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MECH468 : Modern Control Engineering
MECH509 : Controls

L5 : Solution to continuous-time LTI SS model

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Zoom lecture to be recorded and posted on Canvas

Course plan

Topics	CT	DT
Modeling	→	
Stability		
Controllability/observability		
Realization		
State feedback/observer		
LQR/Kalman filter		

Acronyms and notation

- SS : State-space
- CT : Continuous-Time
- DT : Discrete-Time
- LTI : Linear Time-Invariant
- LTV : Linear Time-Varying
- A:=B : A is defined by B.
- $\mathbb{R}$: Set of real numbers
- $\mathbb{C}$: Set of complex numbers

Today's topic

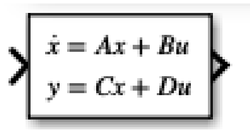
- Analytically solve LTI state-space model equation

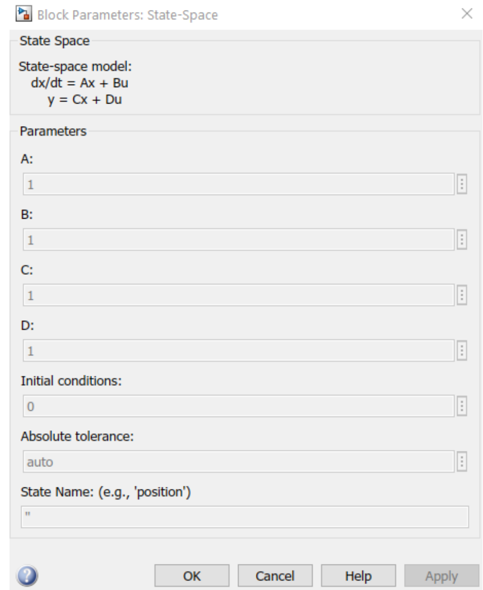
$$\begin{cases} \dot{x}(t) &= Ax(t) + Bu(t), \quad x(0) = x_0 \\ y(t) &= Cx(t) + Du(t) \end{cases}$$

- We never solve SS equation analytically in practice.
- Why do we need to know how to solve analytically?
  - To derive theoretical analysis and design results by using the explicit solution to the state-space model.
  - Useful in discretization (next lecture)
  - To interpret (or debug) what Matlab simulates. (next slide)

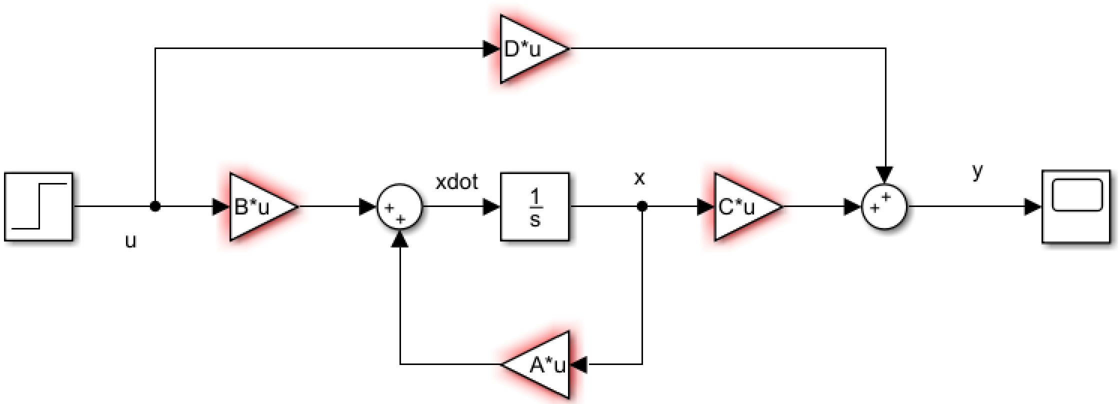
Simulation for state-space model in Simulink

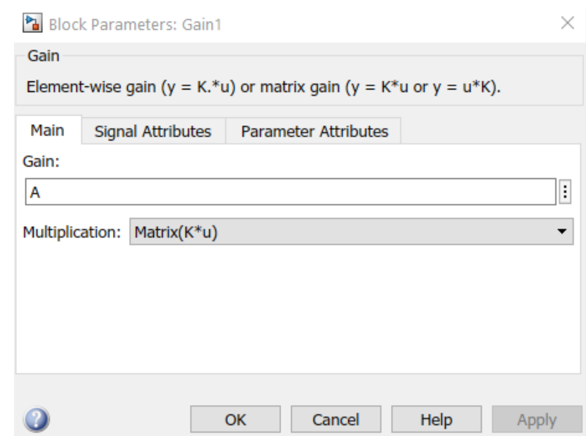
- Using state-space block





- Using integrator block





Solution to CT LTI SS model

- CT LTI state-space model $\begin{cases} \dot{x}(t) &= Ax(t) + Bu(t), \quad x(0) = x_0 \\ y(t) &= Cx(t) + Du(t) \end{cases}$

- Solution

Memorize this!

$$
\begin{aligned}
x(t) &= e^{At}x_0 + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau \\
y(t) &= Ce^{At}x_0 + C\int_0^t e^{A(t-\tau)}Bu(\tau)d\tau + Du(t)
\end{aligned}
$$

where the matrix exponential is defined as

$$
e^{At} := I + At + \frac{1}{2!}A^2t^2 + \frac{1}{3!}A^3t^3 + \cdots
$$

Remarks

- We can see the linearity of the system from this equation. (Proof: next slide)
- Verification : $x(0) = x_0$ since $e^{A0} = I$

$$\begin{aligned}
\dot{x}(t) &= Ae^{At}x_0 + \frac{d}{dt}\left(e^{At}\int_0^t e^{-A\tau}Bu(\tau)d\tau\right) \\
&= Ae^{At}x_0 + Ae^{At}\int_0^t e^{-A\tau}Bu(\tau)d\tau + e^{At}e^{-At}Bu(t) \\
&= Ax(t) + Bu(t)
\end{aligned}$$

- For LTV case, the solution looks similar, but more complicated. (not covered in this course)

Proof of linearity

- Suppose $\left.\begin{array}{r} x(0) = x_{i0} \\ u_i(t), t \geq 0 \end{array}\right\} \Rightarrow y_i(t), t \geq 0, \quad i = 1, 2$

  i.e.

$$y_i(t) \;=\; Ce^{At}x_{i0} + C\int_0^t e^{A(t-\tau)}Bu_i(\tau)d\tau + Du_i(t)$$

- Now, we take new initial condition and input as

$$\begin{array}{l} x(0) = \alpha_1 x_{10} + \alpha_2 x_{20} \\ u(t) = \alpha_1 u_1(t) + \alpha_2 u_2(t), t \geq t_0 \end{array}$$

- Then, output is $\begin{array}{l} Ce^{At}(\alpha_1 x_{10} + \alpha_2 x_{20}) \\ +C\int_0^t e^{A(t-\tau)}B(\alpha_1 u_1(\tau) + \alpha_2 u_2(\tau))d\tau \\ +D(\alpha_1 u_1(t) + \alpha_2 u_2(t)) = \cdots = \alpha_1 y_1(t) + \alpha_2 y_2(t) \end{array}$

Matrix exponential

- Definition $e^{At} := I + At + \frac{1}{2!}A^2t^2 + \frac{1}{3!}A^3t^3 + \cdots$
- Property $\frac{d}{dt}(e^{At}) = Ae^{At} = e^{At}A$
- How to compute analytically?
    1. Definition of matrix exponential
    2. Laplace transform
    3. Diagonal form (or Jordan form)
    4. (Cayley-Hamilton Theorem)

    (Numerically, in Matlab, use "expm.m", NOT "exp.m")

1. Definition of matrix exponential

- Nilpotent matrix $(A^q = 0 \text{ for some } q)$

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow A^2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad A^3 = A^4 = \cdots = 0$$

$$\Rightarrow e^{At} := I + At + \frac{(At)^2}{2!} + \underbrace{\cdots}_{=0} = \begin{bmatrix} 1 & t & t^2/2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix}$$

- Diagonal matrix

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \Rightarrow A^k = \begin{bmatrix} 2^k & 0 \\ 0 & 3^k \end{bmatrix}$$

$$\Rightarrow e^{At} := I + At + \frac{(At)^2}{2!} + \cdots = \begin{bmatrix} e^{2t} & 0 \\ 0 & e^{3t} \end{bmatrix}$$

Shift matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \ddots & & \vdots \\ 0 & 0 & 0 & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & \ddots & 0 \\ \vdots & & & \ddots & \ddots & 1 \\ 0 & \cdots & \cdots & \cdots & 0 & 0 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \ddots & & \vdots \\ 0 & 0 & 0 & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & \ddots & 1 \\ \vdots & & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & \cdots & 0 & 0 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 0 & 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & 0 & \ddots & & \vdots \\ 0 & 0 & 0 & \ddots & \ddots & 1 \\ \vdots & & \ddots & \ddots & \ddots & 0 \\ \vdots & & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & \cdots & 0 & 0 \end{bmatrix}$$

2. Laplace transform

- Formula $e^{At} = \mathcal{L}^{-1}\left\{(sI - A)^{-1}\right\}$ $\left(\begin{array}{c}\text{cf. scalar case}\\ e^{at} = \mathcal{L}^{-1}\left\{\dfrac{1}{s-a}\right\}\end{array}\right)$

- Ex:

Partial fraction expansion (next slide)

$$A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \Rightarrow (sI-A)^{-1} = \cdots = \frac{1}{s+1}\underbrace{\begin{bmatrix} 2 & 1 \\ -2 & -1 \end{bmatrix}}_{K_1} + \frac{1}{s+2}\underbrace{\begin{bmatrix} -1 & -1 \\ 2 & 2 \end{bmatrix}}_{K_2}$$

$$\Rightarrow e^{At} = \mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\}K_1 + \mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\}K_2 = e^{-t}K_1 + e^{-2t}K_2$$

- Ex:

$$A = \begin{bmatrix} \sigma & \omega \\ -\omega & \sigma \end{bmatrix} \Rightarrow (sI - A)^{-1} = \frac{1}{(s-\sigma)^2 + \omega^2}\begin{bmatrix} s-\sigma & \omega \\ -\omega & s-\sigma \end{bmatrix}$$

$$\Rightarrow e^{At} = e^{\sigma t}\begin{bmatrix} \cos\omega t & \sin\omega t \\ -\sin\omega t & \cos\omega t \end{bmatrix}$$

Partial fraction expansion

$$(sI - A)^{-1} = \begin{bmatrix} s & -1 \\ 2 & s+3 \end{bmatrix}^{-1}$$

$$= \frac{1}{s^2 + 3s + 2} \begin{bmatrix} s+3 & 1 \\ -2 & s \end{bmatrix} = \frac{1}{s+1} K_1 + \frac{1}{s+2} K_2$$

$$\Rightarrow \quad \begin{bmatrix} s+3 & 1 \\ -2 & s \end{bmatrix} = (s+2)K_1 + (s+1)K_2$$

$$\Rightarrow \quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} s + \begin{bmatrix} 3 & 1 \\ -2 & 0 \end{bmatrix} = (K_1 + K_2)s + 2K_1 + K_2$$

$$\Rightarrow \quad \begin{cases} K_1 + K_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ 2K_1 + K_2 = \begin{bmatrix} 3 & 1 \\ -2 & 0 \end{bmatrix} \end{cases}$$

3. Diagonal form

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- Suppose that we have distinct eigenvalues and corresponding eigenvectors as

$$Ax_i = \lambda_i x_i, \;\; i = 1, \ldots, n$$

- Then, $e^{At} = \underline{T} e^{Dt} \underline{T^{-1}}$ $\quad T := [x_1, \cdots, x_n] \in \mathbb{C}^{n \times n}$

Similarity transformation

$$D := \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} \in \mathbb{C}^{n \times n}$$

- Remark: Diagonalization of *A* by nonsingular (i.e. invertible) matrix *T* is NOT always possible!

Diagonal form: proof

- Suppose that we have eigenvalues/vectors as

$$Ax_i = \lambda_i x_i, \;\; i = 1, \ldots, n$$

which can be written in a matrix form as

$$A \underbrace{[x_1, \cdots, x_n]}_{T} = \underbrace{[x_1, \cdots, x_n]}_{T} \underbrace{\begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}}_{D}$$

Here, *T* is nonsingular. $A = TDT^{-1}, \;\; A^2 = TD^2T^{-1}, \cdots$

$$A^n = (TD\cancel{T^{-1}})(\cancel{T}D\cancel{T^{-1}}) \cdots (\cancel{T}DT^{-1}) = TD^nT^{-1}$$

Diagonal form: proof (cont'd)

- By definition,

$$
\begin{aligned}
e^{At} &:= I + At + \frac{A^2t^2}{2!} + \frac{A^3t^3}{3!} + \cdots \\
&= \underline{I} + TDT^{-1}t + \frac{TD^2T^{-1}t^2}{2!} + \frac{TD^3T^{-1}t^3}{3!} + \cdots \\
&= \underline{TT^{-1}} + TDT^{-1}t + \frac{TD^2T^{-1}t^2}{2!} + \frac{TD^3T^{-1}t^3}{3!} + \cdots \\
&= T\left\{\underline{I + Dt + \frac{D^2t^2}{2!} + \frac{D^3t^3}{3!} + \cdots}\right\}T^{-1} \\
&= T\underline{e^{Dt}}T^{-1}
\end{aligned}
$$

Diagonal form: Example $A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$

- Eigenvalue

$$\det(\lambda I - A) = 0 \Longrightarrow \lambda = -1, -2$$

- Eigenvector

$$\lambda_1 = -1 \Rightarrow (\lambda_1 I - A)x_1 = 0 \Rightarrow x_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\lambda_2 = -2 \Rightarrow (\lambda_2 I - A)x_2 = 0 \Rightarrow x_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

- Matrix exponential

$$e^{At} = Te^{Dt}T^{-1} = \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} e^{-t} & \\ & e^{-2t} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix}^{-1} = \cdots$$

Summary

- Solution to CT LTI systems
- Computation of matrix exponential
  - By definition
  - By Laplace transform
  - By diagonal form (or Jordan form)
  - (By Cayley-Hamilton Theorem)
- Next,
  - Discretization
  - Solution to discrete-time LTI systems

Appendix: Jordan form

- For any matrix *A*, there is a nonsingular *T* s.t.

$$J = T^{-1}AT = \begin{bmatrix} J_1 & & \\ & \ddots & \\ & & J_k \end{bmatrix} \qquad J_j := \underbrace{\begin{bmatrix} \lambda_j & 1 & & \\ & \lambda_j & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_j \end{bmatrix}}_{d_j}$$

- Matrix exponential $e^{At} = T \begin{bmatrix} e^{J_1 t} & & \\ & \ddots & \\ & & e^{J_k t} \end{bmatrix} T^{-1}$

$$\begin{aligned} e^{J_j t} &= e^{(\lambda_j I + S_j)t} \\ &= e^{\lambda_j t} \begin{bmatrix} 1 & t & t^2/2 & \cdots & t^{d_j-1}/(d_j-1)! \\ & 1 & t & \ddots & \vdots \\ & & \ddots & \ddots & t^2/2 \\ & & & \ddots & t \\ & & & & 1 \end{bmatrix} \end{aligned} \qquad S_j := \underbrace{\begin{bmatrix} 0 & 1 & & \\ & 0 & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{bmatrix}}_{d_j}$$

Cayley-Hamilton Theorem (optional)

- For an *n*-by-*n* matrix *A*, the following holds:

$$\Delta(A) = A^n + \alpha_1 A^{n-1} + \cdots + \alpha_{n-1} A + \alpha_n I_n = 0$$

where *characteristic polynomial* of *A* is

$$\Delta(\lambda) := \det(\lambda I - A) = \lambda^n + \alpha_1 \lambda^{n-1} + \cdots + \alpha_{n-1}\lambda + \alpha_n$$

This implies that every polynomial of *A* (*n-by-n*) can be expressed as a linear combination of $\{I, A, \cdots, A^{n-1}\}$

$$\underline{f}(A) = \beta_0 I_n + \beta_1 A + \cdots + \beta_{n-1} A^{n-1}$$

This can be a general polynomial of very high order.

C-H Theorem: example (optional)

- Ex $A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \Rightarrow e^{At} = \underline{\beta_0(t)}I + \underline{\beta_1(t)}A$

How to compute these?

- Compute eigenvalues of *A* $\lambda(A) = \underbrace{-1}_{\lambda_1}, \underbrace{-2}_{\lambda_2}$
- Solve the following linear equation w.r.t. β_i

$$\begin{aligned} e^{\lambda_1 t} &= \beta_0 + \beta_1 \lambda_1 \\ e^{\lambda_2 t} &= \beta_0 + \beta_1 \lambda_2 \end{aligned}$$

(If there are repeated eigenvalues, derivative conditions will appear.)

Laplace transform table

$f(t)$	$F(s)$
$\delta(t)$	1
$u(t)$	$\frac{1}{s}$
$tu(t)$	$\frac{1}{s^2}$
$t^n u(t)$	$\frac{n!}{s^{n+1}}$
$e^{-at}u(t)$	$\frac{1}{s+a}$
$\sin \omega t \cdot u(t)$	$\frac{\omega}{s^2+\omega^2}$
$\cos \omega t \cdot u(t)$	$\frac{s}{s^2+\omega^2}$
$te^{-at}u(t)$	$\frac{1}{(s+a)^2}$

$\mathcal{L}$

$\mathcal{L}^{-1}$

Inverse Laplace Transform

(u(t): unit step function)

Review of linear algebra

- Matrix determinant
  - 2-by-2: $\det \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} = m_{11}m_{22} - m_{12}m_{21}$
  - 3-by-3: $$\det \begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{bmatrix}$$
    $$= m_{11}m_{22}m_{33} + m_{12}m_{23}m_{31} + m_{21}m_{32}m_{13}$$
    $$-m_{13}m_{22}m_{31} - m_{12}m_{21}m_{33} - m_{11}m_{23}m_{32}$$
- Eigenvalues λ and eigenvectors *v* of a matrix *M* $\in \mathbb{C}^{n\times n}$
  - Definition $Mv = \lambda v,\ \lambda \in \mathbb{C},\ v \in \mathbb{C}^{n\times 1},\ v \neq 0$
  - Computation of λ $\det(\lambda I - M) = 0$

Review of linear algebra

- An *n*-by-*n* matrix *M* is called invertible or nonsingular if there exists another *n*-by-*n* matrix *N* s.t.

$$MN = NM = I_n$$

In this case, the matrix *N* is called the inverse of *M*, and denoted by M^{-1}

- Computation: $$M^{-1} = \frac{\mathrm{adj}(M)}{\det M}$$ Adjoint matrix of *M*

$$\begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix}^{-1} = \frac{1}{\det M} \begin{bmatrix} m_{22} & -m_{12} \\ -m_{21} & m_{11} \end{bmatrix}$$