MECH468 : Modern Control Engineering
MECH509 : Controls

L16 : Realization
Controllable canonical form

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Zoom lecture to be recorded and posted on Canvas

Course plan

Topics	CT	DT
Modeling	✓	✓
Stability	✓	✓
Controllability/observability	✓	✓
Realization	→	
State feedback/observer		
LQR/Kalman filter		

Motivation

- Now, we have learned two ways to describe LTI systems, i.e., TF and SS models.

TF: $y(s) = G(s)u(s)$ *SS:* $\begin{cases} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t) \end{cases}$

- To use analysis & design techniques for SS models, one may want to transform a TF model to an equivalent SS model. How?
- More generally, what is the relationship between two models?

From SS to TF (review)

- CT LTI SS model $\begin{cases} \dfrac{dx(t)}{dt} &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t) \end{cases}$

- Laplace transform with *x(0)=0*

$$\begin{cases} sX(s) - \cancel{x(0)} &= AX(s) + BU(s) \\ Y(s) &= CX(s) + DU(s) \end{cases}$$

$$\Rightarrow \begin{cases} X(s) &= (sI - A)^{-1}BU(s) \\ Y(s) &= CX(s) + DU(s) \end{cases}$$

Memorize this!

$$\Rightarrow Y(s) = \underbrace{\boxed{\left\{C(sI - A)^{-1}B + D\right\}}}_{=:G(s)} U(s)$$

Realization: From TF to SS

- Given a rational proper transfer matrix $G(s)$ find matrices (*A*,*B*,*C*,*D*) s.t.

$$G(s) = C(sI - A)^{-1}B + D$$

  - Rationality: (polynomial)/(polynomial)

    Rational $\frac{s+1}{s+2}$ Non-rational e^{-s}

  - Properness: deg(num)<=deg(den)

    Proper $\frac{s+1}{s+2}, \frac{1}{s+1}$ Non-proper $s+1, \frac{s^2+2s+3}{s+1}$

Remarks

- For one SS model, there is a unique TF.
- For one TF model, there are infinitely many SS!

$$\begin{tikzcd} (A,B,C,D) \arrow[r, bend left, "\text{unique}"] & G(s) \arrow[l, bend left, "\text{infinitely many}"] \end{tikzcd}$$

$$\left(\begin{array}{c} \text{Coordinate transformation } T \\ \text{Order difference} \end{array} \right)$$

- After realization, check the correctness by recovering the original transfer matrix.

First step for realization

- Always extract *D*-matrix first!

$$G(s) = \underbrace{G(\infty)}_{\text{Constant}} + \underbrace{G_{sp}(s)}_{\text{Strictly proper}}$$

$$\begin{aligned} G(\infty) &= D \\ G_{sp}(s) &= C(sI - A)^{-1}B \end{aligned}$$

$$\deg(\text{num}) < \deg(\text{den})$$

- Ex $\begin{bmatrix} \frac{s+2}{s+1} & \frac{1}{s+2} \\ \frac{s}{s/2+1} & 1 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}}_{\text{Constant}} + \underbrace{\begin{bmatrix} \frac{1}{s+1} & \frac{1}{s+2} \\ \frac{-4}{s+2} & 0 \end{bmatrix}}_{\text{Strictly proper}}$

- After extracting *D*, find (*A*,*B*,*C*) s.t.

$$G_{sp}(s) = C(sI - A)^{-1}B$$

Controllable canonical form

- SISO example $G(s) = \dfrac{n_1 s^2 + n_2 s + n_3}{s^3 + \alpha_1 s^2 + \alpha_2 s + \alpha_3}, \quad n_i \in \mathbb{R}$

$$\boxed{\begin{cases} \dot{x}(t) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\alpha_3 & -\alpha_2 & -\alpha_1 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(t) \\ y(t) = \begin{bmatrix} n_3 & n_2 & n_1 \end{bmatrix} x(t) \end{cases}}$$

or equivalently
$$\begin{cases} \dot{x}(t) = \begin{bmatrix} -a_1 & -a_2 & -a_3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u(t) \\ y(t) = \begin{bmatrix} n_1 & n_2 & n_3 \end{bmatrix} x(t) \end{cases}$$

Companion matrix

- The following form (and its transpose) of a matrix is called *companion matrix (form):*

$$A := \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \cdots & 1 \\ -\alpha_n & -\alpha_{n-1} & \cdots & -\alpha_2 & -\alpha_1 \end{bmatrix} \in \mathbb{R}^{n\times n}$$

- Important property of a companion matrix

$$\det(sI - A) = s^n + \alpha_1 s^{n-1} + \cdots + \alpha_{n-1} s + \alpha_n$$

SISO examples

- Ex.1 $$G(s) = \frac{5s^2 + 3s + 2}{s^3 + 11s^2 + 14s + 6}$$

- Ex.2 $$G(s) = \frac{4s^2 + 3s + 2}{2s^2 + 5s + 6}$$

Controllable canonical form for MIMO cases

$$G(s) = \frac{N_1 s^{r-1} + N_2 s^{r-2} + \cdots + N_{r-1} s + N_r}{\underbrace{s^r + \alpha_1 s^{r-1} + \cdots + \alpha_{r-1} s + \alpha_r}_{\text{Least common denominator}}}, \quad N_i \in \mathbb{R}^{q \times p}$$

Least common denominator

$$\dot{x} = \overbrace{\begin{bmatrix} 0 & I_p & 0 & \cdots & 0 \\ 0 & 0 & I_p & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & I_p \\ -\alpha_r I_p & -\alpha_{r-1} I_p & \cdots & -\alpha_2 I_p & -\alpha_1 I_p \end{bmatrix}}^{A \in \mathbb{R}^{rp \times rp}} x + \underbrace{\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ I_p \end{bmatrix}}_{B \in \mathbb{R}^{rp \times p}} u$$

$$y = \underbrace{\begin{bmatrix} N_r & N_{r-1} & \cdots & N_2 & N_1 \end{bmatrix}}_{C \in \mathbb{R}^{q \times rp}} x$$

Least common denominator (LCD)

- Least common multiple of denominators of a set of fractions
  - Examples

$$\frac{1}{2}+\frac{2}{3}=\frac{3}{6}+\frac{4}{6}$$

LCD of 2 and 3

$$\frac{1}{s+1}+\frac{1}{s+2}=\frac{s+2}{(s+1)(s+2)}+\frac{s+1}{(s+1)(s+2)}$$

$$\frac{1}{s(s+1)}+\frac{1}{s^2(s+2)}=\frac{(*)}{s^2(s+1)(s+2)}$$

MIMO example

- Transfer matrix $$G(s) = \begin{bmatrix} \frac{1}{s^2+4s+3} & \frac{1}{s+3} \end{bmatrix}$$ $$= \frac{1}{s^2+4s+3} \begin{bmatrix} 1 & s+1 \end{bmatrix}$$ $$= \frac{1}{s^2+4s+3} \left\{ \begin{bmatrix} 0 & 1 \end{bmatrix} s + \begin{bmatrix} 1 & 1 \end{bmatrix} \right\}$$

$$\Rightarrow \left\{ \begin{aligned} \dot{x} &= \left[\begin{array}{cc|cc} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \hline -3 & 0 & -4 & 0 \\ 0 & -3 & 0 & -4 \end{array} \right] x + \left[\begin{array}{cc} 0 & 0 \\ 0 & 0 \\ \hline 1 & 0 \\ 0 & 1 \end{array} \right] u \\ y &= \left[\begin{array}{cc|cc} 1 & 1 & 0 & 1 \end{array} \right] x \end{aligned} \right.$$

Note that the size of *A*-matrix is four.

Remark

- Controllable canonical realization is always controllable (but not always observable). Why?

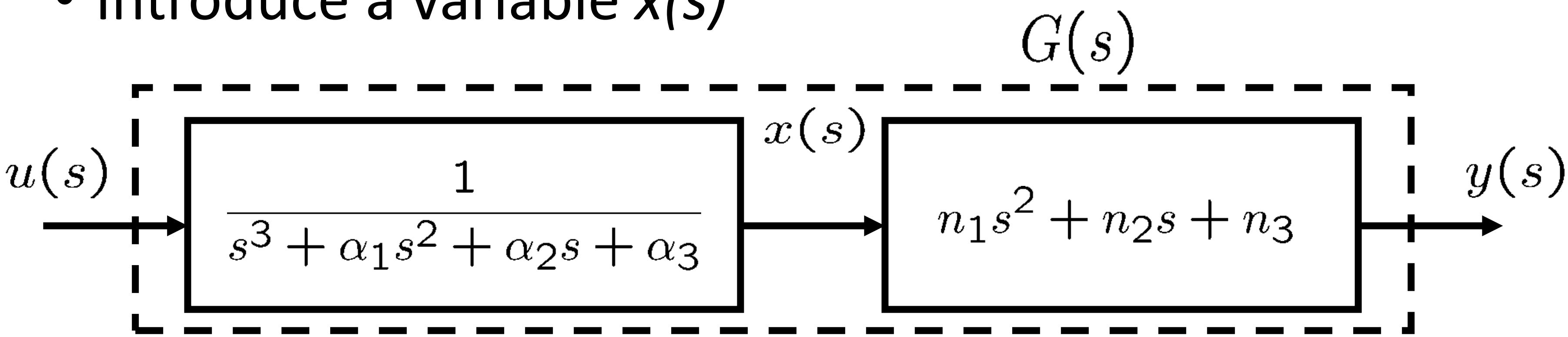
$$\dot{x} = \begin{bmatrix} 0 & I_p & 0 & \cdots & 0 \\ 0 & 0 & I_p & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & I_p \\ -\alpha_r I_p & -\alpha_{r-1} I_p & \cdots & -\alpha_2 I_p & -\alpha_1 I_p \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ I_p \end{bmatrix} u$$

$\Rightarrow$ $\mathcal{C} =$

Derivation of controllable canonical form

- TF $$G(s) = \frac{n_1 s^2 + n_2 s + n_3}{s^3 + \alpha_1 s^2 + \alpha_2 s + \alpha_3}, \quad n_i \in \mathbb{R}$$

- Introduce a variable *x(s)*



- We have $$(s^3 + \alpha_1 s^2 + \alpha_2 s + \alpha_3)x(s) = u(s)$$ $$y(s) = (n_1 s^2 + n_2 s + n_3)x(s)$$

Derivation (cont'd)

- Introduce state variables $\begin{cases} x_1(s) := x(s) \\ x_2(s) := sx(s) \\ x_3(s) := s^2 x(s) \end{cases}$
- Then,

$$(s^3 + \alpha_1 s^2 + \alpha_2 s + \alpha_3)x(s) = u(s)$$

$$\Rightarrow \quad sx_3(s) = -\alpha_1 x_3(s) - \alpha_2 x_2(s) - \alpha_3 x_1(s) + u(s)$$

$$y(s) = (n_1 s^2 + n_2 s + n_3)x(s)$$

$$\Rightarrow \quad y(s) = n_1 x_3(s) + n_2 x_2(s) + n_3 x_1(s)$$

- By inverse Laplace transform, done!

Summary

- Realization (In Matlab, ss.m)
  - Controllable canonical form
- Next,
  - Observable canonical form
  - Connections of state-space models
    - Parallel connection
    - Series connection

Project for MECH509

- MECH509 students should start thinking about their projects now!
- Either individual work or group work (up to 2 students)
- Requirements for an eligible project
  - The theory that you learned in this course is usable.
  - The project should be a control or estimation problem.
    - What are inputs (actuators), outputs (sensors), disturbances, and control/estimation goals?
  - A state-space model with all the parameters should be available/obtainable.
  - The project should not be just a “copy-and-paste” of a problem in textbooks.
  - Obtain an approval of your project proposal (written in a given template) from the instructor by mid March.