University of British Columbia
Department of Mechanical Engineering

MECH468 Modern Control Engineering
MECH522 Foundations in Control Engineering
Final exam: Solutions

Examiner: Dr. Ryozo Nagamune
April 8 (Monday), 2019, 8:30-11am

Last name, First name

Name: Student #:

Signature:

Exam policies

- **Allowed:** Lecture note print-outs, hand-written notes, homework assignments and solutions, past exams and solutions.
- **Not-allowed:** PC, calculators, mobile phones, textbooks.
- Write all your answers on the provided exam booklet. This question sheet will be collected but not evaluated.
- Motivate your answers properly. (No chance to defend your answers orally.)
- 100 points in total. Mark will be scaled later.

Before you start ...

- Use washroom before the exam.
- Turn off your mobile phone.
- Questions are NOT allowed.

If you finish early ...

- If you would like to leave the room **before 10:50am**, **raise your hand with this booklet**, and **wait at your seat** until the invigilator comes to you and collects your exam booklet.

To be filled in by the instructor/marker

Problem #	Mark	Full mark
1		30
2		20
3		30
4		20
Total		100

1. Answer the following true-or-false questions. Write **True** or **False**. **No need to motivate your answers.** (2pt each)

   (a) **(F)** A spring system model $y(t) = (1/k)f(t)$, where f is the force input, y is the displacement output and k is the spring constant, is a dynamical model.

   (b) **(F)** For a linear dynamic system, if the input $u(t)$ becomes twice, then the output $y(t)$ will become twice, regardless of the initial condition.

   (c) **(F)** To track a sinusoidal reference signal by feedback control, it is necessary to have an integrator in the feedback controller.

   (d) **(T)** McMillan degree of the transfer matrix $G(s) = \begin{bmatrix} 1/s & 0 \\ 0 & 1/s \end{bmatrix}$ is two.

   (e) **(T)** If a system is controllable but not observable, by the Kalman decomposition, there are always some states $z_{c\bar{o}}$ which are controllable and unobservable.

   (f) **(F)** For a single-input single-output transfer function $G(s)$, its controllable canonical form realization is always minimal.

   (g) **(F)** In observer-based controller design, in general, the eigenvalues of $A - BK$ should be placed far left compared to the eigenvalues of $A - LC$ in the complex plane.

   (h) **(T)** By applying Kalman filter for state estimation, the error covariance of *a priori* estimate is larger than or equal to the error covariance of the subsequent *a posteriori* estimate.

   (i) **(F)** By solving the finite-horizon linear quadratic regulator problem, we will obtain the time-varying state feedback controller which stabilizes the feedback system.

   (j) **(F)** Duality between controllability and observability implies that a controllable system is always observable.

   (k) **(T)** The following system is asymptotically stable.

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ -1 & -2 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & -3 & -4 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -5 & -6 \end{bmatrix} x(t)$$

   (l) **(F)** If a linear time-invariant system is detectable, then it is observable.

   (m) **(T)** If a linear time-invariant system is stable, then it is detectable.

   (n) **(T)** For the transfer matrix $G(s) = \begin{bmatrix} \dfrac{1}{s^2 + 2s + 2} & \dfrac{2}{s^2 + 2s + 2} \end{bmatrix}$, the size of A-matrix of its observable canonical form realization is two.

   (o) **(F)** The observer was invented by Rudolf E. Kalman around year 1960.

2. For each of the following transfer matrices, obtain the minimal realization. Check if your obtained realization is indeed minimal. (10pt each)

$$\text{(a) } G_1(s) = \begin{bmatrix} \dfrac{2}{s+10} & 0 \\ 0 & \dfrac{4}{s+10} \end{bmatrix} \qquad \text{(b) } G_2(s) = \begin{bmatrix} \dfrac{2}{s+10} & \dfrac{4}{s+10} \\ \dfrac{2}{s+10} & \dfrac{4}{s+10} \end{bmatrix}$$

Solution

(a)

$$\begin{cases} \dot{x} &= \begin{bmatrix} -10 & 0 \\ 0 & -10 \end{bmatrix} x + u \\ y &= \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix} x \end{cases}$$

This realization is both controllable and observable, and thus minimal.

(b)

$$\begin{cases} \dot{x} &= -10x + \begin{bmatrix} 2 & 4 \end{bmatrix} u \\ y &= \begin{bmatrix} 1 \\ 1 \end{bmatrix} x \end{cases}$$

This realization is both controllable and observable, and thus minimal.

3. Consider the following **discrete-time** linear time-invariant state equation:

$$x[k+1] = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} x[k] + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u[k]. \tag{1}$$

(a) Design the gain matrix K of the state feedback control law $u(t) = -Kx(t)$ so that all the closed-loop poles are placed at the origin of the complex plane, by **using the canonical form method**. (10pt)

(b) Solve the infinite-horizon discrete-time LQR problem with the state equation (1) and the following cost function:

$$J(u[\cdot]) := \sum_{k=0}^{\infty} \left\{ 2x_1[k]^2 + u[k]^2 \right\}.$$

Also, obtain the closed-loop pole locations. (20pt)

Solution

(a) The system (1) is controllable. The characteristic equation of the open-loop system is

$$\det(zI - A) = z^2 - 1.$$

The matrix T is

$$T^{-1} = \mathcal{C}W = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = I_2$$

The desired characteristic polynomial is z^2. Thus,

$$K = \begin{bmatrix} 0-(-1) & 0-0 \end{bmatrix} T = \begin{bmatrix} 1 & 0 \end{bmatrix}.$$

(b) Discrete algebraic Riccati equation is

$$\underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}_{A^T} \underbrace{\begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix}}_{P} \underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}_{A} - \underbrace{\begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix}}_{P} + \underbrace{\begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}}_{Q}$$
$$- \underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}_{A^T} \underbrace{\begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix}}_{P} \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{B} (\underbrace{1}_{R} + \underbrace{p_3}_{B^TPB})^{-1} B^T P A = 0$$

$$\begin{aligned} (1,1): &\quad p_3 - p_1 + 2 - \frac{p_3^2}{1+p_3} = 0 \\ (1,2): &\quad p_2 - p_2 + 0 - \frac{p_2 p_3}{1+p_3} = 0 \\ (2,2): &\quad p_1 - p_3 + 0 - \frac{p_2^2}{1+p_3} = 0 \end{aligned}$$

From the second equation, $p_2 = 0$ because p_3 must be positive for positive definiteness of P. Then, from the third equation, $p_1 = p_3$.
The first equation becomes

$$p_3^2 - 2p_3 - 2 = 0 \ \Rightarrow \ p_3 = 1 + \sqrt{3}$$

Thus,

$$P = \begin{bmatrix} 1+\sqrt{3} & 0 \\ 0 & 1+\sqrt{3} \end{bmatrix}$$

The control law is

$$u[k] = -(1+p_3)^{-1} B^T P A x[k] = -\frac{1}{2+\sqrt{3}} \begin{bmatrix} 1+\sqrt{3} & 0 \end{bmatrix} x[k]$$

The closed-loop A-matrix is

$$A - BK = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix} \frac{1}{2+\sqrt{3}} \begin{bmatrix} 1+\sqrt{3} & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \dfrac{1}{2+\sqrt{3}} & 0 \end{bmatrix}$$

So, the eigenvalues (i.e., closed-loop poles) are computed as

$$\det(\lambda I - (A - BK)) = 0 \ \Rightarrow \ \lambda^2 - \frac{1}{2+\sqrt{3}} = 0 \ \Rightarrow \ \lambda = \pm\sqrt{\frac{1}{2+\sqrt{3}}}$$

4. Let us consider to estimate a **constant scalar unknown** x from measurements with noises. The discrete-time state-space model can be written as

$$\begin{aligned} x[k+1] &= x[k] \\ y[k] &= x[k] + v[k], \end{aligned}$$

where k is the time-index, $y[k]$ is the measurement at time k, and $v[k]$ is the measurement noise at time k. Assume that the mean value and the variance of $v[k]$ are given as $E\{v[k]\} = 0$ and $E\{v[k]^2\} = 2$, respectively. We will use the following standard notations:

$\hat{x}[k|k-1]$ and $P[k|k-1]$: A priori estimate of $x[k]$ and its error variance
$\hat{x}[k|k]$ and $P[k|k]$: A posteriori estimate of $x[k]$ and its error variance

Now, assume that we get the measurements

$$y[0] = 11,\ y[1] = 9,\ y[2] = 10.$$

By using time-varying Kalman filter, fill out the following table. Initial estimate $\hat{x}[0|-1]$ and its error covariance $P[0|-1]$ are given in the table. (20pt)

k	$\hat{x}[k\|k-1]$	$P[k\|k-1]$	$\hat{x}[k\|k]$	$P[k\|k]$
0	10	5	**75/7**	**10/7**
1	**75/7**	**10/7**	**10**	**5/6**
2	**10**	**5/6**	**10**	**10/17**

Solution Kalman filter equations are:

$$\begin{aligned} \text{Correction :}\quad P[k|k] &= P[k|k-1] - P[k|k-1]C^T(CP[k|k-1]C^T + R_v)^{-1}CP[k|k-1] \\ &= P[k|k-1] - \frac{P[k|k-1]^2}{P[k|k-1]+2} = \frac{2P[k|k-1]}{P[k|k-1]+2} \\ \hat{x}[k|k] &= \hat{x}[k|k-1] + P[k|k]C^T R_v^{-1}(y[k] - C\hat{x}[k|k-1]) \\ &= \hat{x}[k|k-1] + \frac{P[k|k]}{2}(y[k] - \hat{x}[k|k-1]) \\ \text{Prediction :}\quad P[k+1|k] &= P[k|k] \\ \hat{x}[k+1|k] &= \hat{x}[k|k] \end{aligned}$$

$$\begin{aligned} P[0|0] &= \frac{2\cdot 5}{5+2} = \frac{10}{7} & \hat{x}[0|0] &= 10 + \frac{10}{7\cdot 2}(11-10) = \frac{75}{7} \\ P[1|0] &= P[0|0] = \frac{10}{7} & \hat{x}[1|0] &= \hat{x}[0|0] = \frac{75}{7} \\ P[1|1] &= \frac{2\cdot 10/7}{10/7+2} = \frac{5}{6} & \hat{x}[1|1] &= \frac{75}{7} + \frac{5}{6\cdot 2}(9 - \frac{75}{7}) = 10 \\ P[2|1] &= P[1|1] = \frac{5}{6} & \hat{x}[2|1] &= \hat{x}[1|1] = 10 \\ P[2|2] &= \frac{2\cdot 5/6}{5/6+2} = \frac{10}{17} & \hat{x}[2|2] &= 10 + \frac{10}{17\cdot 2}(10-10) = 10 \end{aligned}$$