a place of mind

MECH468 : Modern Control Engineering
MECH509 : Controls

L13 : Observability

Dr. Ryozo Nagamune
Department of Mechanical Engineering
University of British Columbia

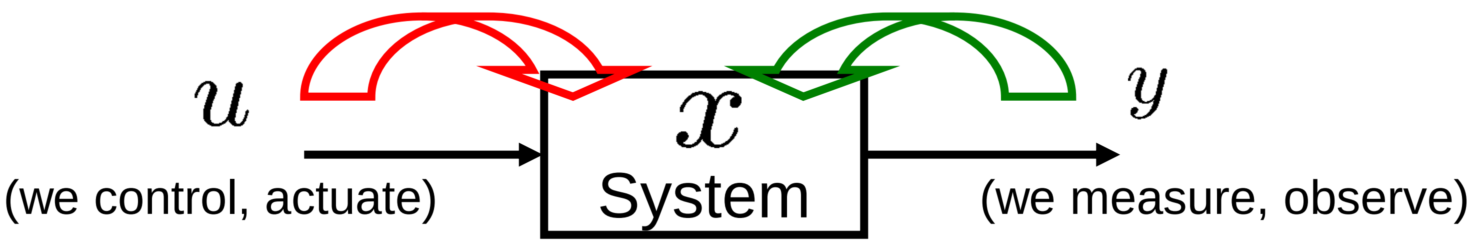
Zoom lecture to be recorded and posted on Canvas

Course plan

Topics	CT	DT
Modeling	✓	✓
Stability	✓	✓
→ Controllability/observability		
Realization		
State feedback/observer		
LQR/Kalman filter		

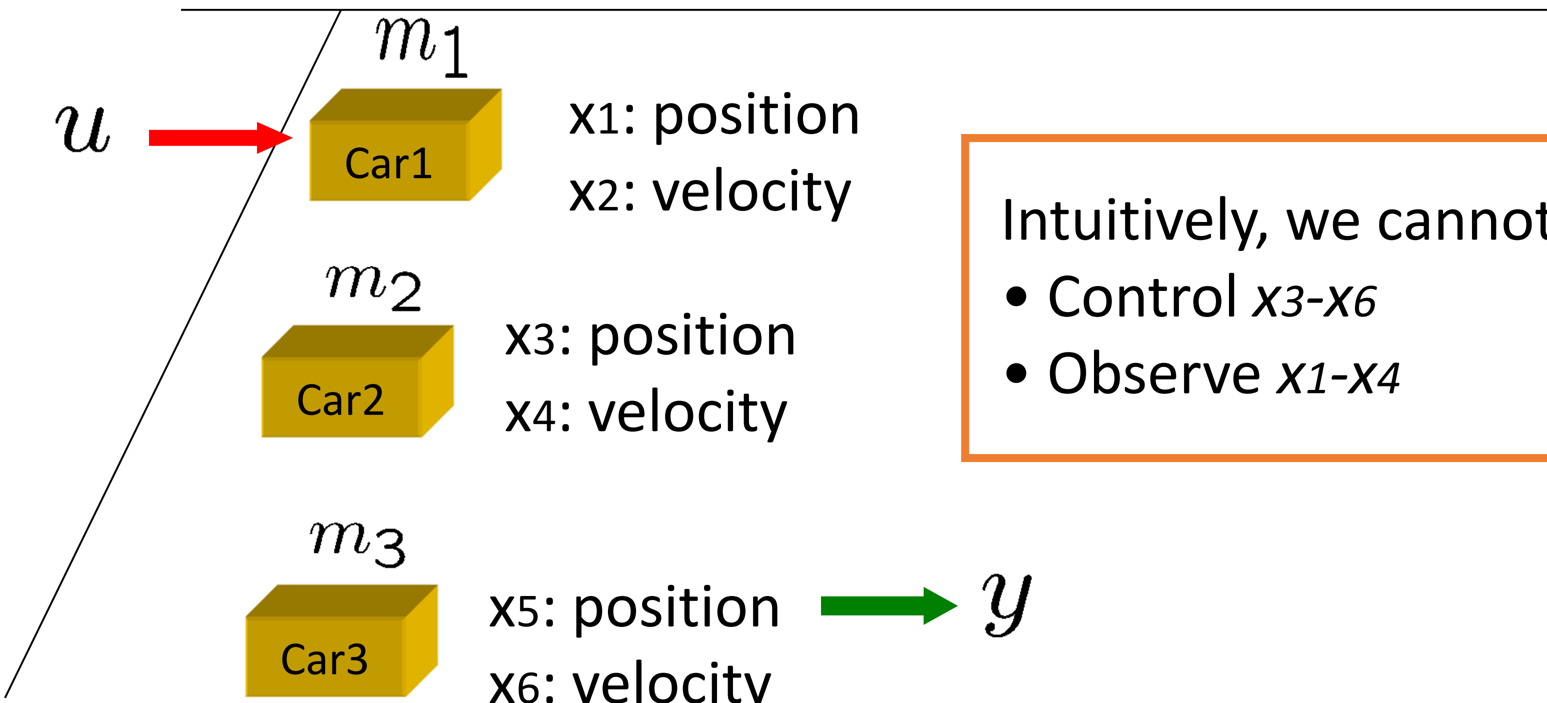
Review & today's topic

- Controllability: How much can we control *x* by manipulating *u*? (Done)
  - Nec. & suf. condition
  - Minimum energy control
  - Decomposition for controllability
- Observability: How much can we observe *x* by measuring *y*? (Today's topic)



Very simple example: revisited

- Three cars with one input and one output



Intuitively, we cannot
- Control *x3-x6*
- Observe *x1-x4*

Model of three car example

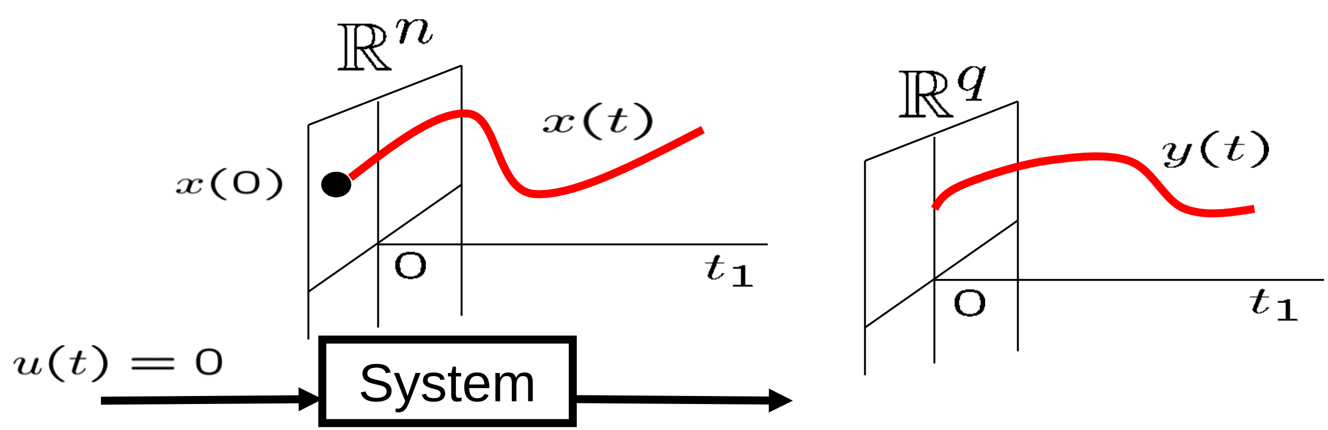
- State-space model

$$\begin{cases} \dot{x}(t) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1/m_1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} u(t) \\ y(t) = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} x(t) \end{cases}$$

- How can we explain observability of the system from (*A*,*B*,*C*,*D*)?

Observability for LTI system

- System equations (no input) $\begin{cases} \dot{x}(t) = Ax(t), & A \in \mathbb{R}^{n \times n} \\ y(t) = Cx(t), & C \in \mathbb{R}^{q \times n} \end{cases}$
- *Assumptions*: *y(t)*: measurable, *x(0)*: unknown.
- *Definition*: The system above, or (*A*,*C*), is called ***observable*** if, there is a finite $t_1>0$ such that *y* over time interval $[0,t_1]$ determines *uniquely* *x(0)*.



Remarks

- If a system is observable, then we can determine $x(0)$, and therefore, $\{x(t), t>0\}$.
- However, this is possible after future time (t_1) comes, and it is not practical.
- In practice, we want to estimate $x(t)$ in real time.
- Later in this course, we will learn design of state estimator, called *observer*.
- Observability is necessary to construct a successful observer.

Condition for observability

- ***Observability matrix***

$$\mathcal{O} := \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} \in \mathbb{R}^{nq\times n}$$

has full column rank, i.e., $\mathrm{rank}\mathcal{O} = n$

Remark: Observability depends only on *A* and *C* matrices.

Simple examples

- Ex. $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \ C = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$

$$\mathcal{O} = \begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \text{rank}\mathcal{O} = 1$$

Unobservable!

- Ex. $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \ C = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$

$$\mathcal{O} = \begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow \text{rank}\mathcal{O} = 3$$

Observable!

Three car example: revisited

- Model

$$\begin{cases} \dot{x}(t) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1/m_1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} u(t) \\ y(t) = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} x(t) \end{cases}$$

- Observability matrix

$$\mathcal{O} := \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \text{rank}\mathcal{O} = 2 < 6$$

Unobservable!

This number indicates the “degree of observability”.

This matrix indicates which states are observable and which are not.

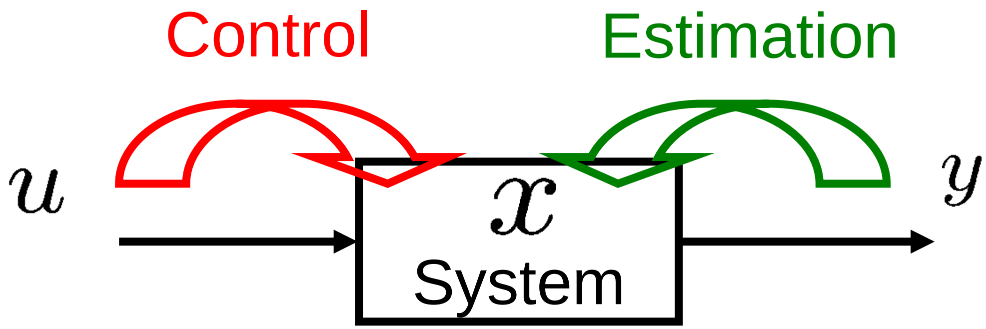
Three car example (cont'd)

- If the measurement is velocity of 3rd car:

$$\begin{cases} \dot{x}(t) &= \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1/m_1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} u(t) \\ y(t) &= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} x(t) \end{cases}$$

$$\Rightarrow \quad \mathcal{O} := \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \Rightarrow \quad \text{rank}\mathcal{O} = 1$$

Duality between “control” and “estimation”



- There is mathematical “duality” between:
  - Controllability & Observability
  - State feedback & Observer
  - Linear quadratic regulator & Kalman filter
- Importance of duality is that results for one will lead to, and will be led by, results for the other.

Mathematical duality

- (A,B) is controllable $\Leftrightarrow$ (A',B') is observable.

  - Proof: Since rank M = rank M',

    (A,B) is controllable

    $\Leftrightarrow$ rank $\left[B, AB, \cdots, A^{n-1}B\right] = n$

    $$\Leftrightarrow \text{rank} \begin{bmatrix} B^T \\ B^T A^T \\ \vdots \\ B^T (A^T)^{n-1} \end{bmatrix} = n$$

    $\Leftrightarrow$ (A',B') is observable

To check observability of (A,C), we can use "ctrb"-code:

```
rank(ctrb(A',C'))
```

Decomposition for observability

- If (A,C) is not observable with $\text{rank}\mathcal{O} = m < n$ then there exists a coordinate transformation (i.e., nonsingular T) that *decomposes* states into observable part and the unobservable part:

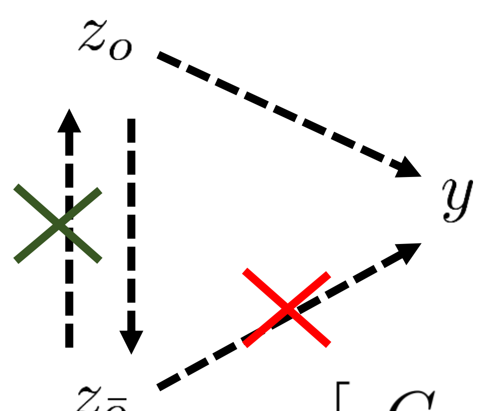
$$\left.\begin{array}{r} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) + Du(t) \\ \begin{bmatrix} z_o(t) \\ z_{\bar{o}}(t) \end{bmatrix} := Tx(t) \end{array}\right\} \longrightarrow \left\{\begin{array}{rcl} \begin{bmatrix} \dot{z}_o(t) \\ \dot{z}_{\bar{o}}(t) \end{bmatrix} & = & \underbrace{\begin{bmatrix} A_o & 0 \\ A_{21} & A_{\bar{o}} \end{bmatrix}}_{TAT^{-1}} \begin{bmatrix} z_o(t) \\ z_{\bar{o}}(t) \end{bmatrix} + \underbrace{\begin{bmatrix} B_o \\ B_{\bar{o}} \end{bmatrix}}_{TB} u(t) \\ y(t) & = & \underbrace{\begin{bmatrix} C_o & 0 \end{bmatrix}}_{CT^{-1}} \begin{bmatrix} z_o(t) \\ z_{\bar{o}}(t) \end{bmatrix} + Du \end{array}\right.$$

$\boxed{A_o \in \mathbb{R}^{m \times m}}$ $\boxed{(A_o, C_o) \text{ is observable}}$

Interpretation

- Unobservable part does not affect directly output.
- Unobservable part does not affect observable part, and thus, does not affect output indirectly either.
- Therefore, unobservable part does not affect output.

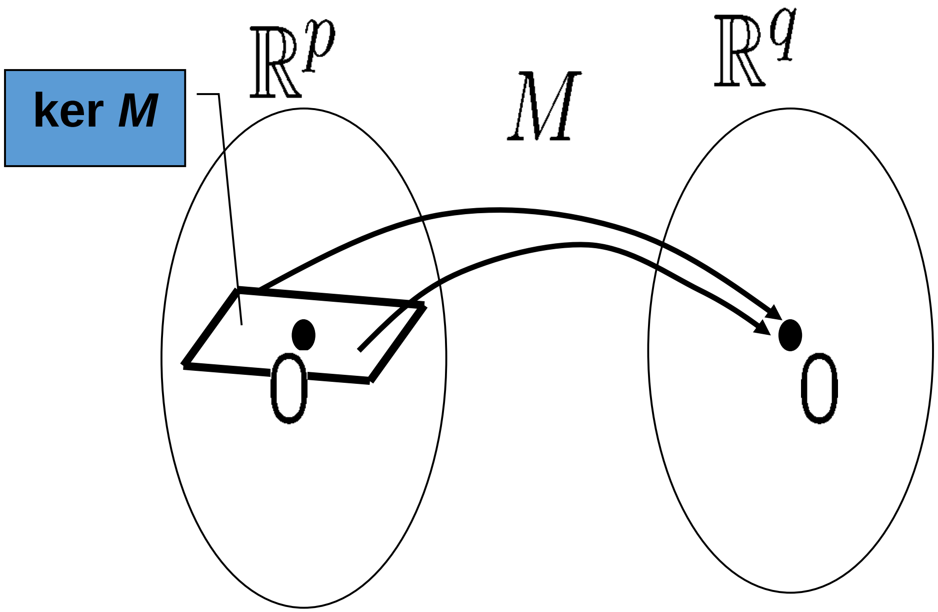
$$\begin{bmatrix} A_o & 0 \\ A_{21} & A_{\bar{o}} \end{bmatrix}$$



$$\begin{bmatrix} C_o & 0 \end{bmatrix}$$

Kernel space

- For a matrix *M* (q-by-p): $\ker M := \{x \in \mathbb{R}^p : Mx = 0\}$



$$M = \begin{bmatrix} 1 & 0 & 2 \\ -2 & 0 & -4 \end{bmatrix}$$

$$\begin{aligned}
\ker M &:= \{x \in \mathbb{R}^3 : Mx = 0\} \\
&= \left\{ x \in \mathbb{R}^3 : \begin{array}{c} x_1 + 2x_3 = 0 \\ -2x_1 - 4x_3 = 0 \end{array} \right\} \\
&= \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} : x_1 = -2x_3,\ x_i \in \mathbb{R} \right\} \\
&= \left\{ x_2 \underline{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}} + x_3 \underline{\begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}}, x_2, x_3 \in \mathbb{R} \right\}
\end{aligned}$$

A basis of ker *M*

How to find *T*?

- We use *kernel space* of observability matrix.

$$T^{-1} := [T_o, T_{\bar{o}}] \quad \begin{cases} T_{\bar{o}} & : \text{A basis of } \ker\mathcal{O} \\ T_o & : \text{any complement of } T_{\bar{o}} \text{ in } \mathbb{R}^n \end{cases}$$

Unobservable subspace

- Ex.

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 2 & 2 & 2 \\ 3 & 0 & -1 \end{bmatrix}, \; C = [0,0,1] \qquad \mathcal{O} = \begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 3 & 0 & -1 \\ 3 & 0 & 1 \end{bmatrix} \Rightarrow \text{rank}\mathcal{O} = 2 < 3$$

Unobservable!

$$\Rightarrow \quad T^{-1} = \begin{bmatrix} \underbrace{\begin{matrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{matrix}}_{T_o} & \underbrace{\begin{matrix} 0 \\ 1 \\ 0 \end{matrix}}_{T_{\bar{o}}} \end{bmatrix} \quad \Rightarrow \quad TAT^{-1} = \begin{bmatrix} 2 & 0 & 0 \\ 3 & -1 & 0 \\ 2 & 2 & 2 \end{bmatrix}$$

$$CT^{-1} = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}$$

Summary

- Observability
  - Definition
  - Condition by observability matrix
  - Duality between controllability and observability
  - Decomposition (Matlab command "obsvf.m")
    (We can use duality for decomposition.)
- Next, Kalman decomposition