2.3. Undamped SDOF Response – 1

MECH 463: Mechanical Vibrations

A. Srikantha Phani
srikanth@mech.ubc.ca

Suggested Readings:

1. Topic 2.3 from notes package.
2. Sections 1.10, 2.2 and 2.3 from the course textbook.

Learning Objectives

1. Determine undamped free vibration response of a SDOF system.
2. Compute the natural frequencies for free vibrations.
3. Recognize the importance of system parameters and operating conditions on the natural frequencies.
4. Visualize harmonic vibrations in a graphical form.

So Far ...

1. We know how to select co-ordinates to describe vibrations.

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2. We know how to compute accelerations and velocities in general planar motion using kinematics.

3. We know how to write equations of motion using force and energy methods.

4. We learned about superposition principle for linear systems.

5. We learned how to deduce equivalent system parameters for continuous ∞ DOF systems.

6. This topic combines all of the above.

Universal SDOF Equations of Motion (NP 2.8)

$$m\ddot{x} + kx = f \quad \text{Translatory vibrations} \tag{1a}$$

$$J_o\ddot{\theta} + k_\theta\theta = M_o \quad \text{Rotatory or torsional vibrations} \tag{1b}$$

Some remarks...

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We focus on harmonic force of the form $f(t) = F_0 \cos \omega t$. Most rotating systems are subjected to forces of this kind. **For general, non harmonic forcing, we make use of superposition principle using Fourier series or in the form of a convolution integral as we shall see in later topics.** See the notes package for detailed mathematical derivations.

2.9. Undamped Vibration Response (NP 2.9)

$$m\ddot{x}_h + kx_h = 0 \quad \text{Homogeneous response/Free vibration} \tag{2a}$$
$$m\ddot{x}_h + kx_h = f \quad \text{Particular solution/Forced vibration.} \tag{2b}$$

Adding the above two equations we have the TOTAL response, from the principle of superposition

$$m\ddot{x} + kx = f, \quad x = x_h + x_p \quad \text{TOTAL response} \tag{3}$$

It is required to specify the initial conditions on the TOTAL response. They can be initial velocity, or initial displacement:

$$x(0) = x_0; \quad \dot{x}(0) = \dot{x}_0 \qquad \text{INITIAL conditions } \underline{\textbf{apply on the TOTAL solution}}. \tag{4}$$

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2.10. Free Vibration Response (NP 2.10, T 2.2.4, T 2.2.5+Notes) — # 1

$$m\ddot{x}_h + kx_h = 0 \quad \text{Homogeneous response/Free vibration} \tag{5}$$

$$\therefore x_h(t) = A_1 \cos \omega_n t + A_2 \sin \omega_n t \tag{6}$$

Notice that we have two unknown constants A_1 and A_2. These can be found using:

1. Initial conditions only when the particular solution is zero, $x_p = 0$.
2. Initial conditions on the TOTAL solution $x = x_h + x_p$, when there is an external force, $x_p \neq 0$.

In free vibration problems, there is no external force $f = 0 \Rightarrow x_p = 0$. Therefore $x = x_h + x_p = x_h$.

2.10. Free Vibration Response (NP 2.10, T 2.2.4, T 2.2.5+Notes) — # 2

Imposing initial conditions in Eq.(4) on the total solution, we have

$$x_h(t) = x_0 \Rightarrow A_1 \cos 0 + A_2 \sin 0 = x_0 \Rightarrow A_1 = x_0$$

$$\dot{x}_h(t) = \dot{x}_0 \Rightarrow [-\omega_n A \sin \omega_n t + \omega_n A_2 \cos \omega_n t]_{t=0} = \dot{x}_0$$

$$\omega_n A_2 = \dot{x}_0 \Rightarrow A_2 = \frac{\dot{x}_0}{\omega_n}$$

Thus, the free vibration response of an undamped SDOF system is obtained as given below.

$$\boxed{x = x_h = x_o \cos \omega_n t + \frac{\dot{x}_0}{\omega_n} \sin \omega_n t, \quad \omega_n = \sqrt{\frac{k}{m}} \qquad (7)}$$

In solving problems it is best not to memorize the above formula, but, instead use the form $x_h(t) = A_1 \cos \omega_n t + A_2 \sin \omega_n t$ and determine A_1 and A_2 as appropriate.

2.10. Free Vibration Response (NP 2.10, T 2.2.4, T 2.2.5+Notes) — # 3

1. **Free vibration takes place at the system's natural frequency**, irrespective of the initial conditions.
2. Natural frequencies depend only on the stiffness and mass properties of the system. Natural frequency increases with an increase in the stiffness or a *decrease* in the mass.
3. One can estimate natural frequencies from static deflections using the formula $\omega_n = \sqrt{\frac{g}{\delta_{st}}}$. Can you show this?

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4. The S.I. unit of natural frequency is Hertz. One Hz is one cycle per second. In most computer programs, and in mathematics, frequencies are measured in rad/s. In engineering practise one also uses rpm. The following conversion may be useful $1Hz = 2\pi rad/s = \frac{1}{60} rpm$. 1200 rpm is thus 20 Hz or 40π rad/s.

Example 7 — # 1

Example 7: Determine the natural frequencies of: (a) simple pendulum; (b) compound pendulum; (c) Example 4; (d) Tutorial problem on rotating spring-mass system. (p.68 of NP)

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Example 7 — # 2

Fill in the class

Example 7 — # 3

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Example 7 — # 4

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Example 8 — # 1

Example 8: An air-conditioning chiller unit weighs 600 kg is to be supported by four air springs as shown below. Design the air springs such that the natural frequency of vibration of the unit lies between 5 rad/s and 10 rad/s. (p.70 of NP)

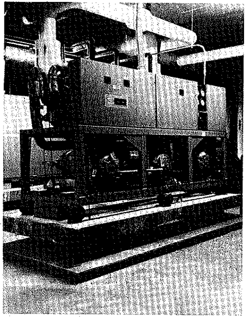

Figure: Figure for example 8.

Example 8 — # 2

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Example 8 — # 3

Fill in the class

Example 9 — # 1

Example 9: A rigid block of mass M is mounted on four elastic supports, as shown below. A mass m drops from a height l and adheres to the rigid block without rebounding. If the spring constant of each support is k, find the natural frequency of vibration of the system (a) without mass m, and (b) with the mass m. Also find the resulting motion of the system in case (b).(p.72 of NP)

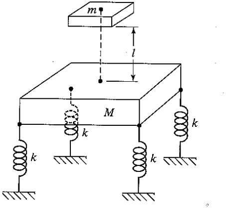

Figure: Figures for example 9.

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Example 9 — # 2

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Example 9 — # 3

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Example 9 — # 4

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Example 9 — # 5

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Example 9 — # 6

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2.11. Different Representations of Response (T 2.2.5, T 1.10+Notes) — # 1

Amplitude-Phase Form

$$\boxed{x = x_h = x_o \cos \omega_n t + \frac{\dot{x}_0}{\omega_n} \sin \omega_n t, \quad \omega_n = \sqrt{\frac{k}{m}}}$$

Phase-lag form: $x(t) = A \cos (\omega_n t - \phi_0), A = \sqrt{x_0^2 + \left(\frac{\dot{x}_0}{\omega_n}\right)^2}, \phi_0 = \tan^{-1}\left(\frac{\dot{x}_0}{x_0 \omega_n}\right)$

(8a)

Phase-lead form: $x(t) = A \cos (\omega_n t + \phi_0), A = \sqrt{x_0^2 + \left(\frac{\dot{x}_0}{\omega_n}\right)^2}, \phi_0 = \tan^{-1}\left(-\frac{\dot{x}_0}{x_0 \omega_n}\right)$

(8b)

2.11. Different Representations of Response (T 2.2.5, T 1.10+Notes) — # 2

Complex variable representation

$$x(t) = Re\left[Ce^{j\omega_n t}\right] \tag{9}$$

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Both the amplitude-phase forms and the complex variable representation of free vibration offers a graphical representation of the vibrations. This graphical method can be used to our great advantage to gain insight into vibration.

Rotating Vector Representation — # 1

The free vibration response can be visualised as a sum of projections of two rotating vectors of amplitudes A_1 and A_2, respectively, as shown below.(p.77 of NP)

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Rotating Vector Representation — # 2

The response forms mentioned in Eq.(8a) and Eq.(8b) are particularly suitable to be visualised as rotating vectors. The response in Eq.(8a) can be visualised as the horizontal projection of a rotating vector of amplitude *A* and phase lag ϕ_0 as shown below. (p.78 of NP)

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Rotating Vector Representation — # 3

Similarly, the response in Eq.(8b) can be visualised as the horizontal projection of a rotating vector of amplitude A and phase lead ϕ_0 as shown above.(p.78 of NP)

Rotating Vector Representation — # 4

Another visualisation is provided by a rotating vector in a complex plane. This rotating vector is called a phasor. Let us represent the displacement, velocity, and acceleration as complex numbers, and then as rotating vectors in the complex plane. We will take the form given Eq.(8a) for the displacement:

$$x(t) = A\cos(\omega_n t - \phi_0) = Re\left[Ae^{j(\omega_n t - \phi_0)}\right]$$

$$\dot{x}(t) = -A\omega_n \sin(\omega_n t - \phi_0) = A\omega_n \cos\left(\omega_n t - \phi_0 + 90^0\right)$$

$$= Re\left[A\omega_n e^{j(\omega_n t - \phi_0 + 90^0)}\right] = Re\left[j\omega_n A e^{j(\omega_n t - \phi_0)}\right]$$

$$\ddot{x}(t) = -A\omega_n^2 \cos(\omega_n t - \phi_0) = A\omega_n^2 \cos\left(\omega_n t - \phi_0 + 180^0\right)$$

$$= Re\left[A\omega_n^2 e^{j(\omega_n t - \phi_0 + 180^0)}\right] = Re\left[(j\omega_n)^2 A e^{j(\omega_n t - \phi_0)}\right]$$

we used $j = e^{j\frac{\pi}{2}}$ and $-1 = j^2 = e^{j\pi}$; Note $\frac{\pi}{2} rad = 90^0, \pi rad = 180^0$.

Rotating Vector Representation — # 5

We can represent the above in the following graphical form. (p.79 of NP)

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We note that, it is sufficient to know the displacement amplitude to deduce the velocity and acceleration amplitudes for a harmonic motion. **You will find this result useful in the Shaky table experiment.**

Summary — # 1

1. Undamped free vibration is specified by the second order, linear, ODE: $m\ddot{x} + kx = 0$ along with the initial conditions: an initial displacement $x(0) = x_0$ and an initial velocity $\dot{x}(0) = \dot{x}_0$.
2. Undamped free vibration response is given by $x = x_h = x_o \cos \omega_n t + \frac{\dot{x}_0}{\omega_n} \sin \omega_n t$, where the *natural frequency*, ω_n, is given by $\omega_n = \sqrt{\frac{k}{m}}$.
3. Undamped free vibration response can also be represented in amplitude-phase form $x(t) = A \cos(\omega_n t - \phi_0)$, which lends itself into a rotating vector representation of harmonic motion.
4. In a harmonic motion at frequency ω rad/s and phase lag ϕ_0, whose displacement is given by $x(t) = A \cos(\omega t - \phi_0)$, the velocity and acceleration amplitudes are related to the displacement amplitude, A, via $A_{velocity} = \omega A$ and $A_{acceleration} = \omega^2 A$. The phase lags are related via $\phi_{0,velocity} = \phi_0 - 90^0$ and $\phi_{0,acceleration} = \phi_0 - 180^0$.
5. The essential features of the free-vibration of a undamped system are shown in the sketch below.

Summary — # 2

Q: Can you explain the meaning of phase-lag from the figure shown? How will the response sketch change if the phase is a lead? (p.81 of NP)

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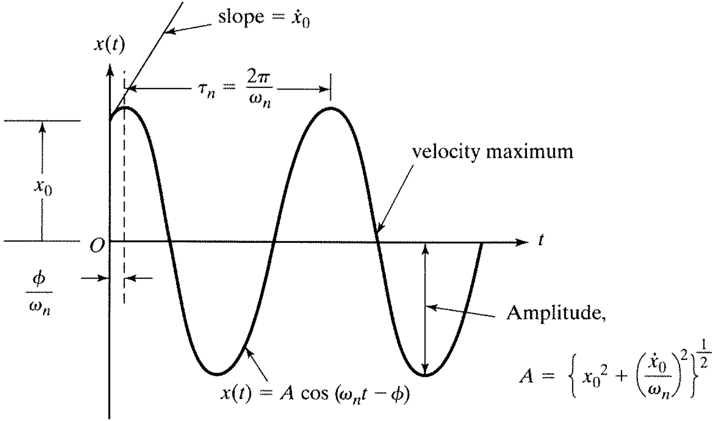

Summary — # 3

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Summary — # 4

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