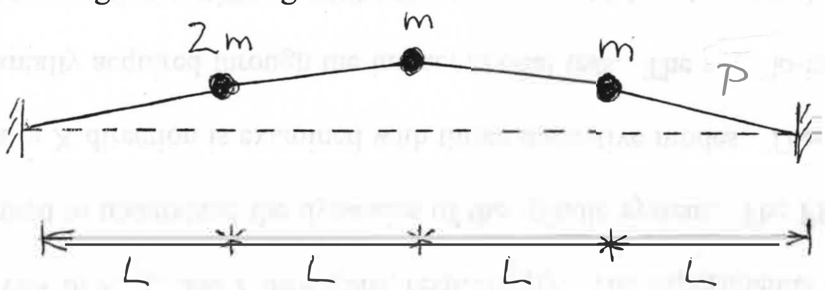
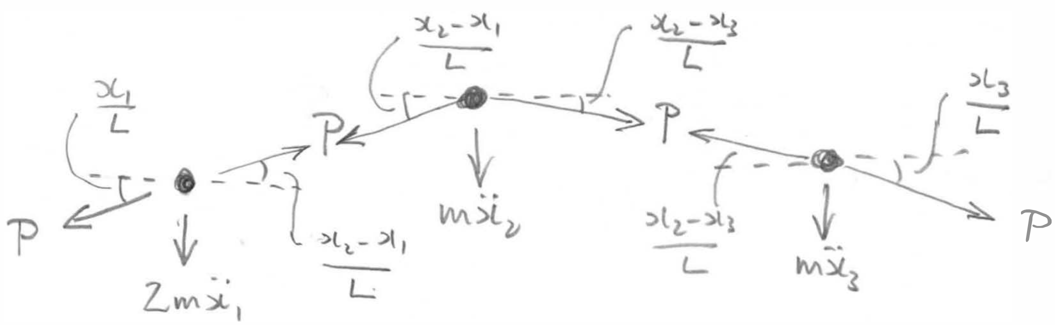
MECH 463 -- Homework 10

1. Three concentrated masses 2m, m, and m are fixed at equal intervals L along the length of a stretched string, of total length 4L, and tension P. The masses can vibrate perpendicular to the length of the string.



(a) Draw free-body diagrams and formulate the equations of motion in matrix format. For convenience, you may write k = P/L.



Use small-angle approximations to define the angles in FBDs.

Vertical equilibrium:

$$-P\frac{x_1}{L} - 2m\ddot{x}_1 + P\left(\frac{x_2-x_1}{L}\right) = 0 \qquad P\left(\frac{x_2-x_3}{L}\right) - m\ddot{x}_3 - P\frac{x_3}{L} = 0$$

$$-P\left(\frac{x_2-x_1}{L}\right) - m\ddot{x}_2 - P\left(\frac{x_2-x_3}{L}\right) = 0$$

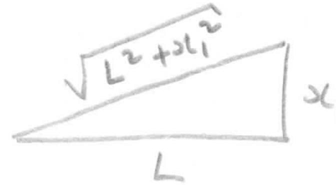
Rearranging and putting into matrix form: $k = \frac{P}{L}$

$$\begin{bmatrix} 2m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{bmatrix} + \begin{bmatrix} 2k & -k & 0 \\ -k & 2k & -k \\ 0 & -k & 2k \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

(b) Use Lagrange's equations to formulate the equations of motion, and confirm that the result is the same as in part (a).

The initial tension P is sufficiently large so that small deflections do not cause a significant tension change. Under these conditions, using the equilibrium state as the zero datum, the potential energy $V = P \times$ string extension.

Consider the leftmost segment:



$$V = P \times \left(\sqrt{L^2+x_1^2} - L \right)$$

$$= PL \left(\sqrt{1+\left(\frac{x_1}{L}\right)^2} - 1 \right)$$

$$\simeq PL \left(1 + \frac{x_1^2}{2L^2} + \dots - 1 \right) \quad \text{using binomial series}$$

$$\simeq \frac{P}{2L} x_1^2 = \frac{k}{2} x_1^2 \quad \text{putting } k = \frac{P}{L}$$

Combining similar results for all segments:

$$V = \frac{k}{2} \left(x_1^2 + (x_2-x_1)^2 + (x_2-x_3)^2 + x_3^2 \right)$$

Kinetic energy: $T = \frac{m}{2} \left(2\dot{x}_1^2 + \dot{x}_2^2 + \dot{x}_3^2 \right)$

Lagrange Equation: $\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} + \frac{\partial R}{\partial \dot{q}_i} + \frac{\partial V}{\partial q_i} = Q_i$

$i=1 \rightarrow q_i = x_1 \rightarrow \frac{d}{dt}\left(2m\dot{x}_1\right) - 0 + 0 + k\left(x_1 - (x_2-x_1)\right) = 0$

$i=2 \rightarrow q_i = x_2 \rightarrow \frac{d}{dt}\left(m\dot{x}_2\right) - 0 + 0 + k\left((x_2-x_1) + (x_2-x_3)\right) = 0$

$i=3 \rightarrow q_i = x_3 \rightarrow \frac{d}{dt}\left(m\dot{x}_3\right) - 0 + 0 + k\left(-(x_2-x_3) + x_3\right) = 0$

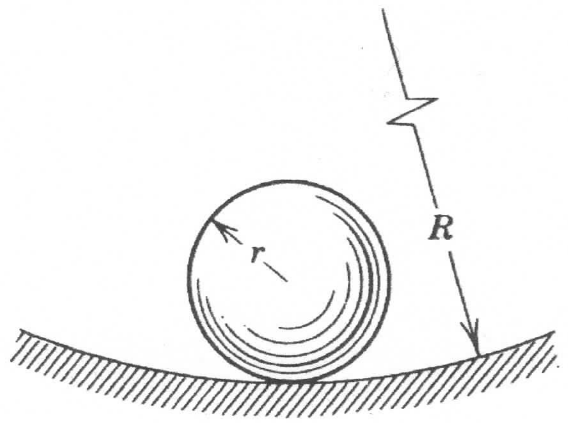
In matrix form:

$$\begin{bmatrix} 2m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{bmatrix} + \begin{bmatrix} 2k & -k & 0 \\ -k & 2k & -k \\ 0 & -k & 2k \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

2. A ball of mass m, radius r and moment of inertia $I = {}^2/_5\, m r^2$ rolls without slipping in a bowl of radius R.

   (a) Draw a free-body diagram and formulate the equations of motion for small vibrations.

   (b) Use Lagrange's Equation to formulate the equations of motion for small vibrations. Confirm that the result is the same as in part (a).

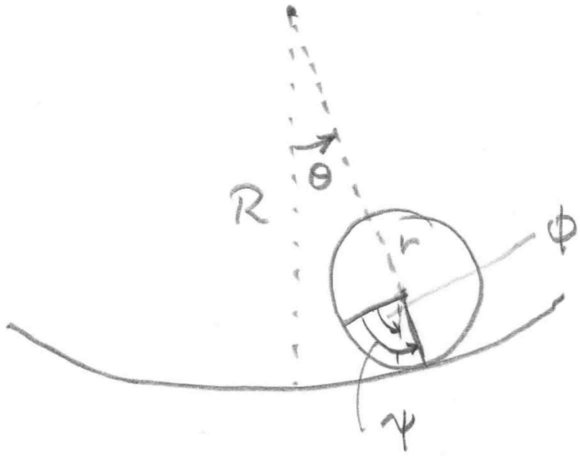


Ball rolls without slipping

$\rightarrow$ arc length along bowl

= arc length around ball

$\rightarrow R\theta = r\psi$



Absolute ball rotation

$\phi = \psi - \theta = \frac{R}{r}\theta - \theta$

$\phi = \left(\frac{R-r}{r}\right)\theta$

θ = rotation along bowl surface

ψ = rotation of ball relative to bowl surface

ϕ = absolute ball rotation $= \psi - \theta$

(a) Free-body diagram

Take moments about contact point to eliminate contact forces:

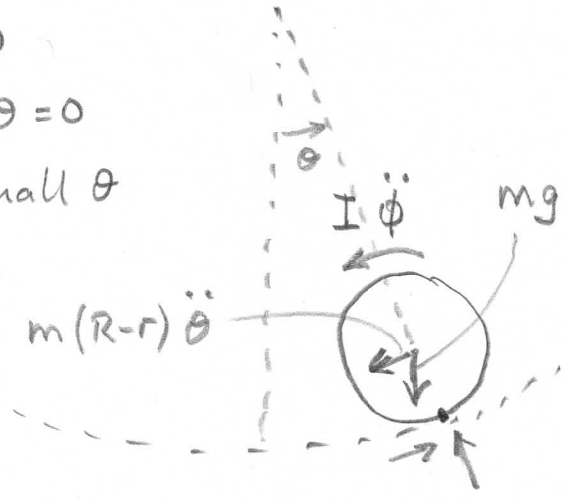
$m(R-r)\ddot\theta \cdot r + I\ddot\phi + mgr\sin\theta = 0$

$m(R-r)\ddot\theta r + \frac{2}{5}mr^2\left(\frac{R-r}{r}\right)\ddot\theta + mgr\theta = 0$ for small θ

$\div\, mr$

$\frac{7}{5}(R-r)\ddot\theta + g\theta = 0$

$\omega^2 = \frac{5}{7}\frac{g}{R-r}$



(b) Lagrange's Equation

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}}\right) - \frac{\partial T}{\partial q} + \frac{\partial R}{\partial \dot{q}} + \frac{\partial V}{\partial q} = Q$$

Here $q = \theta$, $Q = 0$ $R = 0$

$$T = \tfrac{1}{2} m \left((R-r)\dot{\theta}\right)^2 + \tfrac{1}{2} I \dot{\psi}^2$$

$$= \tfrac{1}{2} m (R-r)^2 \dot{\theta}^2 + \tfrac{1}{2} \tfrac{2}{5} m r^2 . \left(\frac{R-r}{r}\right)^2 \dot{\theta}^2$$

$$= \tfrac{1}{2} \cdot \tfrac{7}{5} m (R-r)^2 \dot{\theta}^2$$

$$V = mg(R-r)(1-\cos\theta) = mg(R-r)\left(1-\left(1-\frac{\theta^2}{2}+\ldots\right)\right)$$

$$= \tfrac{1}{2} m (R-r) g \theta^2 \longrightarrow \text{sub. in Lagrange's Eqn.}$$

$$\frac{d}{dt}\left(\frac{\partial}{\partial \dot{\theta}}\left(\tfrac{1}{2}\cdot\tfrac{7}{5} m (R-r)^2 \dot{\theta}^2\right)\right) - \frac{\partial}{\partial \theta}\left(\tfrac{1}{2}\cdot\tfrac{7}{5} m (R-r)^2 \dot{\theta}^2\right)$$

$$+ \frac{\partial}{\partial \dot{\theta}}(0) + \frac{\partial}{\partial \theta}\left(\tfrac{1}{2} m (R-r) g \theta^2\right) = 0$$

$$= \tfrac{7}{5} m (R-r)^2 \ddot{\theta} - 0 + 0 + m(R-r) g \theta = 0$$

$$\div\, m(R-r) \longrightarrow \underline{\tfrac{7}{5}(R-r)\ddot{\theta} + g\theta = 0}$$

same as before!