1

Prove that in any Eulerian graph, if there are k edge-disjoint paths from a to b, there are k edge-disjoint paths from b to a:

First define a flow network G in terms of the input graph. Assign a capacity of 1 to each edge in G. This graph has some maximum flow of capacity m. Using the argument from page 353 of the Algorithms Textbook, m will also be the maximum number of edge-disjoint paths from a to b, and $m \geq k$.

By the maxflow mincut theorem, the minimum (a, b) cut of the graph is of capacity m.

Since this cut is a partitioning into 2 sections of an Eulerian graph, the number of edges from L to R, or m, is equal to the number of edges from R to L. This same partitioning is also a (b, a) cut with capacity m. It is the smallest (b, a) cut because if there was a smaller one, it would have capacity less than m, meaning it would have less than m incoming or outgoing edges, which would imply there was a smaller (a, b) cut than the one we found.

Then, again by the maxflow-mincut theorem, the maximum (b, a) flow is m, and by the textbook argument, there exists a set of m edge-disjoint paths in the input graph from b to a. Since $m \geq k$, there is also a set of k edge-disjoint paths from b to a.

2

Proof that any d-regular bipartite graph has a perfect matching.

Define a flow network G based on the graph, by adding a vertex a and b, connecting a to all vertices in L, b to all vertices in R, and setting all edge capacities to 1.

Note that in a d-regular bipartite graph, there are $\frac{|V|}{2} = k$ vertices in each of L and R. If there were more vertices on one side, there would be too many outgoing edges to cover the fewer vertices on the other side, assuming $d > 0$.

Let m be defined as the maximum (a, b) flow of G.

By the maxflow-mincut theorem, m is also the size of the minimum (a, b) cut.

I claim that the smallest cut of the graph is of size k.

Any other cut has at least one vertex besides a and b in each of the partition sets.

Let set L_a be the set of vertices in the left side of the bipartite graph which are in a partition set with a, and similarly for L_b, R_a, R_b.

if $|L_a| \geq |R_a|$, there will be at least $d(|L_a| - |R_a|)$ edges from L_a to R_b.

We also know that $|R_b| \geq |L_a|$, since $|L_a| + |L_b| = |R_a| + |R_b| = k$.

We want to find the minimum number of edges from R_a to L_b. We know that there are at least $d(|L_a| - |R_a|)$ edges from L_a to R_b. This leaves $d(|R_b| - |L_a| + |R_a|)$ connections left that R_b must fill. We then get $d(|L_b| - |R_b| + |L_a| - |R_a|)$ edges between R_a and L_b. There will then be a total of $d(|L_a| - |R_a|) + d(|L_b| - |R_b| + |L_a| - |R_a|) + |R_a| + |L_b|$ edges.

Because of the relationship $|L_a|+|L_b| = |R_a|+|R_b| = k$, this can be simplified to $(d-1)(|L_a| - |R_a|) + k$. This will always be at least k because we assumed $|L_a| \geq |R_a|$, if $d \geq 1$.

The same argument applies for $|L_a| \leq |R_a|$ to show that the minimum possible cut is k.

Since k is the minimum cut capicity, it is also the maximum flow capacity, and this flow can be separated into k edge-disjoint paths. These edge-disjoint paths correspond to a matching of size k in the original graph. Each left vertex has only one edge going to it, and each right vertex has only one edge coming from it in these paths, no vertices are shared between edges that were in the original graph. There are $k = \frac{|V|}{2}$ edges aside from the extra edges we added, so it is a perfect matching.