2.3. Undamped SDOF Response – 2

MECH 463: Mechanical Vibrations

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Suggested Readings:

1. Topic 2.3 from notes package.
2. Sections 1.10, 2.2 and 2.3 from the course textbook.

Learning Objectives

1. Determine forced vibration response of a SDOF system.

2. Apply the rotating vector technique to identify three regimes of steady forced vibration response.

3. Deduce design guidelines to mitigate forced vibration response.

2.12 Forced Vibration Response (T 3.3+Notes) — # 1

The particular solution, x_p, when the spring-mass system is subjected to a harmonic force $f(t) = F_0 \cos \omega t$ is governed by

$$m\ddot{x}_p + kx_p = F_0 \cos \omega t \qquad (1)$$

Assuming

$$x_p(t) = X \cos \omega t \qquad (2)$$

in Eq.(1), we find (p.82 of NP)

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2.12 Forced Vibration Response (T 3.3+Notes) — # 2

$$x_p = \frac{F_0}{k - m\omega^2} \cos \omega t, \quad X = \frac{F_0}{k - m\omega^2} \tag{3}$$

Therefore, the total response is given by

$$x(t) = x_h + x_p = A_1 \cos \omega_n t + A_2 \sin \omega_n t + \frac{F_0}{k - m\omega^2} \cos \omega t, \tag{4}$$

1. The homogeneous solution is harmonic at *natural* frequency ω_n. The forced vibration or particular solution is harmonic at the *forcing* frequency ω.
2. **The two unknown constants, A_1 and A_2, are to be determined from the initial conditions applied to the total response.**

Substituting the initial conditions $x(0) = x_0$ and $\dot{x}(0) = \dot{x}_0$ in Eq.(4) gives the total response

2.12 Forced Vibration Response (T 3.3+Notes) — # 3

(p.84 of NP)

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2.12 Forced Vibration Response (T 3.3+Notes) — # 4

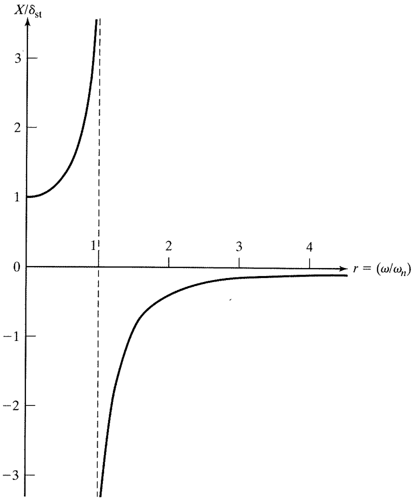
$$x(t) = \left(x_0 - \frac{F_0}{k - m\omega^2}\right)\cos\omega_n t + \frac{\dot{x}_0}{\omega_n}\sin\omega_n t + \frac{F_0}{k - m\omega^2}\cos\omega t, \quad (5)$$

It is useful to represent $\frac{F_0}{k-m\omega^2}$ in terms of a non-dimensional parameter, called *Dynamic Magnification Factor* (DMF), which is defined as the ratio of the displacement amplitudes in the dynamic and static case as follows

$$\frac{X}{\delta_{st}} = \frac{X}{\frac{F_0}{k}} = \frac{\frac{F_0}{k-m\omega^2}}{\frac{F_0}{k}} = \frac{k}{k - m\omega^2} = \frac{1}{1 - \frac{m\omega^2}{k}} = \frac{1}{1 - \left(\frac{\omega}{\omega_n}\right)^2} \quad (6)$$

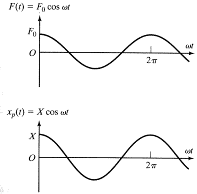
The DMF is the factor by which the static displacement needs to be multiplied with in order to obtain the dynamic displacement in the steady state, <u>ignoring the homogeneous solution</u>. A plot of the DMF as a function of the non-dimensional frequency ratio $r = \frac{\omega}{\omega_n}$ is shown below:

2.12 Forced Vibration Response (T 3.3+Notes) — # 5

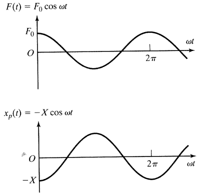


(p.85 of NP)

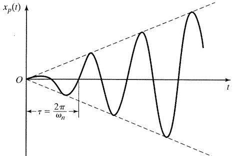
Case 1:



Case 2:

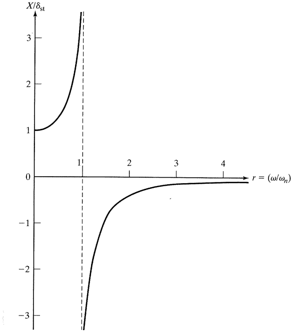


Case 3:



2.12 Forced Vibration Response (T 3.3+Notes) — # 6

Q: List the important features of the DMF curve? (p.86 of NP)



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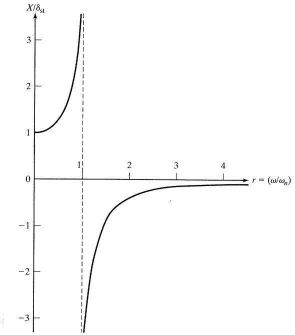
2.12 Forced Vibration Response (T 3.3+Notes) — # 7



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2.12 Forced Vibration Response (T 3.3+Notes) — # 8

Q: Can you explain why X increases with frequency in Case 1, while X decreases with forcing frequency in Case 2? (p.87 of NP)

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2.12 Forced Vibration Response (T 3.3+Notes) — # 9

Case 3: $\omega = \omega_n$

The total response in Eq.(5) can be expressed as follows

$$x(t) = x_0 \cos \omega_n t + \frac{\dot{x}_0}{\omega_n} \sin \omega_n t + \delta_{st} \left[\frac{\cos \omega t - \cos \omega_n t}{1 - \left(\frac{\omega}{\omega_n}\right)^2} \right] \tag{7}$$

In the limit $\omega \to \omega_n$ the factor in [] reduces to an indeterminate form $\frac{0}{0}$. Recall from your Calculus that in such cases we use L'Hospital rule. That is, we differentiate the numerator and denominator with respect to ω until such point where the limit is determinate. Let us do this (p.88 of NP)

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2.12 Forced Vibration Response (T 3.3+Notes) — # 10

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2.12 Forced Vibration Response (T 3.3+Notes) — # 11

Thus the total response when the forcing frequency approaches the natural frequency is given by

$$x(t) = x_0 \cos \omega_n t + \frac{\dot{x}_0}{\omega_n} \sin \omega_n t + \delta_{st} \frac{\omega_n t}{2} \sin \omega_n t \tag{8}$$

The above result says that the **response grows linearly with time at resonance. In other words, the system becomes unstable!**

Q: Can you tell the phase relationship between the force and displacement associated with the particular solution at resonance? (p.89 of NP)

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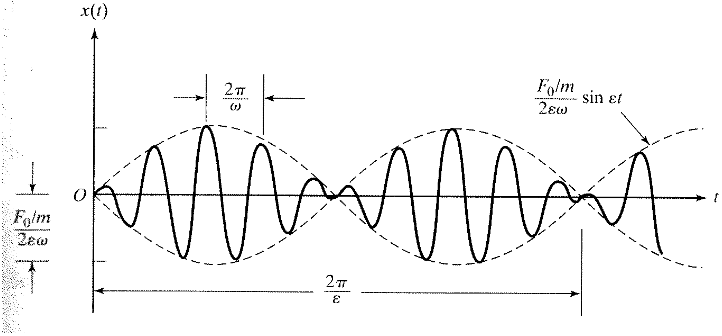
2.12 Forced Vibration Response (T 3.3+Notes) — # 12

Another important phenomenon is observed as the forcing frequency is brought close to resonance leading to *beats*. The response for zero initial velocity and displacement is given by

$$x(t) = \frac{F_0/m}{\omega_n^2 - \omega^2}\left[2\sin\frac{\omega+\omega_n}{2}t\sin\frac{\omega-\omega_n}{2}t\right] = \frac{F_0/m}{2\epsilon\omega}\sin\epsilon t\sin\omega t,$$

$$\epsilon = \frac{\omega_n - \omega}{2}. \quad (9)$$



Summary of Forced Response

1. The total response of an undamped system subjected to a harmonic force $f(t) = F_0 \cos \omega t$ is given by $x(t) = \left(x_0 - \frac{F_0}{k - m\omega^2} \right) \cos \omega_n t + \frac{\dot{x}_0}{\omega_n} \sin \omega_n t + \frac{F_0}{k - m\omega^2} \cos \omega t$.
2. Free vibration takes place at the natural frequency ω_n while the forced vibration is at ω.
3. With increasing forcing frequency from zero, the response increases reaching an instability at resonance $\omega = \omega_n$ and then decreases for forcing frequencies above resonances. The forced vibration response grows linearly with time at resonance $\omega = \omega_n$.
4. The response is in-phase with the force for $\omega < \omega_n$; a phase lag of 90^0 at resonance $\omega = \omega_n$; and the response lags behind the force by 180^0 above resonance. The displacement is in exactly the opposite direction to the force. **This is the first counter-intuitive feature we observe in vibration!**
5. The amplitude of the forced vibration can be evaluated from $DMF = \frac{X}{\delta_{st}} = \frac{1}{1 - \left(\frac{\omega}{\omega_n} \right)^2}$
6. Beating arises due to the interaction between the free and forced vibration.

Example 10 — # 1

Example 10: A portable shredder used to shred bark, tree branches, and shrub clippings, has a mass of 200 kg resting on tires and support system with an elastic constant of 460 N/mm. The amplitude of the vertical sinusoidal force shown below is 3 kN. Find the maximum vertical displacement, if the shredder operates at 1200 rpm. (p.92 of NP)

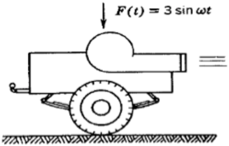


Figure : Figure for example 10.

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Example 10 — # 2

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Example 10 — # 3

Fill in the class

Example 11 — # 1

Example 11: Deduce the expression for forced vibration amplitude X, by using the rotating vector representation. Which forces are dominant below, at, and above the resonant frequency in the vector diagram of forces? (p.94 of NP)

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Example 11 — # 2

Fill in the class

Example 11 — # 3

Fill in the class

Summary of Topic 2.3 — # 1

1. Undamped free vibration is specified by the second order, linear, ODE: $m\ddot{x} + kx = 0$ along with the initial conditions: an initial displacement $x(0) = x_0$ and an initial velocity $\dot{x}(0) = \dot{x}_0$.
2. The undamped free vibration response is given by $x = x_h = x_o \cos \omega_n t + \frac{\dot{x}_0}{\omega_n} \sin \omega_n t$, where the *natural frequency*, ω_n, is given by $\omega_n = \sqrt{\frac{k}{m}}$.
3. Undamped free vibration response can also be represented in term of the amplitude-form $x(t) = A \cos(\omega_n t - \phi_0)$, which lends itself into a rotating vector representation of harmonic motion.
4. In a harmonic motion at frequency ω rad/s and phase lag ϕ_0, whose displacement is given by $x(t) = A \cos(\omega t - \phi_0)$, the velocity and acceleration amplitudes are related to the displacement amplitude, A, via $A_{velocity} = \omega A$ and $A_{acceleration} = \omega^2 A$. The phase lags are related via $\phi_{0,velocity} = \phi_0 - 90^0$ and $\phi_{0,acceleration} = \phi_0 - 180^0$.

Summary of Topic 2.3 — # 2

5. The total response of an undamped system subjected to a harmonic force $f(t) = F_0 \cos \omega t$ is given by $x(t) = \left(x_0 - \frac{F_0}{k - m\omega^2} \right) \cos \omega_n t + \frac{\dot{x}_0}{\omega_n} \sin \omega_n t + \frac{F_0}{k - m\omega^2} \cos \omega t$. Free vibration takes place at the natural frequency ω_n while the forced vibration is at ω.
6. With increasing forcing frequency from zero, the response increases reaching an instability at resonance $\omega = \omega_n$ and then decreases for forcing frequencies above resonances. The forced vibration response grows linearly with time at resonance $\omega = \omega_n$.
7. The response is in-phase with the force for $\omega < \omega_n$; a phase lag of 90^0 at resonance $\omega = \omega_n$; and the response lags behind the force by 180^0 above resonance. The displacement is in exactly the opposite direction to the force. **This is the first counter-intuitive feature we observe in vibration!**
8. The amplitude of the forced vibration can be evaluated from $DMF = \frac{X}{\delta_{st}} = \frac{1}{1 - \left(\frac{\omega}{\omega_n} \right)^2}$
9. Beating arises due to the interaction between the free and forced vibration.

Summary of Topic 2.3 — # 3

10. Elastic forces dominate below resonance while inertial forces dominate above the resonance. Thus, low frequency forced vibrations can be reduced by stiffening the system, while reducing high frequency forced vibration requires considerable addition of mass. **Adding stiffness has little influence on the DMF well above resonance!**