varying frames $\underline{C}_0(t)$, $\underline{C}_1(t)$, with $\underline{C}_1(t) = \underline{C}_0 Q(t)$, and assume that $Q(t)$ is differentiable (we let ${}^0C_1(t) = Q(t)$ to simplify notation). Then,

$$\dot{Q}(t) \;=\; \dot{Q}(t)\underbrace{Q(t)^T Q(t)}_{I} = [\dot{Q}(t)Q(t)^T]\,Q(t) = \Omega(t)Q(t) \tag{74}$$

Now

$$
\begin{aligned}
\Omega(t) &= \dot{Q}Q(t)^T = -Q(t)\dot{Q}(t)^T \;\text{ because }\; Q(t)Q(t)^T = I && (75)\\
&= -[\dot{Q}(t)Q(t)^T]^T \stackrel{\Delta}{=} -\Omega(t)^T && (76)
\end{aligned}
$$

so $\Omega(t)$ is a *skew-symmetric* matrix:

$$\Omega(t) \;=\; \begin{bmatrix} 0 & -c_\omega(t) & b_\omega(t) \\ c_\omega(t) & 0 & -a_\omega(t) \\ -b_\omega(t) & a_\omega(t) & 0 \end{bmatrix} \stackrel{\Delta}{=} {}^0\omega_{1,0}(t)\times \quad . \tag{77}$$

The vector $\underline{\omega}_{1,0}(t)$ defined by $\underline{\omega}_{1,0}(t) = \underline{C}_0(t)^0\omega_{1,0}(t)$ is called the *angular velocity* of the frame $\underline{C}_1(t)$ with respect to $\underline{C}_0(t)$. To simplify notation, let $\underline{\omega}_{1,0} \stackrel{\Delta}{=} \underline{\omega}$ for the remainder of this subsection.

The physical interpretation is familiar. Consider the motion of a rigid body hinged at a point $\underset{\sim}{o}$, and let $\underline{C}_0$ be a stationary frame. Then,

$$\underset{\sim}{y}(t) \;=\; \underset{\sim}{o} + \underline{\underline{Q}}(t)(\underset{\sim}{x} - \underset{\sim}{o}) = \underset{\sim}{o} + \underline{C}_0 Q(t)x \ , \tag{78}$$

and therefore

$$\underline{v}(t) \triangleq \dot{\underset{\sim}{y}}(t) = \underline{C}_0\dot{Q}(t)x = \underline{C}_0(\omega(t)\times)Q(t)x = \underline{C}_0 v(t) \tag{79}$$

$$= (\underline{C}_0\omega(t)) \times \underline{C}_0 Q(t)x = \underline{\omega}(t) \times (\underset{\sim}{y}(t) - \underset{\sim}{o}) \ . \tag{80}$$

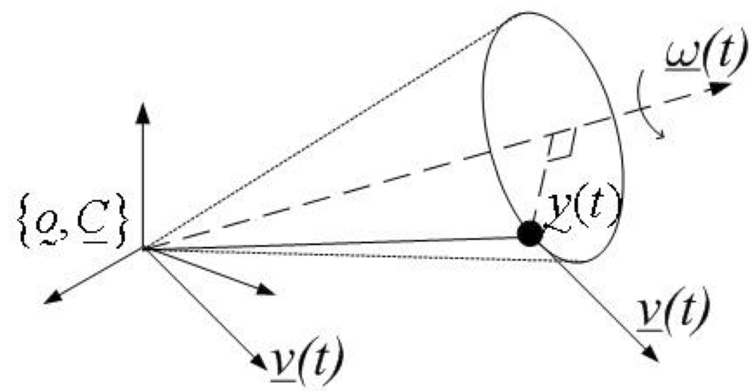

Figure 11: Velocity of a point on a rotating rigid body.

Caution: What are the coordinates of $\underline{v}(t)$, $\underline{\omega}(t)$ in "body-attached" frame $\underline{C}_1 = \underline{C}_0 Q(t)$? The temptation is to say zero, since the coordinates of points in frame $\underline{C}_0 Q(t)$ are not changed by a rigid motion. This is not so. The coordinates of $\underline{\omega}$ in $\underline{C}_0 Q(t)$ are simply $Q(t)^T\omega(t)$, $Q(t)^T v(t)$, neither of which can be zero if $\omega(t)$ and $v(t)$ are not zero.

Note:
1. The kinematic relationship

$$\dot{Q} = \omega \times Q = Q \ \ (Q^T\omega)\times \tag{81}$$

is easy to remember and important. It allows one to compute orientation from knowledge of the angular velocity and an initial condition (note that integration keeps $Q(t)^T Q(t) = Q(0)^T Q(0) = I$).

General velocity and acceleration terms are easily obtained from the same expression. For example, if

$$\underset{\sim}{y}(t) = \underset{\sim}{o} + \underline{C}_0 Q(t) x \quad (82)$$

$$\begin{aligned} \dot{\underset{\sim}{y}}(t) &= \dot{\underset{\sim}{o}} + \underline{C}_0(\omega\times)Qx && (83)\\ &= \dot{\underset{\sim}{o}} + \underline{C}_0(\omega\times)\underline{C}_0^T\underline{C}_0 Qx && (84)\\ &= \dot{\underset{\sim}{o}} + \underline{\omega} \times (\underset{\sim}{y} - \underset{\sim}{o}) && (85) \end{aligned}$$

$$\begin{aligned} \ddot{\underset{\sim}{y}}(t) &= \ddot{\underset{\sim}{o}} + \underline{C}_0(\dot{\omega}\times)Qx + \underline{C}_0(\omega\times)\dot{Q}x && (86)\\ &= \ddot{\underset{\sim}{o}} + \underline{C}_0(\dot{\omega}\times)Qx + \underline{C}_0(\omega\times)(\omega\times)Qx && (87)\\ &= \ddot{\underset{\sim}{o}} + \underbrace{\dot{\underline{\omega}} \times (\underset{\sim}{y} - \underset{\sim}{o})}_{\text{transverse acceleration}} + \underbrace{(\underline{\omega}\times)[(\underline{\omega}\times)(\underset{\sim}{y} - \underset{\sim}{o})]}_{\text{centripetal acceleration}} && (88) \end{aligned}$$

2. If $\underline{C}_1(t) = \underline{C}_0 Q(t) = \underline{C}_0 e^{\theta(t)s(t)\times}$, we know that $\underline{C}_0 s(t)$ is the *absolute* normalized axis of rotation. The angular velocity vector is aligned with the *instantaneous* axis of rotation.

Without loss of generality, we will show this for $t = 0$. Consider the definition of the angular velocity of $\underline{C}_1$ with respect to $\underline{C}_0$. Using that $Q(0) = I$ and $\theta(0) = 0$,

$$\begin{aligned} {}^0\omega_{1,0}(0)\times &= \dot{Q}(0)Q(0)^T && (89)\\ &= \lim_{\Delta t\to 0} \frac{1}{\Delta t}[Q(\Delta t) - I]Q(0)^T && (90)\\ &= \lim_{\Delta t\to 0} \frac{1}{\Delta t}[e^{\theta(\Delta t)s(\Delta t)\times} - I] && (91)\\ &= \left[\frac{d}{dt}\theta(t)s(t)\times\right]|_{t=0} && (92)\\ &= \dot{\theta}(0)s(0)\times \quad . && (93) \end{aligned}$$

It follows that the angular velocity equals the instantaneous normalized axis of rotation scaled by the rate of change of the instantaneous angle of rotation.

1.6 Addition of Angular Velocities

Consider two time-varying frames $\underline{C}_{i+1}$ and $\underline{C}_i$. If we know the angular velocity $\underline{\omega}_{i,i-1}$ of $\underline{C}_i$ with respect to some frame $\underline{C}_{i-1}$, and the angular velocity $\underline{\omega}_{i+1,i}$ of $\underline{C}_{i+1}$ with respect to $\underline{C}_i$, the angular velocity of $\underline{C}_{i+1}$ with respect to $\underline{C}_{i-1}$ is simply $\underline{\omega}_{i+1,i}+\underline{\omega}_{i,i-1}$.

Indeed, let $\underline{C}_{i+1} = \underline{C}_i{}^i C_{i+1}$ and $\underline{C}_i = \underline{C}_{i-1}{}^{i-1}C_i$. Then, by definition

$$
\begin{aligned}
{}^{i-1}\omega_{i,i-1}\times &= {}^{i-1}\dot{C}_i \; {}^{i-1}C_i^T & (94)\\
{}^{i}\omega_{i+1,i}\times &= {}^{i}\dot{C}_{i+1} \; {}^{i}C_{i+1}^T & (95)
\end{aligned}
$$

and therefore

$$
\begin{aligned}
{}^{i-1}\omega_{i+1,i-1}\times &= \frac{d}{dt}[\,{}^{i-1}C_{i+1}] \; {}^{i-1}C_{i+1}^T & (96)\\
&= \frac{d}{dt}[\,{}^{i-1}C_i{}^{i}C_{i+1}] \; {}^{i}C_{i+1}^T \, {}^{i-1}C_i^T & (97)\\
&= [\,{}^{i-1}\dot{C}_i \; {}^{i}C_{i+1} + {}^{i-1}C_i \; {}^{i}\dot{C}_{i+1}] \; {}^{i}C_{i+1}^T \, {}^{i-1}C_i^T & (98)\\
&= {}^{i-1}\dot{C}_i \, {}^{i-1}C_i^T + {}^{i-1}C_i \; [\,{}^{i}\dot{C}_{i+1} \; {}^{i}C_{i+1}^T] \; {}^{i-1}C_i^T & (99)\\
&= {}^{i-1}\omega_{i,i-1}\times + {}^{i-1}C_i \; [{}^{i}\omega_{i+1,i}\times] \; {}^{i-1}C_i^T & (100)\\
&= ({}^{i-1}\omega_{i,i-1}\times) + ({}^{i-1}\omega_{i+1,i}\times) & (101)\\
&= ({}^{i-1}\omega_{i,i-1} + {}^{i-1}\omega_{i+1,i})\times \; . & (102)
\end{aligned}
$$

Example 1.5 *Spherical Wrist.*

$$
\begin{array}{ll}
\underline{C}_1 = \underline{C}_0 \, {}^{0}C_1 = \underline{C}_0 e^{\theta_1 k\times} & {}^{0}\omega_{1,0} = \dot{\theta}_1 k \\
\underline{C}_2 = \underline{C}_1 \, {}^{1}C_2 = \underline{C}_1 e^{\theta_2 j\times} & {}^{1}\omega_{2,1} = \dot{\theta}_2 j \\
\underline{C}_3 = \underline{C}_2 \, {}^{2}C_3 = \underline{C}_2 e^{\theta_3 i\times} & {}^{2}\omega_{3,2} = \dot{\theta}_3 i
\end{array} \qquad (103)
$$

Therefore,

$$
\begin{aligned}
\underline{\omega}_{3,0} &= \underline{C}_0\dot{\theta}_1 k + \underline{C}_1\dot{\theta}_2 j + \underline{C}_2\dot{\theta}_3 i & (104)\\
&= \underline{C}_0\dot{\theta}_1 k + \underline{C}_0 e^{\theta_1 k\times}\dot{\theta}_2 j + \underline{C}_0 e^{\theta_1 k\times} e^{\theta_2 j\times}\dot{\theta}_3 i \; . & (105)
\end{aligned}
$$

This could have been obtained, in an obviously painful way, by direct differentiation of ${}^{0}C_3$.

$$
\begin{aligned}
{}^{0}C_3 &= e^{\theta_1 k\times} e^{\theta_2 j\times} e^{\theta_3 i\times} & (106)\\
{}^{0}\dot{C}_3 &= \dot{\theta}_1(k\times)e^{\theta_1 k\times} e^{\theta_2 j\times} e^{\theta_3 i\times} + e^{\theta_1 k\times}\dot{\theta}_2(j\times)e^{\theta_2 j\times} e^{\theta_3 i\times} + \\
&\quad e^{\theta_1 k\times} e^{\theta_2 j\times}\dot{\theta}_3(i\times)e^{\theta_3 i\times} & (107)\\
&= \dot{\theta}_1(k\times)e^{\theta_1 k\times} e^{\theta_2 j\times} e^{\theta_3 i\times} \\
&\quad +\dot{\theta}_2[(e^{\theta_1 k\times} j)\times]e^{\theta_1 k\times} e^{\theta_2 j\times} e^{\theta_3 i\times} \\
&\quad +\dot{\theta}_3[(e^{\theta_1 k\times} e^{\theta_2 j\times} i)\times]e^{\theta_1 k\times} e^{\theta_2 j\times} e^{\theta_3 i\times} \\
&= [\dot{\theta}_1 k + \dot{\theta}_2 e^{\theta_1 k\times} j + \dot{\theta}_3 e^{\theta_1 k\times} e^{\theta_2 j\times} i] \times \; {}^{0}C_3 & (108)
\end{aligned}
$$

and therefore

$$ {}^{0}\omega_{3,0} \;=\; \dot{\theta}_1 k + \dot{\theta}_2 e^{\theta_1 k\times} j + \dot{\theta}_3 e^{\theta_1 k\times} e^{\theta_2 j\times} i \qquad (109) $$

which is the same as before.

1.7 Summary

Geometric Object / Transformation				**Coordinate Representation**
Vector	$\underline{x}$	$=$	$\underline{C}x$	x in frame $\underline{C}$
Point	$\underset{\sim}{x}$	$=$	$\underset{\sim}{o} + \underline{C}x$	x in coord. system $\{\underset{\sim}{o}, \underline{C}\}$
Scalar Product	$\underline{y}$	$=$	$\underline{s}^T \underline{x}$	$y = s^T x$
Vector Product	$\underline{y}$	$=$	$\underline{s} \times \underline{x}$	$y = \underline{C}^T[(\underline{C}s)\times]\underline{C}x = s \times x$ in frame $\underline{C}$
Rotation	$\underline{y}$	$=$	$\underline{\underline{Q}}\,\underline{x} = e^{\theta \underline{s}\times} \underline{x}$	$y = \underline{C}^T \underline{\underline{Q}}\,\underline{C}\, x = Qx = e^{\theta s\times}\, x$ in frame $\underline{C}$

Rigid Motion

$\underset{\sim}{y} = \underset{\sim}{o} + \underline{d} + \underline{\underline{Q}}(\underset{\sim}{x} - \underset{\sim}{o})$ — $y = Qx + d$ in

first rotation then translation) — coordinate system $\{\underset{\sim}{o}, \underline{C}\}$