Lesson 7-1 – Compound Interest – Uniform Series

Special Acknowledgment to Dr Ron Mackinnon and Dr Tamara Etmannski who helped with the development of this material.

Review of Key Compound Interest Terms

- Terms:
  - P = Present Value, money at time = now
  - F = Future Value, money at a specified point in the future
  - i = interest rate per compounding period
  - n = number of compounding periods

- Compound Interest Single Payment Formulae

- $F = P(1+i)^n$
- $P = F(1+i)^{-n}$

Review of Equivalence

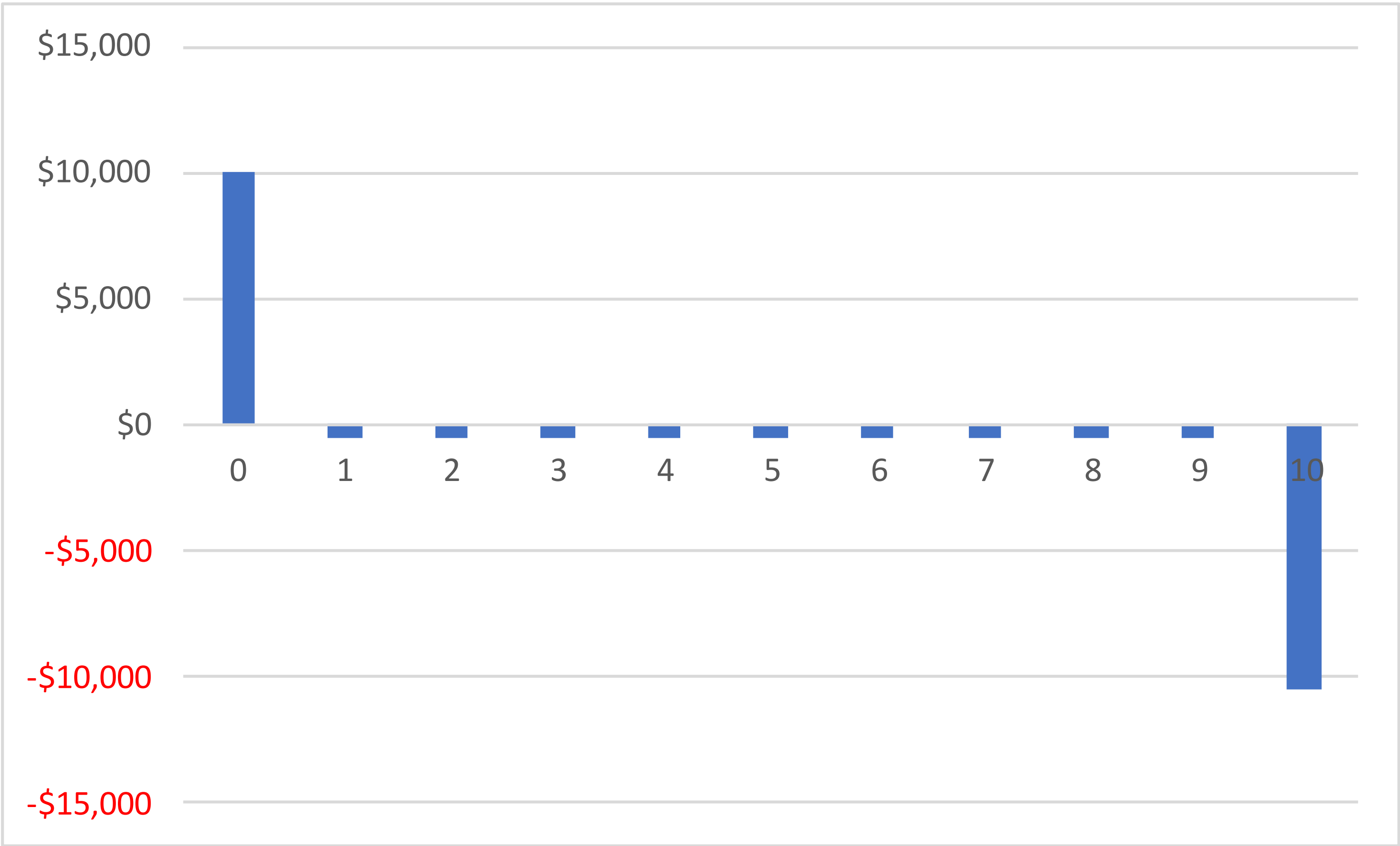
- Equivalence implies that a sum of money in one time period has the same “value” to a different sum in another time period with respect to an interest rate.
- Answers the question “how much is $X now worth to me at some point in the future?”
- Example: $1000 now is equivalent to:
    - $1100 one year from now at 10% per year
    - $1050 one year from now at 5% per year
    - $1210 two years from now at 10% per year
    - $1102 two years from now at 5% per year

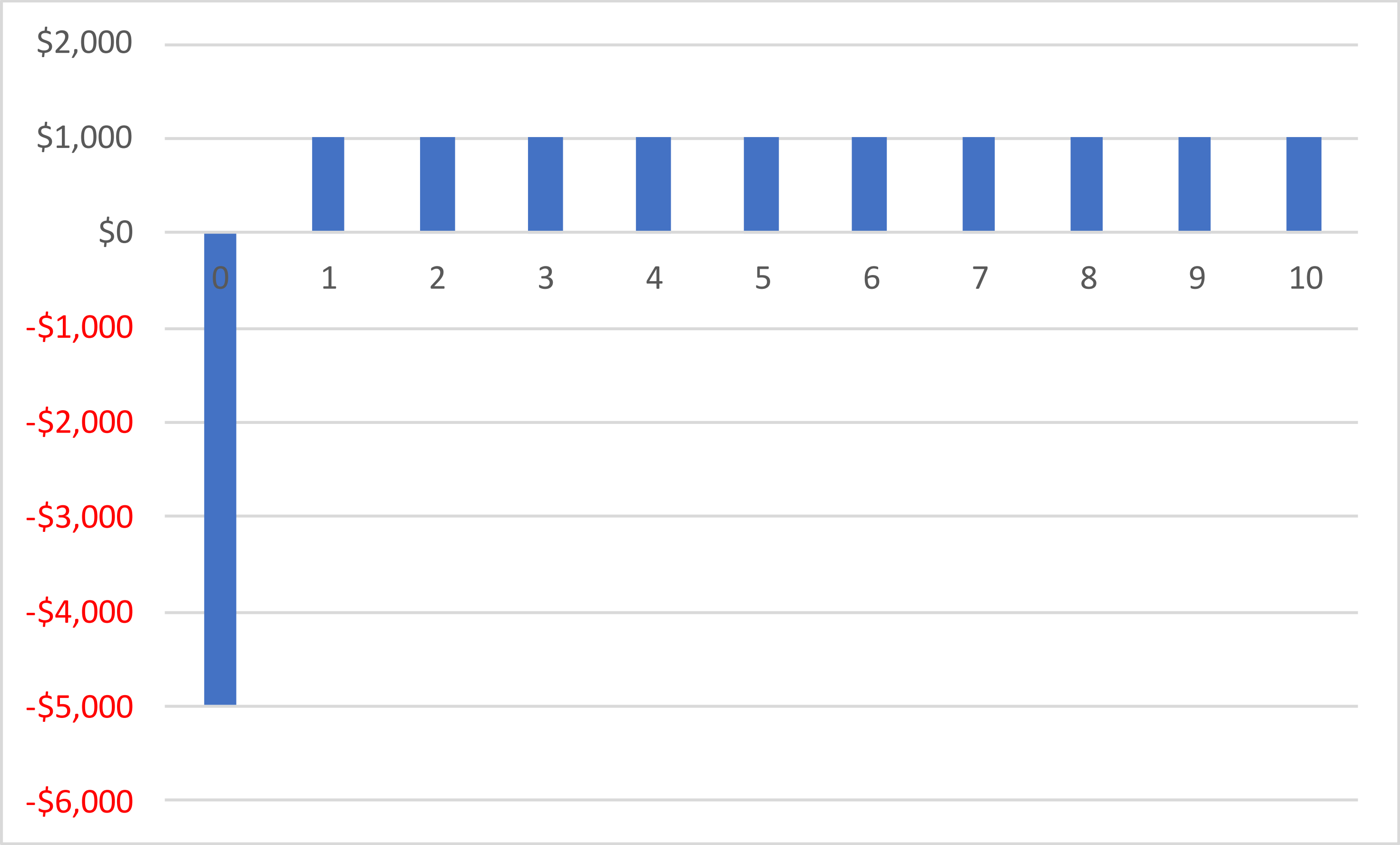
Chapter 4: Learning Objectives

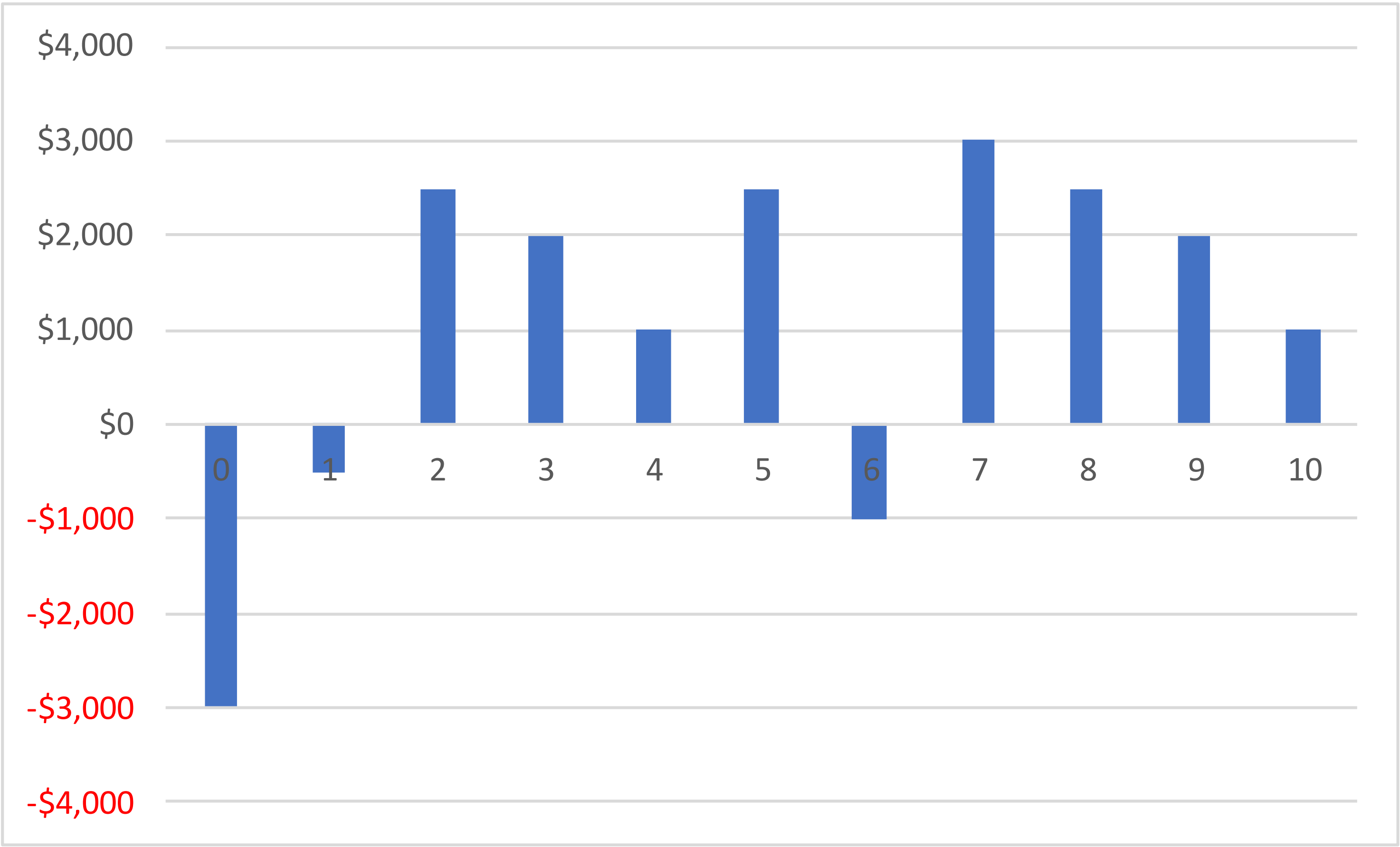
- Solve problems modelled by the uniform series compound interest formulas
- Use arithmetic gradients and geometric gradients to solve series of cash flow problems
- Evaluate non-standard series cash flows:
    a. With begin-period payments
    b. With different payment and compounding periods
    c. With infinite series of payments (perpetual annuities)

More About Compound Interest

- Calculation of present value (P), future value (F) and periodic cash flows (A) are fundamental to engineering economic analysis.
- Some problems are more complex and require an understanding of added components:
  - uniform series
  - arithmetic or geometric gradients
  - non-standard series


$15,000
$10,000
$5,000
$0
-$5,000
-$10,000
-$15,000
0
1
2
3
4
5
6
7
8
9
10


$2,000
$1,000
$0
-$1,000
-$2,000
-$3,000
-$4,000
-$5,000
-$6,000
0
1
2
3
4
5
6
7
8
9
10


$4,000
$3,000
$2,000
$1,000
$0
-$1,000
-$2,000
-$3,000
-$4,000
0
1
2
3
4
5
6
7
8
9
10

Consider the equivalence

- What would the equivalent PV of a single payment of \$1,000,000 one year from today be, given an interest rate of 2%?
  - Many ways to solve – formula, interest tables, spreadsheets
  - $P = F(1+i)^{-n}$ = \$1,000,000 $(1+0.02)^{-1}$ = \$1,000,000(0.98) = \$980,000
- What would the equivalent PV of **two** payments of \$1,000,000 be, the first one year from today, the second two years from today. i=2%
  - $P1 = F(1+i)^{-n}$ = \$1,000,000 $(1+0.02)^{-1}$ = \$1,000,000(0.98) = \$980,000
  - $P2 = F(1+i)^{-n}$ = \$1,000,000 $(1+0.02)^{-2}$ = \$1,000,000(0.96) = \$960,000
- P_{total} = P1+P2 = \$980,000 + \$960,000 = \$1,940,000

And generalize it

- P_{total} = P1 + P2
- $P_{total} = F(1+i)^{-1} + F(1+i)^{-2}$
- $P_{total} = F\ ((1+i)^{-1} + (1+i)^{-2})$
- Let's rename "F" to "A" for annuity, and extend indefinitely
- $P = A((1+i)^{-1} + (1+i)^{-2} +\ldots. +(1+i)^{-n}))$
- This can (with great difficulty) be simplified to the Uniform Series Present Worth Factor

$$P = A\left[\frac{(1+i)^{n}-1}{i(1+i)^{n}}\right]$$

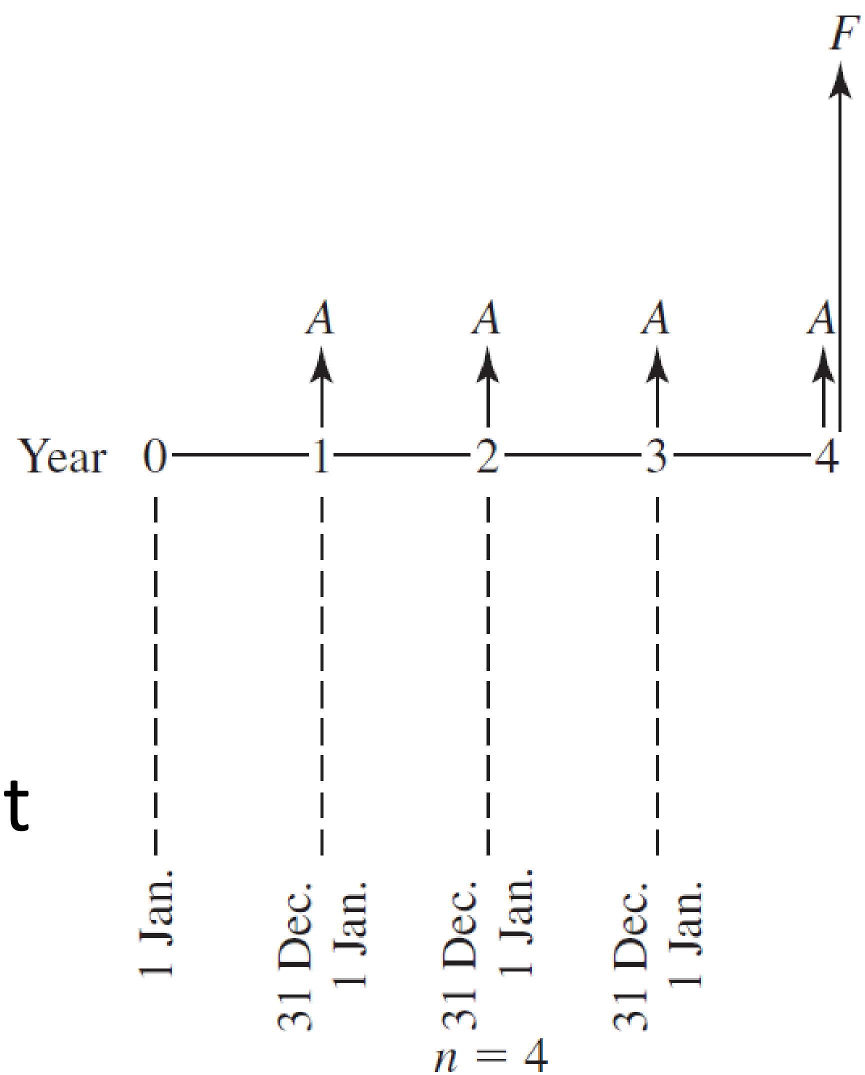
Uniform Series Cashflows

- Uniform Series
  - An end-of-period cash receipt or disbursement (A) in equal succession and amount, continuing for n periods, with the entire series equivalent to P or F, at an interest rate I

- Often called an 'annuity' – be careful not to confuse with the financial product of the same name.

Uniform Series

- Annuities have equally-spaced and equal-valued cash flows during a period of time.
- Cash inflows are positive — toward the firm.
- Cash outflows are negative — away from the firm.
- Ordinary annuities have a cash flow at the end of each payment period



Uniform Series Formulas

The “uniform series compound amount factor” is:

$$F = A\left[\frac{(1+i)^n - 1}{i}\right]$$

Also noted as (F/A, i, n), as in F = A(F/A, i, n)

Rearrange and solving for A is finding the “uniform series sinking fund factor”:

$$A = F\left[\frac{i}{(1+i)^n - 1}\right]$$

- Noted as (A/F, i, n)

Uniform Series Formulas Continued…

Taking the 'sinking fund formula' and substitute P = F(1+i)-n to get the 'Capital Recovery Factor'

$$A = P\left[\frac{i(1+i)^n}{(1+i)^n - 1}\right]$$

(A/P, i, n)

Solving the capital recovery formula for PV results in the "uniform series present worth factor":

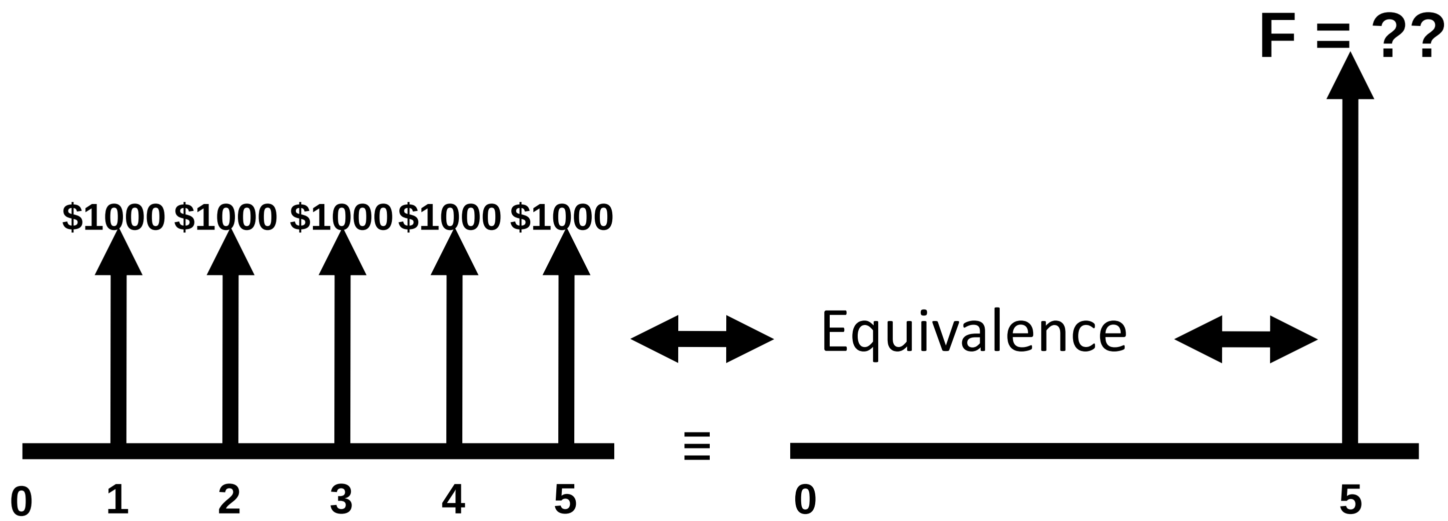
$$P = A\left[\frac{(1+i)^n - 1}{i(1+i)^n}\right]$$

- (P/A, i, n)

Relationships between Compound Interest Factors

- Single Payment (Chapter 3)
  - Compound amount factor = 1/Present Worth Factor
  - (F/P, i, n) = $(1+i)^n$
  - Single Payment Present Worth Factor:
    (P/f, i, n) = $(1+i)^{-n}$
- Uniform Series (Chapter 4)
  - Compound amount factor = 1/Sinking Fund Factor
  - Capital recovery factor = 1/Present Worth Factor

Uniform Series: Future Example 1



- What is the value in five years of five end-of-year deposits of $1000 beginning one year from today if interest is 10% compounded annually?

Uniform Series: Future Example 1

- From the cash flow diagram we see that this is a series of compound interest single payments:

$F = F1 + F2 + F3 + F4 + F5$ where

$Fn = Pn(1+i)^n$ (Single payment compound interest formula, note end of period payment)

Year	Beginning Balance	Interest for Period	Payment	Ending Balance
1	$ -	$ -	$1,000.00	$1,000.00
2	$1,000.00	$ 100.00	$1,000.00	$2,100.00
3	$2,100.00	$ 210.00	$1,000.00	$3,310.00
4	$3,310.00	$ 331.00	$1,000.00	$4,641.00
5	$4,641.00	$ 464.10	$1,000.00	$6,105.10

Uniform Series: Future Example 1

- In general, we can use the series formula:

$$FV = A\left[\frac{(1+i)^n - 1}{i}\right]$$

- which in this case gives:

$$FV = \$1000\left[\frac{(1+0.10)^5 - 1}{0.10}\right] = \$6105.10$$

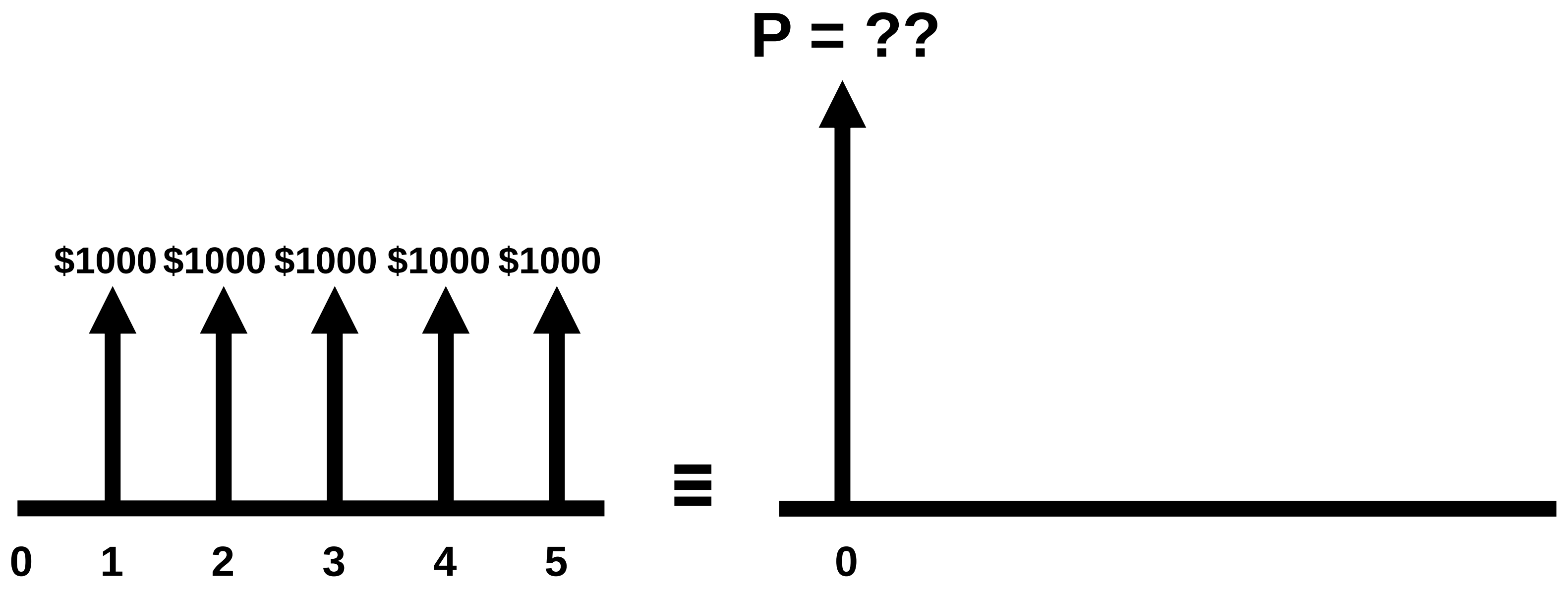
Uniform Series: Future Example 2

- Find the balance in ten years of annual deposits of \$1500 into a fund that pays interest of 8% compounded annually.

$$FV = \$1500\left[\frac{(1+0.08)^{10}-1}{0.08}\right] = \$21{,}729.84$$

- All other things being equal, \$1500 at the end of each year for ten years is equivalent to \$21,729.84 ten years from today at 8% annual interest

Uniform Series: Present Example 1



- What is the value today of five end-of-year deposits of $1000 beginning one year from today if interest is 10% compounded annually?

Uniform Series: Present Example 1

- From the cash flow diagram, we see that

$$P = F1(1+i)^{-n1} + F2(1+i)^{-n1} \ldots..$$

$$= \$1000(1.10^{-1} + 1.10^{-2} + 1.10^{-3} + 1.10^{-4} + 1.10^{-5})$$

$$= \$3790.79$$

- In general, we can use the series formula

$$P = A\left[\frac{(1+i)^n - 1}{i(1+i)^n}\right]$$

- which in this case gives:

$$P = \$1000\left[\frac{(1+0.10)^5 - 1}{0.10(1+0.10)^5}\right] = \$3790.79$$

Uniform Series: PV Example 1 - Spreadsheet

	A	B	C	D	E	F	G
1			**Rate:**	10%	Per Period		
2			**Payment**	$1,000	Per Period		
3			**Number of Periods**	5			
4							
5			**Period**	**Cashflow**	**Present Value**		
6			0	$0	$0		
7			1	$1,000	$909.09	=PV(D1, C5, 0, -D5)	
8			2	$1,000	$826.45	=PV(D1, C6, 0, -D6)	
9			3	$1,000	$751.31		
10			4	$1,000	$683.01		
11			5	$1,000	$620.92	=PV(D1, C9, 0, -D9)	
12							
13				**Total**	$3,790.79	=SUM(E4:E9)	
14							
15				**Present Value**	$3,790.79	=PV(D1, D3, -D2)	
16							
17		**Function:**	=PV(rate, nper, pmt, [fv], [type])				

Uniform Series: PV Example 2

- A minor change in testing procedures at your facility would save \$200 per month over five years. Your company uses an interest rate of 18% per year compounded monthly ($i = 1.5\%/\text{month}$). Would you pay \$7,450 today for these improvements?

- $P = \$200(P/A, 1.5\%, 60) = \$200(39.380) = \$7,876$

- The PV of the procedure is greater than PV of the \$7,450 in cash it would cost to implement it, therefore, we should do it.

Uniform Series: PV Example 2

- A minor change in testing procedures at your facility would save $200 per month over five years. Your company … year compoun… would you pay $7,45… ?

$1\frac{1}{2}\%$ — Compound Interest Factors

	Single Payment		Uniform Payment Series			
n	Compound Amount Factor Find F Given P F/P	Present Worth Factor Find P Given F P/F	Sinking Fund Factor Find A Given F A/F	Capital Recovery Factor Find A Given P A/P	Compound Amount Factor Find F Given A F/A	Present Worth Factor Find P Given A P/A
1	1.015	.9852	1.0000	1.0150	1.000	0.985
2	1.030	.9707	.4963	.5113	2.015	1.956
3	1.046	.9563	.3284	.3434	3.045	2.912
4	1.061	.9422	.2444	.2594	4.091	3.854
5	1.077	.9283	.1941	.2091	5.152	4.783
48	2.043	.4894	.0144	.0294	69.565	34.042
50	2.105	.4750	.0136	.0286	73.682	35.000
52	2.169	.4611	.0128	.0278	77.925	35.929
60	2.443	.4093	.0104	.0254	96.214	39.380
70	2.835	.3527	.00817	.0232	122.363	43.155

- P = $200(… $7,876

- The PV of the procedure is greater than PV of the $7,450 in cash it would cost to implement it, therefore, we should do it.

What if?

- What if the corporate interest rate changed to 24% per year (2% per month?).
- P = \$200(P/A, 2%, 60) = \$200(34.761) = \$6,952
- This is less than the \$7,450 you would pay today for these improvements. \$7,450 in cash has a greater PV than the PV of the procedure. Ergo, we should not implement it.
- What's going on?