2.6. General Excitation

MECH 463: Mechanical Vibrations

A. Srikantha Phani
srikanth@mech.ubc.ca

Suggested Readings:

1. Topic 2.6 from notes package for detailed derivations.
2. Sections 4.2–4.5 from the course textbook.

Learning Objectives

1. Determine the response to general forcing function: step, impulse, shocks.
2. Recognize the importance of harmonic response.
3. Appreciate interrelations among different forms of response.
4. Apply Fourier series.

Fill in the class

Introduction

Q: Which aspects of the solution (response) will be different for a non-harmonic force? p.154 in NP

Fill in the class

Available methods — # 1

Approach 1: Use ODE Theory (Look up table)

Fill in the class

Particular integrals

For linear differential equations with constant coefficients:

Right-hand side	Trial P.I.
constant	a
x^n (n integer)	$a\,x^n + b\,x^{n-1} + \ldots$
e^{kx}	$a\,e^{kx}$
$x\,e^{kx}$	$(a\,x + b)\,e^{kx}$
$x^n\,e^{kx}$	$(a\,x^n + b\,x^{n-1} + \ldots)\,e^{kx}$
$\left.\begin{array}{l}\sin px\\ \cos px\end{array}\right\}$	$a \sin px + b \cos px$
$\left.\begin{array}{l}e^{kx}\sin px\\ e^{kx}\cos px\end{array}\right\}$	$e^{kx}(a \sin px + b \cos px)$

A dictionary of Particular Solutions for linear ODEs. Note that we replace x with t and k with ω for vibration problems, since time t is the independent variable.

Available methods — # 2

Fill in the class

Available methods — # 3

Approach 2: Use Fourier Series

In Fourier analysis we express the force in the following form

$f(t) = \frac{a_0}{2} + \sum_{n=1}^{n=\infty} [a_n \cos n\omega t + b_n \sin n\omega t]$, $\omega = \frac{2\pi}{T}$, where T is the period of the force.

This method works for periodic forces and periodisable forces. We treat each term in the Fourier series as a force and find the associated response. **Using the principle of superposition:**

$x_p(t) = x_{a_0} + \sum_{n=1}^{n=\infty} [x_{a_n} + x_{b_n}]$.

Fill in the class

Available methods — # 4

Available methods — # 5

Approach 3: Superposition principle with impulse response
This can deal with any forcing. Here, we break up any time series representing the force (periodic or not) in to a sum of shifted impulse functions (or Dirac delta functions).
The response of the system due to the original force is given by the sum of the responses of the system, calculated for each shifted impulse.

If $h(t)$ is the impulse response then the response for any force $f(t)$ is:

$$x_p(t) = \int_{t=0^+}^{t} h(t-\tau) f(\tau) d\tau$$

Fill in the class

Available methods — # 6

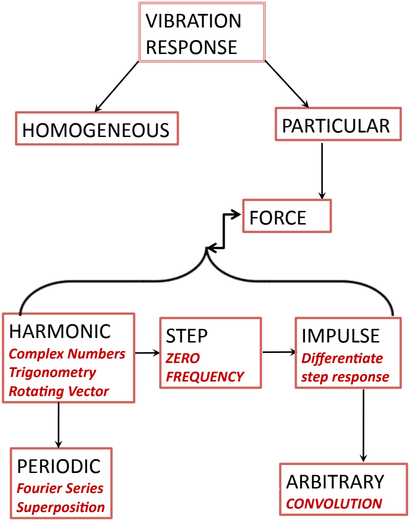
Fill in the class

Available methods — # 7

Q: List the advantages and drawbacks of each of the above methods? p.157 in NP

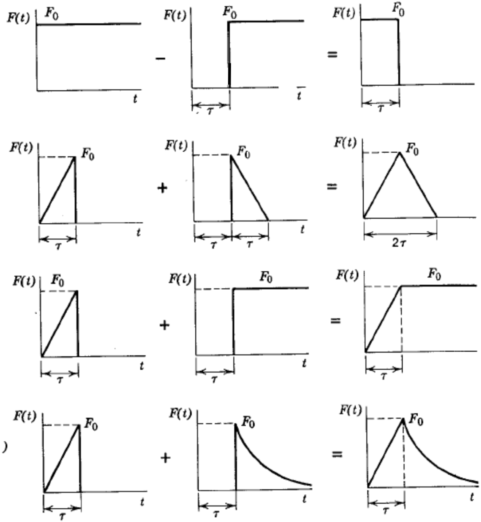
Fill in the class

Interrelations



Relationship among different forced vibration responses.

Application of Superposition Principle — # 1



Superposition is a powerful principle for linear systems.

Application of Superposition Principle — # 2

Fill in the class

2.15. Step Response — # 1

Following the map presented earlier in Fig.(11), we can evaluate the step response from the harmonic response by setting $\omega = 0$:

$$
\begin{aligned}
m\ddot{x} + c\dot{x} + kx = f = F_0 = [F_0 \cos(\omega t)]_{\omega=0} \\
x(t) = e^{-\zeta\omega_n t}\left[C_1 \cos\omega_d t + C_2 \sin\omega_d t\right] + \frac{F_0}{k} \\
C_1 = x_0 - \frac{F_0}{k}; \quad C_2 = \frac{\dot{x}_0 + \zeta\omega_n\left[x_0 - \frac{F_0}{k}\right]}{\omega_d}
\end{aligned}
\tag{1}
$$

Fill in the class

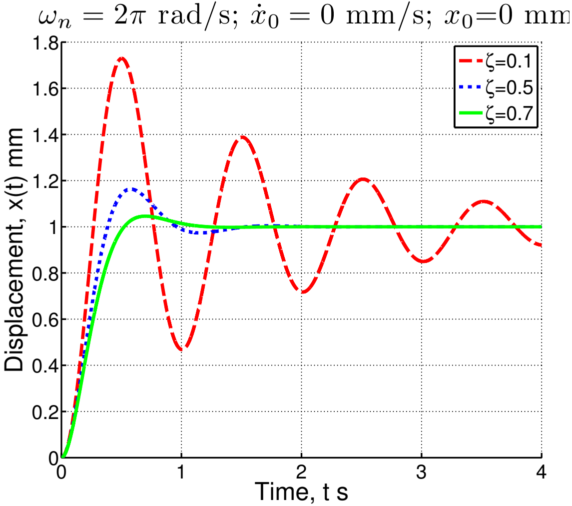
2.15. Step Response — # 2

Note that C_1 and C_2 were found from the initial conditions imposed on the TOTAL response. This must always be followed.

The unit step response $F_0 = 1$ tells us three things:

1. Rise time: Time taken from 10% to 90% of final value (0.1 to 0.9)
2. Settling time: Response reaches within 1% or 5% of final value (0.99 or 0.95).
3. Overshoot: % of final value by which the response rises initially.

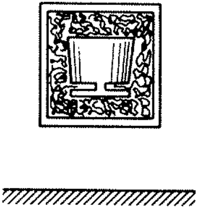
2.15. Step Response — # 3

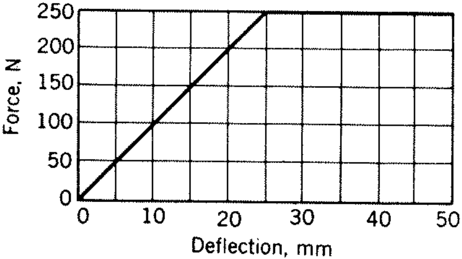


Unit step response of a SDOF system, initially at rest, for different damping ratios. Notice that higher is the damping ratio faster is the settling time, lower overshoot, but increased rise time!

Example 24 — # 1

Q: A 5-kg fragile glass vase is packed in chopped sponge rubber and placed in a cardboard box that has negligible mass. It is then accidentally dropped from a height of h m. This particular sponge rubber exhibits the force-deflection curve sketched below. Determine the maximum acceleration experienced by the vase. *(p.162 in NP)*





Example 24 — # 2

Fill in the class

Example 24 — # 3

Fill in the class

Example 24 — # 4

Fill in the class

Example 24 — # 5

Fill in the class

Example 24 — # 6

Fill in the class

Example 25 — # 1

Q: In ejection seat experiments, the torso is modelled as a spring and mass system. The head is a single mass weighing 5.44 kg. It is supported by the spinal column, with an elastic modulus of 87.56 N/mm. If ejection follows the acceleration time curve shown, determine the peak acceleration of the head. Does it match the experimental result?

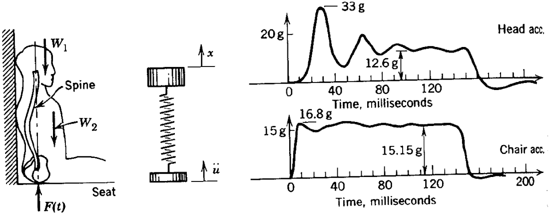


Figure : Figure for example 25.

Example 25 — # 2

Fill in the class

Example 25 — # 3

Fill in the class

Example 25 — # 4

Fill in the class

Example 25 — # 5

Fill in the class

Example 25 — # 6

Fill in the class

2.16. Impulse Response (T4.5,NP2.16) — # 1

A large force applied over a small interval of time is impulse. Units of impulse are N-s or kg-m/s. A unit impulse has a magnitude of 1 N-s. Mathematically, an ideal impulse function is denoted by $\delta(t)$ and it has the following properties

$$\int_0^\infty \delta(t)dt = 1 \quad \int_0^\infty \delta(t - t_0)f(t)dt = f(t_0) \tag{2}$$

An impulse forcing function can be defined by shrinking a rectangular pulse of width δt and height $\frac{1}{\delta t}$, in other words of unit area, to an infinitesimally small interval of time: $\delta t \rightarrow 0$.

2.16. Impulse Response (T4.5,NP2.16) — # 2

Fill in the class

Example 26 — # 1

Q: Using impulse-momentum theorem show that the response of a SDOF at rest subjected to unit impulse is of the form $h(t) = \frac{1}{m\omega_d} e^{-\zeta\omega_n t} \sin \omega_d t$.

Fill in the class

Fill in the class

Example 26 — # 2

Fill in the class

Example 26 — # 3

Fill in the class

Example 26 — # 4

Fill in the class

Example 26 — # 5

Fill in the class

Example 27 — # 1

Q: Using the unit impulse response $h(t) = \frac{1}{m\omega_d} e^{-\zeta\omega_n t} \sin \omega_d t$ *, and the principle of superposition, show that the response of a SDOF system subjected to arbitrary force* $f(t)$ *is given by the integral* $x_p(t) = \int_{t=0^+}^{t} h(t-\tau) f(\tau) d\tau$ *. Discuss the role of initial conditions.*

Fill in the class

Fill in the class

Example 27 — # 2

Fill in the class

Example 27 — # 3

Fill in the class

Example 27 — # 4

Fill in the class

Example 27 — # 5

Fill in the class

Summary — # 1

1. Harmonic response is the fundamental response. Setting $\omega = 0$ in the harmonic response gives the step response. Differentiating step response gives impulse response. Convolution gives reponse for any force.

2. The step response is given by

$$
\begin{aligned}
m\ddot{x} + c\dot{x} + kx = f = F_0 = [F_0 \cos(\omega t)]_{\omega=0} \\
x(t) = e^{-\zeta\omega_n t}\left[C_1 \cos\omega_d t + C_2 \sin\omega_d t\right] + \frac{F_0}{k} \\
C_1 = x_0 - \frac{F_0}{k}; \quad C_2 = \frac{\dot{x}_0 + \zeta\omega_n\left[x_0 - \frac{F_0}{k}\right]}{\omega_d}.
\end{aligned}
\tag{3}
$$

   Higher damping ratios ζ lead to lower overshoot, faster settling time, but result in slow rise time.

3. The impulse response of a SDOF system is given by $h(t) = \frac{1}{m\omega_d} e^{-\zeta\omega_n t} \sin\omega_d t$.

4. The response of a SDOF system to any arbitrary force is given by the convolution integral: $x_p(t) = \int_{t=0^+}^{t} h(t-\tau) f(\tau) d\tau$. This integral is usually evaluated numerically using a computer. Hand calculations are possible only for the simpler cases.

Summary — # 2

5. The homogenous response must be added to the above in order to determine the two unknown constants. Thus, we have for the most general case, the total response: $x(t) = e^{-\zeta\omega_n t}\left[C_1 \cos\omega_d t + C_2 \sin\omega_d t\right] + \int_{t=0^+}^{t} h(t-\tau)f(\tau)d\tau$. C_1 and C_2 must be found by imposing initial conditions on the total response $x(t)$.