MECH 421: Mechatronics System Instrumentation
2020/21 Winter Session – Term 2

Homework 1 – Solution

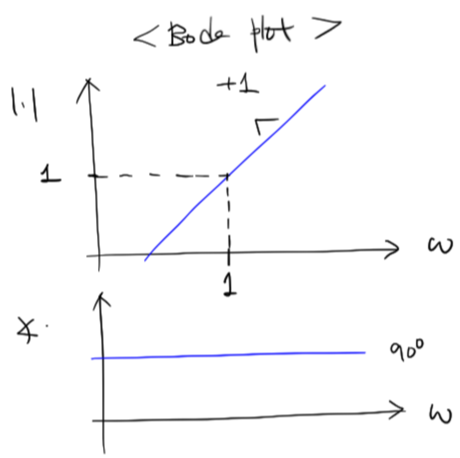
Assigned: Jan 15, 2021
Due: Jan 22, 2021

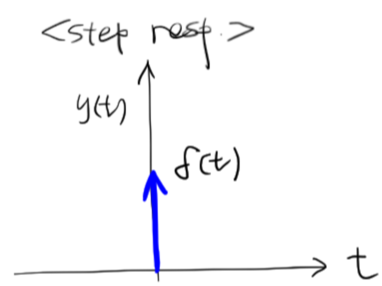
Problem 1

Manually draw the 1) step response and 2) Bode plot of the following transfer functions. For the step responses, clearly show the first-order time constants (if exist), and the initial response and steady state response. For the Bode plots, clearly show the break frequencies (if exist) and asymptotes for the gain and phase curves, and make the transition between the asymptotes properly.

(a) $H(s) = s$

Answer:





Note that a differentiator is the inversion of an integrator. The impulse response of an integrator is a unit step, and therefore the step response of a differentiator is an impulse.

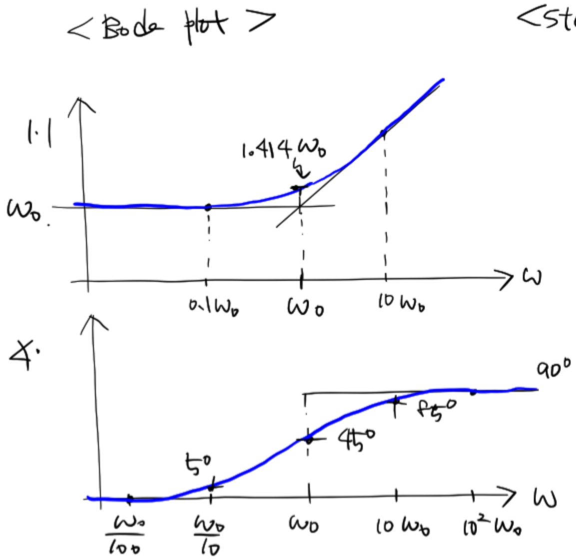
Pure differentiator, i.e., s, is not a proper transfer function because the order of numerator is higher than that of denominator. Also, strictly speaking it is not a causal system because the differentiation requires information in the future as well as the past, i.e.,

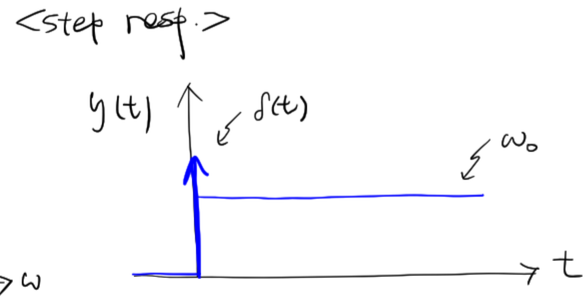
$$\frac{\mathrm{d}y}{\mathrm{d}t} = \lim_{\Delta \to 0} \frac{y(t+\Delta) - y(t)}{\Delta} = \lim_{\Delta \to 0} \frac{y(t) - y(t-\Delta)}{\Delta}$$

Therefore, we cannot implement it in real time.

(b) $H(s) = s + \omega_o \qquad (\omega_o > 0)$

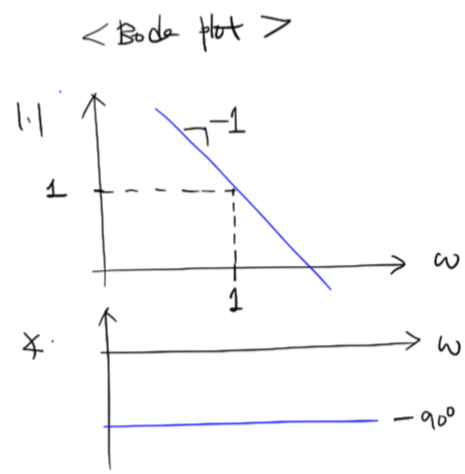
Answer:

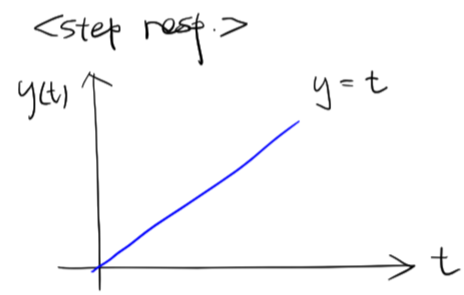




(c) $H(s) = \dfrac{1}{s}$

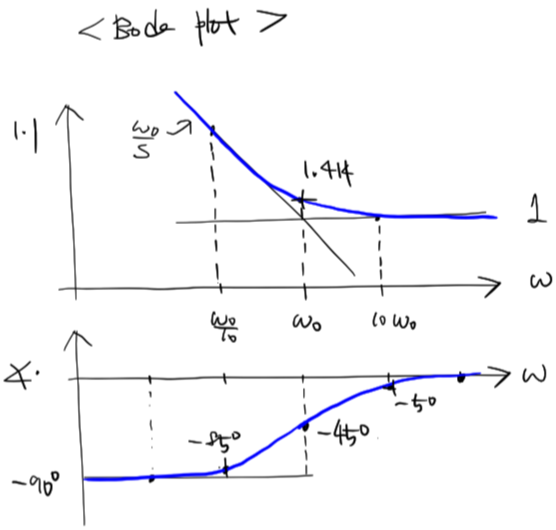
Answer:

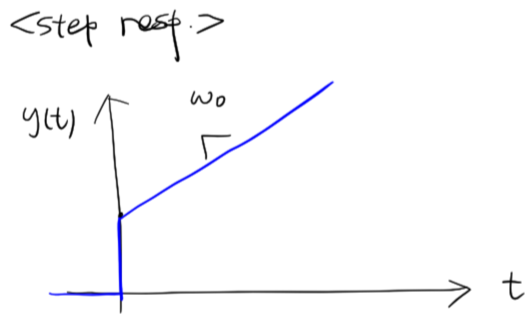




(d) $H(s) = \dfrac{\omega_o}{s} + 1 \qquad (\omega_o > 0)$

Answer:

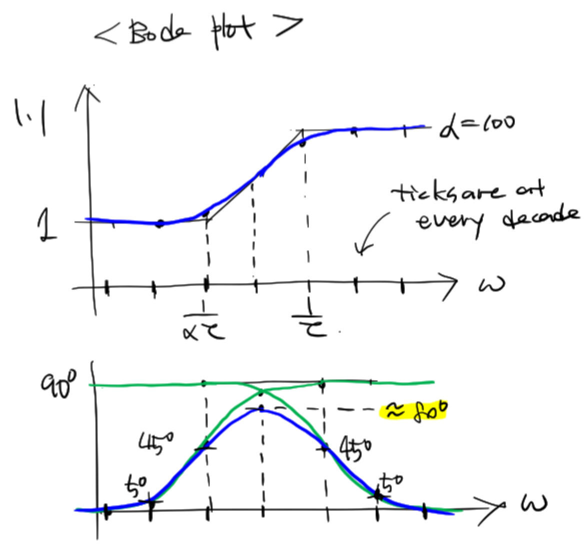


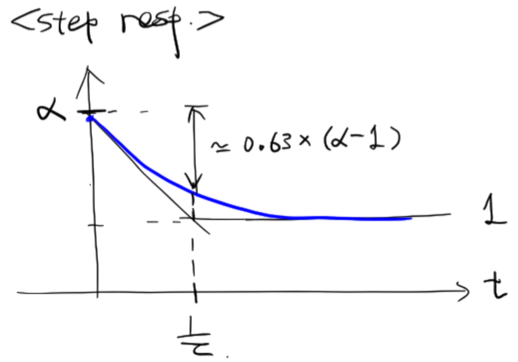


When a transfer function has the same order of numerator and denominator, the high-frequency gain becomes a flat. This means a unit step can directly pass through the system and the response makes a jump at $t = 0$. .

(e) $H(s) = \dfrac{\alpha\tau s + 1}{\tau s + 1} \qquad (\alpha = 100, \tau > 0)$

Answer:





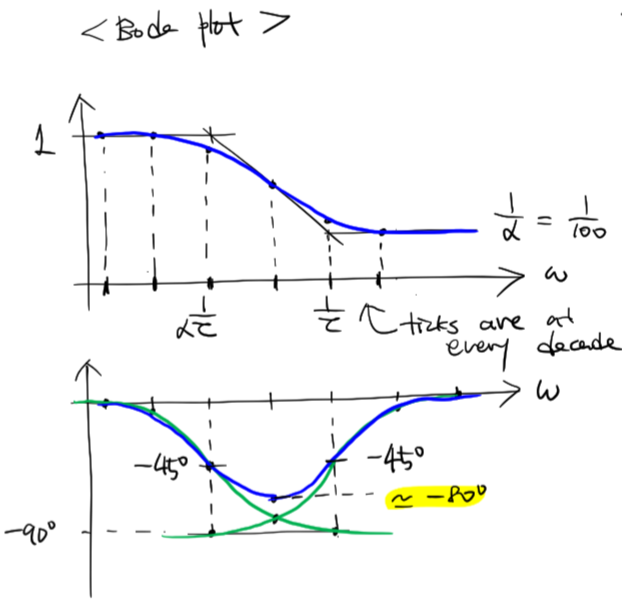
This transfer function is typical of a lead compensator. High-frequency gain is $\alpha = 100$, which becomes the initial value of the step response. DC gain is 1, which becomes the final value of the step response. The exponential transition between the initial and final value is governed by the time constant of the pole τ.

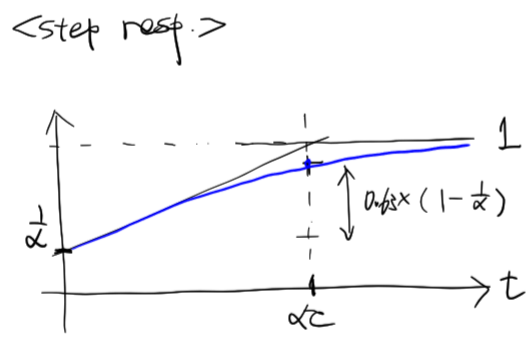
Alternatively, the step response can be understood by taking a partial fraction expansion of $H(s)$

$$H(s) = \frac{\alpha\tau s + 1}{\tau s + 1} = \frac{\alpha(\tau s + 1) - \alpha + 1}{\tau s + 1} = \alpha - \frac{\alpha + 1}{\tau s + 1}$$

and consider the response as the sum of two separate step responses.

(f) $H(s) = \dfrac{\tau s + 1}{\alpha \tau s + 1} \qquad (\alpha = 100, \tau > 0)$





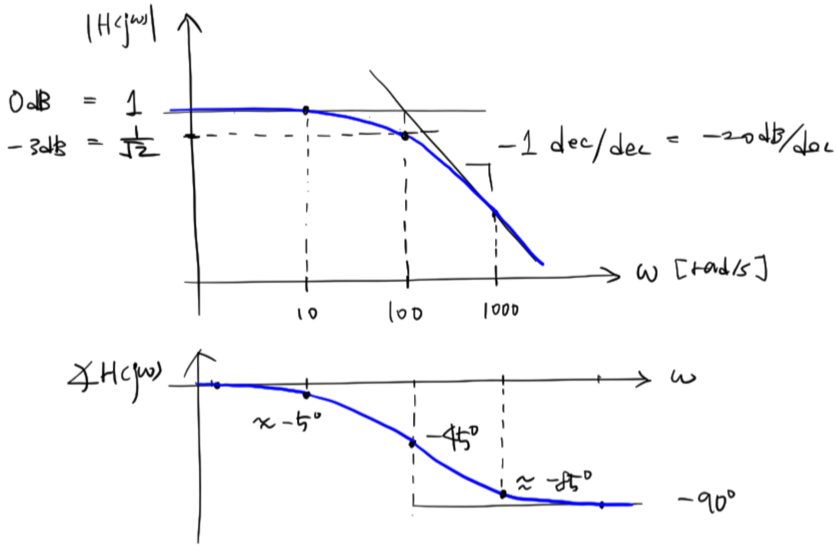
The high-frequency gain is $1/\alpha$, which sets the initial value of the step response. The dc gain is 1, which becomes the steady-state value of the step response. The exponential transition is governed by the time constant of the pole $\alpha\tau$.

Problem 2

For the following transfer function

$$H(s) = \frac{1}{0.01s + 1}$$

(a) Manually draw the Bode plot of $H(s)$.



(b) Find the response $y(t)$ to an input sinusoid $x(t) = \sin(10t + \pi/3)$.

Answer: Note that the input is a persistent sinusoid existing for all time, including $t < 0$. Therefore, the output is the same sinusoid whose magnitude and phase are altered by the frequency response of the system at the frequency of the input sinusoid.

$$|H(j\omega)|_{\omega=10} = 0.995$$
$$\angle H(j\omega)|_{\omega=10} = -0.0997$$

Therefore,

$$\boxed{y(t) = 0.995 \sin(10t + \pi/3 - 0.0997)}$$

(c) Find the response $y(t)$ to an input sinusoid $x(t) = \sin(100t + 2\pi/3)$.

Answer: The magnitude and gain of the given system at $\omega = 100$ is

$$|H(j\omega)||_{\omega=100} = 0.7071$$
$$\angle H(j\omega)|_{\omega=100} = -0.7854$$

Therefore,

$$\boxed{y(t) = 0.7071 \sin(100t + 2\pi/3 - 0.7854)}$$

(d) Find the response $y(t)$ to an input sinusoid $x(t) = \sin(1000t + \pi)$.

Answer: The magnitude and gain of the system at $\omega = 1000$ is

$$\begin{aligned} |H(j\omega)||_{\omega=1000} &= 0.0995 \\ \angle H(j\omega)|_{\omega=1000} &= -1.4711 \end{aligned}$$

Therefore,

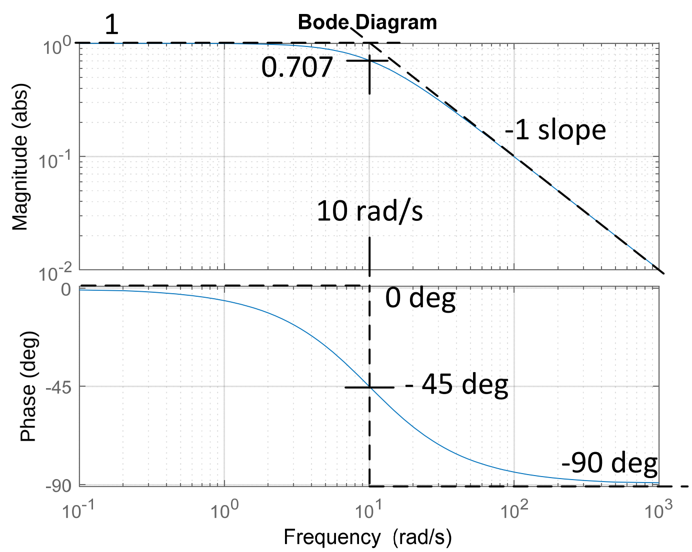
$$\boxed{y(t) = 0.0995 \sin(1000t + \pi - 1.4711)}$$

The inputs are persistent sinusoids, that is, they exist for all time $(-\infty < t < \infty)$.

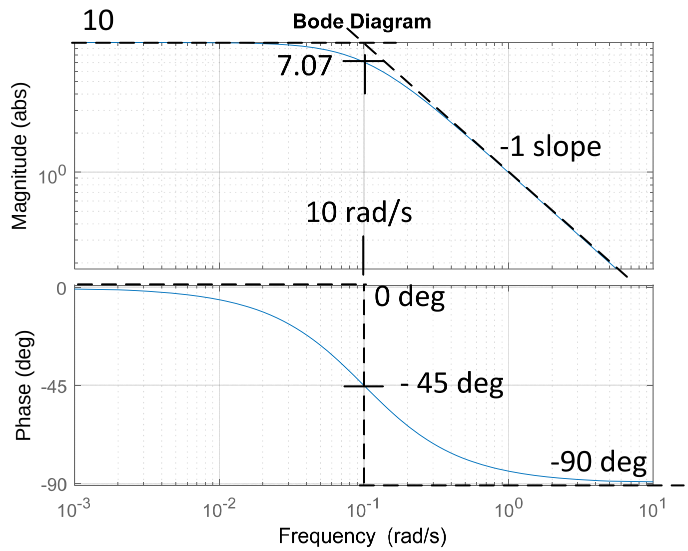
Problem 3

Manually draw the Bode plot of the following transfer functions. Clearly show the break frequencies (if exist) and asymptotes for the gain and phase curves, and make the transition between the asymptotes properly. Make sure the starting point of the Bode phase curve, i.e., $\angle H(j\omega)|_{\omega \to 0}$, is within $\pm 180°$ by adding or subtracting an integer multiple of 360°.

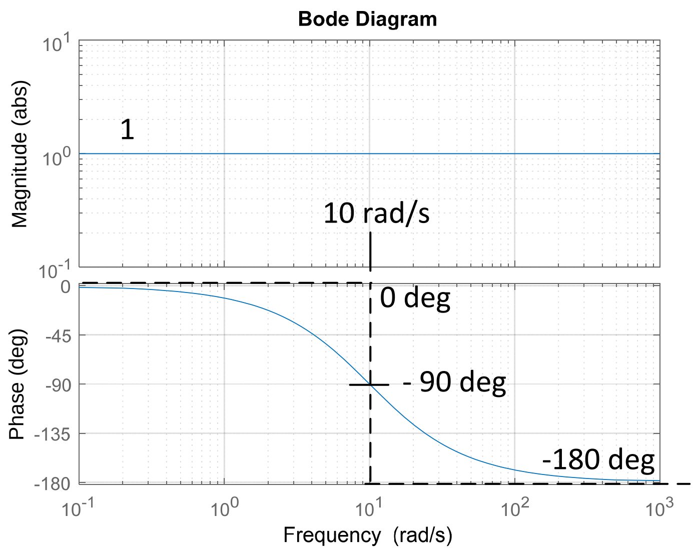
(a) $H(s) = \dfrac{1}{0.1s + 1}$



(b) $H(s) = \dfrac{1}{s + 0.1}$

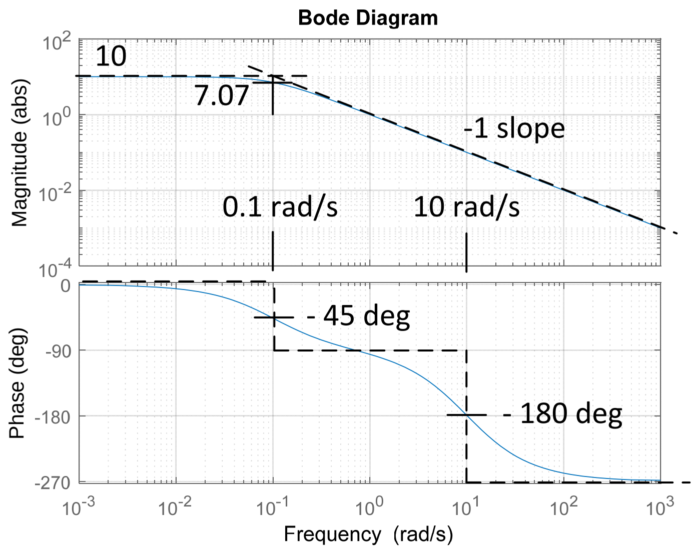


(c) $H(s) = \dfrac{1 - 0.1s}{1 + 0.1s}$



Note that the magnitude curve is unity for all frequencies. This is an example of *all-pass* system. The phase curve is negative for all frequencies. This system can be used as a first-order approximation of a pure time delay $e^{-sT} \approx \frac{1-sT/2}{1+sT/2}$ for $T = 0.2\,\mathrm{s}$ (Padé approximation). Check the `pade` command in MATLAB.

(d) $H(s) = \left(\frac{1}{s+0.1}\right)\left(\frac{1-0.1s}{1+0.1s}\right)$



This is an example of *non-minimum phase* system, where the Bode's gain-phase relation does not hold. Note that the phase curve assymptode cannot be derived from the gain curve assymptode via the relation $\phi \approx 90° \times n$ where n is the gain curve slope. In general, a non-minimum phase system can be factorized into a minimum-phase part (e.g., $\frac{1}{s+0.1}$) and all-pass part (e.g., $\frac{1-0.1s}{1+0.1s}$).