1

Prove that in any Eulerian graph, if there are k edge-disjoint paths from a to b, there are k edge-disjoint paths from b to a:

First define a flow network G in terms of the input graph. Assign a capacity of 1 to each edge in G. This graph has some maximum flow of capacity m. Using the argument from page 353 of the Algorithms Textbook, m will also be the maximum number of edge-disjoint paths from a to b, and $m \geq k$.

By the maxflow mincut theorem, the minimum (a, b) cut of the graph is of capacity m.

Since this cut is a partitioning into 2 sections of an Eulerian graph, the number of edges from L to R, or m, is equal to the number of edges from R to L. This same partitioning is also a (b, a) cut with capacity m. It is the smallest (b, a) cut because if there was a smaller one, it would have capacity less than m, meaning it would have less than m incoming or outgoing edges, which would imply there was a smaller (a, b) cut than the one we found.

Then, again by the maxflow-mincut theorem, the maximum (b, a) flow is m, and by the textbook argument, there exists a set of m edge-disjoint paths in the input graph from b to a. Since $m \geq k$, there is also a set of k edge-disjoint paths from b to a.

2

Proof that any d-regular bipartite graph has a perfect matching.

Define a flow network G based on the graph, by adding a vertex a and b, connecting a to all vertices in L, b to all vertices in R, and setting all edge capacities to 1.

Note that in a d-regular bipartite graph, there are $\frac{|V|}{2} = k$ vertices in each of L and R. If there were more vertices on one side, there would be too many outgoing edges to cover the fewer vertices on the other side, assuming $d > 0$.

Let m be defined as the maximum (a, b) flow of G.

By the maxflow-mincut theorem, m is also the size of the minimum (a, b) cut.

I claim that the smallest cut of the graph is of size k.

The cut $(\{a\}, G \setminus \{a\})$ is of size k.

The cut $(G \setminus \{b\}, \{b\})$ is of size k.

Any other cut is size $\geq k$ because it will include at least one of the original (not a or b) vertices in the left partition and in the right partition, as those are the remaining cases.

For any vertex in the right side of the bipartite graph, it will contribute 1 to the cut size. If it is in the a side of the cut, the edge neighboring it with b will be counted. If it is in the b side of the cut, if it is connected to a vertex in the left side of the bipartite graph which is in the a side of the cut, the