PhD Bootcamp: Calculus and Analysis

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Geometric Series

- **Finite Geometric Series:** If $r \neq 1$, then

$$\sum_{n=0}^{N-1} ar^n = \frac{a(1-r^N)}{1-r}$$

- **Infinite Geometric Series:** If $-1 < r < 1$, then

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$$

Taylor Series

- For an arbitrary function f, the Taylor Series about 0 is

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = f(0) + f'(0)x + \frac{f''(0)}{2} x^2 + \mathcal{O}(x^3)$$

- The Taylor series for e^x is, for all $x \in \mathbb{R}$,

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \mathcal{O}(x^4)$$

p-Series

- **p-Series:** The series $\sum_{n=1}^{\infty} \frac{1}{n^p}$
  - Diverges if $p \leq 1$
  - Converges if $p > 1$
- **Harmonic Series:** In the important special case where $p = 1$,

$$\sum_{n=1}^{\infty} \frac{1}{n} = \infty$$

  - *Fun Fact:* It can be shown that $\sum_{n=1}^{k} \frac{1}{n} = \ln(k) + \gamma + \mathcal{O}(1/k)$.

p-Integrals

- If $p > -1$, then

$$\int_0^1 x^p \, \mathrm{d}x = \frac{1}{p+1}$$

Otherwise, the integral diverges.

- If $p < -1$, then

$$\int_1^\infty x^p \, \mathrm{d}x = -\frac{1}{p+1}$$

Otherwise, the integral diverges.

Gamma Function

- **Gamma Function:** The Gamma function, Γ, is defined to be

$$\Gamma(z) = \int_0^\infty x^{z-1} e^{-x} \, \mathrm{d}x$$

- For $n \in \mathbb{N}$, $\Gamma(n) = (n-1)!$
- For all $z > 0$, $\Gamma(z+1) = z\Gamma(z)$
- For $z = \frac{1}{2}$, $\Gamma(\frac{1}{2}) = \sqrt{\pi}$
- $\Gamma'(1) = -\gamma$, where $\gamma \approx 0.577$ is the Euler-Mascheroni constant.

Gamma Function II

- The below integral arises often in statistics, for $a, b \in \mathbb{R}^+$

$$\int_0^\infty x^{a-1} e^{-bx} \, \mathrm{d}x = \frac{\Gamma(a)}{b^a}$$

- **Gamma Distribution:** $X \sim \text{Gamma}(\alpha, \beta)$, for $\alpha, \beta \in \mathbb{R}^+$, if X has PDF

$$f(x) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x}$$

Beta Function

- **Beta Function:** The Beta function, B, is defined to be

$$\mathrm{B}(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1}\,\mathrm{d}t$$

- Relationship to Gamma function: $\mathrm{B}(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$
- **Beta Distribution:** $X \sim \text{Beta}(\alpha, \beta)$, for $\alpha, \beta \in \mathbb{R}^+$, if X has PDF

$$f(x) = \frac{1}{\mathrm{B}(\alpha, \beta)} x^{\alpha-1}(1-x)^{\beta-1}$$

Kernels

- **Kernel of a Distribution:** Form of the PDF or PMF of a distribution with factors that are not functions of any variables of the domain omitted.
- Often, tricky-looking integrals and sums encountered in statistics can be easily computed by recognizing the integral as the kernel of a known distribution.
- *Example:* Finding the expectation of $X \sim \text{Exp}(\lambda)$.

$$E[X] = \int_0^\infty x\,\lambda e^{-\lambda x} dx = \lambda \int_0^\infty \underbrace{x^{2-1} e^{-\lambda x}}_{\text{Gamma}(2,\lambda)} dx = \lambda \cdot \frac{\Gamma(2)}{\lambda^2} = \frac{1}{\lambda}$$

Limits

- **Definition of e^x:** e^x can be expressed via the following limit

$$e^x = \lim_{n \to \infty} \left(1 + \frac{x}{n}\right)^n$$

- **L'Hospital's Rule:** If $\lim_{x \to c} f(x) = \lim_{x \to c} g(x) = 0$ or $\pm\infty$, then

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)}$$

  - *Technical Sidenote:* f and g must be differentiable on an open interval, I, except possibly at $c \in I$

Completing the Square (One Dimension)

- **Completing the Square:** A technique for converting an equation of the form $ax^2 + bx + c$ to the form $a(x-h)^2 + k$
- *Example:* For $a = 3$, $b = 12$, and $c = 27$,

$$\begin{aligned} 3x^2 + 12x + 27 &= 3(x^2 + 4x + 9) \\ &= 3(x^2 + 4x + 4 + 5) \\ &= 3(x+2)^2 + 15 \end{aligned}$$

- In general, $h = -\frac{b}{2a}$ and $k = c - \frac{b^2}{4a}$

Differentiation under the Integral Sign

Under certain regularity assumptions of f, we have that

$$\frac{\partial}{\partial x}\int_a^b f(x,t)dt = \int_a^b \frac{\partial}{\partial x} f(x,t)dt$$

This can be useful in dealing with difficult integrals. For instance,

$$\begin{aligned}\int_0^\infty e^{-t}\ln(t)dt &= \int_0^\infty \frac{\partial}{\partial x}[e^{-t}t^{x-1}]\big|_{x=1}dt \\ &= \frac{\partial}{\partial x}\left[\int_0^\infty e^{-t}t^{x-1}dt\right]\Big|_{x=1} \\ &= \Gamma'(1) = -\gamma.\end{aligned}$$

(Recall that $t^{x-1} = e^{(x-1)\ln(t)}$)

Differentiation under the Summation Sign

The same theorem also applies with summations. For example, we have that (for $|x| < 1$)

$$\begin{aligned}
\sum_{n=0}^{\infty} nx^n &= x \sum_{n=0}^{\infty} nx^{n-1} \\
&= x \sum_{n=0}^{\infty} \frac{\partial}{\partial x}[x^n] \\
&= x \frac{\partial}{\partial x}\left[\frac{1}{1-x}\right] \\
&= \frac{x}{(1-x)^2}
\end{aligned}$$

Calculus Exercises I

1. Let $X \sim \text{Beta}(\alpha, \beta)$. Find $E[X]$ and simplify as much as possible.
2. Evaluate the following limit in TWO different ways (using methods covered in these slides)

$$\lim_{x \to 0} \frac{e^x - (1+x)}{x^2}$$

3. Let $X \sim \text{Exp}(\lambda)$. Find a general form for $E[X^k]$.
4. Identify which of the following series and integrals converge

$$1.\ \int_0^1 x^2 \, dx \qquad 3.\ \int_1^\infty x^2 \, dx \qquad 5.\ \sum_{n=1}^\infty n^2$$

$$2.\ \int_0^1 x^{-2} \, dx \qquad 4.\ \int_1^\infty x^{-2} \, dx \qquad 6.\ \sum_{n=1}^\infty n^{-2}$$

Calculus Exercises II

5. X has a two-sided geometric distribution with parameter p if it has PMF $\mathrm{P}[X = x] = cp^{|x|}$ for $x \in \mathbb{Z}$
    1. Find the normalizing constant, c
    2. Find the CDF, $F(x) = \mathrm{P}[X \leq x]$
    3. Find $E[X]$ and $Var[X]$ (*Hint:* Find $\sum_{n=0}^{\infty} n^2 x^n$)
6. Let $X \sim \mathcal{N}(0, 1)$ and $Y \mid X \sim \mathcal{N}(X, 1)$. Determine the distribution of Y.
    - *Hint:* Recall that $f_Y(y) = \int f_{Y|X}(y|x) f_X(x)\, \mathrm{d}x$
    - Recall that $Z \sim \mathcal{N}(\mu, \sigma^2)$ has PDF

$$f_Z(z) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{1}{2\sigma^2}(z - \mu)^2\right\}$$

Feel free to reach out to me at zekican.kazan@duke.edu with questions or to check your solutions.