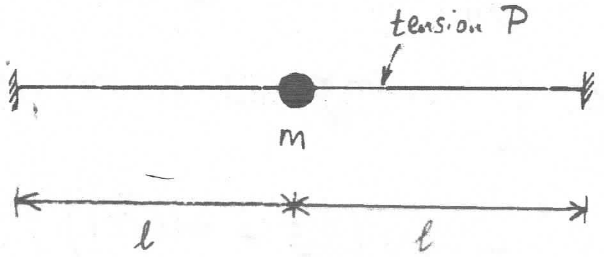
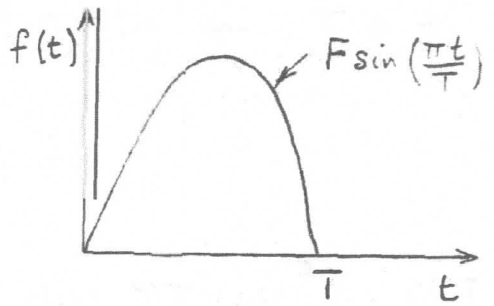
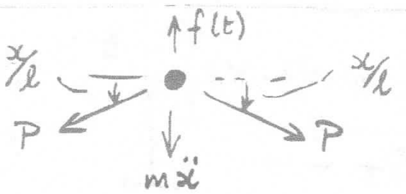
MECH463 -- Tutorial 11

1. A mass m is secured at the centre of a tight string of length 2ℓ. The tension in the string, P, is not significantly altered by the small lateral vibrations of the mass. A half-sinewave pulse force, f(t), shown in the diagram is applied to the mass. Calculate the natural frequency of this 1-DOF system, and its response to the applied force.







Let x = transverse displacement

→ angle of string = x/ℓ

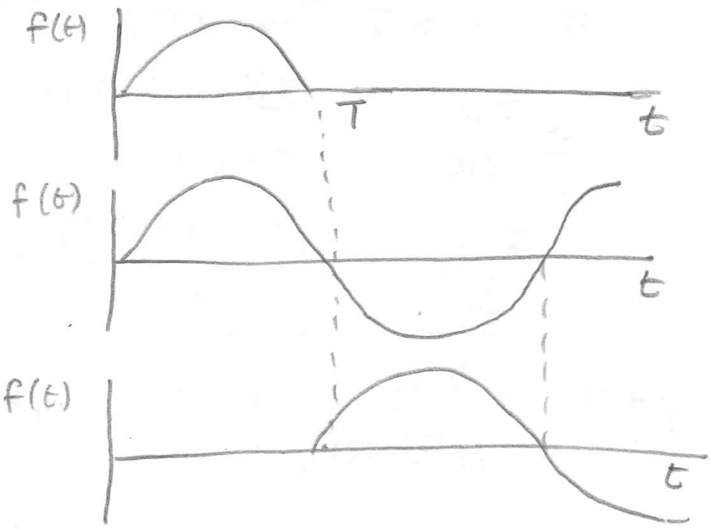
Vertical force balance → $m\ddot{x} + 2P\sin(x/\ell) = f(t)$

Assume x/ℓ is small and P remains constant → $\sin(x/\ell) = x/\ell$

$$\rightarrow \quad m\ddot{x} + 2\frac{P}{\ell}x = f(t)$$

or $m\ddot{x} + kx = f(t)$ where $k = \frac{2P}{\ell}$ → $\boxed{\omega^2 = \frac{k}{m} = \frac{2P}{m\ell}}$

We can find the solution for the half sinewave pulse by summing the solutions for a continuous sine wave and the same sine wave shifted by half a period.



For a continuous sine wave: $m\ddot{x} + kx = F \sin\left(\frac{\pi t}{T}\right)$

Complementary solution is $x_c = A\cos\omega t - B\sin\omega t$

For particular solution try $x_p = C\sin\left(\frac{\pi t}{T}\right)$

$$\rightarrow \left(-\left(\frac{\pi}{T}\right)^2 m + k\right) C\sin\left(\frac{\pi t}{T}\right) = F\sin\left(\frac{\pi t}{T}\right)$$

$$\rightarrow C = \frac{F}{-\left(\frac{\pi}{T}\right)^2 m + k} = \frac{F}{k\left(-\left(\frac{\pi}{T}\right)^2 \frac{m}{k} + 1\right)} = \frac{-F\omega^2}{k\left(\left(\frac{\pi}{T}\right)^2 - \omega^2\right)}$$

General solution $x = x_c + x_p = A\cos\omega t - B\sin\omega t + C\sin\left(\frac{\pi t}{T}\right)$

Initial conditions: $x(0) = 0$ $\quad \dot{x}(0) = 0$

$$x(0) = A - 0 + 0 = 0 \quad \rightarrow A = 0$$

$$\dot{x}(t) = -\omega A\sin\omega t - \omega B\cos\omega t + \frac{\pi}{T} C\cos\left(\frac{\pi t}{T}\right)$$

$$\dot{x}(0) = 0 - \omega B + \frac{\pi}{T} C = 0 \quad \rightarrow B = \frac{\pi}{T}\frac{C}{\omega}$$

$$\rightarrow x = \frac{-F\omega^2}{k\left(\left(\frac{\pi}{T}\right)^2 - \omega^2\right)} \left(-\frac{\pi}{T}\cdot\frac{1}{\omega}\sin\omega t + \sin\left(\frac{\pi t}{T}\right)\right)$$

$$x = \frac{F\omega}{k\left(\left(\frac{\pi}{T}\right)^2 - \omega^2\right)} \left(\frac{\pi}{T}\sin\omega t - \omega\sin\left(\frac{\pi t}{T}\right)\right)$$

Create shifted solution by replacing $t \rightarrow t - T$

Pulse solution is continuous solution plus shifted solution

$$x(t) = \frac{F\omega}{k\left(\left(\frac{\pi}{T}\right)^2 - \omega^2\right)} \left(\frac{\pi}{T}\sin\omega t - \omega\sin\left(\frac{\pi t}{T}\right) + \frac{\pi}{T}\sin\omega(t-T) - \omega\sin\left(\frac{\pi(t-T)}{T}\right)\right)$$

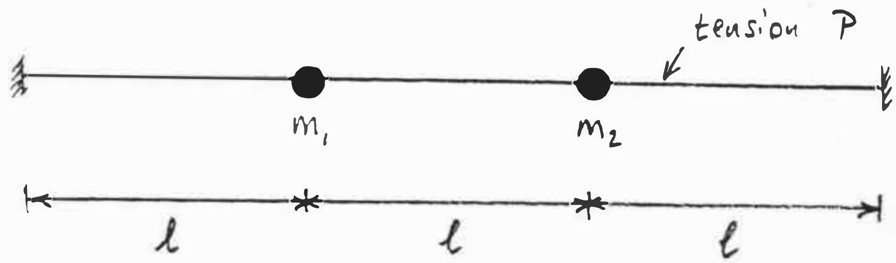
Note that $\sin\left(\frac{\pi(t-T)}{T}\right) = \sin\left(\frac{\pi t}{T} - \pi\right) = -\sin\left(\frac{\pi t}{T}\right)$

$$\rightarrow x(t) = \frac{F\omega}{k\left(\left(\frac{\pi}{T}\right)^2 - \omega^2\right)} \left(\frac{\pi}{T}\sin\omega t + \frac{\pi}{T}\sin\omega(t-T)\right)$$

for $t > T$

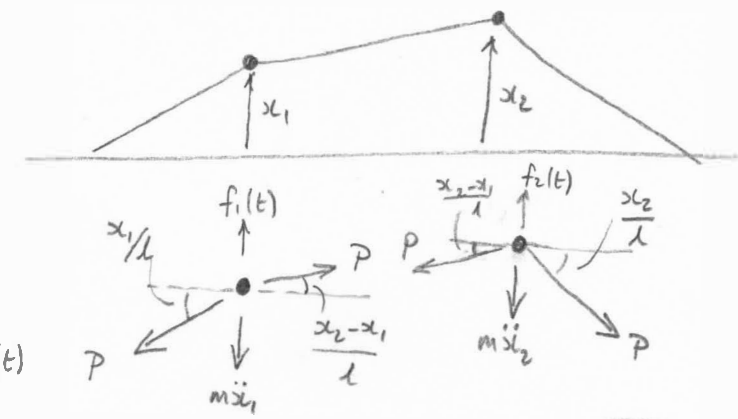
2. Two equal masses m are secured at the one-third points of a tight string of length 3ℓ. As in question 1, the tension in the string, P, is not significantly altered by small lateral vibrations of the masses. The same half-sinewave pulse is applied to m_1 only. Calculate the natural frequencies and mode shapes of this 2-DOF system, and its response to the applied force.

$m_1 = m_2 = m$



Let x_1 and x_2 be the transverse displacements of the masses m_1 and m_2

Assume that the displacements and string angles are small, and that P remains constant.



Vertical force balances:

$$m\ddot{x}_1 + P\sin\left(\frac{x_1}{l}\right) - P\sin\left(\frac{x_2-x_1}{l}\right) = f_1(t)$$

$$m\ddot{x}_2 + P\sin\left(\frac{x_2-x_1}{l}\right) + P\sin\left(\frac{x_2}{l}\right) = f_2(t)$$

In matrix form (for small angles) and put $k = P/l$ ⟵ different from Q.2.

$$\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} 2k & -k \\ -k & 2k \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} f_1(t) \\ f_2(t) \end{bmatrix} \rightarrow \underset{\sim}{M}\,\underline{\ddot{x}} + \underset{\sim}{K}\,\underline{x} = \underline{f}(t)$$

For natural frequency determination, we consider the free vibration case, i.e. $f_1(t) = f_2(t) = 0$. Try a solution of form $\underline{x} = \underline{X}\cos\omega t$

$$\rightarrow \begin{bmatrix} 2k - \omega^2 m & -k \\ -k & 2k - \omega^2 m \end{bmatrix}\begin{bmatrix} X_1 \\ X_2 \end{bmatrix}\cos\omega t = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

This must be true for all time t $\rightarrow \cos\omega t \neq 0$. For a non-trivial solution

$$\rightarrow \begin{vmatrix} 2k - \omega^2 m & -k \\ -k & 2k - \omega^2 m \end{vmatrix} = 0$$

$$\rightarrow (2k - \omega^2 m)(2k - \omega^2 m) - k^2 = 0$$

$$m^2\omega^4 - 4mk\omega^2 + 3k^2 = 0 \longrightarrow$$

$$\boxed{\omega_1^2 = \frac{k}{m} = \frac{P}{ml} \qquad \omega_2^2 = \frac{3k}{m} = \frac{3P}{ml}}$$

Let $\underline{u} = \begin{bmatrix} 1 \\ u \end{bmatrix}$ be the mode shape

$$\rightarrow \begin{bmatrix} 2k - \omega^2 m & -k \\ -k & 2k - \omega^2 m \end{bmatrix} \begin{bmatrix} 1 \\ u \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

From the first equation $\rightarrow 2k - \omega^2 m - uk = 0$

$$\rightarrow u = 2 - \frac{m}{k}\omega^2 \rightarrow \boxed{u_1 = 1, \quad u_2 = -1} \qquad \underline{u}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \ \underline{u}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

The modal matrix $\underset{\sim}{U} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

Define principal coordinates $\underline{p}$, such that $\underline{x} = \underset{\sim}{U}\,\underline{p}$

Pre-multiply equation of motion by $\underset{\sim}{U}^T$, and substitute $\underline{p}$

$$\rightarrow \underset{\sim}{U}^T \underset{\sim}{M} \underset{\sim}{U} \ddot{\underline{p}} + \underset{\sim}{U}^T \underset{\sim}{K} \underset{\sim}{U} \underline{p} = \underset{\sim}{U}^T \underline{f}(t) \rightarrow \underset{\sim}{M}^* \ddot{\underline{p}} + \underset{\sim}{K}^* \underline{p} = \underset{\sim}{U}^T \underline{f}(t)$$

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} \ddot{p}_1 \\ \ddot{p}_2 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 2k & -k \\ -k & 2k \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} f_1(t) \\ f_2(t) \end{bmatrix}$$

$$= \begin{bmatrix} 2m & 0 \\ 0 & 2m \end{bmatrix}\begin{bmatrix} \ddot{p}_1 \\ \ddot{p}_2 \end{bmatrix} + \begin{bmatrix} 2k & 0 \\ 0 & 6k \end{bmatrix}\begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = \begin{bmatrix} f_1(t) + f_2(t) \\ f_1(t) - f_2(t) \end{bmatrix}$$

These are two uncoupled equations $\rightarrow$

$$2m\ddot{p}_1 + 2k\,p_1 = f_1(t) + f_2(t)$$
$$2m\ddot{p}_2 + 6k\,p_2 = f_1(t) - f_2(t)$$

Here, $f_2(t) = 0$ and $f_1(t)$ is the same as in question 2.

Both equations for p_1 and p_2 have exactly analogous forms to question 2, with only minor differences in the stiffness term k. Hence, we can use the solution from Q2 to determine p_1 and p_2. Then find x_1 and x_2 from

$$\underline{x} = \underset{\sim}{U}\,\underline{p} \rightarrow x_1 = p_1 + p_2, \quad x_2 = p_1 - p_2$$

For $t < T$

$$x_1 = p_1 + p_2 = \frac{F\omega_1}{2k\left(\left(\frac{\pi}{T}\right)^2 - \omega_1^2\right)}\left(\frac{\pi}{T}\sin\omega_1 t - \omega_1 \sin\left(\frac{\pi t}{T}\right)\right) + \frac{F\omega_2}{6k\left(\left(\frac{\pi}{T}\right)^2 - \omega_2^2\right)}\left(\frac{\pi}{T}\sin\omega_2 t - \omega_2 \sin\left(\frac{\pi t}{T}\right)\right)$$

$x_2 = p_1 - p_2 =$ - - - - - - - - - - $-$ - - - - - - -

minus sign

For $t > T$

$$x_1 = p_1 + p_2 = \frac{F\omega_1}{2k\left(\left(\frac{\pi}{T}\right)^2 - \omega_1^2\right)}\left(\frac{\pi}{T}\sin\omega_1 t - \frac{\pi}{T}\sin\omega(t-T)\right) + \frac{F\omega_2}{6k\left(\left(\frac{\pi}{T}\right)^2 - \omega_2^2\right)}\left(\frac{\pi}{T}\sin\omega_2 t - \frac{\pi}{T}\sin\omega(t-T)\right)$$

$x_2 = p_1 - p_2 =$ - - - - - - - - - - $-$ - - - - - - -

minus sign