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Solution: In order to prove there is a $|f'|-|f|$ valued (s,t)-flow in G_f, it is first useful to bound these quantities. Starting with a bound on the flow of G_f, which will be referred to abstractly as f_{G_f}, the value of this flow will be the sum of the outbound flow from s in G_f minus the sum of the inbound flow to s, e.g. $|f_{G_f}| = \Sigma_w f(s \to w) - \Sigma_u f(u \to s)$.

$$|f_{G_f}| = \Sigma_w f(s \to w) - \Sigma_u f(u \to s) \tag{1}$$
$$\Sigma_w f(s \to w) \leq \Sigma_w c(s \to w) - |f| \tag{2}$$
$$\Sigma_u f(u \to s) \leq -|f| \tag{3}$$
$$-|f| \leq |f_{G_f}| \leq \Sigma_w c(s \to w) - |f| \tag{4}$$

The justification for (2) is by the definition of residual graphs - namely $c_f(s \to w) = c(s \to w) - f(s \to w)$.The justification for (3) is also by the definition of residual graphs, where all $u \to s \in E_f$ are backtrack edges such that $c_f(u \to s) = f(s \to u)$. Note for this equation I dropped the value of outbound edges, meaning I minimized full quantity in the case where none of the residual edges have a flow along them. Now we proceed to the bounds on the quantity $|f'|-|f|$ below.

$$0 \leq |f'| \leq \Sigma_w c(s \to w) \tag{5}$$
$$-|f| \leq |f'| - |f| \leq \Sigma_w c(s \to w) \tag{6}$$

The justification for (1) above is just satisfying the capacity constraint, and (2) just plugs in the bounds of $|f'|$ into $|f'|-|f|$. Seeing clearly now that the bounds of the quantities $|f'|-|f|$ and f_{G_f} are the same, it is sufficient to prove that the upper and lower bound values for f_{G_f} are attainable for all the values in the range to be attainable. Starting with the easy case, the quantity $-|f|$ is just flow in G_f that saturates all backtracking edges $u \to s$. This is clearly attainable because for any edge $u \to s$ with capacity c_f, there is a set of vertices s.t. $\Sigma_a a \to u = c_f$ since the original flow f satisfied the conservation constraint. This identity can be applied outwards to those vertices a and connected edges all the way to t. For the upper bound $\Sigma_w c(s \to w) - |f|$, it is clear that the capacity constraint can be kept to reach this quantity. Chiefly, the flow that saturates all edges $s \to w$ in G_f produces exactly this sum. It should be noted that the equality can most certainly be satisfied up to the sum $|f'|$, because if there was a bottleneck in some edge such that its flow in G_f and its value in f could not sum to its value f', f' could not exist because it would be impossible that f' doesn't run into a bottleneck at the same edge.

Using the above knowledge, it can be said that f is a maximum flow iff there is no $s-t$ path in G_f in the following way. Consider the case where f is a maximum flow but there IS an $s-t$ path in G_f. In thie case, this $s-t$ path means there exist $|f_{G_f}|$ with positive flow, or in other words, some f' such that $|f'|-|f| > 0$. But if that's the case, the $|f'| > |f|$, meaning f was not the maximum flow and we have a contradiction. Similarly, assuming there is some f that is not the maximum flow for which G_f has no $s-t$ path, it is clear that there is no such positive flow in G_f, or in other words no f' such that $|f'|-|f| > 0$. However if this is the case, then clearly $|f|$ is the maximum flow and we have another contradiction. ■