1

Algorithm to check if a given flow f is a maximum flow in flow network G:

Compute the residual of f. This operates on each edge a constant number of times and takes $O(m)$.

Perform a DFS on the residual starting from s and return true if it reaches t and false otherwise. This takes $O(n+m)$ time.

The whole algorithm will take $O(n+m)$ time.

2

Prove that a flow f' exists such that $|f| = |f'|$ and $f'(e) \leq f(e) - 1$ given e is in a directed cycle in G'.

Since there is a cycle in G' such that e is in the cycle, and G' is defined as containing only edges which have a positive flow, e is a member of a cycle C in the flow f where every edge is positive.

Let F be the smallest flow weight in C. F will be at least 1 because the edge weights are positive integers.

Define f' as $f'(e) = \begin{cases} f(e) - F & e \in C \\ f(e) & e \notin C \end{cases}$

In other words, cancel a flow cycle at C from f.

Since canceling a cycle does not change the value of the (s,t)-flow, and $f'(e) = f(e) - F \leq f(e) - 1$, f' satisfies the requirements.

3

Given a flow f, network graph G, and edge $e \in G$, give an algorithm to find the maximum flow in F with $c_F(e) = c_G(e) - 1$, with all else the same between F and G.

First, note that in the case where $f(e) < c(e)$, the solution is trivial, since there is no need to change the flow.

In the case that $f(e) = c(e)$, we must find a new flow. Cancel 1 unit of flow off a path from s to t going through e. To do this, label u and v such that e is the edge from u to v. Find the path with the fewest edges from s to u using a breadth-first search. Find the shortest path from v to t similarly. Do these BFS searches such that an edge is only followed when it has positive flow. We also know that these two paths exist because the > 1 unit of flow going through e has to come from and go to somewhere. Subtract one flow unit from each of these edges. This new flow is of value at least $|f| - 1$.

Since a single path augmentation is guaranteed to increase the flow by some positive integer value, by augmenting this new path, we are guaranteed to get a flow of value $|f|$.