MECH468 : Modern Control Engineering
MECH509 : Controls

L14 : Kalman Decomposition

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Zoom lecture to be recorded and posted on Canvas

Review & Today's topic

- So far, we learned controllability and observability:
  - Definition
  - Condition
  - Decomposition
  - Duality
- Today, we will study the combination of decompositions for controllability and observability, called *Kalman decomposition*.

Rudolf Emil Kalman (1930-2016)

Hungarian-born American

Electrical engineer & mathematician

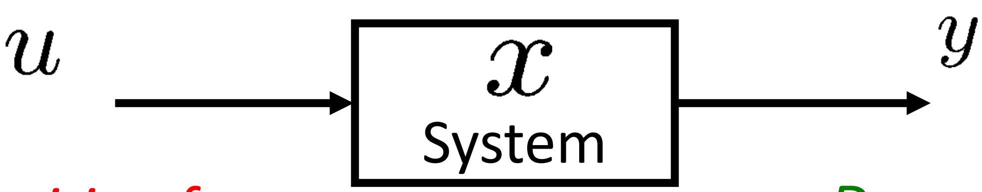
During 1950s & 60s, he developed *state-space control theory*

- Controllability, observability
- Linear quadratic regulator
- Kalman filter

Most of the theory in this course have been established by Kalman!

Two decompositions

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t), & A \in \mathbb{R}^{n \times n},\ B \in \mathbb{R}^{n \times p} \\ y(t) = Cx(t) + Du(t), & C \in \mathbb{R}^{q \times n},\ D \in \mathbb{R}^{q \times p} \end{cases}$$



Decomposition for controllability

Decomposition for observability

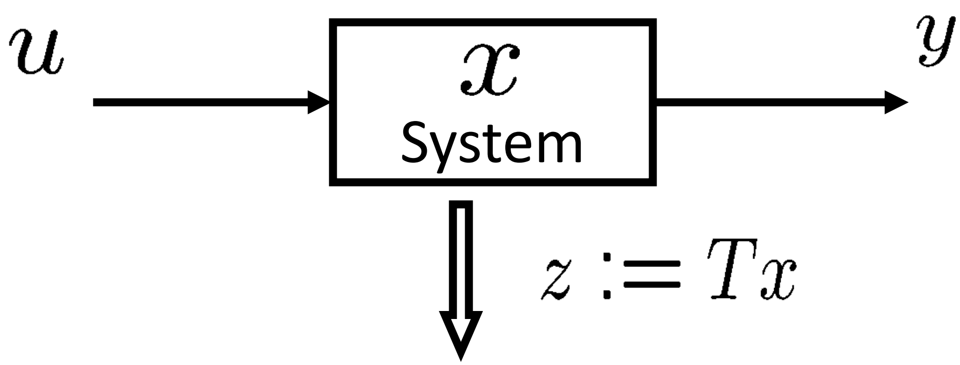
$$z := Tx$$

Controllable part → $\begin{bmatrix} z_c \\ z_{\bar{c}} \end{bmatrix}$ ← Uncontrollable part

$\begin{bmatrix} z_o \\ z_{\bar{o}} \end{bmatrix}$ ← Observable part / Unobservable part

Kalman decomposition (idea)

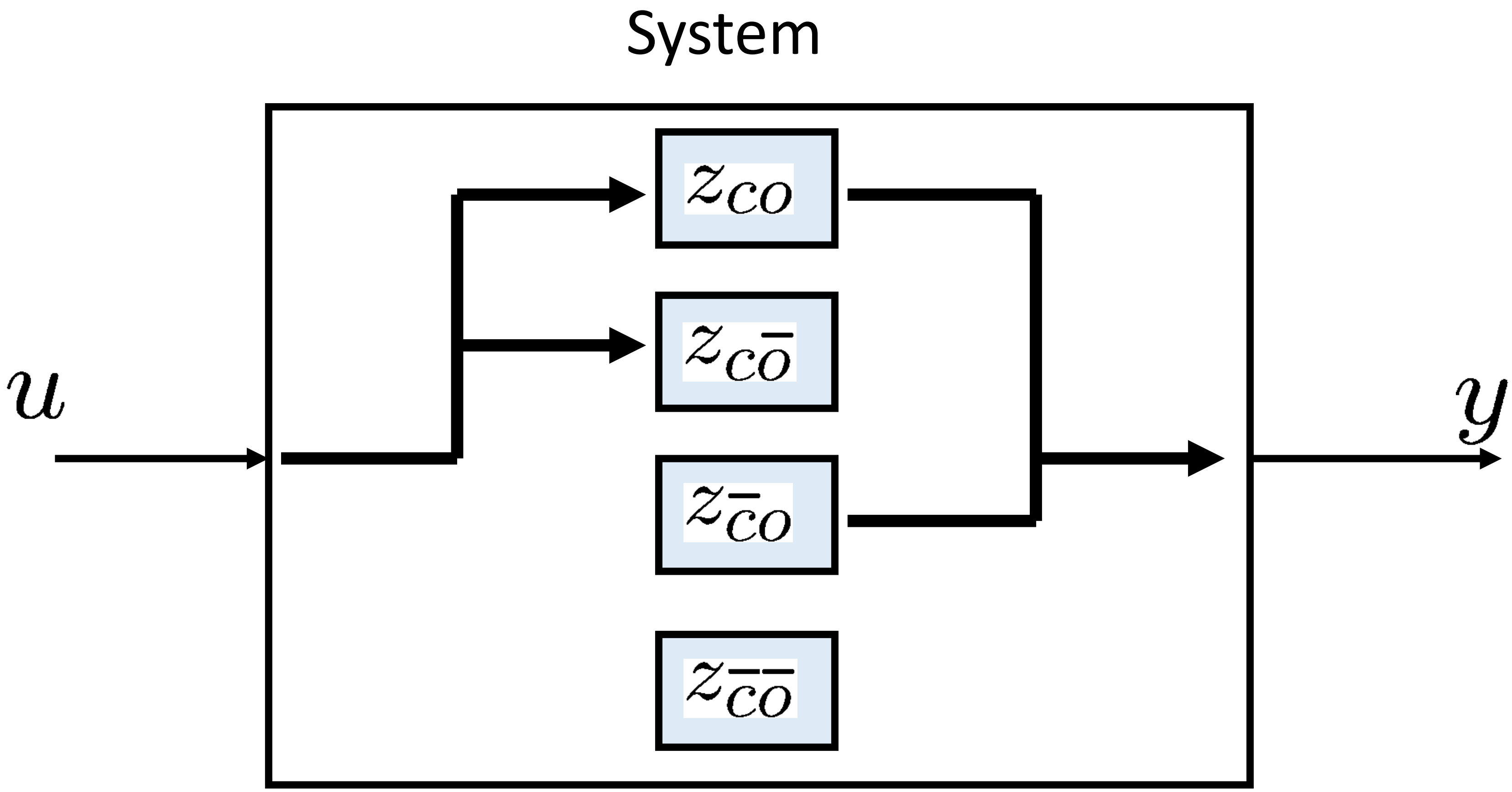
$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t), & A \in \mathbb{R}^{n \times n},\ B \in \mathbb{R}^{n \times p} \\ y(t) = Cx(t) + Du(t), & C \in \mathbb{R}^{q \times n},\ D \in \mathbb{R}^{q \times p} \end{cases}$$



$$\begin{bmatrix} z_{co} \\ z_{c\bar{o}} \\ z_{\bar{c}o} \\ z_{\bar{c}\bar{o}} \end{bmatrix}$$

- z_{co} ← Controllable & observable part
- $z_{c\bar{o}}$ ← Controllable & unobservable part
- $z_{\bar{c}o}$ ← Uncontrollable & observable part
- $z_{\bar{c}\bar{o}}$ ← Uncontrollable & unobservable part

Kalman decomposition
Conceptual figure (Not block-diagram)



Kalman decomposition

- Every SS model can be transformed by *z=Tx* for some appropriate *T* into a canonical form:

$$\begin{cases} \begin{bmatrix} \dot z_{co} \\ \dot z_{c\bar o} \\ \dot z_{\bar c o} \\ \dot z_{\bar c \bar o} \end{bmatrix} = \underbrace{\begin{bmatrix} A_{co} & 0 & A_{13} & 0 \\ A_{21} & A_{c\bar o} & A_{23} & A_{24} \\ 0 & 0 & A_{\bar c o} & 0 \\ 0 & 0 & A_{43} & A_{\bar c \bar o} \end{bmatrix}}_{TAT^{-1}} \begin{bmatrix} z_{co} \\ z_{c\bar o} \\ z_{\bar c o} \\ z_{\bar c \bar o} \end{bmatrix} + \underbrace{\begin{bmatrix} B_{co} \\ B_{c\bar o} \\ 0 \\ 0 \end{bmatrix}}_{TB} u \\ y = \underbrace{\begin{bmatrix} C_{co} & 0 & C_{\bar c o} & 0 \end{bmatrix}}_{CT^{-1}} \begin{bmatrix} z_{co} \\ z_{c\bar o} \\ z_{\bar c o} \\ z_{\bar c \bar o} \end{bmatrix} + Du \end{cases}$$

Note the decomposition structure for controllability.

Kalman decomposition 2

- If the second and the third state vectors are exchanged, one can get another form:

$$
\begin{cases}
\begin{bmatrix} \dot{z}_{co} \\ \dot{z}_{c\bar{o}} \\ \dot{z}_{\bar{c}o} \\ \dot{z}_{\bar{c}\bar{o}} \end{bmatrix} = \underbrace{\begin{bmatrix} A_{co} & 0 & A_{13} & 0 \\ A_{21} & A_{c\bar{o}} & A_{23} & A_{24} \\ 0 & 0 & A_{\bar{c}o} & 0 \\ 0 & 0 & A_{43} & A_{\bar{c}\bar{o}} \end{bmatrix}}_{TAT^{-1}} \begin{bmatrix} z_{co} \\ z_{c\bar{o}} \\ z_{\bar{c}o} \\ z_{\bar{c}\bar{o}} \end{bmatrix} + \underbrace{\begin{bmatrix} B_{co} \\ B_{c\bar{o}} \\ 0 \\ 0 \end{bmatrix}}_{TB} u \\
y = \underbrace{\begin{bmatrix} C_{co} & 0 & C_{\bar{c}o} & 0 \end{bmatrix}}_{CT^{-1}} \begin{bmatrix} z_{co} \\ z_{c\bar{o}} \\ z_{\bar{c}o} \\ z_{\bar{c}\bar{o}} \end{bmatrix} + Du
\end{cases}
$$

Note the decomposition structure for observability.

Remark

- It may happen that some states are missing.
    - Ex. No controllable-and-unobservable part

$$
\begin{cases}
\begin{bmatrix} \dot{z}_{co} \\ \dot{z}_{c\bar{o}} \\ \dot{z}_{\bar{c}o} \\ \dot{z}_{\bar{c}\bar{o}} \end{bmatrix} = \underbrace{\begin{bmatrix} A_{co} & 0 & A_{13} & 0 \\ A_{21} & A_{c\bar{o}} & A_{23} & A_{24} \\ 0 & 0 & A_{\bar{c}o} & 0 \\ 0 & 0 & A_{43} & A_{\bar{c}\bar{o}} \end{bmatrix}}_{TAT^{-1}} \begin{bmatrix} z_{co} \\ z_{c\bar{o}} \\ z_{\bar{c}o} \\ z_{\bar{c}\bar{o}} \end{bmatrix} + \underbrace{\begin{bmatrix} B_{co} \\ B_{c\bar{o}} \\ 0 \\ 0 \end{bmatrix}}_{TB} u \\
y = \underbrace{\begin{bmatrix} C_{co} & 0 & C_{\bar{c}o} & 0 \end{bmatrix}}_{CT^{-1}} \begin{bmatrix} z_{co} \\ z_{c\bar{o}} \\ z_{\bar{c}o} \\ z_{\bar{c}\bar{o}} \end{bmatrix} + Du
\end{cases}
$$

Remarks

- (A_{co}, B_{co}): controllable & (A_{co}, C_{co}): observable
- Transfer function is determined by ONLY controllable & observable parts.

$$(CT^{-1})(sI - TAT^{-1})^{-1}(TB) + D = C_{co}(sI - A_{co})^{-1}B_{co} + D$$

- If uncontrollable and/or unobservable parts are unstable, we need to change the structure (actuators/sensors) of the system, because no output feedback control can stabilize the system. (next slide)

Eigenvalues of A-matrix

$\sigma(M)$: set of eigenvalues of M

$$\sigma\left(TAT^{-1}\right) = \sigma\left(\left[\begin{array}{cc:cc} A_{co} & 0 & A_{13} & 0 \\ A_{21} & A_{c\bar{o}} & A_{23} & A_{24} \\ \hdashline 0 & 0 & A_{\bar{c}o} & 0 \\ 0 & 0 & A_{43} & A_{\bar{c}\bar{o}} \end{array}\right]\right)$$

$$= \sigma\left(\begin{bmatrix} A_{co} & 0 \\ A_{21} & A_{c\bar{o}} \end{bmatrix}\right) \cup \sigma\left(\begin{bmatrix} A_{\bar{c}o} & 0 \\ A_{43} & A_{\bar{c}\bar{o}} \end{bmatrix}\right)$$

$$= \sigma(A_{co}) \cup \underbrace{\sigma(A_{c\bar{o}}) \cup \sigma(A_{\bar{c}o}) \cup \sigma(A_{\bar{c}\bar{o}})}_{\text{These have to be stable for output feedback control.}}$$

These have to be stable
for output feedback control.

How to find *T*? (review)

- For controllability, we used *image space* of controllability matrix.

$$T^{-1} := [T_c, T_{\bar{c}}] \quad \begin{cases} T_c & : \text{A basis of } \mathrm{Im}\,\mathcal{C} \\ T_{\bar{c}} & : \text{any complement of } T_c \text{ in } \mathbb{R}^n \end{cases}$$

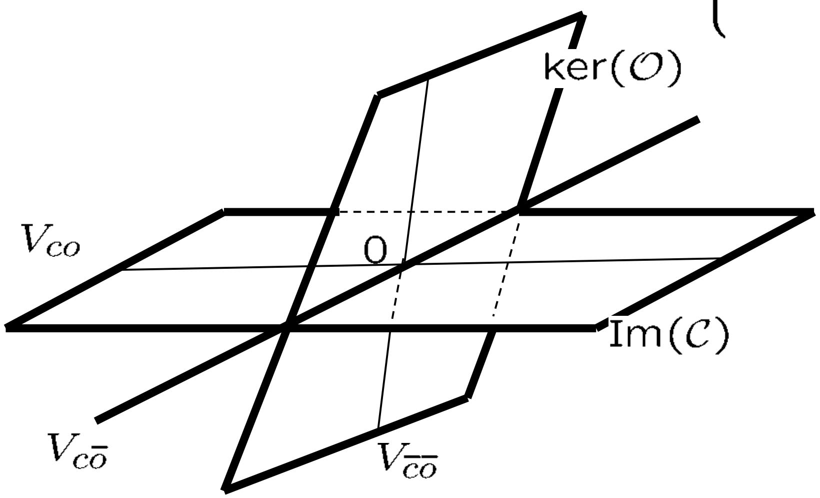
- For observability, we used *kernel space* of observability matrix.

$$T^{-1} := [T_o, T_{\bar{o}}] \quad \begin{cases} T_{\bar{o}} & : \text{A basis of } \ker\mathcal{O} \\ T_o & : \text{any complement of } T_{\bar{o}} \text{ in } \mathbb{R}^n \end{cases}$$

How to find T for Kalman decomposition?

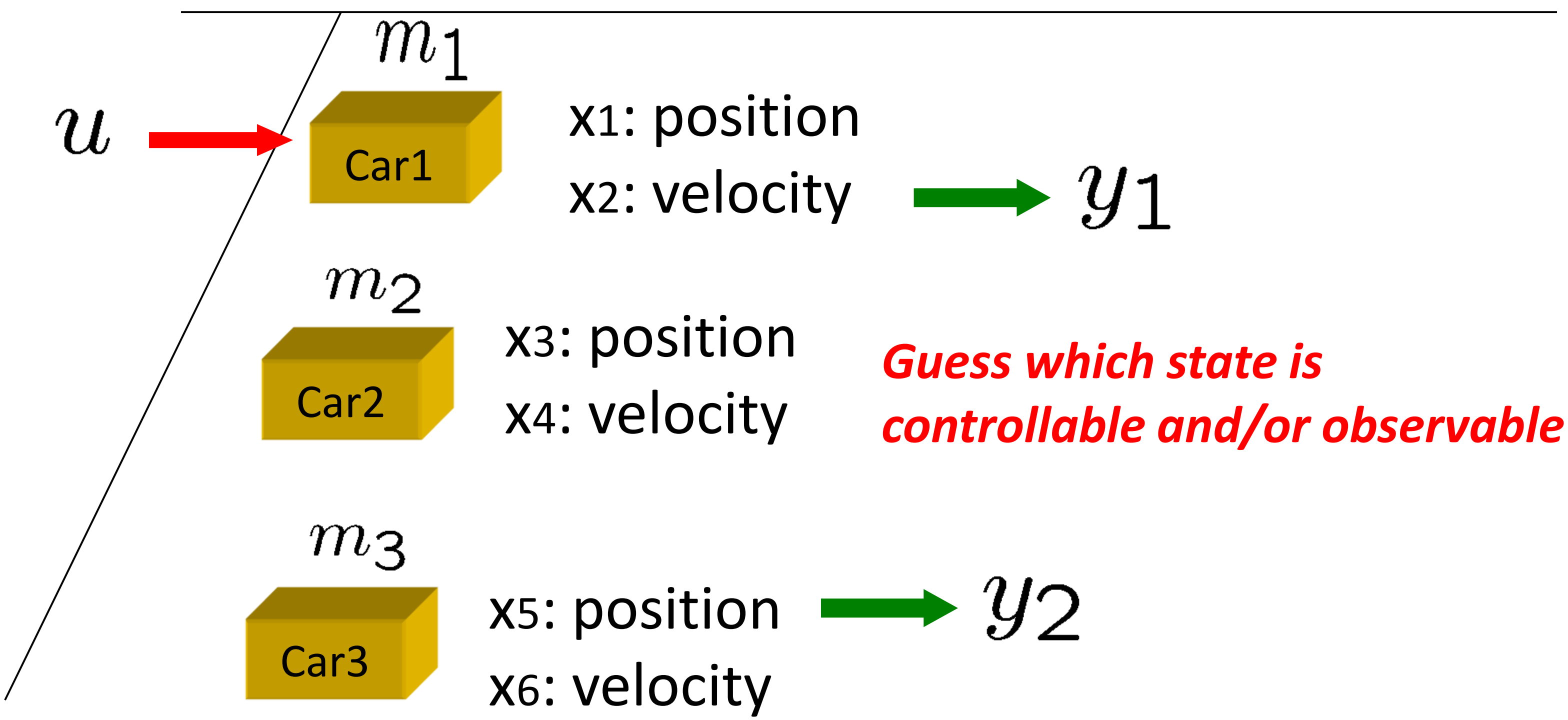
$$T^{-1} := [T_{co}, T_{c\bar{o}}, T_{\bar{c}o}, T_{\bar{c}\bar{o}}]$$

$$\begin{cases}
T_{c\bar{o}} & : \text{ basis for the subspace} \\
 & V_{c\bar{o}} := \mathrm{Im}(\mathcal{C}) \cap \ker(\mathcal{O}) \\
T_{co} & : \text{ basis for any complement of } V_{c\bar{o}} \text{ in } \mathrm{Im}(\mathcal{C}) \\
 & \mathrm{Im}(\mathcal{C}) = V_{c\bar{o}} \oplus V_{co} \\
T_{\bar{c}\bar{o}} & : \text{ basis for any complement of } V_{c\bar{o}} \text{ in } \ker(\mathcal{O}) \\
 & \ker(\mathcal{O}) = V_{c\bar{o}} \oplus V_{\bar{c}\bar{o}} \\
T_{\bar{c}o} & : \text{ basis for the subspace } V_{\bar{c}o} \text{ s.t.} \\
 & \mathbb{R}^n = V_{co} \oplus V_{c\bar{o}} \oplus V_{\bar{c}\bar{o}} \oplus V_{\bar{c}o}
\end{cases}$$



Very simple example: revisited

- Three cars with one input and two outputs



Guess which state is controllable and/or observable!

SS model of three car example

$$
\begin{cases}
\dot{x}(t) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} u(t) \\
y(t) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} x(t)
\end{cases}
$$

Velocity of 1st mass

Position of 3rd mass

Derivation of T

- Controllability matrix

$$\mathcal{C} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \mathrm{Im}\,\mathcal{C} = \mathrm{span}\{e_1, e_2\}$$

- Observability matrix

$$\mathcal{O} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \ker\mathcal{O} = \mathrm{span}\{e_1, e_3, e_4\}$$

- (Inverse of T)=[e_2,e_1,{e_5,e_6},{e_3,e_4}](=T in this case)

$$T_{co}\ T_{c\bar{o}}\quad T_{\bar{c}o}\qquad T_{\bar{c}\bar{o}}$$

Example (cont'd)

- After transformation …

$$\begin{array}{c} z_{co} \\ z_{c\bar{o}} \\ z_{\bar{c}o} \\ \\ z_{\bar{c}\bar{o}} \\ \\ \end{array} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \\ z_6 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}}_{T} x = \begin{bmatrix} x_2 \\ x_1 \\ x_5 \\ x_6 \\ x_3 \\ x_4 \end{bmatrix}$$

$$\begin{cases} \dot{z}(t) = \underbrace{\left[\begin{array}{c|c|cc|cc} 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]}_{TAT^{-1}} z(t) + \underbrace{\left[\begin{array}{c} 1 \\ \hline 0 \\ \hline 0 \\ 0 \\ \hline 0 \\ 0 \end{array}\right]}_{TB} u(t) \\ y(t) = \underbrace{\left[\begin{array}{c|c|cc|cc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{array}\right]}_{CT^{-1}} z(t) \end{cases}$$

Matlab command “minreal.m”

```
>> help minreal
 MINREAL  Minimal realization and pole-zero cancellation.


   For a state-space model SYS=SS(A,B,C,D),
      [MSYS,U] = MINREAL(SYS)
   also returns an orthogonal matrix U such that (U*A*U',U*B,C*U')
   is a Kalman decomposition of (A,B,C).
```

Remark: Matlab may return matrices corresponding the states in an order different from the order in Slide 7.

Summary

- Kalman decomposition
  - Combination of
    - Decomposition for controllability
    - Decomposition for observability
  - How to find coordinate transformation matrix T
    - Image space of controllability matrix
    - Kernel space of observability matrix
- Next, controllability and observability for discrete-time systems