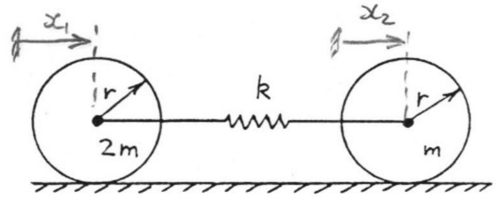
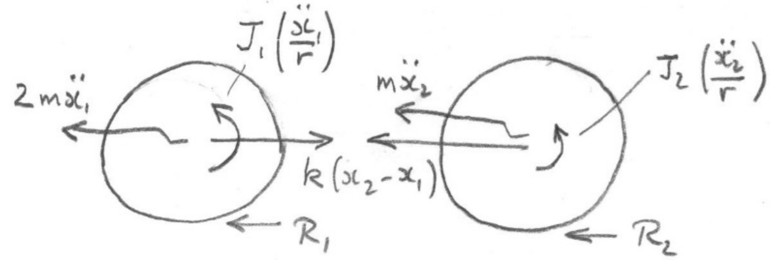
# MECH 463 -- Tutorial 5

1. A vibrating system consists of two circular cylinders, both of radius r, rolling on a rough horizontal surface. One cylinder has mass 2m and the other has mass m. A spring of stiffness k joins the two cylinders. Choose a convenient coordinate system and derive the matrix equation of motion. Solve for natural frequencies and mode shapes. Sketch the mode shapes.




$J_1 = \frac{1}{2}\cdot(2m)r^2 \qquad J_2 = \frac{1}{2}mr^2$

Let $x_1$ and $x_2$ be the lateral displacements of the two cylinders.
$\rightarrow$ rotations are $\frac{x_1}{r}$ and $\frac{x_2}{r}$




Take moments about contact points (to avoid needing reaction forces)

$$2m\ddot{x}_1 . r + J_1\frac{\ddot{x}_1}{r} - k(x_2 - x_1)r = 0$$

$$m\ddot{x}_2 . r + J_2\frac{\ddot{x}_2}{r} + k(x_2 - x_1)r = 0$$

Divide by $r$ $\rightarrow$ $\begin{bmatrix} 3m & 0 \\ 0 & 3/2\,m \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} k & -k \\ -k & k \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Try solution $\underline{x} = \underline{X}\cos(\omega t + \phi) \rightarrow (\underset{\sim}{K} - \omega^2 \underset{\sim}{M})\underline{X}\cos(\omega t + \phi) = \underline{0}$

For non-trivial solution valid for all $t$ $\rightarrow \det(\underset{\sim}{K} - \omega^2\underset{\sim}{M}) = 0 \rightarrow \begin{vmatrix} k - 3m\omega^2 & -k \\ -k & k - \frac{3}{2}m\omega^2 \end{vmatrix} = 0$

$\rightarrow (k - 3m\omega^2)(k - 3/2\,m\omega^2) - k^2 = 0 \longrightarrow \frac{9}{2}m^2\omega^4 - \frac{9}{2}mk\omega^2 = 0$

$$\rightarrow \underline{\omega_1^2 = 0 \quad \text{or} \quad \omega_2^2 = k/m}$$

For mode shapes, put $\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = C\begin{bmatrix} 1 \\ u \end{bmatrix} \longrightarrow \begin{bmatrix} k - 3m\omega^2 & -k \\ -k & k - 3/2\,m\omega^2 \end{bmatrix}\begin{bmatrix} 1 \\ u \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

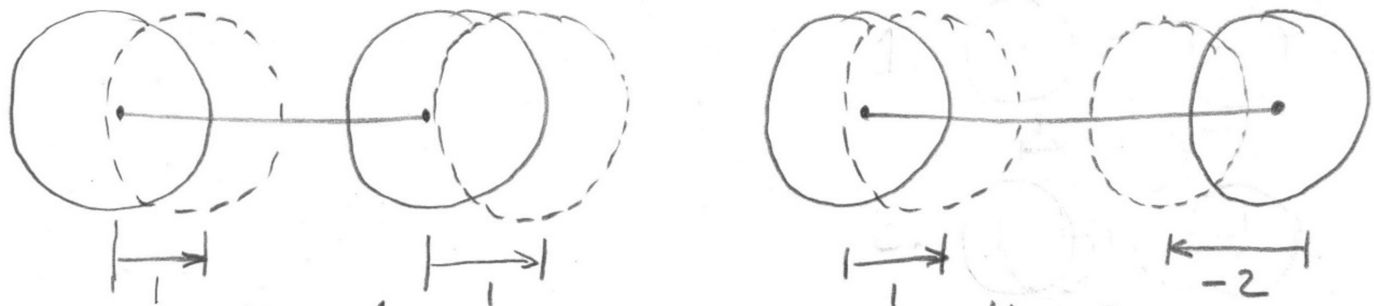
First equation $\rightarrow k - 3m\omega^2 - ku = 0 \rightarrow u = 1 - 3\frac{m}{k}\omega^2$

When $\omega_1^2 = 0 \rightarrow \underline{u_1 = 1}$ $\qquad$ When $\omega_2^2 = \frac{k}{m} \rightarrow \underline{u_2 = -2}$

Mode 1 is a "rigid" body translation (the spring does not stretch).

$\omega_1^2 = 0$ . This is a semi-definite system.

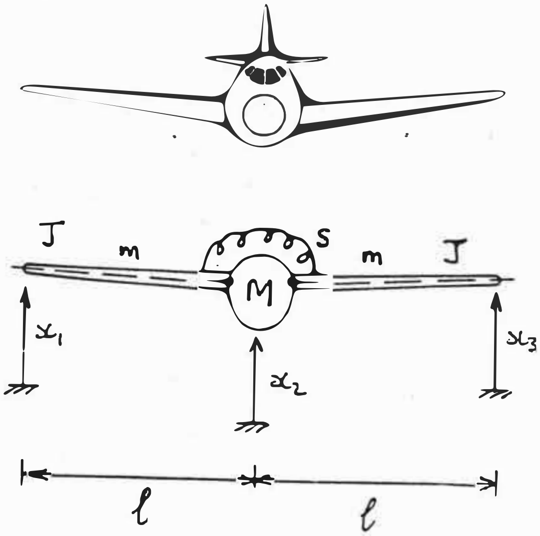
Mode 2 is a vibration about the overall centre of mass.




Mode 1 Mode 2.

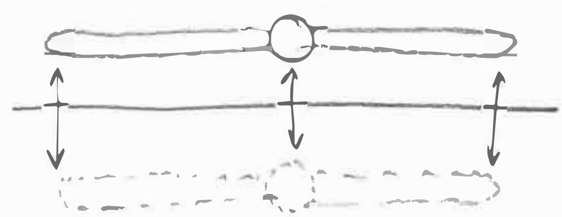
2. The diagram shows a highly idealized model of an airplane in flight. The fuselage is represented by a concentrated mass M and the wings by uniform beams of mass m, length $\ell$, and centroidal polar moment of inertia $J = m\ell^2/12$. For simplicity, the flexibility of the wings is assumed to be concentrated in a central spring of angular stiffness s.

**Guess** the three mode shapes. Interpret the meanings of your guesses. Use your guessed mode shapes to simplify the analysis of the system. Find the three natural frequencies.




Vertical rigid-body motion is possible because the plane can move up and down without relative deformation

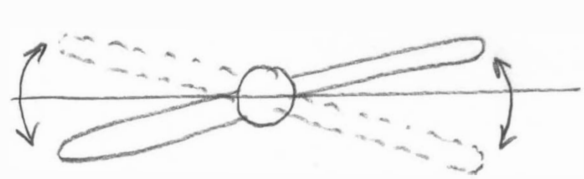
$$\underline{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$



$\longrightarrow \omega_1 = 0$

Rigid-body rotation is possible because the plane can rotate without relative deformation

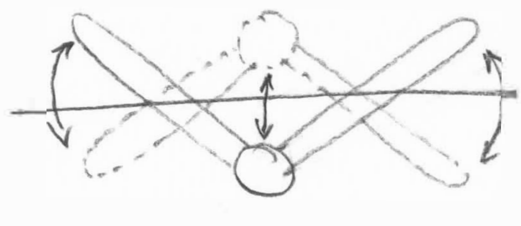
$$\underline{u}_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$



$\longrightarrow \omega_2 = 0$

The third mode is a wing flapping vibration about the centre of mass

$$\underline{u}_3 = \begin{bmatrix} 1 \\ u_3 \\ 1 \end{bmatrix}$$



$\longrightarrow \omega_3 > 0$

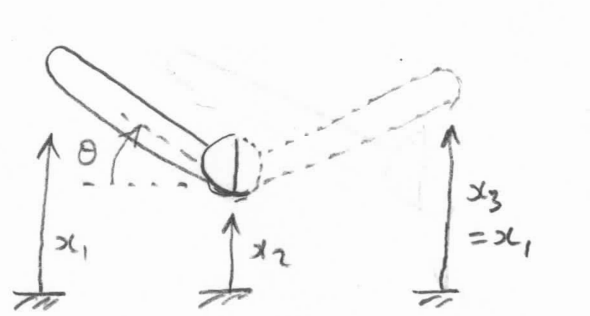
This 3-DOF has two rigid-body motions, and is semi-definite twice. There is only one non-zero natural frequency.

We note that the airplane is symmetrical left-to-right.

For vibration modes 1 and 3, $x_3 = x_1$.

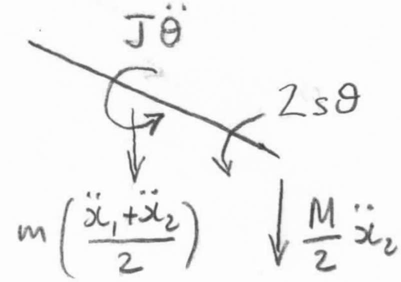
With this substitution, we can reduce the system to 2-DOF.

Since $x_3 = x_1$, we need consider only half of the plane (an unfortunate consequence of buying a half-price ticket!)




Angle $\theta = \dfrac{x_1 - x_2}{l}$

$$\Sigma F = 0 \rightarrow m\left(\frac{\ddot{x}_1 + \ddot{x}_2}{2}\right) + \frac{M}{2}\ddot{x}_2 = 0$$




$$\Sigma M = 0 \rightarrow J\left(\frac{\ddot{x}_1 - \ddot{x}_2}{l}\right) + 2s\left(\frac{x_1 - x_2}{l}\right) - \frac{M}{2}\cdot\frac{l}{2}\ddot{x}_2 = 0$$

In matrix form: (1st equ. x2, 2nd equ. $\div l$)

$$\begin{bmatrix} m & M+m \\ \frac{m}{12} & -\frac{M}{4} - \frac{m}{12} \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 2s/l^2 & -2s/l^2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \longrightarrow \underset{\sim}{M}\underline{x} + \underset{\sim}{K}\underline{x} = \underline{0}$$

Just to prove that it is possible to solve this equation without making it symmetrical, we shall proceed directly.

(Real reason = I don't know how to make the equations symmetrical.)

Try solution $\underline{x} = \underline{X}\cos(\omega t + \phi)$. For a non-trivial solution valid for all $t \longrightarrow \det(\underset{\sim}{K} - \omega^2 \underset{\sim}{M}) = 0$

$$\begin{vmatrix} -\omega^2 m & -\omega^2(M+m) \\ 2s/l^2 - \omega^2\frac{m}{12} & -2s/l^2 + \omega^2\left(\frac{M}{4} + \frac{m}{12}\right) \end{vmatrix} = 0 \rightarrow \begin{matrix} -\omega^2 m\left(2s/l^2 + \omega^2\left(\frac{M}{4} + \frac{m}{12}\right)\right) \\ +\omega^2(M+m)\left(2s/l^2 - \omega^2\frac{m}{12}\right) = 0 \end{matrix}$$

$$\rightarrow \omega^2\left(\omega^2\left(-\frac{Mm}{4} - \frac{m^2}{12} - \frac{Mm}{12} - \frac{m^2}{12}\right) + 2s/l^2\,(m + M + m)\right) = 0$$

$$= \omega^2\left(-\frac{m\omega^2}{6}(2M+m) + 2s/l^2\,(M+2m)\right) \rightarrow \begin{matrix} \omega_1^2 = 0 \\ \omega_3^2 = \dfrac{12(M+2m)s}{(2M+m)\,m\,l^2} \end{matrix}$$

(We can also show that $u_1 = 1$ and $u_3 = -\dfrac{m}{M+m}$)