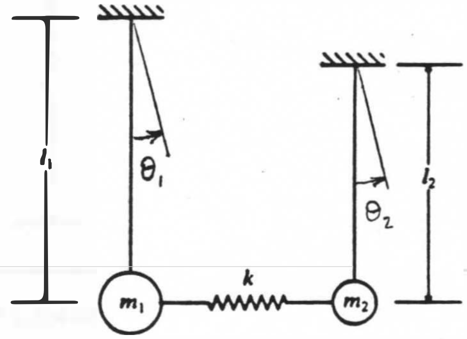
MECH 463 -- Homework 11

1. Two pendulums of lengths ℓ_1 and ℓ_2, and masses m_1 and m_2 are coupled together by a spring of stiffness k. In the particular case considered here, $\ell_1 = 2\ell$, $\ell_2 = \ell$, $m_1 = m_2 = m$, and $mg/\ell = 3k$.



The matrix equation of motion for the coupled pendulum system is

$$\begin{bmatrix} m_1\ell_1^2 & 0 \\ 0 & m_2\ell_2^2 \end{bmatrix}\begin{bmatrix} \ddot\theta_1 \\ \ddot\theta_2 \end{bmatrix} + \begin{bmatrix} mg\ell_1+k\ell_1^2 & -k\ell_1\ell_2 \\ -k\ell_1\ell_2 & mg\ell_2+k\ell_2^2 \end{bmatrix}\begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Use the Raleigh Quotient to estimate the lowest natural frequency of the pendulum system. To get a good natural frequency estimate, use a guessed mode shape $\underline{v} = [\,1 \quad v_2\,]^T$, where v_2 is a variable. Find a value of v_2 to give a good natural frequency result. Justify your procedure.

For $l_1 = 2l$, $l_2 = l$, $m_1 = m_2 = m$ and $\frac{mg}{l} = 3k$,

the matrix equation of motion is:

$$\begin{bmatrix} m(2l)^2 & 0 \\ 0 & ml^2 \end{bmatrix}\begin{bmatrix} \ddot\theta_1 \\ \ddot\theta_2 \end{bmatrix} + \begin{bmatrix} mg(2l)+k(2l)^2 & -k(2l)l \\ -k(2l)l & mgl+kl^2 \end{bmatrix}\begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\div l^2 \rightarrow \begin{bmatrix} 4m & 0 \\ 0 & m \end{bmatrix}\begin{bmatrix} \ddot\theta_1 \\ \ddot\theta_2 \end{bmatrix} + \begin{bmatrix} 10k & -2k \\ -2k & 4k \end{bmatrix}\begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Raleigh Quotient: $\omega_R^2 = \dfrac{\underline{v}^T \underset{\sim}{K} \underline{v}}{\underline{v}\, \underset{\sim}{M}\, v}$ where $\underline{v}$ is a guessed mode shape. $\underline{v} = \begin{bmatrix} 1 \\ v_2 \end{bmatrix}$

$$\omega_R^2 = \frac{[1 \quad v_2]\begin{bmatrix} 10k & -2k \\ -2k & 4k \end{bmatrix}\begin{bmatrix} 1 \\ v_2 \end{bmatrix}}{[1 \quad v_2]\begin{bmatrix} 4m & 0 \\ 0 & m \end{bmatrix}\begin{bmatrix} 1 \\ v_2 \end{bmatrix}} = \frac{k[10-2v_2 \quad -2+4v_2]\begin{bmatrix} 1 \\ v_2 \end{bmatrix}}{m[4 \quad v_2]\begin{bmatrix} 1 \\ v_2 \end{bmatrix}}$$

$$= \frac{k\left(10 - 2V_2 + V_2(-2 + 4V_2)\right)}{m\,(4 + V_2^2)} = \frac{(4V_2^2 - 4V_2 + 10)\,k}{(V_2^2 + 4)\quad m}$$

The best estimate of ω^2 is the minimum value of ω_R^2.
We can find this minimum from $\frac{d\omega_R^2}{dV_2} = 0$

$$\frac{d\omega_R^2}{dV_2} = \frac{(V_2^2+4)(8V_2-4) - (4V_2^2-4V_2+10)(2V_2)}{(V_2^2+4)^2} \cdot \frac{k}{m} = 0$$

$$\rightarrow \cancel{8V_2^3} - 4V_2^2 + 32V_2 - 16 - \cancel{8V_2^3} + 8V_2^2 - 20V_2 = 0$$

$$\rightarrow 4V_2^2 + 12V_2 - 16 = 0 \qquad \rightarrow 4(V_2 - 1)(V_2 + 4) = 0$$

$$\rightarrow V_2 = 1 \quad \text{or} \quad V_2 = -4$$

The lower natural frequency has the positive V_2 value
(no nodes) $\rightarrow V_2 = 1$

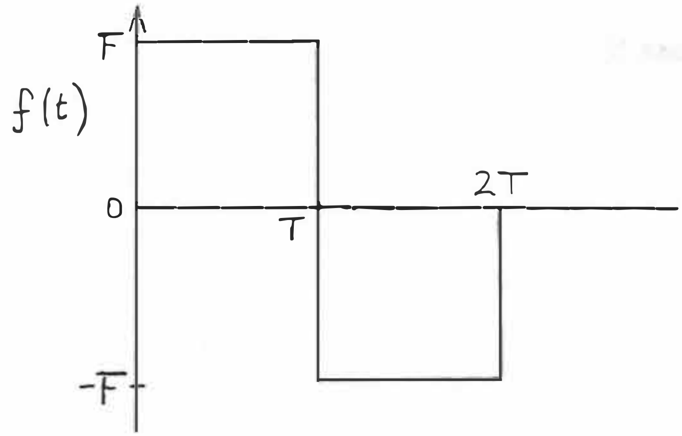
$$\rightarrow \omega_R^2 = \frac{(4V_2^2 - 4V_2 + 10)}{(V_2^2+4)} \cdot \frac{k}{m} = \underline{2\ k/m}$$

Since this is the minimum possible ω_R^2 value,
it equals the exact ω^2.

The higher natural frequency is the largest possible
value of ω_R^2. This also occurs when $\frac{d\omega_R^2}{dV_2} = 0$.
This frequency corresponds to the second root $V_2 = -4$

$$\rightarrow \omega_R^2 = \frac{(4V_2^2 - 4V_2 + 10)}{(V_2^2+4)} \cdot \frac{k}{m} = \underline{4\tfrac{1}{2}\ k/m}$$

2. A pulsed square wave force f(t) = F for 0 < t < T, f(t) = –F for T < t < 2T and f(t) = 0 for t > 2T is applied to a 1-DOF vibrating system. Starting from the equation mx + kx = f(t), calculate the response of the system for t > 2T, i.e., after completion of the pulse force. Assume the system is at rest before the force application. Confirm that if T = $2n\pi/\omega$, where n is a positive integer, then the response after completion of the pulse force is zero.



Use Duhamel's Integral

$$x(t) = x_0 \cos \omega t + \frac{\dot{x}}{\omega} \sin \omega t + \int_0^t f(\tau)\, h(t-\tau)\, d\tau$$

where $h(t) = \omega/k \sin \omega t$

For $t > 2T$

$$x(t) = \underbrace{0 + 0}_{} + \int_0^T F \,\omega/k \sin \omega(t-\tau)\, d\tau$$

zero initial conditions

$$+ \int_T^{2T} -F \,\omega/k \sin \omega(t-\tau)\, d\tau + \int_{2T}^t 0 \, d\tau$$

$$= F \,\omega/k \left[\frac{1}{\omega} \cos \omega(t-\tau)\right]_0^T - F\,\omega/k \left[\frac{1}{\omega} \cos \omega(t-\tau)\right]_T^{2T} + 0$$

$$\rightarrow \boxed{x(t) = F/k \left(2 \cos \omega(t-T) - \cos \omega t - \cos \omega(t-2T)\right)}$$

If $T = 2n\pi/\omega \rightarrow x(t) = F/k \left(2\cos(\omega t - 2n\pi) - \cos \omega t - \cos(\omega t - 4n\pi)\right)$

$$= F/k \left(2 \cos \omega t - \cos \omega t - \cos \omega t\right)$$

$$= 0$$