MECH 364: ASSIGNMENT 5

Requires course text book: MECHANICAL VIBRATIONS BY S.S. RAO (4TH EDITION). Solutions will appear approximately ten days after the assignment is posted on VISTA.

Q1. (T4.15, 4th Edition) Sandblasting is a process in which an abrasive material, entrained in a jet, is directed onto a surface of a casting to clean its surface. In a particular setup for sandblasting, the casting mass m is placed on a flexible support of stiffness k as shown below. If the force exerted on the casting due to the sandblasting operation varies as shown below, find the response of the casting.

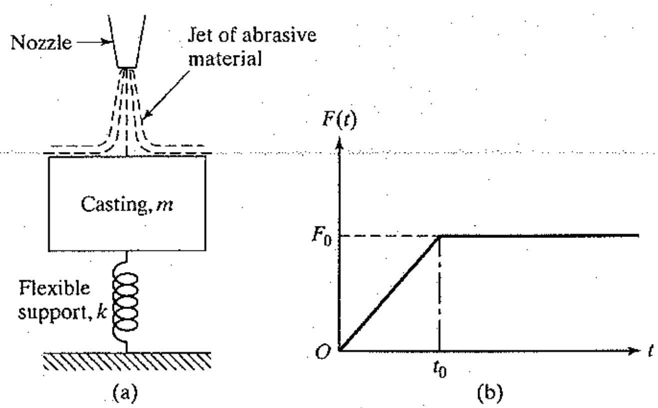


Figure A5.1: Figure for Question 1.

Note: You can use convolution integral, or, use the principle of superposition to solve this problem.

Answer:

$x(t) = \frac{F_0}{k}\left[\frac{t}{t_0} - \frac{\sin\omega_n t}{\omega_n t_0}\right]$ for $0 \le t \le t_0$

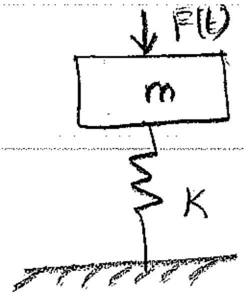
$x(t) = \frac{F_0}{k}\left[1 + \frac{\sin(\omega_n(t-t_0)) - \sin\omega_n t}{t_0\omega_n}\right]$ for $t \ge t_0$;

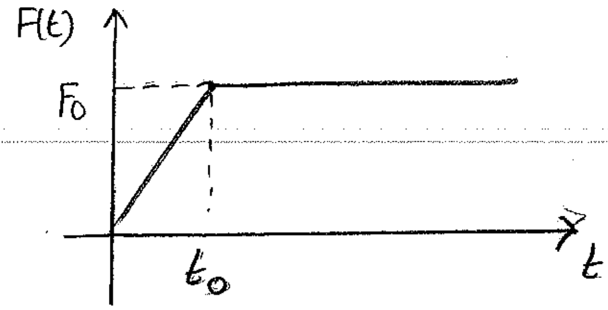
Notice that at $t = t_0$ we have the same solution: $x(t_0) = \frac{F_0}{k}\left[1 - \frac{\sin\omega_n t_0}{\omega_n t_0}\right]$ from both of the above solutions, valid for different time intevals.

MECH 364

ASSIGNMENT #5 : SOLUTION

Q1)





METHOD 1: USING CONVOLUTION INTEGRAL

$$x_p(t) = \int_{\tau=0^+}^{\tau=t} h(t-\tau)\, f(\tau)\, d\tau \qquad f(t) = \frac{F_0 t}{t_0} \quad 0 \le t \le t_0$$
$$= F_0 \quad t \ge t_0$$

$$h(t) = \frac{e^{-\zeta\omega_n t}}{m\omega_d} \sin \omega_d t$$

FOR UNDAMPED SYSTEM $\zeta = 0$; $\omega_d = \omega_n$

$$\therefore h(t) = \frac{1}{m\omega_n} \sin \omega_n t$$

$$x_p(t) = \int_0^t \frac{1}{m\omega_n} \sin \omega_n (t-\tau) \frac{F_0 \tau}{t_0} d\tau \qquad 0 < t < t_0$$

$$= \int_0^t \frac{1}{m\omega_n} \sin \omega_n (t-\tau) F_0\, d\tau \qquad t_0 \le t \le \infty \text{ (OR) } t \ge t_0$$

*** THUS WE HAVE TWO SOLUTIONS FOR THE PARTICULAR INTEGRAL DEPENDING ON THE TIME INTERVAL.

$0 \le t \le t_o$

$$x_p(t) = \int_0^t \frac{F_o}{m\omega_n t_o} \tau \sin\omega_n(t-\tau)\, d\tau$$

INTEGRATE BY PARTS $\int fg\, dt = f\int g\,dt - \int \frac{df}{dt} \int g\,dt\; dt$

CHOSE TO DIFFERENTIATE $\tau \Rightarrow f = \tau;\quad g = \sin\omega_n(t-\tau)$

$$x_p(t) = \frac{F_o}{m\omega_n t_o}\left\{ \left[\frac{-\tau \cos\omega_n(t-\tau)}{-\omega_n}\right]_{\tau=0}^{\tau=t} - \int_0^t \left[\frac{\cos\omega_n(t-\tau)}{-\omega_n}\right] \frac{d}{d\tau}(\tau)\, d\tau \right\}$$

$$= \frac{F_o}{m\omega_n t_o}\left\{ \frac{-t\cos 0}{-\omega_n} - \frac{0 \cos\omega_n t}{-\omega_n} - \int_0^t \frac{\cos\omega_n(t-\tau)\, d\tau}{\omega_n} \right\}$$

$$= \frac{F_o}{m\omega_n t_o}\left\{ \frac{t}{\omega_n} - 0 - \left[\frac{\sin\omega_n(t-\tau)}{-\omega_n^2}\right]_{\tau=0}^{\tau=t} \right\}$$

$$= \frac{F_o}{m\omega_n t_o}\left\{ \frac{t}{\omega_n} - 0 - \left\{ \frac{\sin\omega_n(t-t)}{-\omega_n^2} - \frac{\sin\omega_n(t-0)}{-\omega_n^2} \right\} \right\}$$

$$= \frac{F_o}{m\omega_n t_o}\left\{ \frac{t}{\omega_n} - 0 + \frac{\sin 0}{\omega_n^2} - \frac{\sin\omega_n t}{\omega_n^2} \right\}$$

$$= \frac{F_o}{m\omega_n}\left\{ \frac{t}{t_o\omega_n} - 0 + 0 - \frac{\sin\omega_n t}{\omega_n^2 t_o} \right\} = \frac{F_o}{m\omega_n^2}\left\{ \frac{t}{t_o} - \frac{\sin\omega_n t}{\omega_n t_o} \right\}$$

$$= \frac{F_o}{K}\left\{ \frac{t}{t_o} - \frac{\sin\omega_n t}{\omega_n t_o} \right\}$$

NOTE: YOU NEED NOT SHOW ALL STEPS FOR INTEGRATION BY PARTS IN AN EXAM!

$t \geq t_0$

IN THIS CASE WE HAVE TO CAREFULLY EVALUATE THE CONVOLUTION INTEGRAL. THE RESPONSE IS DUE TO BOTH THE FORCES

$\frac{F_0 t}{t_0}$ $0 \leq t \leq t_0$ & F_0 $t \geq t_0$

$$\therefore x_p(t) = \int_{\tau=0}^{\tau=t_0} \frac{F_0 \tau}{t_0} \frac{\sin \omega_n (t-\tau)}{m\omega_n} d\tau + \int_{\tau=t_0}^{\tau=t} \frac{F_0}{m\omega_n} \sin \omega_n (t-\tau)\, d\tau$$

$$= I_1 + I_2$$

WE KNOW THAT I_1 IS GIVEN BY:

$$\int \frac{F_0 \tau}{t_0} \frac{\sin \omega_n (t-\tau)}{m\omega_n} d\tau = \frac{F_0}{m\omega_n t_0} \left\{ \frac{\tau \cos \omega_n (t-\tau)}{\omega_n} + \frac{\sin \omega_n (t-\tau)}{\omega_n^2} \right\}$$

FROM PREVIOUS CONVOLUTION INTEGRAL.

WE NEED TO SET THE APPROPRIATE LIMITS FOR τ IN THE ABOVE, NAMELY $\tau = 0$ & $\tau = t_0$

$$\therefore \int_{\tau=0}^{\tau=t_0} \frac{F_0 \tau}{t_0} \frac{\sin \omega_n (t-\tau)}{m\omega_n} d\tau = \frac{F_0}{m\omega_n t_0} \left[\frac{\tau \cos \omega_n (t-\tau)}{\omega_n} + \frac{\sin \omega_n (t-\tau)}{\omega_n^2} \right]_{\tau=0}^{\tau=t_0}$$

$$= \frac{F_0}{m\omega_n t_0} \left[\frac{t_0 \cos \omega_n (t-t_0)}{\omega_n} - 0 + \frac{\sin \omega_n (t-t_0)}{\omega_n^2} - \frac{\sin \omega_n t}{\omega_n^2} \right]$$

$$= I_1 \quad \left(\text{DUE TO } \frac{F_0 t}{t_0} \right)$$

$$I_2 = \int_{\tau=t_0}^{\tau=t} \frac{F_0}{m\omega_n} \sin \omega_n (t-\tau)\, d\tau = \frac{F_0}{m\omega_n^2} \Big[\cos \omega_n (t-\tau) \Big]_{t_0}^{t}$$

$$I_2 = \frac{F_0}{m\omega_n^2}\left[\cos\omega_n(t-t) - \cos\omega_n(t-t_0)\right]$$

$$= \frac{F_0}{m\omega_n^2}\left[1 - \cos\omega_n(t-t_0)\right]$$

$$\therefore\ x_p(t) = I_1 + I_2 = \frac{F_0}{m\omega_n t_0}\left[\frac{t_0\cos\omega_n(t-t_0)}{\omega_n} + \frac{\sin\omega_n(t-t_0)}{\omega_n^2} - \frac{\sin\omega_n t}{\omega_n^2}\right]$$

$$+ \frac{F_0}{m\omega_n^2}\left[1 - \cos\omega_n(t-t_0)\right]$$

$$= \frac{F_0}{m\omega_n^2}\left[\cancel{\frac{t_0\cos\omega_n(t-t_0)}{t_0}} + \frac{\sin\omega_n(t-t_0) - \sin\omega_n t}{\omega_n t_0} + 1 - \cancel{\cos\omega_n(t-t_0)}\right]$$

COS TERMS CANCEL!

$$= \frac{F_0}{m\omega_n^2}\left[\frac{\sin\omega_n(t-t_0) - \sin\omega_n t}{\omega_n t_0} + 1\right] \quad t \geq t_0$$

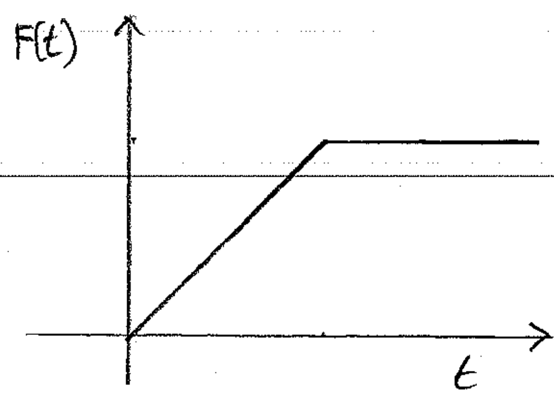
Note: $m\omega_n^2 = k$

PARTICULAR SOLUTION IS:

$$x_p(t) = \frac{F_0}{k}\left[\frac{t}{t_0} - \frac{\sin\omega_n t}{\omega_n t_0}\right] \quad 0 \leq t \leq t_0$$

$$= \frac{F_0}{k}\left[1 + \frac{\sin\omega_n(t-t_0) - \sin\omega_n t}{\omega_n t_0}\right] \quad t \geq t_0$$

USING PRINCIPLE OF SUPERPOSITION: (EASIER OPTION)



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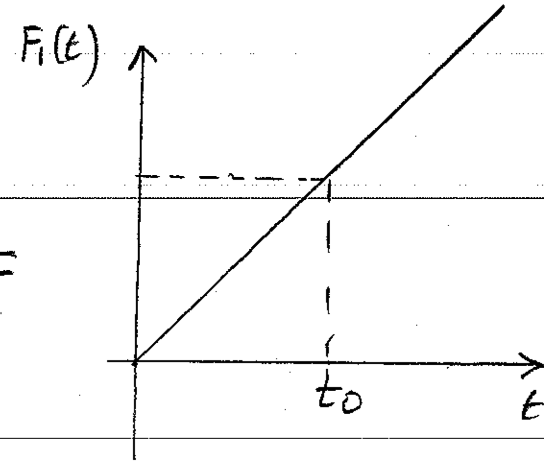

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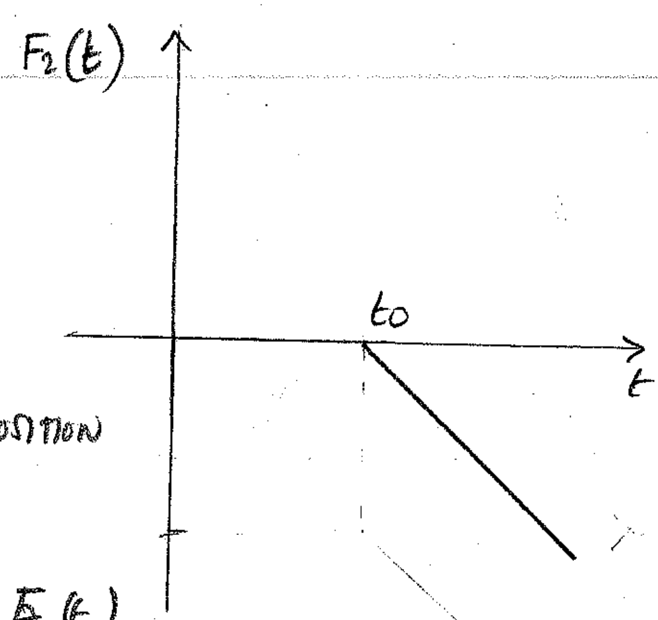
$$F(t) = F_1(t) + F_2(t)$$

$$F_1(t) = \frac{F_0 t}{t_0} \quad t \geq 0$$

$$F_2(t) = 0 \quad t \leq t_0$$

$$= -\frac{F_0(t-t_0)}{t} \quad t \geq t_0$$



ACCORDING TO PRINCIPLE OF SUPERPOSITION

x_p DUE TO $F(t)$

$= x_p$ DUE TO $F_1(t) + x_p$ DUE TO $F_2(t)$

$$F_1(t) \Rightarrow x_{p1} = \frac{F_0}{K}\left[\frac{t}{t_0} - \frac{\sin\omega_n t}{\omega_n t_0}\right] = \int_0^t \frac{F_0 t}{t_0}\frac{1}{m\omega_n}\sin\omega_n(t-\tau)d\tau$$

$$F_2(t) \Rightarrow x_{p2} = 0 \quad t \leq t_0$$

$$= -\frac{F_0}{K}\left[\frac{t-t_0}{t} - \frac{\sin\omega_n(t-t_0)}{\omega_n t_0}\right]$$

ADDING

$$x_p(t) = x_{p1} + x_{p2} = \frac{F_0}{K}\left[\frac{t}{t_0} - \frac{\sin\omega_n t}{\omega_n t_0}\right] \quad t \leq t_0$$

$$x_p(t) = \frac{F_0}{K}\left[\frac{t}{t_0} - \frac{\sin\omega_n t}{\omega_n t_0}\right] \quad t \le t_0$$

$$= \frac{F_0}{K}\left[\frac{t}{t_0} - \frac{\sin\omega_n t}{\omega_n t_0}\right] - \frac{F_0}{K}\left[\frac{t-t_0}{t_0} - \frac{\sin\omega_n(t-t_0)}{\omega_n t_0}\right]$$

$$= \frac{F_0}{K}\left[1 + \frac{\sin\omega_n(t-t_0) - \sin\omega_n t}{\omega_n t_0}\right] \quad t \ge t_0$$

SAME AS THE RESULT OBTAINED USING CONVOLUTION INTEGRAL!!

IN SUMMARY WE OBTAINED x_p USING CONVOLUTION INTEGRAL AND THE PRINCIPLE OF SUPERPOSITION. WE FOUND THE SAME PARTICULAR (FORCED VIBRATION) RESPONSE!

Note: (1) PERHAPS THIS IS THE ONLY TIME YOU WILL SEE CONVOLUTION INTEGRAL WORKED OUT IN DETAIL.

(2) HOWEVER PRINCIPLE OF SUPERPOSITION IS POWERFUL ONCE WE KNOW THE PARTICULAR RESPONSE FOR AN ELEMENTARY FUNCTION SUCH AS $F(t) = t$. IN EXAM YOU WILL BE GIVEN x_p FOR SIMPLE FUNCTIONS AND MAY BE ASKED TO DETERMINE RESPONSE FOR COMPOSITE FUNCTION USING SUPERPOSITION PRINCIPLE!