MECH 467 - Tutorial 11 - Sample Problems for Final Exam

Solutions

1) Based on the FRF given, we have:

$$\begin{aligned}
\frac{v(s)}{F_d(s)} &= \frac{1}{Ms+B} = \frac{1/B}{(M/B)s+1} \rightarrow \frac{v(j\omega)}{F_d(j\omega)} = \frac{1/B}{j\omega(M/B)+1} \\
\left|\frac{1}{j\omega M+B}\right| &= \frac{1/B}{\sqrt{\omega^2(M/B)^2+1}} \measuredangle -\tan^{-1}\frac{M}{B}\omega \\
\phi(\omega &= 2rad/s) = -\frac{\pi}{4} = -\tan^{-1}\frac{M}{B}2 \rightarrow \frac{2M}{B} = 1 \rightarrow \frac{M}{B} = 0.5 \\
\left|\frac{1}{j\omega M+B}\right|_{\omega=0.1} &= 0.1 = \frac{1/B}{\sqrt{0.1^2(0.5)^2+1}} \\
B &= \frac{1}{0.1\sqrt{0.1^2(0.5)^2+1}} = 10[N/m/s] \rightarrow M = 0.5\cdot 10 = 5[kg]
\end{aligned}$$

2) The total dynamic force needed from the drive:

$$F_{total} = M\frac{d^2x}{dt^2} + B\frac{dx}{dt} + F_{cut} + F_{fr} = 50+10+100+0.3 = 160.3[N] \rightarrow i_{\max} = \frac{F_{total}}{K_t} = \frac{160.3}{20} = 8.015[A]$$
$$F_{cont} = B\frac{dx}{dt} + F_{cut} + F_{fr} = 10+100+0.3 = 110.3[N] \rightarrow i_{cont} = \frac{F_{cont}}{K_t} = \frac{110.3}{20} = 5.515[A]$$

3) The closed-loop transfer function would be:

$$G_{cl}(s) = \frac{y(s)}{x(s)} = \frac{\frac{D(s)K_aK_iK_e}{s(Ms+B)}}{1+\frac{D(s)K_aK_iK_e}{s(Ms+B)}} = \frac{D(s)K_aK_iK_e}{s(Ms+B)+D(s)K_aK_iK_e}$$

If we use a proportional controller $D(s) = K_p$:

$$\begin{aligned}
G_{cl}(s) &= \frac{y(s)}{x(s)} = \frac{K_pK_aK_iK_e}{s(Ms+B)+K_pK_aK_iK_e} = \frac{K}{s(Ms+B)+K} = \frac{K/M}{s^2+(B/M)s+K/M} \\
G_{cl}(s) &= \frac{K/M}{s^2+2\zeta\omega_n s+K/M} \rightarrow 2\zeta\omega_n = \frac{B}{M} \rightarrow \omega_n = \frac{B}{2\zeta M} = \frac{10}{2\cdot 0.8\cdot 5} = 1.25[rad/s] \\
\frac{K}{M} &= \omega_n^2 \rightarrow K = \omega_n^2 M = 1.25^2\cdot 5 = 7.8125 \rightarrow K_p = \frac{K}{K_aK_iK_e} = \frac{7.8125}{1\cdot 20\cdot 1} = 0.39063[V/m]
\end{aligned}$$

Root locus:

$$s(Ms+B)+K=0 \;\rightarrow\; p_{1,2}=\frac{-B\pm\sqrt{B^2-4KM}}{2M}$$

$$K=0 \;\rightarrow\; p_1=0,\;\; p_2=-\frac{B}{M}$$

$$B^2-4KM=0 \;\rightarrow\; K=\frac{B^2}{4M}=\frac{10^2}{4\times 5}=5,\;\; p_{1,2}=-\frac{B}{2M}$$

$$B^2-4KM<0 \;\rightarrow\; p_{1,2}=\frac{-B\pm j\sqrt{4KM-B^2}}{2M} \;\rightarrow\; K>5$$

4) Phase margin of the open-loop system at the given frequency is:

$$\begin{aligned}
G_0(s) &= \frac{K_aK_iK_e}{s(Ms+B)}=\frac{20}{s(5s+10)} \\
G_0(j\omega) &= \frac{20}{j\omega(j\omega 5+10)} \rightarrow |G_0(j\omega)|=\frac{20}{\omega\sqrt{(5\omega)^2+100}} \rightarrow \phi=-\frac{\pi}{2}-\tan^{-1}0.5\omega \\
\phi &= \left(-\frac{\pi}{2}-\tan^{-1}(0.5\cdot 50)\right)\frac{180}{\pi}=-177.71\,\text{deg}
\end{aligned}$$

An additional phase lag of $\phi_m=60-(180-177.71)=57.3$ is needed.

$$\alpha=\frac{1+\sin(57.7^\circ)}{1-\sin(57.7^\circ)}=11.925,\;\; T=\frac{1}{50\sqrt{11.925}}=0.0057916$$

$$\begin{aligned}
G_{ol}(s) &= \frac{D(s)K_aK_iK_e}{s(Ms+B)}=K\frac{1+\alpha Ts}{1+Ts}\frac{K_aK_iK_e}{s(Ms+B)} \\
G_{ol}(j\omega) &= K\frac{1+j\alpha T\omega}{1+jT\omega}\frac{K_aK_iK_e}{j\omega(j\omega M+B)}
\end{aligned}$$

At the frequency where the PM is calculated, the magnitude of the open loop transfer function must be unity.

$$\begin{aligned}
|G_{ol}(j\omega)| &= K\frac{\sqrt{1+(\alpha T\omega)^2}}{\sqrt{1+(T\omega)^2}}\frac{K_aK_iK_e}{\omega\sqrt{(M\omega)^2+(B)^2}}=1 \\
&= K\frac{\sqrt{1+(11.925\cdot 0.0057916\cdot 50)^2}}{\sqrt{1+(0.0057916\cdot 50)^2}}\frac{20}{50\sqrt{(5\cdot 50)^2+(10)^2}}=1 \\
&= 0.0057916K=1 \rightarrow K=\frac{1}{0.0057916}=172.66
\end{aligned}$$

5) The transfer functions can be expressed as:

$$
\begin{aligned}
e(s) &= x(s) - y(s) \\
y(s) &= \{[x(s)-y(s)]\, D(s) K_a K_t - F_d(s)\} \frac{K_e}{s(Ms+B)} \\
&= \frac{D(s)K_aK_tK_e}{s(Ms+B)} x(s) - \frac{D(s)K_aK_tK_e}{s(Ms+B)} y(s) - \frac{K_e}{s(Ms+B)} F_d(s) \\
\left[1 + \frac{D(s)K_aK_tK_e}{s(Ms+B)}\right] y(s) &= \frac{D(s)K_aK_tK_e}{s(Ms+B)} x(s) - \frac{K_e}{s(Ms+B)} F_d(s) \\
y(s) &= \frac{K\frac{1+\alpha Ts}{1+Ts}\frac{K_aK_tK_e}{s(Ms+B)}}{1+K\frac{1+\alpha Ts}{1+Ts}\frac{K_aK_tK_e}{s(Ms+B)}} x(s) - \frac{\frac{K_e}{s(Ms+B)}}{1+K\frac{1+\alpha Ts}{1+Ts}\frac{K_aK_tK_e}{s(Ms+B)}} F_d(s) \\
y(s) &= \frac{K_aK_tK_eK\,(1+\alpha Ts)}{s(Ms+B)(1+Ts) + KK_aK_tK_e(1+\alpha Ts)} x(s) \\
&\quad - \frac{K_e(1+Ts)}{s(Ms+B)(1+Ts) + KK_aK_tK_e(1+\alpha Ts)} F_d(s)
\end{aligned}
$$

$$
\begin{aligned}
e(s) &= x(s) - \frac{K_aK_tK_eK\,(1+\alpha Ts)}{s(Ms+B)(1+Ts) + KK_aK_tK_e(1+\alpha Ts)} x(s) \\
&\quad + \frac{K_e(1+Ts)}{s(Ms+B)(1+Ts) + KK_aK_tK_e(1+\alpha Ts)} F_d(s) \\
&= \left[1 - \frac{K_aK_tK_eK\,(1+\alpha Ts)}{s(Ms+B)(1+Ts) + KK_aK_tK_e(1+\alpha Ts)}\right] x(s) \\
&\quad + \frac{K_e(1+Ts)}{s(Ms+B)(1+Ts) + KK_aK_tK_e(1+\alpha Ts)} F_d(s) \\
&= \left[\frac{s(Ms+B)(1+Ts)}{s(Ms+B)(1+Ts) + KK_aK_tK_e(1+\alpha Ts)}\right] x(s) \\
&\quad + \frac{K_e(1+Ts)}{s(Ms+B)(1+Ts) + KK_aK_tK_e(1+\alpha Ts)} F_d(s)
\end{aligned}
$$

The steady-state error based on the Final Value Theorem would be:

$$
\begin{aligned}
e_{ss} &= \lim_{s=0} s \left[\frac{s(Ms+B)(1+Ts)}{s(Ms+B)(1+Ts) + KK_aK_tK_e(1+\alpha Ts)}\right] \frac{f}{s^2} \\
&\quad + \lim_{s=0} s \frac{K_e(1+Ts)}{s(Ms+B)(1+Ts) + KK_aK_tK_e(1+\alpha Ts)} \frac{F_0}{s} \\
&= \lim_{s=0} \left[\frac{(Ms+B)(1+Ts)}{s(Ms+B)(1+Ts) + KK_aK_tK_e(1+\alpha Ts)}\right] f \\
&\quad + \lim_{s=0} \frac{K_e(1+Ts)}{s(Ms+B)(1+Ts) + KK_aK_tK_e(1+\alpha Ts)} F_0 \\
&= \frac{B}{KK_aK_tK_e} f + \frac{K_e}{KK_aK_tK_e} F_0
\end{aligned}
$$

6) The zero-order hold equivalent of the system is:

$$ZOH\left(G_p(s)\right) = (1 - z^{-1})Z\left(\frac{G_p(s)}{s}\right) = (1 - z^{-1})Z\left(\frac{K_a K_t K_e}{s^2(Ms + B)}\right)$$

$$\frac{K_a K_t K_e}{s^2(Ms + B)} = \frac{A}{s^2} + \frac{B}{s} + \frac{C}{Ms + B}$$

$$A = \lim_{s \to 0} s^2 \left(\frac{K_a K_t K_e}{s^2(Ms + B)}\right) = \frac{K_a K_t K_e}{B}$$

$$B = \lim_{s \to 0} \frac{d}{ds}\left(s^2 \left(\frac{K_a K_t K_e}{s^2(Ms + B)}\right)\right) = \lim_{s \to 0} \frac{-MK_a K_t K_e}{(Ms + B)^2} = \frac{-MK_a K_t K_e}{B^2}$$

$$C = \lim_{s \to \frac{-B}{M}} (Ms + B)\left(\frac{K_a K_t K_e}{s^2(Ms + B)}\right) = \frac{M^2 K_a K_t K_e}{B^2}$$

$$Z\left(\frac{K_a K_t K_e}{s^2(Ms + B)}\right) = Z(\frac{\frac{K_a K_t K_e}{B}}{s^2} + \frac{\frac{-MK_a K_t K_e}{B^2}}{s} + \frac{\frac{M^2 K_a K_t K_e}{B^2}}{Ms + B})$$

$$= \frac{K_a K_t K_e T z^{-1}}{B(1 - z^{-1})^2} - \frac{MK_a K_t K_e}{B^2(1 - z^{-1})} + \frac{MK_a K_t K_e}{B^2(1 - e^{-\frac{BT}{M}} z^{-1})}$$

$$ZOH\left(G_p(s)\right) = (1 - z^{-1})\left(\frac{K_a K_t K_e T z^{-1}}{B(1 - z^{-1})^2} - \frac{MK_a K_t K_e}{B^2(1 - z^{-1})} + \frac{MK_a K_t K_e}{B^2(1 - e^{-\frac{BT}{M}} z^{-1})}\right) = \frac{z^{-1}(b_0 + b_1 z^{-1})}{1 + a_1 z^{-1} + a_2 z^{-2}}$$

$$b_0 = \frac{K_a K_t K_e T}{B} + \frac{\left(1 + e^{-\frac{BT}{M}}\right) MK_a K_t K_e}{B^2} - \frac{2MK_a K_t K_e}{B^2}$$

$$b_1 = \frac{MK_a K_t K_e}{B^2} - \frac{MK_a K_t K_e e^{-\frac{BT}{M}}}{B^2} - \frac{K_a K_t K_e T e^{-\frac{BT}{M}}}{B}$$

$$a_1 = -\left(1 + e^{-\frac{BT}{M}}\right), \qquad a_2 = e^{-\frac{BT}{M}}$$

7) To get the desired values with the pole placement controller we have:

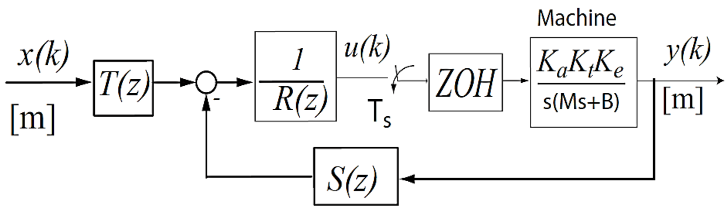
$$
\begin{aligned}
A_m(z^{-1}) &= 1 - 2e^{-\zeta_m \omega_m T} cos(\omega_m \sqrt{1-\zeta_m^2} T) z^{-1} + e^{-2\zeta_m \omega_m T} z^{-2} = 1 + m_1 z^{-1} + m_2 z^{-2} \\
\mathbf{deg}(B) &= 1, d = 1, \mathbf{deg}(A) = 2 \\
\mathbf{deg}(R) &= d + \mathbf{deg}(B) - 1 = 1 + 1 - 1 = 1 \rightarrow R(z) = 1 + r_1 z^{-1} \\
\mathbf{deg}(S) &= \mathbf{deg}(A) - 1 = 2 - 1 = 1 \rightarrow S(z) = s_0 + s_1 z^{-1} \\
\left[\frac{z^{-d} B(z^{-1}) b_m}{A_m(z^{-1})} \right]_{z=1} &= 1 \rightarrow \left[\frac{z^{-1}(b_0 + b_1 z^{-1}) b_m}{1 + m_1 z^{-1} + m_2 z^{-2}} \right] = 1 \rightarrow b_m = \frac{1 + m_1 + m_2}{b_0 + b_1} = t_0 \\
AR + z^{-d} BS &\equiv A_m(z^{-1}) \\
& \left(1 + a_1 z^{-1} + a_2 z^{-2}\right) \left(1 + r_1 z^{-1}\right) + z^{-1} \left(b_0 + b_1 z^{-1}\right) \left(s_0 + s_1 z^{-1}\right) \\
&= 1 + m_1 z^{-1} + m_2 z^{-2}
\end{aligned}
$$

$$
\left(1 + a_1 z^{-1} + a_2 z^{-2}\right) r_1 z^{-1} + z^{-1} \left(b_0 + b_1 z^{-1}\right) \left(s_0 + s_1 z^{-1}\right) = 1 + m_1 z^{-1} + m_2 z^{-2} - 1 - a_1 z^{-1} - a_2 z^{-2}
$$

$$
\begin{aligned}
r_1 z^{-1} + a_1 r_1 z^{-2} + r_1 a_2 z^{-3} + b_0 s_0 z^{-1} + b_0 s_1 z^{-2} + b_1 s_0 z^{-2} + b_1 s_1 z^{-3} &= (m_1 - a_1) z^{-1} + (m_2 - a_2) z^{-2} \\
(r_1 + b_0 s_0) z^{-1} + (a_1 r_1 + b_0 s_1 + b_1 s_0) z^{-2} + (b_1 s_1 + r_1 a_2) z^{-3} &= (m_1 - a_1) z^{-1} + (m_2 - a_2) z^{-2} \\
r_1 + b_0 s_0 &= m_1 - a_1 \\
a_1 r_1 + b_1 s_0 + b_0 s_1 &= m_2 - a_2 \\
r_1 a_2 + b_1 s_1 &= 0
\end{aligned}
$$

$$
\begin{aligned}
\begin{bmatrix} 1 & b_0 & 0 \\ a_1 & b_1 & b_0 \\ a_2 & 0 & b_1 \end{bmatrix} \begin{Bmatrix} r_1 \\ s_0 \\ s_1 \end{Bmatrix} &= \begin{Bmatrix} m_1 - a_1 \\ m_2 - a_2 \\ 0 \end{Bmatrix} \\
\begin{Bmatrix} r_1 \\ s_0 \\ s_1 \end{Bmatrix} &= \begin{bmatrix} 1 & b_0 & 0 \\ a_1 & b_1 & b_0 \\ a_2 & 0 & b_1 \end{bmatrix}^{-1} \begin{Bmatrix} m_1 - a_1 \\ m_2 - a_2 \\ 0 \end{Bmatrix} \\
\begin{Bmatrix} r_1 \\ s_0 \\ s_1 \end{Bmatrix} &= \frac{1}{a_2 b_0^2 - a_1 b_0 b_1 + b_1^2} \begin{bmatrix} b_1^2 & -b_0 b_1 & b_0^2 \\ -a_1 b_1 + a_2 b_0 & b_1 & -b_0 \\ -a_2 b_1 & a_2 b_0 & b_1 - a_1 b_0 \end{bmatrix} \begin{Bmatrix} m_1 - a_1 \\ m_2 - a_2 \\ 0 \end{Bmatrix} \\
r_1 &= \frac{b_1^2 (m_1 - a_1) - b_0 b_1 (m_2 - a_2)}{a_2 b_0^2 - a_1 b_0 b_1 + b_1^2} \\
s_0 &= \frac{(-a_1 b_1 + a_2 b_0)(m_1 - a_1) + b_1 (m_2 - a_2)}{a_2 b_0^2 - a_1 b_0 b_1 + b_1^2} \\
s_1 &= \frac{-a_2 b_1 (m_1 - a_1) + a_2 b_0 (m_2 - a_2)}{a_2 b_0^2 - a_1 b_0 b_1 + b_1^2}
\end{aligned}
$$

8) The control law would be as follows:



$$
\begin{aligned}
\frac{T(z)x(k) - S(z)y(k)}{R(z)} &= u(k) \rightarrow \left(1 + r_1 z^{-1}\right) u(k) = t_0 x(k) - s_0 y(k) - s_1 y(k-1) \\
u(k) &= -r_1 u(k-1) + t_0 x(k) - s_0 y(k) - s_1 y(k-1)
\end{aligned}
$$

9) The steady-state error for the ramp input is calculated as follows:

$$
\begin{aligned}
e(z) &= x(z) - y(z) \\
y(k) &= \frac{T(z)x(k) - S(z)y(k)}{R(z)} \frac{z^{-1}B(z)}{A(z)} \rightarrow RAy(k) = z^{-1}BTx(k) - z^{-1}BSy(k) \\
\frac{y(k)}{x(k)} &= \frac{z^{-1}B(z)T(z)}{R(z)A(z) + z^{-1}B(z)S(z)} \qquad e(z) = x(z) - y(z) = x(z) - \frac{z^{-1}B(z)T(z)}{R(z)A(z) + z^{-1}B(z)S(z)} x(z) \\
e(z) &= \left[\frac{R(z)A(z) + z^{-1}B(z) - z^{-1}B(z)T(z)}{R(z)A(z) + z^{-1}B(z)S(z)}\right] x(z) = \left[\frac{R(z)A(z) + z^{-1}B(z) - z^{-1}B(z)T(z)}{R(z)A(z) + z^{-1}B(z)S(z)}\right] \frac{fTz^{-1}}{(1-z^{-1})^2} \\
e_{ss} &= \lim_{z=1}(1 - z^{-1}) \left[\frac{\left(1 + r_1 z^{-1}\right)\left(1 + a_1 z^{-1} + a_2 z^{-2}\right) + z^{-1}\left(b_0 + b_1 z^{-1}\right)\left(1 - t_0\right)}{\left(1 + r_1 z^{-1}\right)\left(1 + a_1 z^{-1} + a_2 z^{-2}\right) + z^{-1}\left(b_0 + b_1 z^{-1}\right)\left(s_0 + s_1 z^{-1}\right)}\right] \frac{fTz^{-1}}{(1-z^{-1})^2}
\end{aligned}
$$

Steady-state error of the step input would be as follows:

$$
e(z) = x(z) - y(z) = \left[1 - \frac{z^{-d}B_m(z^{-1})}{A_m(z^{-1})}\right] \frac{U}{1 - z^{-1}}
$$

$$
e_{ss} = \lim_{z=1}(1 - z^{-1}) \left[\frac{A_m(z^{-1}) - z^{-d}B(z^{-1})b_m}{A_m(z^{-1})}\right] \frac{U}{1 - z^{-1}}
$$

$$
= \left[\frac{1 + m_1 + m_2 - (b_0 + b_1)\dfrac{1 + m_1 + m_2}{b_0 + b_1}}{1 + m_1 + m_2}\right] U = 0
$$