University of British Columbia
Department of Mechanical Engineering

MECH 463. Midterm 1, October 2, 2020

Allowed Time: 50 min

Materials admitted: Pen, pencil, eraser, straightedge, simple scientific calculator without programming or communication capabilities, personal hand-written notes within one letter-size sheet of paper (one side), timer and document copier apps on your phone (all other phone functionalities are **not** allowed).

There are 2 questions in this exam. You are asked to answer all questions.

The purpose of this test is to evaluate your knowledge of the course material. Orderly presentation demonstrates your knowledge most clearly. Marks are assigned accordingly. A bonus of up to 2 marks will be given for exemplary presentation.

Honour Code: You are asked to behave honourably during this exam and to obey all instructions carefully. Please write and sign the following promise in the space below: "I promise to work honestly on this exam, to obey all instructions carefully, and not to have any unfair advantage over any other students."

Promise:

Signed: Name:

	Mark Received	**Maximum Mark**
1		10
2		10
Presentation		2 bonus
Total		20+2

Start Time running

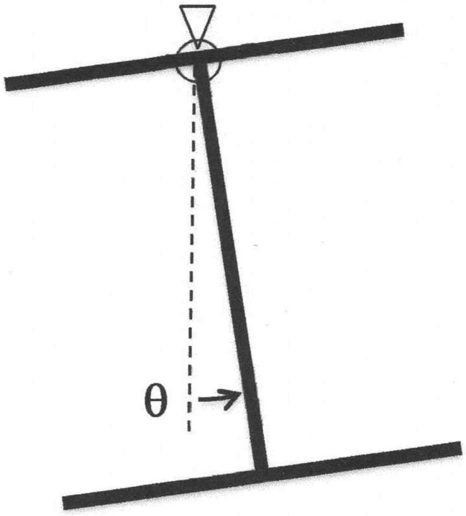
Finish Time running

1. An anchor has the shape of a horizontal letter "H". It hangs from a frictionless pivot at its upper connection. Each of the three main sections of the anchor are made of uniform rods of length L and mass m, with centroidal moment of inertia $I_0 = mL^2/12$.

   (a) Draw a labeled free-body diagram of the vibrating system.

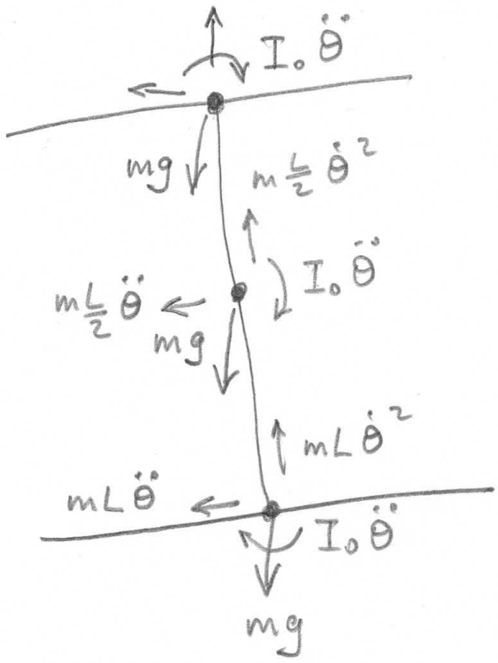
   (b) Use your free-body diagram to formulate the equation of motion.

   (c) Solve your equation of motion to determine the natural frequency of vibration of the anchor. Show the needed steps in detail.



   (d) The anchor was found to be too heavy, so the lower horizontal section was cut off to make a vertical letter "T". Determine the new natural frequency of the anchor.

   (e) Comment on and explain your result.



(a) Draw FBD with gravity and inertia forces based on the centres of mass of the three parts.

(b) Take moments about pivot (to eliminate support forces)

$$3 \times I_o\ddot{\theta} + \frac{mL}{2}\ddot{\theta}\frac{L}{2} + mL\ddot{\theta}\cdot L + mg\cdot\frac{L}{2}\sin\theta + mgL\sin\theta = 0$$

for small θ

$$3\times\frac{1}{12}mL^2\ddot{\theta} + \frac{mL^2}{4}\ddot{\theta} + mL^2\ddot{\theta} + mg\frac{L}{2}\theta + mgL\theta = 0$$

$$\frac{3}{2}mL^2\ddot{\theta} + \frac{3}{2}mgL\theta = 0 \xrightarrow{\div\frac{3}{2}mL^2} \ddot{\theta} + \frac{g}{L}\theta = 0$$

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(c) Try solution $\theta = C\cos(\omega t + \phi)$

$\ddot{\theta} = -\omega^2 C\cos(\omega t + \phi)$

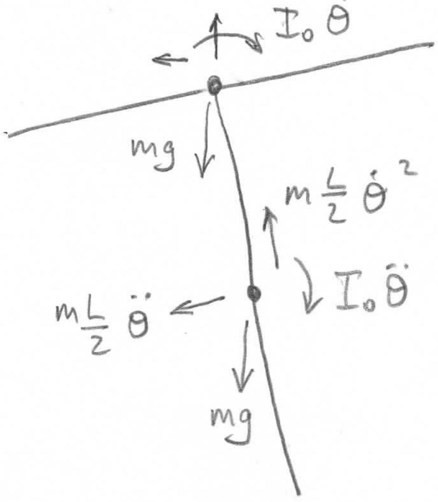
Sub in $\ddot{\theta} + \frac{g}{L}\theta = 0 \rightarrow \left(-\omega^2 + \frac{g}{L}\right) C\cos(\omega t + \phi) = 0$

For non-trivial solution valid for all t

$\rightarrow C \neq 0$ and $\cos(\omega t + \phi) \neq 0$

$\rightarrow -\omega^2 + \frac{g}{L} = 0 \rightarrow \underline{\omega = \sqrt{\frac{g}{L}}}$

(d) Following same procedure:



$\Sigma M = 2\times I_o\ddot{\theta} + \frac{mL}{2}\ddot{\theta}\frac{L}{2} + mg\frac{L}{2}\sin\theta = 0$

$\simeq 2\times\frac{1}{12}mL^2\ddot{\theta} + \frac{mL^2}{4}\ddot{\theta} + mg\frac{L}{2}\theta = 0$

$= \frac{5}{12}mL^2\ddot{\theta} + \frac{1}{2}mgL\theta = 0$

$\div \frac{5}{12}mL^2 \rightarrow \ddot{\theta} + \frac{6}{5}\frac{g}{L}\theta = 0$

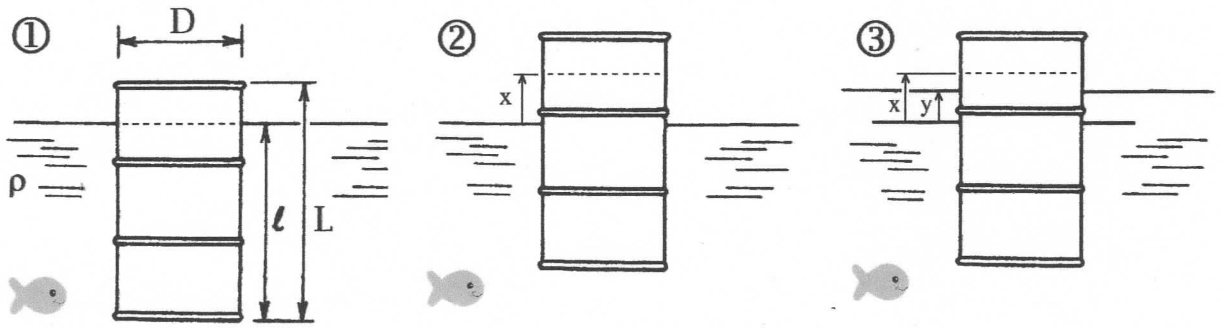
Characteristic equ: $-\omega^2 + \frac{6}{5}\frac{g}{L} = 0 \rightarrow \underline{\omega = \sqrt{\frac{6}{5}\frac{g}{L}}}$

(e) Mass appears in the denominator of $\omega^2 = \frac{k}{m}$ formulas, so a decrease in mass causes an increase in natural frequency. For a pendulum type system, this is not universally true because mass also contributes to the "stiffness" term.

For example, if the mass of the middle section were reduced, the natural frequency would have decreased!

2. A cylindrical oil barrel of length L and diameter D floats upright in the sea, where the water has density ρ. In calm conditions, the barrel floats at equilibrium with a submerged length ℓ, as shown in diagram ①. When disturbed, the barrel bobs up and down with vertical displacement x, as shown in diagram ②. It is observed that the water viscosity causes the barrel to have a damping factor ζ. In windy conditions, waves form that cause the sea surface to go up and down with vertical displacement $y = Y \cos \omega f t$, as shown in diagram ③.

(Archimedes' Principle: The upward buoyant force on a body immersed in a fluid equals the weight of the displaced fluid.

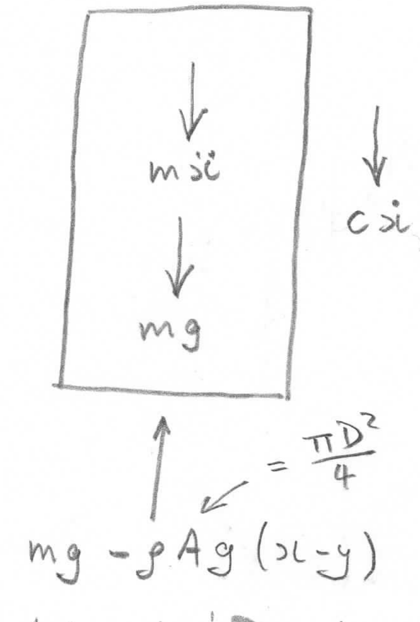


(a) Draw a fully labeled free-body diagram for the situation shown in diagram ③. Use it to derive an equation of motion for the barrel.

(b) Solve your equation to determine the undamped natural frequency. Show the needed steps in detail.

(c) Solve your equation to determine a compact formula for the amplitude of vibration caused by the water waves. Again, show the needed steps in detail. Sketch and label the corresponding response curve.

(a) When the equilibrium position is level with the water surface, the buoyancy force equals the barrel weight mg. When the equilibrium position (dotted line) is above the water surface, the buoyancy force reduces by an amount equal to the weight of the difference in displaced water. → Archimedes' Principle.



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If the barrel moves up x and the water surface y, then the change in volume of displaced water is $A(x-y)$, corresponding to a change in buoyant force $\rho A g(x-y)$.

FBD: $\Sigma F = mg - \rho A g(x-y) - mg - m\ddot{x} - c\dot{x} = 0$

$$\rightarrow \quad m\ddot{x} + c\dot{x} + \rho A g x = \rho A g y$$

At equilibrium in diagram ①, using Archemedes' principle $\rightarrow mg = \rho A l g \quad \rightarrow \quad \rho A = \frac{m}{l}$

$$\rightarrow \quad m\ddot{x} + c\dot{x} + \frac{mg}{l}x = \frac{mg}{l}y$$

$\div m$ and put into standard form:

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = \omega_n^2 y$$

where $\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{g}{l}} \quad \zeta = \frac{c}{2\sqrt{km}} \quad k = \frac{mg}{l}$

(b) For the undamped natural frequency, we need consider the homogeneous equation with $c=0$

$$m\ddot{x} + \frac{mg}{l}x = 0$$

Try solution $x = C\cos(\omega t + \phi) \rightarrow \ddot{x} = -\omega^2 C\cos(\omega t+\phi)$

Sub: $\left(-m\omega^2 + \frac{mg}{l}\right)C\cos(\omega t+\phi) = 0$

For non-trivial solution valid for all t C and $(\cos\omega t+\phi) \neq 0$

$\rightarrow \left(-m\omega^2 + \frac{mg}{l}\right) = 0 \quad \rightarrow \quad \underline{\omega_n = \sqrt{g/l}}$

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(c) For the steady-state (particular) solution,

try $x = \mathrm{Re}(\bar{X} e^{i\omega_f t})$ and sub. in EOM.

$$(-\omega_f^2 + i\,2\zeta\omega_n\omega_f + \omega_n^2)\bar{X} e^{i\omega_f t} = \omega_n^2 Y e^{i\omega_f t}$$

$$\rightarrow \bar{X} = \frac{\omega_n^2 Y}{\omega_n^2 - \omega_f^2 + i\,2\zeta\omega_n\omega_f}$$

$$\bar{X} = \frac{Y}{(1-r^2) + i\,2\zeta r}$$

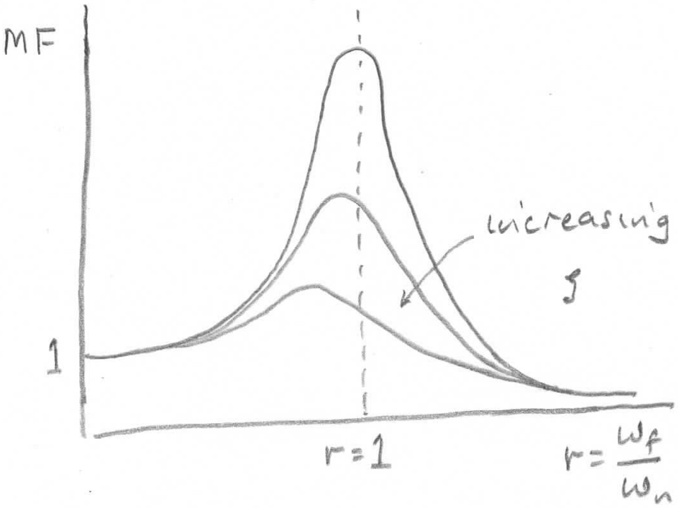
dividing by ω_n^2 and putting $r = \frac{\omega_f}{\omega_n}$

The response ampitude is

$$|\bar{X}| = \frac{Y}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}}$$

The magnification factor is

$$MF = \frac{|\bar{X}|}{Y} = \frac{1}{\sqrt{(1-r^2) + (2\zeta r)^2}}$$



Note: If we had used $c(\dot{x} - \dot{y})$ instead of $c\dot{x}$, we would get

$$MF = \frac{|\bar{X}|}{Y} = \frac{\sqrt{1 + (2\zeta r)^2}}{\sqrt{(1-r^2) + (2\zeta r)^2}}$$

which has a similar but not identical MF curve