MAGNETOSTÁTICA

TEMAS

1. Corrientes estacionarias
2. Interacciones magnéticas
3. Fuentes de campo magnético
4. Magnetismo en presencia de la materia

1.1 CONDICIÓN DE CORRIENTE ESTACIONARIA

- Supongamos que estamos en algún punto P observando cargas pasar a lo largo de un alambre conductor o partículas cargadas moviéndose a través del espacio

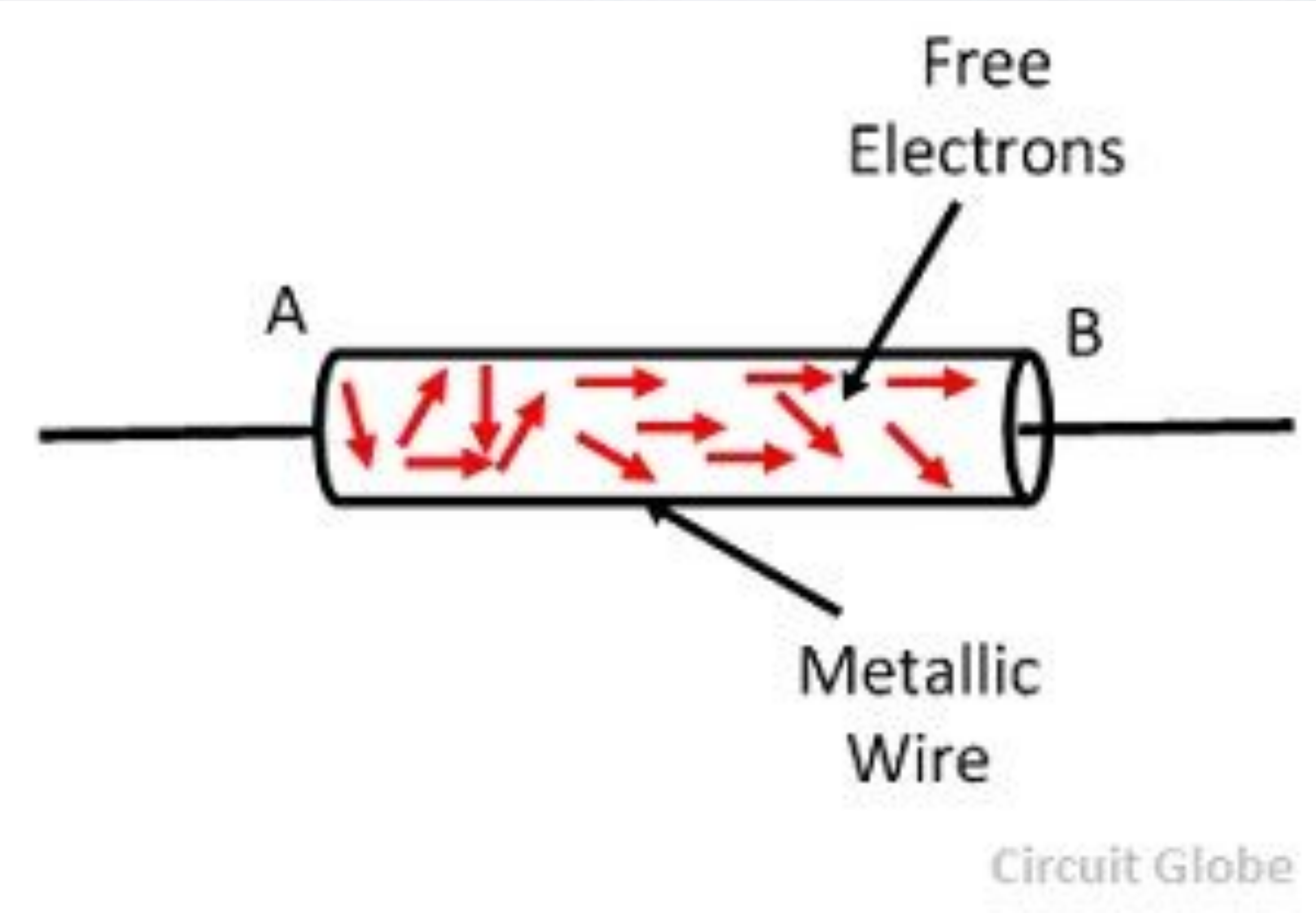
  - En un intervalo Δt de tiempo unas carga Δq pasa por el punto P

  - Se define la corriente promedio $\langle I \rangle = \dfrac{\Delta q}{\Delta t}$

  - Si el flujo de carga no es uniforme en el tiempo, podemos definir una corriente instantánea $I = \dfrac{dq}{dt}$

Unidades: Amperes $1A = 1C/s$

Las corrientes que son constantes en tiempo, $I=const$, es llamada <u>corriente estacionaria</u>



1.1 CONDICIÓN DE CORRIENTE ESTACIONARIA

- Ahora considere la situación en la que el flujo de carga está distribuida a través de un volumen o una superficie
    - Debemos introducir el concepto de densidad de corriente
        - Densidad de corriente volumétrico $\vec{j}$: corriente por unidad de área a través de un área perpendicular al flujo

$$\Delta q = \langle I \rangle \Delta t = \langle j \rangle \Delta a \Delta t$$

y $\Delta q = \rho \Delta l \Delta a$

$$\langle j \rangle = \rho \frac{\Delta l}{\Delta t} = \rho \langle v \rangle$$

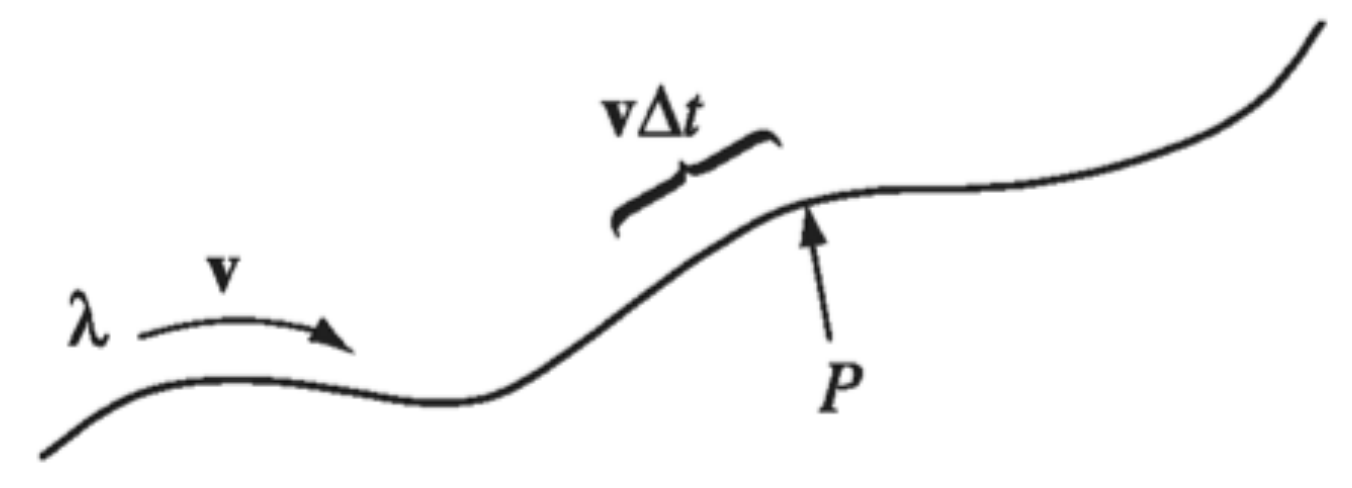


FIGURE 5.9

Esta relación es válida para valores instantáneos de tiempo

$$\vec{j} = \rho \vec{v} \qquad \text{o} \qquad \vec{K} = \sigma \vec{v}$$

Si las cargas son diferentes y se mueven a distinta velocidad se puede definir

$$\vec{j} = \sum_i \rho_i \vec{v_i}$$

De la definición podemos ver

$$\left(\frac{dq}{dt}\right)_{d\vec{S}} = \vec{j} \cdot d\vec{S}, \qquad \left(\frac{dq}{dt}\right)_{d\vec{l}} = \vec{K} \cdot d\vec{l}$$

1.1 CONDICIÓN DE CORRIENTE ESTACIONARIA

- Sabemos que en un sistema cerrado la carga se conserva
  - Supongamos que una superficie S es una superficie estacionaria encerrada por un volumen V
  - Si Q, la carga total fluyendo por un volumen V es constante

$$-\frac{dQ}{dt} = \int_S \vec{j} \cdot d\vec{S}$$

$$\Longrightarrow$$

$$-\frac{d}{dt}\int_V \rho dV = -\int_V \frac{\partial \rho}{\partial t} dV = \int_V \nabla \cdot \vec{j} dV$$

$$\Longrightarrow$$

$$\int_V \left(\nabla \cdot \vec{j} + \frac{\partial \rho}{\partial t} \right) dV = 0$$

Debido a que la carga debe conservarse en todo los puntos del espacio

$$\boxed{\nabla \cdot \vec{j}(\vec{r}, t) + \frac{\partial \rho(\vec{r}, t)}{\partial t} = 0}$$

Ecuación de continuidad

En la condición de corriente estacionario

$$\nabla \cdot \vec{j}(\vec{r}) = 0$$

1.2 TEORÍA POTENCIAL DE MATERIALES OHMICOS

- La ecuación del flujo de corriente para un material ohmico para un medio son

$$\nabla \cdot \vec{j}(\vec{r}) = 0$$

$$\vec{j}(\vec{r}) = \sigma(\vec{r})\vec{E}(\vec{r})$$

$$\nabla \times \vec{E}(\vec{r}) = 0$$

$$\Downarrow$$

$$\sigma(\vec{r})\nabla \cdot \vec{E}(\vec{r}) + \vec{E}(\vec{r}) \cdot \nabla\sigma(\vec{r}) = 0$$

Si la conductividad es constante

$$\Downarrow$$

$$\left.\begin{aligned} &\sigma(\vec{r})\nabla \cdot \vec{E}(\vec{r}) = 0 \\ &\text{pero} \\ &\nabla \cdot \vec{E}(\vec{r}) = \rho(\vec{r})/\epsilon_0 \end{aligned}\right\} \Longrightarrow \rho(\vec{r}) = 0$$

$$\nabla^2\phi(\vec{r}) = 0$$

Con condiciones de frontera

$$\hat{n} \cdot (\vec{j_1} - \vec{j_2}) = 0$$

$$\sigma_1 \frac{\partial \phi_1}{\partial n}\bigg|_S = \sigma_2 \frac{\partial \phi_2}{\partial n}\bigg|_S$$

$$\hat{n} \times (\vec{E_1} - \vec{E_2}) = 0$$

$$\phi_1(\vec{r}_S) = \phi_2(\vec{r}_S)$$

1.3 DEFINICIÓN DE FUERZA ELECTROMOTRIZ

- Una corriente directa (estacionaria) sucede en un medio ohmico si una fuente de energía está presente para contrarrestar la pérdida de energía por resistencia del medio

- $\vec{E} = \vec{j}/\sigma$ es un buen candidato porque el campo electrostático en un un circuito cerrado no hace trabajo

- Las corrientes estacionarias son posibles sólo si hay fuentes de campo eléctrico conocidas como <u>fuerza electromotriz</u> que produce campos no irrotacionales

- Asumimos que el campo electromotriz existe $\vec{E}'$

$$\vec{j} = \sigma(\vec{E} + \vec{E}')$$

Definimos $\mathscr{E} = \int_A^B \vec{E}' \cdot d\vec{l}$

Usando $I = jA$ a lo largo de L_{AB}

$$R_{AB} = \frac{1}{I\sigma}\int_A^B \vec{j} \cdot d\vec{l} = \frac{L_{AB}}{A\sigma}$$

$$IR_{AB} = \phi_A - \phi_B + \mathscr{E}_{AB}$$

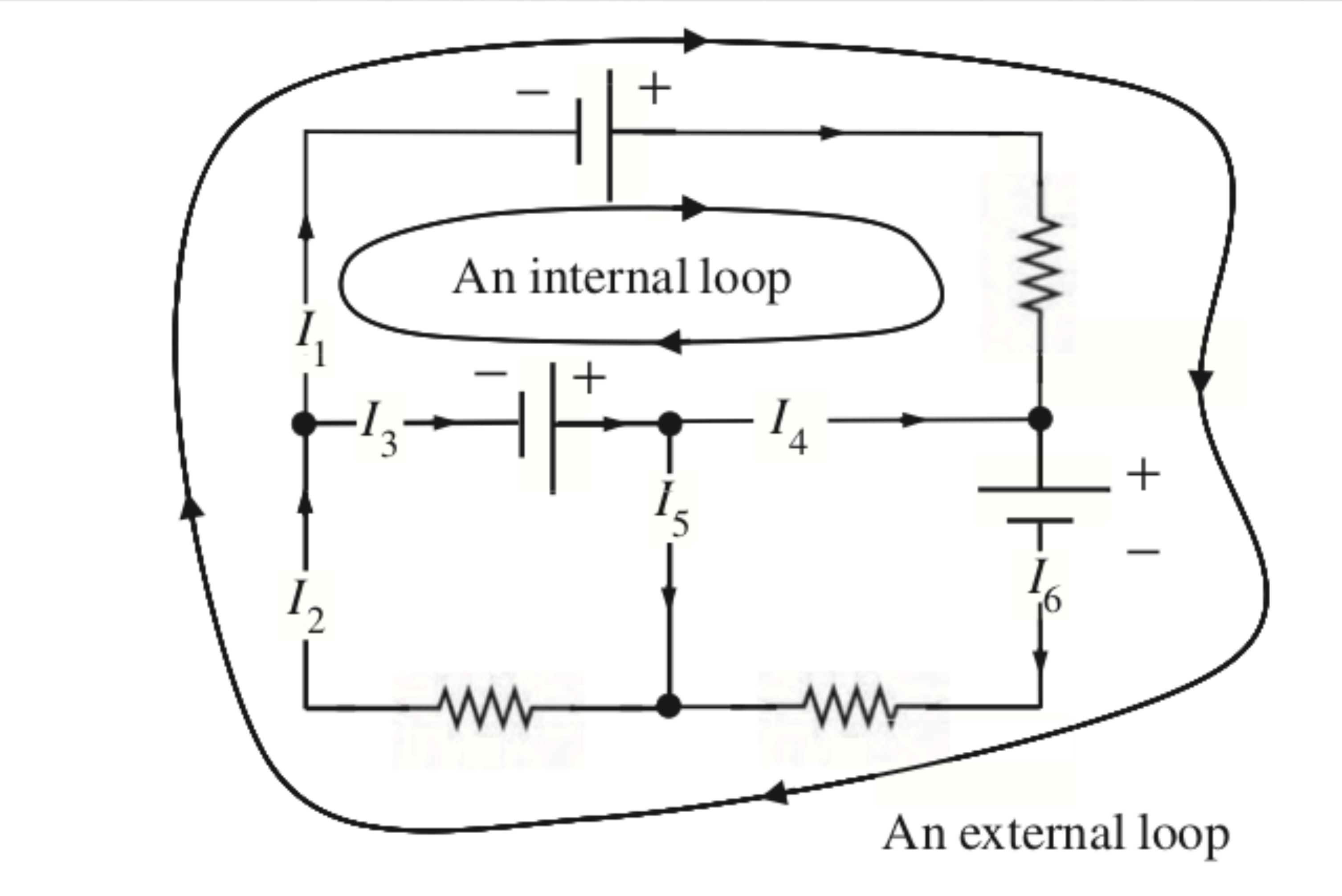
Para circuitos cerrados $\mathscr{E} = \oint (\vec{E} + \vec{E}') \cdot d\vec{l} = \oint \frac{\vec{j} \cdot d\vec{l}}{\sigma}$

$$\boxed{\mathscr{E} = \oint \vec{E}' \cdot d\vec{l} = \frac{1}{q}\oint \vec{F} \cdot d\vec{l} = IR}$$

1.4 LEYES DE KIRCHHOFF

- Las leyes de Kirchhoff son consecuencia de las leyes diferenciales

$$\nabla \cdot \vec{j} = 0, \quad \text{y} \quad \nabla \times \vec{E} = 0$$



- Para la condición de corriente estacionaria

  - La primera condición $\nabla \cdot \vec{j} = 0$, implica que las corrientes deben fluir dentro y fuera de cada nodo y no debe haber acumulación de cargas

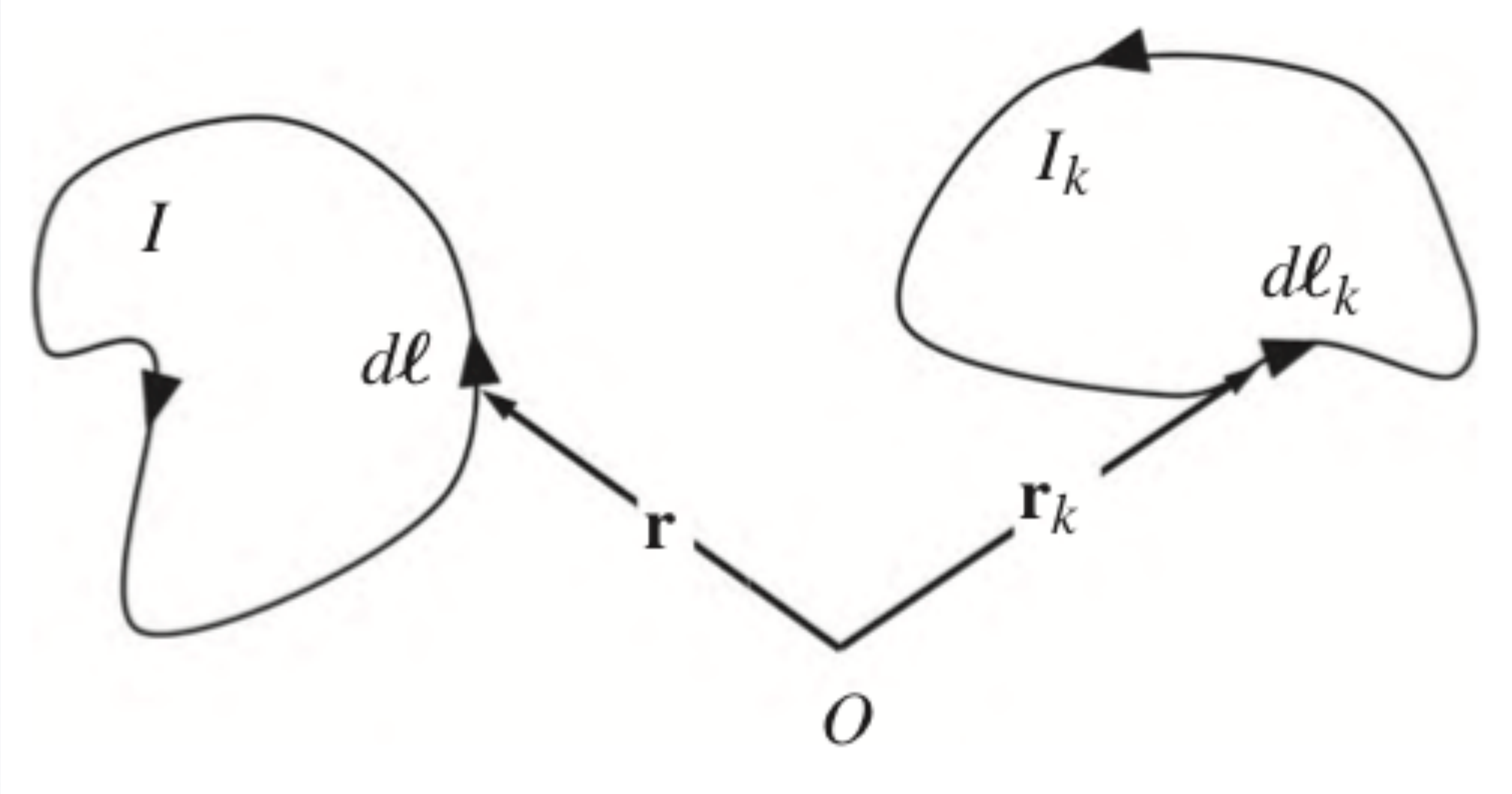
$$\sum_k I_k = 0 \quad \text{Ley de corrientes de Kirchhoff}$$

  - La segunda condición $\nabla \times \vec{E} = 0$ implica que si hay una corriente I_n que fluye a través del resistor R_n, entonces la EMF es

$$\sum_k \mathscr{E}_k = \sum_n I_n R_n \quad \text{Ley de voltajes de Kirchhoff}$$

2.1 FUERZA ENTRE DOS CIRCUITOS COMPLETOS

- Uno de los grandes resultados publicados por Ampère fue el cálculo de la fuerza sobre un circuito cerrado, que lleva una corriente I, debido a la presencia de otros N circuitos que llevan corrientes I_k



- Si $\boldsymbol{r}$ apunta al elemento de línea $d\boldsymbol{l}$ del circuito con I y $\boldsymbol{r}_k$ apunta al elemento $d\boldsymbol{l}_k$ del circuito k, la fórmula de Ampère para la fuerza sobre I es

$$\vec{F} = -\frac{\mu_0}{4\pi}\oint I \sum_{k=1}^{N} \oint I_k \frac{\vec{r}-\vec{r}_k}{|\vec{r}-\vec{r}_k|^3} d\vec{l}_k \cdot d\vec{l}$$

$\mu_0 \longleftarrow$ Permeabilidad del vacío

$$\mu_0 = 4\pi \times 10^{-7}\, N/A^2$$

Usando $\vec{A}\times(\vec{B}\times\vec{C}) = \vec{B}(\vec{A}\cdot\vec{C}) - \vec{C}(\vec{A}\cdot\vec{B})$

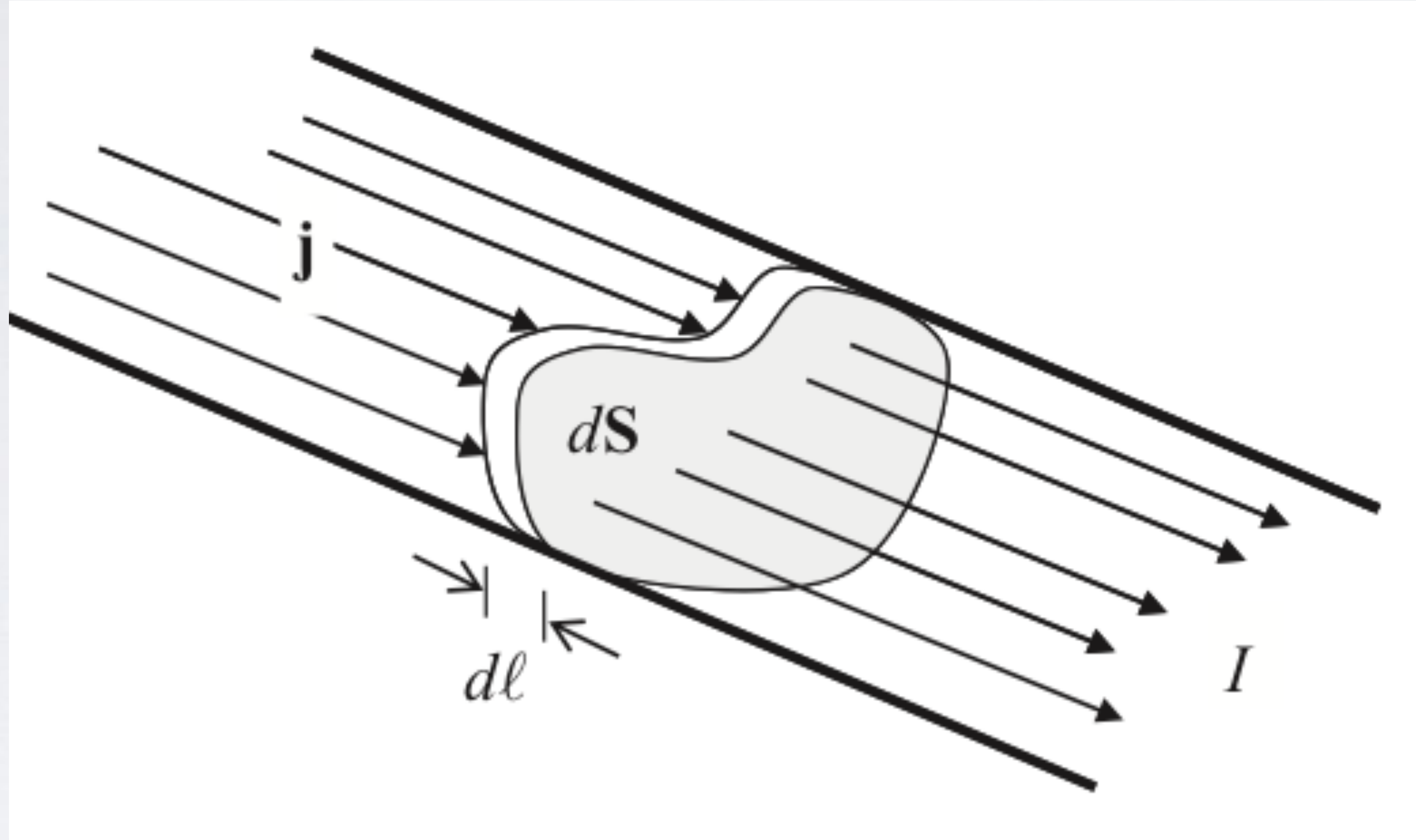
y $$\oint \frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3}\cdot d\vec{l} = 0$$

$$\Downarrow$$

$$\vec{F} = \frac{\mu_0}{4\pi}\oint I d\vec{l} \times \left(\sum_{k=1}^{N} \oint I_k d\vec{l}_k \times \frac{\vec{r}-\vec{r}_k}{|\vec{r}-\vec{r}_k|^3}\right)$$

2.1 FUERZA ENTRE DOS CIRCUITOS COMPLETOS

- En el caso que $d\overrightarrow{S} \parallel \vec{j}$ es un vector cuya magnitud es un elemento diferencial de la sección transversal de área



  - Podemos transformar circuitos lineales en distribuciones de corrientes volumétricas

$$I\int d\vec{l} = \int \vec{j}\cdot d\overrightarrow{S}\int d\vec{l} = \int\int \vec{j}\,d\overrightarrow{S}\cdot d\vec{l} = \int \vec{j}\,dV$$

Para corrientes estacionarias

$$\overrightarrow{F} = \int_V \vec{j} \times \left(\frac{\mu_0}{4\pi}\int_{V'} \frac{\vec{j}' \times (\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3} dV' \right) dV$$

2.2 DEFINICIÓN DE INDUCCIÓN MAGNÉTICA

- La separación del campo de fuerzas puede hacerse considerando

$$\vec{F} = \int_V \vec{j} \times \vec{B}\, dV$$

- Donde $\vec{B}$ es campo de inducción magnética producida por un circuito primado en la posición de uno sin primar es

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} \int_{V'} \frac{\vec{j}\,' \times (\vec{r} - \vec{r}\,')}{|\vec{r} - \vec{r}\,'|^3} dV' \longleftarrow$$

Ley de Biot-Savart

- El campo de inducción magnética determina la fuerza que actúa sobre un elemento del circuito

2.3 LA LEY DE BIOT-SAVART

- Desde un punto de vista más formal, podemos definir el campo de inducción magnético a través del rotacional

$$\vec{B}(\vec{r}) = \nabla \times \frac{\mu_0}{4\pi}\int_{V'} \frac{\vec{j}(\vec{r}')}{|\vec{r}-\vec{r}'|}dV'$$

Esto nos dice que

$$\nabla \cdot \vec{B}(\vec{r}) = 0$$

$$\nabla \times \vec{B}(\vec{r}) = \mu_0 \vec{j}(\vec{r})$$

Para una densidad de corriente volumétrica $\vec{j}(\vec{r})$

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\int_{V'} \frac{\vec{j}(\vec{r}') \times (\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3}dV'$$

Para densidad de corriente superficial $\vec{K}(\vec{r})$

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\int_{S'} \frac{\vec{K}(\vec{r}') \times (\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3}dS'$$

Para una corriente filamentaria I

$$\vec{B}(\vec{r}) = \frac{\mu_0 I}{4\pi}\int_{C} \frac{d\vec{l} \times (\vec{r}-\vec{l})}{|\vec{r}-\vec{l}|^3}$$

2.3 LEY DE BIOT-SAVART

- Considere una corriente filamentaria I y encontramos el campo magnético producido en cualquier punto del espacio

    - De la ley de Biot-Savart

$$\vec{B}(\vec{r}) = \frac{\mu_0 I}{4\pi} \int \frac{d\vec{l} \times (\vec{r} - \vec{l})}{|\vec{r} - \vec{l}|^3}$$

$$\vec{l} = l\hat{z} \qquad \vec{r} = \begin{pmatrix} \rho\cos\varphi \\ \rho\sin\varphi \\ z \end{pmatrix}$$

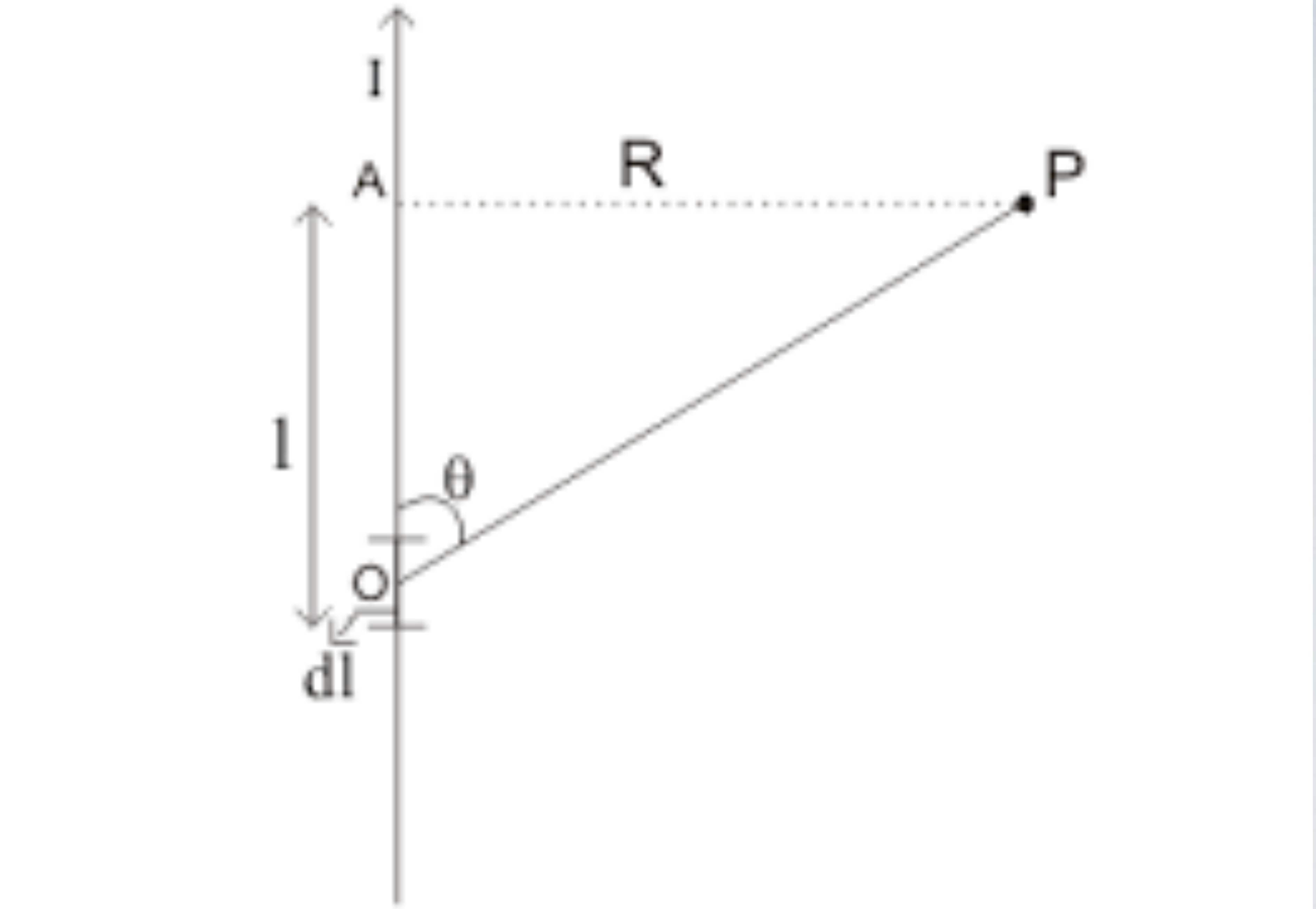


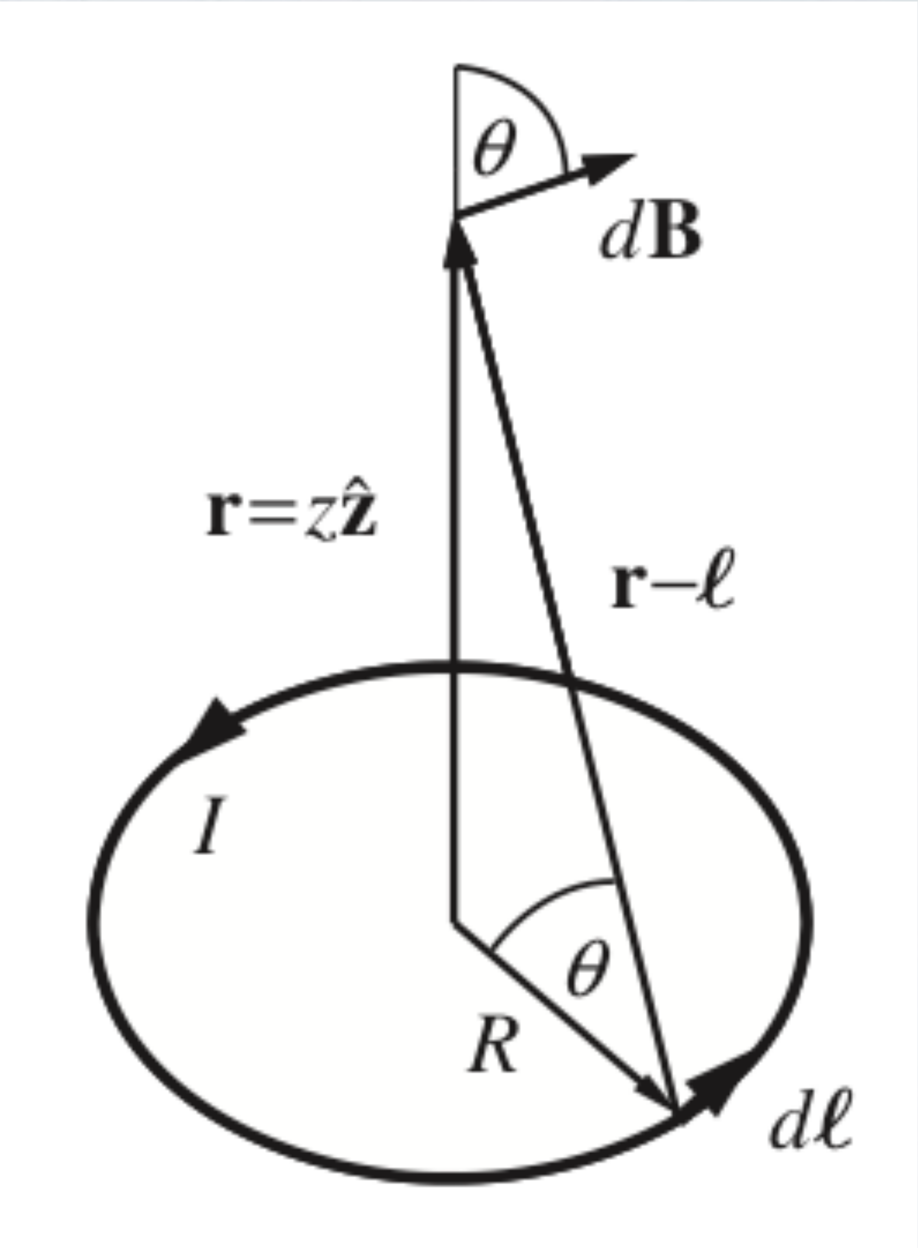
Figure 2. Infinitely long wire carrying current I

$$\Longrightarrow \vec{B}(\vec{r}) = \frac{\mu_0 I}{4\pi}\hat{\varphi} \int_{-\infty}^{\infty} \frac{\rho\, dl}{[\rho^2 + (z - l)^2]^{3/2}}$$

$$\boxed{\vec{B}(\vec{r}) = \frac{\mu_0 I}{2\pi\rho}\hat{\varphi}}$$

2.3 LEY DE BIOT-SAVART

- Ahora consideremos una corriente I circulando a lo largo de una circunferencia de un círculo de rado R.
  - Queremos encontrar el campo magnético en un punto del eje de simetría



$$\vec{B}(\vec{r}) = \frac{\mu_0 I}{4\pi}\int \frac{d\vec{l}\times(\vec{r}-\vec{l})}{|\vec{r}-\vec{l}|^3}$$

$$\vec{l} = R\hat{\rho} \qquad \vec{r} = z\hat{z} \quad \Longrightarrow \quad d\vec{l} = R\hat{\varphi}d\varphi$$

$$\Longrightarrow \quad \vec{B}(\vec{r}) = \frac{\mu_0 I}{4\pi}\int_0^{2\pi} \frac{R\hat{\varphi}\times(z\hat{z}-R\hat{\rho})d\varphi}{(z^2+R^2)^{3/2}}$$

$$\Longrightarrow \quad \vec{B}(\vec{r}) = \frac{\mu_0 I}{4\pi}\int_0^{2\pi} \frac{R(z\hat{\rho}+R\hat{z})d\varphi}{(z^2+R^2)^{3/2}}$$

$$\boxed{\vec{B}(z) = \hat{z}\frac{\mu_0 I}{2}\frac{R^2}{(z^2+R^2)^{3/2}}}$$

2.3 LEY DE BIOT-SAVART

- Ahora consideremos una distribución de corriente continua sobre un plano infinito con densidad de corriente $\overrightarrow{K} = K\hat{z}$
  - El plano coincide con el plano yz

$$\overrightarrow{B}(\vec{r}) = \frac{\mu_0}{4\pi}\int_{S'} \frac{\overrightarrow{K}(\vec{r}\,') \times (\vec{r} - \vec{r}\,')}{|\vec{r} - \vec{r}\,'|^3} dS'$$

$$\vec{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \quad \vec{r}\,' = \begin{pmatrix} 0 \\ y' \\ z' \end{pmatrix}$$

$$\Longrightarrow \overrightarrow{B}(\vec{r}) = \frac{\mu_0 K}{4\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{x\hat{y} - (y-y')\hat{x}}{[x^2 + (y-y')^2 + (z-z')^2]^{3/2}} dy'dz'$$

$$\overrightarrow{B}(\vec{r}) = \frac{\mu_0 K}{4\pi}\hat{y}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{x d\xi d\eta}{[x^2 + \xi^2 + \eta^2]^{3/2}}$$

$$\overrightarrow{B}(\vec{r}) = sign(x)\frac{\mu_0 K}{4\pi}\hat{y}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{|x| d\xi d\eta}{[x^2 + \xi^2 + \eta^2]^{3/2}}$$

$$\overrightarrow{B}(\vec{r}) = sign(x)\frac{\mu_0 K}{2}\hat{y}$$

2.4 FUERZA DE LORENTZ

- De la definición de fuerza entre corrientes y la ley de Biot-Savart podemos encontrar la fuerza que ejerce un campo magnético sobre cargas en movimiento

$$\vec{F} = \int_V \vec{j} \times \vec{B}\, dV$$

- Ahora, consideremos la densidad de corriente en el límite de cargas individuales q_k que se mueven con velocidades $\dot{\vec{r}}_k(t)$

$$\vec{j}(\vec{r}, t) = \sum_{k=1}^{N} q_k \vec{v}_k \delta(\vec{r} - \vec{r}_k)$$

Fuerza magnética sobre cargas puntuales

$$\boxed{\vec{F} = \sum_{k=1}^{N} q_k \vec{v}_k \times \vec{B}(\vec{r}_k)}$$

La fuerza sobre una partícula q y velocidad $\vec{v}$ en presencia de un campo eléctrico $\vec{E}$ y magnético $\vec{B}$ es

Fuerza de Lorentz

$$\boxed{\vec{F} = q\left(\vec{E} + \vec{v} \times \vec{B}\right)}$$

Nota: La fuerza magnética no realiza trabajo sobre una partícula en un campo magnético

2.4 FUERZA DE LORENTZ

- El campo magnético $\vec{B}$ ejerce fuerza y torque sobre una densidad de corriente

$$\Longrightarrow \quad \vec{F} = \int_V \vec{j}(\vec{r}) \times \vec{B}(\vec{r}) dV$$

$$\vec{N} = \int_V \vec{r} \times \left[\vec{j}(\vec{r}) \times \vec{B}(\vec{r}) \right] dV$$

Si $\quad \vec{j}(\vec{r},t) = \sum_{k=1}^{N} q_k \vec{v}_k \delta(\vec{r} - \vec{r}_k)$

$$\vec{F} = \sum_{k=1}^{N} q_k \vec{v_k} \times \vec{B}(\vec{r_k})$$

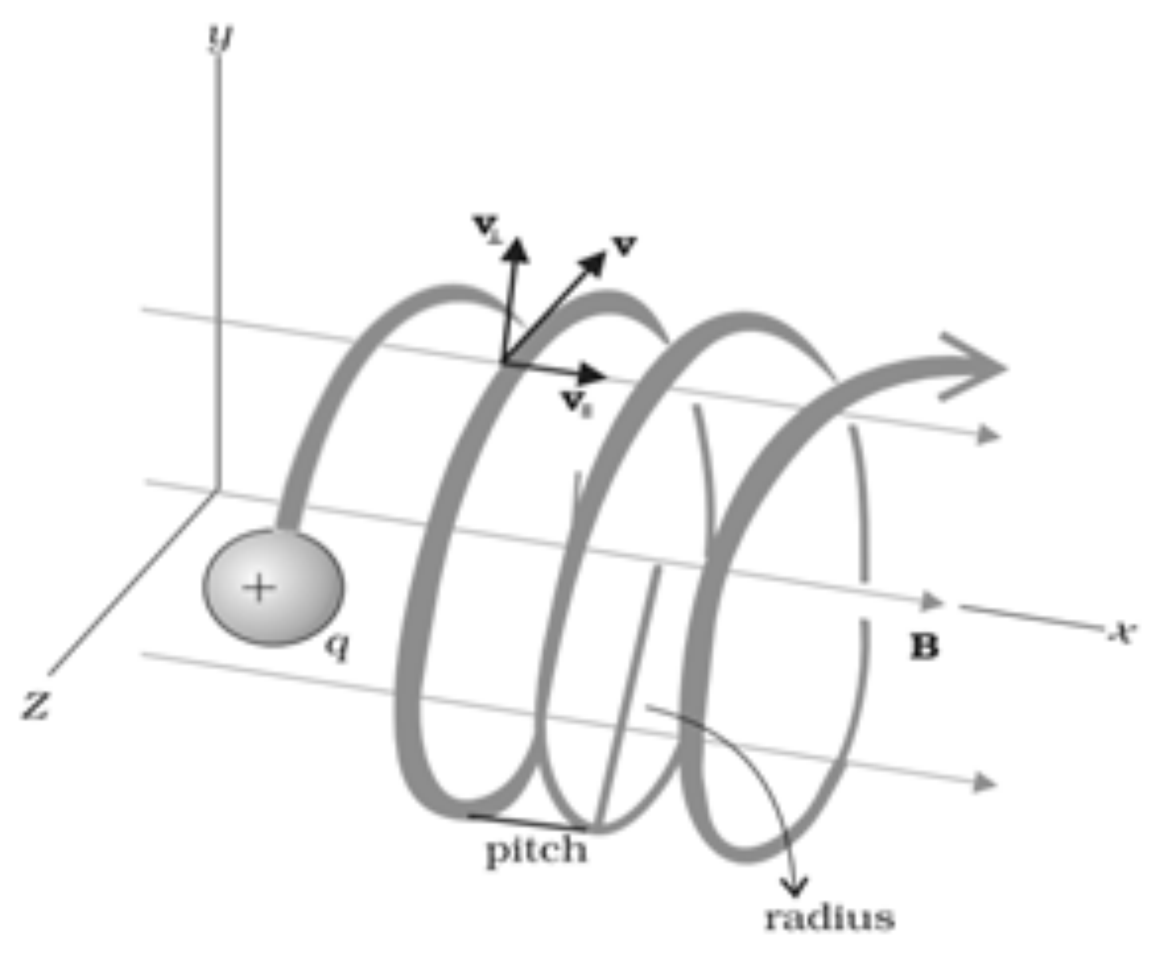
Para una partícula cargada

$$m\frac{d\vec{v}}{dt} = q\vec{v} \times \vec{B}(\vec{r})$$

con $\quad \vec{v} = \vec{v}_\perp + \vec{v}_\parallel$

$$\Longrightarrow \quad v_\parallel = const.$$

$$\vec{r}_\perp = \begin{pmatrix} x_0 + R\cos\omega_c t \\ y_0 + R\sin\omega_c t \end{pmatrix}$$



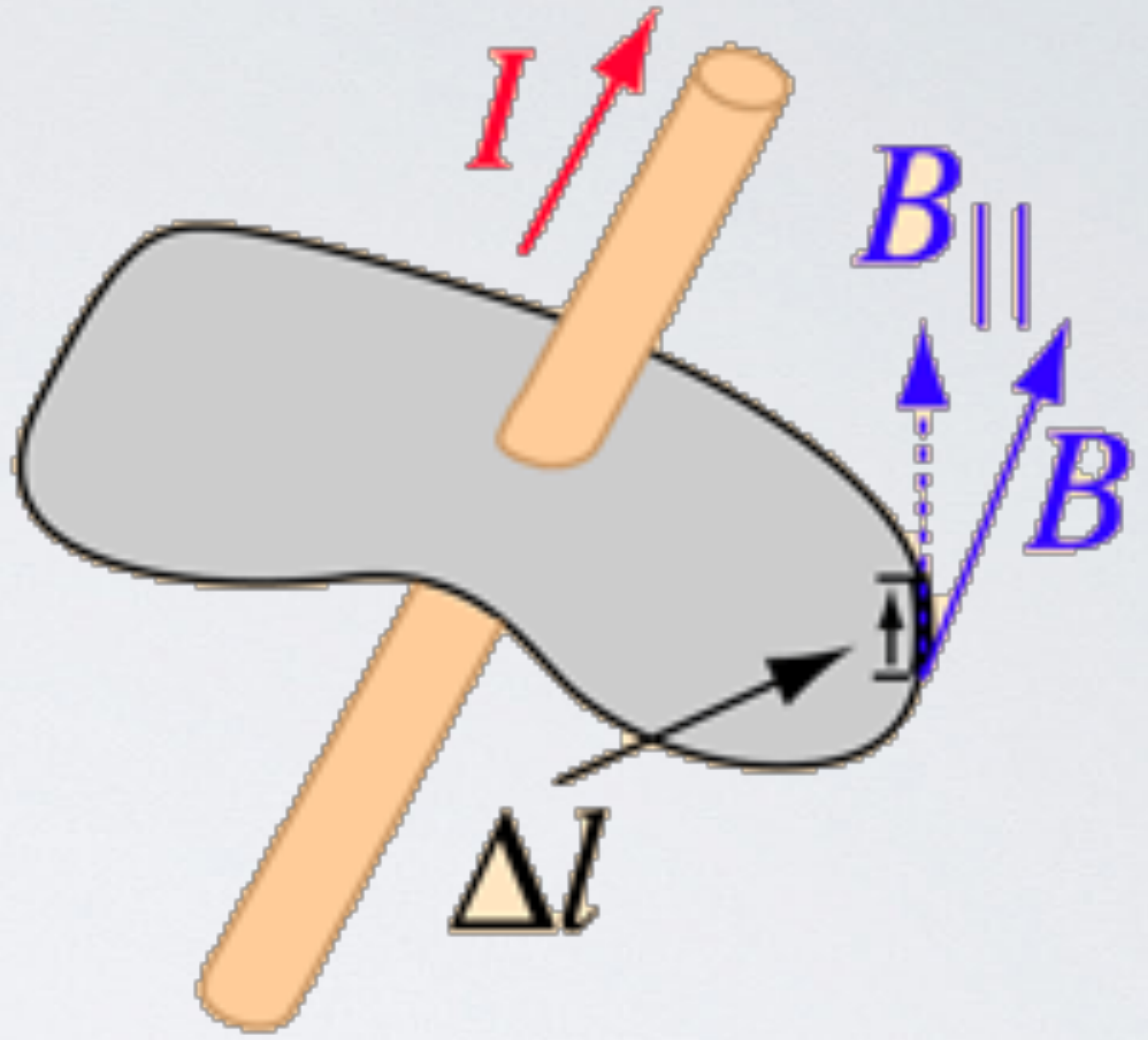
$$\omega_c = \frac{qB}{m} \quad \text{frecuencia de ciclotrón}$$

$$R = \frac{v_\perp}{|\omega_c|} \quad \text{radio de ciclotrón}$$

2.5 LEY DE AMPÈRE

- Una relación fundamental en magnetostática, derivada de la ley de Biot-Savart es $\nabla \times \overrightarrow{B}(\vec{r}) = \mu_0 \vec{j}(\vec{r})$

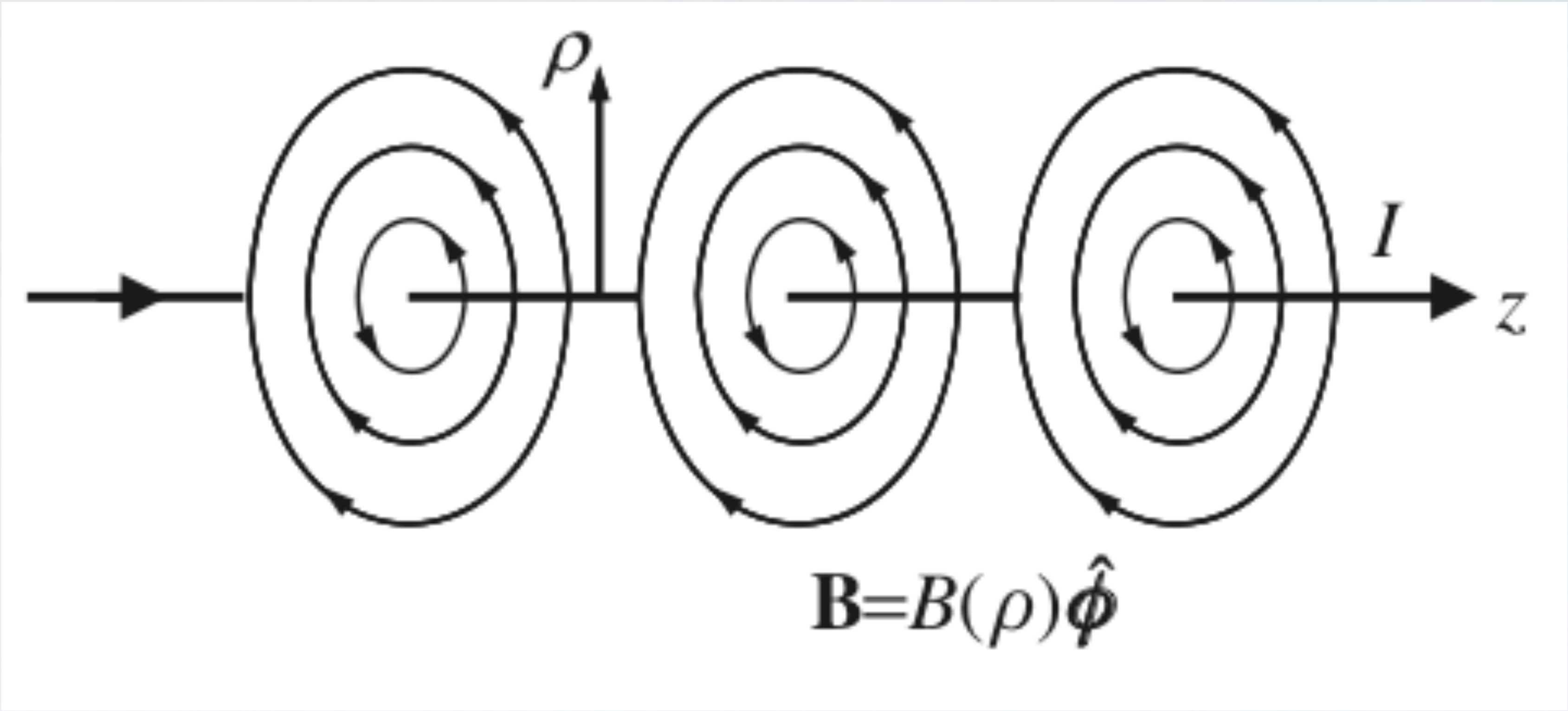
- Integrando ambos lados sobre la superficie S y tomando el teorema de Stokes, encontramos



$$\oint_C \overrightarrow{B} \cdot d\vec{l} = \mu_0 \int_S \vec{j} \cdot d\overrightarrow{S} = \mu_0 I_C$$

El signo de la corriente está fijado por la regla de la mano derecha; cerrando los dedos en la dirección de la curva C parametrizada

2.5 LEY DE AMPÈRE



- El campo magnético producido por una corriente estacionaria es fácil de encontrar
  - Una línea infinita de corriente

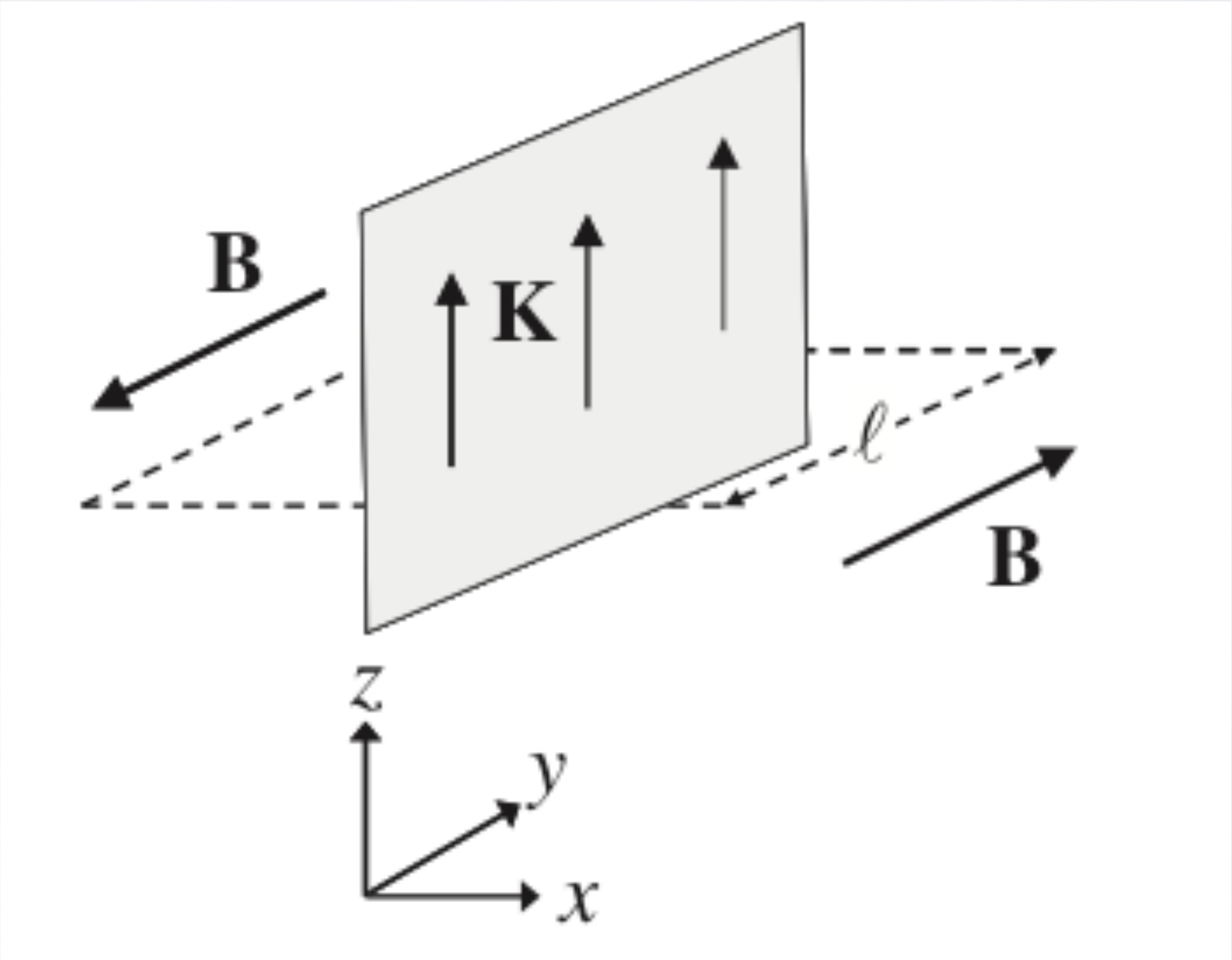
$$\oint_C \vec{B}\cdot d\vec{l} = \mu_0 I_C$$

$$\int_0^{2\pi} B\rho d\varphi = \mu_0 I_C \quad \Rightarrow \quad \vec{B}(\rho) = \frac{\mu_0 I_C}{2\pi\rho}\hat{\varphi}$$

  - Un plano infinito de corriente

$$\oint_C \vec{B}\cdot d\vec{l} = \mu_0 \int_S \vec{j}\cdot d\vec{S}, \quad \vec{j} = K\delta(x)\hat{z}$$

$$\int_{-l/2}^{l/2} 2Bdy = \mu_0 \int_{-l/2}^{l/2}\int_{x'/2}^{x'/2} K\delta(x)dxdy = \mu_0 Kl \quad \Rightarrow \quad \vec{B} = \frac{1}{2}\mu_0 \vec{K}\times\hat{S}$$

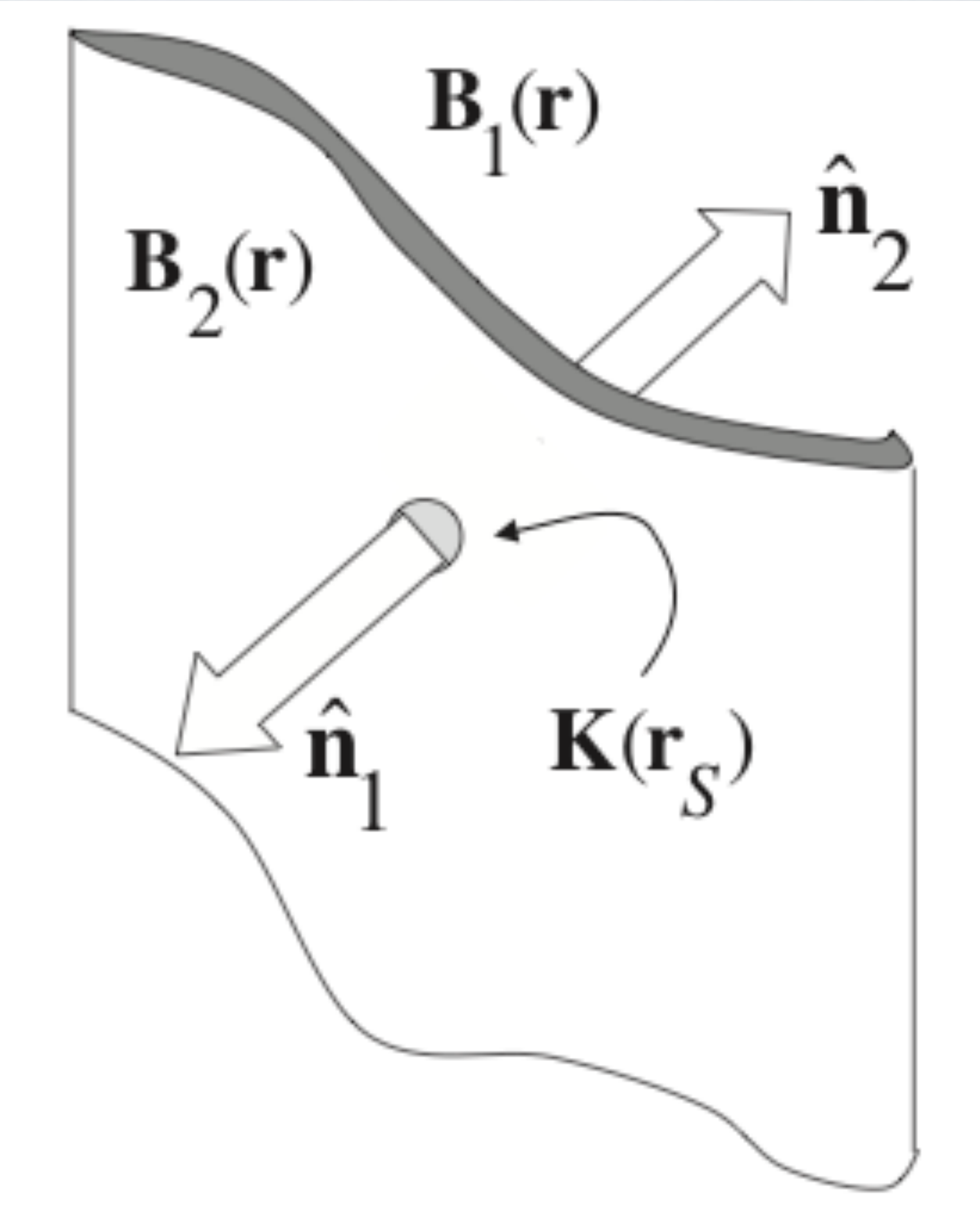


3.1 CONDICIONES DE FRONTERA

- Usando las expresiones para la divergencia y el rotacional del campo magnético,

$$\nabla \cdot \vec{B}(\vec{r}) = 0$$
$$\nabla \times \vec{B}(\vec{r}) = \mu_0 \vec{j}(\vec{r})$$

- Inmediatamente podemos encontrar las condiciones de frontera a satisfacer



$$\hat{n} \cdot \left[\vec{B}_1 - \vec{B}_2\right] = 0$$

$$\hat{n} \times \left[\vec{B}_1 - \vec{B}_2\right] = \mu_0 \vec{K}(\vec{r}_S)$$

3.2 POTENCIAL ESCALAR MAGNÉTICO

- Una importante consecuencia de la ley de Ampère es que en cualquier volumen del espacio donde la densidad de corriente es idénticamente cero se cumple $\nabla \times \overrightarrow{B} = 0$

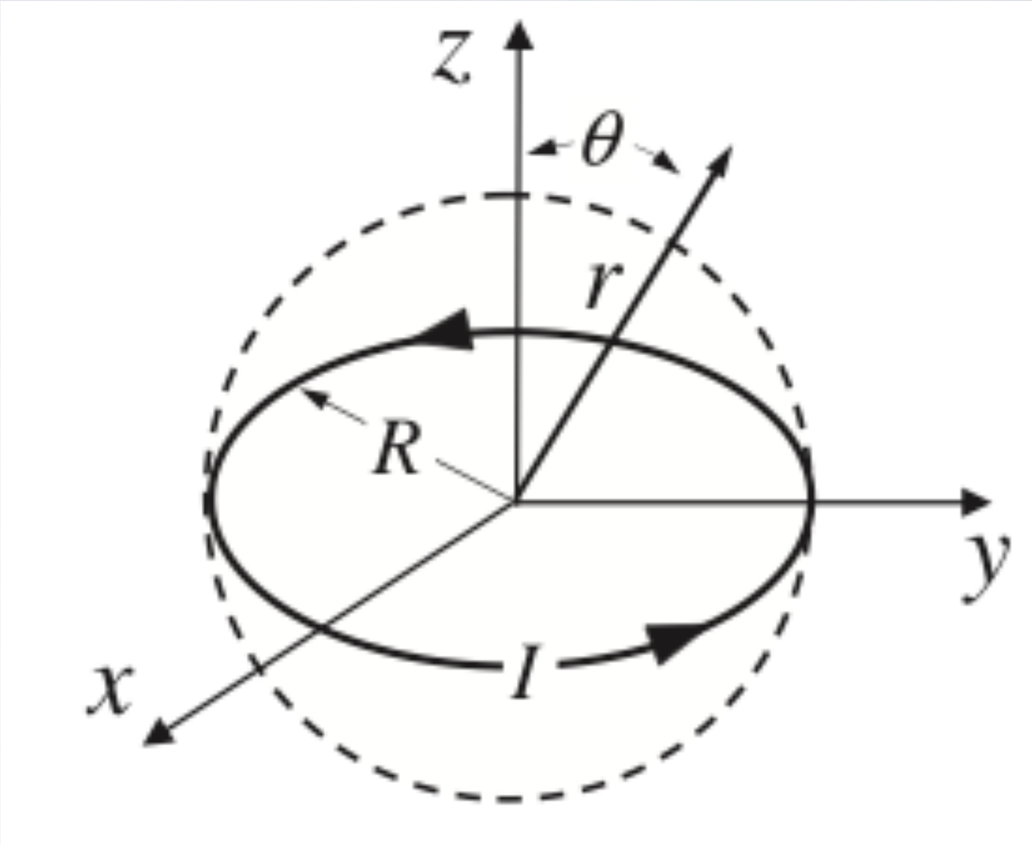
  - En el interior de ese volumen podemos definir un potencial escalar magnético $\psi(\vec{r})$ tal que $\overrightarrow{B}(\vec{r}) = -\nabla\psi(\vec{r}), \quad \vec{r} \in V$

  - Además, dado que $\nabla \cdot \overrightarrow{B}(\vec{r}) = 0$ en cualquier parte, tenemos $\nabla^2\psi(\vec{r}) = 0, \quad \vec{r} \in V$

Satisface la ecuación de Laplace

Podemos usar los métodos de la teoría de potencial para encontrar una solución a cualquier problema de magnetostática donde la corriente es nula

3.2 POTENCIAL ESCALAR MAGNÉTICO



- Ejemplo: Consideremos un anillo que tiene una corriente I

Solución.

En analogía con los problemas de electrostática de una anillo con densidad lineal de carga uniforme

$$\psi(r,\theta) = \begin{cases} \sum_{\ell=1}^{\infty} A_\ell \left(\frac{r}{R}\right)^\ell P_\ell(\cos\theta) & r < R, \\ \sum_{\ell=1}^{\infty} B_\ell \left(\frac{R}{r}\right)^{\ell+1} P_\ell(\cos\theta) & r > R. \end{cases} \qquad B_\ell = -\frac{\ell}{\ell+1} A_\ell .$$

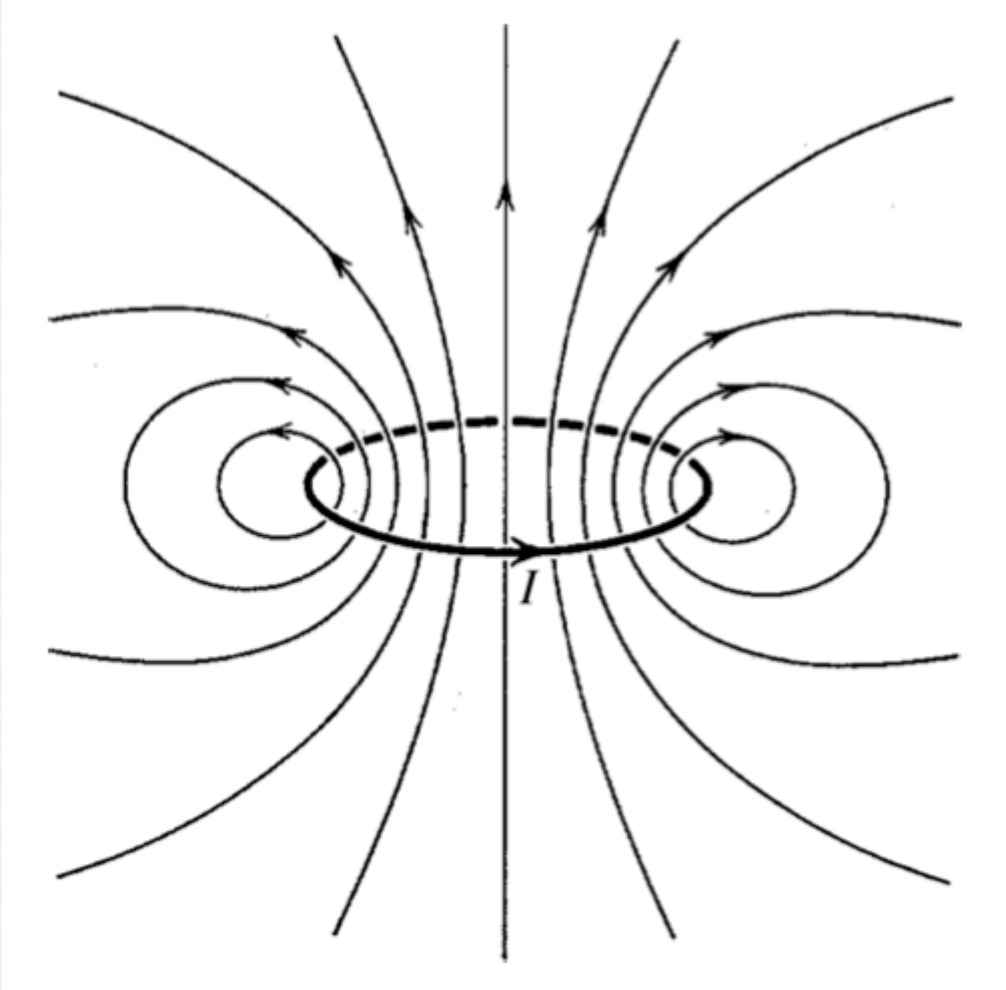
$$\psi(A) - \psi(B) = \pm\mu_0 I$$

Usando el hecho que $B_z(z) = \frac{\mu_0 I}{2}\frac{R^2}{(R^2+z^2)^{3/2}}$ y $\psi(A) - \psi(B) = \int_A^B d\boldsymbol{\ell}\cdot\mathbf{B}.$

$\Longrightarrow$

$$\psi(z) = -\frac{\mu_0 I}{2}\frac{z}{\sqrt{R^2+z^2}}. \Longrightarrow \psi(z) = -\frac{\mu_0 I}{2}\sum_{\ell=1}^{\infty}\left(\frac{z}{R}\right)^\ell P_{\ell-1}(0). \qquad z < R$$

$$\Longrightarrow \psi(r,\theta) = \begin{cases} -\frac{\mu_0 I}{2}\sum_{\ell=1,3,\ldots}^{\infty}\left(\frac{r}{R}\right)^\ell P_{\ell-1}(0) P_\ell(\cos\theta) & r < R, \\ -\frac{\mu_0 I}{2}\sum_{\ell=1,3,\ldots}^{\infty}\left(\frac{R}{r}\right)^{\ell+1} P_{\ell+1}(0) P_\ell(\cos\theta) & r > R. \end{cases}$$



3.3 POTENCIAL VECTORIAL

- El formalismo del potencial escalar magnético $\overrightarrow{B} = -\nabla\psi$ no es válido en puntos del espacio donde $\vec{j} \neq \overrightarrow{0}$

    - Podemos usar la condición de divergencia cero para asumir que existe un potencial vectorial que cumple con $\overrightarrow{B}(\vec{r}) = \nabla \times \overrightarrow{A}(\vec{r})$

Una consecuencia inmediata de la existencia de un potencial vectorial es que se puede expresar el flujo magnético en términos de una integral de línea

$$\Phi_B = \int_S \overrightarrow{B} \cdot d\overrightarrow{S} = \int_S (\nabla \times \overrightarrow{A}) \cdot d\overrightarrow{S} = \int_C \overrightarrow{A} \cdot d\vec{l}$$

3.3 POTENCIAL VECTORIAL

- Podemos ver que el potencial vectorial no está definido de manera única
  - Si cualquier función escalar $\chi(\vec{r})$ es añadida al potencial vectorial $\overrightarrow{A}(\vec{r})$ produce exactamente el mismo campo magnético que el potencial

$$\boxed{\overrightarrow{A}'(\vec{r}) = \overrightarrow{A}(\vec{r}) + \nabla\chi(\vec{r})}$$

Esto nos sugiere una idea muy poderosa: Encontrar una norma $\chi(\vec{r})$ que simplifique los cálculos

Podemos imponerle algunas constricciones sobre los potenciales vectoriales

Norma de Coulomb

$$\nabla \cdot \overrightarrow{A}(\vec{r}) = 0$$

$$\nabla \cdot \overrightarrow{A} \neq 0, \quad \nabla \cdot \overrightarrow{A}' = 0$$

$$\nabla^2\chi(\vec{r}) = -\nabla \cdot \overrightarrow{A}(\vec{r})$$

3.4 LA ECUACIÓN DE POISSON VECTORIAL

- De la expresión $\nabla \times \overrightarrow{B} = \mu_0 \vec{j}$ y usando el hecho que $\overrightarrow{B} = \nabla \times \overrightarrow{A}$ $\Longrightarrow$

$$\nabla \times \left(\nabla \times \overrightarrow{A} \right) = \mu_0 \vec{j}$$

$$\Longrightarrow \; \partial_i \partial_j A_j - \partial_j \partial_j A_i = \mu_0 j_i$$

$$\nabla \left(\nabla \cdot \overrightarrow{A} \right) - \nabla^2 \overrightarrow{A} = \mu_0 \vec{j}$$

Si pedimos que el potencial vectorial cumpla con la norma de Coulomba

$$\nabla^2 \overrightarrow{A} = - \mu_0 \vec{j}$$

Los componentes del potencial vectorial satisfacen la ecuación de Poisson

$$A_k = \frac{\mu_0}{4\pi} \int_{V'} \frac{j_k}{|\vec{r} - \vec{r}'|} dV'$$

$$\overrightarrow{A} = \frac{\mu_0}{4\pi} \int_{V'} \frac{\vec{j}}{|\vec{r} - \vec{r}'|} dV'$$

Densidad de corriente volumétrica

$$\overrightarrow{A} = \frac{\mu_0}{4\pi} \int_{S'} \frac{\overrightarrow{K}}{|\vec{r} - \vec{r}'|} dS'$$

Densidad de corriente superficial

$$\overrightarrow{A} = \frac{\mu_0}{4\pi} \int_{C} \frac{I}{|\vec{r} - \vec{r}'|} d\vec{l}$$

Corriente filamentaria

3.4 LA ECUACIÓN DE POISSON VECTORIAL

- Ejemplo: Encontrar el potencial vectorial y el campo magnético producido por una corriente lineal infinita

Solución:

Usando la definición de potencial vectorial

$$\overrightarrow{A}(\vec{r}) = \frac{\mu_0}{4\pi}\int_C \frac{I}{|\vec{r}-\vec{r}'|}d\vec{l}$$

con

$$d\vec{l} = \hat{z}dl \qquad \vec{r}' = l\hat{z}$$

$$\overrightarrow{A}(\vec{r}) = \frac{I\mu_0}{4\pi}\hat{z}\int_{-L_1}^{L2}\frac{1}{\sqrt{\rho^2+(z-l)^2}}dl$$

$$\overrightarrow{A} = \frac{I\mu_0}{4\pi}\hat{z}\ln\left\{\frac{\left[\rho^2+(z+L_1)^2\right]^{1/2}+z+L_1}{\left[\rho^2+(z-L_2)^2\right]^{1/2}+z-L_2}\right\}$$

$$\overrightarrow{B} = \nabla\times\overrightarrow{A}$$

$$B_\varphi = \frac{\partial A_z}{\partial\rho}$$

$$\overrightarrow{B} = \frac{I\mu_0}{4\pi\rho}\hat{\varphi}\left\{\frac{z+L_1}{\left[\rho^2+(z+L_1)^2\right]^{1/2}} - \frac{z-L_2}{\left[\rho^2+(z-L_2)^2\right]^{1/2}}\right\}$$

$$L_i \to \infty$$

$$\overrightarrow{B}(\vec{r}) = \frac{\mu_0 I}{2\pi\rho}\hat{\varphi}$$

3.5 EXPANSIÓN MULTIPOLAR MAGNÉTICA

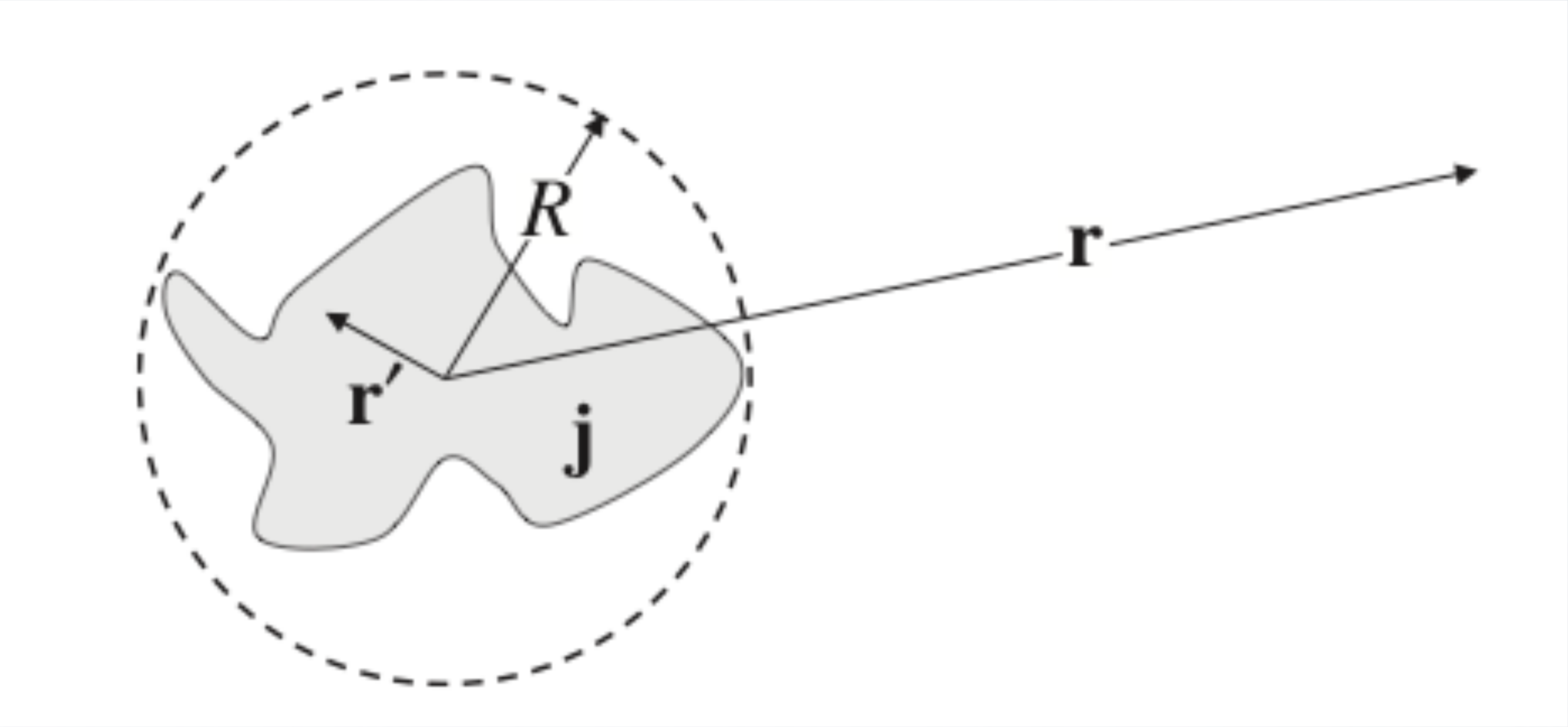
- La expresión del potencial vectorial sugiere hacer una aproximación multipolar

$$\overrightarrow{A}(\vec{r}) = \frac{\mu_0}{4\pi}\int_{V'} \frac{\vec{j}(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}dV'$$

- De esta relación es claro que se puede hacer una expansión exterior o interior
  - La expansión exterior surge cuando el punto de observación se encuentra fuera de un volumen finito que contiene una fuente de corriente
  - La expansión interior surge cuando el punto de observación se encuentra dentro de un volumen finito que excluye la fuente de la corriente

3.5 EXPANSIÓN MULTIPOLAR MAGNÉTICA

- Empecemos con el caso donde el punto de observación se encuentra fuera de una esfera de radio R que contiene una distribución de corriente



$$\frac{1}{|\vec{r}-\vec{r}'|} = \frac{1}{r} + \frac{\vec{r}'\cdot\vec{r}}{r^3} + \cdots$$

$$A_k(\vec{r}) = \frac{\mu_0}{4\pi}\left[\frac{1}{r}\int_{V'} j_k(\vec{r}')dV' + \frac{\vec{r}}{r^3}\cdot\int_{V'}\vec{r}' j_k(\vec{r}')dV' + \cdots\right]$$

3.5 EXPANSIÓN MULTIPOLAR MAGNÉTICA

- El primer término de la expansión magnética (término monopolar) siempre es idénticamente cero
  - The reason is the constraint imposed on the three components of the current density j_k by the steady-current condition

$$\nabla \cdot \vec{j}(\vec{r}) = 0$$

Debemos notar que

$$\nabla' \cdot (r_k' \vec{j}) = r_k' (\nabla' \cdot \vec{j}) + \vec{j} \cdot \nabla' r_k' = j_k$$

$$\int_{\mathbb{R}^3} j_k(\vec{r}')dV' = \int_{\mathbb{R}^3} \nabla'(r_k \vec{j})dV' = \int_{\partial\mathbb{R}^3} (r_k' \vec{j}) \cdot d\vec{S} = 0$$

3.6 MOMENTO DIPOLAR MAGNÉTICO

- From the expansion, we have that for long distances, the second term gives the behavior of the vector potential

$$A_k(\vec{r}) = \frac{\mu_0}{4\pi}\frac{r_l}{r^3}\left[\int_{V'} r'_l j_k(\vec{r}')dV'\right] = \frac{\mu_0}{4\pi}T_{kl}\frac{r_l}{r^3}$$

with

$$T_{kl} = \int_{V'} r'_l j_k(\vec{r}')dV'$$

Nine integrals that constitutes the magnetic dipole term

using

$$\nabla' \cdot (r'_l r'_k \vec{j}) = r'_l r'_k \nabla' \cdot \vec{j} + r'_l j_k + r'_k j_l = r'_l j_k + r'_k j_l$$

and $\epsilon_{lki}(\vec{r}' \times \vec{j})_i = r'_l j_k - r'_k j_l$

$$r'_l j_k = \frac{1}{2}\left[\nabla' \cdot \left(r'_l r'_k \vec{j}\right) + \epsilon_{lki}\left(\vec{r}' \times \vec{j}\right)\right]$$

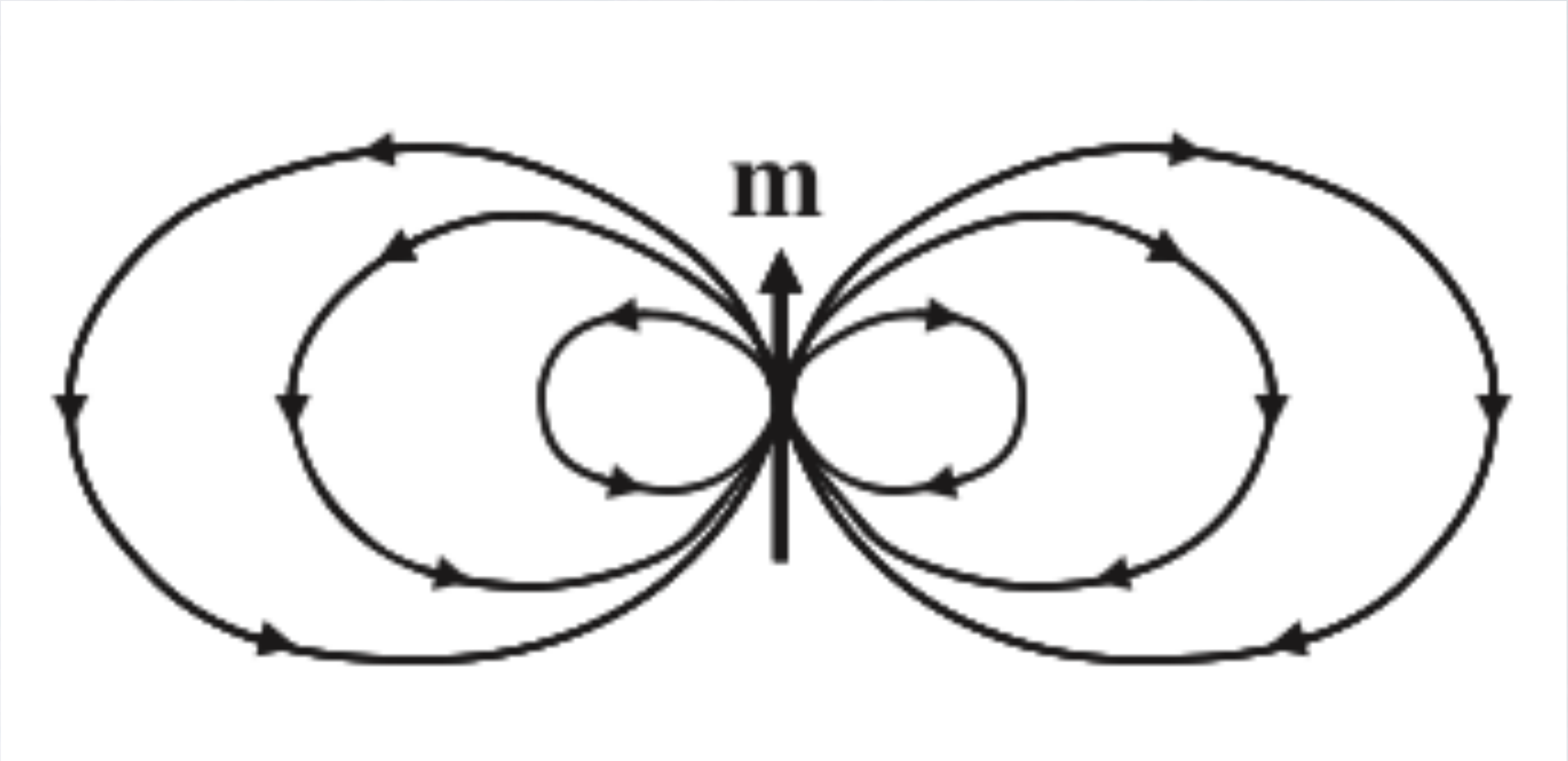
$$\boxed{T_{kl} = \int_{V'} j_k r'_l dV' = \frac{1}{2}\epsilon_{kil}\int\left(\vec{r}' \times \vec{j}\right)_i dV' = \epsilon_{kil} m_i}$$

$$\boxed{\vec{m} = \frac{1}{2}\int_V \vec{r} \times \vec{j}(\vec{r})dV}$$

Magnetic dipole moment vector

3.6 MOMENTO DIPOLAR MAGNÉTICO

- Entonces, el potencial vectorial para $r>>R$



$$\boxed{\overrightarrow{A}(\vec{r}) = \frac{\mu_0}{4\pi}\frac{\overrightarrow{m} \times \vec{r}}{r^3}, \quad r \gg R}$$

$$\Downarrow$$

$$\overrightarrow{B}(\vec{r}) = \nabla \times \left(\frac{\mu_0}{4\pi}\frac{\overrightarrow{m} \times \vec{r}}{r^3}\right) = \frac{\mu_0}{4\pi}\left[\overrightarrow{m}\left(\nabla \cdot \frac{\vec{r}}{r^3}\right) - \left(\overrightarrow{m} \cdot \nabla\right)\frac{\vec{r}}{r^3}\right]$$

$$\Downarrow$$

$$\boxed{\overrightarrow{B}(\vec{r}) = \frac{\mu_0}{4\pi}\frac{3\hat{r}\left(\hat{r} \cdot \overrightarrow{m}\right) - \overrightarrow{m}}{r^3}}$$

Campo magnético de dipolo magnético

3.6 MOMENTO DIPOLAR MAGNÉTICO

- From the expansion, we have that for long distances, the second term gives the behavior of the vector potential

$$A_k(\vec{r}) = \frac{\mu_0}{4\pi}\frac{r_l}{r^3}\left[\int_{V'} r'_l j_k(\vec{r}')dV'\right] = \frac{\mu_0}{4\pi}T_{kl}\frac{r_l}{r^3}$$

with

$$T_{kl} = \int_{V'} r'_l j_k(\vec{r}')dV' \longleftarrow$$

Nine integrals that constitutes the magnetic dipole term

using

$$\nabla'\cdot(r'_l r'_k \vec{j}) = r'_l r'_k \nabla'\cdot\vec{j} + r'_l j_k + r'_k j_l = r'_l j_k + r'_k j_l$$

and $\epsilon_{lki}(\vec{r}'\times\vec{j})_i = r'_l j_k - r'_k j_l$

$$r'_l j_k = \frac{1}{2}\left[\nabla'\cdot\left(r'_l r'_k \vec{j}\right) + \epsilon_{lki}\left(\vec{r}'\times\vec{j}\right)\right]$$

$$\boxed{T_{kl} = \int_{V'} j_k r'_l dV' = \frac{1}{2}\epsilon_{kil}\int\left(\vec{r}'\times\vec{j}\right)_i dV' = \epsilon_{kil}m_i}$$

$$\boxed{\vec{m} = \frac{1}{2}\int_V \vec{r}\times\vec{j}(\vec{r})dV}$$

Magnetic dipole moment vector

3.6 MOMENTO DIPOLAR MAGNÉTICO

- We found the tensor T_{kl} in the vector potential and we ended with three component os the magnetic dipole moment

  - From the definition

$$T_{kl} = \epsilon_{kil} m_i \implies T_{kl} = -T_{lk}$$

  - We can find

$$m_j = \frac{1}{2}\epsilon_{jkl} T_{lk}$$

The asymmetric part of a tensor

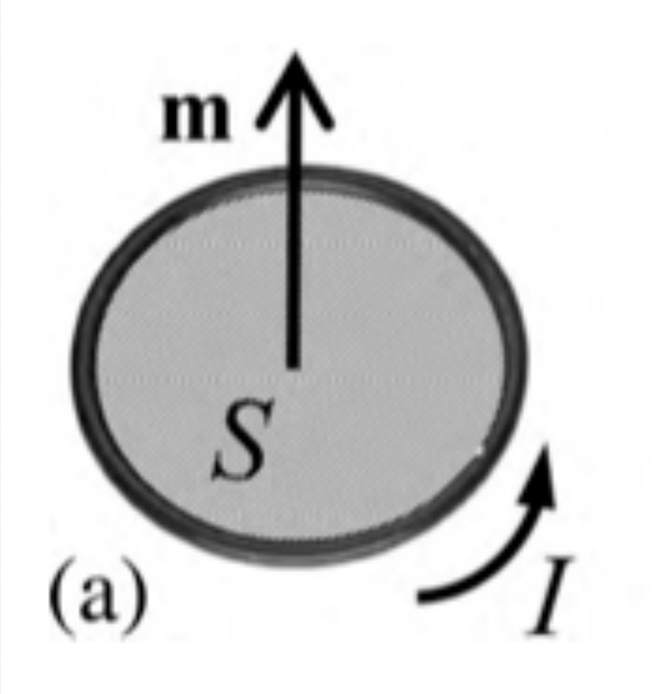
For a filamentary loop of arbitrary shape which carries steady current I around its circuit C

$$\vec{m} = \frac{1}{2} I \oint \vec{r} \times d\vec{l}$$

Using Stoke's theorem for $\vec{F}(\vec{r}) = \vec{A}(\vec{r}) \times \vec{c}$

$$\implies \oint d\vec{l} \times \vec{A} = \int_S \nabla A_k dS_k - \int_S \left(\nabla \cdot \vec{A}\right) d\vec{S}$$

$$\mathbf{m} = \tfrac{1}{2} I \int_S d\mathbf{S}\, \nabla \cdot \mathbf{r} - \tfrac{1}{2} I \int_S dS_k \nabla r_k = I \int_S d\mathbf{S} \equiv I\mathbf{S}.$$

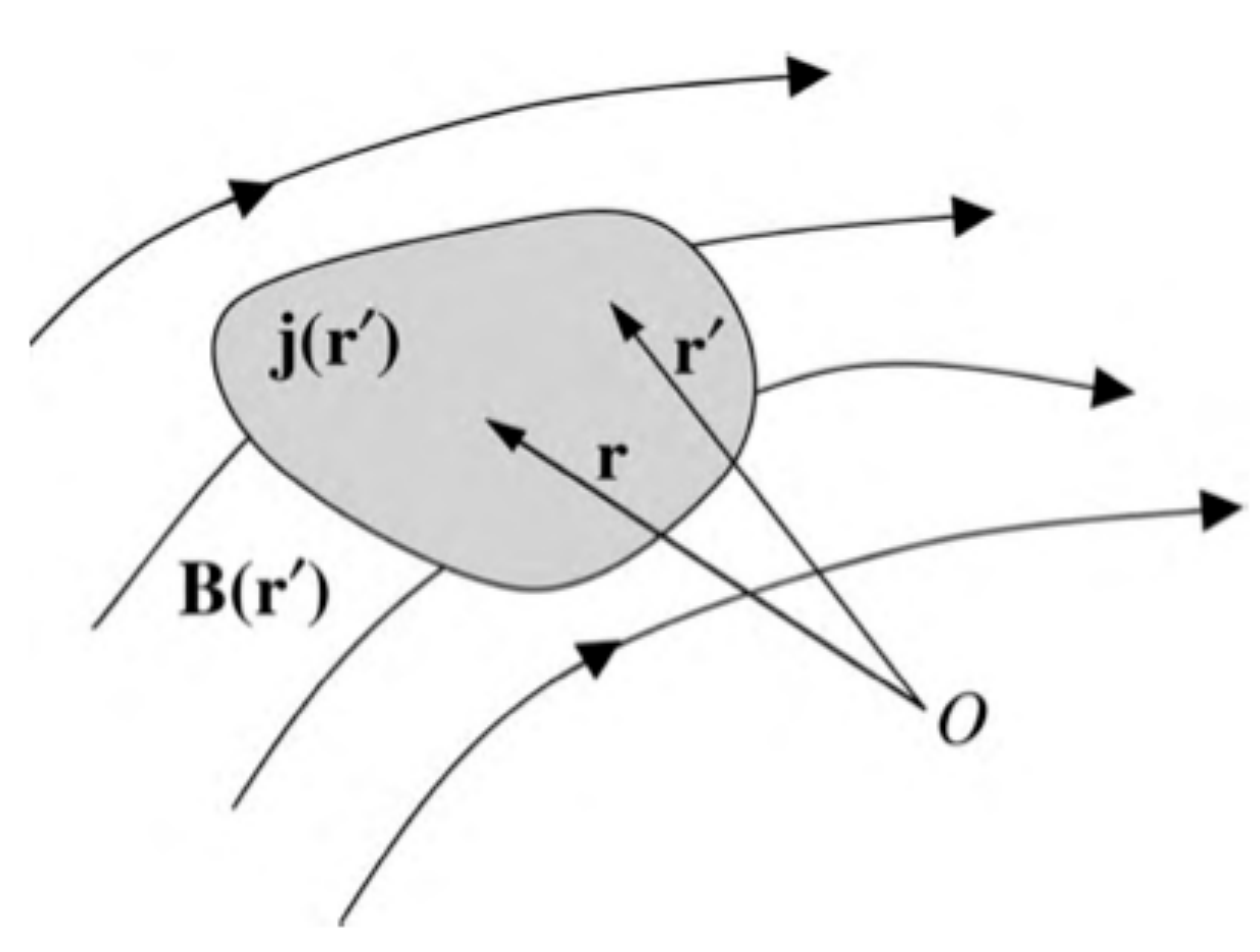


$\implies$ For a planar loop

$$\vec{m} = IA\hat{n}$$

FORCE ON A MAGNETIC DIPOLE

- Considere a current distribution $\vec{j}(\vec{r}')$ in presence of a magnetic field $\vec{B}(\vec{r}')$



  - The Lorentz force which the field exerts on the current is

$$\vec{F} = \int \vec{j}(\vec{r}') \times \vec{B}(\vec{r}')dV'$$

- If the magnitude and direction do not change very much over the volume of the distribution

  - Taylor expansion around reference point

$$\vec{B}(\vec{r}') = \vec{B}(\vec{r}) + [(\vec{r} - \vec{r}') \cdot \nabla]\vec{B}(\vec{r}) + \cdots$$

using $\int j_k r'_l dV' = -\frac{1}{2}\epsilon_{kli}\int (\vec{r}' \times \vec{j})_i dV'$

$$\vec{m} = \frac{1}{2}\int \vec{r} \times \vec{j}(\vec{r})dV$$

$$\vec{F} = m_k \nabla B_k - \vec{m}\nabla \cdot \vec{B}$$

$$\boxed{\vec{F} = m_k \nabla B_k}$$

If $\vec{m}$ is constant

$$\Longrightarrow \vec{F} = \nabla(\vec{m} \cdot \vec{B}) \quad \Longrightarrow \quad \boxed{V_B = -\vec{m} \cdot \vec{B}(\vec{r})}$$

Magnetic potential energy

SPHERICAL EXPANSION FOR $\psi(\vec{r})$

- In spherical coordinates, it is simplest to generate an exterior spherical multipole expansion for the magnetic scalar potential

- Using the fact $\overrightarrow{B} = -\nabla\psi$

$$\implies \vec{r}\cdot\overrightarrow{B} = -r\frac{\partial\psi}{\partial r}$$

using $\nabla\times\overrightarrow{B} = \mu_0\vec{j}$

and

$$\nabla\times\left(\nabla\times\vec{f}\right) = \nabla(\nabla\cdot\vec{f}) - \nabla^2\vec{f}$$

$$\implies \nabla^2\left(\vec{r}\cdot\overrightarrow{B}\right) = -\mu_0\vec{r}\cdot\nabla\times\vec{j}$$

Poisson's equation

$$\vec{r}\cdot\overrightarrow{B} = \frac{\mu_0}{4\pi}\int\frac{\vec{r}'\cdot\nabla'\times\vec{j}(\vec{r}')}{|\vec{r}-\vec{r}'|}dV'$$

Using

$$\frac{1}{|\vec{r}-\vec{r}'|} = \frac{1}{r}\sum_{l_0}^{\infty}\sum_{m=-l}^{l}\frac{4\pi}{2l+1}\left(\frac{r'}{r}\right)^l Y^*_{lm}(\Omega')\,Y_{lm}(\Omega)$$

$$r' < r$$

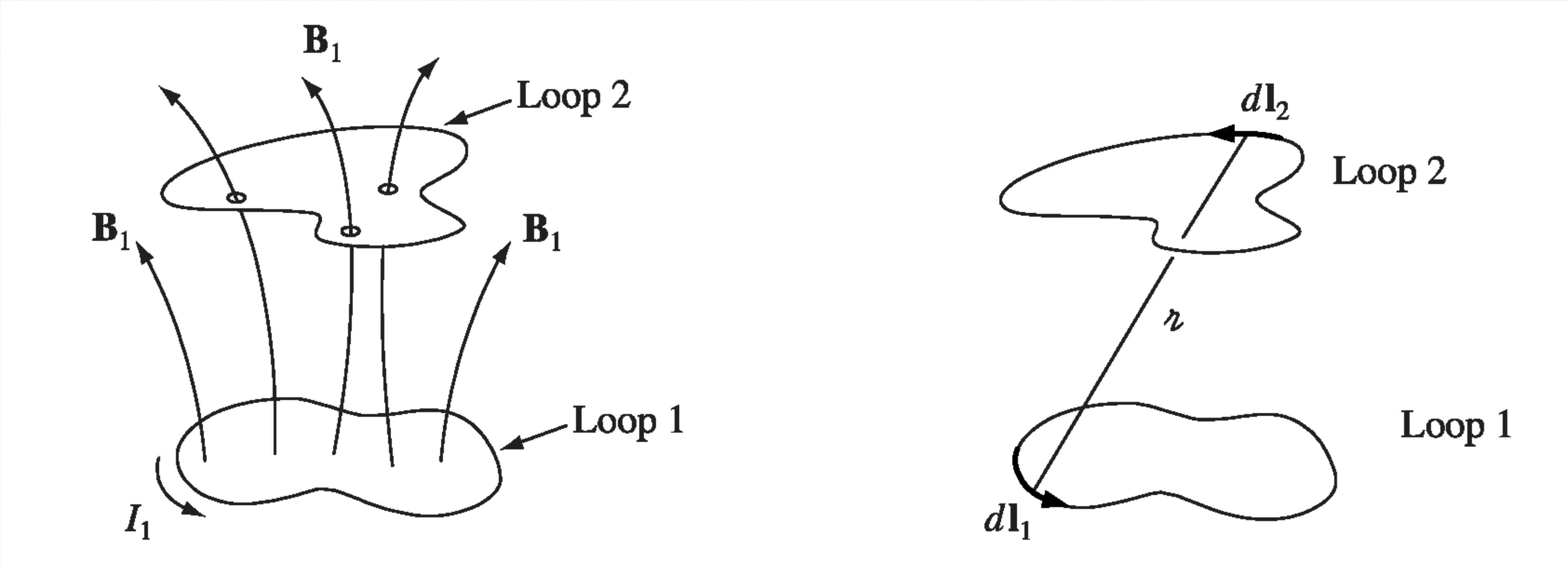
$$\psi(\vec{r}) = \frac{\mu_0}{4\pi}\sum_{l=1}^{\infty}\sum_{m=-l}^{l}\frac{4\pi}{2l+1}M_{lm}\frac{Y_{lm}(\Omega)}{r^{l+1}}$$

with

$$M_{lm} = \frac{1}{l+1}\int r^l Y^*_{lm}(\Omega)\vec{r}\cdot\left[\nabla\times\vec{j}(\vec{r})\right]dV$$

Spherical magnetic multipole moment

MUTUAL INDUCTANCE



- Suppose that we have two loops of wire C_j and C_k with respective currents I_j and I_k
  - The steady current I_j around loop j produces a magnetic field $\boldsymbol{B_j}$
    - The field lines of $\boldsymbol{B_j}$ pass through loop k
    - By means of the Biot-Savart law we can calculate the magnetic field and the flux throughout loop k
    - In order to simplify the expression we can use the vector potential

$$\vec{B}_j = \frac{\mu_0}{4\pi} I_j \oint \frac{d\vec{l}_j \times (\vec{r} - \vec{l})}{|\vec{r} - \vec{l}|^3}$$

$$\Phi_{j\to k} = \oint_{C_k} \vec{A}_j(\vec{r}_k) \cdot d\vec{l}_k = \frac{\mu_0}{4\pi} \oint_{C_k} \oint_{C_j} \frac{I_j d\vec{l}_k \cdot d\vec{l}_j}{|\vec{r}_j - \vec{r}_k|}$$

MUTUAL INDUCTANCE

- We can define a important geometrical property of a system known as <u>inductance</u>
  - Inductance is related with the capacity of produce an electromotive force by means of the magnetic field
  - The change in the current of one circuit affects the second circuit and viceversa
  - The mutual inductance can be either positive or negative depending on the chosen orientations of the circuits

$$\Phi_{j\to k} = \frac{\mu_0}{4\pi}\oint_{C_k}\oint_{C_j}\frac{I_j d\vec{l}_k \cdot d\vec{l}_j}{|\vec{r}_j - \vec{r}_k|}$$

$$\Downarrow$$

$$\Phi_{j\to k} = I_j M_{kj}$$

with

$$M_{kj} = \frac{\mu_0}{4\pi}\oint_{C_k}\oint_{C_j}\frac{d\vec{l}_k \cdot d\vec{l}_j}{|\vec{r}_j - \vec{r}_k|} = M_{jk}$$

Mutual inductance of circuits j and k

$$\Downarrow$$

$$\mathscr{E}_{j\to k} = -\frac{d\Phi_{j\to k}}{dt} = -M_{kj}\frac{dI_j}{dt}$$

If there is more than one circuit producing the resulting field B

$$\Phi_k = \sum_j \Phi_{j\to k} = \sum_j I_k M_{kj}$$

$$\Downarrow$$

$$\mathscr{E}_{k,mutual} = -\frac{d\Phi_k}{dt} = -\sum_j M_{kj}\frac{dI_j}{dt}$$

SELF-INDUCTANCE

- It is possible for the circuit to produce a flux that passes through itself
  - The coefficient of proportionality that arises is called the self-inductance or inductance

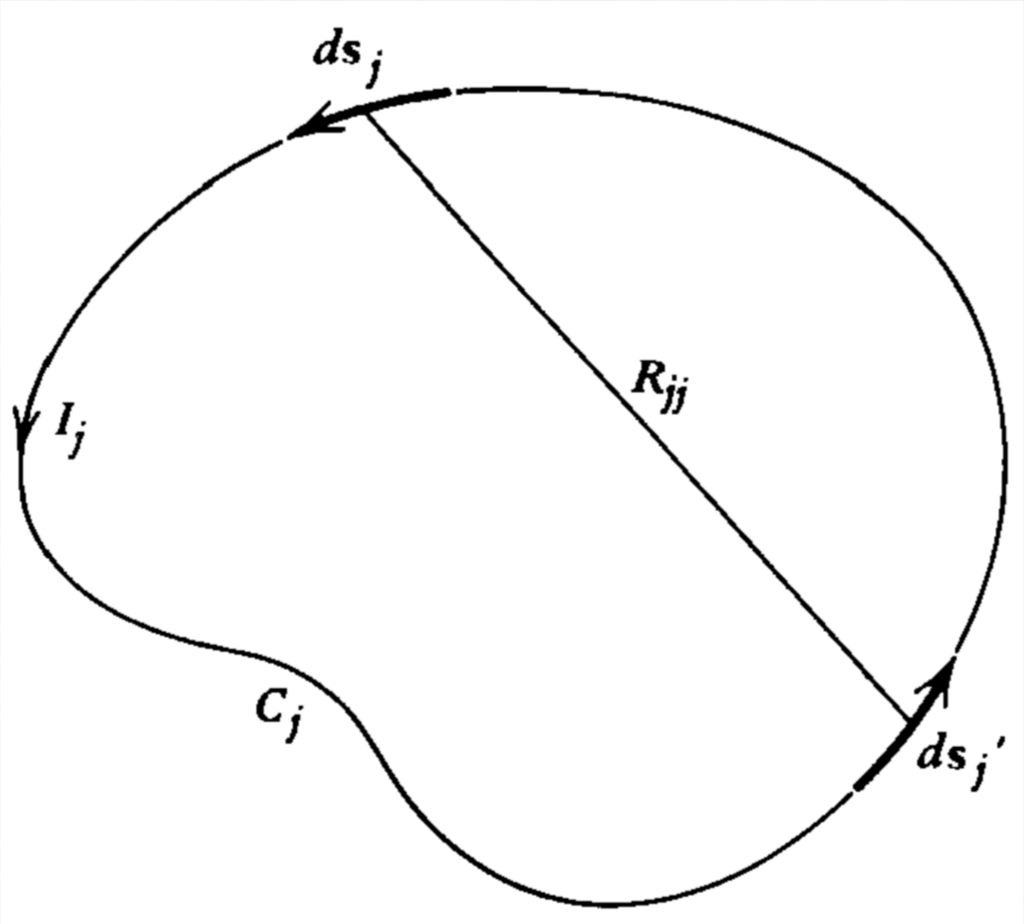
$$\Phi_{j \to j} = L_{jj} I_j$$

with

$$L_{jj} = L_j = L = \frac{\mu_0}{4\pi} \oint_{C_j} \oint_{C_j} \frac{d\vec{l}_j \cdot d\vec{l}'_j}{|\vec{r}_j - \vec{r}'_j|}$$

- The current I_j is changing then there will be an emf induced in the circuit because of its own changing flux
  - This is called self-inductance emf or back emf

$$\mathscr{E}_j = -L \frac{dI_j}{dt}$$



ENERGY OF A SYSTEM OF FREE CURRENTS

- Lets assume that at any instant t, each current is the same fraction λ and is increasing quasistatically from zero to a final value I

$$I(\lambda) = \lambda I, \quad 0 \le \lambda \le 1$$

$$U_m = \sum_{j=1}^{N} \sum_{k=1}^{N} M_{kj} I_j I_k \int_0^1 \lambda d\lambda$$

$$U_m = \frac{1}{2} \sum_{j=1}^{N} \sum_{k=1}^{N} M_{kj} I_j I_k$$

$$\boxed{U_m = \frac{1}{2} \sum_j I_j \Phi_j}$$

$$U_B = \frac{1}{2} \int \vec{j} \cdot \overrightarrow{A} dV$$

$$U_B = \frac{\mu}{8\pi} \int_V \int_{V'} \frac{\vec{j}(\vec{r}) \cdot \vec{j}(\vec{r}')}{|\vec{r} - \vec{r}'|} dV' dV$$

ENERGY IN TERMS OF MAGNETIC INDUCTION

- We can use the Ampère's law to write the energy in terms of the magnetic induction field

$$\vec{j}_f = \frac{1}{\mu_0}\nabla \times \vec{B}$$

$$U_B = \frac{1}{2\mu_0}\int_V \left(\nabla \times \vec{B}\right) \cdot \vec{A}\,dV$$

Using the identity

$$\vec{A} \cdot \left(\nabla \times \vec{B}\right) = \vec{B} \cdot \left(\nabla \times \vec{A}\right) - \nabla \cdot \left(\vec{A} \times \vec{B}\right)$$

$$U_B = \frac{1}{2\mu_0}\int_V B^2 dV - \frac{1}{2\mu_0}\int_S \left(\vec{A} \times \vec{B}\right) \cdot d\vec{S}$$

Integrating all over the space

$$\boxed{U_B = \frac{1}{2\mu_0}\int_{\text{all space}} B^2 dV}$$

MAGNETIZATION

- Magnetization refers to the rearrangement of internal currents that occurs when matter is exposed to an external magnetic field

- Magnetization is the function$\vec{M}(\vec{r})$ used to characterize the details of the rearrangement

  - This functions represents the dipole moment per unit volume

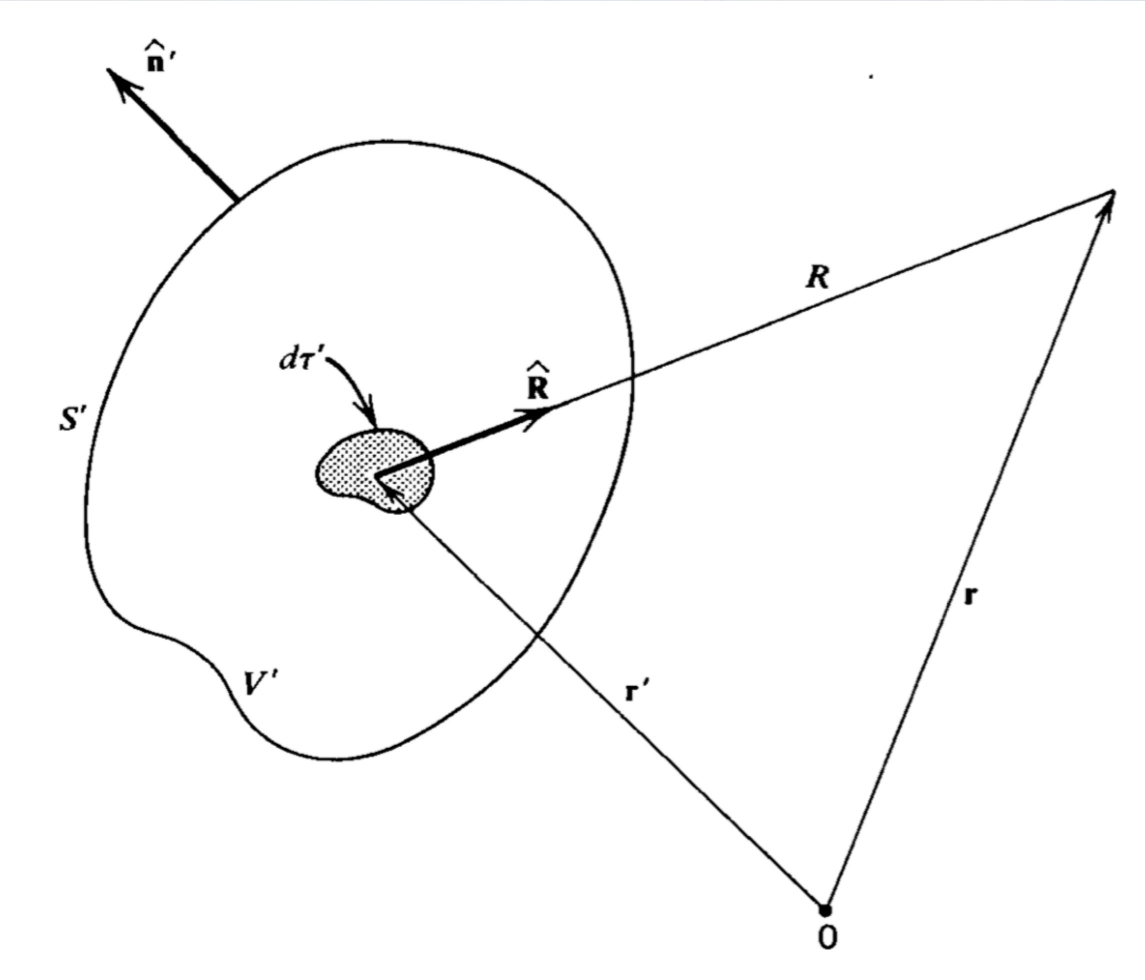
Magnetization is a macroscopic description of the material

$$d\vec{m} = \vec{M}(\vec{r})dV \qquad \Longrightarrow \qquad \vec{m} = \int_V \vec{M}(\vec{r})dV$$

$$[M] = A/m$$

THE FIELD PRODUCED BY MAGNETIZED MATTER

- Let's assume that we have a magnetized object
  - We can find the vector potential produced at any point outside the body
  - The contribution to the vector potential of a dipole moment per unit of volume at point is $\vec{r}$



$$d\vec{A} = \frac{\mu_0}{4\pi}\frac{d\vec{m} \times (\vec{r} - \vec{r}\,')}{|\vec{r} - \vec{r}\,'|^3} = \frac{\mu_0}{4\pi}\frac{\vec{M}(\vec{r}\,') \times (\vec{r} - \vec{r}\,')}{|\vec{r} - \vec{r}\,'|^3}dV'$$

The total vector potential

$$\vec{A} = \frac{\mu_0}{4\pi}\int_V \frac{\vec{M}(\vec{r}\,') \times (\vec{r} - \vec{r}\,')}{|\vec{r} - \vec{r}\,'|^3}dV' = \frac{\mu_0}{4\pi}\int_{V'} \vec{M}(\vec{r}\,') \times \nabla'\left(\frac{1}{|\vec{r} - \vec{r}\,'|}\right) dV'$$

$$\nabla \times \left(\psi\vec{A}\right) = \psi\left(\nabla \times \vec{A}\right) + \left(\nabla\psi\right) \times \vec{A}$$

$$\vec{A} = \frac{\mu_0}{4\pi}\int_V \frac{\nabla' \times \vec{M}(\vec{r}\,')}{|\vec{r} - \vec{r}\,'|}dV' + \frac{\mu_0}{4\pi}\int_{V'} \nabla' \times \left(\frac{\vec{M}(\vec{r}\,')}{|\vec{r} - \vec{r}\,'|}\right) dV'$$

$$\boxed{\vec{A} = \frac{\mu_0}{4\pi}\int_V \frac{\nabla' \times \vec{M}(\vec{r}\,')}{|\vec{r} - \vec{r}\,'|}dV' + \frac{\mu_0}{4\pi}\int_{S'} \frac{\vec{M}(\vec{r}\,') \times \hat{n}'}{|\vec{r} - \vec{r}\,'|}dS'}$$

THE FIELD PRODUCED BY MAGNETIZED MATTER

- Then, we obtain a integral over the volume and one surface integral
  - This is exactly the vector potential that would be produced by a volume current density $\vec{j}_m$ distributed throughout the volume and a surface current density $\vec{K}_m$ on the bounding surface

$$\vec{A}_m = \frac{\mu_0}{4\pi}\int_V \frac{\vec{j}_m(\vec{r}')}{|\vec{r}-\vec{r}'|}dV' + \frac{\mu_0}{4\pi}\int_{S'} \frac{\vec{K}_m(\vec{r}')}{|\vec{r}-\vec{r}'|}dS'$$

$\vec{B} = \nabla \times \vec{A}$

$$\vec{B}_m = \frac{\mu_0}{4\pi}\int_V \frac{\vec{j}_m(\vec{r}')\times(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3}dV' + \frac{\mu_0}{4\pi}\int_{S'} \frac{\vec{K}_m(\vec{r}')\times(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3}dS'$$

$$\left.\begin{aligned}\vec{j}_m &= \nabla'\times\vec{M}(\vec{r}')\\ \vec{K}_m &= \vec{M}(\vec{r})\times\hat{n}'\end{aligned}\right\}\ \text{Magnetization current densities}$$

THE FIELD PRODUCED BY MAGNETIZED MATTER

- Example: Consider an infinitely long uniformly magnetized cylinder. Magnetization is parallel to the axis of the cylinder
  $\vec{M} = M\hat{z}$
  Find the magnetic field in every point of the space

Solution:

Constant magnetization, then

$$\vec{j}_m = \nabla \times \vec{M} = 0$$

$$\vec{K}_m = \vec{M} \times \hat{\rho}' = M\hat{\varphi}' \longleftarrow$$

Surface current equivalent to an infinitely long ideal solenoid

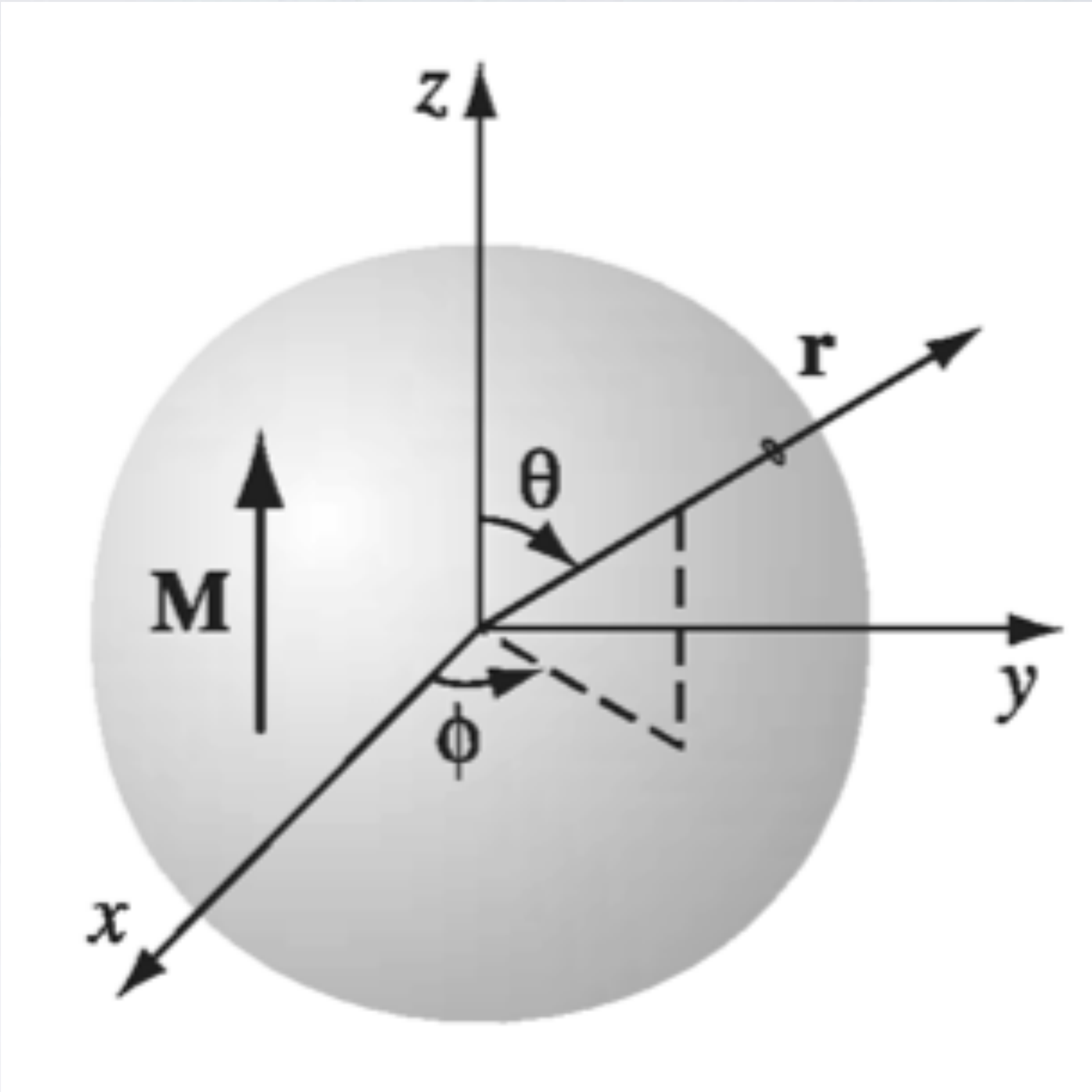
$$\Downarrow$$

$$\vec{B}_{out} = 0$$

$$\vec{B}_{in} = \mu_0 \vec{M}$$

THE FIELD PRODUCED BY MAGNETIZED MATTER

- Consider a uniformly magnetized sphere $\vec{M} = M\hat{z}$. Find the magnetic field in all points of the space.



Solution:

Then, the current densities

$$\vec{j}_m = \nabla \times \vec{M} = 0$$

$$\vec{K}_m = M\hat{z} \times \hat{r}' = M \sin\theta' \hat{\varphi}'$$

$$\Longrightarrow$$

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\int_{S'} \frac{\vec{K}_m(\vec{r}') \times \left(\vec{r} - \vec{r}'\right)}{|\vec{r} - \vec{r}'|^3} dS' = \frac{\mu_0 M a^2}{4\pi}\int_0^{2\pi}\int_0^{\pi} \frac{\hat{\varphi}' \times \left(\vec{r} - \vec{r}'\right)}{|\vec{r} - \vec{r}'|^3} \sin^2\theta' d\theta' d\varphi'$$

We can consider $\vec{r} = r\hat{z}$

$$\Longrightarrow \quad \hat{\varphi} \times (\vec{r} - \vec{r}') = z \sin\theta' \hat{r}' + (z\cos\theta' - a)\hat{\theta}'$$

$$\Longrightarrow B_z(z) = \frac{\mu_0 M a^3}{4\pi}\int_0^{2\pi}\int_0^{\pi} \frac{\sin^3\theta' d\theta' d\varphi'}{\left(z^2 + a^2 - 2az\cos\theta'\right)^{3/2}}$$

$$B_{zo}(z) = \frac{2\mu_0 M a^3}{3z^3}$$

$$B_{zi}(z) = \frac{2}{3}\mu_0 M$$

THE TOTAL MAGNETIC FIELD

- Ampère's law points out that the magnetic field is determined by currents of all kinds

$$\nabla \times \vec{B} = \mu_0 \vec{j}$$

- We can write the total current density as the sum of

$$\vec{j} = \vec{j}_f + \vec{j}_m$$

$$\Big\Downarrow \quad \vec{j}_m = \nabla \times \vec{M}$$

$$\nabla \times \vec{B} = \mu_0 \left(\vec{j}_f + \vec{j}_m\right) = \mu_0 \left(\vec{j}_f + \nabla \times \vec{M}\right)$$

$$\Big\Downarrow$$

$$\nabla \times \left(\frac{\vec{B}}{\mu_0} - \vec{M}\right) = \vec{j}_f$$

It is useful to define a new vector field $\vec{H}(\vec{r})$

$$\boxed{\vec{H}(\vec{r}) = \frac{\vec{B}(\vec{r})}{\mu_0} - \vec{M}(\vec{r})}$$

Magnetic field

$$\Longrightarrow \quad \vec{B}(\vec{r}) = \mu_0 \left[\vec{H}(\vec{r}) + \vec{M}(\vec{r})\right]$$

$$\Big\Downarrow$$

$$\boxed{\nabla \times \vec{H} = \vec{j}_f}$$

Ampère's law

THE TOTAL MAGNETIC FIELD

- From the definition of the auxiliary field it is easy to see that

Fictitious magnetic charge

$$\nabla \cdot \vec{H} = -\nabla \cdot \vec{M} = \rho^*$$

This suggest to use potential theory.
Then by Helmholtz theorem

$$\psi_M = \frac{1}{4\pi}\int_{V'} \frac{\rho^*(\vec{r}')}{|\vec{r}-\vec{r}'|}dV' + \frac{1}{4\pi}\int_{S'} \frac{\sigma^*(\vec{r}')}{|\vec{r}-\vec{r}'|}dS'$$

with $\sigma^*(\vec{r}_S) = \vec{M}(\vec{r}_S)\cdot\hat{n}(\vec{r}_S)$

$$\vec{H}(\vec{r}) = \frac{1}{4\pi}\int_{V'} \frac{\vec{j}_f(\vec{r}')\times\left(\vec{r}-\vec{r}'\right)}{|\vec{r}-\vec{r}'|^3} - \nabla\psi_M(\vec{r})$$

MATCHING CONDITIONS

- The matching conditions for the auxiliary and magnetic fields are given by the conditions previously found

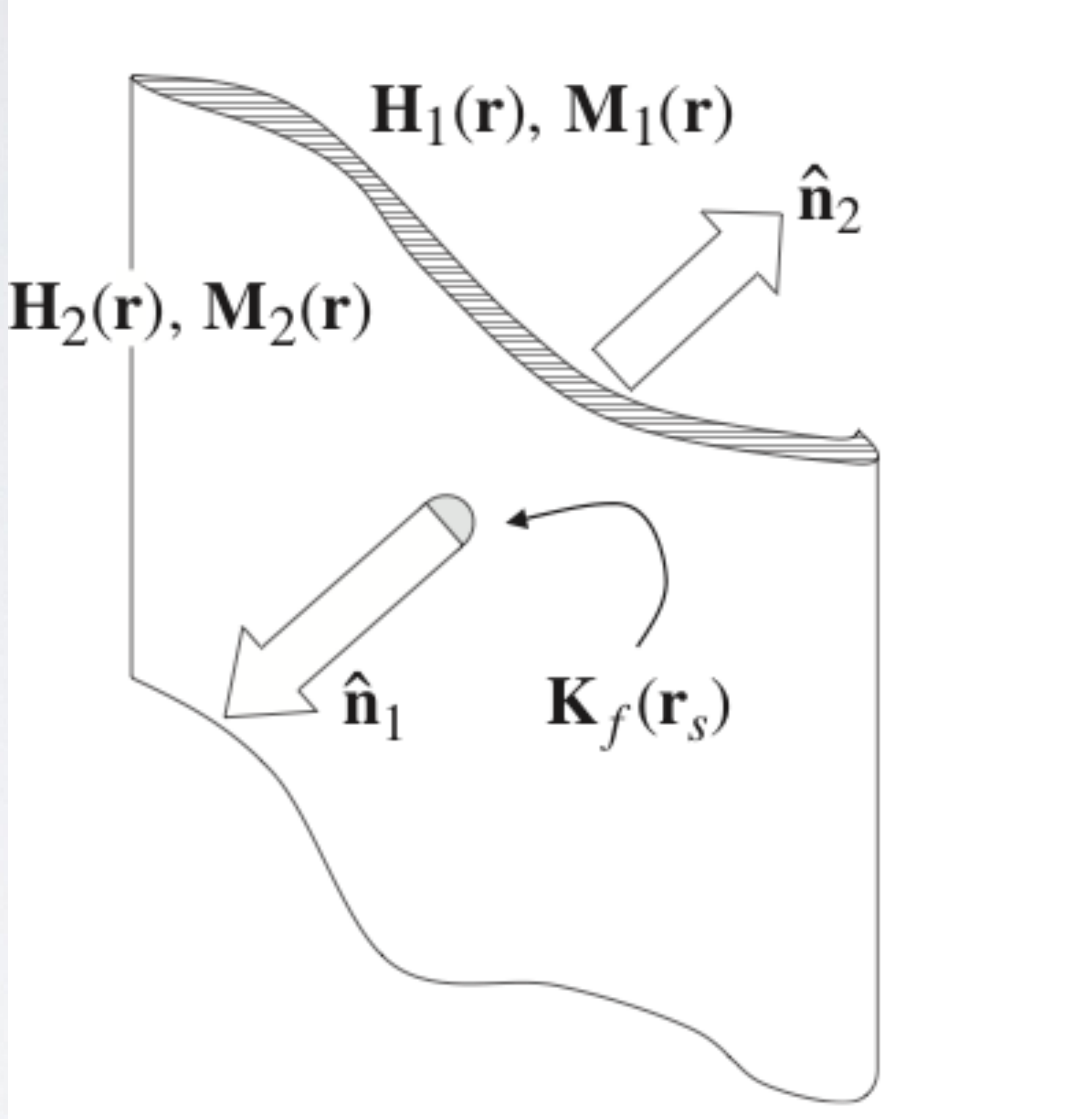
$$\hat{n}_2 \times \left[\vec{H}_1 - \vec{H}_2\right] = \vec{K}_f$$

and

$$\hat{n}_2 \cdot \left[\vec{H}_1 - \vec{H}_2\right] = \left[\vec{M}_2 - \vec{M}_1\right] \cdot \hat{n}_2$$

and

$$\hat{n}_2 \cdot \left[\vec{B}_1 - \vec{B}_2\right] = 0$$

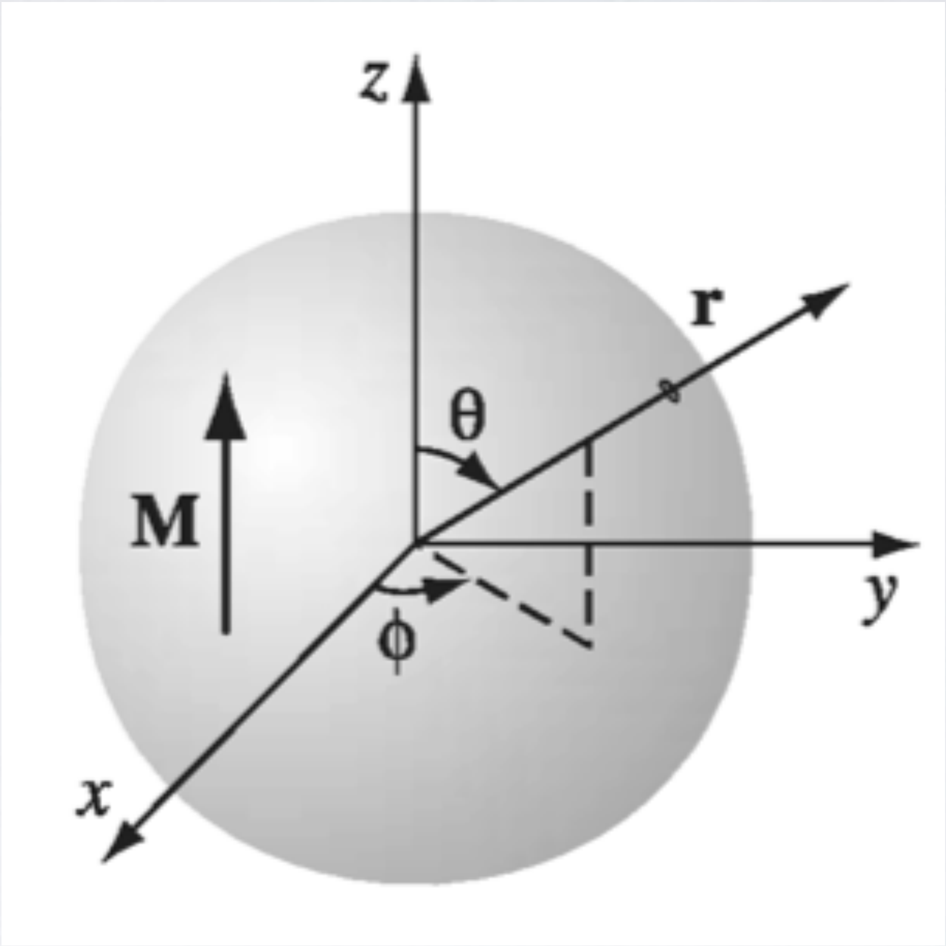


The vector potential is always continuous at interfaces where the material properties change discontinuously

$$\vec{A}_1(\vec{r}_S) = \vec{A}_2(\vec{r}_S)$$

MATCHING CONDITIONS

- Example: Consider a uniformly magnetized sphere $\overrightarrow{M} = M\hat{z}$. Find the magnetic and auxiliary field everywhere in the space.



Solution:

Since we have don't have free current density and there is a constant magnetization, then

$$\vec{j}_f = 0 \quad \vec{j}_m = 0 \quad \rho^* = -\nabla\cdot\overrightarrow{M} = 0$$

$$\sigma^* = \overrightarrow{M}\cdot\hat{r} = M\cos\theta'$$

These conditions are telling us that the solution can be find using potential theory

$$\psi(\vec{r}) = \frac{1}{4\pi}\int_{S'}\frac{\sigma^*(\vec{r}')}{|\vec{r}-\vec{r}'|}dS' \quad \text{with} \quad \left[\frac{\partial\psi_{in}}{\partial r} - \frac{\partial\psi_{out}}{\partial r}\right]_{r=R} = \sigma^*$$

Because of the azimuthal symmetry, the solution is given by an expansion in terms of Legendre's polynomials

$$\psi(\vec{r}) = \begin{cases} \frac{1}{3}Mz & r<R, \\ \frac{1}{3}MR^3\frac{\cos\theta}{r^2} & r>R \end{cases}$$

$$\overrightarrow{H}(\vec{r}) = \begin{cases} -\frac{1}{3}\overrightarrow{M} & r<R, \\ \frac{R^3}{3}\left[\frac{3(\hat{r}\cdot\overrightarrow{M})\hat{r}-\overrightarrow{M}}{r^3}\right] & r>R \end{cases}$$

$$\overrightarrow{B}(\vec{r}) = \begin{cases} \frac{2}{3}\mu_0\overrightarrow{M} & r<R, \\ \frac{\mu_0R^3}{3}\left[\frac{3(\hat{r}\cdot\overrightarrow{M})\hat{r}-\overrightarrow{M}}{r^3}\right] & r>R \end{cases}$$

CONSTITUTIVE RELATIONS

- The equations of the total magnetic field cannot be solved unless
  - We specify $\vec{M}(\vec{r})$
  - We invoke a constitutive relation which relates $\vec{M}$ to $\vec{H}$
- The general rule revealed by experiment that

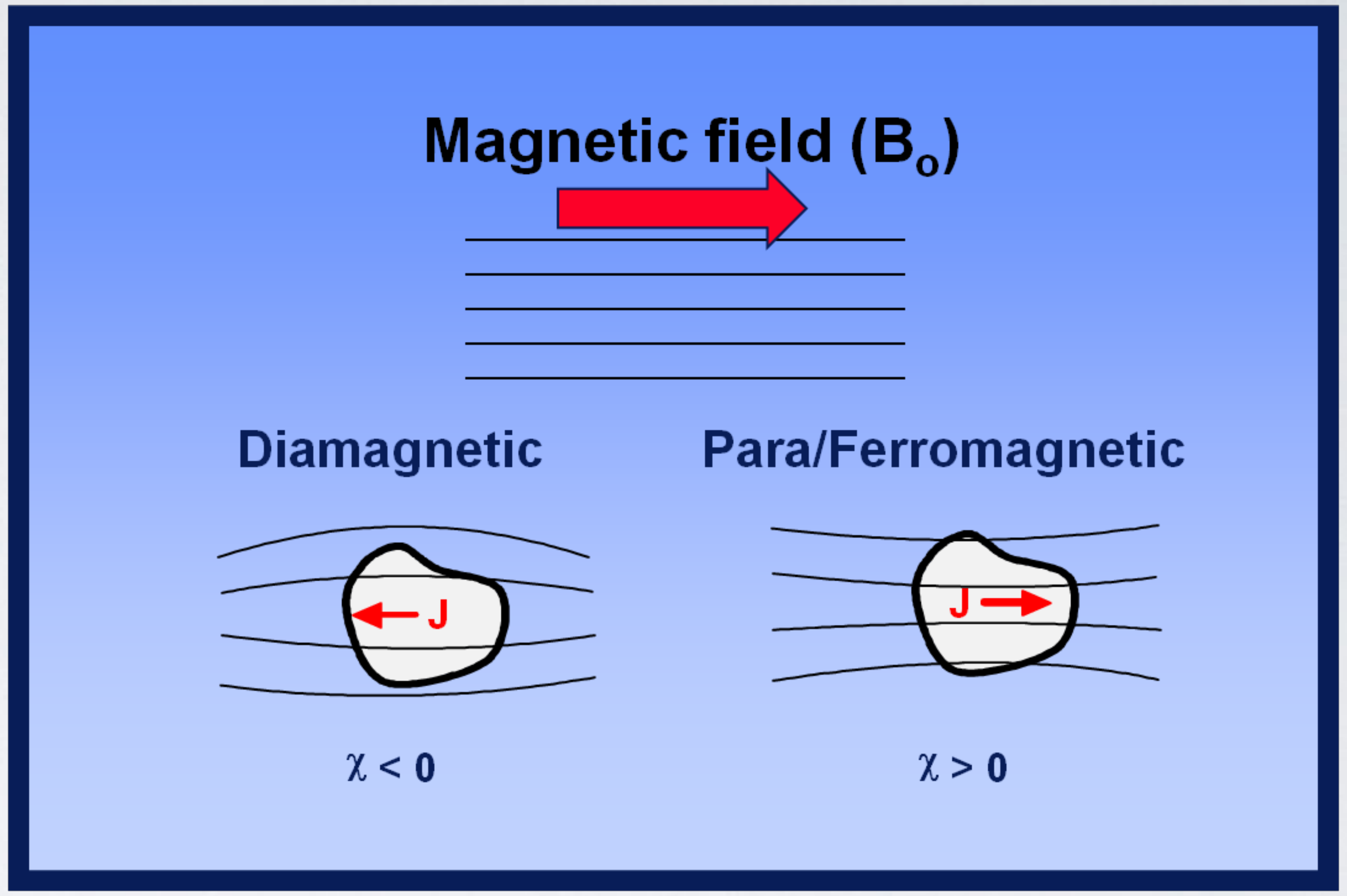
$$M_i = \chi_{ij} H_j + \chi^{(2)}_{ijk} H_j H_k + \cdots$$

The field is not altered in the presence of matter provided that we kept the free current constant and that we have an appropriate geometry

For most substances the magnetization is a functions of the field.
If the field is zero and the magnetization is different from zero then we have a *permanent magnetization*

The tensor character of the constants allows the possibility that $\vec{M}$ is not parallel to $\vec{H}$

SIMPLE MAGNETIC MATTER



- The first term in the expansion for the magnetization is sufficient to describe a simple magnetic matter
  - A linear isotropic homogeneous magnetic material
  - Material which magnetization is proportional and parallel to the magnetic field

$$\vec{M} = \chi_m \vec{H}$$

Magnetic susceptibility

- If $\chi_m > 0$ the material is called paramagnetic
- If $\chi_m < 0$ the material is called diamagnetic
  - All materials have a diamagnetic contribution to their susceptibility arising from the altered orbital motion of their constituent electrons that is produced by an applied field

SIMPLE MAGNETIC MATTER

- From the expression of the total magnetic field we can obtain a relation between the fields

$$\vec{B}(\vec{r}) = \mu_0 \left[\vec{H}(\vec{r}) + \vec{M}(\vec{r})\right]$$

$$\Downarrow$$

$$\vec{B}(\vec{r}) = \mu_0 \left(1 + \chi_m\right) \vec{H}(\vec{r})$$

with

$\kappa_m = 1 + \chi_m$ Relative permeability

$\mu = \kappa_m \mu_0$ Absolute permeability

$$\vec{B} = \mu \vec{H} \quad \Longrightarrow \quad \vec{H} = \frac{1}{\mu}\vec{B}$$

$$\Downarrow$$

$$\vec{M} = \frac{\chi_m}{\kappa_m \mu_0}\vec{B} = \frac{\chi_m}{(1+\chi_m)\mu_0}\vec{B}$$

$$\Downarrow$$

Since μ is constant and $\nabla \cdot \vec{B} = 0$

$$\nabla \cdot \vec{B} = \nabla \cdot \left(\mu \vec{H}\right) = \mu \nabla \cdot \vec{H} = 0$$

$$\Longrightarrow \quad \boxed{\nabla \cdot \vec{H} = 0}$$

$$\Longrightarrow \quad \boxed{\nabla \cdot \vec{M} = 0}$$

SIMPLE MAGNETIC MATTER

Material	Susceptibility	Material	Susceptibility
Diamagnetic:		*Paramagnetic:*	
Bismuth	-1.7×10^{-4}	Oxygen (O_2)	1.7×10^{-6}
Gold	-3.4×10^{-5}	Sodium	8.5×10^{-6}
Silver	-2.4×10^{-5}	Aluminum	2.2×10^{-5}
Copper	-9.7×10^{-6}	Tungsten	7.0×10^{-5}
Water	-9.0×10^{-6}	Platinum	2.7×10^{-4}
Carbon Dioxide	-1.1×10^{-8}	Liquid Oxygen (−200° C)	3.9×10^{-3}
Hydrogen (H_2)	-2.1×10^{-9}	Gadolinium	4.8×10^{-1}

TABLE 6.1 Magnetic Susceptibilities (unless otherwise specified, values are for 1 atm, 20° C). *Data from Handbook of Chemistry and Physics,* 91st ed. (Boca Raton: CRC Press, Inc., 2010) and other references.

SIMPLE MAGNETIC MATTER

- The free and magnetization current densities are also simply related in a material by

If $\vec{j}_f = 0$ there is no sources of $\vec{H}$

within the material

$$\vec{j}_m = \chi_m \vec{j}_f = (\kappa_m - 1)\vec{j}_f$$

$$\vec{j} = (1 + \chi_m)\vec{j}_f = \kappa_m \vec{j}_f$$

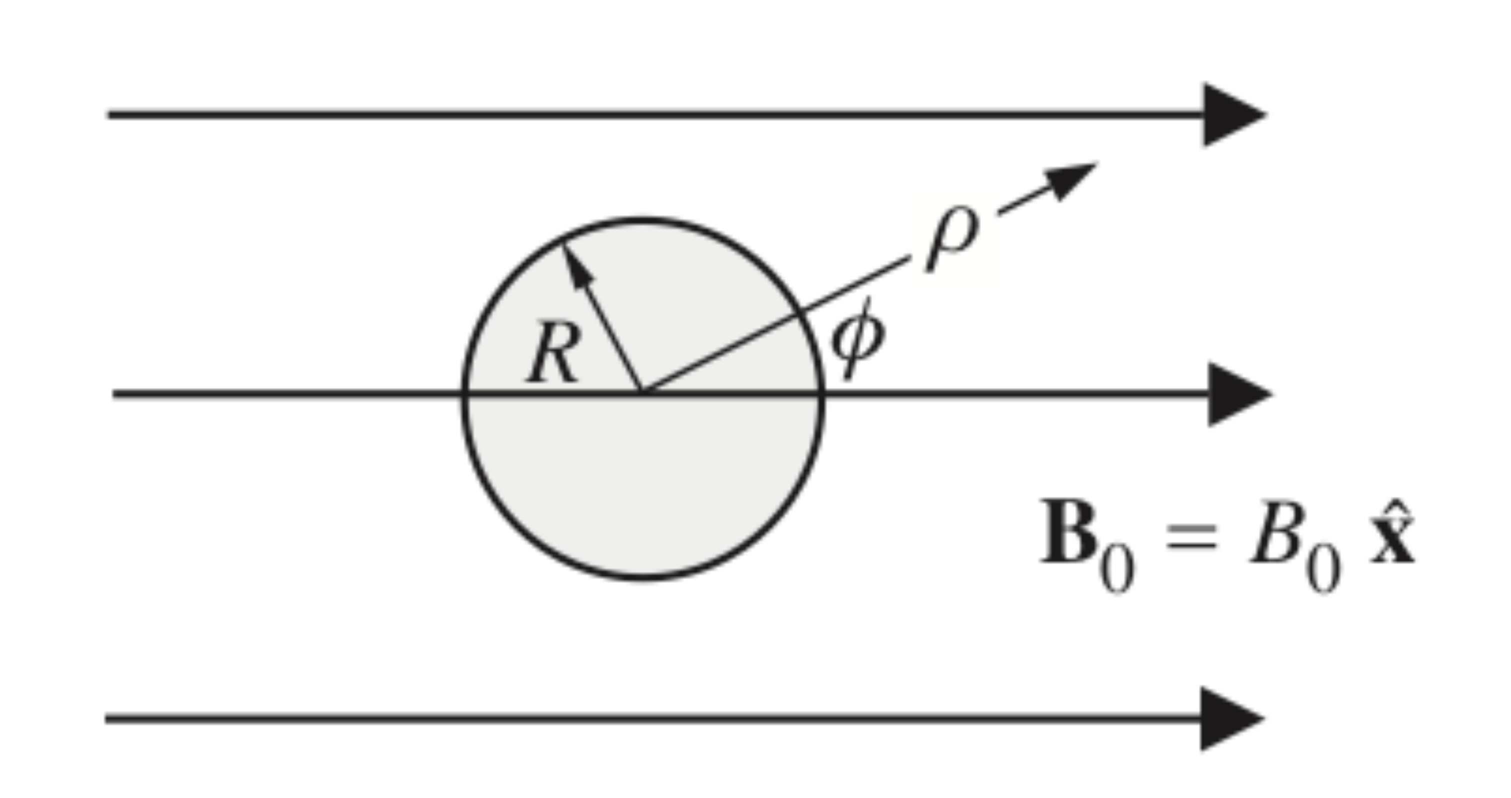
Boundary conditions

$$\vec{n}_2 \cdot \left(\vec{B}_1 - \vec{B}_2\right) = 0 \qquad \vec{n}_2 \times \left(\frac{\vec{B}_1}{\mu_1} - \frac{\vec{B}_2}{\mu_2}\right) = \vec{K}_f$$

$$\vec{n}_2 \cdot \left(\mu_1 \vec{H}_1 - \mu_2 \vec{H}_2\right) = 0 \qquad \vec{n}_2 \times \left(\vec{H}_1 - \vec{H}_2\right) = \vec{K}_f$$

SIMPLE MAGNETIC MATTER

- Example: Consider a magnetizable rod in a transverse external field
  $\vec{B} = B_0\hat{x}$
  Find the magnetic field in all the space.



Solution:

Because there is no free current density we have

$$\left.\begin{array}{l}\nabla \times \vec{H} = 0 \\ \nabla \cdot \vec{H} = 0\end{array}\right\} \Longrightarrow \vec{H} = -\nabla \psi_m \Longrightarrow \nabla^2 \psi_m = 0$$

There is a translational invariance in z axis, then $\psi_m = \psi_m(\rho, \varphi)$

with boundary conditions $\left.\mu_1 \frac{\partial \psi_1}{\partial n}\right|_S = \left.\mu_2 \frac{\partial \psi_2}{\partial}\right|_S$

$$\psi_m(\rho, \varphi) = \begin{cases} A\rho\cos\varphi & r < R \\ (C/\rho - H_0\rho)\cos\varphi & r > R \end{cases}$$

$$A = -\frac{2\mu_0}{\mu + \mu_0}H_0, \quad C = \frac{\mu - \mu_0}{\mu + \mu_0}R^2 H_0$$

- Outside the sphere the field is the sum of a uniform field and a dipole field
- Inside the sphere the field is uniform

$$\vec{B}_{in} = \frac{2\kappa_m}{\kappa_m + 1}\vec{B}_0 \quad \text{with} \quad \begin{cases} B_{in} > B_0, & k_m > 1 \\ B_{in} < B_0, & k_m < 1 \end{cases}$$

ENERGY

- We know that the magnetic energy of a system of free currents

$$U_B = \frac{1}{2}\int_{\text{all space}} \vec{j} \cdot \vec{A}\, dV$$

- After some mathematical manipulation we can finally find the following expression

$$\boxed{U_B = \int_{AS} \frac{1}{2}\vec{H} \cdot \vec{B}\, dV}$$

- It is possible to calculate the change in energy that occurs when a sample of simple magnetic matter is inserted into an external field

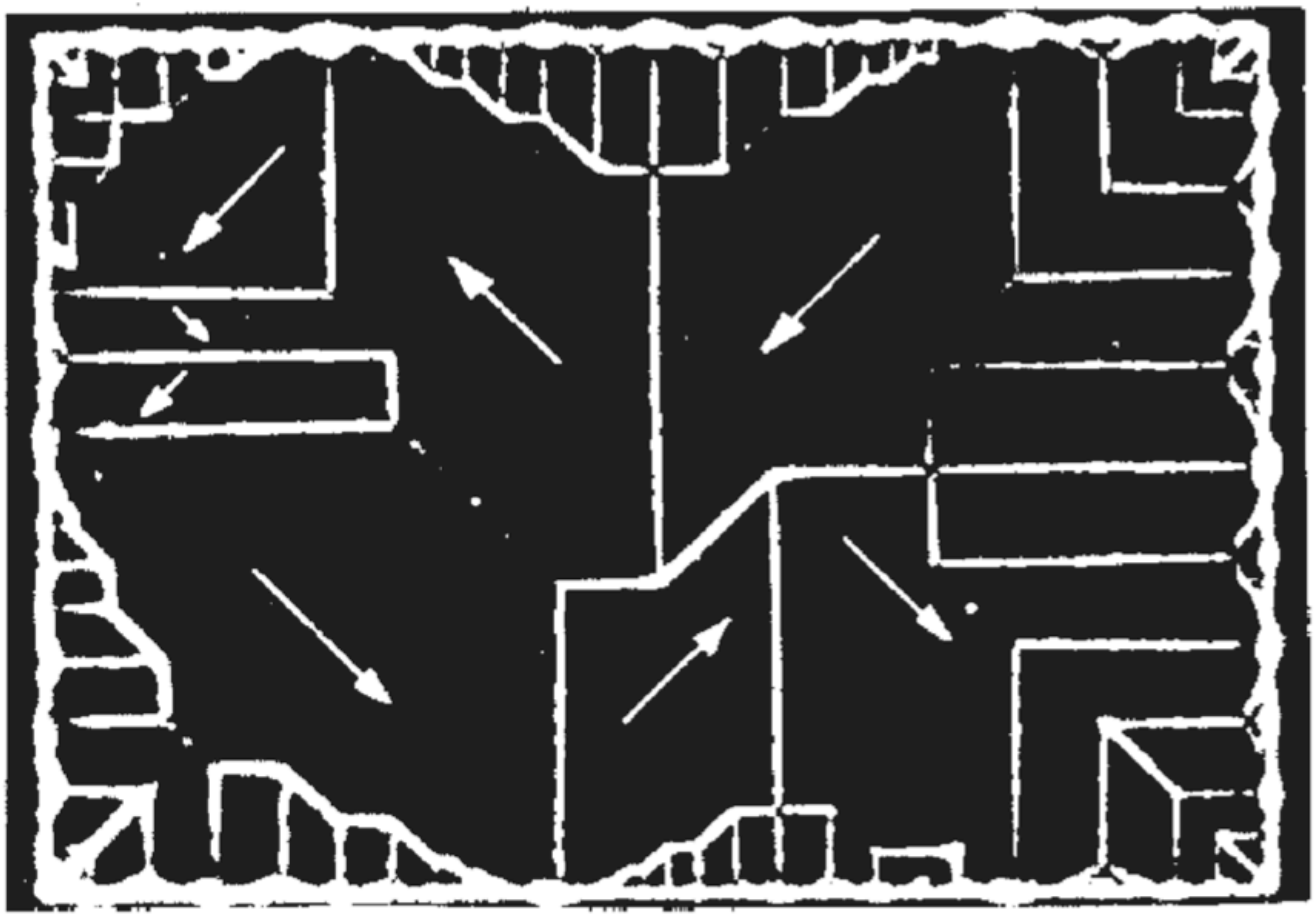
$$\boxed{\Delta U_B = \frac{1}{2}\int \vec{M} \cdot \vec{B}_0\, dV}$$

FERROMAGNETIC MATERIAL

- For quantum mechanical reasons, every microscopic magnetic moment in a ferromagnet spontaneously aligns itself with the magnetic moments in its neighborhood
  - This produce uniformly magnetized domains
  - The direction of M is not the same in every domain
  - In presence of external magnetic field domains grow
    - If the field is strong enough there is finally one entirely domain, and the material is to be said *saturated*

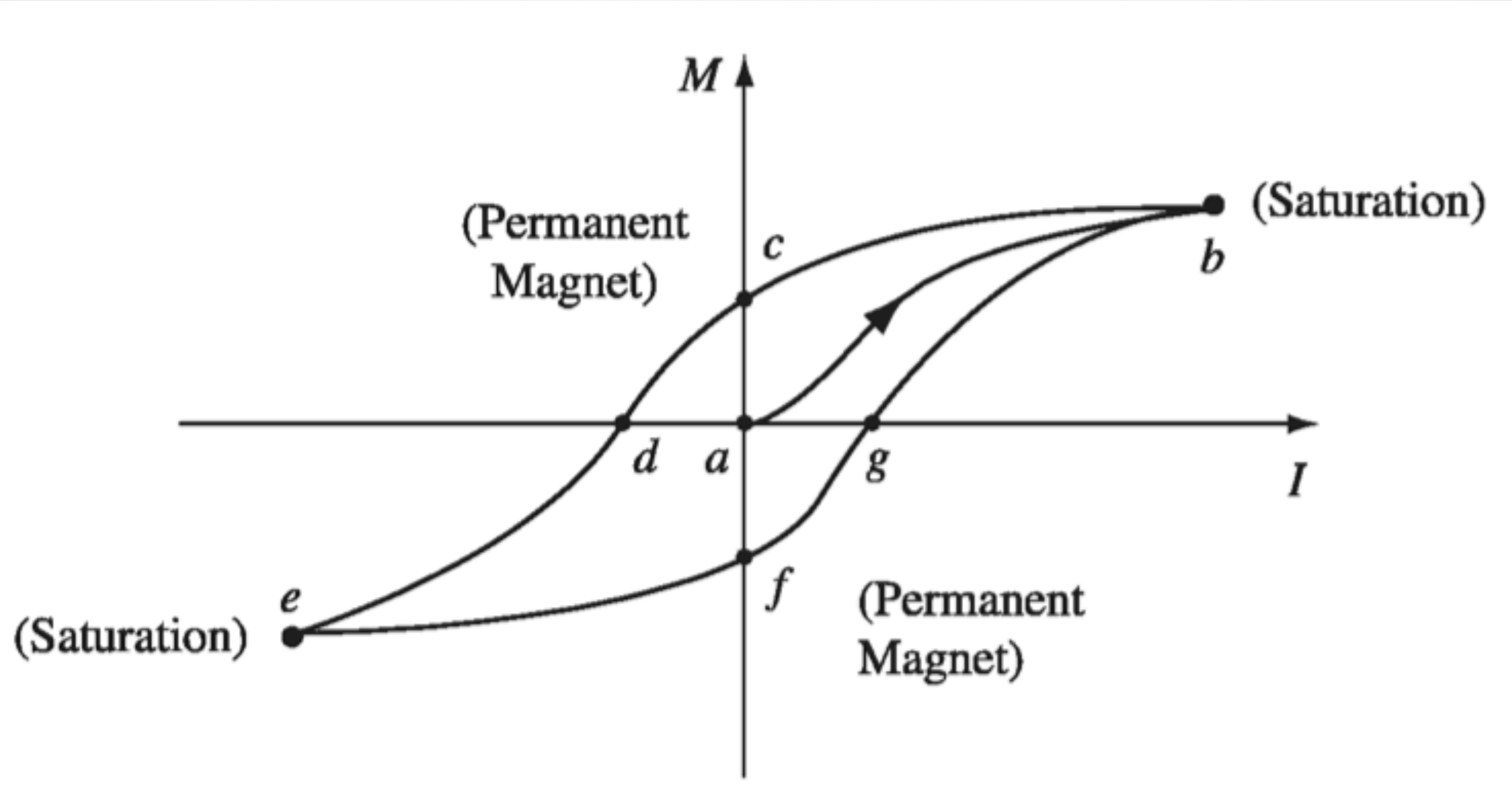
- Now reduce the current
  - Instead of retracting the path back to M=0, there is a partial return to randomly oriented domains
  - In order to eliminate the remaining magnetization we run the current backwards
    - Magnetization drops down to zero
  - We saturate the material in the oposite direction

FERROMAGNETIC MATERIAL



Ferromagnetic domains. (Photo courtesy of R. W. DeBlois)

FERROMAGNETIC MATERIAL



FERROMAGNETIC MATERIAL

- Ferromagnetism depends on temperature
  - For very high temperatures the alignment is destroyed
  - That occurs at a precise temperature known as Curie point
    - For example, iron es ferromagnetic for temperatures below 770ºC
      - Above this temperature iron is paramagnetic
      - This is a phase transition