# Linear Algebra Bootcamp Exercise Solutions

- Show the that the reverse triangle inequality holds for norms. That is, for all $v, w \in V$, $\|v\| - \|w\| \leq \|v - w\|$.

- Show that norms are continuous. (Hint: try a $\delta - \epsilon$ proof)

- Suppose $C \subset \mathbb{R}^n$ is a convex set (i.e., for all $x, y \in C$ and $\theta \in [0,1]$, $\theta x + (1-\theta)y \in C$). We say a function $f : C \to \mathbb{R}$ is **convex** if $f(\theta x + (1-\theta)y) \leq \theta f(x) + (1-\theta)f(y)$ for all $x, y \in C$ and $\theta \in [0,1]$. Show that norms are convex.

- Let $V$ be a vector space, and let $\{v_n\} \subset V$ be a sequence. If $\lim_{n\to\infty} \|v_n\|_2 = 0$, then what is $\lim_{n\to\infty} \|v_n\|_1$?

- Show that $(\mathbb{R}^n, \|\cdot\|_2)$ is a normed space, where $\|\cdot\|_p$ is the $\ell_p$-norm defined by $\|v\|_p := \left(\sum_{j=1}^n v_j^p\right)^{1/p}$.

- Let $V$ be an inner product space. If $\langle v, w \rangle = 0$ for some $v, w \in V$, show that the Pythagorean theorem holds: $\|v + w\|^2 = \|v\|^2 + \|w\|^2$.

- Find $c, C$ explicitly such that $c\|v\|_\infty \leq \|v\|_2 \leq C\|v\|_\infty$ for $v \in \mathbb{R}^n$, where $\|v\|_\infty := \max_{1\leq i\leq n} |v_i|$.

- Show the diagonal elements of a PD matrix are positive.

- Let $A$ be PD. Using the eigenvalue decomposition, find a "square root" $A^{1/2}$, which is a matrix $B$ such that $A = B^2$.

- Take a PD matrix $A \in \mathbb{R}^{n\times n}$. Define the function $f : \mathbb{R}^n \to \mathbb{R}_+$ by $f(x) = \sqrt{x^\intercal A x}$. Show that $f$ is continuous and convex.

- Show that the $\ell_2$-norm is unitarily invariant. That is, if $Q \in \mathbb{R}^{n\times n}$ is an orthogonal matrix, then $\|Qx\|_2 = \|x\|_2$ for all $x \in \mathbb{R}^n$. Interpret this result geometrically.

- Show that the determinant of an orthogonal matrix is either 1 or -1.

- Suppose an $n$-by-$n$ matrix $A$ has a zero eigenvalue. List as many properties as possible that this fact tells you about $A$.

- Prove $\mathrm{tr}(AB) = \mathrm{tr}(BA)$, assuming $A$ is an $m$-by-$n$ matrix and $B$ is an $n$-by-$m$ matrix.

- Prove that if $A = A^\intercal A$, then $A = A^\intercal$ and $A = A^2$

- For any matrix $A$, prove that $C(A)$ and the null space of $A^T$ are orthogonal complements.

- Classify all orthogonal matrices that are also orthogonal projections. Can you deduce from this why both are called orthogonal?

- Compute the gradient $\partial_\beta \|y - X\beta\|_2^2$ (1) directly and (2) using the chain rule. Set the result equal to 0 and use the QR decomposition on $X$ to simplify your expression. Can you guess why this might be favorable to do if we were trying to solve for $\beta$?

- In the slides, we saw that you can find inverse of a diagonalizable matrix using the EVD. Explain why $\Lambda^{-1}$ is well-defined if the matrix $M$ is invertible. Then use this construction to inspire you to find an "inverse" for a rectangular matrix $A \in \mathbb{R}^{m\times n}$ using the SVD. Does it behave like an actual inverse of a square matrix?

- We know that an EVD of a matrix $A$ can be written $A = Q\Lambda Q^\intercal$. Using the fact that matrix multiplication represents composition of linear maps, interpret geometrically what the EVD says about $A$.

- Suppose $A$ is a block diagonal matrix with diagonal blocks $A_1, \ldots, A_n$. Show that $\det(A) = \prod_{j=1}^n \det(A_j)$ using induction.

- Suppose we have a block diagonal matrix with blocks $A_1$ and $A_2$ that are each positive definite. Show that the entire block matrix is also positive definite.

- Consider the rotation matrix
$$R = \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ \sin(\theta) & -\cos(\theta) \end{pmatrix}$$
where $\theta \in \mathbb{R}$. Find the eigenvalues over $\mathbb{R}$ and over $\mathbb{C}$. Explain geometrically what you observe.

- Suppose $A \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{n\times p}$. Show that $N(B) \subset N(AB)$.

- Find the rank of the matrix associated with the shearing transformation.

- Let $V = \mathbb{R}^2$. Sketch a picture of the vector $(1,1)^\intercal$ and its orthogonal complement. Describe the orthogonal complement using the span of a vector.

- Suppose $A$ is symmetric. Show that $\mathrm{tr}(A)$ is the sum of its eigenvalues.

- For compatible vectors, show that $a^\intercal b = \mathrm{tr}(ba^\intercal)$.

- If $M$ is positive semi-definite, what can we say about $I + M$, where $I$ is an appropriately-sized identity matrix?