a place of mind

MECH468 : Modern Control Engineering
MECH509 : Controls

L17 : Realization
Observable canonical form

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Zoom lecture to be recorded and posted on Canvas

Course plan

Topics	CT	DT
Modeling	✓	✓
Stability	✓	✓
Controllability/observability	✓	✓
Realization	→	
State feedback/observer		
LQR/Kalman filter		

Realization (review)

- Given a rational proper transfer matrix $G(s)$ find matrices (*A*,*B*,*C*,*D*) s.t.

$$G(s) = C(sI - A)^{-1}B + D$$

1. Always extract *D*-matrix first!

$$D = G(\infty)$$

2. After extracting *D*, find (*A*,*B*,*C*) s.t

$$G_{sp}(s) = C(sI - A)^{-1}B$$

Review

- In the last lecture, we learned a realization, called a *controllable canonical form*.

$$G(s) = \frac{1}{s^2+4s+3}\left\{\begin{bmatrix} 0 & 1 \end{bmatrix} s + \begin{bmatrix} 1 & 1 \end{bmatrix}\right\}$$

$$\Rightarrow \left\{\begin{array}{l} \dot{x} = \left[\begin{array}{cc|cc} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \hline -3 & 0 & -4 & 0 \\ 0 & -3 & 0 & -4 \end{array}\right] x + \left[\begin{array}{cc} 0 & 0 \\ 0 & 0 \\ \hline 1 & 0 \\ 0 & 1 \end{array}\right] u \\ y = \left[\begin{array}{cc|cc} 1 & 1 & 0 & 1 \end{array}\right] x \end{array}\right.$$

- Today, we study "dual" realization, called an *observable canonical form*.

Observable canonical form

- SISO example $G(s)=\dfrac{n_1s^2+n_2s+n_3}{s^3+\alpha_1s^2+\alpha_2s+\alpha_3},\quad n_i\in\mathbb{R}$

$$\boxed{\begin{cases}\dot{x}(t) &= \begin{bmatrix}0 & 0 & -\alpha_3\\ 1 & 0 & -\alpha_2\\ 0 & 1 & -\alpha_1\end{bmatrix}x(t)+\begin{bmatrix}n_3\\ n_2\\ n_1\end{bmatrix}u(t)\\ y(t) &= \begin{bmatrix}0 & 0 & 1\end{bmatrix}x(t)\end{cases}}$$

or equivalently
$$\begin{cases}\dot{x}(t) &= \begin{bmatrix}-\alpha_1 & 1 & 0\\ -\alpha_2 & 0 & 1\\ -\alpha_3 & 0 & 0\end{bmatrix}x(t)+\begin{bmatrix}n_1\\ n_2\\ n_3\end{bmatrix}u(t)\\ y(t) &= \begin{bmatrix}1 & 0 & 0\end{bmatrix}x(t)\end{cases}$$

Companion matrix (review)

- The following form (and its transpose) of a matrix is called *companion matrix (form):*

$$A := \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \cdots & 1 \\ -\alpha_n & -\alpha_{n-1} & \cdots & -\alpha_2 & -\alpha_1 \end{bmatrix} \in \mathbb{R}^{n\times n}$$

- Important property of a companion matrix

$$\det(sI - A) = s^n + \alpha_1 s^{n-1} + \cdots + \alpha_{n-1} s + \alpha_n$$

SISO examples

- Ex.1 $$G(s) = \frac{5s^2 + 3s + 2}{s^3 + 11s^2 + 14s + 6}$$

- Ex.2 $$G(s) = \frac{4s^2 + 3s + 2}{2s^2 + 5s + 6}$$

Observable canonical form for MIMO cases

$$G(s) = \frac{N_1 s^{r-1} + N_2 s^{r-2} + \cdots + N_{r-1}s + N_r}{\underbrace{s^r + \alpha_1 s^{r-1} + \cdots + \alpha_{r-1}s + \alpha_r}_{\text{Least common denominator}}}, \; N_i \in \mathbb{R}^{q \times p}$$

$$\boxed{\begin{aligned} \dot{x} &= \overbrace{\begin{bmatrix} 0 & 0 & \cdots & 0 & -\alpha_r I_q \\ I_q & 0 & \cdots & 0 & -\alpha_{r-1} I_q \\ 0 & \ddots & \ddots & \vdots & \vdots \\ \vdots & 0 & I_q & 0 & -\alpha_2 I_q \\ 0 & \cdots & 0 & I_q & -\alpha_1 I_q \end{bmatrix}}^{A \in \mathbb{R}^{rq \times rq}} x + \underbrace{\begin{bmatrix} N_r \\ N_{r-1} \\ \vdots \\ N_2 \\ N_1 \end{bmatrix}}_{B \in \mathbb{R}^{rq \times p}} u \\ y &= \underbrace{\begin{bmatrix} 0 & 0 & \cdots & 0 & I_q \end{bmatrix}}_{C \in \mathbb{R}^{q \times rq}} x \end{aligned}}$$

MIMO example

- Transfer matrix
$$G(s) = \begin{bmatrix} \frac{1}{s^2+4s+3} & \frac{1}{s+3} \end{bmatrix} = \frac{1}{s^2+4s+3}\begin{bmatrix} 1 & s+1 \end{bmatrix} = \frac{1}{s^2+4s+3}\left\{\begin{bmatrix} 0 & 1 \end{bmatrix} s + \begin{bmatrix} 1 & 1 \end{bmatrix}\right\}$$

$$\Rightarrow \left\{\begin{aligned} \dot{x} &= \left[\begin{array}{c|c} 0 & -3 \\ \hline 1 & -4 \end{array}\right] x + \left[\begin{array}{cc} 1 & 1 \\ \hline 0 & 1 \end{array}\right] u \\ y &= \left[\begin{array}{c|c} 0 & 1 \end{array}\right] x \end{aligned}\right.$$

Note that the size of A-matrix is two, not four!

Q: What is the smallest size of *A*?

(Minimal realization: next lecture)

Remarks

- Note the duality between controllable and observable canonical form.
- Observable canonical realization is always observable (but not always controllable). Why?

$$\dot{x} = \begin{bmatrix} 0 & 0 & \cdots & 0 & -\alpha_r I_q \\ I_q & 0 & \cdots & 0 & -\alpha_{r-1} I_q \\ 0 & \ddots & \ddots & \vdots & \vdots \\ \vdots & 0 & I_q & 0 & -\alpha_2 I_q \\ 0 & \cdots & 0 & I_q & -\alpha_1 I_q \end{bmatrix} x + \begin{bmatrix} N_r \\ N_{r-1} \\ \vdots \\ N_2 \\ N_1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 0 & 0 & \cdots & 0 & I_q \end{bmatrix} x$$

$$\Rightarrow \mathcal{O} =$$

Derivation of observable canonical form

- TF $$G(s) = \frac{n_1 s^2 + n_2 s + n_3}{s^3 + \alpha_1 s^2 + \alpha_2 s + \alpha_3}, \quad n_i \in \mathbb{R}$$

- Rewrite I/O relation as

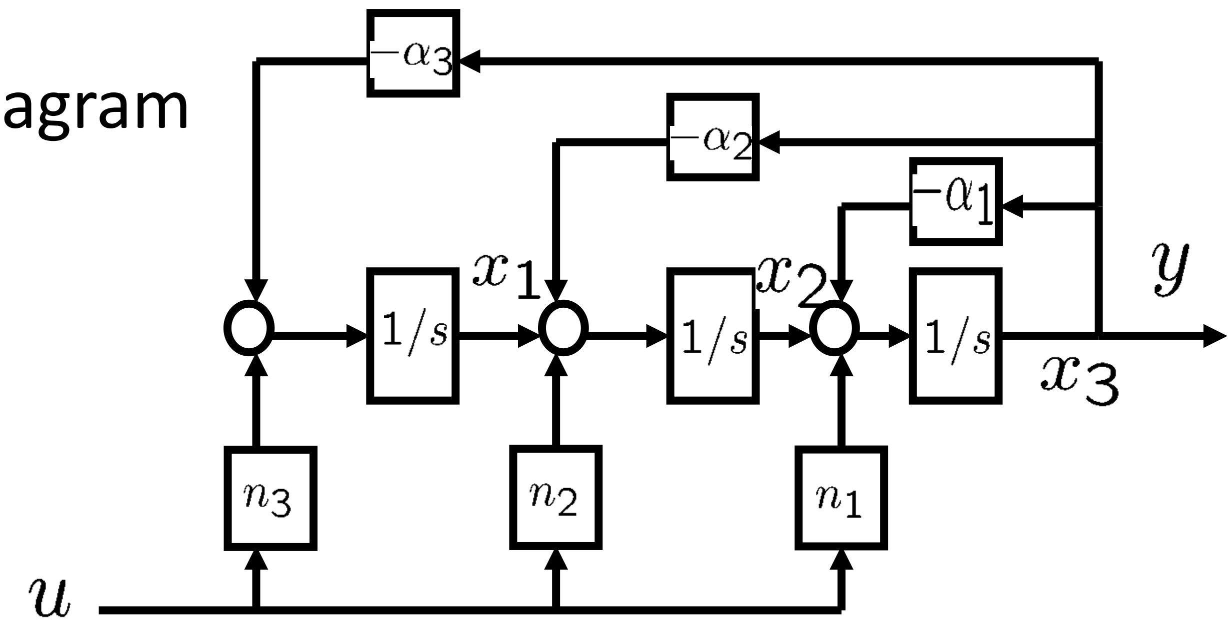
$$y(s) = \frac{n_1 s^2 + n_2 s + n_3}{s^3 + \alpha_1 s^2 + \alpha_2 s + \alpha_3} u(s)$$

$$\Rightarrow \quad (s^3 + \alpha_1 s^2 + \alpha_2 s + \alpha_3) y(s) = (n_1 s^2 + n_2 s + n_3) u(s)$$

$$\Rightarrow \quad y(s) = -\frac{\alpha_1}{s} y(s) - \frac{\alpha_2}{s^2} y(s) - \frac{\alpha_3}{s^3} y(s) + \frac{n_1}{s} u(s) + \frac{n_2}{s^2} u(s) + \frac{n_3}{s^3} u(s)$$

Derivation (cont'd)

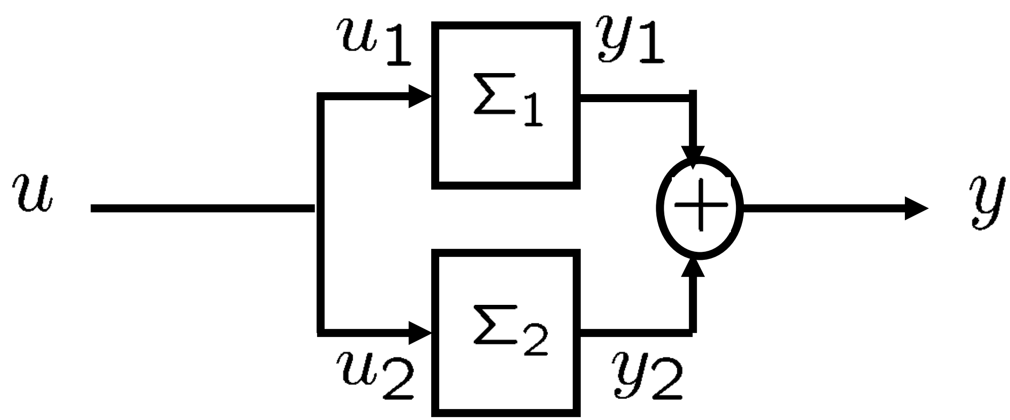
- Draw block-diagram



- By introducing state variables as in the figure, done!

$$\begin{cases} \dot{x}(t) &= \begin{bmatrix} 0 & 0 & -\alpha_3 \\ 1 & 0 & -\alpha_2 \\ 0 & 1 & -\alpha_1 \end{bmatrix} x(t) + \begin{bmatrix} n_3 \\ n_2 \\ n_1 \end{bmatrix} u(t) \\ y(t) &= \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} x(t) \end{cases}$$

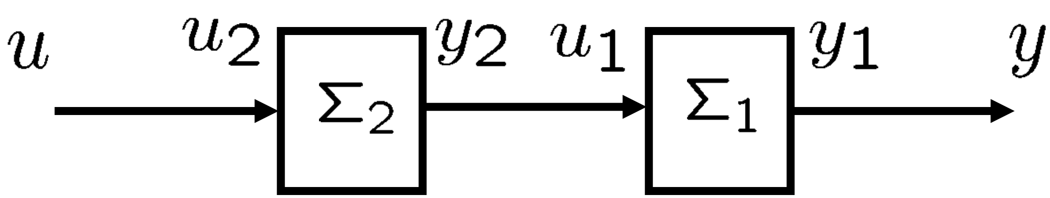
Parallel connection of SS models



$$\Sigma_1 : \begin{cases} \dot{x}_1 = A_1 x_1 + B_1 u_1 \\ y_1 = C_1 x_1 + D_1 u_1 \end{cases} \qquad \Sigma_2 : \begin{cases} \dot{x}_2 = A_2 x_2 + B_2 u_2 \\ y_2 = C_2 x_2 + D_2 u_2 \end{cases}$$

$$\Rightarrow \quad \Sigma_1 + \Sigma_2 : \begin{cases} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u \\ y = \begin{bmatrix} C_1 & C_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + (D_1 + D_2) u \end{cases}$$

Series connection of SS models

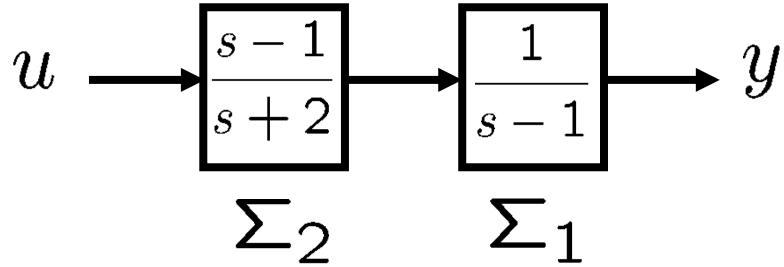


$$\Sigma_2 : \begin{cases} \dot{x}_2 = A_2 x_2 + B_2 u_2 \\ y_2 = C_2 x_2 + D_2 u_2 \end{cases} \qquad \Sigma_1 : \begin{cases} \dot{x}_1 = A_1 x_1 + B_1 u_1 \\ y_1 = C_1 x_1 + D_1 u_1 \end{cases}$$

$$\Rightarrow \quad \Sigma_1 \Sigma_2 : \begin{cases} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} A_1 & B_1 C_2 \\ 0 & A_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} B_1 D_2 \\ B_2 \end{bmatrix} u \\ y = \begin{bmatrix} C_1 & D_1 C_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + D_1 D_2 u \end{cases}$$

An example (Lecture 8, Slide 4)

- Series connection



$$\Sigma_2 : \begin{cases} \dot{x}_2 = -2x_2 - 3u_2 \\ y_2 = x_2 + u_2 \end{cases} \qquad \Sigma_1 : \begin{cases} \dot{x}_1 = x_1 + u_1 \\ y_1 = x_1 \end{cases}$$

$$\Rightarrow \quad \Sigma_1\Sigma_2 : \begin{cases} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ -3 \end{bmatrix} u \\ y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \end{cases}$$

BIBO stable but not internally stable!

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UBC

Summary

- Observable canonical realization
- Parallel and series connections of SS models
- Next, minimal realization