# <u>MECH 420 SENSORS AND ACTUATORS</u>

**Solution Guidelines for Mid-Term Examination, 06 November 2020**

**Solution 1**

**(i)**
We use the two properties of an op-amp: 1. Current into the op-am from each input lead is zero; 2. The potentials at the two input leads of the op-amp are equal.
Current summation at the inverting input (- lead) node of the op-amp:

$$\frac{v_s}{R_a}+\frac{v_1}{R_b}+\frac{v_o}{R_f}=0 \qquad \text{(i)}$$

where, $v_1$ = voltage at the junction between $C$ and $R_b$

Current balance at the junction of $C$ and $R_b$: $C\dfrac{d(v_s-v_1)}{dt}=\dfrac{v_1}{R_b}$ (ii)

Substitute (i) in (ii), to eliminate $v_1$: $C\dfrac{dv_s}{dt}-C\dfrac{d}{dt}R_b\left(-\dfrac{v_o}{R_f}-\dfrac{v_s}{R_a}\right)=-\dfrac{v_o}{R_f}-\dfrac{v_s}{R_a}$

→ $$C\left(1+\frac{R_b}{R_a}\right)\frac{dv_s}{dt}+\frac{v_s}{R_a}=-C\frac{R_b}{R_f}\frac{dv_o}{dt}-\frac{v_o}{R_f}$$

→ $$CR_f\left(R_a+R_b\right)\frac{dv_s}{dt}+R_f v_s=-CR_aR_b\frac{dv_o}{dt}-R_a v_o$$

→ I/O differential equation: $R_bC\dfrac{dv_o}{dt}+v_o=-\dfrac{R_f}{R_a}\left[\left(R_a+R_b\right)C\dfrac{dv_s}{dt}+v_s\right]$

In the Laplace domain ($\dfrac{d}{dt}\Rightarrow s$), we have the circuit transfer function

$$\frac{v_o}{v_s}=G(s)=-\frac{R_f}{R_a}\frac{\left[\left(R_a+R_b\right)Cs+1\right]}{\left(R_bCs+1\right)}=-k\frac{\left(\tau_a s+1\right)}{\left(\tau_b s+1\right)}$$

where, gain (steady-state value, i.e., at $\omega$ = 0) $k=\dfrac{R_f}{R_a}$ ; and time constants $\tau_a=\left(R_a+R_b\right)C$ and $\tau_b=R_bC$.

Now, as usual, ignore the –ve sign in the transfer function.
*Note*: As we know, the sign in a transfer function (of an op-amp circuit) can be reversed in many ways, and it has no significance in the subsequent analysis.

The frequency transfer function of the circuit (obtained by setting $s=j\omega$) is

$$G(j\omega)=k\frac{\left(\tau_a j\omega+1\right)}{\left(\tau_b j\omega+1\right)} \qquad \text{(iii)}$$

The Bode curves (solid lines) and their asymptotes (broken lines) are shown in Figure S1. As usual, first the asymptotes are sketched, with appropriate slopes, and then the actual curves are sketched (roughly) based on the asymptotes.

**Procedure for obtaining the asymptotes:**

**At low frequencies:** We have $\tau_a\omega << 1.0$ and $\tau_b\omega << 1.0$, and they can be neglected wrt 1.0. Hence, $G(j\omega) \to k$ at low frequencies. This is a "real" value. Its magnitude is $k = \frac{R_f}{R_a}$ (this is the steady-state gain) and the phase angle (because the transfer function is "real" now) is 0°.

**At intermediate frequencies between $\omega_a = 1/\tau_a$ and $\omega_b = 1/\tau_b$:** The zero term (i.e., transfer function numerator) $(\tau_a j\omega + 1)$ dominates over the pole term (i.e., transfer function denominator) $(\tau_b j\omega + 1)$. *Note*: $\tau_a > \tau_b$.
This results in a "phase lead" action.
Hence, $G(j\omega) \approx k(\tau_a j\omega + 1)$. This provides a magnitude asymptote of +ve slope 20 dB/decade (in log scale), and a phase angle asymptote at 90° (corresponding to the maximum possible lead action, which is the derivative action).

**At high frequencies:** We have $\tau_a\omega >> 1.0$ and $\tau_b\omega >> 1.0$, and hence 1.0 can be neglected in the pole term and in the zero term. Then, $G(j\omega) \to k\frac{\tau_a}{\tau_b} = \frac{R_f(R_a + R_b)}{R_a R_b} = R_f\left(\frac{1}{R_b} + \frac{1}{R_a}\right)$. This is also a "real" value. Its magnitude is $R_f\left(\frac{1}{R_b} + \frac{1}{R_a}\right)$ and the phase angle (because the transfer function is real now) is 0°.

*Note*: The break frequencies are $\omega_a = 1/\tau_a$ and $\omega_b = 1/\tau_b$, which correspond to the points of intersection of the asymptotes.

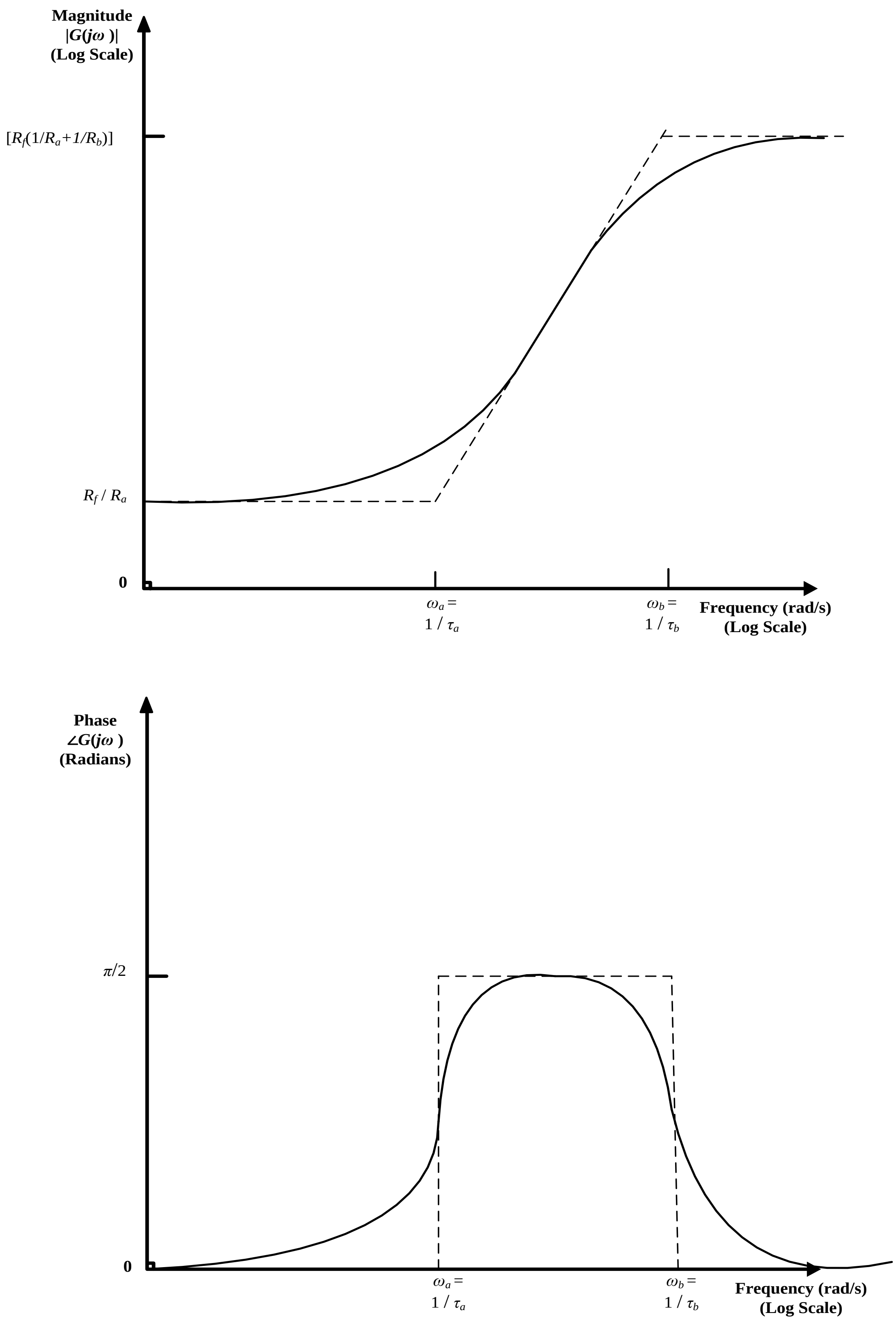


**Figure S1: Bode magnitude and phase angle curves of the circuit.**

**(ii)**

(a) The steady state corresponds to $\omega = 0$. From the transfer function (iii), the corresponding gain (the transfer function magnitude) is $k = \frac{R_f}{R_a}$, as obtained before.

(b) For high frequencies, $\omega \to \infty$. From the transfer function (iii), the high-frequency gain (i.e., the transfer function magnitude for $\omega \to \infty$) is $k\frac{\tau_a}{\tau_b} = \frac{R_f(R_a + R_b)}{R_a R_b} = R_f\left(\frac{1}{R_b} + \frac{1}{R_a}\right)$, as obtained before.

(c) The numerator of $G(j\omega)$ provides a phase lead of $\tan^{-1}\tau_a\omega$. The denominator of $G(j\omega)$ provides a phase lag of $\tan^{-1}\tau_b\omega$. Hence, the overall phase "lead" (in radians) provided by the circuit is $\tan^{-1}\tau_a\omega - \tan^{-1}\tau_b\omega$, where $\tau_a = (R_a + R_b)C$ and $\tau_b = R_b C$.

(d) As noted before, the break points (in radians/s) of the Bode curve are: $\omega_a = \frac{1}{\tau_a}$ and $\omega_b = \frac{1}{\tau_b}$.

**Benefits of the Circuit:**

1. It has two frequency ranges, the low-frequency range and the high frequency range, that provide almost constant signal amplification of gain $\left(\frac{R_f}{R_a}\right)$ and $\left(\frac{R_f(R_a + R_b)}{R_a R_b}\right)$, respectively, and zero phase change, for the sensor signal.
2. In the intermediate frequency range, it becomes a "lead compensator" providing a phase lead, which has the following benefits of "derivative" or "preview" action: suppressing signal overshoots, speeding up the response, improving the system stability, etc.
3. It provides its very high input impedance for the sensor and its very low output impedance to the ADC (due to the op-amp in the analog circuit), thereby considerably reducing electrical loading.

**(iii)**

(a) To realize the best overall sensitivity for the given arrangement, when the sensor generates its full scale voltage (2.0V), the ADC should receive its full-scale input voltage of 6.0 V (which corresponds to $2^8$ = 256 counts). Correspondingly, then, the analog circuit should provide its largest gain. This condition corresponds to the high frequency range of operation, in which the maximum gain (the high-frequency gain) is provided by the circuit, as established before.

Hence, the required gain from the circuit $= \frac{6.0\text{ (V)}}{2.0\text{ (V)}} = 3.0$

The high-frequency gain of the circuit should be equal to this value. Hence,

$$R_f\left(\frac{1}{R_b} + \frac{1}{R_a}\right) = 3.0 = R_f\left(\frac{1}{10.0\times10^3} + \frac{1}{2.0\times10^3}\right)$$

$\rightarrow \quad R_f = 5.0\times10^3\Omega = 5.0\text{ k}\Omega$

The maximum count of the ADC = $2^8$ = 256 counts. This corresponds to 6.0 V into the ADC. The corresponding speed measured by the sensor is 100.0 cm/s (given). Hence, the overall sensitivity of the device under this condition is

$$256/100.0 \text{ counts/cm/s} = 2.56 \text{ counts/cm/s}$$

*Note*: The sensitivity of the ADC alone is $256/6.0 \text{ counts/V} = 42.7 \text{ counts/V}$.

(b) $\tau_a = (R_a + R_b)C = (10.0+2.0)\times 2.0\times 10^3 \times 10^{-6} = 24.0\times 10^{-3}\text{s}$

The corresponding break frequency (see Figure S1) is,

$$\omega_a = \frac{1}{\tau_a} = \frac{1}{24.0\times 10^{-3}} \text{rad/s} = 41.67 \text{ rad/s} = 6.6 \text{ Hz}$$

$\tau_b = R_b C = 2.0\times 2.0\times 10^3 \times 10^{-6} = 4.0\times 10^{-3}\text{s}$

The corresponding break frequency (see Figure S1) is,

$$\omega_b = \frac{1}{\tau_b} = \frac{1}{4.0\times 10^{-3}} \text{rad/s} = 250.0 \text{ rad/s} = 39.8 \text{ Hz}$$

From Figure S1 it is clear that there are three possible frequency ranges of operation.

1. **Frequency Range 0.0 to 6.6 Hz (Low frequency range):**
   In this frequency range, the system (analog circuit in particular) operates in a steady state, with amplification (low-frequency gain) $\dfrac{R_f}{R_a} = \dfrac{5.0\times 10^3}{10.0\times 10^3} = 0.5$.
   The corresponding sensitivity of the overall device is,
   $$\frac{256 \text{ (counts)}}{6.0 \text{ (V)}} \times 0.5 \times \frac{2.0 \text{ (V)}}{100.0 \text{ (cm/s)}} = 0.43 \text{ counts/cm/s}.$$
   So, in this range, the sensitivity of the device is relatively poor.
   As a rule of thumb (or by Shannon's sampling theorem), a suitable sampling rate in this frequency range should be at least 6.6×2 samples/s → 14 samples/s

2. **Frequency Range 6.6 Hz to 39.8 Hz (Intermediate frequency range):**
   In this frequency range, the analog circuit functions as a lead compensator. The circuit conditions are dynamic (not steady) as clear from Figure S1. So, the overall sensitivity will also vary with frequency. However, the device will be more stable (even though dynamic) as the circuit is a lead compensator in this frequency range.

3. **Frequency Range > 39.8 Hz (High frequency range):**
   In this frequency range as well, the system (analog circuit in particular) operates in a steady state. As obtained before, the device will have the best sensitivity then, at $2.56 \text{ counts/cm/s}$.
   As a rule of thumb (or by Shannon's sampling theorem), a suitable sample rate in this frequency range should be at least 39.8×2 samples/s → 80 samples/s. But depending on the high frequency limit of actual operation, the sampling rate should be increased correspondingly (at least to twice the highest frequency of operation).

**Solution 2**

**(i)**

The characteristic curve of the thermistor is sketched in Figure S2.

Consider the thermistor model $R = R_0 \exp\left[\beta\left(\frac{1}{T} - \frac{1}{T_0}\right)\right]$.

Set $T = T_0$. Then the exponent becomes zero, and its exponential value is 1. Hence, then $R = R_0$. This means that the empirical parameter $R_0$ is the resistance of the thermistor at the reference temperature $T_0$.

Next, let $T \to \infty$. Then the exponent becomes $-\frac{\beta}{T_0}$. Hence, the resistance at very large values of temperature is $R_\infty = R_0 \exp\left[-\frac{\beta}{T_0}\right] = \frac{R_0}{\exp\left[\frac{\beta}{T_0}\right]}$.

*Note*: This value is $< R_0$.

These key points are marked in Figure S2.

**(ii)**

**(a)**

Write the sensor equation as: $\ln R - \ln R_0 = \beta\left(\frac{1}{T} - \frac{1}{T_0}\right)$ and take the differentials of the individual terms.

*Note*: There cannot be an error in the reference temperature $T_0$ itself, because it is a value that one is free to select. Also, since $\beta$ is very accurate (given), it cannot have error. Of course, there will be errors in the associated resistance $R_0$.

$$\frac{\delta R}{R} - \frac{\delta R_0}{R_0} = -\beta \frac{\delta T}{T^2} \rightarrow \quad e_R - e_{R_0} = -\frac{\beta}{T} e_T$$

With the absolute method of error combination:

$$e_T = \frac{T}{\beta}(e_R + e_{R_0}) \qquad \text{(i)}$$

*Note*: We use of the "+" sign instead of "−" on the RHS of the error equation since we employ the "absolute" method of error combination (i.e., positive magnitudes are used regardless of the actual algebraic sign). However, each error value is $\pm$.

It is clear from (i) that a larger $T$ will result in larger fractional error in the determined temperature.

**(b)**

Substitute the given numerical values: $e_T = \frac{400}{4200}(0.01 + 0.02) = 0.003$

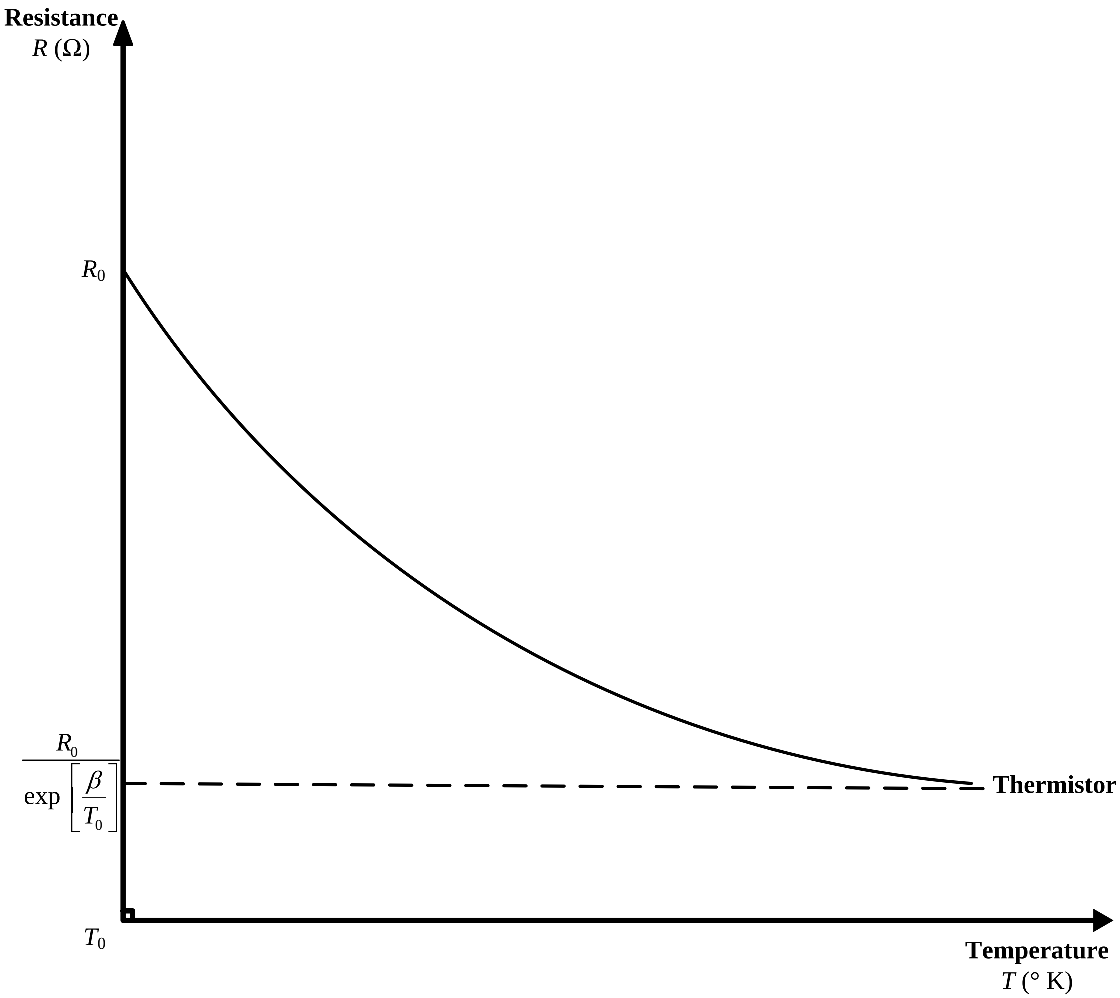


**Figure S2: Characteristic curve of the thermistor.**