CS/ECE 473 Fall 2020
Homework 4 Problem 2

Ryan Prendergast (ryanp4)
Noah Watson (nwatson3)
Lawson Probasco (lawsonp2)

(a) **Solution:**

$$\begin{aligned}
E[X] &= E[X_1 + ... + X_n] = \Sigma_{i=1}^n E[X+i] && \text{linearity of expectation}\\
&= \Sigma_{i=1}^n \Sigma_x xp(1-p)^{i-1} && \text{defn. of E}\\
&= \Sigma i = 1^n \frac{1-p}{p} && \text{expectation for geometric}\\
&= \frac{n(1-p)}{p}
\end{aligned}$$

$$\begin{aligned}
E[X^2] &= E[(X_1 + ... + X_n)^2] && X = \Sigma_i X_i\\
&= E[X_1{}^2 + X_1X_1 + ... + X_n{}^2]\\
&= \Sigma_{i=1}^n E[X_i^2] + \Sigma_{i,j\in|n|,i\neq j}^n E[X_iX_j] && \text{linearity of expectation}\\
E[X]^2 &= (E[X_1] + ... + E[X_2])^2 && \text{linearity of expectation}\\
&= \Sigma i = 1^n E[X+i]^2 + \Sigma_{i,j\in|n|,i\neq j}^n E[X_iX_j] && \text{indep. of } X_i\ X_i\ i \neq j\\
Var(X) &= E[X^2] - E[X]^2 && \text{defn. of var in terms of E}\\
&= \Sigma i = 1^n E[X_i^2] - \Sigma_{i=1}^n E[X_i]^2 && E[X^2] \text{ and } E[X]^2 \text{ from above}\\
&= \Sigma_{i=1}^n (E[X_i^2] - E[X_i]^2)\\
&= \Sigma_{i=1}^n Var(X_i)
\end{aligned}$$

■

(b) **Solution:** When applying Chebyschev's inequality $Pr[|(X - E[X])] \geq \epsilon]$ to $Pr[X \geq c\,E[X]$, because X is always positive (as is $E[X]$ naturally), there are two cases for the absolute value.
$X - E[X]$ where $X >= E[X] >= 0$ is a positive or zero value, and thus would never deviate from $|X - E[X]|$.
$X - E[X]$ where $0 <= X < E[X]$ is a negative value v in the range $E[X] <= v < 0$. Thus, in order for Chebyschev's to be applied to $(X - E[X])$, ϵ must satisfy $\epsilon \geq -E[X]$, such that there is no difference of evaluation of $|X - E[X]| \geq \epsilon$ and $X - E[X] \geq \epsilon$. Because $\epsilon = c\,E[X]$ satisfies this requirement for $c \geq 2$, we get $Pr[X \geq E[X]] \leq \frac{Var(X)}{E[X]^2}$ applying Chebyschev's, and using values from the previous part. ■

(c) **Solution:** The event $X > t$ is describes the occurrence in which $X > t$ flips are required to yield n flips of heads. As a result, by definition this means that the first t flips yieled fewer than n heads, e.g. $Y < n$. As a result, $Pr[X > t] = Pr[Y < n]$. Considering the case where $Pr[X = t]$, there are $\binom{t-1}{n-1}$ ways to partition the t flips into n groups (dilineated by a successful flip of heads, using stars and bars). Each of these ways to partition have probability $p^n(1-p)^{t-n}$ (based on number of successes and failures), therefore $Pr[X = t] = \binom{t-1}{n-1}p^n(1-p)^{t-n}$. $Pr[Y = n]$ is modeled by a binomial distribution, thus $Pr[Y = n] = \binom{t}{n} p^n (1-p)^{t-n}$ This leads to $Pr[X = t] \leq Pr[Y = n]$, and thus $Pr[X \geq t] \leq Pr[Y \leq n]$.

■

(d) **Solution:** In this application, $\epsilon = c - 1$, e.g. is ϵ is over the bound $\epsilon \geq 1$. Applying part 1c i for the first applicable portion of this bound, we can say that $Pr[X \geq c\,\mathrm{E}[X]] \leq e^{-(c-1)^2\,\mathrm{E}[X]/4}$ for $2 \leq c \leq 2 + \ln 4$. Similarly for second portion of the bound of c, we can apply part 1c ii and say that $Pr[X \geq c\,\mathrm{E}[X]] \leq 2^{-(c-1)\,\mathrm{E}[X]/2}$ for $c \geq \ln 4$, and thus we have bound $Pr[X \geq c\,\mathrm{E}[X]]$ for $c \geq 2$. ■