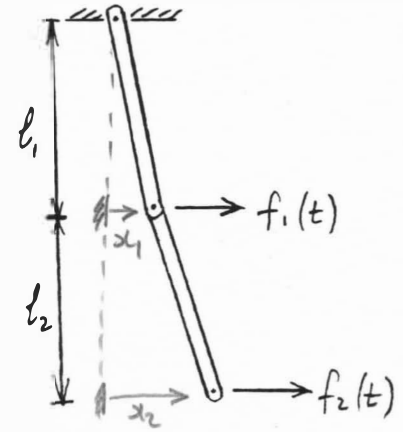
1. A double compound pendulum consists of two uniform rods of lengths ℓ_1 and ℓ_2 and masses m_1 and m_2. Horizontal forces $f_1(t)$ and $f_2(t)$ act at the lower ends of the rods, as shown. Assume small motions.

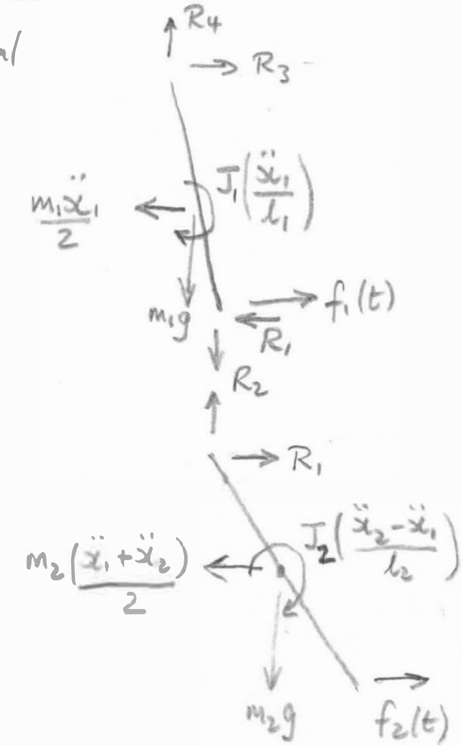
   (a) Draw the free body diagrams of the system and then determine the equations of motion. Express your equations in symmetric matrix form.

   (b) Determine the kinetic and potential energies of the system. Use Lagrange's equations to formulate the matrix equation of motion. Verify that the matrices are symmetric and are the same as found in (a).



Choose coordinates x_1 and x_2 as the lateral displacements of the lower ends of the rods.

Assume small displacements.



(a) Free body diagrams.

We have to be a little careful here to avoid having to evaluate the reaction forces R_1, R_2, R_3 and R_4. (They can be evaluated readily enough by horizontal and vertical equilibrium, but they are not of any real interest to us.)

Consider the lower rod, and take moments about its upper end

$$J_2\left(\frac{\ddot{x}_2-\ddot{x}_1}{l_2}\right) + m_2 g\left(\frac{x_2-x_1}{2}\right) + m_2\frac{\ddot{x}_1+\ddot{x}_2}{2}\cdot\frac{l_2}{2} - f_2 l_2 = 0$$

$$\rightarrow \frac{m_2 l_2}{6}\ddot{x}_1 + \frac{m_2 l_2}{3}\ddot{x}_2 - \frac{m_2 g}{2}x_1 + \frac{m_2 g}{2}x_2 = f_2 l_2$$

putting $J_2 = \frac{1}{12}m_2 l_2^2$

$$\rightarrow \frac{m_2}{6}\ddot{x}_1 + \frac{m_2}{3}\ddot{x}_2 - \frac{m_2 g}{2l_2}x_1 + \frac{m_2 g}{2l_2}x_2 = f_2$$

dividing by l_2

To avoid having to work with R_1 and R_2, consider <u>both</u> rods together and take moments about the upper end. R_1 and R_2 then become "internal" forces, and do not appear explicitly in the calculation.

$$J_1 \frac{\ddot{x}_1}{l_1} + J_2 \frac{\ddot{x}_2 - \ddot{x}_1}{l_2} + \frac{m_1 \ddot{x}_1}{2} \cdot \frac{l_1}{2} + m_2 \frac{(\ddot{x}_1 + \ddot{x}_2)}{2} \cdot \left(l_1 + \frac{l_2}{2}\right)$$

$$+ m_1 g \frac{x_1}{2} + m_2 g \left(\frac{x_1 + x_2}{2}\right) - f_1 \cdot l_1 - f_2 \cdot (l_1 + l_2) = 0$$

$$\rightarrow \tfrac{1}{12} m_1 l_1 \ddot{x}_1 + \tfrac{1}{12} m_2 l_2 \ddot{x}_2 - \tfrac{1}{12} m_2 l_2 \ddot{x}_1 + \tfrac{1}{4} m_1 l_1 \ddot{x}_1 + \tfrac{1}{2} m_2 l_1 \ddot{x}_1 + \tfrac{1}{4} m_2 l_2 \ddot{x}_1$$

$$+ \tfrac{1}{2} m_2 l_1 \ddot{x}_2 + \tfrac{1}{4} m_2 l_2 \ddot{x}_2 + \frac{m_1 g}{2} x_1 + \frac{m_2 g}{2} x_1 + \frac{m_2 g}{2} x_2$$

$$= f_1 l_1 + f_2 (l_1 + l_2)$$

$$= \left(\tfrac{1}{3} m_1 l_1 + \tfrac{1}{6} m_2 l_2 + \tfrac{1}{2} m_2 l_1\right) \ddot{x}_1 + \left(\tfrac{1}{2} m_2 l_1 + \tfrac{1}{3} m_2 l_2\right) \ddot{x}_2$$

$$+ \left(\frac{m_1 g}{2} + \frac{m_2 g}{2}\right) x_1 + \frac{m_2 g}{2} x_2 = f_1 l_1 + f_2 (l_1 + l_2)$$

<u>Ugly!</u> We would really prefer to work with a simpler equation, say with only f_1 on the RHS. To do this, multiply the first equation by $l_1 + l_2$ and subtract:

$$\rightarrow \left(\tfrac{1}{3} m_1 l_1 + \tfrac{1}{3} m_2 l_1\right) \ddot{x}_1 + \left(\tfrac{1}{6} m_2 l_1\right) \ddot{x}_2 + \left(\frac{m_1 g}{2} + m_2 g + \frac{m_2 g l_1}{2 l_2}\right) x_1$$

$$- \left(\frac{m_2 g l_1}{2 l_2}\right) x_2 = f_1 l_1$$

Divide by l_1:

$$\left(\tfrac{1}{3} m_1 + \tfrac{1}{3} m_2\right) \ddot{x}_1 + \tfrac{1}{6} m_2 \ddot{x}_2 + \left(\frac{m_1 g}{2 l_1} + \frac{m_2 g}{l_1} + \frac{m_2 g}{2 l_2}\right) x_1 - \left(\frac{m_2 g}{2 l_2}\right) x_2 = f_1$$

In matrix form:

$$\begin{bmatrix} \frac{m_1}{3}+\frac{m_2}{3} & \frac{m_2}{6} \\ \frac{m_2}{6} & \frac{m_2}{3} \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} \frac{m_1 g}{2l_1} + \frac{m_2 g}{l_1} + \frac{m_2 g}{2l_2} & -\frac{m_2 g}{2l_2} \\ -\frac{m_2 g}{2l_2} & \frac{m_2 g}{2l_2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$$

(b) Lagrange's Equations

Kinetic energy, $T = \frac{1}{2} m_1 \left(\frac{\dot{x}_1}{2}\right)^2 + \frac{1}{2} m_2 \left(\frac{\dot{x}_1+\dot{x}_2}{2}\right)^2 + \frac{1}{2} J_1 \left(\frac{\dot{x}_1}{l_1}\right)^2 + \frac{1}{2} J_2 \left(\frac{\dot{x}_2-\dot{x}_1}{l_2}\right)^2$

$$= \frac{1}{6} m_1 \dot{x}_1^2 + \frac{1}{8} m_2 (\dot{x}_1+\dot{x}_2)^2 + \frac{1}{24} m_2 (\dot{x}_2-\dot{x}_1)^2$$

Potential energy, $V = m_1 g \left[\frac{l_1}{2} - \sqrt{\left(\frac{l_1}{2}\right)^2 - \left(\frac{x_1}{2}\right)^2}\right]$

$$+ m_2 g \left[l_1 - \sqrt{l_1^2 - x_1^2} + \frac{l_2}{2} - \sqrt{\left(\frac{l_2}{2}\right)^2 - \left(\frac{x_2-x_1}{2}\right)^2} \right]$$

In the potential energy equation, the bracketed terms are the changes in heights of the masses. We can simplify the expression by using binomial expansion

$$l - \sqrt{l^2 - x^2} = l\left(1 - \left(1 - \left(\frac{x}{l}\right)^2\right)^{1/2}\right) = l\left(1 - \left(1 - \frac{1}{2}\left(\frac{x}{l}\right)^2 \cdots\right)\right)$$

$$= \frac{x^2}{2l} \quad \text{for small vibrations}$$

Hence, $V = m_1 g \frac{x_1^2}{4l_1} + m_2 g \frac{x_1^2}{2l_1} + m_2 g \frac{(x_2-x_1)^2}{4l_2}$

Undamped system $\rightarrow$ dissipation energy $R = 0$

Generalized force Q_1 is associated with x_1 $\rightarrow$ $Q_1 = f_1(t)$

Generalized force Q_2 is associated with x_2 $\rightarrow$ $Q_2 = f_2(t)$

Recall Lagrange's Equations

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} + \frac{\partial V}{\partial q_i} + \frac{\partial R}{\partial \dot{q}_i} = Q_i$$

$$\frac{\partial T}{\partial \dot{x}_1} = \frac{1}{3} m_1 \dot{x}_1 + \frac{1}{4} m_2 (\dot{x}_1 + \dot{x}_2) - \frac{1}{12} m_2 (\dot{x}_2 - \dot{x}_1)$$

$$\frac{\partial T}{\partial \dot{x}_2} = \frac{1}{4} m_2 (\dot{x}_1 + \dot{x}_2) + \frac{1}{12} m_2 (\dot{x}_2 - \dot{x}_1)$$

$$\frac{\partial V}{\partial x_1} = \frac{m_1 g}{2l_1} x_1 + \frac{m_2 g}{l_1} x_1 - \frac{m_2 g}{2l_2} (x_2 - x_1)$$

$$\frac{\partial V}{\partial x_2} = \frac{m_2 g}{2l_2} (x_2 - x_1)$$

$$\frac{\partial T}{\partial x_1} = \frac{\partial T}{\partial x_2} = 0 \qquad \frac{\partial R}{\partial \dot{x}_1} = \frac{\partial R}{\partial \dot{x}_2} = 0$$

Substituting in Lagrange's equations :

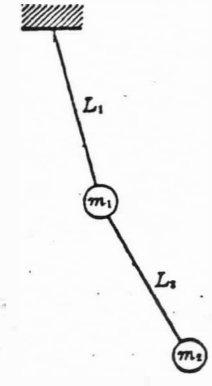
$$\left(\frac{m_1}{3} + \frac{m_2}{3}\right)\ddot{x}_1 + \frac{m_2}{6}\ddot{x}_2 + \left(\frac{m_1 g}{2l} + \frac{m_2 g}{l_1} + \frac{m_2 g}{2l_2}\right) x_1 - \frac{m_2 g}{2l_2} x_2 = f_1$$

$$\frac{m_2}{6}\ddot{x}_1 + \frac{m_2}{3}\ddot{x}_2 - \frac{m_2 g}{2l_2} x_1 + \frac{m_2 g}{2l_2} x_2 = f_2$$

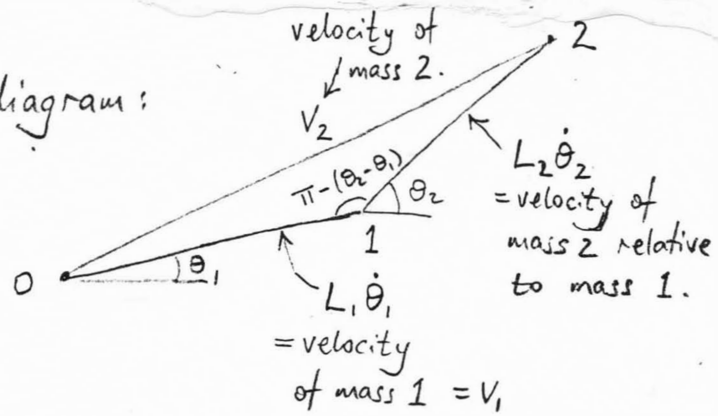
$$\rightarrow \begin{bmatrix} \frac{m_1}{3} + \frac{m_2}{3} & \frac{m_2}{6} \\ \frac{m_2}{6} & \frac{m_2}{3} \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} \frac{m_1 g}{2l} + \frac{m_2 g}{l_1} + \frac{m_2 g}{2l_2} & -\frac{m_2 g}{2l_2} \\ -\frac{m_2 g}{2l_2} & \frac{m_2 g}{2l_2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$$

This is the same as before, but is guaranteed to be symmetric "straight out of the box". We only managed to make the previous matrices symmetric because we recognized f_1 and f_2 as the generalized forces, and arranged to have them on the RHS.

2. The diagram shows the double pendulum considered previously. Use Lagrange's equations to formulate the full non-linear equations of motion. Linearize your result and verify that it is the same as found before.



Velocity diagram:



From cosine formula:

$$V_2^2 = (L_1\dot\theta_1)^2 + (L_2\dot\theta_2)^2 - 2L_1\dot\theta_1 L_2\dot\theta_2\cos\left(\pi-(\theta_2-\theta_1)\right)$$
$$= L_1^2\dot\theta_1^2 + L_2^2\dot\theta_2^2 + 2L_1L_2\cos(\theta_2-\theta_1)\,\dot\theta_1\dot\theta_2$$

Kinetic energy, $T = \tfrac12 m_1 L_1^2\dot\theta_1^2 + \tfrac12 m_2\left(L_1^2\dot\theta_1^2 + L_2^2\dot\theta_2^2 + 2L_1L_2\cos(\theta_2-\theta_1)\dot\theta_1\dot\theta_2\right)$

Potential energy, $V = m_1 g L_1(1-\cos\theta_1) + m_2 g\left(L_1(1-\cos\theta_1) + L_2(1-\cos\theta_2)\right)$.

Lagrange's equations $\quad \dfrac{d}{dt}\left(\dfrac{\partial T}{\partial \dot q_i}\right) - \dfrac{\partial T}{\partial q_i} + \dfrac{\partial V}{\partial q_i} = 0$

For $i=1$, $q_i \to \theta_1$

$$\frac{d}{dt}\left((m_1+m_2)L_1^2\dot\theta_1 + m_2L_1L_2\cos(\theta_2-\theta_1)\dot\theta_2\right) - m_2L_1L_2\sin(\theta_2-\theta_1)\dot\theta_1\dot\theta_2$$
$$+ (m_1+m_2)gL_1\sin\theta_1 = 0$$

$$= (m_1+m_2)L_1^2\ddot\theta_1 + m_2L_1L_2\left(\cos(\theta_2-\theta_1)\ddot\theta_2 + \cancel{\sin(\theta_2-\theta_1)\dot\theta_1\dot\theta_2} - \sin(\theta_2-\theta_1)\dot\theta_2^2\right)$$
$$- m_2L_1L_2\cancel{\sin(\theta_2-\theta_1)\dot\theta_1\dot\theta_2} + (m_1+m_2)gL_1\sin\theta_1 = 0$$

$$= \boxed{(m_1+m_2)L_1^2\ddot\theta_1 + m_2L_1L_2\left(\cos(\theta_2-\theta_1)\ddot\theta_2 - \sin(\theta_2-\theta_1)\dot\theta_2^2\right) + (m_1+m_2)gL_1\sin\theta_1 = 0}$$

For $i=2$, $q_i \to \theta_2$

$$\frac{d}{dt}\left(m_2 L_2^2 \dot{\theta}_2 + m_2 L_1 L_2 \cos(\theta_2-\theta_1)\,\dot{\theta}_1 \right) + m_2 L_1 L_2 \sin(\theta_2-\theta_1)\,\dot{\theta}_1\dot{\theta}_2 + m_2 g L_2 \sin\theta_2 = 0$$

$$= m_2 L_2^2 \ddot{\theta}_2 + m_2 L_1 L_2 \left(\cos(\theta_2-\theta_1)\,\ddot{\theta}_1 \cancel{- \sin(\theta_2-\theta_1)\,\dot{\theta}_1\dot{\theta}_2} + \sin(\theta_2-\theta_1)\,\dot{\theta}_1^2 \right) \cancel{+ m_2 L_1 L_2 \sin(\theta_2-\theta_1)\,\dot{\theta}_1\dot{\theta}_2} + m_2 g L_2 \sin\theta_2 = 0$$

$$= \boxed{m_2 L_2^2 \ddot{\theta}_2 + m_2 L_1 L_2 \left(\cos(\theta_2-\theta_1)\,\ddot{\theta}_1 + \sin(\theta_2-\theta_1)\,\dot{\theta}_1^2 \right) + m_2 g L_2 \sin\theta_2 = 0}$$

Linearizing, $\theta \to 0$, $\sin\theta \to \theta$, $\cos\theta \to 1$, $\theta^2 \to 0$

$$\to \quad (m_1+m_2) L_1^2 \ddot{\theta}_1 + m_2 L_1 L_2 \ddot{\theta}_2 + (m_1+m_2) g L_1 \theta_1 = 0$$

$$m_2 L_1 L_2 \ddot{\theta}_1 + m_2 L_2^2 \ddot{\theta}_2 + m_2 g L_2 \theta_2 = 0$$

In matrix form:

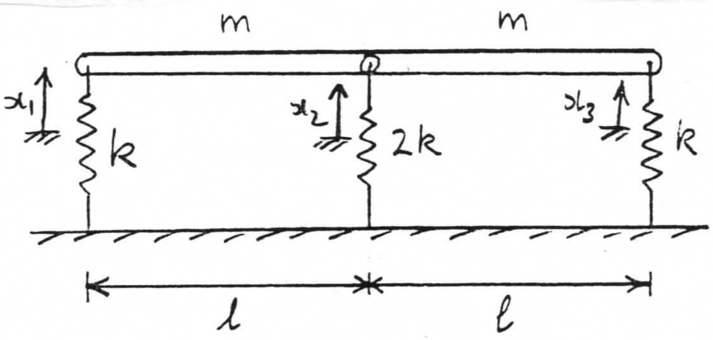
$$\begin{bmatrix} (m_1+m_2) L_1^2 & m_2 L_1 L_2 \\ m_2 L_1 L_2 & m_2 L_2^2 \end{bmatrix} \begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix} + \begin{bmatrix} (m_1+m_2) g L_1 & 0 \\ 0 & m_2 g L_2 \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

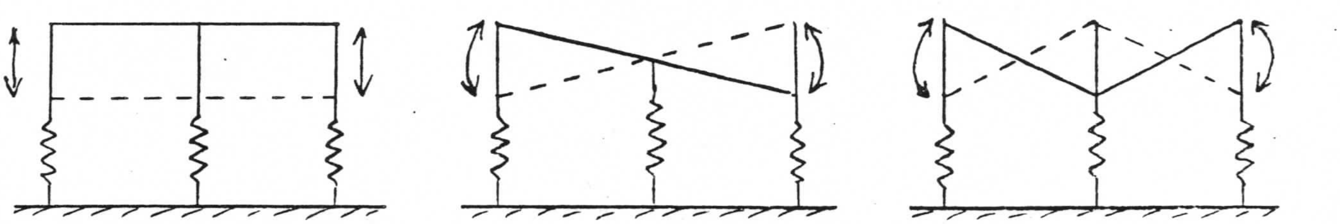
For $m_1 = m_2 = m$, $L_1 = L_2 = L$

$$\boxed{\to \begin{bmatrix} 2mL^2 & mL^2 \\ mL^2 & mL^2 \end{bmatrix} \begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix} + \begin{bmatrix} 2mgL & 0 \\ 0 & mgL \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}}$$

This is the same equation we got before. Notice that Lagrange's equations automatically give us symmetric equations. All terms from $\frac{\partial T}{\partial q_i}$ are non-linear, and disappear on linearization. Typically, we would linearize at the start, and avoid all the ugly algebra.

3. The diagram shows the vibrating system considered in Homework 1, question 2(b). Choose a convenient coordinate system and use Lagrange's equation to formulate the equations of motion in matrix form. Express the illustrated mode shapes in terms of your coordinate system. Verify that these mode shapes are orthogonal. Transform your equations of motion into the principal coordinates, with diagonal mass and stiffness matrices $\underset{\sim}{M}^*$ and $\underset{\sim}{K}^*$.





mode 1, $\omega^2 = \frac{2k}{m}$ mode 2, $\omega^2 = \frac{3k}{m}$ mode 3, $\omega^2 = \frac{6k}{m}$

For a change, choose a coordinate system based on the springs. We therefore expect to have no static coupling. (There seems no obvious way to eliminate dynamic coupling)

Potential energy, $V = \frac{1}{2}kx_1^2 + \frac{1}{2}(2k)x_2^2 + \frac{1}{2}kx_3^2$

Kinetic energy, $T = \frac{1}{2}m\left(\frac{\dot{x}_1+\dot{x}_2}{2}\right)^2 + \frac{1}{2}m\left(\frac{\dot{x}_2+\dot{x}_3}{2}\right)^2 + \frac{1}{2}J\left(\frac{\dot{x}_2-\dot{x}_1}{l}\right)^2 + \frac{1}{2}J\left(\frac{\dot{x}_3-\dot{x}_2}{l}\right)^2$

where $J = \frac{1}{12}ml^2$

Use Lagrange's equations

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} + \frac{\partial V}{\partial q_i} = 0$$

For $i=1$, $q_i \to x_1$

$$\to \frac{d}{dt}\left(m\left(\frac{\dot{x}_1+\dot{x}_2}{2}\right)\cdot\frac{1}{2} + J\left(\frac{\dot{x}_2-\dot{x}_1}{l}\right)\left(-\frac{1}{l}\right)\right) - 0 + kx_1 = 0$$

For $i=2$, $q_i \to x_2$

$$\to \frac{d}{dt}\left(m\left(\frac{\dot{x}_1+\dot{x}_2}{2}\right)\cdot\frac{1}{2} + m\left(\frac{\dot{x}_2+\dot{x}_3}{2}\right)\cdot\frac{1}{2} + J\left(\frac{\dot{x}_2-\dot{x}_1}{l}\right)\cdot\frac{1}{l} + J\left(\frac{\dot{x}_3-\dot{x}_2}{l}\right)\left(-\frac{1}{l}\right)\right) - 0 + 2kx_2 = 0$$

For $i=3$, $q_i \to x_3$

$$\frac{d}{dt}\left(m\left(\frac{\dot{x}_2+\dot{x}_3}{2}\right)\cdot\frac{1}{2} + J\left(\frac{\dot{x}_3-\dot{x}_2}{l}\right)\cdot\left(\frac{1}{l}\right)\right) - 0 + kx_3 = 0$$

$$\to \frac{m}{4}\left(\ddot{x}_1+\ddot{x}_2\right) + \frac{J}{l^2}\left(\ddot{x}_1-\ddot{x}_2\right) + kx_1 = 0$$

$$\frac{m}{4}\left(\ddot{x}_1+2\ddot{x}_2+\ddot{x}_3\right) + \frac{J}{l^2}\left(-\ddot{x}_1+2\ddot{x}_2-\ddot{x}_3\right) + 2kx_2 = 0$$

$$\frac{m}{4}\left(\ddot{x}_2+\ddot{x}_3\right) + \frac{J}{l^2}\left(\ddot{x}_3-\ddot{x}_2\right) + kx_3 = 0$$

In matrix form, putting $J = \frac{1}{12}ml^2$

$$\underbrace{\begin{bmatrix} \frac{m}{3} & \frac{m}{6} & 0 \\ \frac{m}{6} & \frac{2m}{3} & \frac{m}{6} \\ 0 & \frac{m}{6} & \frac{m}{3} \end{bmatrix}}_{\underset{\sim}{M}} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{bmatrix} + \underbrace{\begin{bmatrix} k & 0 & 0 \\ 0 & 2k & 0 \\ 0 & 0 & k \end{bmatrix}}_{\underset{\sim}{K}} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

In terms of the $x_1 - x_2 - x_3$ coordinate system, the mode shape vectors and modal matrix are:

$$\underline{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \qquad \underline{u}_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \qquad \underline{u}_3 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \qquad \underset{\sim}{U} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 1 \end{bmatrix}$$

The diagonalized equation of motion in terms of the principal coordinates $\underline{p}$ is

$$\underset{\sim}{M}^*\ddot{\underline{p}} + \underset{\sim}{K}^*\underline{p} = \underline{0} \qquad \text{where} \quad \underset{\sim}{M}^* = \underset{\sim}{U}^T\underset{\sim}{M}\underset{\sim}{U}$$

$$\underset{\sim}{K}^* = \underset{\sim}{U}^T\underset{\sim}{K}\underset{\sim}{U}$$

$$\underline{x} = \underset{\sim}{U}\underline{p}$$

$$\underset{\sim}{M}^* = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 1 \end{bmatrix}\begin{bmatrix} \frac{m}{3} & \frac{m}{6} & 0 \\ \frac{m}{6} & \frac{2m}{3} & \frac{m}{6} \\ 0 & \frac{m}{6} & \frac{m}{3} \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 1 \end{bmatrix}\begin{bmatrix} \frac{m}{2} & \frac{m}{3} & \frac{m}{6} \\ m & 0 & -\frac{m}{3} \\ \frac{m}{2} & -\frac{m}{3} & \frac{m}{6} \end{bmatrix} = \begin{bmatrix} 2m & 0 & 0 \\ 0 & \frac{2m}{3} & 0 \\ 0 & 0 & \frac{2m}{3} \end{bmatrix}$$

$$\underset{\sim}{K}^* = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 1 \end{bmatrix}\begin{bmatrix} k & 0 & 0 \\ 0 & 2k & 0 \\ 0 & 0 & k \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 1 \end{bmatrix}\begin{bmatrix} k & k & k \\ 2k & 0 & -2k \\ k & -k & k \end{bmatrix} = \begin{bmatrix} 4k & 0 & 0 \\ 0 & 2k & 0 \\ 0 & 0 & 4k \end{bmatrix}$$

The diagonal matrices decouple the equations of motion to:

$$2m\,\ddot{p}_1 + 4k\,p_1 = 0 \quad \rightarrow \quad \omega_1^2 = 2\,k/m$$

$$\frac{2m}{3}\ddot{p}_2 + 2k\,p_2 = 0 \quad \rightarrow \quad \omega_2^2 = 3\,k/m$$

$$\frac{2m}{3}\ddot{p}_3 + 4k\,p_3 = 0 \quad \rightarrow \quad \omega_3^2 = 6\,k/m$$