Lecture 16

Mode Shape Orthogonality

$$(\omega_r^2 - \omega_s^2)\vec{u}_r^T [M] \vec{u}_s = 0 \qquad \text{In general, } \omega_s \neq \omega_r$$

$$\Rightarrow \vec{u}_r^T [M] \vec{u}_s = 0 \quad \text{for } r \neq s$$

Recall that $T = \dot{\vec{q}}^T [M] \dot{\vec{q}} > 0$ for $\dot{\vec{q}} \neq 0$

Positive definite

$$\Rightarrow \vec{u}_r^T [M] \vec{u}_s > 0 \quad \text{for } r = s$$

Similarly, $\vec{u}_r^T [K] \vec{u}_s = 0$ for $r \neq s$ for stable systems

> 0 for $r = s$

Principal Coordinates

$\vec{q} = [U]\vec{p}$ Note $[U]$ vs $\vec{u}$.

$$\begin{bmatrix} q_1 \\ q_2 \\ \vdots \end{bmatrix} = \begin{bmatrix} \uparrow & \uparrow & \\ \vec{u}_1 & \vec{u}_2 & \cdots \\ \downarrow & \downarrow & \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ \vdots \end{bmatrix} \Rightarrow \vec{q}_1 = p_1 \vec{u}_1 + p_2 \vec{u}_2 + \ldots$$

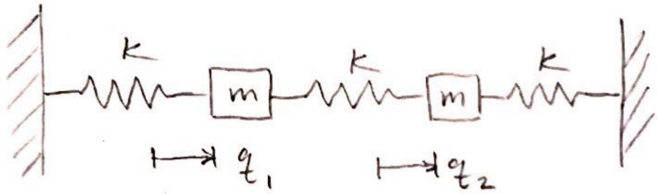
Premultiply by $\vec{u}_r^T [M]$

$$\Rightarrow \vec{u}_r^T [M] \vec{q} = p_1 \vec{u}_r^T [M] \vec{u}_1 + p_2 \vec{u}_r^T [M] \vec{u}_2 + \ldots$$

$$= p_r \vec{u}_r^T [M] \vec{u}_r$$

$$\Rightarrow p_r = \frac{\vec{u}_r^T [M] \vec{q}}{\vec{u}_r^T [M] \vec{u}_r} = \frac{\vec{u}_r^T [K] \vec{q}}{\vec{u}_r^T [K] \vec{u}_r}$$

Ex:



$\vec{u}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \vec{u}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$$\vec{q} = [U]\vec{p} \Rightarrow \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \end{bmatrix}$$

Inverting: $\begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}$

Say $\vec{q} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 1 \\ -1 \end{bmatrix} \quad (\vec{q} = p_1\vec{u}_1 + p_2\vec{u}_2)$

Fancy way: $p_1 = \dfrac{\vec{u}_1^T [M] \vec{q}}{\vec{u}_1^T [M] \vec{u}_1} = \dfrac{[1 \ \ 1]\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix}}{[1 \ \ 1]\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix}} = \dfrac{m}{2m} = \dfrac{1}{2}$

Similarly for p_2.

$\boxed{\text{Proportional Damping}}$ Approx. [C] to give it good properties

In general, $[M]\ddot{\vec{q}} + [C]\dot{\vec{q}} + [K]\vec{q} = \vec{f}$

For proportional damping to occur: $[C] = \alpha[M] + \beta[K]$

Sub. $\vec{q} = [U]\vec{p}$ into equation of motion:

$$[M][U]\ddot{\vec{p}} + (\alpha[M] + \beta[K])[U]\dot{\vec{p}} + [K][U]\vec{p} = \vec{f}$$

Premultiply by $[U]^T$

$$[U]^T[M][U]\ddot{\vec{p}} + (\alpha[U]^T[M][U] + \beta[U]^T[M][U])\dot{\vec{p}} + [U]^T[K][U]\vec{p} = \ldots$$
$$= [U]^T\vec{f}$$

Here, $[U]^T[M][U] = [M^*]$ and is diagonal. Same for $[K]$.

Then $$[M^*]\ddot{\vec{p}} + \left(\alpha[M^*] + \beta[K^*]\right)\dot{\vec{p}} + [K^*]\vec{p} = [U]^T f$$

Similarly, $[C^*] = \alpha[M^*] + \beta[K^*]$

Finally, $[M^*]\ddot{\vec{p}} + [C^*]\dot{\vec{p}} + [K]\vec{p} = [U]^T\vec{f}$ is fully diagonal

$\Rightarrow$ all equations are uncoupled.

$$\Rightarrow \begin{cases} m_{11}^* \ddot{p}_1 + c_{11}^* \dot{p}_1 + k_{11}^* p_1 = \vec{u}_1^T \vec{f} \\ m_{22}^* \ddot{p}_2 + c_{22}^* \dot{p}_2 + k_{22}^* p_2 = \vec{u}_2^T \vec{f} \end{cases}$$

For a proportionally damped system, the damped mode shapes are the same as the undamped ones. So, find mode shapes for undamped system first.