Lesson 7-2– Arithmetic Series

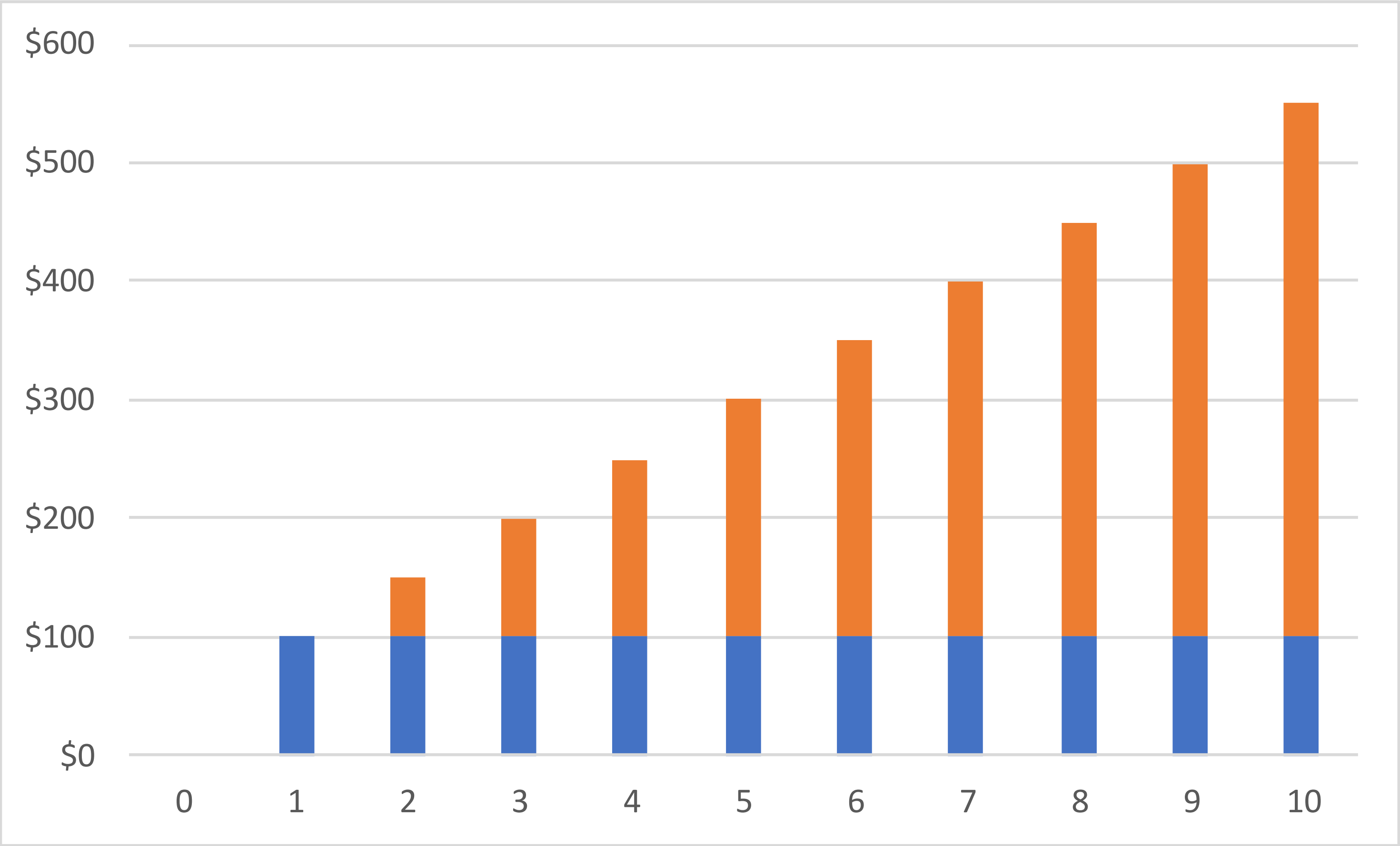
Special Acknowledgment to Dr Ron Mackinnon and Dr Tamara Etmannski who helped with the development of this material.

Uniform Series

- Last lesson we looked at a uniform series – series of cashflows with a constant nominal value for n periods an interest rate i.
- Figured out how to solve for P, F, A and move between them

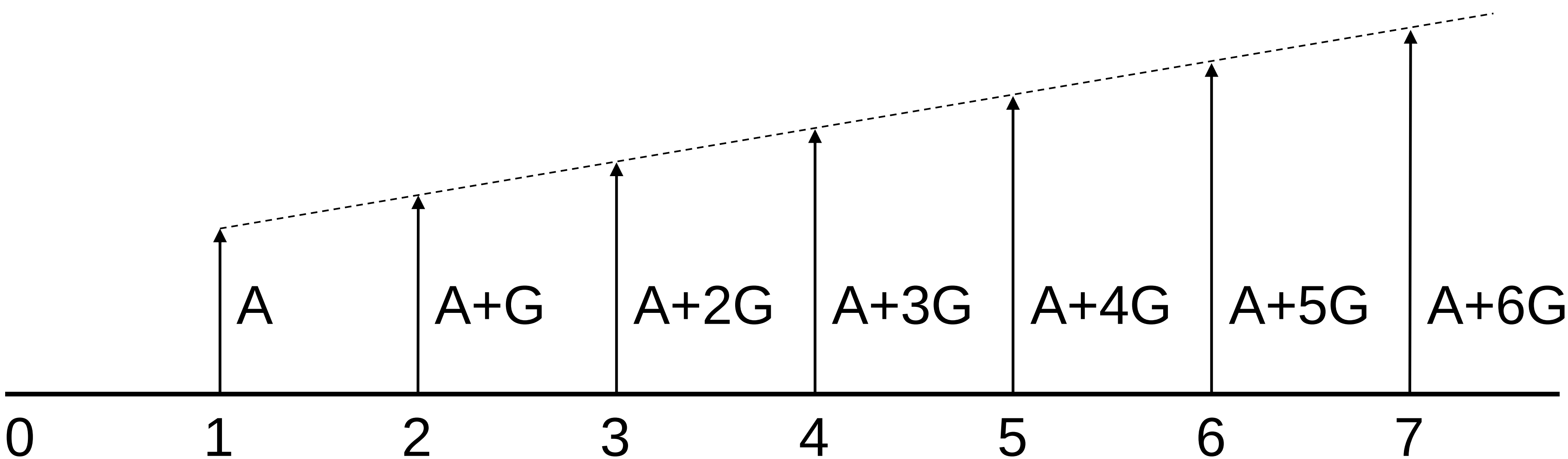
- What if the cashflow series isn't constant?

Arithmetic Gradient



Arithmetic Gradient Series

- A uniformly increasing series:
  - Consists of two components:
  - Uniform Series component (*A*)
  - Gradient component (*G*)



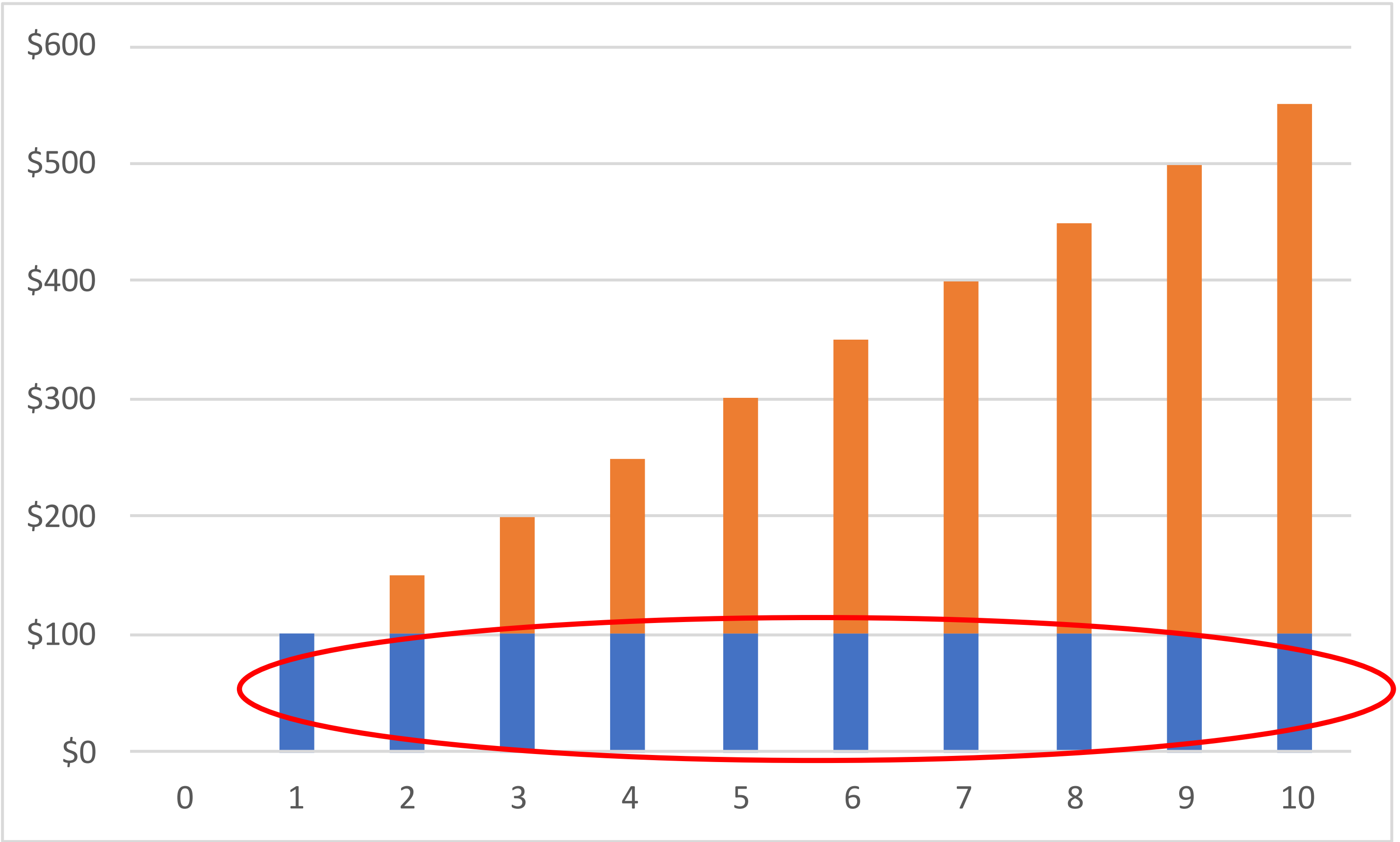
Arithmetic Gradient – Present Value

- P = P' + P'', where
- P' = A(P/A, i, n) – Uniform Series Present Worth Factor
- P'' = G(P/G, i, n) – Gradient Present Worth Factor
- We already know the Uniform Series Present Worth Factor, from before

$$P = A\left[\frac{(1+i)^n - 1}{i(1+i)^n}\right]$$

Arithmetic Gradient – Uniform Component

- P' = A(P/A, i, n) – Uniform Series Present Worth Factor



A A A A A A A A A A

Gradient Present Worth Factor

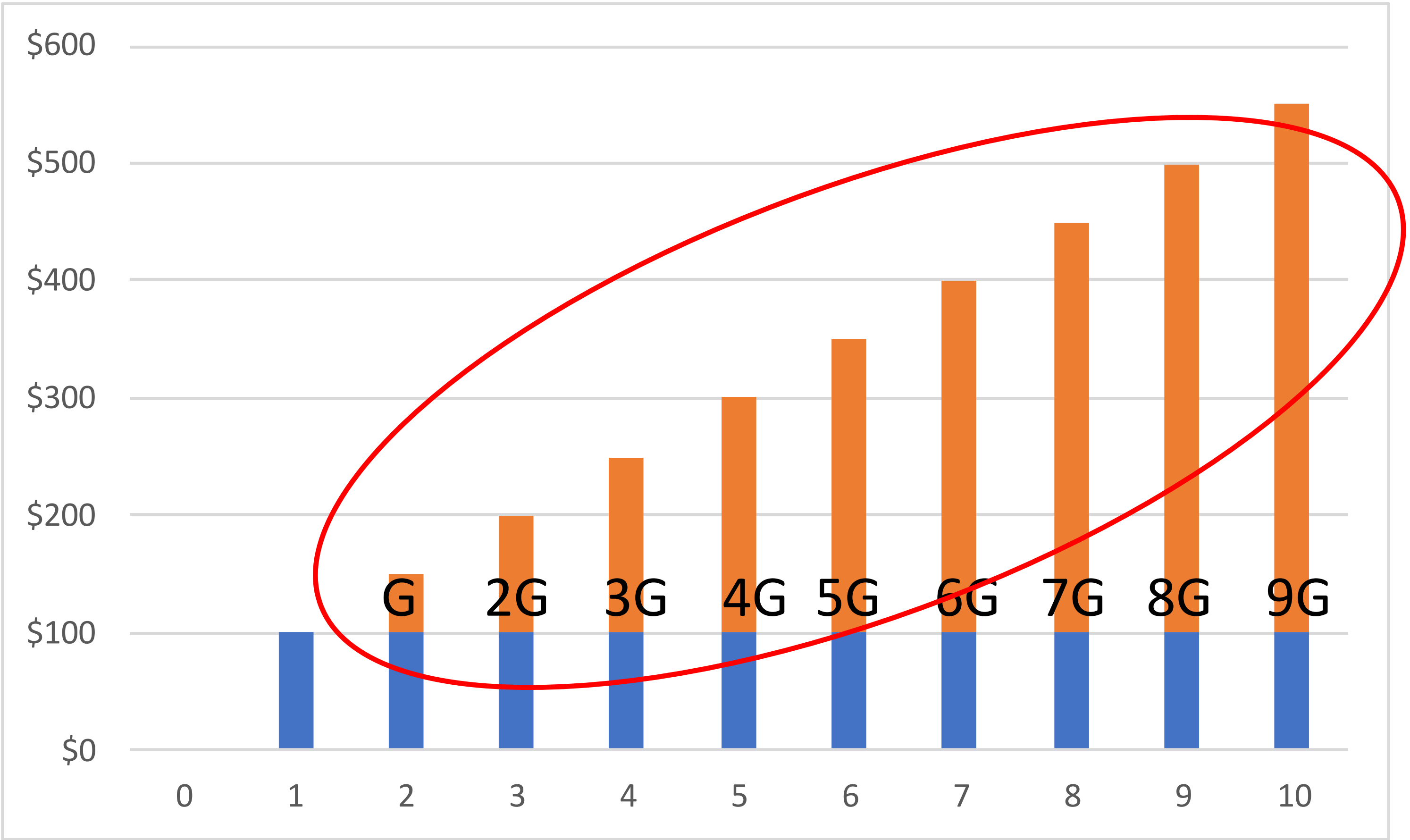
- “arithmetic gradient present worth factor”:

$$P = G\left[\frac{(1+i)^n - in - 1}{i^2(1+i)^n}\right]$$

- IMPORTANT – This calculate the present value of the GRADIENT PORTION OF THE CASHFLOW ONLY

Arithmetic Gradient – Uniform Component

- P’’ = G(P/G, i, n) – Arithmetic Gradient Present Worth Factor



Arithmetic Gradient Series: Example 1

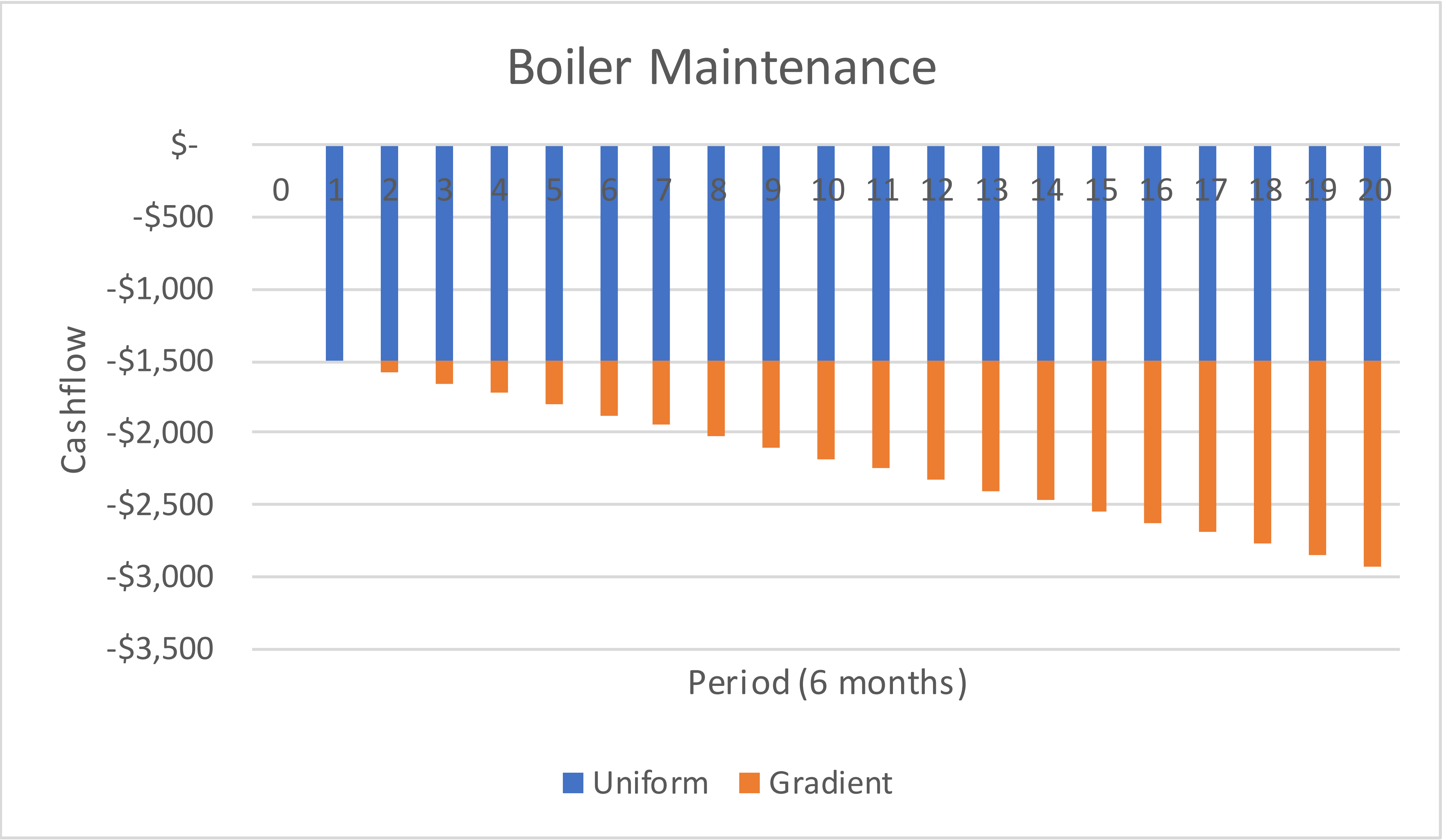
- A company has maintenance costs that will be \$1500 six months from today and will grow by \$75 every six months. Find the value today of the maintenance costs over a ten-year period if the rate of interest is 11.25% compounded semi-annually.

G = \$75   i = 0.1125/2 = 0.05625   n = 10(2) = 20

$$PV = P' + P''$$

$$P' = A\left[\frac{(1+i)^n - 1}{i(1+i)^n}\right]$$

$$P'' = G\left[\frac{(1+i)^n - in - 1}{i^2(1+i)^n}\right]$$


Boiler Maintenance
$-
-$500
-$1,000
-$1,500
-$2,000
-$2,500
-$3,000
-$3,500
Cashflow
0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20
Period (6 months)
Uniform
Gradient

Arithmetic Gradient Series: Example 1

$$P' = A[\frac{(1+i)^n - 1}{i(1+i)^n}]$$

$$P' = 1500[\frac{(1+0.05625)^{20} - 1}{0.05625(1+0.05625)^{20}}]$$

$$P' = 17741.13$$

$$P'' = G\left[\frac{(1+i)^n - in - 1}{i^2(1+i)^n}\right]$$

$$P'' = G\left[\frac{(1+0.05625)^{20} - 0.05625 * 20 - 1}{0.05625^2(1+0.05625)^{20}}\right]$$

$$P'' = 6844.36$$

$$PV = 24585.49$$

Arithmetic Gradient Equivalent Annuity

- At times it can be more useful to consider the gradient series as an annuity instead of a PV or an FV

- Once we solve for the PV or FV of a gradient, we can use the capital recovery factor or sinking fund factor (respectively) to convert that those values to an annuity.

- This is called an 'equivalent annuity' or A_{eq}
- The factor is called the 'arithmetic gradient uniform series factor'

$$A_{eq} = G\left[\frac{1}{i} - \frac{n}{(1+i)^n - 1}\right]$$

Arithmetic Gradient Series: Example 1

- Alternative method – convert G into Aeq, and treat problem as a uniform series with payment A+Aeq

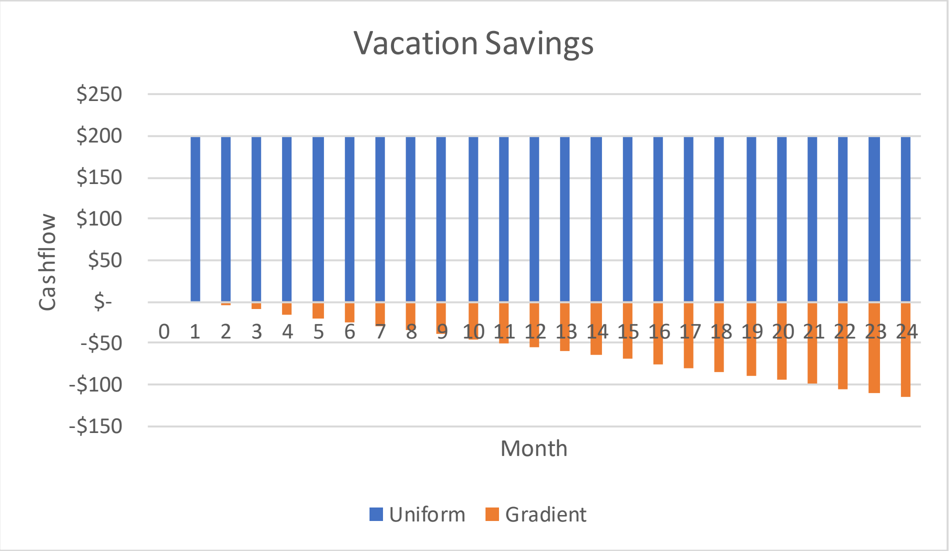
$$A_{eq} = \$75\left[\frac{1}{0.05625} - \frac{20}{(1+0.05625)^{20}-1}\right] = \$578.6856361$$

$$PV = (\$1500 + \$578.6856361)\left[\frac{1-1.05625^{-20}}{0.05625}\right] = \$24{,}585.49$$

Arithmetic Gradient Series: Example 2

- You will save for a vacation by depositing \$200 in one month then \$5 less each month for two years. Determine the amount you will have saved after two years if the nominal interest rate is 4.5% compounded monthly. $G = -\$5$ $i = 0.045/12 = 0.00375$ $n = 2(12) = 24$

- $A = \$200$
- $G = -\$5$
- $i = 4.5\%/12 = 0.375\%$
- $N = 24$



Arithmetic Example 2

$$A_{eq} = -\$5\left[\frac{1}{0.00375} - \frac{24}{(1+0.00375)^{24}-1}\right] = -\$56.60336367$$

$$FV = (\$200 - \$56.60336367)\left[\frac{1.00375^{24}-1}{0.00375}\right] = \$3594.10$$

Arithmetic Example - Spreadsheets

- Tabulate cashflows (a la making a cashflow diagram) and calculate present values of each periodic cashflow

- Calculate the equivalent annuity using the formula or Gradient Uniform Series Factor (A/G, i, n). Equivalent annuity can then be added to the PMT term in the PV or FV function.