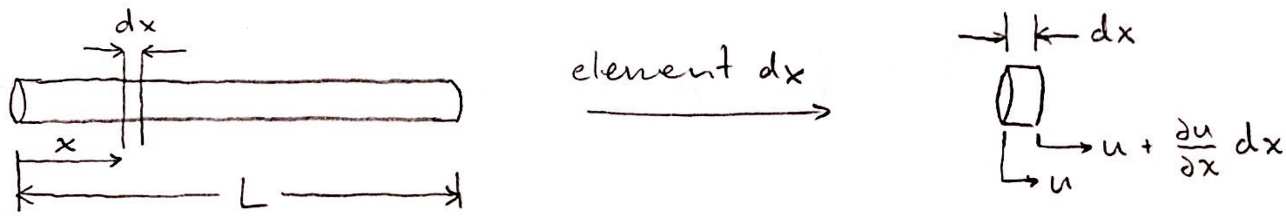
MECH 463
Nov 21

Lecture 20

Longitudinal Vibration of a Rod



u = longitudinal vibration displacement

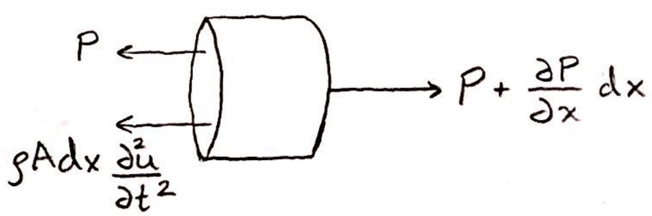
ρ = volumetric density

A = cross-sectional area

E = Young's Modulus

P = longitudinal force

FBD of element dx



$$\text{strain} = \frac{\left(u + \frac{\partial u}{\partial x} dx\right) - u}{dx} = \frac{du}{dx}$$

$$\text{stress} = \frac{P}{A}$$

$$E = \frac{(P/A)}{(\partial u/\partial x)} \Rightarrow P = AE \frac{\partial^2 u}{\partial x}$$

Force balance on element:

$$\Sigma F = P + \frac{\partial P}{\partial x} dx - P - \rho A dx \frac{\partial^2 u}{\partial t^2} = 0$$

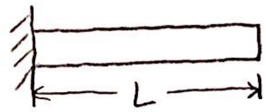
$$\Rightarrow \frac{\partial^2 u}{\partial t^2} = \frac{1}{\rho A} \frac{\partial P}{\partial x} = \frac{E}{\rho} \frac{\partial^2 u}{\partial x^2}$$

Let $\frac{E}{\rho} = c^2 \Rightarrow \frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x} \Rightarrow$ Wave speed $c = \sqrt{\frac{E}{\rho}}$

Boundary conditions

$\omega = \beta c$

Fixed - Free:



$$u(x,t) = X(x)\,T(t) = \left(C\cos(\beta x) - D\sin(\beta x)\right)\left(A\cos(\omega t) - B\sin(\omega t)\right)$$

B.C.s : $u(0,t) = 0$

$$P(L,t) = 0 \Rightarrow AE\frac{\partial u}{\partial x} = 0 \Rightarrow \frac{\partial u}{\partial x} = 0$$

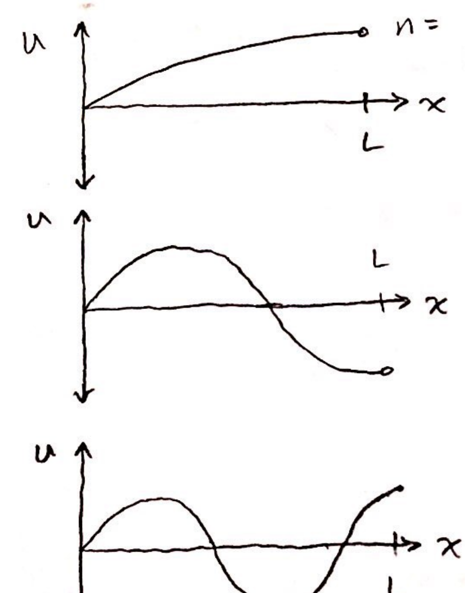
1) $u(0,t) = C - D(0) = 0 \Rightarrow C = 0$

2) $\frac{\partial u}{\partial x}(L,t) = -\beta C\sin(\beta L) - \beta D\cos(\beta L) = 0$

$$\Rightarrow \beta D\cos(\beta L) = 0 \Rightarrow \cos(\beta L) = 0$$

So $\beta L = (n - \frac{1}{2})\pi \Rightarrow \omega_N = \beta_c = (n - \frac{1}{2})\frac{\pi c}{L}$

Plots:



Other Boundary Conditions

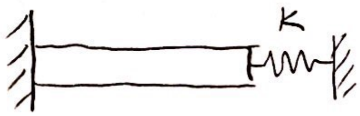
Fixed - Fixed

$$X(0,t)=0 \qquad X(L,t)=0 \quad \Rightarrow \quad \sin(\beta L)=0$$

Fixed - Free

$$X'(L)=0 \quad \Rightarrow \quad \cos(\beta L)=0$$

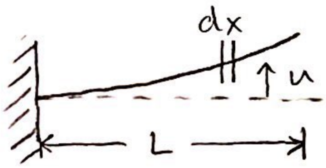
Fixed - Spring



$$X'(L) = -\frac{K}{EA}\,X(L)$$

$$\Rightarrow \tan(\beta L) = -\frac{EA}{KL}(\beta L)$$

Beam Vibrations

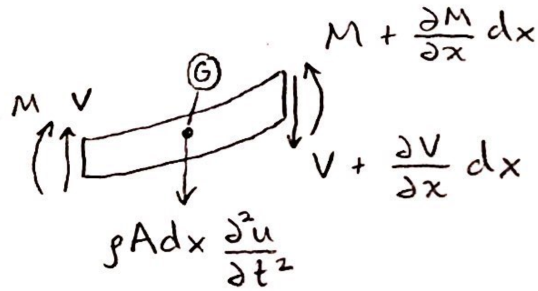


EI = flexural rigidity

M = Bending moment

V = Shear force

Element dx :



Moments balance : $\circlearrowleft + \sum M_G = 0$

$$M - \left(M + \frac{\partial M}{\partial x}dx\right) + V\frac{dx}{2} + \left(V + \frac{\partial V}{\partial x}dx\right)\frac{dx}{2} = 0$$

$$\Rightarrow \frac{-\partial M}{\partial x}dx + V dx + \underbrace{\cancel{\frac{1}{2}\frac{\partial V}{\partial x}dx^2}}_{0\,;\ 2^{nd}\text{ order}} = 0$$

$$\Rightarrow V = \frac{\partial M}{\partial x}$$

Vertical Force Balance : $+\uparrow \Sigma F_y = 0$

$$-\rho A \, dx \frac{\partial^2 u}{\partial x^2} + V - \left(V + \frac{\partial V}{\partial x} dx\right) = 0$$

$$\Rightarrow \rho A \, dx \frac{\partial^2 u}{\partial x^2} = -\frac{\partial V}{\partial x} = -\frac{\partial^2 M}{\partial x^2}$$

From beam theory : $M = EI \dfrac{\partial^2 u}{\partial x^2}$

Substitute: $\dfrac{\partial^2 u}{\partial t^2} + c^2 \dfrac{\partial^4 u}{\partial x^4} = 0$ where $c = \sqrt{\dfrac{EI}{\rho A}}$

Not a wave equation, $c \neq$ wave speed.