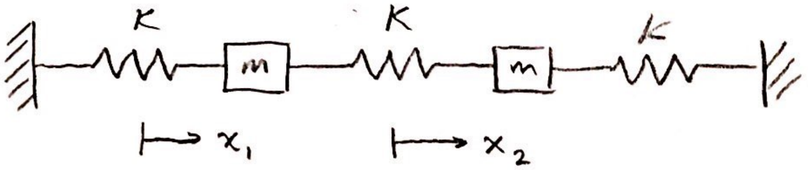
MECH 463
Oct 10

Lecture 10

Principal Coordinates



(mass-based)

$$\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} 2k & -k \\ -k & 2k \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{cases} \omega_1^2 = \dfrac{k}{m} \\ \omega_2^2 = \dfrac{3k}{m} \end{cases} \quad \begin{cases} u_1 = 1 \\ u_2 = -1 \end{cases}$$

$[M]$ $\qquad$ $[k]$

Let $[U] = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ and $\vec{x} = [U]\vec{p}$

Then $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} p_1 \\ p_2 \end{bmatrix} \Rightarrow \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

So $p_1 = \dfrac{x_1 + x_2}{2}$ and $p_2 = \dfrac{x_1 - x_2}{2}$

Let $[M^{*}] = [U]^T[M][U]$

$$[M^{*}] = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 2m & 0 \\ 0 & 2m \end{bmatrix}$$

And $[K^{*}] = [U]^T[K][U]$

$$[K^{*}] = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 2k & -k \\ -k & 2k \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 2k & 0 \\ 0 & 6k \end{bmatrix}$$

Equation with principal coordinates:

$$[M^*]\ddot{\vec{p}} + [K^*]\vec{p} = \vec{0}$$

$$\begin{bmatrix} 2m & 0 \\ 0 & 2m \end{bmatrix}\begin{bmatrix} \ddot{p}_1 \\ \ddot{p}_2 \end{bmatrix} + \begin{bmatrix} 2k & 0 \\ 0 & 6k \end{bmatrix}\begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Uncoupled equations:

$$\left.\begin{aligned} 2m\ddot{p}_1 + 2kp_1 = 0 &\Rightarrow \omega_1^2 = \frac{2k}{2m} = \frac{k}{m} \\ 2m\ddot{p}_2 + 6kp_2 = 0 &\Rightarrow \omega_2^2 = \frac{6k}{2m} = \frac{3k}{m} \end{aligned}\right\} \text{As before}$$

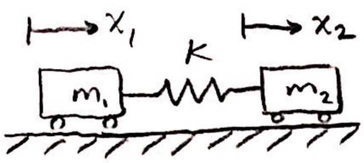
Vibration Stability

This system has a higher ω_N than a flatter bowl. $(\omega_N^2 > 0)$

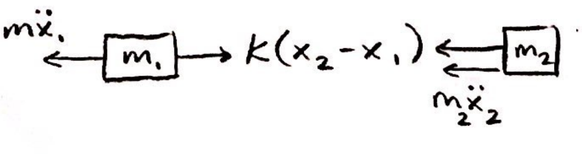
As the bowl becomes flatter $\omega_N^2 \to 0$.
For perfectly flat plane $\omega_N^2 = 0$.

There is no frequency here. The ball rolls off and does return. $(\omega_N^2 < 0)$

Semi-Definite Systems



$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} k & -k \\ -k & k \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$



Try solution $$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \overset{\text{constants}}{\begin{bmatrix} X_1 \\ X_2 \end{bmatrix}} \cos(\omega t + \phi)$$

Equation of motion: $$\begin{bmatrix} k-\omega^2 m_1 & -k \\ -k & k-\omega^2 m_2 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \cos(\omega t + \phi) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Non-trivial solution: $$\det \begin{bmatrix} k-\omega^2 m_1 & -k \\ -k & k-\omega^2 m_2 \end{bmatrix} = 0$$

Then $m_1 m_2 \omega^4 - k(m_1+m_2)\omega^2 + k^2 - k^2 = 0$

$$\Rightarrow \omega_1^2 = 0 \qquad \omega_2^2 = \frac{k(m_1+m_2)}{m_1 m_2}$$

Mode shapes $$\begin{bmatrix} k-\omega^2 m_1 & -k \\ -k & k-\omega^2 m_2 \end{bmatrix} \begin{bmatrix} 1 \\ u \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

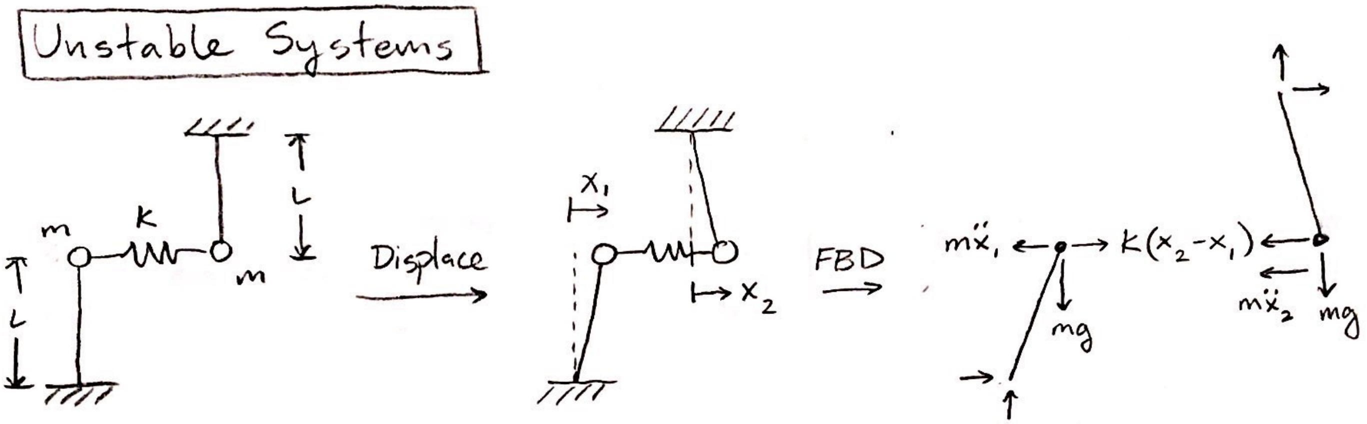
When $\omega_1^2 = 0 \Rightarrow k - ku_1 = 0 \Rightarrow \boxed{u_1 = 1}$ (top line)

$\omega_2^2 = \frac{k(m_1+m_2)}{m_1 m_2} \Rightarrow -k\frac{m_1}{m_2} - ku_2 = 0 \Rightarrow \boxed{u_2 = -\frac{m_1}{m_2}}$ (top line)

Physically, for $u_1 = 1$, both amplitudes are equal, so the train moves forever. The period of vibration is infinite. Frequency is period^{-1} → $\omega = 0$. This is rigid body translation.

For $u_2 = -\frac{m_1}{m_2}$ the cars vibrate around the center of mass of the whole system.

Unstable Systems



Moments at ends $\circlearrowleft \Sigma M = 0$ (small angles)

$$\Rightarrow \begin{cases} m\ddot{x}_1 l - mgx_1 - k(x_2-x_1)l = 0 \\ m\ddot{x}_2 l + mgx_2 + k(x_2-x_1)l = 0 \end{cases}$$

Matrices

$$\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} k - \frac{mg}{l} & -k \\ -k & k + \frac{mg}{l} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Require

$$\det\begin{bmatrix} k - \frac{mg}{l} - m\omega^2 & -k \\ -k & k + \frac{mg}{l} - m\omega^2 \end{bmatrix} = 0$$

Solve $m^2\omega^4 - 2mk\omega^2 - \frac{m^2g^2}{l^2} = 0 \Rightarrow \omega^2 = \frac{k}{m} \pm \sqrt{\left(\frac{k}{m}\right)^2 + \left(\frac{g}{l}\right)^2}$

Since $\sqrt{\left(\frac{k}{m}\right)^2 + \left(\frac{g}{l}\right)^2} > \left(\frac{k}{m}\right) \Rightarrow \underset{\substack{\uparrow \\ \text{Collapse}}}{\omega_1^2 < 0}$ and $\underset{\substack{\uparrow \\ \text{Vibration around C.O.M.}}}{\omega_2^2 > 0}$

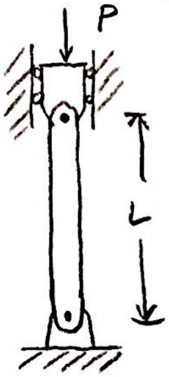
Meaning of $\omega^2 < 0$

Let $\omega^2 = -\lambda^2 \Rightarrow \lambda = i\omega$. If $\omega^2 < 0 \Rightarrow \lambda^2 > 0$

Then $x = C_1e^{i\omega t} + C_2e^{-i\omega t}$

$$x = \underset{\substack{\uparrow \\ \text{Unstable portion}}}{C_1e^{\lambda t}} + C_2e^{-\lambda t}$$

Buckling



Pinned-Pinned Column

Frequency [Hz] $f = \frac{1}{2\pi L}\sqrt{\left(\frac{EI}{\rho A}\right)\left(\frac{\pi^4}{L^2} - \frac{P\pi^2}{EI}\right)}$

The column buckles when $f = 0$.

$\Rightarrow$ when $\frac{\pi^4}{L^2} = \frac{P\pi^2}{EI} \Rightarrow P = \frac{\pi^2 EI}{L^2}$

This is the Euler buckling load.