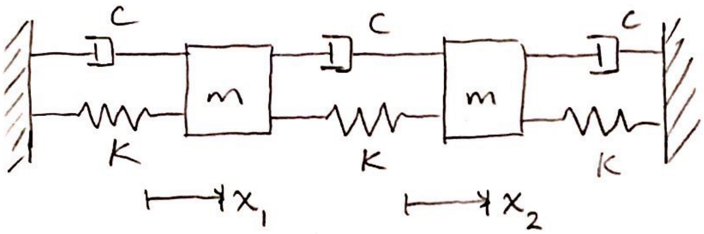
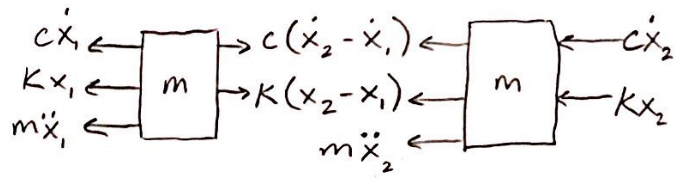
Lecture 12

2-DOF Damped (Transient)



FBD:



Matrix: $\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} 2c & -c \\ -c & 2c \end{bmatrix}\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} + \begin{bmatrix} 2k & -k \\ -k & 2k \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$\Rightarrow [M]\ddot{\vec{x}} + [C]\dot{\vec{x}} + [K]\vec{x} = \vec{0}$$

Use solution 3: $\vec{x} = \vec{X} e^{\lambda t}$

$$\Rightarrow \left(\lambda^2 [M] + \lambda [C] + [K]\right) \vec{X} e^{\lambda t} = \vec{0}$$

$$\Rightarrow \det\left(\lambda^2 [M] + \lambda [C] + [K]\right) = 0$$

Expand: $\det \begin{bmatrix} \lambda^2 m + 2\lambda c + 2k & -\lambda c - k \\ -\lambda c - k & \lambda^2 m + 2\lambda c + 2k \end{bmatrix} = 0$

Simplify: $m^2 \lambda^4 + 4mc\lambda^3 + (4mk + 3c^2)\lambda^2 + 6ck\lambda + 3k^2 = 0$

Factorize: $(m\lambda^2 + c\lambda + k)(m\lambda^2 + 3c\lambda + 3k) = 0$ (2 quad. eqns.)

First equation: $\lambda = -\frac{c}{2m} \pm \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}}$

$$\Rightarrow \lambda = \zeta_1 \omega_{N1} \pm i\omega_{N1}\sqrt{1 - \zeta^2}$$

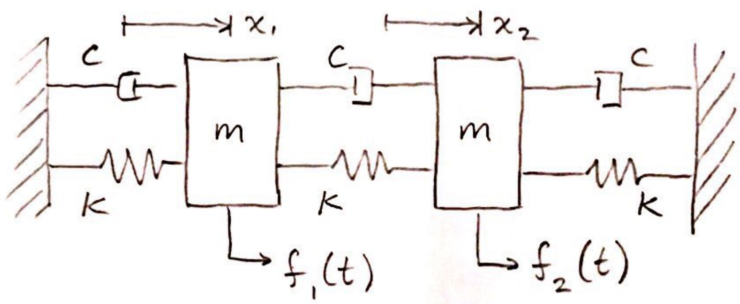
First equation: $\omega_{N1} = \sqrt{\frac{k}{m}}$ $\quad \zeta_1 = \frac{c}{2\sqrt{km}}$

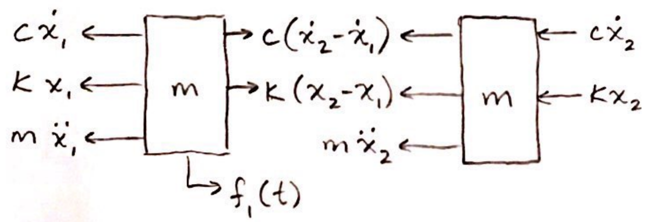
Second equation: $\lambda = \frac{3c}{2m} \pm \sqrt{\left(\frac{3c}{2m}\right)^2 - \frac{3k}{m}}$

$$\Rightarrow \lambda = \zeta_2 \omega_{N2} \pm i\,\omega_{N2}\sqrt{1-\zeta_2}$$

where $\omega_{N2} = \sqrt{\frac{3k}{m}}$ $\quad \zeta_2 = \frac{3c}{2\sqrt{3km}}$

$\boxed{\text{2-DOF Damped and Forced}}$ (Steady State)





$$\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} 2c & -c \\ -c & 2c \end{bmatrix}\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} + \begin{bmatrix} 2k & -k \\ -k & 2k \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} f_1(t) \\ f_2(t) \end{bmatrix}$$

Harmonic excitation: $f_1(t) = F_1\cos(\omega_F t) = \mathbb{R}e\left(F_1 e^{i\omega_F t}\right)$

$\mathbb{R}e \equiv$ "real part of"

$$f_2(t) = F_2\cos(\omega_F t) = \mathbb{R}e\left(F_2 e^{i\omega_F t}\right)$$

$$\Rightarrow \vec{f}(t) = \mathbb{R}e\left(\vec{F} e^{i\omega_F t}\right)$$

Solution 4: $\vec{x} = \mathbb{R}e\left(\vec{X} e^{i\omega_F t}\right)$

Equation: $[M]\ddot{\vec{x}} + [C]\dot{\vec{x}} + [K]\vec{x} = \vec{F} e^{i\omega_F t}$

$$\left(-\omega_F^2[M] + i\omega_F[C] + [K]\right)\vec{X} = \vec{F}$$

Expand:

$$\begin{bmatrix} 2k - \omega_F^2 m + 2i\omega_F c & -k - i\omega_F c \\ -k - i\omega_F c & 2k - \omega_F^2 m + 2i\omega_F c \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} F \\ 0 \end{bmatrix}$$

Let $F_2 = 0$

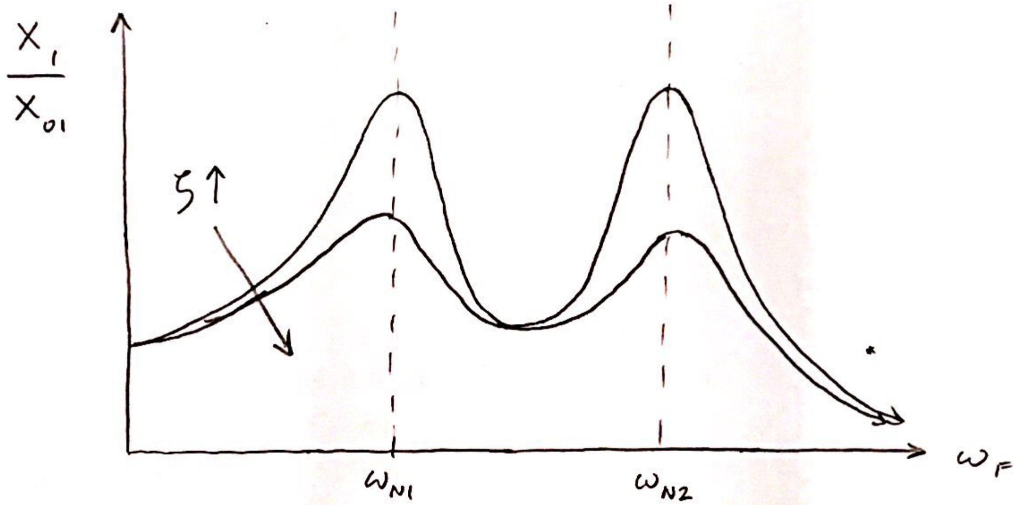
Cramer's Rule

$$X_1 = \frac{F(2k - \omega_F^2 m + 2i\omega_F c)}{\Delta}$$

$$X_2 = \frac{F(k - i\omega_F c)}{\Delta}$$

$$\Delta = (2k - \omega_F^2 m)^2 - 4\omega_F^2 c^2 - k^2 + \omega_F^2 c^2 + i\left[4\omega_F(2k - \omega_F^2 m) - 2\omega_F ck\right.$$

Plot:



small $\zeta \Rightarrow$ high, narrow peaks

high $\zeta \Rightarrow$ vice versa