MECH468 : Modern Control Engineering
MECH509 : Controls

L11 : Minimum energy control

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Zoom lecture to be recorded and posted on Canvas

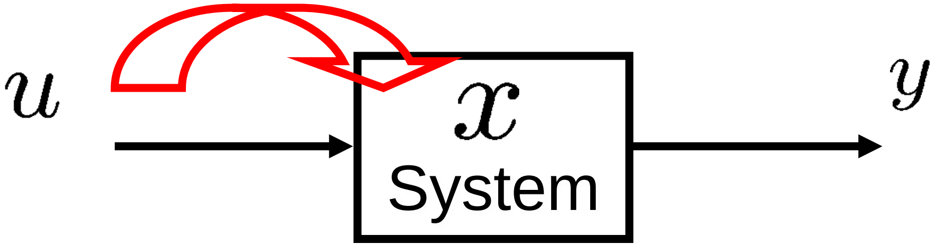
Course plan

Topics	CT	DT
Modeling	✓	✓
Stability	✓	✓
→ Controllability/observability		
Realization		
State feedback/observer		
LQR/Kalman filter		

Review

- Controllability

$$\dot{x}(t) = Ax(t) + Bu(t)$$



- (A,B) is *controllable* if, for *any* x_0 and *any* x_1, there is $u(t)$ which transfers from x_0 to x_1 in a (any) specified finite time.

- Nec. and suf. condition for controllability
  - ***Controllability matrix*** has full row rank, i.e.,

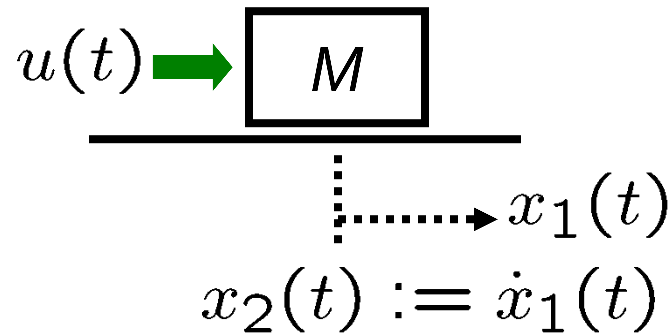
$$\mathcal{C} := \left[B, AB, \cdots, A^{n-1}B \right] \qquad \text{rank}\,\mathcal{C} = n$$

Today's topics

- If *(A,B)* is controllable
  - Minimum energy control
- If *(A,B)* is not controllable
  - Decomposition into:
    - controllable part
    - uncontrollable part

Example

- Consider a controllable system:



$$\frac{d}{dt}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1/M \end{bmatrix} u(t)$$

- Suppose that we want to transfer states as

$$x(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightarrow x(1) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Start time

Final time

At rest at position 0

At rest at position 1

Intuitively, there are many inputs achieving this objective!
What is the minimum energy control input?

Necessary & sufficient conditions for controllability

- Controllability matrix has full row rank, i.e.,

$$\mathcal{C} := \left[B, AB, \cdots, A^{n-1}B\right] \qquad \text{rank}\mathcal{C} = n$$

- Equivalently, the following *controllability Grammian* is nonsingular (in fact, positive definite) for any *t>0*:

$$W_c(t) := \int_0^t e^{A\tau} B B^T e^{A^T\tau} d\tau$$

Note: The proof is in a note posted on Canvas.

Minimum energy control

- If (A,B) is controllable, then there exist, in general, many inputs that transfer
$$x(0) = x_0 \to x(t_f) = x_f$$
where x_0, x_f, t_f are specified.
- Find the input with *minimum energy*, i.e., solve
$$\min_{u(\cdot)} \int_0^{t_f} u^T(t)u(t)dt \quad \text{subj. to} \begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ x(0) = x_0, \ x(t_f) = x_f \end{cases}$$

Ans. $$\boxed{u^*(t) = B^T e^{A^T(t_f - t)} W_c^{-1}(t_f) \left[x_f - e^{At_f} x_0 \right]}$$

(Proof is posted on Canvas.)

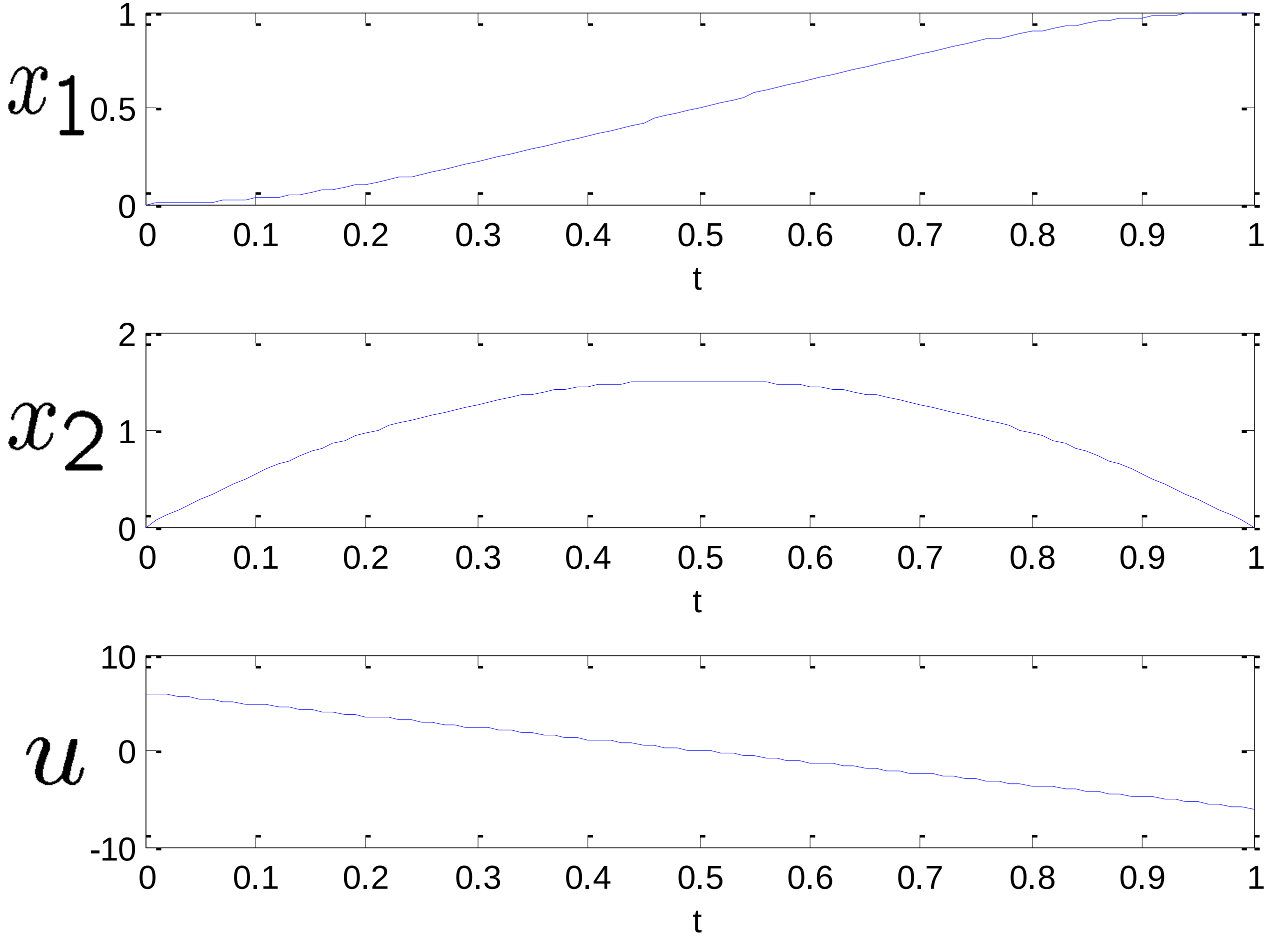
Example revisited

- System: $$\frac{d}{dt}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1/M \end{bmatrix} u(t) \qquad (M=1)$$
- Initial & final conditions: $$x(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightarrow x(1) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$
- Minimum energy control:

$$\begin{aligned} u^*(t) &= B^T e^{A^T(t_f - t)} W_c^{-1}(t_f)\left[x_f - e^{At_f} x_0\right] \\ &= \cdots \\ &= 6 - 12t \end{aligned}$$

$$\begin{cases} e^{At} = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \\ \begin{aligned} W_c(t) &= \int_0^t e^{A\tau} B B^T e^{A^T\tau} d\tau \\ &= \int_0^t \begin{bmatrix} \tau \\ 1 \end{bmatrix} \begin{bmatrix} \tau & 1 \end{bmatrix} d\tau \\ &= \begin{bmatrix} t^3/3 & t^2/2 \\ t^2/2 & t \end{bmatrix} \end{aligned} \end{cases}$$

Example (cont'd)



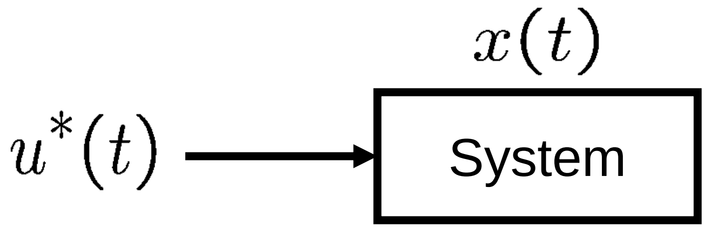
position

velocity

acceleration

Remarks

- Minimum energy control is an open-loop control.



- Open-loop control works only when there is no significant uncertainty in the real system and its environment, and therefore, not practical!
- Later in this course, we will learn optimal feedback controller design.

Another remark

- Controllability matrix of the example

$$\mathcal{C} = \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} 0 & 1/M \\ 1/M & 0 \end{bmatrix}$$

  - As *M* becomes larger and larger, the (1,2) and (2,1)-entries become smaller and smaller.
  - When the controllability matrix is almost losing its rank, it means that the system is “almost uncontrollable” or “weakly controllable”.
  - Physically, in such cases, we need input with huge amplitude to control states.

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Coordinate transformation

- System $\begin{cases} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t) \end{cases}$

- Coordinate transformation

$$z(t) := Tx(t) \qquad T\text{: any nonsingular matrix}$$

$$\Rightarrow \begin{cases} \dot{z}(t) &= TAT^{-1}z(t) + TBu(t) \\ y(t) &= CT^{-1}z(t) + Du(t) \end{cases}$$

  - Does not change transfer function, stability, controllability, observability. (Next slide)
  - Can be used to clarify the structure, and to improve numerical property.

Some proofs

- Transfer function is not affected by *T*.

$$(CT^{-1})(sI-TAT^{-1})^{-1}(TB)+D=\cdots=C(sI-A)^{-1}B+D$$

- Stability is not affected by *T*.

$$\begin{aligned}\det(\lambda I-A) &= \det(T^{-1}(\lambda I-TAT^{-1})T)\\ &= (\det T)^{-1}\det(\lambda I-TAT^{-1})\det T\\ &= \det(\lambda I-TAT^{-1})\end{aligned}$$

- Controllability is not affected by *T*.

$$\left[TB,(TAT^{-1})TB,\cdots,(TAT^{-1})^{n-1}TB\right]=T\left[B,AB,\cdots,A^{n-1}B\right]$$

Decomposition for controllability

- If (A,B) is not controllable with $\text{rank}\mathcal{C} = m < n$ then there exists a coordinate transformation (i.e., nonsingular T) that *decomposes* states into controllable part and the uncontrollable part:

$$\left.\begin{array}{l}\dot{x}(t) = Ax(t) + Bu(t) \\ \begin{bmatrix} z_c(t) \\ z_{\bar{c}}(t) \end{bmatrix} := Tx(t)\end{array}\right\} \longrightarrow \begin{bmatrix} \dot{z}_c(t) \\ \dot{z}_{\bar{c}}(t) \end{bmatrix} = \underbrace{\begin{bmatrix} A_c & A_{12} \\ 0 & A_{\bar{c}} \end{bmatrix}}_{TAT^{-1}} \begin{bmatrix} z_c(t) \\ z_{\bar{c}}(t) \end{bmatrix} + \underbrace{\begin{bmatrix} B_c \\ 0 \end{bmatrix}}_{TB} u(t)$$

$$\boxed{A_c \in \mathbb{R}^{m \times m}} \qquad \boxed{(A_c, B_c) \text{ is controllable}}$$

Summary

- Controllability
  - If controllable, minimum energy control
  - If not controllable, decomposition
- Next, decomposition for controllability