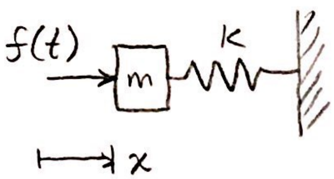
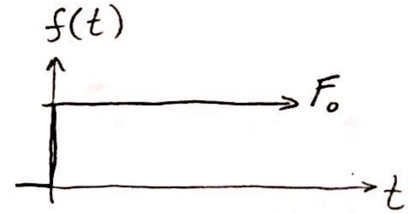
Lecture 17

Non-Harmonic Forced Vibrations



$f(t)$ is a step function:



Equation of motion: $m\ddot{x} + kx = F_0$

Solution: $x(t) = \underbrace{A\cos(\omega t) - B\sin(\omega t)}_{} + \underbrace{\frac{F_0}{k}}_{}$

$= \text{complementary} + \text{particular}$

Initial conditions: $x(0)$ and $\dot{x}(0)$ equal 0.

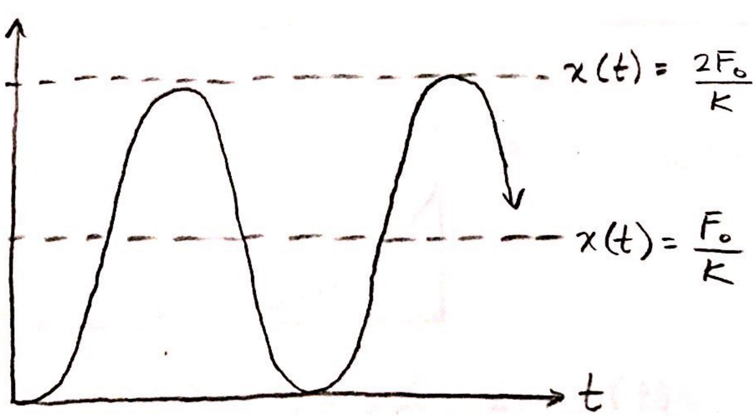
$\Rightarrow x(0) = A - 0 + \frac{F_0}{k}$

$\Rightarrow \dot{x}(0) = -\omega A \sin(\omega t) - \omega B \cos(\omega t)$

$\Rightarrow A = -\frac{F_0}{k}$ and $B = 0$

Full solution: $x(t) = \frac{F_0}{k}\left(1 - \cos(\omega t)\right)$

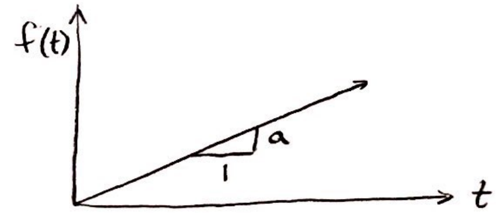
Plot:



If the system is damped, it oscillates around, and settles on, $\frac{F_0}{k}$.

Ex: $f(t)$ is ramp function:

$\Rightarrow f(t) = at$



Equation of motion: $m\ddot{x} + kx = at$

Solution: $x(t) = A\cos(\omega t) - B\sin(\omega t) + \frac{at}{k}$

Initial conditions: $x(0)$ and $\dot{x}(0)$ are 0.

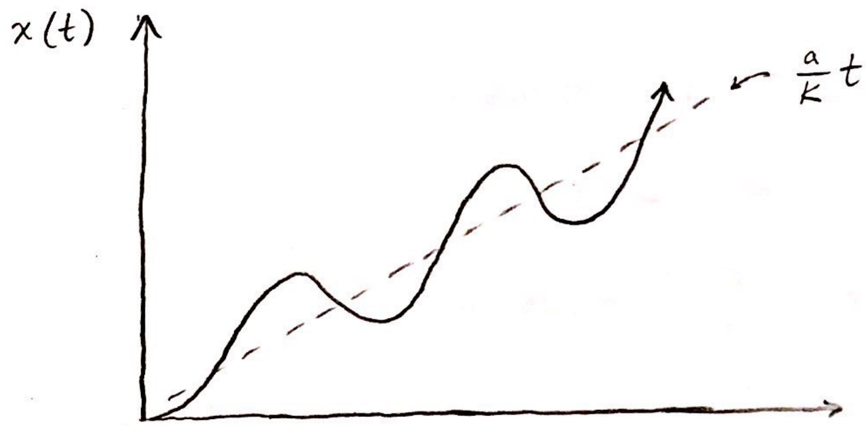
$\Rightarrow x(0) = A - 0 + 0$

$\Rightarrow \dot{x}(0) = 0 - \omega B + \frac{a}{k}$

$\Rightarrow A = 0$ and $B = \frac{a}{\omega k}$

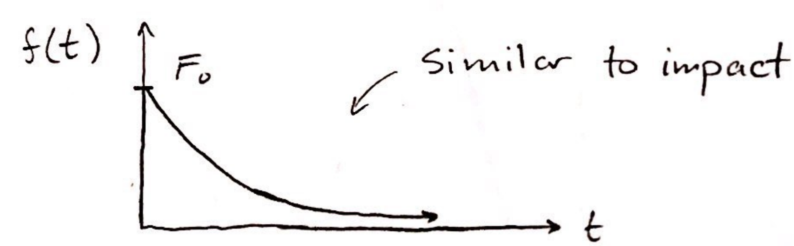
Full solution: $x(t) = \frac{a}{\omega k}\left(\omega t - \sin(\omega t)\right)$

Plot:



Ex: Exponential decay

$\Rightarrow f(t) = F_0 e^{-at}$



Equation of motion: $x(t) = m\ddot{x} + kx = F_0 e^{-at}$

Solution: $x(t) = A\cos(\omega t) - B\sin(\omega t) + \frac{F_0 e^{-at}}{ma^2 + k}$

Initial conditions: $x(0)$ and $\dot{x}(0)$ are 0.

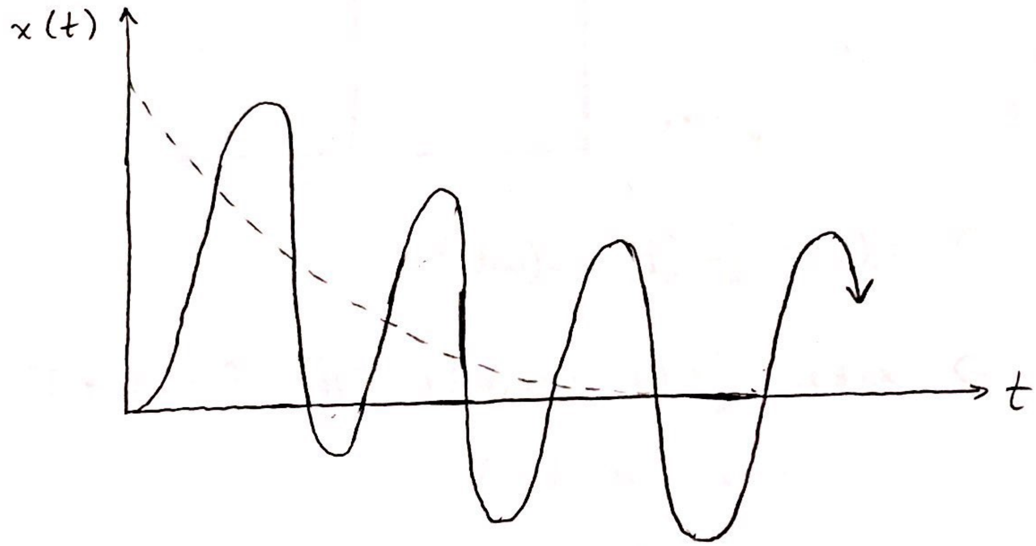
$$\Rightarrow x(0) = A - 0 + \frac{F_0}{ma^2+k}$$

$$\Rightarrow \dot{x}(0) = 0 - \omega B - \frac{F_0 a}{ma^2+k}$$

$$\Rightarrow A = -\frac{F_0}{ma^2+k} \quad \text{and} \quad B = \frac{-F_0 a}{\omega(ma^2+k)}$$

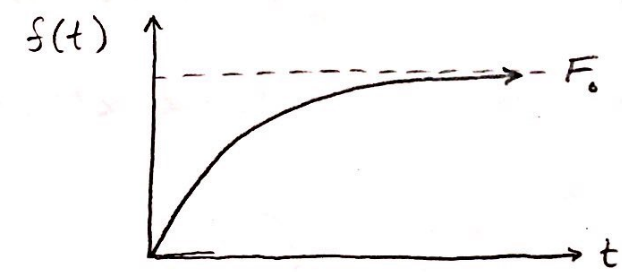
Full solution: $x(t) = \frac{F_0}{ma^2+k}\left(\frac{a}{\omega}\sin(\omega t) - \cos(\omega t) + e^{-at}\right)$

Plot:



Ex: Exponential step

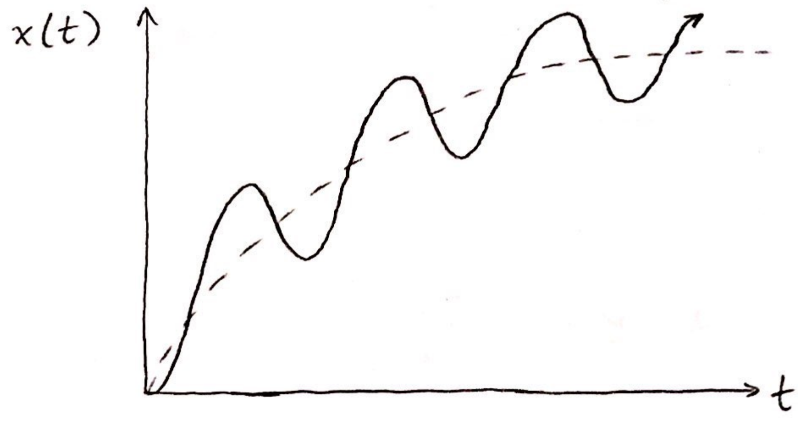
$$\Rightarrow f(t) = F_0\left(1 - e^{-at}\right)$$



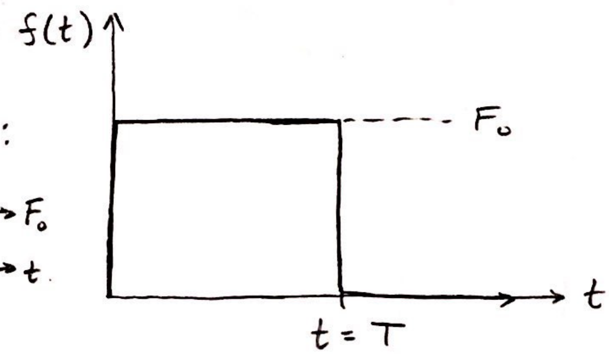
Use superposition. Response is sum of step & -exponential

$$\Rightarrow x(t) = \frac{F_0}{k}\left(1 - \cos(\omega t)\right) - \frac{F_0}{ma^2+k}\left(\frac{a}{\omega}\sin(\omega t) - \cos(\omega t) + e^{-at}\right)$$

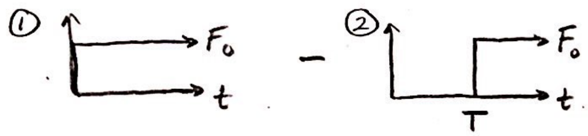
Plot:



Ex: Pulse function



$\Rightarrow$ Sum of two pulses:



For pulse ①: $x_1(t) = \frac{F_0}{K}\left(1 - \cos(\omega t)\right)$

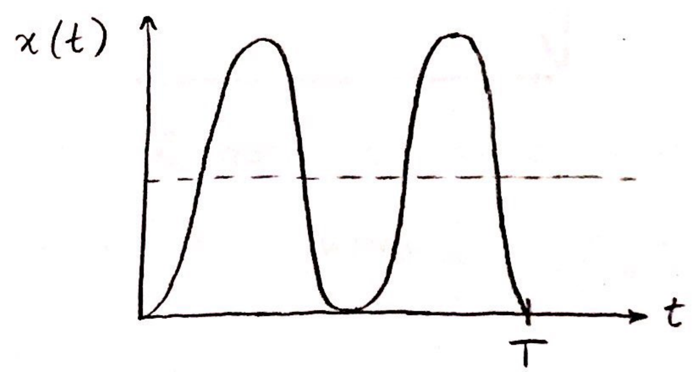
For pulse ②: $x_2(t) = -\frac{F_0}{K}\left(1 - \cos(\omega(t-T))\right)$ for $t > T$

$= 0$ for $t < T$

Full solution: $x(t) = \frac{F_0}{K}\left(\cos(\omega(t-T)) - \cos(\omega t)\right)$ for $t > T$

Notice that if $T = \frac{2n\pi}{\omega} \Rightarrow x(t) = 0$ for $t > T$

Plot:

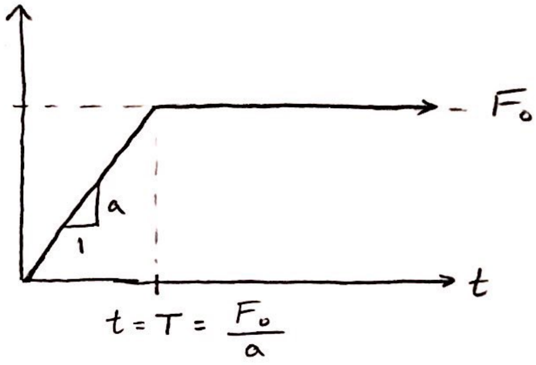


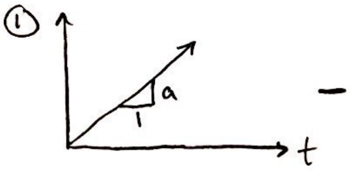
If we time pulses correctly the system stops oscillating.

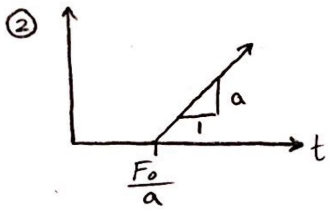
Ex: Ramped step

Sum of two pulses:

$f(t)$



① 

② 

For $t > T$: $x(t) = \frac{a}{k\omega}\left(\omega t - \sin(\omega t)\right) - \frac{a}{k\omega}\left(\omega(t-T) - \sin(\omega(t-T))\right)$

Trig identities: $x(t) = \frac{a}{k\omega}\left(\omega T + \sin(\omega(t-T)) - \sin(\omega(t-T))\right)$

If $T = \frac{2n\pi}{\omega} \Rightarrow x(t) = \frac{aT}{k} = \frac{F_0}{k}$