MECH468 : Modern Control Engineering
MECH509 : Controls

L23 : Observer

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Zoom lecture to be recorded and posted on Canvas

Course plan

Topics	CT	DT
Modeling	✓	✓
Stability	✓	✓
Controllability/observability	✓	✓
Realization	✓	✓
→ State feedback/observer		
LQR/Kalman filter		

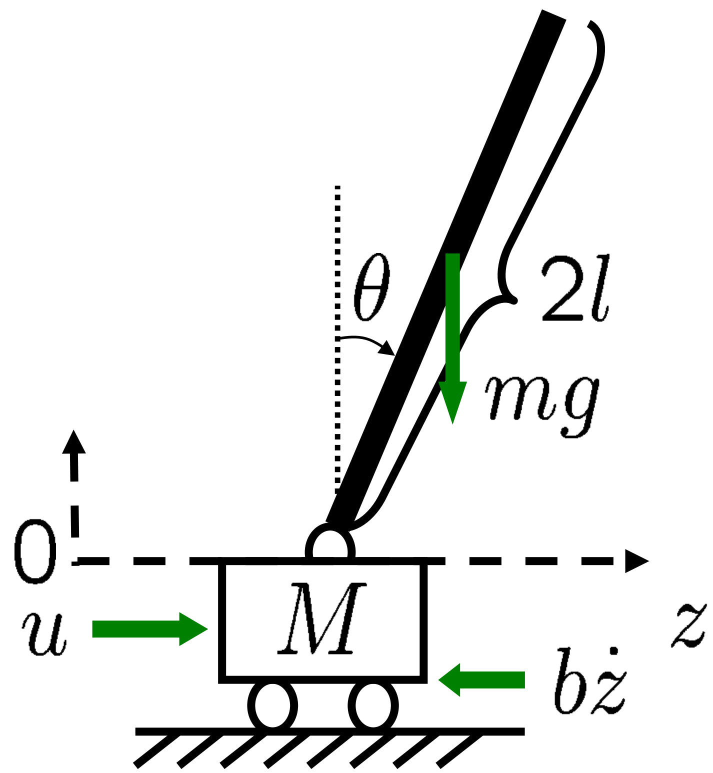
Review & today's topic

- Review
  - State feedback
  - Pole placement theorem
  - Methods to compute the state feedback gain K
  - Servo control
  - Assumption: x is available for control (Unrealistic!)
- Today
  - Assumption: x is NOT available
  - How to estimate x in real time?
  - Observer (developed by D. G. Luenberger in 1963)

Example: Inverted pendulum

- State vector $x := \left[z, \dot{z}, \theta, \dot{\theta}\right]^T$
- SS model around *x=0*

$$
\begin{cases}
\dot{x} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -\frac{(I+ml^2)b}{d} & -\frac{m^2l^2g}{d} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{mlb}{d} & \frac{(M+m)mgl}{d} & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ \frac{I+ml^2}{d} \\ 0 \\ -\frac{ml}{d} \end{bmatrix} u \\
\qquad d := I(M+m) + Mml^2 \\
y = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} x
\end{cases}
$$



Not all the states are measurable!

Observer: Motivation

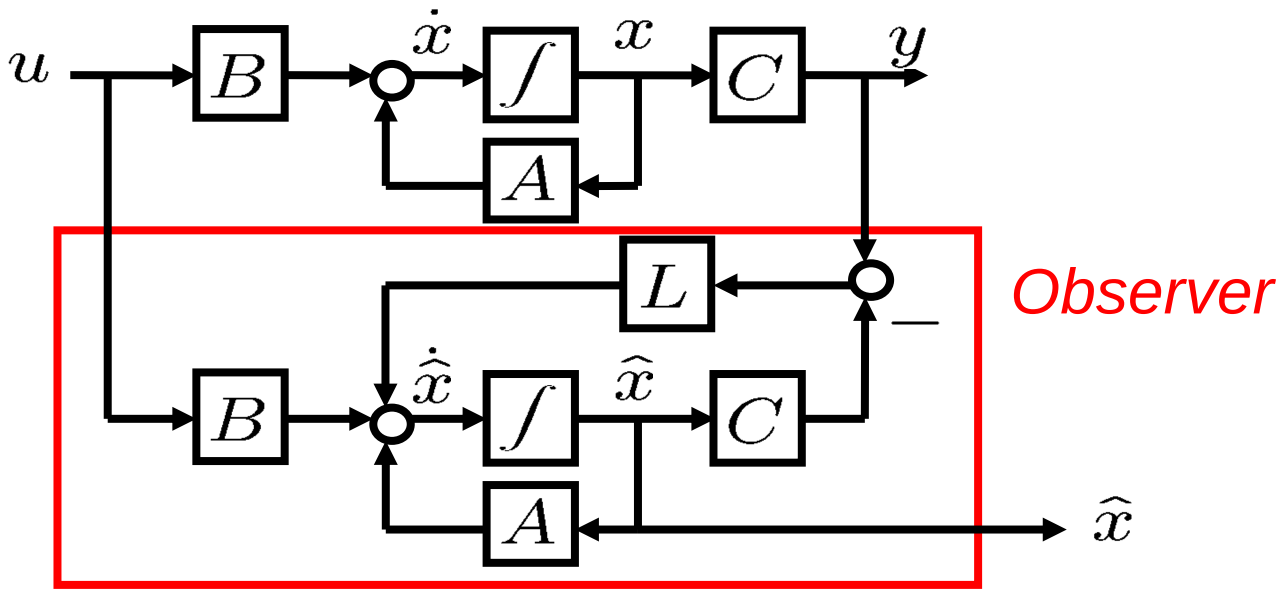
- Given an LTI system

$$\Sigma : \begin{cases} \dot{x}(t) & = & Ax(t) + Bu(t) \\ y(t) & = & Cx(t) \end{cases}$$

  design a state feedback controller.
- If the state vector *x* is NOT available, we can build a state estimator, called *observer*, to estimate *x*, and use the estimate for feedback.
- How to construct an observer?

Full dimensional observer

- LTI system $\Sigma : \begin{cases} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) \end{cases}$
- Observer $\frac{d}{dt}\hat{x}(t) = A\hat{x}(t) + Bu(t) + L(y(t) - C\hat{x}(t))$



Analysis of the observer

- LTI system $\Sigma : \begin{cases} \dot{x}(t) & = & Ax(t) + Bu(t) \\ y(t) & = & Cx(t) \end{cases}$
- Observer $\frac{d}{dt}\hat{x}(t) = A\hat{x}(t) + Bu(t) + L(y(t) - C\hat{x}(t))$

$$\begin{aligned} \Rightarrow \quad \dot{x} - \dot{\hat{x}} &= Ax + Bu - \{A\hat{x} + Bu + L(y - C\hat{x})\} \\ &= A(x - \hat{x}) - LC(x - \hat{x}) \\ &= (A - LC)\underbrace{(x - \hat{x})}_{e} \end{aligned}$$

Estimation error

$$\Rightarrow \quad \dot{e} = (A - LC)e$$

If A-LC is stable, then e goes to zero asymptotically!

Design of the observer gain L

- *If (A,C) is observable, then eigenvalues of (A-LC) can be assigned arbitrarily.* Why?
  - If (A,C) is observable, by duality, (A',C') is controllable.
  - Then, by pole placement theorem,
    - Eigenvalues of $A'-C'K$ can be assigned arbitrarily.
    - Thus, eigenvalues of $A-K'C$ can be assigned arbitrarily.
- Design procedure
  1. Design state feedback K for (A',C'). (direct method, canonical form method, place.m, acker.m, etc.)
  2. Define $L:=K'$.

Example

- Design an observer for the system

$$\begin{cases} \dot{x}(t) &= \begin{bmatrix} 0 & 20 \\ 1 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \\ y(t) &= \begin{bmatrix} 0 & 1 \end{bmatrix} x(t) \end{cases}$$

s.t. eigenvalues of the error system are -10,-10.

- Observability analysis $\mathcal{O} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \Rightarrow \text{rank}\mathcal{O} = 2$

- Observer

$$\frac{d}{dt}\hat{x}(t) = \begin{bmatrix} 0 & 20 \\ 1 & 0 \end{bmatrix} \hat{x}(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) + L\left\{ y(t) - \begin{bmatrix} 0 & 1 \end{bmatrix} \hat{x}(t) \right\}$$

Example (cont'd)

- Direct method $A-LC = \begin{bmatrix} 0 & 20 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} l_1 \\ l_2 \end{bmatrix} \begin{bmatrix} 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 20-l_1 \\ 1 & -l_2 \end{bmatrix}$

$$\Rightarrow \quad \begin{aligned} \det(\lambda I - (A - LC)) &= \lambda(\lambda + l_2) - (20 - l_1) \\ &= \lambda^2 + l_2\lambda + l_1 - 20 \\ &= \lambda^2 + 20\lambda + 100 \end{aligned}$$

$$\Rightarrow \quad L = \begin{bmatrix} 120 \\ 20 \end{bmatrix}$$

- Canonical method $(A^T, C^T) = \left(\begin{bmatrix} 0 & 1 \\ 20 & 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right)$ C.C.F! → $T=I$

$$K := [\alpha_n - a_n, \cdots, \alpha_1 - a_1]\, T$$

$s^2 + 20s + 100$: desired characteristic poly.

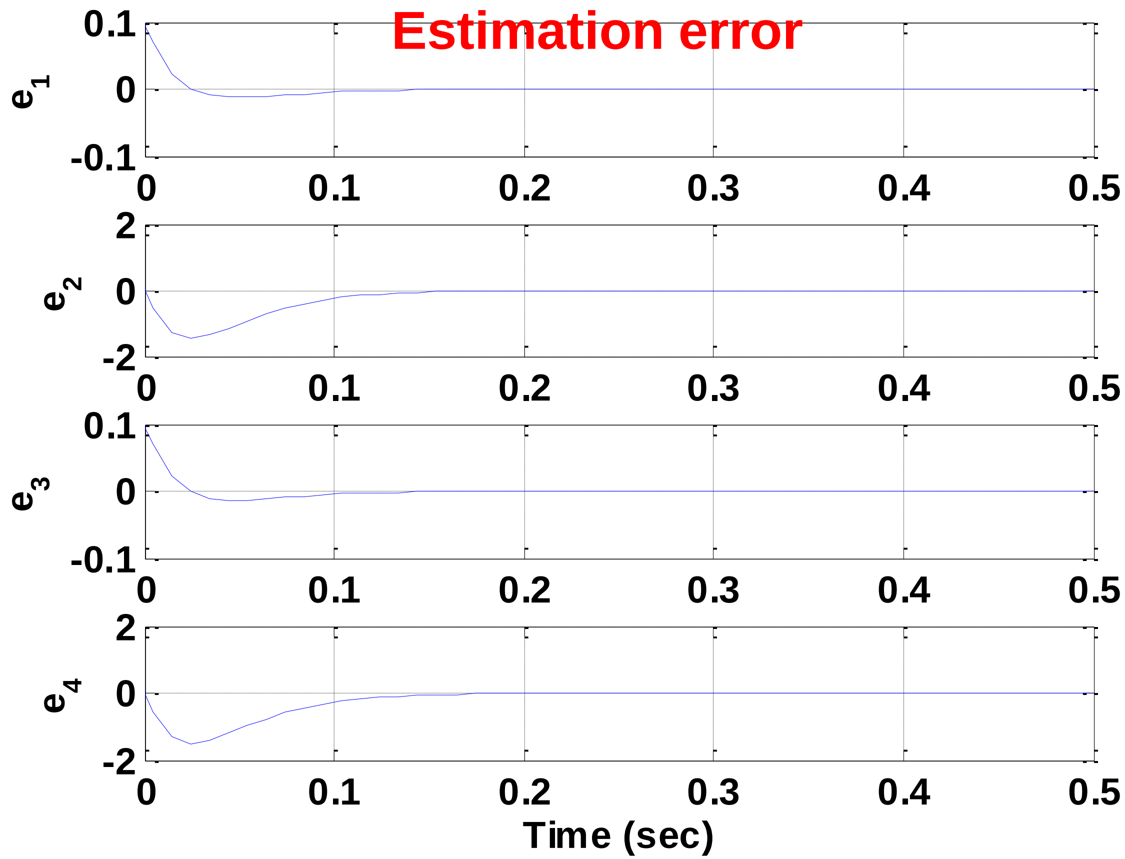
$s^2 + 0s - 20$: original characteristic poly.

$$\Rightarrow \quad K := [120, 20] \quad \Rightarrow \quad L = K^T$$

Inverted pendulum example

- Eigenvalues of error system $-41, -42, -43, -44$

$$e(0) = \begin{bmatrix} 0.1 \\ 0.1 \\ 0.1 \\ 0.1 \end{bmatrix}$$



Remarks

- Full dimensional observer has the same number of states as the plant has.
- If some states are accurately measurable, we do not need to estimate those states.
  - Inverted pendulum example: $$y = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} x$$

We do not need to estimate x_1 and x_3!

- Reduced dimensional observer: Not covered in this course, but a pdf-file explaining such observer is posted on Canvas.

Detectability (Dual concept of stabilizability)

- Suppose that (A,C) is NOT observable.
- If the "unobservable part" of A-matrix is stable, then the system is called *detectable*.

$$\begin{cases} \dot{x}(t) &= Ax(t) \\ y(t) &= Cx(t) \end{cases}$$

Observable pair

$z(t) := Tx(t)$

$$\begin{cases} \begin{bmatrix} \dot{z}_o(t) \\ \dot{z}_{\bar{o}}(t) \end{bmatrix} &= \begin{bmatrix} A_o & 0 \\ A_{12} & A_{\bar{o}} \end{bmatrix} \begin{bmatrix} z_o(t) \\ z_{\bar{o}}(t) \end{bmatrix} \\ y(t) &= \begin{bmatrix} C_o & 0 \end{bmatrix} \begin{bmatrix} z_o(t) \\ z_{\bar{o}}(t) \end{bmatrix} \end{cases}$$

Eigenvalues of this cannot be changed by observer design.

Detectability (cont'd)

$$
\begin{aligned}
A - LC &= T^{-1}\begin{bmatrix} A_o & 0 \\ A_{21} & A_{\bar{o}} \end{bmatrix} T - L\begin{bmatrix} C_o & 0 \end{bmatrix} T \\
&= T^{-1}\left\{\begin{bmatrix} A_o & 0 \\ A_{21} & A_{\bar{o}} \end{bmatrix} - TL\begin{bmatrix} C_o & 0 \end{bmatrix}\right\} T \\
&\quad \left(TL =: \begin{bmatrix} L_o \\ L_{\bar{o}} \end{bmatrix}\right) \\
&= T^{-1}\begin{bmatrix} A_o - L_o C_o & 0 \\ A_{21} - L_{\bar{o}} C_o & A_{\bar{o}} \end{bmatrix} T
\end{aligned}
$$

$$
\Rightarrow \quad \operatorname{eig}(A - LC) = \underbrace{\operatorname{eig}(A_o - L_o C_o)}_{\text{Arbitrarily assignable}} \cup \underbrace{\operatorname{eig}(A_{\bar{o}})}_{\text{Not movable!}}
$$

Remarks on detectability

- If a system is observable, it is detectable.
- Detectability is necessary (but may not be sufficient) for successful output feedback control.
- The real plant needs to be modified (e.g. by adding sensors, changing the sensor locations, or changing the types of sensors) if
  - the system is not detectable, or
  - the unobservable part is stable, but limits the closed-loop performance (e.g., the un-movable eigenvalues are too close to the imaginary axis).

Summary

- Observer
  - Structure
  - Design methods (dual to state feedback controller design)
  - Inverted pendulum example (Matlab files “pendulum.m” and “pendulum_observer.slx” are posted on Canvas.)
- Detectability (dual to stabilizability)
- Next, observer-based control