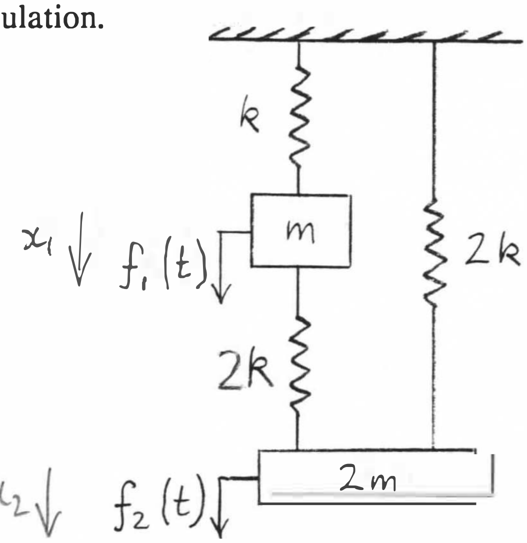
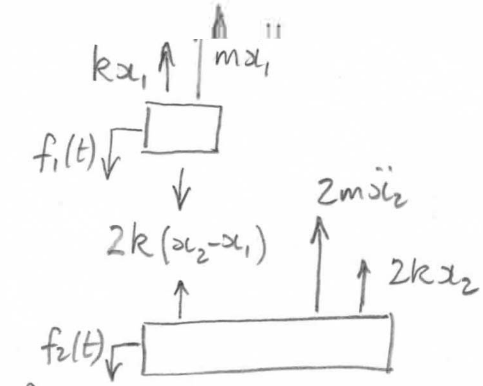
MECH 463 -- Tutorial 6

1. The diagram shows two masses connected by three springs. Assume that the masses only translate and do not rotate. Draw the free-body diagram of the system, including the two applied forces $f_1(t)$ and $f_2(t)$. Formulate the equations of motion and find the natural frequencies and mode shapes. Comment on your results. Reformulate the equations of motion into uncoupled form, including the terms due to the applied forces. Be careful to explain all significant steps in your formulation.





free body diagrams.

Vertical force balance:

$$m\ddot{x}_1 + kx_1 - 2k(x_2 - x_1) = f_1(t)$$
$$2m\ddot{x}_2 + 2kx_2 + 2k(x_2 - x_1) = f_2(t)$$

(no dynamic coupling) $\underset{\sim}{M}$ (static coupling) $\underset{\sim}{K}$ $\underline{x}$

$$\rightarrow \begin{bmatrix} m & 0 \\ 0 & 2m \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} 3k & -2k \\ -2k & 4k \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} f_1(t) \\ f_2(t) \end{bmatrix}$$

For natural frequency of free vibration calculation, set RHS = 0.

$$\rightarrow \underset{\sim}{M}\ddot{\underline{x}} + \underset{\sim}{K}\underline{x} = \underline{0}$$

Try solution $\underline{x} = \underline{X}\cos(\omega t - \phi)$ $\rightarrow (-\omega^2 \underset{\sim}{M} + \underset{\sim}{K})\underline{X}\cos(\omega t - \phi) = \underline{0}$

This must be true for all time t $\rightarrow \cos(\omega t - \phi) \neq 0$

$\rightarrow (-\omega^2 \underset{\sim}{M} + \underset{\sim}{K})\underline{X} = \underline{0}$ $\rightarrow$ For non-trivial solution $|-\omega^2 \underset{\sim}{M} + \underset{\sim}{K}| = 0$

$$\rightarrow \begin{vmatrix} 3k - \omega^2 m & -2k \\ -2k & 4k - 2\omega^2 m \end{vmatrix} = 0 \rightarrow (3k - \omega^2 m)(4k - 2\omega^2 m) - 4k^2 = 0$$

$$\rightarrow 2m^2\omega^4 - 10mk\omega^2 + 8k^2 = 0 \rightarrow \boxed{\begin{aligned} \omega^2 &= \frac{10 \pm 6}{4}\, k/m \\ &= k/m \text{ and } 4k/m \end{aligned}}$$

Let $\underline{x} = \begin{bmatrix} 1 \\ u \end{bmatrix} C\cos(\omega t - \phi)$ be a vibration mode.

For non-trivial $C\cos(\omega t-\phi)$ $\longrightarrow$ $\begin{bmatrix} 3k-\omega^2 m & -2k \\ -2k & 4k-2\omega^2 m \end{bmatrix}\begin{bmatrix} 1 \\ u \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

From first equation $\rightarrow$ $3k - \omega^2 m - 2ku = 0$ $\rightarrow$ $u = \frac{3}{2} - \omega^2 \frac{m}{2k}$

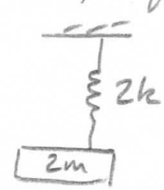
$$= \frac{3}{2} - \frac{m}{2k}\left(\frac{10 \pm 6}{4}\right) k/m = \frac{1}{4} \mp \frac{3}{4} = 1 \text{ or } -\frac{1}{2}$$

Mode shapes are $\begin{bmatrix} 1 \\ u \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ for $\omega^2 = k/m$

and $\begin{bmatrix} 1 \\ u \end{bmatrix} = \begin{bmatrix} 1 \\ -\frac{1}{2} \end{bmatrix}$ for $\omega^2 = 4k/m$

The first mode does not involve any stretching/compression of the spring between the masses. Thus, the natural frequency of this mode does not depend on this spring stiffness. This behaviour occurs because the natural frequencies of the two end subsystems

and



are the same, $\omega^2 = k/m = 2k/2m$.

We can write the equations of motion in uncoupled form

$\underset{\sim}{M}^* \ddot{\underline{p}} + \underset{\sim}{K}^* \underline{p} = \underset{\sim}{U}^T \underline{f}$ where $\underset{\sim}{M}^* = \underset{\sim}{U}^T \underset{\sim}{M}\, \underset{\sim}{U}$, $\underset{\sim}{K}^* = \underset{\sim}{U}^T \underset{\sim}{K}\, \underset{\sim}{U}$,

$\underline{x} = \underset{\sim}{U}\underline{p}$ $\quad$ $\underset{\sim}{U} = \begin{bmatrix} 1 & 1 \\ 1 & -\frac{1}{2} \end{bmatrix}$ = modal matrix $\quad$ $\underline{p}$ = principal coords.

$$\underset{\sim}{M}^* = \begin{bmatrix} 1 & 1 \\ 1 & -\frac{1}{2} \end{bmatrix}\begin{bmatrix} m & 0 \\ 0 & 2m \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -\frac{1}{2} \end{bmatrix}\begin{bmatrix} m & m \\ 2m & -m \end{bmatrix} = \begin{bmatrix} 3m & 0 \\ 0 & \frac{3}{2}m \end{bmatrix}$$

$$\underset{\sim}{K}^* = \begin{bmatrix} 1 & 1 \\ 1 & -\frac{1}{2} \end{bmatrix}\begin{bmatrix} 3k & -2k \\ -2k & 4k \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -\frac{1}{2} \end{bmatrix}\begin{bmatrix} k & 4k \\ 2k & -4k \end{bmatrix} = \begin{bmatrix} 3k & 0 \\ 0 & 6k \end{bmatrix}$$

$$\longrightarrow \begin{bmatrix} 3m & 0 \\ 0 & \frac{3}{2}m \end{bmatrix}\begin{bmatrix} \ddot{p}_1 \\ \ddot{p}_2 \end{bmatrix} + \begin{bmatrix} 3k & 0 \\ 0 & 6k \end{bmatrix}\begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = \begin{bmatrix} f_1(t) + f_2(t) \\ f_1(t) - \frac{1}{2}f_2(t) \end{bmatrix}$$

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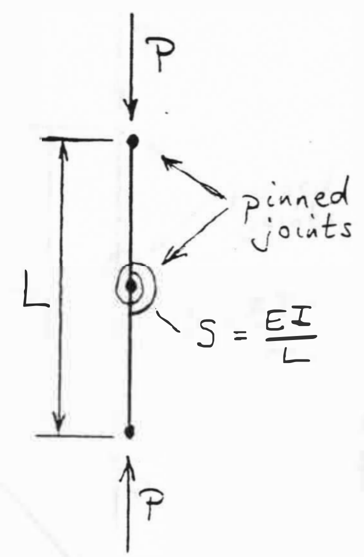
2. The diagram shows an idealized model of a pin-ended strut under axial compression. The strut is divided into two parts connected by a torsional spring of stiffness S = EI/L. The mass density of the strut is ρ and the cross-section area is A. Formulate the equation of motion of this 1-DOF system, and determine the natural frequency as a function of the axial force P. Hence find the critical load for buckling.



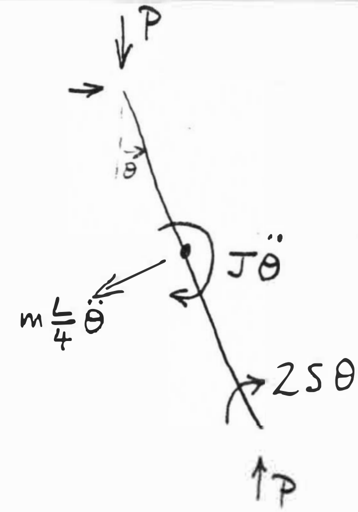
$$\omega = \frac{1}{L}\sqrt{\frac{EI}{\rho A}\left(\frac{48}{L^2} - \frac{12P}{EI}\right)} \qquad P_{cr} = 4EI/L^2$$

The system is symmetrical, so we need only consider half of it.

Let θ = rotation of each half.

The rotation of the spring is 2θ.

For one half, $m = \rho A \frac{L}{2}$, $J = m\frac{(L/2)^2}{12} = \frac{\rho A L^3}{96}$



By symmetry, there is no horizontal reaction force at the spring.

Take moments about upper end:

$$J\ddot{\theta} + m\frac{L}{4}\ddot{\theta}\cdot\frac{L}{4} + 2S\theta - P\frac{L}{2}\theta = 0$$

putting $S = \frac{EI}{L}$

$$\rightarrow \left(\frac{\rho A L^3}{96} + \frac{\rho A L^3}{32}\right)\ddot{\theta} + \left(2\cdot\frac{EI}{L} - P\frac{L}{2}\right)\theta = 0$$

$$= \frac{\rho A L^3}{24}\ddot{\theta} + \left(\frac{2EI}{L} - \frac{PL}{2}\right)\theta = 0$$

By inspection, the natural frequency. $\omega^2 = \dfrac{\frac{2EI}{L} - \frac{PL}{2}}{\rho A L^3/24}$

$$\rightarrow \boxed{\omega = \frac{1}{L}\sqrt{\frac{EI}{\rho A}\left(\frac{48}{L^2} - \frac{12P}{EI}\right)}}$$

The critical load for buckling occurs when ω becomes zero.

$$\rightarrow \boxed{P_{CR} = 4EI/L^2}$$