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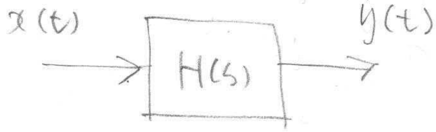
# < Root Locus & Nyquist Test >

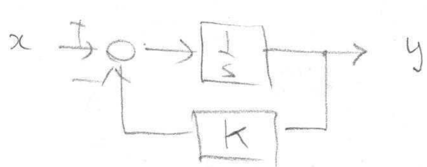
- Objective
  - Stability assessmet of LTI feedback systems.
  - Root Locus review.
  - Nyquist test and stability margin.

- Stability of LTI systems




Stable $\Longleftrightarrow$ $Re\{p_i\} < 0$.

- Stability assessment of feedback systems.
  - Directly finding the closed-loop poles.

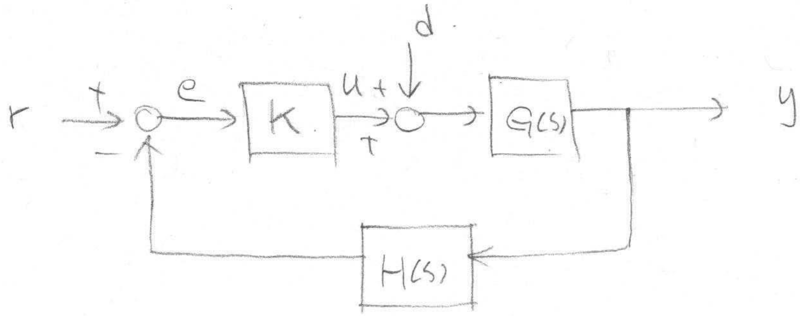



$$\frac{Y}{X} = \frac{\frac{1}{s}}{1+\frac{K}{s}} = \frac{1}{s+K} \qquad \text{stable if } K>0.$$

  - Inferring the closed-loop poles from $L(s)$
    ① Root locus : the locations of CL poles (explicit)
    ② Nyquist test : the number of unstable CL poles. (implicit)
  - Note that both methods are based on "Loop analysis"

∘ Characteristic Equation : $f(s) = 0$.

Consider a feedback system.




$$\begin{bmatrix} Y \\ U \end{bmatrix} = \begin{bmatrix} \frac{KG}{1+KGH} & \frac{G}{1+KGH} \\ \frac{K}{1+KGH} & \frac{-KGH}{1+KGH} \end{bmatrix} \begin{bmatrix} R \\ D \end{bmatrix}$$

$$= \frac{1}{1+KGH} \begin{bmatrix} KG & G \\ K & -KGH \end{bmatrix} \begin{bmatrix} R \\ D \end{bmatrix}$$

- The closed-loop system is stable if
  $\frac{1}{1+KGH} = S(s)$ does not have RHP poles.

<u>Q</u>. What if $G(s)$ has RHP poles? For example at $s = p_0$?

$$\left.\frac{G}{1+KGH}\right|_{s=p_0} = \frac{\cancel{G(p_0)}^{\to\infty}}{1 + K\cancel{G(p_0)}^{\to\infty} H(p_0)} \Rightarrow \left|\frac{1}{K \cdot H(p_0)}\right| < \infty$$

<u>A</u>. $p_0$ is not the pole of $\frac{G}{1+KGH}$ anymore.
"Feedback" moved it around.

- poles of $\frac{1}{1+KGH}$ $\Longleftrightarrow$ Zeros of $1+KGH = 1 + L(s) \triangleq f(s)$

- For stability, $\underbrace{f(s) = 0}_{}$ should not have any <u>roots</u> on RHP
  "Characteristic Equation"

- Root Locus

(Roots of $f(s)=0$.)

- Infers the locations of CL poles from $L(s)$.
- Shows how the roots of $f(s)=0$ move with respect to a parameter (e.g. K)

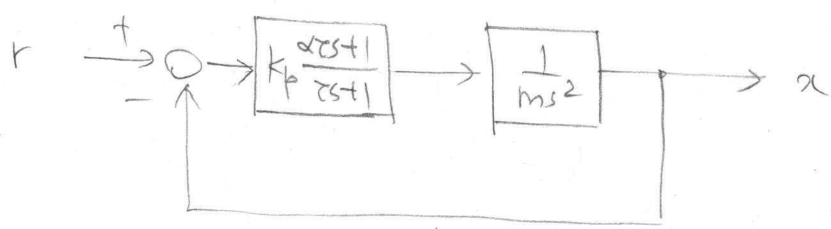
$$1+K\,G(s)H(s)=0 \quad \Rightarrow \quad G(s)H(s)=-\frac{1}{K}.$$

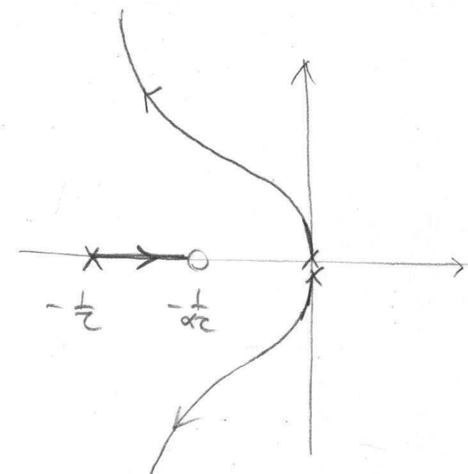
① When $K\to\infty$, roots of $f(s)$ → zeros of $L(s)$.

② When $K\to 0$, roots of $f(s)$ → poles of $L(s)$

③ When $0<K<\infty$, roots of $f(s)$ → $\measuredangle G(s_0)H(s_0)=180^\circ$

<u>Example</u> Free mass position control.




$$L(s)=K_p\,\frac{\alpha\tau s+1}{\tau s+1}\cdot\frac{1}{ms^2}$$

<u>Remark</u>

- Limited to $L(s)=\frac{a(s)}{b(s)}$ "Rational function"
- Cannot handle other types, such as $e^{-sT}$ or $\sqrt{s}$
- Gives intuition for simple systems, but difficult to use for complex systems having many poles and zeros

<u>Motivations for Nyquist test</u>

- We don't need to keep track of the exact locations of CL poles to check the stability
- We just need to check the "existence" of unstable CL poles

- Nyquist test (Roots of $f(s)=0$ on RHP)

  - Infers the number of unstable CL poles from $L(s)$
  - Require less information on the loop $\begin{cases} ① \ L(s)|_{s=j\omega} \\ ② \ \text{No. of unstable poles of } L(s) \end{cases}$
  - It is the foundation of loop shaping design.
  - The concepts of gain margin & phase margin are derived from the Nyquist plot.
  - When we meet challenging control design problem, (e.g., multiple cross-over frequencies), we need to go back to the Nyquist test as the first principle

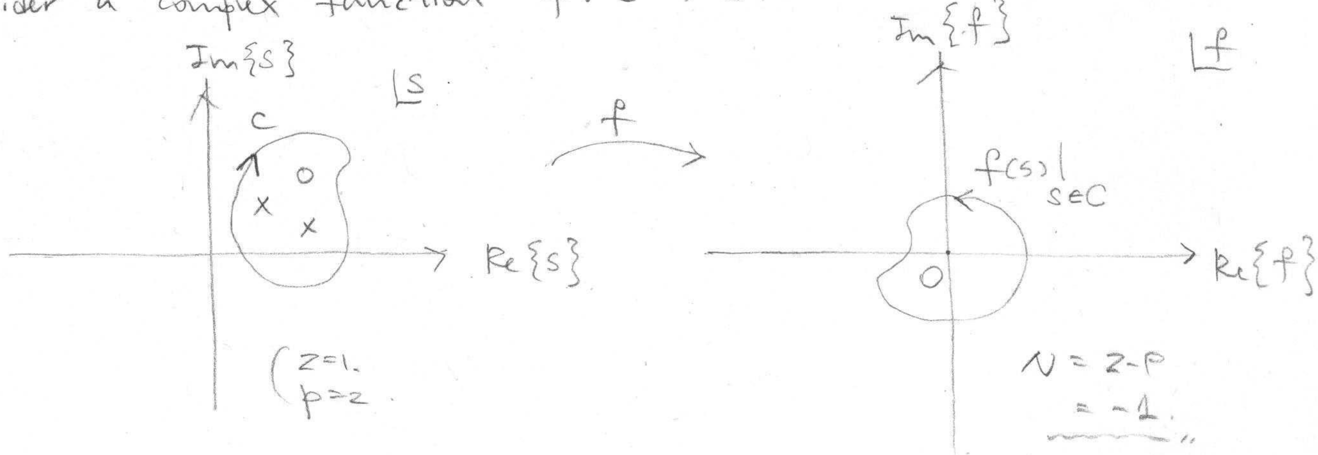
Key Idea

- CL poles $\Leftrightarrow$ zeros of $f(s) \triangleq 1 + L(s)$
- Nyquist test tells us the number of zeros of $f(s)$ in the RHP. ($Z$)
- The mathematical basis is Cauchy's argument principle.

- Argument principle

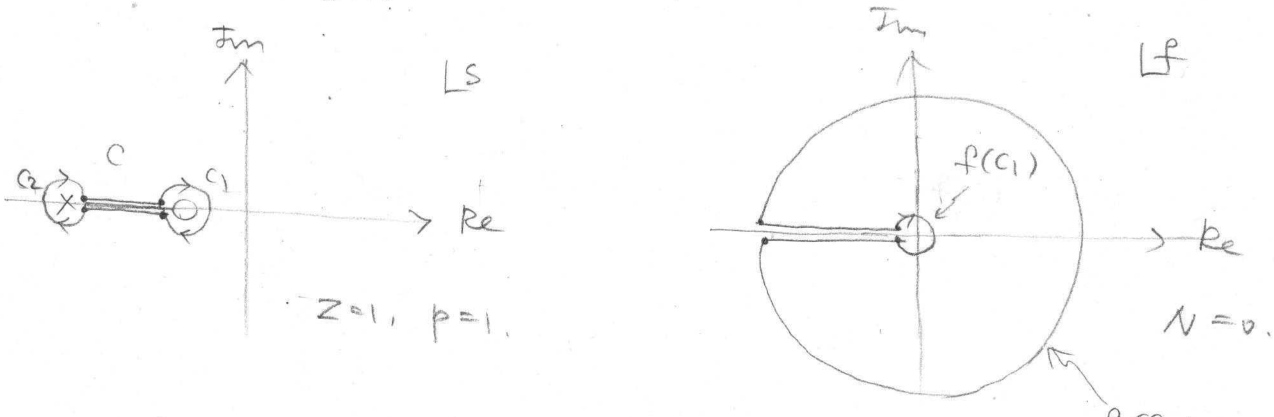
Analytiz on and Inside C
exept at poles & Zeros.

- Consider a complex function $f: \mathbb{C} \rightarrow \mathbb{C}$




- A contour C in the s-plane that captures Z numbers of zeros of f(s) and P numbers of poles of f(s) maps to an image contour $f(s)\big|_{s \in \mathbb{C}}$ that encircles the origin by $\boxed{N = Z - P}$ times.

Example  $f(s) = \dfrac{s+1}{s+10}$



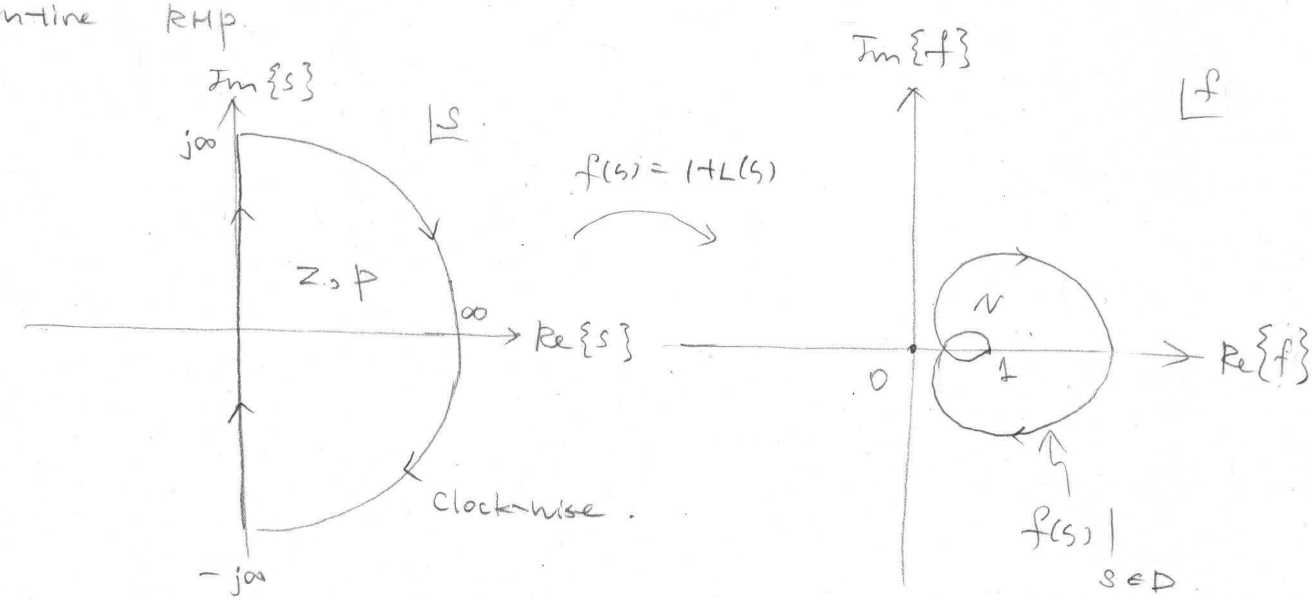

$C_1: -1 + re^{j\theta} \longrightarrow f(C_1) \simeq \dfrac{r \cdot e^{j\theta}}{9}$

$C_2: -10 + re^{j\theta} \longrightarrow f(C_2) \simeq \dfrac{-9}{re^{j\theta}} = \dfrac{9}{r} e^{j(\pi-\theta)}$

- Application to $f(s) = 1+L(s)$.

· We are interested in whether $f(s)$ has zeros in RHP.

· Let's draw a big D-shape contour to capture the entire RHP.




$Z$: # of zeros of $f(s)$ inside $D$ of the s-plane

$P$: # of poles of $f(s)$ inside $D$ of the s-plane

$N$: # of the clock-wise encirclements of the image contour $f(s)|_{s\in D}$ about the origin of $f(s)$ plane.

- From the argument principle

$$N = Z - P \iff \boxed{Z = N + P}$$

- $Z$ tells us $\begin{cases} \text{\# of RHP zeros of } f(s). \\ \text{\# of RHP poles of } \frac{1}{f(s)} \end{cases}$ $Z=0 \Rightarrow$ stability!

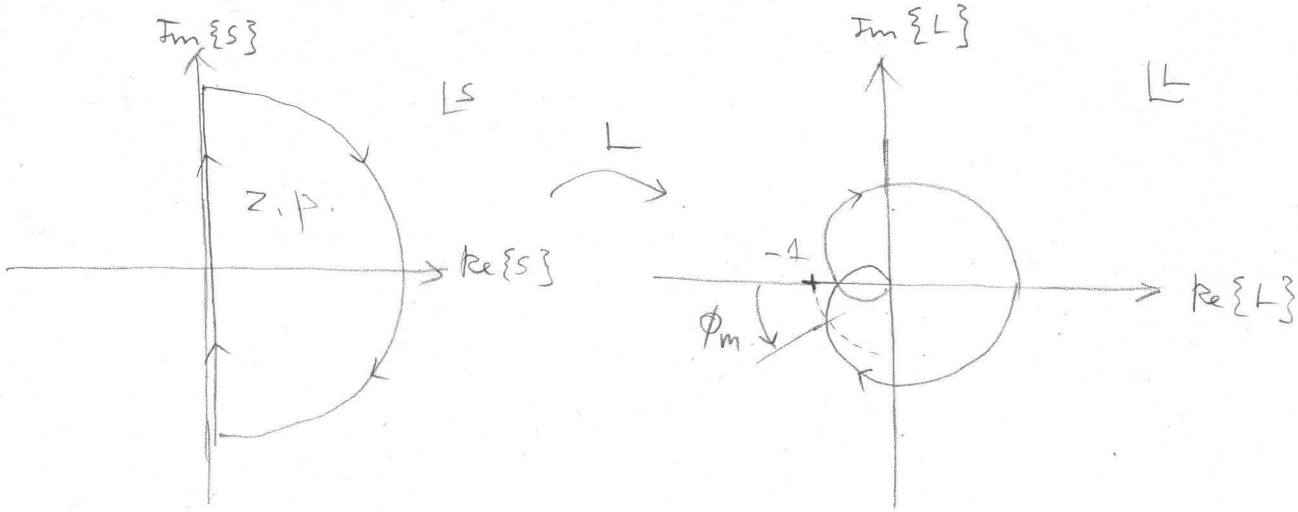
- $N$ can be obtained by counting # of encirclements about "0"

- How about $P$?

$\left.\begin{cases} \text{Since } f(s) = 1+L(s) \\ f(s_0) \to \infty \iff 1+L(s_0) \to \infty \end{cases}\right\} \Rightarrow$ poles of $f(s)$ = poles of $L(s)$. So, we can instead count RHP poles of $L(s)$.

- Nyquist Test

- Slight modification on the previous → "shift the image by $-1$."




- This gives us an alternative way to evaluate $N$

  { # of encirclement of $f(D)$ about the origin.

  # of encirclement of $\underbrace{L(s)}_{\text{Nyquist plot}}$ about $\underbrace{\text{the } -1 \text{ point}}_{\text{Nyquist point}}$

- Therefore,

$$\underbrace{Z}_{\text{End result}} = N + p$$

$Z$: # of unstable CL poles.

- $N$ : # of the clockwise encirclement of the image countour $L(s)|_{s \in D}$ about the $-1$ point of $L(s)$ plane.

- $p$ : # of unstable poles of $L(s)$

- Note that both $N$ and $p$ can be obtained from $L(s)$.

- In particular, $L(j\omega)$ can be drawn from the loop Bode plot.