MECH468 : Modern Control Engineering
MECH509 : Controls

L10 : Controllability

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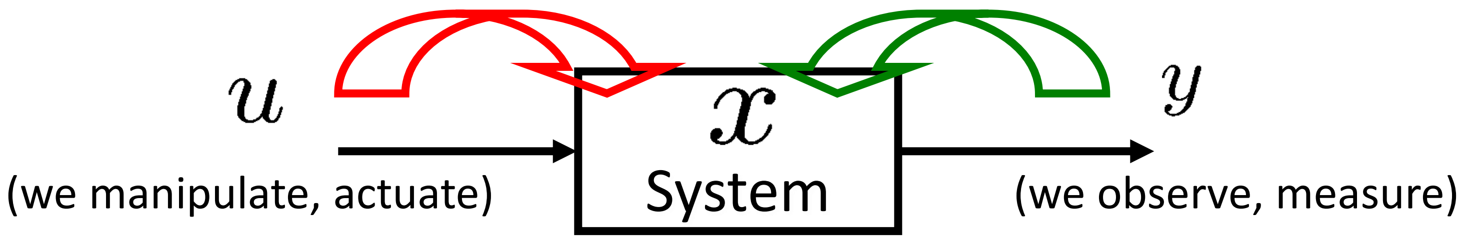
Zoom lecture to be recorded and posted on Canvas

Course plan

Topics	CT	DT
Modeling	✓	✓
Stability	✓	✓
→ Controllability/observability		
Realization		
State feedback/observer		
LQR/Kalman filter		

Controllability & observability

- Consider a system with a state vector:

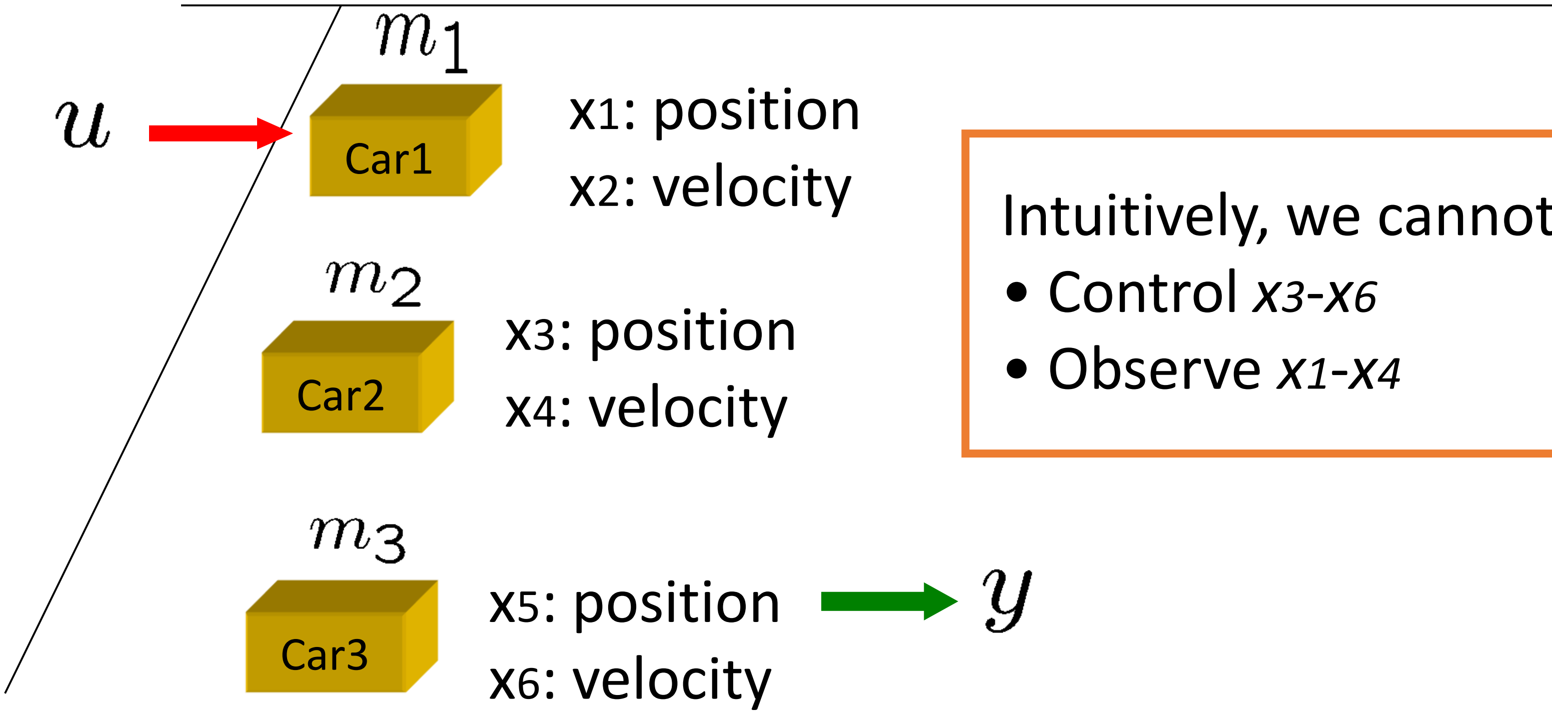


x : neither controllable nor observable directly in general

- *Controllability:* How much can we control x by manipulating u?
- *Observability:* How much can we observe x by observing y?

Very simple example

- Three cars with one input and one output



Intuitively, we cannot
- Control *x3-x6*
- Observe *x1-x4*

Model of three car example

- State-space model

$$\begin{cases} \dot{x}(t) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1/m_1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} u(t) \\ y(t) = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} x(t) \end{cases}$$

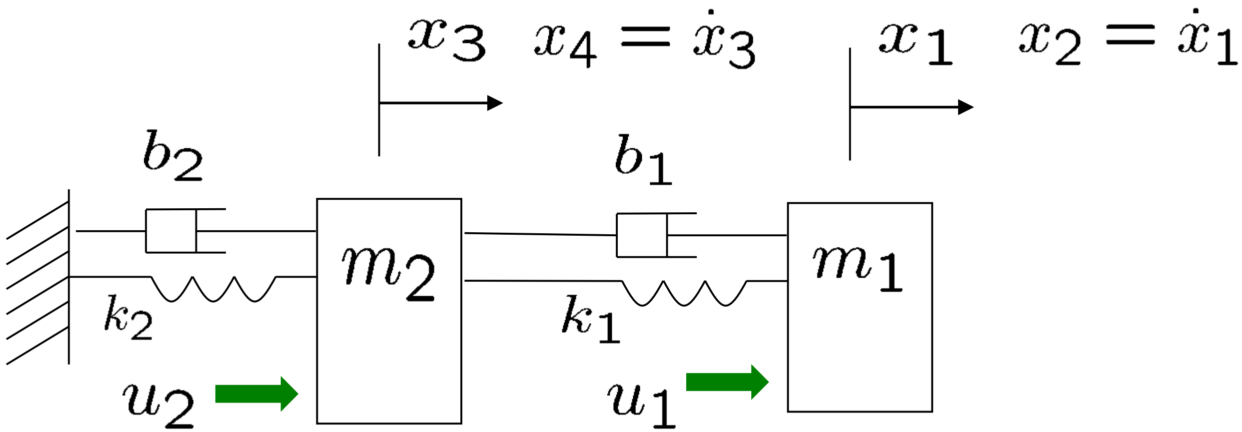
- How can we explain controllability & observability of the system from (*A*,*B*,*C*,*D*)?

Why controllability & observability important?

- Clarify essential/redundant actuators/sensors
- Clarify the "structure" of the system
  - Which state is controllable, uncontrollable, observable, unobservable?
  - Which state is redundant from input-output viewpoint? (Minimality of a realization)
- Clarify possibilities and limitations in
  - Control (state-feedback, linear quadratic regulator)
  - Estimation (observer, Kalman filter)

A mechanical example

- Mass-spring-damper



$$\dot{x}(t) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -k_1/m_1 & -b_1/m_1 & k_1/m_1 & b_1/m_1 \\ 0 & 0 & 0 & 1 \\ k_1/m_2 & b_1/m_2 & -(k_1+k_2)/m_2 & -(b_1+b_2)/m_2 \end{bmatrix} x(t) + \begin{bmatrix} 0 & 0 \\ 1/m_1 & 0 \\ 0 & 0 \\ 0 & 1/m_2 \end{bmatrix} u(t)$$

Some questions

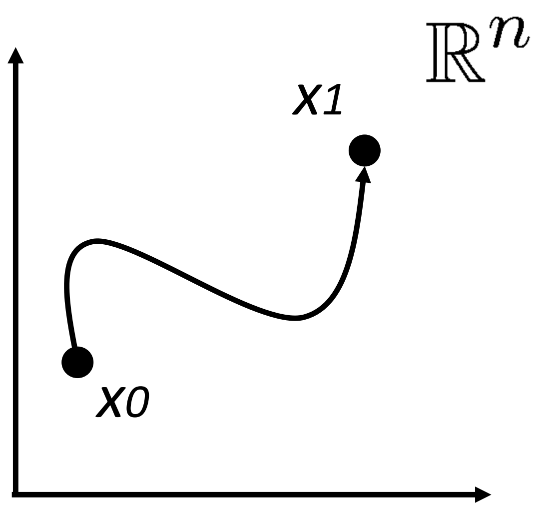
- If we want to transfer any initial positions & velocities to any final positions & velocities in a specified time, is it possible?
  - What if we use only one actuator? (because the other actuator broke down)
  - What if we set some spring and/or damper to be zero?
  - Can you answer these questions by intuition?
  - If yes, how about the system that has 100 (instead of two) mass-spring-damper subsystems? In this case, if you have only a limited number of (say 10) actuators, where to attach the actuators?

Controllability for LTI system

- Consider a state equation

$$\dot{x}(t) = Ax(t) + Bu(t), \quad A \in \mathbb{R}^{n\times n}, \; B \in \mathbb{R}^{n\times p}$$

- *Definition*: The system above, or (A,B), is called ***controllable*** if, for *any* initial state x_0 and *any* final state x_1, there is an input u which transfers from x_0 to x_1 in a (any) finite time.



Condition for controllability

- *Controllability matrix* $\mathcal{C} := \left[B, AB, \cdots, A^{n-1}B\right] \in \mathbb{R}^{n \times np}$ has full row rank, i.e.,

$$\text{rank}\mathcal{C} = n$$

Note: One can use "rank.m" to compute rank of a matrix in Matlab. However, be careful when computing rank of a matrix with computers!!!

$$\begin{bmatrix} 1 & 0 \\ 0 & 0.0000001 \end{bmatrix} \longleftarrow$$

Rank = 1 or 2 ???

Examples

- Ex $$A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Size of *A*-matrix

$$\mathcal{C} = [B, AB] = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \Rightarrow \text{rank}\mathcal{C} = 1 < 2$$

Not controllable!

- Ex $$A = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$$

$$\mathcal{C} = [B, AB] = \begin{bmatrix} 1 & 2 & -1 & -2 \\ 1 & 2 & -2 & -4 \end{bmatrix} \Rightarrow \text{rank}\mathcal{C} = 2$$

Controllable!

- Ex $$A = \begin{bmatrix} 2 & 0 & 0 \\ 2 & 2 & 2 \\ 3 & 0 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

Not controllable!

$$\mathcal{C} = \left[B, AB, A^2B\right] = \begin{bmatrix} 1 & 2 & 4 \\ -2 & 0 & 8 \\ 1 & 2 & 4 \end{bmatrix} \Rightarrow \text{rank}\mathcal{C} = 2 < 3$$

Three car example

- State-space model (*m1=1*)

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1/m_1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} u(t)$$

- Controllability matrix

$$\mathcal{C} := \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Not controllable!

$$\Rightarrow \quad \text{rank}\,\mathcal{C} = 2 < 6$$

This number indicates the "degree of controllability".

This matrix indicates which states are controllable and which are not. (next lecture)

Mechanical example: revisited

- Controllability test
  - $m_1=m_2=1$
  - Various k_1 & k_2
  - Various b_1 & b_2
  - With and without u_1 & u_2

Matlab code

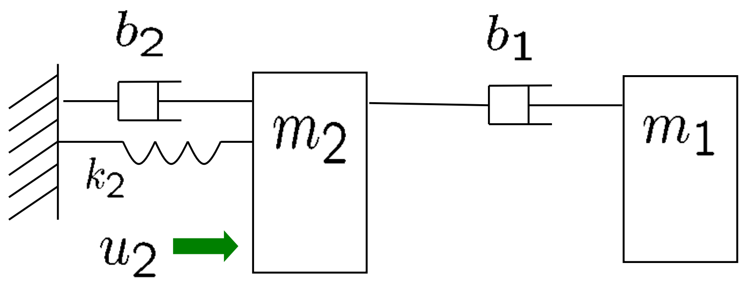
```
b1=1; b2=1;
k1=1; k2=1;
A=[0 1 0 0;
   -k1/m1 -b1/m1 k1/m1 b1/m1;
   0 0 0 1;
   k1/m2 b1/m2 -(k1+k2)/m2 -(b1+b2)/m2];
B=[0 0;
   1/m1 0;
   0 0;
   0 1/m2];
Rboth = rank(ctrb(A,B))
Ru1 = rank(ctrb(A,B(:,1)))
Ru2 = rank(ctrb(A,B(:,2)))
```

Controllability test: results

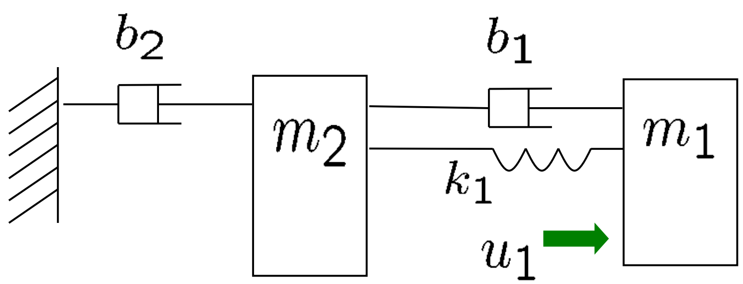
b1	b2	k1	k2	Rboth	Ru1	Ru2
1	1	1	1	4	4	4
0	1	1	1	4	4	4
1	0	1	1	4	4	4
1	1	0	1	4	4	3
1	1	1	0	4	3	4
0	1	1	0	4	4	4
1	0	0	1	4	4	3

Uncontrollable cases

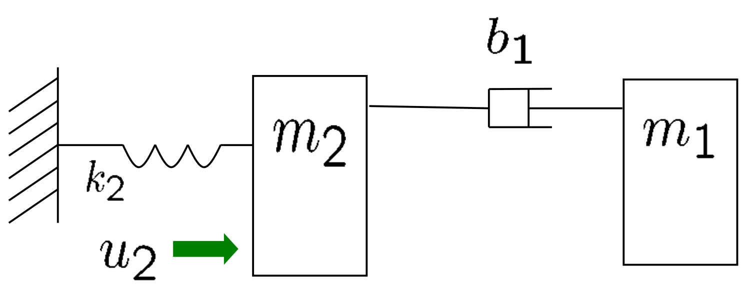
- $k_1=0,\ u_1=0$



- $k_2=0,\ u_2=0$



- $b_2=k_1=0,\ u_1=0$



Summary

- Controllability
  - Definition
  - Necessary and sufficient condition
  - Rank computation
  - Mechanical example
- Next,
  - when (A,B) is controllable, minimum energy control
  - when (A,B) is not controllable, which states can we control and which states cannot we control?