< Digital Control Design >

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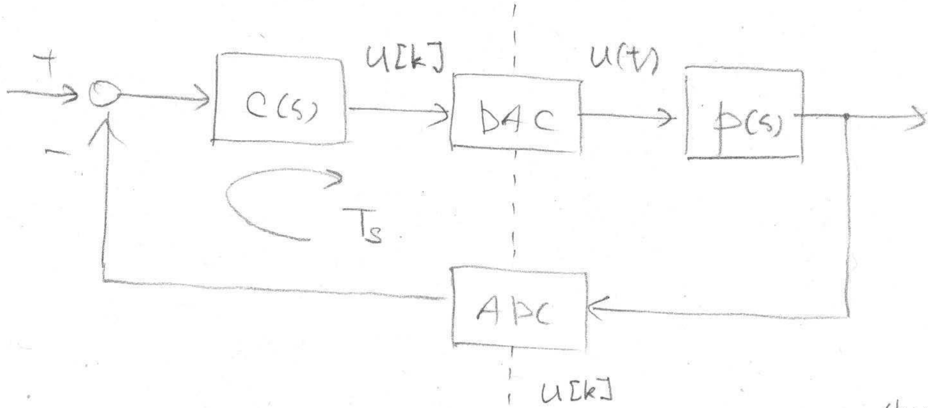
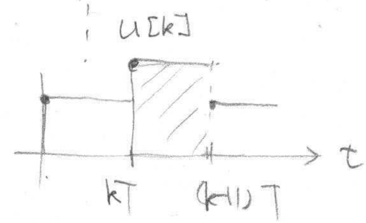
- Objectives
  - Indirect design via DT approximation of CT controls.
  - Understand the difference between DT approximation methods.

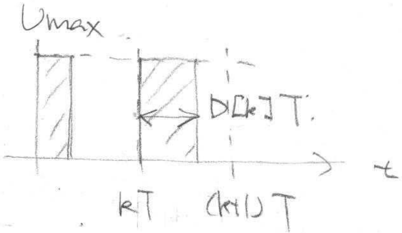


- ADC ① PAM



$$\int_{kT}^{(k+1)T} u(t)\, dt = u[k] \cdot T$$

② PWM



$$\int_{kT}^{(k+1)T} u(t)\, dt = U_{max} D[k]\, T$$

- If $u[k]\, T = U_{max} D[k]\, T$ $\left(D[k] = \dfrac{u[k]}{U_{max}} \right)$

  the two analog signals are similar in low-frequency contents.

- PWM works well for inductive plants (e.g. power convertes, motors)
- It can be directly connected to class D (switched-mode) power amplifiers.

- Sampling rate : Select $f_s > 10 f_c$ at least.
  $(f_s = \frac{1}{T_s})$

- Two methods to design DT control.

① Direct design for DT-equivalent plant (ZOH equivalent)

- Find $P(z)$ and design $C(z)$. MECH 467.

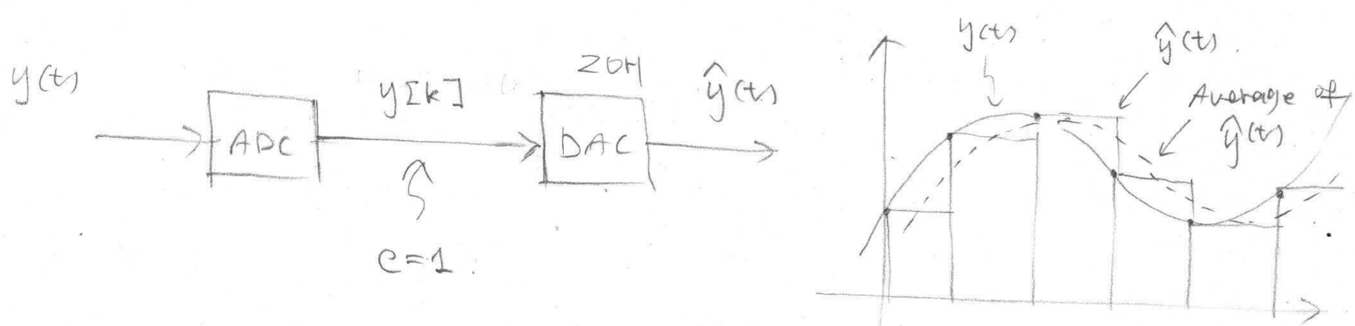
② Indirect design via DT approximation of CT control.

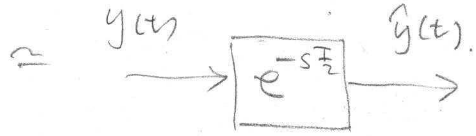
- Design $C(s)$ and implement approximate $C(z)$.

- This is called "Emulation"

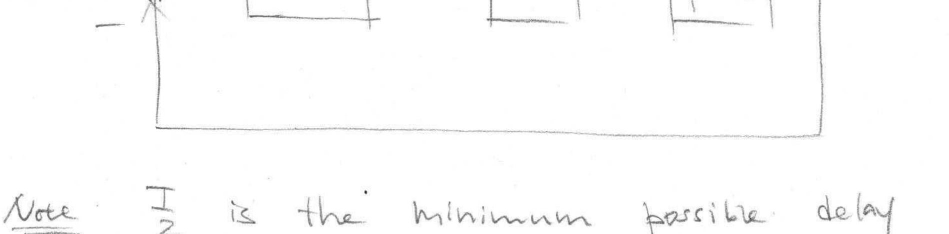
- It gives a satisfactory result when the delay is accounted for.

- The ZOH DAC can be modeled as a half-sample delay





- This delay can be absorbed to the plant and we design $C(s)$ for $p(s)\, e^{-s\frac{T}{2}}$.



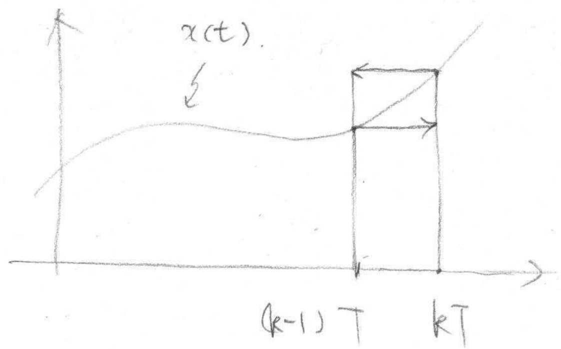
depending on the read/write algorithm.

Note $\frac{T}{2}$ is the minimum possible delay

It is common to have more delay. (e.g. $\frac{T}{2}+T$)

- Discrete-time Approximation Methods.

① Numerical Integration.



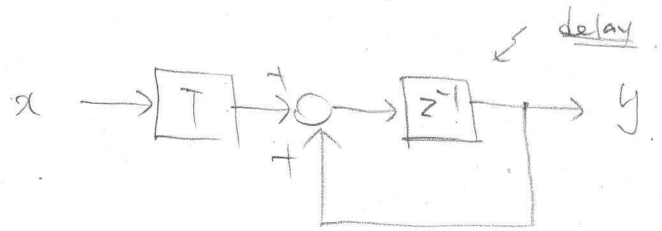
- $y(t) = \int_{-\infty}^{t} x(t)\, dt$

- Find $y[k]$ that approximates $y(t)$:

- Forward rectangular method (Euler method).

$$y[k] = y[k-1] + T \cdot x[k-1].$$

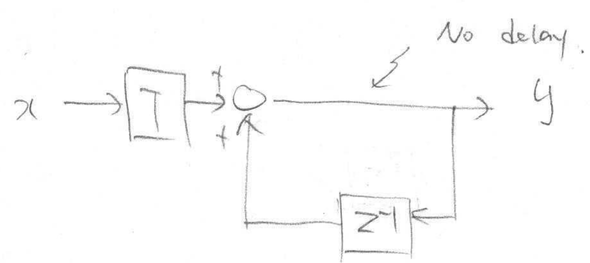
$$Y(z)\,(1 - z^{-1}) = T \cdot z^{-1} \cdot X(z) \qquad \therefore \frac{Y(z)}{X(z)} = T\left(\frac{z^{-1}}{1 - z^{-1}}\right)$$



- Backward rectangular method

$$y[k] = y[k-1] + T\, x[k]$$

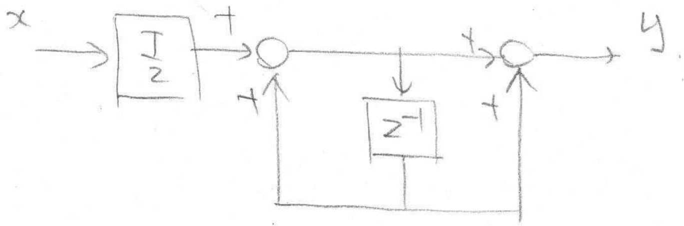
$$Y(z)\,(1 - z^{-1}) = T \cdot X(z) \qquad \therefore \frac{Y}{X} = T\left(\frac{1}{1 - z^{-1}}\right)$$



- Bilinear / Trapezoidal / Tustin method.

$$y[k] = y[k-1] + \frac{T}{2}\left(x[k] + x[k-1]\right)$$

$$Y(z)\left(1-z^{-1}\right) = \frac{T}{2}X(z)\left(1+z^{-1}\right). \quad \therefore \frac{Y}{X} = \frac{T}{2}\left(\frac{1+z^{-1}}{1-z^{-1}}\right)$$



∘ Summary

	Substitution Rule		Mapping Rule
①. Forward Rect. (Euler)			
$\frac{1}{s} = T\frac{z^{-1}}{1-z^{-1}}$ $\rightarrow$	$s = \frac{z-1}{T}$	$\rightarrow$	$z = 1+Ts$
② Backward Rect			
$\frac{1}{s} = T\frac{1}{1-z^{-1}}$ $\rightarrow$	$s = \frac{z-1}{Tz}$	$\rightarrow$	$z = \frac{1}{1-Ts}$
③ Tustin.			
$\frac{1}{s} = \frac{T}{2}\frac{1+z^{-1}}{1-z^{-1}}$ $\rightarrow$	$s = \frac{2}{T}\frac{z-1}{z+1}$	$\rightarrow$	$z = \frac{1+2Ts}{1-2Ts}$

· Substitution rules give us ways to approximate $C(s)$ with $C(z)$. __Note__: Simulink uses __Euler method__ by default!

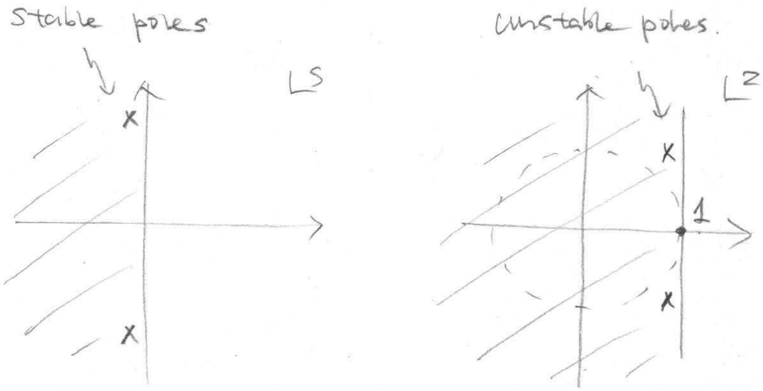
· Once $C(z)$ is obtained, one can

i) Directly implement it using Simulink or LabVIEW.

ii) Convert it to the difference equation and implement it using text-based programming language.

- Effect on stability.
  - Each method maps the left half plane (LHP) of the s-plane to a different region in the z-plane.
  - This affects the stability of the approximate DT system.

① Forward rect.

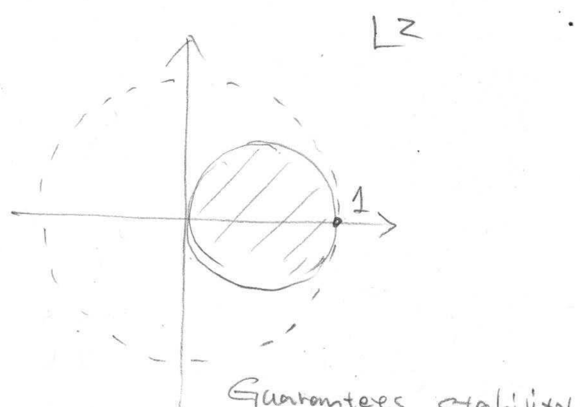


$z = 1 + Ts$.

$$\begin{cases} s = 0 \rightarrow z = 1 \\ s = j\omega \rightarrow z = 1 + jT\omega \end{cases}$$

- The LHP of s-plane is scaled by T & shifted by 1.
- Stable $H(s)$ can turn into unstable $H(z)$.

② Backward rect.



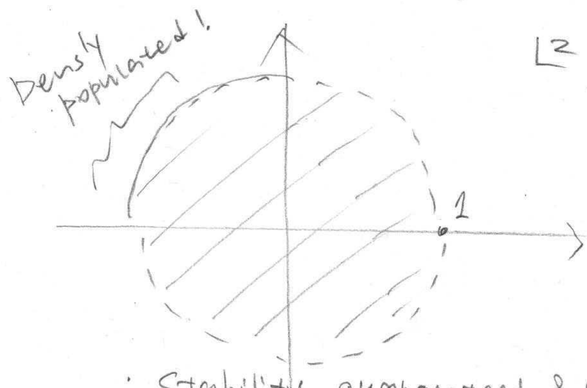
$$z = \frac{1}{1-Ts}$$

$$\begin{cases} s = 0 \rightarrow z = 1 \\ s = j\omega \rightarrow z = \frac{1}{1 - jT\omega} \end{cases}$$

As $\omega \rightarrow \infty$, $z \simeq \frac{1}{-jT\omega}$ $\begin{cases} |z| = 0 \\ \angle z = \frac{\pi}{2} \end{cases}$.

Guarantees stability but distorts dynamics.

③ Tustin method. (Recommended)



$$z = \frac{1+2Ts}{1-2Ts}$$

$$\begin{cases} s = 0 \rightarrow z = 1 \\ s = j\omega \rightarrow z = \frac{1+j2T\omega}{1-j2T\omega} \end{cases}$$

As $\omega \rightarrow \infty$, $z \simeq \frac{j2T\omega}{-j2T\omega}$ $\begin{cases} |z| = 1 \\ \angle z = \pi \end{cases}$

- Stability guaranteed & exact.
- High-freq. distortion