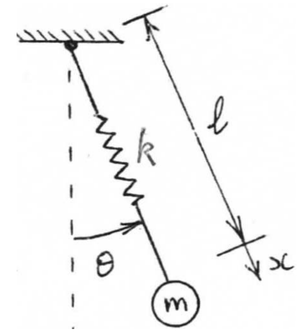
MECH 463 -- Tutorial 10

1. A flexible pendulum consists of a light spring of unstretched length ℓ, stiffness k, with a concentrated mass m attached at its lower end. Use **Lagrange's equations** to formulate the equations of motion of the pendulum in terms of the spring extension x and the pendulum angle θ. Your equations will be non-linear; you are not asked to put them in matrix form nor solve them. Do not make any "small vibration" assumptions yet. Explain the meanings of the various terms in your equations. When done, simplify your equations for the case of small vibration amplitude. Give a physical explanation of your results.



(a)

Kinetic energy, $T = \frac{1}{2} m \left(\dot{x}^2 + (l+x)^2 \dot{\theta}^2 \right)$

Potential energy, $V = \frac{1}{2} k x^2 + mg \left(l - (l+x)\cos\theta \right)$

Dissipation function, $R = 0$

Generalized forces, $Q_i = 0$

Generalized coordinates $q_1 = x \qquad q_2 = \theta$

Recall Lagranges Equations:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) \quad \frac{\partial T}{\partial q_i} + \frac{\partial R}{\partial \dot{q}_i} + \frac{\partial V}{\partial q_i} = Q_i$$

$$\frac{\partial T}{\partial \dot{q}_1} = \frac{\partial T}{\partial \dot{x}} = m x \qquad\qquad \frac{\partial T}{\partial \dot{q}_2} = \frac{\partial T}{\partial \dot{\theta}} = m(l+x)^2 \dot{\theta}$$

$$\frac{\partial T}{\partial q_1} = \frac{\partial T}{\partial x} = m(l+x)\dot{\theta}^2 \qquad\qquad \frac{\partial T}{\partial q_2} = \frac{\partial T}{\partial \theta} = 0$$

$$\frac{\partial R}{\partial \dot{q}_1} = \frac{\partial R}{\partial x} = 0 \qquad\qquad \frac{\partial R}{\partial \dot{q}_2} = \frac{\partial R}{\partial \dot{\theta}} = 0$$

$$\frac{\partial V}{\partial q_1} = \frac{\partial V}{\partial x} = kx - mg\cos\theta \qquad\qquad \frac{\partial V}{\partial q_2} = \frac{\partial V}{\partial \theta} = mg(l+x)\sin\theta$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_1}\right) = m x \qquad\qquad \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_2}\right) = m(l+x)^2\ddot{\theta} + 2m(l+x)\dot{x}\dot{\theta}$$

1st. eqn. $\quad \underset{\text{inertia}}{\underset{\uparrow}{m\ddot{x}}} - \underset{\text{centripetal}}{\underset{\uparrow}{m(l+x)\dot{\theta}^2}} + \underset{\text{spring}}{\underset{\uparrow}{kx}} = \underset{\text{gravity force}}{\underset{\uparrow}{mg\cos\theta}}$

2nd. eqn. $\div (l+x)$ $\quad \underset{\text{inertia}}{\underset{\uparrow}{m(l+x)\ddot{\theta}}} + \underset{\text{coriolis}}{\underset{\uparrow}{2m\dot{x}\dot{\theta}}} + \underset{\text{gravity force}}{\underset{\uparrow}{mg\sin\theta}} = 0$

(b) For small vibrations, $\sin\theta \to \theta \quad \cos\theta \to 1$

$\dot{\theta}^2$ and $\dot{x}\dot{\theta}$ are second order, $\dot{\theta}^2 \to 0 \quad \dot{x}\dot{\theta} \to 0$

$$\longrightarrow \quad \begin{aligned} m\ddot{x} + kx &= mg \\ m\ddot{\theta} + \frac{mg}{l+x}\theta &= 0 \end{aligned}$$

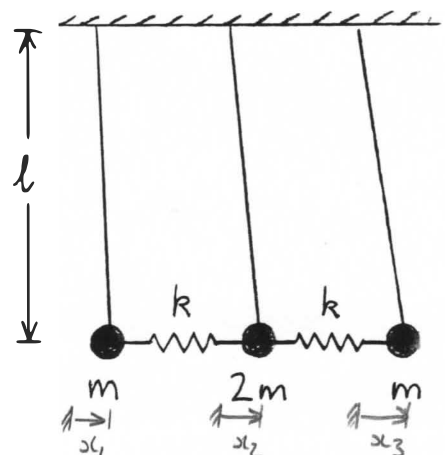
The solution to the first equation is $x = A\cos\omega_1 t - B\sin\omega_1 t + \frac{mg}{k}$. The first two terms describe the vibration, and the third term the static (gravitational) displacement. For small vibrations, the first two terms are small, but the static displacement $\frac{mg}{k}$ is not necessarily small compared with l. Thus, the second equation above should include this term.

Substituting:

$$\boxed{\begin{aligned} m\ddot{x} + kx &= mg \\ m\ddot{\theta} + \frac{mg}{l+\frac{mg}{k}}\theta &= 0 \end{aligned}}$$

The linearized vibrations are uncoupled. One mode is an extensional vibration with $\omega_1^2 = \frac{k}{m}$. The other mode is a pendulum vibration with $\omega_2^2 = \frac{g}{l+\frac{mg}{k}}$.

2. Three simple pendulums of equal length ℓ are connected at their lower ends by springs of stiffness k. The mass of the centre pendulum is 2m and of each of the other two pendulums is m. By inspection, determine the first two mode shapes of the pendulum system. Justify your choices. Determine the mass and stiffness matrices of the system. Use the orthogonality relations to determine the third mode shape.



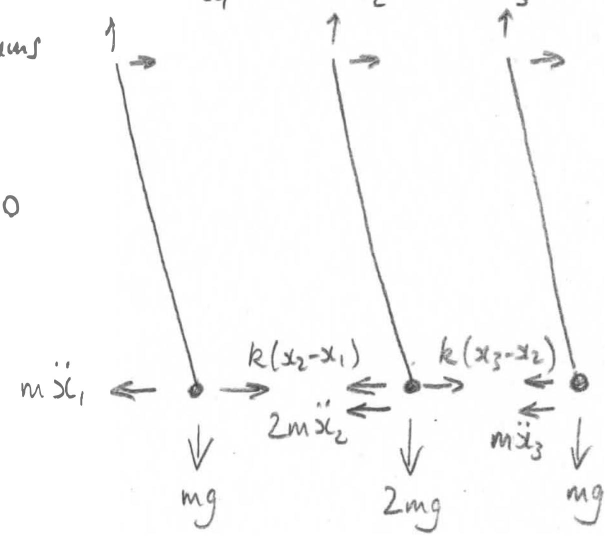
Take moments about tops of pendulums

$$m l \ddot{x}_1 + m g x_1 - k l (x_2 - x_1) = 0$$

$$2 m l \ddot{x}_2 + 2 m g x_2 + k l (x_2 - x_1) - k l (x_3 - x_2) = 0$$

$$m l \ddot{x}_3 + m g x_3 + k l (x_3 - x_2) = 0$$

where x_1, x_2 and x_3 are the lateral displacements of the pendulums (assuming small vibrations)



In matrix format, after dividing by l

$$\underbrace{\begin{bmatrix} m & 0 & 0 \\ 0 & 2m & 0 \\ 0 & 0 & m \end{bmatrix}}_{\underset{\sim}{M}} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{bmatrix} + \underbrace{\begin{bmatrix} k + mg/l & -k & 0 \\ -k & 2k + 2mg/l & -k \\ 0 & -k & k + mg/l \end{bmatrix}}_{\underset{\sim}{K}} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

All three pendulums have the same length. Thus, they all would have the same natural frequency if they were not connected together. Additionally, if each pendulum had the same amplitude and phase, there would be no relative motion of the masses, and it would make no difference whether or not the pendulums were connected. Thus, one mode shape is $\underline{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. This is independent of the value of k. By symmetry, a second mode shape is $\underline{u}_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ which has a nodal point at the centre mass. The third mode shape $\underline{u}_3 = \begin{bmatrix} 1 \\ \alpha \\ \beta \end{bmatrix}$ may be found using the orthogonality relations:

$$\underline{u}_1^T \underset{\sim}{M} \underline{u}_3 = 0$$

$$\underline{u}_2^T \underset{\sim}{M} \underline{u}_3 = 0$$

or similarly using $\underset{\sim}{K}$, but $\underset{\sim}{M}$ is simpler here.

$$\rightarrow \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} m & 0 & 0 \\ 0 & 2m & 0 \\ 0 & 0 & m \end{bmatrix} \begin{bmatrix} 1 \\ \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} m & 2m & m \end{bmatrix} \begin{bmatrix} 1 \\ \alpha \\ \beta \end{bmatrix} = m(1 + 2\alpha + \beta) = 0$$

$$\begin{bmatrix} 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} m & 0 & 0 \\ 0 & 2m & 0 \\ 0 & 0 & m \end{bmatrix} \begin{bmatrix} 1 \\ \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} m & 0 & -m \end{bmatrix} \begin{bmatrix} 1 \\ \alpha \\ \beta \end{bmatrix} = m(1 - \beta) = 0$$

$$\rightarrow \beta = 1, \quad \alpha = -1 \quad \rightarrow \boxed{\underline{u}_3 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}}$$