< Motion Control Design via Loop Shaping >

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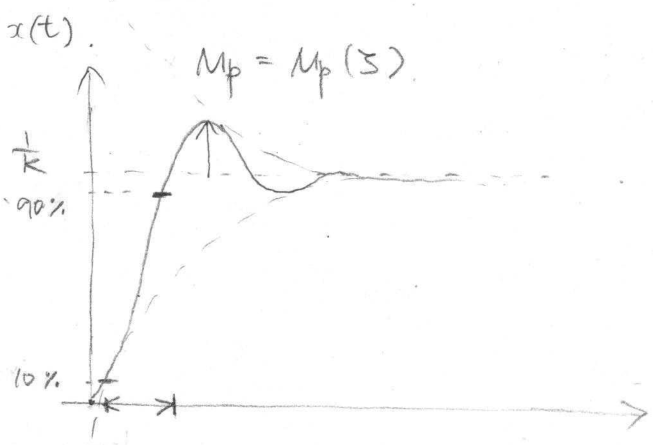
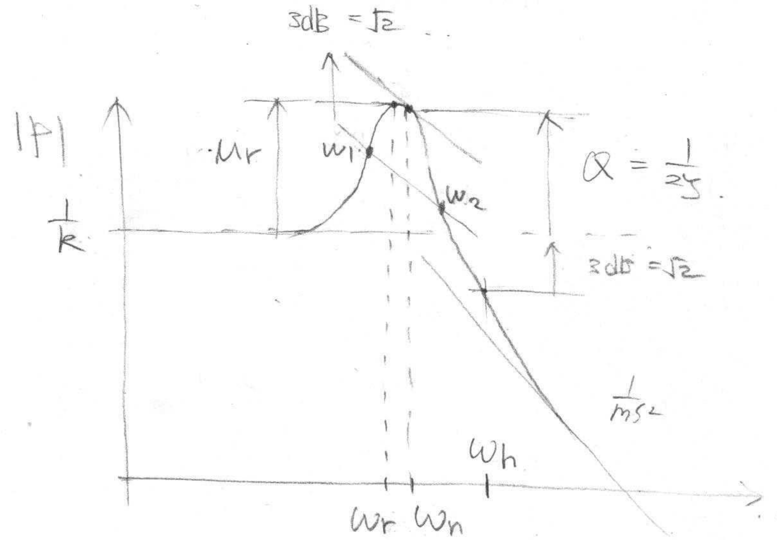
∘ Objective

- 2nd order system frequency response vs. Step resp.
- Loop shaping : $\omega_n \simeq \omega_c$, $\zeta \simeq \frac{\phi_m\,[\deg]}{100}$.

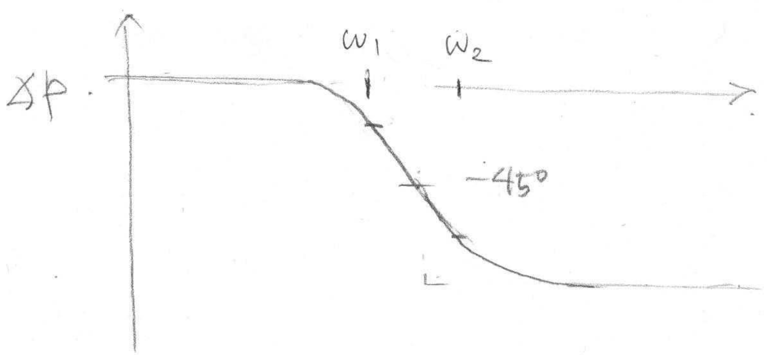
∘ Step Resp Vs. Freq Resp.

$$P(s) = \frac{1}{ms^2+bs+k} = \frac{1}{k}\,\frac{1}{\left(\frac{s}{\omega_n}\right)^2 + 2\zeta\left(\frac{s}{\omega_n}\right) + 1}$$



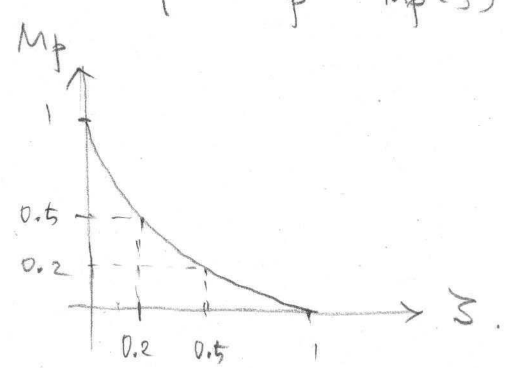


$t_r \simeq \frac{2.2}{\omega_h} < \frac{2.2}{\omega_n}$



< Summary >.

- Rise time : $t_r \simeq \frac{2.2}{\omega_h} < \frac{2.2}{\omega_n}$
- overshoot : $M_p = M_p(\zeta)$



$\omega_n = \sqrt{\frac{k}{m}}$: natural freq.

$Q = \frac{1}{2\zeta}$: quality factor

$(\zeta < \frac{1}{\sqrt{2}})$

$\omega_r = \omega_n \sqrt{1-2\zeta^2}$: resonance freq

$M_r = \frac{1}{2\zeta}\,\frac{1}{\sqrt{1-\zeta^2}}$: resonant peak.

Note $\zeta^2 \ll 1$ $\boxed{M_r \simeq Q = \frac{1}{2\zeta}}$ $\boxed{\omega_r \simeq \omega_n = \sqrt{\frac{k}{m}}}$

- Remark.

- 2nd-order system is completely determined with G_0, ω_n, ζ.

- In particular, ω_n and ζ dictates the dynamics.

< Step Resp >	< Freq Resp >
Overshoot : $M_p = M_p(\zeta)$	Resonant peak : $M_r \simeq \frac{1}{2\zeta}$
Rise time : $t_r \simeq \frac{2.2}{\omega_b} < \frac{2.2}{\omega_n}$	Band width : $\omega_b > \omega_n$

- Why we review this?

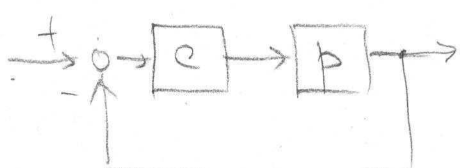
  This is our "template" to design closed-loop position control via "Loop Shaping"

- We will make the following connection shortly.

$$L(s) \qquad\qquad T(s) = \frac{L}{1+L}$$

$$\begin{Bmatrix} \omega_c \\ \phi_m \end{Bmatrix} \longleftrightarrow \begin{Bmatrix} \omega_n \\ \zeta \end{Bmatrix} \qquad \begin{aligned} \omega_n &\simeq \omega_c \\ \zeta &\simeq \frac{\phi_m[\deg]}{100} \end{aligned}$$

- Why do we shape $L(s)$ instead of shaping $T(s)$ directly?



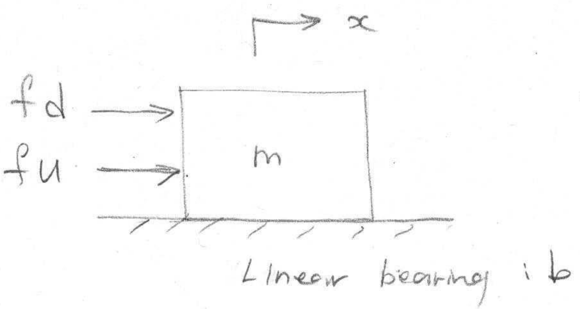
$T = \frac{CP}{1+CP}$ Nonlinear with C

$L = CP$ linear with C

It is much easier to shape L with C.

ω_c and ϕ_m give us good estimates of ω_n and ζ.

Example :



fu [N] : Control effort

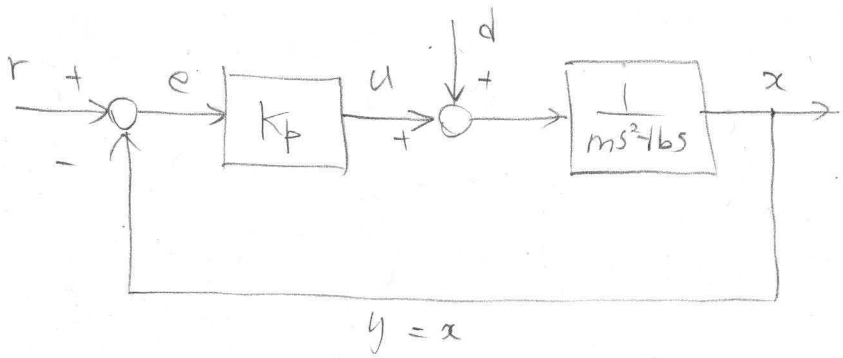
fd [N] : Disturbance force

$$m\ddot{x} = \Sigma f$$
$$= fu + fd - b\dot{x} \Rightarrow (ms^2 + bs)X = f_u + f_d$$

- Plant

$$p(s) = \frac{X}{F} = \frac{1}{ms^2 + bs} \quad \text{for } m = 1\text{kg}, \; b = 10\,\text{Ns/m} \quad p(s) = \frac{1}{s(s+10)}.$$

- Control : $C(s) = K_p$.

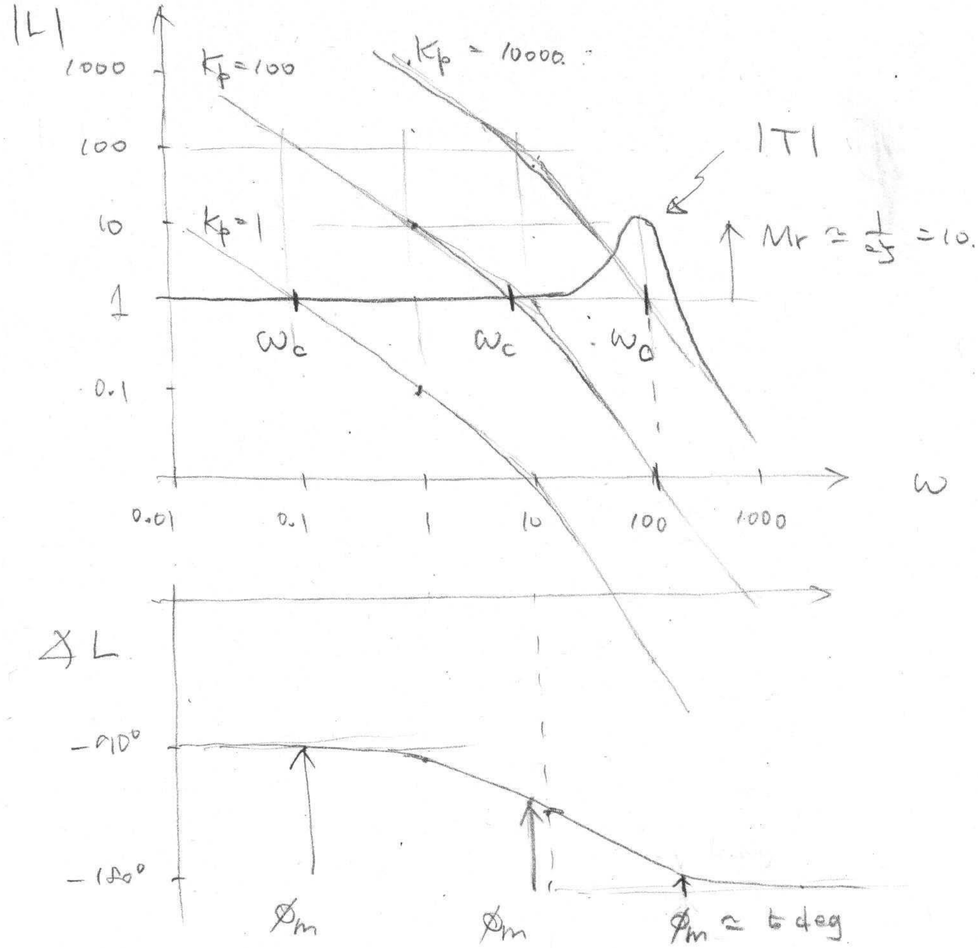


- Loop : $L(s) = \dfrac{K_p}{ms^2 + bs} = \dfrac{K_p}{s^2 + 10s}$

- Closed-loop : $\dfrac{X}{D} = \dfrac{\frac{1}{ms^2+bs}}{1 + \frac{K_p}{ms^2+bs}} = \dfrac{1}{ms^2 + bs + \underbrace{K_p}_{}}$ "Apparent stiffness"

$$\frac{X}{R} = \frac{L}{1+L} = \frac{\frac{K_p}{ms^2+bs}}{1 + \frac{K_p}{ms^2+bs}} = \frac{K_p}{ms^2 + bs + K_p}. = T(s)$$

• Bode plot of $L(s)$ and $T(s)$.



$$L(s) = \frac{K_p}{s^2 + 10s}$$

$$T(s) = \frac{K_p}{s^2 + 10s + K_p}$$

$K_p = 1,\ 100,\ 10000$

• When $K_p = 10000$, $T(s) = \frac{10000}{s^2 + 10s + 1000}$

$$\begin{cases} \omega_n = \sqrt{\frac{K_p}{m}} = 100 \text{ rad/s} \\ \zeta = \frac{b}{2\sqrt{mK_p}} = 0.05 \end{cases}$$

• $T(s)$ shows a resonant peak

$$\begin{cases} \omega_r \simeq \omega_n \\ M_r \simeq \frac{1}{2\zeta} \end{cases}$$

• These parameters can be estimated from $L(s)$

$$\begin{cases} \omega_c = 99.7 \text{ rad/s} \\ \phi_m = 5.72 \text{ deg} \end{cases} \rightarrow \begin{cases} \omega_n = 100 \text{ rad/s} \\ \zeta = 0.05 \simeq \frac{\phi_m[\text{deg}]}{100} \end{cases}$$

- Remark

- The estimation rule $\zeta \simeq \frac{\phi_m[\deg]}{100}$ is widely used among servo designers.

- It is a rule of thumb to estimate the closed-loop performance (e.g., damping ratio ζ & resonant peak M_r) based on a loop parameter.

- We use it even for closed-loop systems that are not strictly 2nd-order.

$$\phi_m[\deg] \rightarrow \zeta \simeq \frac{\phi[\deg]}{100} \rightarrow M_r \simeq \frac{1}{2\zeta}$$

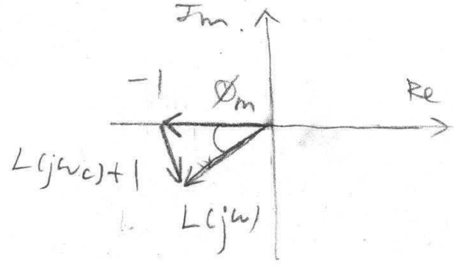
- Similarly, we use $t_r \simeq \frac{2.2}{\omega_h}$ derived from 1st-order systems for general systems as well

- $\omega_r \simeq \omega_n \simeq \omega_c < \omega_h \Rightarrow t_r < \frac{2.2}{\omega_c}$

- Justification for $\zeta \simeq \frac{\phi_m[\deg]}{100}$.

- $M_r \simeq |T(j\omega)|_{\omega=\omega_c} = \left|\frac{L}{1+L}\right|_{\omega=\omega_c} = \frac{1}{|1+L|_{\omega=\omega_c}}$

At $\omega=\omega_c$



for small ϕ_m.

$|1+L| \simeq \phi_m[\text{rad}]$.

Thus, $M_r \simeq \frac{1}{\phi_m} \simeq \frac{1}{2\zeta}$.

Therefore, $\zeta \simeq \frac{\phi_m}{2} = \frac{\phi_m[\deg]}{2\cdot\frac{180°}{\pi}} \simeq \frac{\phi_m[\deg]}{100}$.