

We define the graph  $G = (V, E)$  for all three subproblems as follows: create  $n$  vertices, one for each currency, and there are  $N^2$  directed edges such that the edge  $v_i v_j = -\log(\text{Exch}[i][j])$  and  $v_j v_i = -\log(\text{Exch}[j][i])$ .

We see that taking the negative log of the exchange rates means we add rather than multiply the edge weights to find the amount of money earned. Let  $A$  be the initial amount of currency,  $A'$  be the amount after exchanging and  $R_1 \dots R_k$  the exchange rates.

$$\begin{aligned} A' &= A_0 * \prod_{i=1}^k R_k \\ -\log(A') &= -\log(A_0 * \prod_{i=1}^k R_k) \\ -\log(A') &= -\log(A_0) + \sum_{i=1}^k -\log(R_k) \end{aligned}$$

Each edge is one element of the summation. Note that taking the negative log of an exchange rate  $>1$  will result in a negative number, and  $<1$  will result in a positive number.

### 3.a

Describe an algorithm that returns an array  $\text{MaxAmt}[1..n]$ , where  $\text{MaxAmt}[i]$  is the maximum amount of currency  $i$  that you can obtain by trading, starting with one unit of currency 1, assuming there are no arbitrage cycles.

**Solution:** If we start an amount of currency  $A$ , and we trade to get more currency (i.e an exchange rate  $>1$ ), this means we traverse a negative-length edge in our graph  $G$ . Thus, the problem of finding the maximum amount of currency you can obtain starting with one unit of currency  $i$  is equivalent to the single source shortest paths problem in  $G$  starting at  $v_i$ . From the equation above, we see that  $-\log(A') = -\log(A) + \sum_{i=1}^k -\log(R_k)$ . The last node in the shortest path is the node at which we have the most currency.

We start the path-finding problem with  $-\log(A)$  units. Each edge we add the neative log of the exchange rate at that edge. The total . Since we have negative edge lengths, we use the Bellman-Ford algorithm which runs in  $O(VE)$  time. There are no arbitrage cycles which means there are no negative length cycles (see part b).

To get the array  $\text{MaxAmt}[1..n]$ , we solve the shortest path problem starting at index 1, then get the distance to each node. See Algorithm 1 bellow.

**Time Complexity** Constructing the graph takes  $O(V + E) = O(N^2)$  time. Bellman Ford runs in  $O(VE) = O(N^3)$  time, which dominates the total runtime of  $O(N^3)$ .

---

**Algorithm 1** MaxAmt[1...n]

---

Run Bellman Ford SSSP on  $G$  from  $v_1$  with initial value of  $-\log(1)$ .

**for**  $i = 1$  to  $i = n$  **do**

$L \leftarrow$  distance from  $v_1$  to  $v_n$  from the Bellman-Ford above

$MaxAmt[i] = 10^{-L}$

**end for**

---

**Space Complexity** Space complexity is dominated by the exchange rates matrix of size  $O(N^2)$  used to construct the graph.

■

### 3.b

Describe an algorithm to determine whether the given matrix of currency exchange rates creates an arbitrage cycle.

**Solution:** We will construct the same graph  $G$  as used in part (a).

We recognize that an arbitrage cycle is equivalent to the existence of a negative length cycle in  $G$ . We show this as follows: Suppose we start with  $A$  units of currency  $i$ . An arbitrage cycle is a cycle of exchange rates  $R_1 \dots R_k$  ending back at currency  $i$  such that  $A * \prod_{i=1}^k R_i > A$ . We must end at the same currency we started with. In our graph  $G$ , multiplying by a positive exchange rate is equivalent to adding a negative value. A series of rates  $R_1 * R_2 * \dots * R_k > 1$  that defines an arbitrage cycle, when we take the negative log of it, is a sequence of negative values  $-\log(R_1) - \log(R_2) - \dots - \log(R_k) < 0$ ,

To detect arbitrage cycle, we simply run Bellman Ford on our graph  $G$ . If it detects a negative cycle, there is an arbitrage cycle; if it does not detect a negative cycle there is no arbitrage.

**Time Complexity** The runtime is dominated by the Bellman Ford runtime, which is  $O(VE) = O(V^3)$ .

**Space Complexity** The space is dominated by the graph, which requires an adjacency matrix of size  $O(N^2)$ .

■

### 3.c

Modify your algorithm from part (b) to actually return an arbitrage cycle, if it exists.

**Solution:** We will construct the same graph  $G$  as used in part (a).

We run Bellman Ford on our graph  $G$ , as in part (b), to detect an arbitrage cycle. However, we modify the cycle detection process to walk through the cycle and return the nodes in it. See Algorithm 2 below:

This algorithm modifies the negative cycle detection routine in the Bellman Ford algorithm. Before we run this routine, we must run the  $O(VE)$  path distance routine. Since this part is unchanged from the standard Bellman-Ford algorithm, we do not reprint it here. Note that  $d[v]$  is the Bellman-Ford distance at node  $v$ , and  $p[v]$  is the parent node of the node  $v$ . Both of these are part of the standard implementation of Bellman-Ford.

---

**Algorithm 2** Bellman Ford Negative Cycle Modification

---

```
cycle is a vector of vertices
for each edge  $e = (u,v)$  in  $G$  do
  if  $\text{dist}[u] + \text{weight}(e) < \text{dist}[v]$  then
    Negative cycle detected containing node  $v$ 
     $\text{curr} \leftarrow p[v]$ 
    while  $\text{curr} \neq v$  do
       $\text{cycle.push}(\text{curr})$ 
       $\text{cycle} \leftarrow p[\text{curr}]$ 
    end while
  end if
end for
 $\text{cycle.reverse}()$ 
return  $\text{cycle}$ 
```

---

**Time Complexity** The modified cycle detection routine runs in  $O(VE)$  time. This is the same as the runtime for the standard Bellman-Ford routine, so the total runtime is  $O(VE)$ .

**Space Complexity** The space is dominated by the graph, which requires an adjacency matrix of size  $O(N^2)$ .

