MECH468 : Modern Control Engineering
MECH509 : Controls

L9 : Lyapunov Theorem

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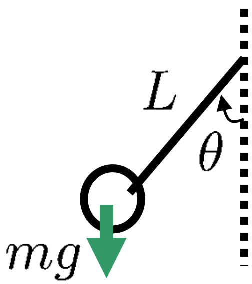
Zoom lecture to be recorded and posted on Canvas

Course plan

Topics	CT	DT
Modeling	✓	✓
→ Stability		
Controllability/observability		
Realization		
State feedback/observer		
LQR/Kalman filter		

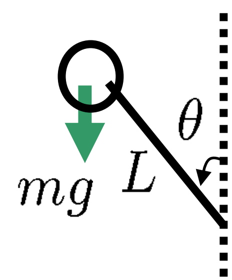
Review & examples

- Last time, eigenvalue criteria for internal stability.
  - CT : asymptotically stable $\Longleftrightarrow$ $\mathrm{Re}\lambda_i(A) < 0, \forall i$
  - DT : asymptotically stable $\Longleftrightarrow$ $|\lambda_i(A)| < 1, \forall i$



Pendulum

$$x(t) := \begin{bmatrix} \theta(t) \\ \dot{\theta}(t) \end{bmatrix}$$



Inverted pendulum

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -\frac{g}{L} & 0 \end{bmatrix} x(t)$$

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ \frac{g}{L} & 0 \end{bmatrix} x(t)$$

$\Rightarrow$ $\lambda(A) = \pm j\sqrt{\dfrac{g}{L}}$ *Marginally stable*

$\Rightarrow$ $\lambda(A) = \pm\sqrt{\dfrac{g}{L}}$ *Unstable*

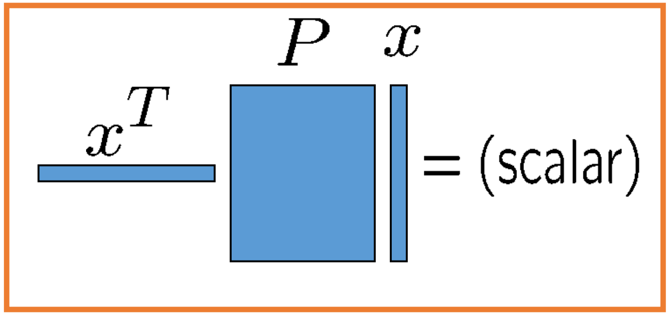
Today's topic

- We will study another condition (equivalent to eigenvalue criteria) for internal stability, called *Lyapunov Theorem*.
- Outline
  - Positive definite matrix
  - Lyapunov Theorem
    - Theorem
    - Example
    - Idea of the theorem

Positive definite matrix

- A symmetric matrix P (n-by-n) is called *positive definite* if

$$x^T P x > 0, \ \forall x \in \mathbb{R}^n, \ x \neq 0$$



We write "*P>0*" to mean "*P* is positive definite".

- Example $P = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$

$$\Rightarrow \quad x^T P x = 2x_1^2 + x_2^2 > 0, \ \text{ if } x \neq 0$$

Facts on positive definite matrix

- Fact 1: $P > 0 \Leftrightarrow \lambda_i(P) > 0, \forall i$
- Fact 2 (Sylvester's criterion):

(Leading principal minor)

$$P > 0 \Leftrightarrow \det P(1:i, 1:i) > 0, \forall i$$

- Ex. $P = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \Rightarrow \begin{array}{l} p_{11} = 2 > 0 \\ \det P = 1 > 0 \end{array} \Rightarrow P > 0$
- Ex. $P = \begin{bmatrix} 2 & 3 \\ 3 & 1 \end{bmatrix} \Rightarrow \begin{array}{l} p_{11} = 2 > 0 \\ \det P = -7 \not> 0 \end{array} \Rightarrow P \not> 0$

Note: Positive entries do NOT mean P.D.!

Facts on positive definite matrix

- **Fact 3**: $P > 0 \Rightarrow p_{ii} > 0, \forall i \quad (p_{ii} = e_i^T P e_i)$

- Ex. $P = \begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix} \not> 0$

$$e_i = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \leftarrow \text{i-th entry}$$

Check diagonal elements first!

- If there is non-positive diagonal element, the matrix is NOT positive definite.
- If all the diagonal elements are positive, the matrix may or may not be positive definite.

Lyapunov Theorem (CT case)

- All the eigenvalues of a matrix *A* have negative real parts (in the open left-half plane)

  if and only if

  the solution *P* of the following *Lyapunov equation*

$$A^T P + PA = -Q$$

  is positive definite for any (and for any one) $Q>0$.

Note: We solve the Lyapunov equation by solving the corresponding linear equation. Use "lyap.m" in Matlab.

Lyapunov Theorem (DT case)

- All the eigenvalues of a matrix *A* have absolute values less than one (inside the unit disc)

  if and only if

  the solution P of the following *discrete Lyapunov equation*

$$A^T P A - P = -Q$$

  is positive definite for any (and for any one) $Q>0$.

Note: Use "dlyap.m" in Matlab.

Remarks

- Normally, Q is taken to be the identity matrix.
- For LTI SS model, Lyapunov Theorem has no advantage over eigenvalue criteria.
- In this course, we study *idea* of Lyapunov theorem, which will be useful in studying advanced control.
  - Nonlinear control
  - Time-varying control
  - Robust control
  - Switching control
  - Control of delay systems

Examples $A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \Rightarrow \lambda = -1, -2$

- CT case $A^T P + PA = -I$

$$\begin{bmatrix} 0 & -2 \\ 1 & -3 \end{bmatrix}\begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} + \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} = -\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow P = \frac{1}{4}\begin{bmatrix} 5 & 1 \\ 1 & 1 \end{bmatrix} > 0$$

➡ Eigs. are all in the open left-half plane!

- DT case $A^T P A - P = -I$

$$\begin{bmatrix} 0 & -2 \\ 1 & -3 \end{bmatrix}\begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} - \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} = -\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

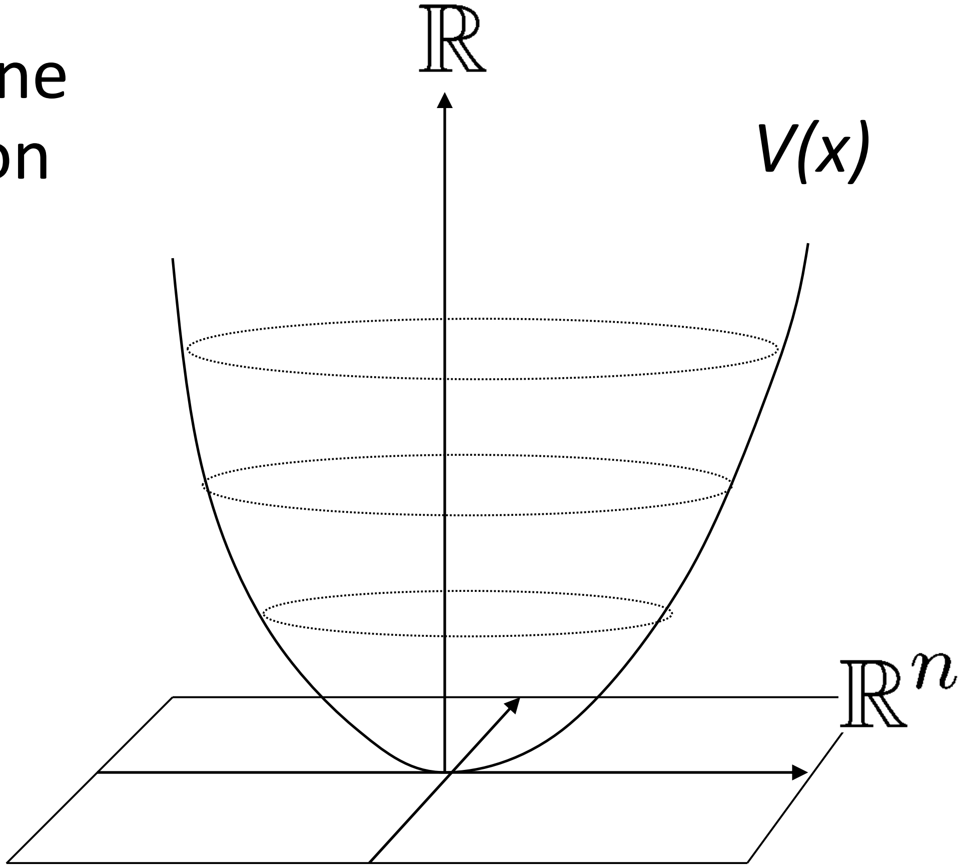
➡ $\begin{bmatrix} 4p_3 - p_1 & -3p_2 + 6p_3 \\ -3p_2 + 6p_3 & p_1 - 6p_2 + 8p_3 \end{bmatrix} = -\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ No solution!

➡ NOT all eigs. are in the unit disc!

Idea of Lyapunov Theorem

1. For a fixed *P>0*, define a (Lyapunov) function

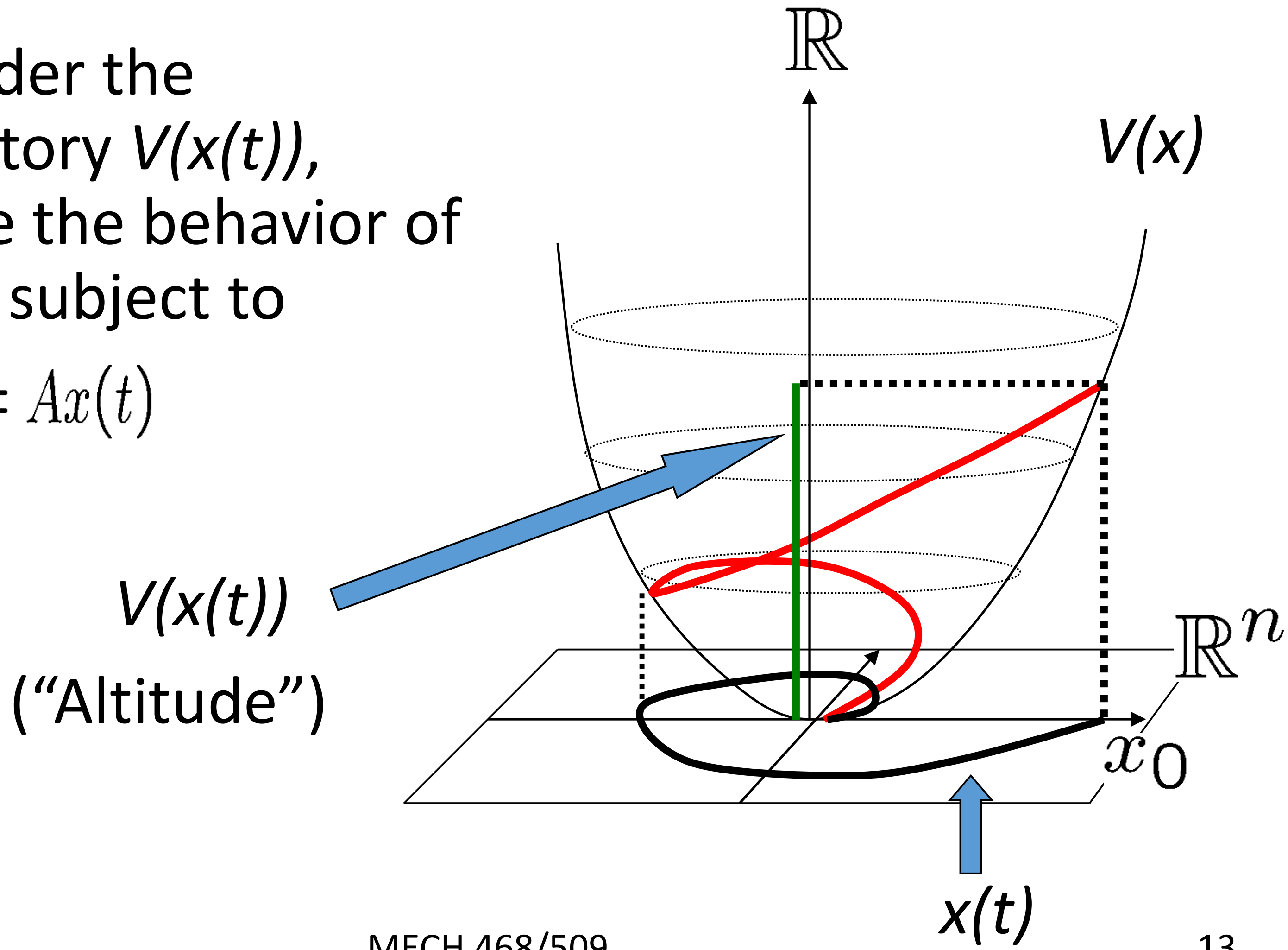
$$V(x) := x^T P x, \;\; x \in \mathbb{R}^n$$



Idea of Lyapunov Th. (cont'd)

2. Consider the trajectory *V(x(t))*, where the behavior of *x(t)* is subject to

$$\dot{x}(t) = Ax(t)$$

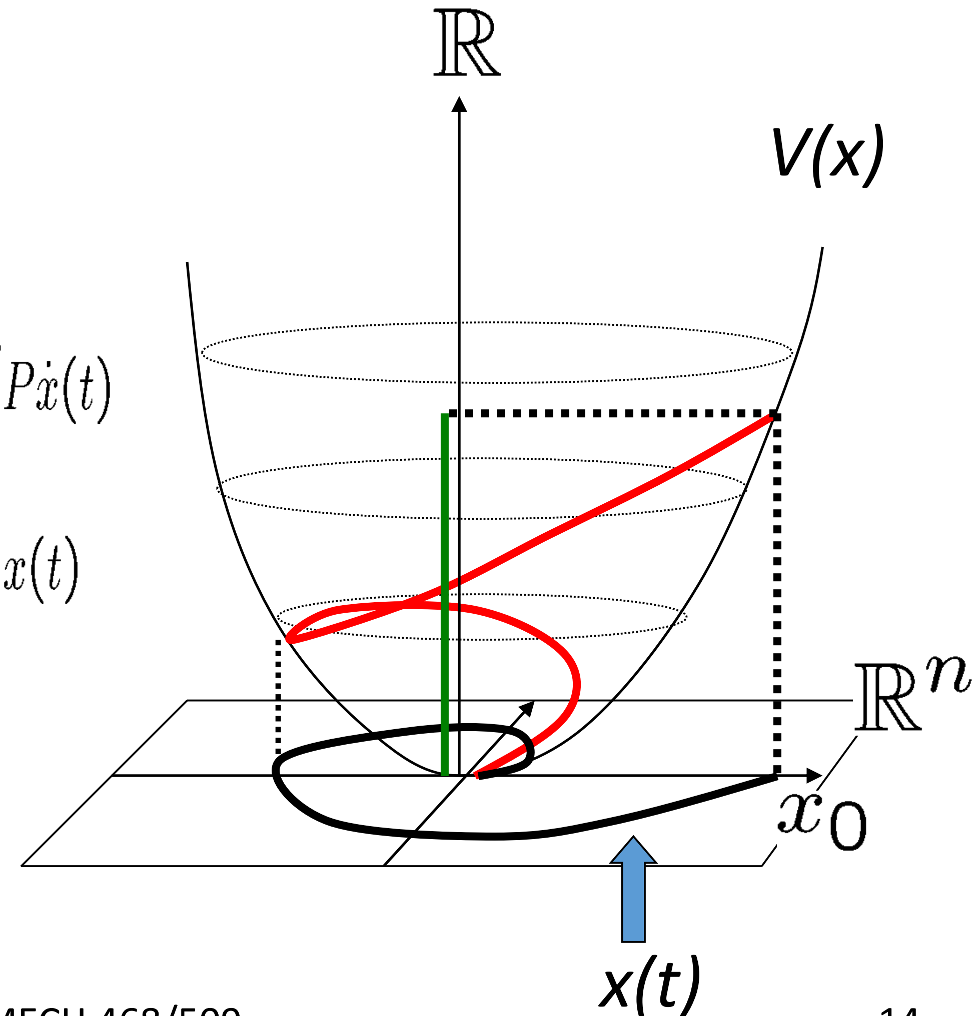


Idea of Lyapunov Th. (cont'd)

3. Take a derivative of *V(x(t))* w.r.t. t:

$$\begin{aligned}\frac{d}{dt}V(x(t)) &= \frac{d}{dt}\left(x(t)^T P x(t)\right)\\ &= \dot{x}(t)^T P x(t) + x(t)^T P \dot{x}(t)\\ &\quad (\dot{x}(t) = Ax(t))\\ &= x(t)^T \left\{A^T P + PA\right\} x(t)\end{aligned}$$

(In DT case, take $V(x[k+1]) - V(x[k])$)



Idea of Lyapunov Th. (cont'd)

4. *If A'P+PA=:-Q<0*, then

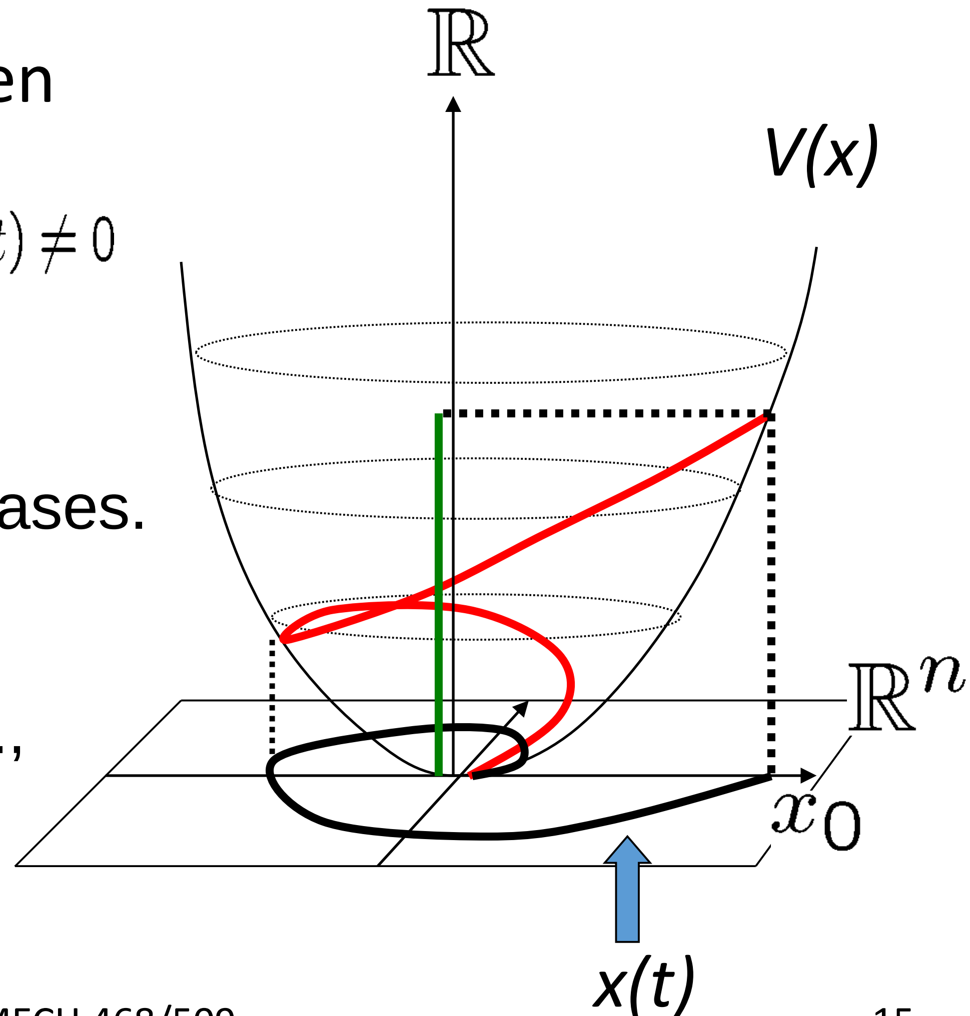
$$\frac{d}{dt}V(x(t)) = x(t)^T\{-Q\}x(t) < 0, \forall x(t) \neq 0$$

⬇

V(x(t)) goes down as *t* increases.

⬇

x(t) approaches to zero, i.e., asymptotically stable!



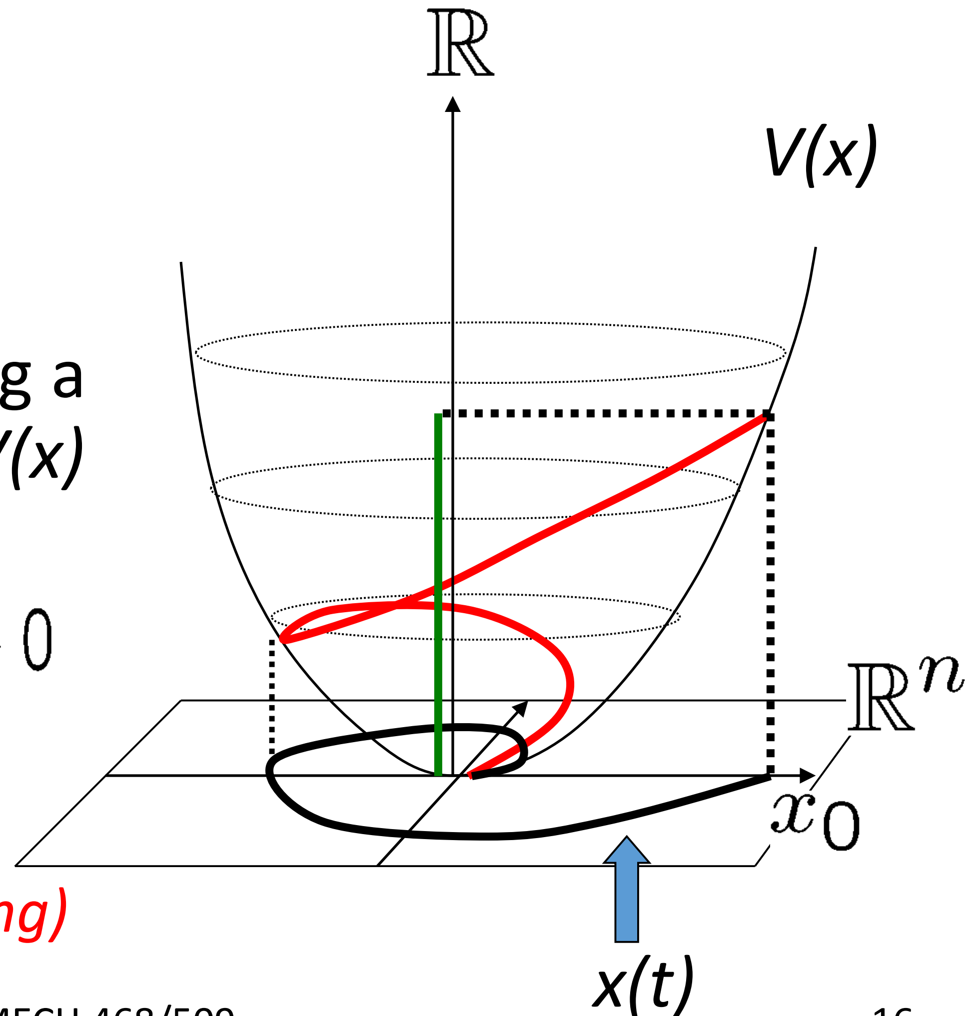
Idea of Lyapunov Th. (summary)

- Solving the Lyapunov equation w.r.t. *P*

$$A^T P + PA = -Q$$

is equivalent to finding a (Lyapunov) function *V(x)* s.t.

$$x(t) \to 0 \Leftrightarrow V(x(t)) \to 0$$



Useful in advanced control!
(nonlinear/robust/time-varying)

An example: Nonlinear system

- A nonlinear system (difficult to solve analytically)

$$\begin{aligned}\dot{x}_1(t) &= x_2(t) - x_1(t)(x_1^2(t) + x_2^2(t)) \\ \dot{x}_2(t) &= -x_1(t) - x_2(t)(x_1^2(t) + x_2^2(t))\end{aligned}$$

- Lyapunov function $V(x) := x_1^2 + x_2^2 > 0, \forall x \neq 0$
- Derivative of *V(x(t))* w.r.t. *t*, where *x(t)* is a trajectory of the nonlinear system

$$\begin{aligned}\frac{d}{dt}V(x(t)) &= 2x_1(t)\dot{x}_1(t) + 2x_2(t)\dot{x}_2(t) \\ &= -2(x_1^2(t) + x_2^2(t))^2 < 0, \forall x(t) \neq 0\end{aligned}$$

Summary

- Lyapunov Theorem
  - Positive definite matrix
  - Lyapunov equation
  - Main idea of the theorem
- Next, controllability & observability