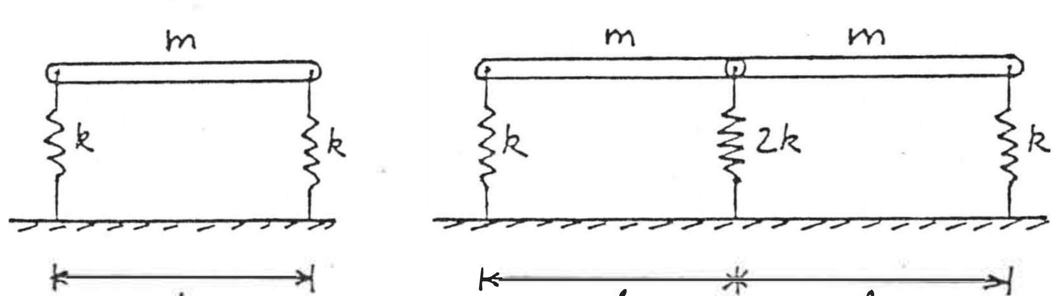
MECH 463 -- Homework 4

1. (a) A uniform rod of total mass m and length ℓ is supported at each end by a spring of stiffness k. By inspection, draw the two mode shapes and determine the natural frequencies for small oscillations.

   (b) Two uniform rods, each of mass m and length ℓ are pinned together and are supported at their free ends by springs of stiffness k, and at the pin joint by a spring of stiffness 2k. By inspection, draw the three mode shapes and determine the natural frequencies for small oscillations.

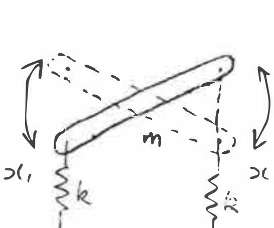
   Ans. $\omega^2 = 2k/m,\ 3k/m,\ 6k/m$



(a) By symmetry, the two mode shapes are:

$\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

and

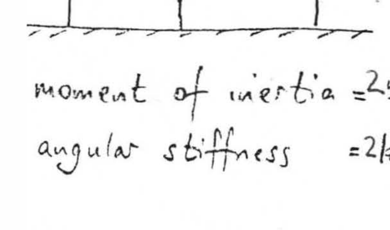


$\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

mass = m
stiffness = 2k $\rightarrow$ $\boxed{\omega_1 = \sqrt{2k/m}}$

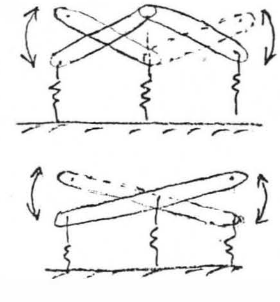
moment of inertia $= \dfrac{m\ell^2}{12}$
angular stiffness $= \dfrac{k\ell^2}{2}$ $\rightarrow$ $\boxed{\omega_2 = \sqrt{\dfrac{6k}{m}}}$

(b) The three spring system is similar to two of the two-spring systems placed side by side. Thus, two of the vibration modes are the same. By symmetry, the third mode is:



moment of inertia $= \dfrac{2m\ell^2}{3}$
angular stiffness $= 2k\ell^2$ $\rightarrow$ $\boxed{\omega_3 = \sqrt{\dfrac{3k}{m}}}$

Three mode shapes are

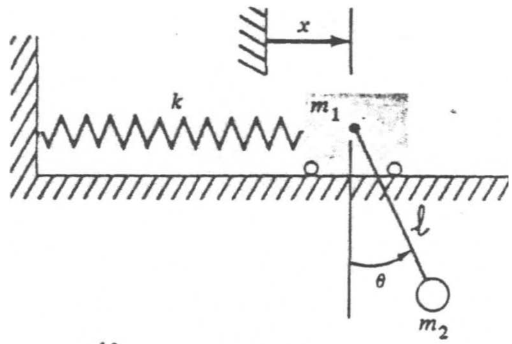


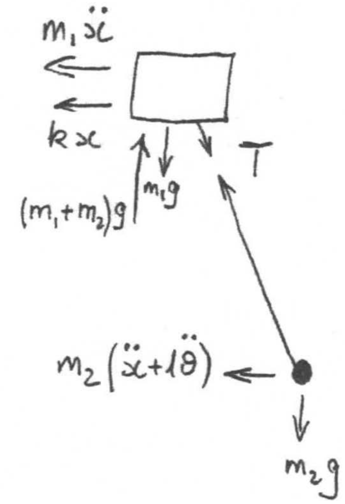
$\begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$\begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

$\begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$

2. A part of a machine can be idealized as a mass m_1 which is free to slide along a horizontal surface. It is attached horizontally through a spring of stiffness k. A pendulum component of mass m_2 and length ℓ is attached to the first mass. Using the coordinate system shown, formulate the matrix equation of motion, and make it symmetrical if necessary. For the case $m_1 = m_2 = m$, $k = mg/\ell$, solve each of the matrix equations for the natural frequencies and mode shapes. Confirm that the results are equivalent.
Ans. $\omega^2 = \frac{1}{2}(3 \pm \sqrt{5})\,k/m$





Let T = tension in pendulum component.
Take a horizontal force balance for m_1 and m_2 together (avoids having to include T explicitly)

$$m_1\ddot{x} + kx + m_2(\ddot{x} + l\ddot{\theta}) = 0$$

Take moments about m_1 $\qquad m_2 l(\ddot{x}+l\ddot{\theta}) + m_2 g l\theta = 0$

In matrix form: (after dividing 2nd eqn. by l)

$$\begin{bmatrix} m_1+m_2 & m_2 \\ m_2 & m_2 \end{bmatrix}\begin{bmatrix} \ddot{x} \\ l\ddot{\theta} \end{bmatrix} + \begin{bmatrix} k & 0 \\ 0 & \frac{m_2 g}{l} \end{bmatrix}\begin{bmatrix} x \\ l\theta \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

For the case $m_1 = m_2 = m$, $k = mg/l$

$$\rightarrow \begin{bmatrix} 2m & m \\ m & m \end{bmatrix} \begin{bmatrix} \ddot{x} \\ \ddot{y} \end{bmatrix} + \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

where $y = l\theta$ = horizontal displacement of m_2 w.r.t m_1

Try solution $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} X \\ Y \end{bmatrix} \cos(\omega t + \phi)$

$$\rightarrow \begin{bmatrix} k - 2m\omega^2 & -m\omega^2 \\ -m\omega^2 & k - m\omega^2 \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix} \cos(\omega t + \phi) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

For a non-trivial solution, valid for all t,

$$\begin{vmatrix} k - 2m\omega^2 & -m\omega^2 \\ -m\omega^2 & k - m\omega^2 \end{vmatrix} = 0$$

$$\rightarrow (k - 2m\omega^2)(k - m\omega^2) - (m\omega^2)^2 = 0$$

$$= k^2 - 3mk\omega^2 + m^2\omega^4 = 0$$

$$\rightarrow \omega^2 = \frac{3mk \pm \sqrt{9m^2k^2 - 4m^2k^2}}{2m^2} = \underline{\tfrac{1}{2}(3 \pm \sqrt{5})\, k/m}$$

Substitute mode shape $\begin{bmatrix} X \\ Y \end{bmatrix} = C \begin{bmatrix} 1 \\ u \end{bmatrix}$

$$\rightarrow \begin{bmatrix} k - 2m\omega^2 & -m\omega^2 \\ -m\omega^2 & k - m\omega^2 \end{bmatrix} \begin{bmatrix} 1 \\ u \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

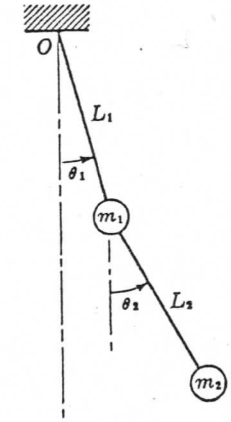
From 1st equation $k - 2m\omega^2 - m\omega^2 u = 0$

$$\rightarrow u = \frac{k}{m\omega^2} - 2. = \frac{2}{3-\sqrt{5}} - 2 = \underline{\frac{-1+\sqrt{5}}{2}} \quad \text{when } \omega^2 = \tfrac{1}{2}(3-\sqrt{5})\, k/m$$

$$= \frac{2}{3+\sqrt{5}} - 2 = \underline{\frac{-1-\sqrt{5}}{2}} \quad \text{when } \omega^2 = \tfrac{1}{2}(3+\sqrt{5})\, k/m$$

3. Formulate the equations of motion and the characteristic equation for the double pendulum shown in the diagram, with $m_1 = m_2 = m$ and $L_1 = L_2 = L$. Assume small displacements. Determine the natural frequencies and mode shapes.

Ans. $\omega^2 = (2 \mp \sqrt{2})\, g/L \quad u = \pm\sqrt{2}$

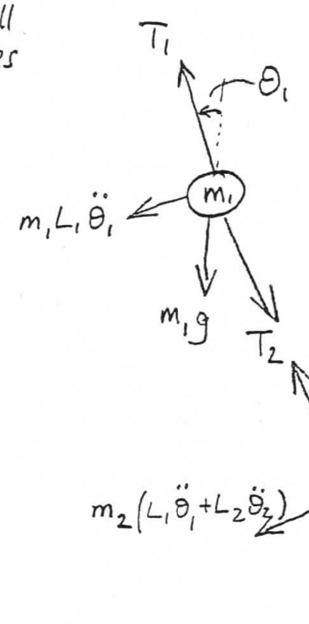


Force balances // string directions

$$T_2 = m_2 g \cos\theta_2 \simeq m_2 g$$
$$T_1 = m_1 g \cos\theta_2 + T_2\cos(\theta_2-\theta_1) \simeq (m_1+m_2) g$$
for small angles

Force balances $\perp$ string directions

$$m_2(L_1\ddot\theta_1 + L_2\ddot\theta_2) + m_2 g \sin\theta_2 = 0$$
$$m_1 L_1 \ddot\theta_1 + m_1 g \sin\theta_1 - T_2 \sin(\theta_2-\theta_1) = 0$$



Free body diagrams

For small angles

$$m_2 L_1 \ddot\theta_1 + m_2 L_2 \ddot\theta_2 + m_2 g\theta_2 = 0$$
$$m_1 L_1 \ddot\theta_1 + m_1 g\theta_1 - m_2 g(\theta_2-\theta_1) = 0$$

In matrix form, substituting $m_1 = m_2 = m$, $L_1 = L_2 = L$, and dividing by mL

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} \ddot\theta_1 \\ \ddot\theta_2 \end{bmatrix} + \begin{bmatrix} 0 & g/L \\ 2g/L & -g/L \end{bmatrix}\begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

and use trial solution $\begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} 1 \\ u \end{bmatrix}\cos(\omega t - \phi)$

For a non-trivial harmonic solution, the characteristic determinant $= 0$

$$\rightarrow \begin{vmatrix} -\omega^2 & g/L - \omega^2 \\ 2g/L - \omega^2 & -g/L \end{vmatrix} = 0 \quad \rightarrow \omega^2 g/L - (2g/L - \omega^2)(g/L - \omega^2) = 0$$

$$\rightarrow \omega^4 - 4\, g/L\, \omega^2 + 2(g/L)^2 = 0 \quad \rightarrow \boxed{\omega^2 = (2 \mp \sqrt{2})\, g/L}$$

For mode shape $\begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} 1 \\ u \end{bmatrix}\cos(\omega t - \phi) \rightarrow \begin{bmatrix} -\omega^2 & g/L - \omega^2 \\ 2g/L - \omega^2 & -g/L \end{bmatrix}\begin{bmatrix} 1 \\ u \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

From second equation $2g/L - \omega^2 - g/L\, u = 0 \rightarrow u = 2 - \dfrac{\omega^2}{g/L} = 2 - (2 \mp \sqrt{2}) \boxed{= \pm\sqrt{2}}$