Tutorial 2

Review

time domain	s domain
$\delta(t)$	1
$u(t)$	$\frac{1}{s}$
t	$\frac{1}{s^2}$
$e^{-\alpha t}$	$\frac{1}{s+\alpha}$
$\sin(\omega t)$ *	$\frac{\omega}{s^2+\omega^2}$
$\cos(\omega t)$ *	$\frac{s}{s^2+\omega^2}$

"Table 3.1, Chapter 3, Lecture Notes"

*) $e^{i\omega t} = \cos(\omega t) + i\sin(\omega t)$

$$L\{e^{i\omega t}\} = \int_0^\infty e^{i\omega t} e^{-st}\,dt = \int_0^\infty e^{(-st+i\omega t)}\,dt = \frac{1}{s-i\omega} =$$

$$\frac{1}{s-i\omega} \times \frac{s+i\omega}{s+i\omega} = \frac{s+i\omega}{s^2+\omega^2} = \underbrace{\frac{s}{s^2+\omega^2}}_{L\{\cos(\omega t)\}} + i\,\underbrace{\frac{\omega}{s^2+\omega^2}}_{L\{\sin(\omega t)\}}$$

$L\{\cos(\omega t)\} + i\,L\{\sin(\omega t)\}$

Other Cases:

Time domain	S domain
$e^{-\alpha t}\sin(wt)$	$\frac{w}{(s+\alpha)^2+w^2}$
$e^{-\alpha t}\cos(wt)$	$\frac{s+\alpha}{(s+\alpha)^2+w^2}$
$t\,e^{-\alpha t}$	$\frac{1}{(s+\alpha)^2}$
$\dot{f}(t),\ f(0)$	$sF(s)-f(0)$
$\int f(t)$	$\frac{F(s)}{s}$

"Table 3.1, Chapter 3, Lecture Notes"

1.a) $G(s) = \frac{3}{3+s}$, $r(t)=6$

$$G(s) = \frac{Y(s)}{R(s)} \Rightarrow Y(s) = G(s)\,R(s)$$

$$r(t)=6 \longrightarrow R(s) = \frac{6}{s} \Rightarrow Y(s) = \frac{3}{s+3}\,\frac{6}{s}$$

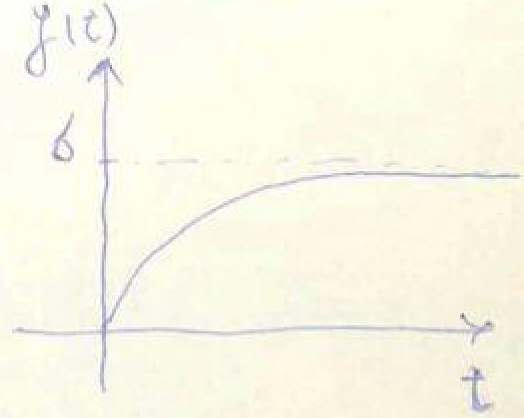
Partial Fractions :

$$F(s) = \frac{A}{(s+P_1)} + \frac{B}{(s+P_2)} + \cdots + \frac{Z}{(s+P_z)}$$

$$A = \lim_{s \to -P_1} \left\{ (s+P_1) F(s) \right)$$

$$B = \lim_{s \to -P_2} \left((s+P_2) F(s) \right)$$

$$Y(s) = \frac{3}{s+3} \times \frac{6}{s} = \frac{A}{s+3} + \frac{B}{s}$$



$$A = \lim_{s \to -3} \left((s+3) Y(s) \right) = \lim_{s \to -3} \frac{18}{s} = -6$$

$$B = \lim_{s \to 0} \lim \left(s\, Y(s) \right) = \lim_{s \to 0} \frac{18}{s+3} = 6$$

$$Y(s) = \frac{6}{s} - \frac{6}{s+3} = L^{-1}\{Y(s)\} = y(t) = \boxed{6 - 6e^{-3t}}$$

1.b) $G(s) = \dfrac{3}{3+s}$, $r(t) = 6$, $y(0) = 10$

$$G(s) = \frac{Y(s)}{R(s)} = \frac{3}{3+s} = \frac{Y(s)}{R(s)} \Rightarrow \dot{y} + 3y = 3r(t) \Rightarrow$$

$$\mathcal{L}\{\dot{y} + 3y\} = \mathcal{L}\{3r(t)\} = sY(s) - y(0) + 3Y(s) = 3R(s) \Rightarrow$$

$$Y(s)(s+3) = \frac{18}{s} + 10 \Rightarrow Y(s)(s+3) = \frac{10s+18}{s} \Rightarrow Y(s) = \frac{10s+18}{s(s+3)}$$

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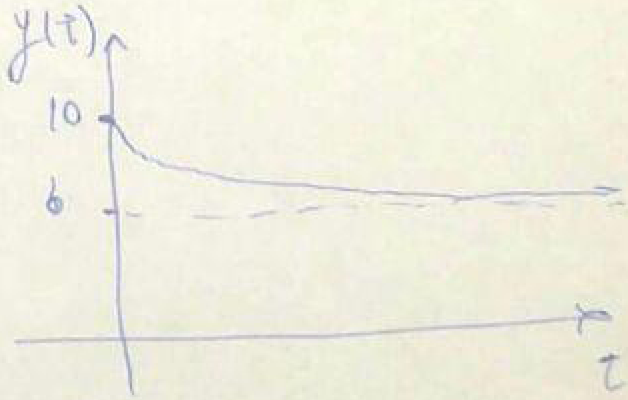
*) $$\frac{3}{3+s} = \frac{Y(s)}{R(s)} \rightsquigarrow 3R(s) = 3Y(s) + sY(s) \Rightarrow$$

$$\mathcal{L}^{-1}\{3R(s)\} = \mathcal{L}^{-1}\{3Y(s) + sY(s)\} \Rightarrow 3r(t) = 3y(t) + \dot{y}(t)$$

$$Y(s) = \frac{10s+18}{s(s+3)} = \frac{A}{s} + \frac{B}{(s+3)}$$



$$A = \lim_{s \to 0} (sY(s)) = \lim_{s \to 0} \left(\frac{10s+18}{s+3}\right) = 6$$

$$B = \lim_{s \to -3} \left((s+3)Y(s)\right) = \lim_{s \to -3} \left(\frac{10s+18}{s}\right) = \frac{-30+18}{-3} = 4$$

$$Y(s) = \frac{6}{s} + \frac{4}{s+3} \Rightarrow y(t) = \boxed{6 + 4e^{-3t}}$$

2. $G(s) = \dfrac{1}{s+10}$, $y(0) = 0$, $r(t) = 3\cos(10t)$

$$G(s) = \frac{Y(s)}{R(s)} \Rightarrow Y(s) = G(s)R(s) = \frac{1}{(s+10)} \times \frac{3s}{s^2+100} \Rightarrow$$

$$Y(s) = \frac{A}{s+10} + \frac{Bs+C}{s^2+100}$$

$$A = \lim_{s \to -10} \left((s+10)\, Y(s) \right) = \lim_{s \to -10} \left(\frac{3s}{s^2+100} \right) = -0.15$$

$$\frac{3s}{(s+10)(s^2+100)} = \frac{A}{s+10} + \frac{Bs+C}{s^2+100} = \frac{A(s^2+100) + (Bs+C)(s+10)}{(s+10)(s^2+100)} \Rightarrow$$

$$3s = As^2 + 100A + Bs^2 + (C+10B)s + 10C \Rightarrow 3s \equiv (A+B)s^2 +$$

$$(C+10B)s + (10C + 100A)$$

$$\begin{cases} A + B = 0 \\ 10B + C = 3 \\ 10C + 100A = 0 \end{cases} \Rightarrow \begin{array}{l} B = 0.15 \\ \\ C = +1.5 \end{array}$$

$$Y(s) = \frac{-0.15}{s+10} + \frac{0.15\,S + 1.5}{S^2+100} = \frac{-0.15}{S+10} + \frac{0.15\,s}{S^2+100} + \frac{1.5}{S^2+100} \times \frac{10}{10}$$

$$y(t) = -0.15\,e^{-10t} + 0.15\cos(10t) + \frac{1.5}{10}\sin(10t) \Rightarrow$$

$$y(t) = -0.15\,e^{-10t} + 0.15\cos(10t) + 0.15\sin(10t)$$

$$A\sin(wt) + B\cos(wt) = \sqrt{A^2+B^2}\left(\underbrace{\frac{A}{\sqrt{A^2+B^2}}}_{\sin(\theta)}\sin(wt) + \underbrace{\frac{B}{\sqrt{A^2+B^2}}}_{\cos(\theta)}\cos(wt)\right)$$

$$\theta = tg^{-1}\left(\frac{A}{B}\right)$$

$$= \sqrt{A^2+B^2}\cos(wt-\theta)$$

$$\boxed{y(t) = -0.15\,e^{-10t} + \underbrace{\sqrt{0.15^2+0.15^2}}_{0.212}\cos\left(10t - \frac{\pi}{4}\right)}$$

3. $G(s) = \dfrac{-4s+20}{s+300}$, $y(0)=0$, $r(t)=10$

$$\Rightarrow Y(s) = G(s)\,R(s) = \frac{-4s+20}{s+300} \times \frac{10}{s} = \frac{A}{s} + \frac{B}{s+300}$$

$$A = \lim_{s\to 0} \left(Y(s)\, s\right) = \lim_{s\to 0} \left(\frac{10 \times (-4s+20)}{s+300}\right) = \frac{2}{3}$$

$$B = \lim_{s\to -300} \left((s+300)\, Y(s)\right) = \lim_{s\to -300} \left(\frac{10 \times (-4s+20)}{s}\right) = \frac{12200}{-300} = -40.6$$

$$\Rightarrow Y(s) = \frac{2}{3s} - \frac{40.67}{s+300} \Rightarrow \boxed{y(t) = \frac{2}{3} - 40.67\, e^{-300t}}$$

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4. $G(s) = \dfrac{3}{s^2+0.5s+4}$, $y(0)=\dot{y}(0)=0$, $r(t)=2$

$$Y(s) = G(s)\,R(s) = \frac{3}{s^2+0.5s+4} \times \frac{2}{s}$$

Trick : $G(s) = \dfrac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$, $R(s) = \dfrac{1}{s}$

$$y(t) = 1 - \frac{1}{\sqrt{1-\zeta^2}}\, e^{-\zeta\omega_n t} \sin(\omega_d t + \varphi)$$

$$\begin{cases} \omega_d = \omega_n\sqrt{1-\zeta^2} \\ \varphi = \tan^{-1} \dfrac{\sqrt{1-\zeta^2}}{\zeta} \end{cases}$$

$$\omega_n = 2 \quad , \quad 2\zeta\omega_n = 0.5 \quad \Rightarrow \quad \zeta = 0.125$$

$$Y(s) = \frac{6}{4s} \times \left[\frac{4}{s^2 + 0.5s + 4} \right] \Rightarrow$$

$$y(t) = \frac{6}{4} \left[1 - \frac{1}{\sqrt{1 - 0.125^2}} \times e^{-0.125 \times 2 \times t} \sin\left(2 \times \sqrt{1 - 0.125^2}\, t + \tan^{-1} \frac{\sqrt{1 - 0.125^2}}{0.125} \right) \right]$$

$$\Rightarrow y(t) = \boxed{1.5 - 1.51\, e^{-0.25t} \sin(1.98t + 1.4455)}$$