Small-area estimation under a nonlinear transformed area-level model

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Abstract

Methods for mean estimation in small area from survey data such as the Fay-Herriot model are intensively studied since a few decades but are often relevant only for linear models. Some methods in the literature improve the estimation for skew outcomes but they lack a general approach. Herein, a transformation is able to handle skewed and bounded outcomes by combining existing nonlinear transformations. The bias correction for the back-transformation is discussed in a general setting by adding a multiplicative term to the related mean estimators. The Box-Cox transformation allows to illustrate the proposed approach with perspectives.

1 Introduction

Small area [1] are often seen as a way to reduce the variance for the estimator of a mean from data in comparison to the more usual synthetic estimators. They uses independent variables which bring information to the estimator and allows the reduction of the variance at the cost to need some bias correction approach in order to avoid a bad estimator. Thus, authors spend a lof of time into accuracte estimators. Here, the case includes nonlinear transformation of the outcome and also eventually a bounded outcome, such that $y_i \in [a, b]$ where y_i is an outcome while a and b are the bounds. This seems not studied in the related literature such that [2, 3, 4], except in [5] and some alternative distributions such as the Gamma one may be also relevant but out of the scope herein. Such missing constraints are very sad because lower and upper bounds may be required in some domains of applied research such that in psychology which often introduces data surveys where some answers have limited values, typically zero to ten or minus hundred to hundred for instance. This justifies the need for a bounded transformation, the one considered herein is based on the logistic transformation bounded [6] associated to the box-cox one [7] but any other transformation may be preferred. By adding a parameter, it is retrieved directly the logarithmic transformation such as the framework is more general than the three transformations alone. Next after this is often this function called generalized logistic-bounded box-cox transformation which is involved in the estimations.

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2 FH model under transformation

The area-level model is reviewed, from its definition to the estimation of the parameters, before the nonlinear transformation is introduced. The resulting back-transformation is explained and the correction of the bias discussed.

2.1 Small area estimation (SAE)

When there are predictive variables, it has been introduced in the literature the mixed models in survey theory. This allows to reduce the variance of the mean for an outcome, by fitting a model with these additional variables. There are mainly two models, one directly of the area, and one at the level of the units.

Model

The general model comes from the bayesian statistics. Matricially,

$$\mathbf{z}_i = \mathbf{X}\boldsymbol{\beta} + \mathbf{v} + \mathbf{e}\,.$$

It is written for the Fay-Herriot model [8], the following particular case:

$$z_i = \mathbf{x}_i^T\boldsymbol{\beta} + u_i + \epsilon_i$$

For the matricial notation, one denotes the vector of regression coefficients $\boldsymbol{\beta} = (\beta_1, \cdots, \beta_p)^T$, the vector of random effect $\mathbf{u} = (u_1, \cdots, u_m)^T$, and the vector of sampling noises $\mathbf{e} = (\epsilon_i, \cdots, \epsilon_m)$. Thus, the random effects are $u_i \sim N(0, \sigma_u^2)$ and the sampling errors are $\epsilon_i \sim N(0, \sigma_{e_i}^2)$, both are independent. Matricially, this leads to the covariance matrix for $\mathbf{y}$ equal to $\boldsymbol{\Sigma}(\sigma_u^2) = diag(\sigma_u^2 + \sigma_{e_1}^2, \cdots, \sigma_u^2 + \sigma_{e_m}^2)$, where σ_u^2 is unknown. For the estimation of $\boldsymbol{\beta}$ and σ_u^2, several approaches have been proposed in the literature, called REML and ML for the two main methods currently used most of the time. Another way to see the model is via the distribution of the data: $\mu_i \sim N(\mathbf{x}_i^T\boldsymbol{\beta}, \sigma_u^2)$, $z_i|\mu_i \sim N(\mu_i, \sigma_{e_i}^2)$, and thus, $z_i \sim N(\mathbf{x}_i^T\boldsymbol{\beta}, \sigma_{e_i}^2 + \sigma_u^2)$. The problem with this notation is to not show the random effect explicitely.

Estimators

Let denote the fraction:

$$\gamma_i = \gamma_i(\sigma_u^2) = \sigma_{e_i}^2(\sigma_u^2 + \sigma_{e_i}^2)^{-1}\,.$$

The best (or Bayes) estimator of μ_i is under squared error loss as follows, for σ_u^2 given, say the mean estimator from small area:

$$\vartheta_{iz} = E(\mu_i|z_i) = z_i - \frac{\sigma_{e_i}^2}{\sigma_u^2 + \sigma_{e_i}^2}(z_i - \mathbf{x}_i^T\boldsymbol{\beta}) = (1-\gamma_i)z_i + \gamma_i\mathbf{x}_i^T\boldsymbol{\beta}\,.$$

Here, the bayesian theory gave that,

$$\mu_i|z_i \sim N(\vartheta_{iz}, g_{1i}(\sigma_u^2)) \text{ where } g_1(\sigma_u^2) = (1-\gamma_i)\sigma_{e_i}^2\,.$$

Following [9], it is deduced the theoritical mean square error for the linear model:

$$MSE(\vartheta_i) \;\;=\;\; g_{1i}(\sigma_u^2)\,.$$

This induces that the variance of the mean in the area is reduced by a factor $(1-\gamma_i)$, with the value $g_{1i}(\sigma_u^2)$ instead of $\sigma_{e_i}^2$, justifying the approach of the small areas.

In practice with a data sample, the parameters $\boldsymbol{\beta}$ and σ_u^2 are unknown, such that they are estimated. It is usual to denote then $\tilde{\boldsymbol{\beta}} = \tilde{\boldsymbol{\beta}}(\sigma_u^2)$ as a function of the parameter σ_u^2, and thus the parameter vector of regression coefficients is denoted $\hat{\boldsymbol{\beta}}$ when σ_u^2 is replaced by an estimation of σ_u^2, say $\hat{\boldsymbol{\beta}} = \tilde{\boldsymbol{\beta}}(\hat{\sigma}_u^2)$. When $\tilde{\boldsymbol{\beta}}$ replaces $\boldsymbol{\beta}$ the best (or Bayes) estimator ϑ_{iz} becomes the first-step empirical best (or empirical Bayes) estimator of μ_i such that it is denoted $\tilde{\vartheta}_i$. When this is the estimation of the $\tilde{\sigma}^2$ replaces σ_u^2 in this first step estimator, with the notation $\hat{\gamma}_i = \gamma_i(\hat{\sigma}_u^2)$, one gets the final EB estimator,

$$\hat{\vartheta}_{iz} \;\; = (1-\hat{\gamma}_i)z_i + \hat{\gamma}_i \mathbf{x}_i^T \hat{\boldsymbol{\beta}}\,.$$

This is called more generally the empirical best linear unbiased predictor (EBLUP). This is the area-level estimator involved herein while unit level models are for a future work.

Algorithm for parameters inference

For the estimation, several methods exist. For a given σ_u^2, one computes the vector of regression coefficients as follows, as in an usual weighted regression, thus with heterodescacity. From the estimation $\hat{\sigma}_u^2$, it is deduced $\hat{\boldsymbol{\Sigma}} = \boldsymbol{\Sigma}(\hat{\sigma}_u^2)$, such that:

$$\begin{aligned}
\hat{\boldsymbol{\beta}} &= (\mathbf{X}^T\hat{\boldsymbol{\Sigma}}\mathbf{X})^{-1}\mathbf{X}^T\hat{\boldsymbol{\Sigma}}\mathbf{y} \\
&= \left\{\sum_{i=1}^m \frac{1}{\hat{\sigma}_u^2+\sigma_{e_i}^2}\mathbf{x}_i\mathbf{x}_i^T\right\}^{-1} \sum_{i=1}^m \frac{1}{\hat{\sigma}_u^2+\sigma_{e_i}^2}\mathbf{x}_i y_i\,.
\end{aligned}$$

Let denote $\mathbf{xx}_i = \mathbf{x}_i^T(\mathbf{X}^T\mathbf{X})^{-1}\mathbf{x}_i$, and $\hat{\boldsymbol{\beta}}_{LS}$ the solution from the usual unweighted least squared regression with $\mathbf{X}$ and $\mathbf{y}$ as respectively the design matrix and the vector of outcomes. For the estimation of σ_u^2, the usual different approaches are as follows.

- Method of Prasad-Rao with an exact solution $\hat{\sigma}^{*2}_{u,\mathrm{PR}}$,

$$\hat{\sigma}^{*2}_{u,\mathrm{PR}} \;=\; \frac{1}{m-p}\sum_{i=1}^m \Big[(y_i - \mathbf{x}_i^T\hat{\boldsymbol{\beta}}_{LS})^2 - \sigma_{e_i}^2(1-\mathbf{xx}_i)\Big]\,.$$

- Method of Fay-Herriot with a numerical solution $\hat{\sigma}^{*2}_{u,\mathrm{FH}}$,

$$\sum_{i=1}^m \frac{1}{\sigma_u^2+\sigma_{e_i}^2}\Big[y_i - \mathbf{x}_i^T\tilde{\boldsymbol{\beta}}(\sigma_u^2)\Big]^2 = m-p\,.$$

- Method of ML with a numerical solution $\hat{\sigma}^{*2}_{u,\mathrm{ML}}$,

$$\sum_{i=1}^m \frac{\Big[y_i - \mathbf{x}_i^T\tilde{\boldsymbol{\beta}}(\sigma_u^2)\Big]^2}{(\sigma_u^2+\sigma_{e_i}^2)^2} - \frac{1}{(\sigma_u^2+\sigma_{e_i}^2)} = 0\,.$$

- Method of REML with a numerical solution $\hat{\sigma}^{*2}_{u,\mathrm{REML}}$,

$$\sum_{i=1}^{m} \frac{\left[y_i - \mathbf{x}_i^T \tilde{\boldsymbol{\beta}}(\sigma_u^2)\right]^2 - (\sigma_u^2 + \sigma_{e_i}^2) + \mathbf{x}_i^T \left(\sum_l \frac{1}{\sigma_u^2 + \sigma_{e_l}^2} \mathbf{x}_l \mathbf{x}_l^T\right)^{-1} \mathbf{x}_i}{(\sigma_u^2 + \sigma_{e_i}^2)^2} = 0\,.$$

These different methods for the estimation of σ_u^2 were proposed in the literature by different authors, see [9] for instance. In the four cases, one ends with a positive solution by choosing respectively, $\hat{\sigma}^2_{u,\mathrm{PR}} = max(0, \hat{\sigma}^{*2}_{u,\mathrm{PR}})$, $\hat{\sigma}^2_{u,\mathrm{FH}} = max(0, \hat{\sigma}^{*2}_{u,\mathrm{FH}})$, $\hat{\sigma}^2_{u,\mathrm{ML}} = max(0, \hat{\sigma}^{*2}_{u,\mathrm{ML}})$ and $\hat{\sigma}^2_{u,\mathrm{REML}} = max(0, \hat{\sigma}^{*2}_{u,\mathrm{REML}})$. Once the estimation $\hat{\sigma}_u^2$ is available, this leads to $\hat{\boldsymbol{\beta}}$ which enters the empirical estimator of the mean, as explained above. As these estimators are all random variable depending on the available sample, the variability is measure by computing here the mean squared error. When the linear model is not enough, a nonlinear transformation is introduced as explained next after.

2.2 Nonlinear transformations with eventual bounds

The transformation, back-transformation and bias correction are presented in this part for several functions when eventually some constrained are required for the minimum and the maximum of the outcome variable. A typical example is for a probability which belongs to the interval $[0, 1]$, such that one do not want to predict values outside.

Transformation of the target variable

When κ_ξ^{-1} denotes a transformation with parameters ξ, typically herein the box-cox. Transformations without parameters such as squared-root for instance may be prefered. Let write the transformation as follows for the usual linear regression with fixed effects.

$$z_i = \kappa_\xi^{-1}\left(\frac{y_i - a}{b - c\, y_i}\right).$$

Hence, this is the transformed outcome variable which is a linear combination of the independent variables instead of the untransformed one,

$$z_i = \beta_0 + \sum_j \beta_j x_{ij} + \epsilon_i\,.$$

With bounds, for instance, in the case logarithmique, it is retrieved the bounds: for infinite positive values of $\hat{z}_i$, the predicted outcome $\hat{z}_i$ becomes equal to b, while for infinite negative values, it becomes equal to a, as expected. This remains true for the bounds in $[0; 1]$ such that the case of a probability or rescaled value, $y_i/\max_y - \min_y \in [0; 1]$, when $\max_y$ and $\min_y$ are respectively the minimum and maximum values of the outcome rescaled. With $[a, b]$ the interval for the bounds, this induces for the inverse of the Box-Cox transformation for $\lambda \neq 0$ and of the log when $\lambda = 0$,

$$\hat{y}_i = \begin{cases} \dfrac{a + be^{\hat{z}_i}}{1 + ce^{\hat{z}_i}} & \text{for } \lambda = 0 \\ \dfrac{a + b(1 + \lambda \hat{z}_i)^{1/\lambda}}{1 + c(1 + \lambda \hat{z}_i)^{1/\lambda}} & \text{for } \lambda \neq 0 \end{cases}$$

Table 1: Generalized transformation for skewed and bounded outcomes

Name	a	b	c	$\kappa_\xi^{-1}(y)$	$\kappa_\xi(z)$ for a=b=0, c=1
Logarithmic	0	1	0	$\log(y)$	$\exp(z)$
Box-cox	0	1	0	$\frac{y^\lambda-1}{\lambda}$	$(1+\lambda z)^{1/\lambda}$
Box-cox (bis)	0	1	0	y^λ	$z^{1/\lambda}$
Logistic	0	1	1	$\log(y)$	$\exp(z)$
Logistic-bounded	a	b	1	$\log(y)$	$\exp(z)$
Log-shift	0	1	0	$\log(y+s)$	$\exp(z)-s$
Dual-shift	0	1	0	$\frac{(y+s)^\lambda-(y+s)^{-\lambda}}{2\lambda}$	$(\lambda z+\sqrt{1+\lambda^2z^2})^{1/\lambda}-s$
Box-Cox-shift	0	1	0	$\frac{(y+s)^\lambda-1}{\lambda}$	$(1+\lambda z)^{1/\lambda}-s$
Inv-sinus-sqrt	0	1	0	$\sin^{-1}(\sqrt{y})$	$\sinh^2(z)$
Sinh-sinh	0	1	0	$\sinh(s_2\sinh^{-1}(y)-s_1)$	$\sinh(\frac{\sinh^{-1}(z)+s_1}{s_2})$

For our proposed generalized review, examples of transformations from the literature [10, 11] are listed in Table 1 with the bounds added for limited outcomes. Next after, the transformation is denoted κ^{-1} with $z_i = \kappa^{-1}(y_i)$, such that the bounds are implicit and included in the nonlinear function for a lighter notation.

Back-transformation and bias correction

When a transformation is involved, the inverse of the transformation leads to another estimator suitable for the former outcome:

$$\vartheta_{iy} = \kappa(\vartheta_{iz}) \text{ and } \hat{\vartheta}_{iy} = \kappa(\hat{\vartheta}_{iz})\,.$$

Because, $E[\hat{\vartheta}_{iy}] \neq \vartheta_{iy}$, the back-transformation needs to add a multiplicative bias correction denoted $\hat{\rho}$, such that now the estimator becomes:

$$\hat{\vartheta}^*_{iy} \;=\; \hat{\rho}_i \kappa(\hat{\vartheta}_{iz})\,.$$

This is with a similar notation as for the logarithmic transformation in [3], such that $\hat{\rho}_i$ is an estimator (by replacing the true unknown quantities β and σ_u^2 by their estimations) of ρ_i. The later one is defined with the quotient and estimated quotient:

$$\rho_i = \frac{E(\kappa(\vartheta_{iz}))}{E(\kappa(\hat{\vartheta}_{iz}))} \text{ and } \hat{\rho}_i = \frac{\hat{E}(\kappa(\vartheta_{iz}))}{\hat{E}(\kappa(\hat{\vartheta}_{iz}))}\,.$$

Such correction may do the job because one can write then the wanted result as follows:

$$E(\hat{\vartheta}^*_{iy}) = E(\hat{\rho}_i\kappa(\hat{\vartheta}_{iz})) \doteq E(\hat{\rho}_i)E(\kappa(\hat{\vartheta}_{iz})) \doteq E(\kappa(\vartheta_{iz}))\,.$$

In the case of the logarithmic transformation, the expression for the correction in the back-transformation is computed in [3] as follows,

$$\hat{\rho}_{i,\mathrm{SM}} = \exp(0.5(1-\hat{\gamma}_i)\hat{\sigma}_u^2) \text{ with } \kappa() = \exp()\,.$$

But also more advanced expression in [4], while the case box-cox transformation or the logistic-bounded transformation for instance were not adressed in the literature to our knowledge.

2.3 Bias correction via expectation for the nonlinear transformation

Bias corrections were mostly proposed for unit-level small area model but some attempts exists for area-level models too. Thus, alternative methods for bias correction have been introduced in the literature such as a recent one via integration, but they are often not general on the contrary to our approach.

- Recently, an integral proposed for back-transformation by the authors in [11] was computed numerically via Monte-Carlo or Gaussian-Hermite quadrature for instance. When denoting $\hat{\sigma}_{iz}^2 = (1-\hat{\gamma}_i)\hat{\sigma}_u^2$, an analytical solution may be available for the inverse sine transformation as follows:

$$
\begin{aligned}
\hat{\vartheta}^{**}_{i,\mathrm{SIN}\frac{1}{2}} &= E_{\hat{\vartheta}_{iz}}(\sin^2(\hat{\vartheta}_{i,z})) \\
&= \int_{-\infty}^{+\infty} \sin^2(t)\frac{1}{\sqrt{2\pi\hat{\sigma}_{iz}^2}}\exp\left(-\frac{(t-\hat{\vartheta}_{i,z})^2}{2\hat{\sigma}_{iz}^2}\right)dt \\
&= \sin^2(\hat{\vartheta}_{i,z})e^{-2\hat{\sigma}_{iz}^2} + 0.5\left(1-e^{-2\hat{\sigma}_{iz}^2}\right) .
\end{aligned}
$$

  Here, the closed-form solution is found after classical trigonometry followed by a classical calculus for integrals. One may prefer to avoid a range with the whole real line by restricting the integral to $[0, 0.5\pi]$ as in the available r implementation for this estimator, but the gaussian function may be already near zero for the values outside of the bounds for some parameters. The approach of bias correction via an integral was introduced at first for bias correction for the dual power (and eventually Box-Cox) transformation in [12]. A nearly similar term is retrieved for this alternative transformation in the conditional setting as one may write that for the log-transformation:

$$
\begin{aligned}
\hat{\vartheta}^{**}_{iy,\mathrm{LOG}} &= \int_{-\infty}^{+\infty} \exp(t)\frac{1}{\sqrt{2\pi\hat{\sigma}_{iz}^2}}\exp\left(-\frac{(t-\hat{\vartheta}_{i,z})^2}{2\hat{\sigma}_{iz}^2}\right)dt \\
&= \int_{0}^{+\infty} t\frac{1}{\sqrt{2\pi\hat{\sigma}_{iz}^2}}\frac{1}{t}\exp\left(-\frac{(\log(t)-\hat{\vartheta}_{i,z})^2}{2\hat{\sigma}_{iz}^2}\right)dt \\
&= \exp\left(\hat{\vartheta}_{i,z} + 0.5\hat{\sigma}_{iz}^2\right) \\
&= \hat{\rho}_{i,\mathrm{SM}}\hat{\vartheta}_{iy,\mathrm{LOG}} .
\end{aligned}
$$

  It is interesting to note the exponential term with a variance which appears in both cases. As explained in [12], this leads to the usual bias correction for this different transformation. More generally for other transformations, an approximated closed-form expression of the integral may be computed analytically from a sum of exponential decaying functions with parameters $(\alpha_{\ell,\kappa})$ and $(\omega_{\ell,\kappa})$ herein, with say $\kappa(t) = \sum_\ell \alpha_{\ell,\kappa}e^{-\omega_{\ell,\kappa}t}$. Hence, the back-transformation introduced by these authors may be computed as follows for a strictly positive derivative of the back-transformation:

$$
\begin{aligned}
\hat{\vartheta}^{**}_{iy,\mathrm{GEN}} &= \int_{-\infty}^{+\infty} t \frac{1}{\sqrt{2\pi\hat{\sigma}^2_{iz}}} \exp\left(-\frac{(\kappa^{-1}(t)-\hat{\vartheta}_{i,z})^2}{2\hat{\sigma}^2_{iz}}\right) \left.\frac{d\kappa^{-1}(u)}{du}\right|_{u=t} dt \\
&\doteq \sum_{\ell=1}^{\ell=L} \frac{\alpha_{\ell,\kappa}}{\sqrt{2\pi\hat{\sigma}^2_{iz}}} \int_{-\infty}^{+\infty} \exp\left(-\frac{(t-\hat{\vartheta}_{i,z})^2 + 2\hat{\sigma}^2_{iz}\omega_{\ell,\kappa}\, t}{2\hat{\sigma}^2_{iz}}\right) dt \\
&= \sum_{\ell=1}^{\ell=L} \alpha_{\ell,\kappa} e^{-\omega_{\ell,\kappa}(\hat{\vartheta}_{i,z}-0.5\hat{\sigma}^2_{iz}\omega_{\ell,\kappa})} .
\end{aligned}
$$

For transformations such as the dual power and sinh sinh ones, the derivative may be positive, such that this is relevant. It may be also retrieved the case of the log-transformation when only the first decaying exponential function is kept with $\alpha_{1,\kappa}=1$ and $\omega_{1,\kappa}=-1$, thus the other terms are removed from the sum. This leads to a general bias-corrected estimator for a back-transformation $\kappa()$ when an approximation with the sum is available for the corresponding function.This expression suggests an eventual additional bias correction wanted because of the sampling error on the estimation of the parameters as in [4] but with the estimator above.

- An alternative approximation for a generalized bias correction is with respect to the second order of the transformation function (see also [13]). This is written:

$$
\begin{aligned}
E[\kappa(\hat{\vartheta}_{iz})] &\doteq \kappa(\vartheta_{iz}) + 0 + 0.5\ddot{\kappa}(\vartheta_{iz})E[(\vartheta_{iz}-\hat{\vartheta}_{iz})^2] \\
&= \kappa(\vartheta_{iz}) + 0.5\ddot{\kappa}(\vartheta_{iz})\sigma^2_{iz} \\
&= \left(1 + 0.5\frac{\ddot{\kappa}(\vartheta_{iz})}{\kappa(\vartheta_{iz})}\sigma^2_{iz}\right)\kappa(\vartheta_{iz}) .
\end{aligned}
$$

When noting that the estimator in [3] is actually obtained directly from the statistics of interest for the log-transformation as the expectation of $\exp(\vartheta_{iz})$, hence as the expectation of $\kappa(\vartheta_{iz})$ in the general case and considering the other existing bias-corrected estimator, such as the existing ones which are related to [3] for the unit-level model and also the one just described above, a first relevant estimator is a sample version from the direct expectation of the back-transformation. Instead to look directly for the multiplicative factor ρ as introduced in [3], one may write:

$$
\hat{\vartheta}^{**}_{iy,\mathrm{GEN}} = \hat{E}[\kappa(\hat{\vartheta}_{iz})] = \hat{h}_{iy}\,\kappa(\hat{\vartheta}_{iz}) .
$$

Here, the correcting multiplicative term is defined from above as follows:

$$
\begin{aligned}
\hat{h}_{iy} = \hat{h}_{iy}(\hat{\beta},\hat{\sigma}^2_u) &= 1 + 0.5\frac{\ddot{\kappa}(\hat{\vartheta}_{iz})}{\kappa(\hat{\vartheta}_{iz})}\hat{\sigma}^2_{iz} \\
&\doteq \exp\left(0.5\frac{\ddot{\kappa}(\hat{\vartheta}_{iz})}{\kappa(\hat{\vartheta}_{iz})}\hat{\sigma}^2_{iz}\right) .
\end{aligned}
$$

Note that with $(1+u)^{-1} \approx e^{-u}$, it may be preferred the exponential form of the multiplicative term. The form of the estimator insures that it remains positive, it is for instance in the case of the Box-Cox transformation:

$$
\hat{\vartheta}^{**}_{iy,\mathrm{BC}} = \left(1 + 0.5\frac{(1-\lambda)}{\kappa^{-\lambda/2}(\hat{\vartheta}_{iz})}\hat{\sigma}^2_{iz}\right)\kappa(\hat{\vartheta}_{iz}) .
$$

The estimator $\hat{\vartheta}^*_{iy,\text{GEN}}$ defined above with the term h_{iy} includes the estimator $\hat{\vartheta}^{**}_{iy,\text{BC}}$ as a particular case. It is further bias corrected at the next section with the final definition for a bias corrected back-transformed mean estimator from the involved small area-level model:

$$\hat{\vartheta}^*_{iy} = \hat{\rho}_i \hat{h}_{iy}\, \kappa(\hat{\vartheta}_{iz})\,.$$

Here $\hat{\rho}_i$ is a term different from before because of the additional term $\hat{h}_{iy}$, or $\kappa(\hat{\vartheta}_{iz})$ needs to be replaced implicitely by $\hat{h}_{iy}\kappa(\hat{\vartheta}_{iz})$ in order to keep the same definition. This estimator includes explicitely a first correction from the back-transformation transformation it-self before the one from the sampling error which comes as another source of bias. This is in correspondence with the log-transformation extended to the Box-Cox transformation in order to retrieve their respective expressions of their bias-corrected estimators from the current literature. Hence to define $\hat{h}_{iy}$, one has an exponential function and one has a first order approximation of the exponential function while both include a weighted variance term.

3 Bias correction after the back-transformation

The Fay-Herriot method or area level method is defined directly for the area instead of the units inside the area. This section proposed a general approach for bias correction after a non linear transformation by extending a recent existing work for log-transformation and also by discussing the variance of the resulting back-transformed estimators.

3.1 Reviewing the log-transformation

When the transformation is inverted a bias appears as explained in the literature [3, 4, 12]. For the proposed transformation, the solution for the logarithmic case, may be not enough, hence a new bias correction is required when $\lambda \neq 0$. It may be supposed that λ is not far from zero, such that the same theoritical bias correction may remain relevant, but this is not always true. In the case of the logarithm transformation, the back-transformation leads to:

$$\begin{aligned} \hat{\vartheta}_{iy} &= \kappa(\hat{\vartheta}_{iz}) \\ &= \exp(\hat{\vartheta}_{iz})\,. \end{aligned}$$

For the log transformation, the authors in [4] have suggested these two approaches. The first method may be directly extended to the Box-Cox transformation.

Method 1

When $\hat{\theta} = (\hat{\beta}, \hat{\sigma}_u)^T$ is the estimator of $\theta = (\beta, \sigma_u)^T$,

$$\begin{aligned} \delta_i &= \delta_i(\theta) = x_i^T\beta + \gamma_i(y_i - x_i^T\beta) + 0.5\sigma_u^2(1-\gamma_i) \\ \hat{\delta}_i &= \hat{\delta}_i(\hat{\theta}) = x_i^T\hat{\beta} + \hat{\gamma}_i(y_i - x_i^T\hat{\beta}) + 0.5\hat{\sigma}_u^2(1-\hat{\gamma}_i)\,. \end{aligned}$$

This leads to compute, when $\tau_i = \sigma_u^2 + \sigma_{e_i}^2$ and $\gamma_i = \sigma_u^2 \tau_i^{-1}$, that:

$$
\begin{aligned}
E(\exp(\hat{\delta}_i) &\approx \exp\left[E(\hat{\delta}_i) + 0.5V(\hat{\delta}_i)\right] \\
E(\hat{\delta}_i) &\doteq \delta_i(\theta) \\
V(\hat{\delta}_i) &\doteq (1+\gamma_i)^2 x_i^T V(\hat{\beta}) x_i + B_i^2 V(\hat{\sigma}_u^2) \\
B_i &= \tau_i^{-1}(1-\gamma_i)(y_i - x_i^T\beta) + 0.5(1-\gamma_i)^2 \,.
\end{aligned}
$$

For the expectation this is because the eblup is involved. For the variance, with $V_{\hat{\theta}}$ for the variance of $\hat{\theta}$ a linearization leads to $V(\hat{\delta}_i) \doteq g_\theta^T V_{\hat{\theta}} g_\theta$ where g_θ is the derivative of $\hat{\delta}_i$ w.r.t. $\hat{\theta}$ in [4]. This leads to a first bias correction,

$$
\begin{aligned}
\rho_{i,\text{LOG}}^{(1)} &= \frac{\exp\left[\delta_i(\theta)\right]}{\exp\left[\delta_i(\theta) + 0.5V(\hat{\delta}_i)\right]} \\
&= \exp\left[-0.5V(\hat{\delta}_i)\right] \text{ for the Logarithmic case when } \lambda = 0\,.
\end{aligned}
$$

This is the limiting case from the Box-cox transformation with λ converging towards zero in the new function $\kappa()$. When $\lambda \neq 0$ the involved correction comes directly from the expectation as found in [14]. With the approximation $1 + u \approx \exp(u)$, this leads to the corrective multiplicative term,

$$
\begin{aligned}
\rho_{i,\text{BC}}^{(1)} &= 1 - 0.5C_i V(\hat{\delta}_i) \\
&\doteq \exp\left[-0.5C_i V(\hat{\delta}_i)\right] \text{ for the Box-Cox case when } \lambda \neq 0\,.
\end{aligned}
$$

Note that one may prefer the function from the Box-Cox transformation instead of the exponential for the bias correction. Here, it is denoted C_i for $(1-\lambda)(\lambda E(\hat{\delta}_i) + 1)^{-2}$ while the same expressions for $E(\hat{\delta}_i)$ and $V(\hat{\delta}_i)$ hold, such that one just needs to plug-in these quantities in the intermediate quantity C_i and the multiplicative term above. It is also retrieved the solution for the log-transormation when λ becomes small near zero as expected. Another approach is a direct linearization without using a property of the log-normal distribution, as next after.

Method 2

The direct linearization proceeds as follows after an expansion from a multivariate Taylor serie. When $Tr()$ is the matricial operator for the trace, this leads to the approximation of the bias as follows,

$$
E[\hat{\vartheta}_{iy}] \doteq \vartheta_{iy} + 0.5Tr\left(\frac{d^2[\vartheta_{iy}]}{d\beta d\beta^T} V_{\hat{\beta}}\right) + 0.5\left(\frac{d^2[\vartheta_{iy}]}{d\sigma_u^2} V_{\hat{\sigma}_u}\right)\,.
$$

When $G_i = 2(y_i - x_i^T\beta) + \sigma_{e_i}^2$, in [4] it was proven:

$$
\begin{aligned}
\frac{\partial^2[\vartheta_{iy}]}{\partial\beta\partial\beta^T} &= Tr\left(A_i V_{\hat{\beta}}\right) \vartheta_{iy} \text{ with } A_i = (1-\gamma_i)^2 x_i^T x_i \\
\frac{\partial^2[\vartheta_{iy}]}{\partial\sigma_u^2} &= \left(B_i V_{\hat{\sigma}_u}\right) \vartheta_{iy} \text{ with } B_i = \frac{\sigma_{e_i}^2}{\tau_i^3} G_i \left(\frac{\sigma_{e_i}^2}{4\tau_i} G_i - 1\right)\,.
\end{aligned}
$$

With the approximation $1+u \approx e^u$, this leads to a second multiplicative term for the bias correction:

$$\rho^{(2)}_{i,\mathrm{LOG}} \;=\; \exp\left[-0.5\left\{Tr\left(A_i V_{\hat\beta}\right) + B_i V_{\hat\sigma_u}\right\}\right] \text{ for the Log case}\,.$$

Two remarks are worth to notice. First of all, according to the available experiments from its authors, the previous method has performed less well than this second one in their numerical results, such that a similar linearization could improve the previous resulting estimates, as a bias correction of the bias correction. Moreover, for the second method, each transformation function asks for a new analytical solution which is generalized next.

3.2 Extending to Box-Cox and other nonlinear transformations

For the Box-cox transformation and the logistic-bounded Box-Cox transformation, the former bias correction needs to be updated because the involved function is different. Hence, a generalized solution is developpped in this subsection before numerical experiments. Note also that the method for bias correction leads to nearly similar expression than for the log transformation, which is expected. It is used the first and second order derivative of the function for the back-transformation, say respectively $\dot\kappa = \frac{d\kappa(u)}{d_u}$ and $\ddot\kappa = \frac{d^2\kappa(u)}{d^2u}$ jointly with partial derivarives w.r.t. the parameters, next.

Multiplicative term for $\kappa(\vartheta_{iz})$

For the linearization, one may write the new expression for the derivatives of $\vartheta_{iy} = \kappa(\vartheta_{iz})$, as follows with an usual expectation after a two-order expansion. The derivatives $\partial^2[\vartheta_{iy}]/\partial\beta\partial\beta^T$ and $d^2[\vartheta_{iy}]/\partial\sigma_u^2$ w.r.t. the regression coefficients and the variance component are required. This may lead to a generalized bias correction as follows:

$$\begin{aligned}\rho^{(2,\kappa)}_{i,\mathrm{GEN}} &= \frac{1}{1+\frac{1}{0.5}Tr\left(\frac{\partial^2[\vartheta_{iy}]}{\partial\beta\partial\beta^T}V_{\hat\beta}\right)+\frac{0.5}{\vartheta_{iy}}\left(\frac{\partial^2[\vartheta_{iy}]}{\partial\sigma_u^2}V_{\hat\sigma_u}\right)} \\ &\doteq 1-0.5\left\{\frac{\ddot\kappa(\vartheta_{iz})}{\kappa(\vartheta_{iz})}Tr\left(\frac{\partial[\vartheta_{iz}]}{\partial\beta}\frac{\partial[\vartheta_{iz}]}{\partial\beta^T}V_{\hat\beta}\right)+\left(\frac{\ddot\kappa(\vartheta_{iz})}{\kappa(\vartheta_{iz})}\left(\frac{\partial[\vartheta_{iz}]}{\partial\sigma_u^2}\right)^2+\frac{\dot\kappa(\vartheta_{iz})}{\kappa(\vartheta_{iz})}\frac{\partial^2[\vartheta_{iz}]}{\partial^2\sigma_u^2}\right)V_{\hat\sigma_u}\right\}.\end{aligned}$$

Multiplicative term for $h_{iy}\kappa(\vartheta_{iz})$

For the linearization, one may write the new expression for the derivatives of $\vartheta_{iy} = h_{iy}\kappa(\vartheta_{iz})$, as follows with an usual expectation after a two-order expansion. The new derivatives $\partial^2[h_{iy}\vartheta_{iy}]/\partial\beta\partial\beta^T$ and $\partial^2[\vartheta_{iy}]/\partial\sigma_u^2$ w.r.t. the regression coefficients and the variance component are required. This may lead to an alternative generalized bias correction for this other estimator including the bias from the back-transformation as follows:

$$\rho^{(2,h\kappa)}_{i,\mathrm{GEN}} = \frac{1}{1+\frac{0.5}{h_{iy}\vartheta_{iy}}Tr\left(\frac{d^2[h_{iy}\vartheta_{iy}]}{d\beta d\beta^T}V_{\hat\beta}\right)+\frac{0.5}{h_{iy}\vartheta_{iy}}\left(\frac{d^2[h_{iy}\vartheta_{iy}]}{d\sigma_u^2}V_{\hat\sigma_u}\right)}\,.$$

Example with the Box-Cox transformation

The derivatives at first order $\dot\kappa(\vartheta_{iz})$ and second order $\ddot\kappa(\vartheta_{iz})$ for the inverse transformation functions $\kappa(\vartheta_{iz})$ are in the Table 2. Note that an additive bias correction is also available but usually

Table 2: Examples of inverse transformation functions and their derivatives.

	$\kappa(\vartheta_{iz})$	$\dot{\kappa}(\vartheta_{iz})$	$\ddot{\kappa}(\vartheta_{iz})$
Exponential	$\exp(\vartheta_{iz})$	$\exp(\vartheta_{iz})$	$\exp(\vartheta_{iz})$
Box-Cox	$(1+\lambda\vartheta_{iz})^{\frac{1}{\lambda}}$	$(1+\lambda\vartheta_{iz})^{\frac{1-\lambda}{\lambda}}$	$(1-\lambda)(1+\lambda\vartheta_{iz})^{\frac{1-2\lambda}{\lambda}}$
Inv-sin-sqrt	$\sin^2(\vartheta_{iz})$	$0.5\sin(2\vartheta_{iz})$	$\cos(2\vartheta_{iz})$

not used for outcomes taking only positive values. Thanks to the expansion, only the derivatives for the nonlinear transformation may change with the transformation for this approximation. The derivative for the linear estimator are required too and are written as follows:

$$\begin{aligned}
\vartheta_{iz} &= x_i^T\beta + \gamma_i(y_i - x_i^T\beta) + 0.5\sigma_u^2(1-\gamma_i) \\
\frac{d[\vartheta_{iz}]}{d\beta} &= (1-\gamma_i)x_i \\
\frac{d^2[\vartheta_{iz}]}{d\beta d\beta^T} &= 0 \\
\frac{d[\vartheta_{iz}]}{d\sigma_u^2} &= \tau_i^{-1}(1-\gamma_i)(y_i - x_i^T\beta) + 0.5(1-\gamma_i)^2 \\
\frac{d^2[\vartheta_{iz}]}{d\sigma_u^2} &= -2\tau_i^{-2}(1-\gamma_i)(y_i - x_i^T\beta) - \tau_i^{-1}(1-\gamma_i)^2 \,.
\end{aligned}$$

When replacing in the expectation the expressions from just before, one may get the multiplicative bias correction as follows for the Box-Cox transformation:

$$\begin{aligned}
\rho_{i,\mathrm{BC}}^{(2,\kappa)} &= 1 - \frac{0.5}{\kappa(\vartheta_{iz})^2}\left\{Tr\left(A_i V_{\hat{\beta}}\right) + E_i V_{\hat{\sigma}_u}\right\} \\
&\doteq \exp\left(-\frac{0.5}{\kappa(\vartheta_{iz})^2}\left\{Tr\left(A_i V_{\hat{\beta}}\right) + E_i V_{\hat{\sigma}_u}\right\}\right) .
\end{aligned}$$

Here, it is denoted,

$$\begin{aligned}
E_i &= \left[-2\tau_i^{-2}(1-\gamma_i)(y_i - x_i^T\beta) - \tau_i^{-1}(1-\gamma_i)^2\right] \\
&+ \left[\tau_i^{-1}(1-\gamma_i)(y_i - x_i^T\beta) + 0.5(1-\gamma_i)^2\right]^2 \kappa(\vartheta_{iz}) \,.
\end{aligned}$$

This is expected to be equal to the correction for the log-transformation when lambda becomes small near zero as expected. This allows to compare with the methods from above in order to check how they may be different. An interesting application is to the estimator from an integral and a sum of decaying exponential functions, which may be also be improved too.

3.3 Variance estimation of the bias corrected estimator

In this subsection, a recent approach by Deville via linearizations and influence functions [15] is implemented. To follow this idea, one has to focus on the estimating equations which are written as sums from the observed data vectors, such that it may be written that:

$$\hat{\vartheta}_i = g_i(\hat{\beta}, \hat{\sigma}_u) \,.$$

With $\theta = (\beta^T, \sigma_u^2)$ the following approximation may be involved.

$$\hat{\vartheta}_i - \vartheta_i \doteq \left.\frac{\partial g_i}{\partial \beta}\right|_\theta^T (\hat{\beta} - \hat{\beta}) + \left.\frac{\partial g_i}{\partial \sigma_u^2}\right|_\theta^T (\hat{\sigma}_u^2 - \sigma_u^2)\,.$$

For β and σ_u^2, it is supposed the two following estimating equations, available from the numerical algorithms for the inference,

$$\begin{array}{lcl} \frac{1}{m}\sum_j^m \phi_{j,\beta}(\hat{\beta}, \hat{\sigma}_u^2) & = & 0 \\ \frac{1}{m}\sum_j^m \phi_{j,\sigma_u^2}(\hat{\beta}, \hat{\sigma}_u^2) & = & 0\,. \end{array}$$

Similarly, with $H_\beta = E[\frac{1}{m}\left.\frac{\partial \sum_j \phi_{j,\beta}}{\partial \beta}\right|_\theta^T]$ and $H_{\sigma_u^2} = E[\frac{1}{m}\left.\frac{\partial \sum_j \phi_{j,\sigma_u^2}}{\partial \sigma_u^2}\right|_\theta^T]$ a linearization leads to:

$$\begin{array}{lcl} \sum_j^m \phi_{j,\beta}(\beta, \sigma_u^2) + H_\beta(\hat{\beta} - \hat{\beta}) & \doteq & 0 \\ \sum_j^m \phi_{j,\sigma_u^2}(\beta, \sigma_u^2) + H_{\sigma_u^2}(\hat{\sigma}_u^2 - \sigma_u^2) & \doteq & 0\,. \end{array}$$

This may induce the following linearization:

$$\hat{\vartheta}_i - \vartheta_i \doteq \sum_j^m I_{j,i}^{lin}\,,$$

where,

$$I_{j,i}^{lin} = \left\{ \left.\frac{\partial g_i}{\partial \beta}\right|_\theta^T H_\beta^{-1} \phi_{j,\beta}(\beta, \sigma_u^2) + \left.\frac{\partial g_i}{\partial \sigma_u^2}\right|_\theta^T H_{\sigma_u^2}^{-1} \phi_{j,\sigma_u^2}(\beta, \sigma_u^2) \right\}\,.$$

Finally, an estimation of the variance may be obtained as the following sum,

$$\hat{V}_{\hat{\vartheta}_i}^{lin} = \frac{1}{m(m-1)} \sum_{j=1}^m (\hat{I}_{j,i}^{lin} - \bar{I}_i^{lin})(\hat{I}_{j,i}^{lin} - \bar{I}_i^{lin})^T \quad \text{with} \quad \bar{I}_i^{lin} = \frac{1}{m}\sum_{j=1}^m \hat{I}_{j,i}^{lin}\,.$$

As empirical likelihood [16] is often shown to perform better via weighting, an additional estimating equation for the unknown lagrange multiplier coming from the estimation of the weights is required. The usual method for variance estimation via an unbiased mse [17] is left as a perspective.

4 Conclusion

Herein, it is suggested, for a small area-level model, several multiplicative (and additive) bias corrections for the Box-Cox transformation with a generalized setting which may be relevant for other nonlinear transformations. Perspectives for the reader are an extension to the unit-level model and the multivariate area-level model. Computational bias corrections are required in order to compare with the analytical solutions. An analytical expression or a computational algorithm for the mean squared error after such bias correction is also wanted.

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