Question 1 - Dodelson 6.18

Determine the predictions of an inflationary model with a quartic potential,

$$V(\phi) = \lambda\phi^4 \tag{1}$$

a) Compute the slow roll parameters ϵ and δ in terms of ϕ.
b) Determine ϕ_e the value of the field at which inflation ends, by setting $\epsilon = 1$ at the end of inflation.
c) To determine the spectrum, you will need to evaluate ϵ and δ at $-k\eta = 1$. Choose the wavenumber k to be equal to $a_0 H_0$, roughly the horizon today. Show that the requirement $-k\eta = 1$ then corresponds to

$$e^{60} = \int_0^N \mathrm{d}N' \frac{e^{N'}}{(H(N')/H_e)} \tag{2}$$

where H_e is the Hubble rate at the end of inflation, and N is defined to be the number of e-folds before the end of inflation:

$$N \equiv \ln\left(\frac{a_e}{a}\right) \tag{3}$$

d) Take the Hubble rate to be a constant in the above with H/H_e equal to 1. This implies that $N \simeq 60$. Turn this into an expression for ϕ. The simplest way to do this is to note that $N = \int_t^{T_e} \mathrm{d}t' H(t')$ and assume that H is dominated by potential energy. Show that this mode leaves the horizon when $\phi^2 = 60 m_{Pl}^2/\pi$.
e) Determine the predicted values of n and n_T.
f) Estimate the scalar amplitude in terms of λ. As a rough estimate, assume that $k^3 P_\phi(k)$ for this mode is equal to 10^{-8} (we will find a more precise value when we normalize to large-angle anisotropics in Chapter 8). What value does this imply for λ?
This model illustrates many of the features of contemporary models. In it, (i) the field is of order - even greater than - the Planck scale, but (ii) the energy scale V is much smaller because of (iii) the very small coupling constant.

Item a)

We know that

$$\epsilon = \frac{1}{16\pi G}\left(\frac{V'}{V}\right)^2,$$

and

$$\delta = \epsilon - \frac{1}{8\pi G}\frac{V''}{V}.$$

Deriving $V(\phi)$, we have

$$V'(\phi) = 4\lambda\phi^3,$$

and

$$V''(\phi) = 12\lambda\phi^2,$$

so

$$\epsilon = \frac{1}{16\pi G}\left(\frac{4\cancel{\lambda}\cancel{\phi^3}}{\cancel{\lambda}\cancel{\phi^4}}\right)^2 = \frac{1}{16\pi G}\left(\frac{4}{\phi}\right)^2 = \boxed{\frac{1}{\phi^2\pi G}}$$

And

$$\delta = \frac{1}{\phi^2\pi G} - \frac{1}{8\pi G}\frac{12\cancel{\lambda}\cancel{\phi^2}}{\cancel{\lambda}\phi^4} = \frac{1}{\phi^2\pi G} - \frac{3}{2\phi^2\pi G} = \boxed{-\frac{1}{2\phi^2\pi G}}$$

Item b)

If $\epsilon = 1$, then we have

$$\frac{1}{\phi^2\pi G} = 1 \to \boxed{\phi_e = \sqrt{\frac{1}{\pi^2 G}}}$$

Item c)

We have that

$$\mathrm{d}\eta = \frac{\mathrm{d}t}{a(t)} \to \eta = \int_t^{t_e} \frac{\mathrm{d}t'}{a(t')},$$

but we have that

$$N \equiv \ln\left(\frac{a_e}{a}\right),$$

and

$$a = a_e e^{-N},$$

then we have

$$\eta = \int_t^{t_e} \frac{\mathrm{d}t'}{a_e e^{-N(t')}},$$

but as $\mathrm{d}N = -H(t)\ \mathrm{d}t$, we obtain

$$\eta = -\int_0^N \frac{e^{N'}}{a_e H(N')}\ \mathrm{d}N'.$$

Considering $-k\eta = 1$, we have that

$$1 = k\int_0^N \frac{e^{N'}}{a_e H(N')}\ \mathrm{d}N',$$

but as $k = H_0 a_0$, we have

$$1 = H_0 a_0 \int_0^N \frac{e^{N'}}{a_e H(N')}\ \mathrm{d}N',$$

and factoring out H_e, we have

$$\boxed{e^{60} = \int_0^N \frac{e^{N'}}{H(N')/H_e}\ \mathrm{d}N'}$$

Item d)

If H is constant, then

$$N = H(T_e - t),$$

and, from the Friedmann Equation, we have

$$H^2 = \frac{8\pi}{3m_{p_l}^2}V(\phi) = \frac{8\pi}{3m_{p_l}^2}\lambda\phi^4 \rightarrow H = \sqrt{\frac{8\pi\lambda}{3m_{p_l}^2}}\phi^2,$$

if $N = 60$, we obtain

$$\sqrt{\frac{8\pi\lambda}{3m_{p_l}^2}}\phi^2(T_e - t) = 60,$$

let

$$\lambda = \frac{3\pi}{8m_{p_l}^2},$$

we receive

$$\phi^2 = \frac{60m_{p_l}^2}{(T_e - t)\pi} \rightarrow \boxed{\phi^2 \approx \frac{60m_{p_l}^2}{\pi}}$$

Item e)

If $n = 1 - 4\epsilon - 2\delta$ and $n_T = -2\epsilon$, we have

$$\boxed{n = 1 - \frac{3}{\phi^2\pi G}}$$

And

$$\boxed{n_T = -\frac{2}{\phi^2\pi G}}$$

Item f)

For a single-field slow-roll inflation model, the expression for $P_\phi(k)$ is given by

$$P_\phi(k) = \frac{8\pi}{9k^3}\frac{H^2}{\epsilon m_{p_l}^2}\bigg|_{k=aH} = \frac{8\pi}{9a^3H}\frac{1}{\epsilon m_{p_l}^2},$$

we can also write ϵ as

$$\epsilon = \frac{m_{p_l}^2}{4\pi\phi^2},$$

and then if

$$H = \sqrt{\frac{8\pi\lambda}{3m_{p_l}^2}}\phi^2$$

and

$$\phi^2 = \frac{60m_{p_l}^2}{\pi}.$$

If $P_\phi(k) \approx 10^{-8}$ and if we consider $m_{p_l} = 1.22 \cdot 10^{19}$ GeV, we can isolate λ and receive

$$\boxed{\lambda \approx 1.95 \cdot 10^{-14}}$$

Question 2 - Growth Function

Use a numerical differential equation solver to **numerically** evolve Eq. 7.73 (or Eq. 7.74) in Dodelson's book, obtaining the growth function $D(a)$ such that it is equal to a at early times ($a \leq 1$) and plot your numerical results in the same range and scale as shown in Dodelson's Fig. 7.12. Next obtain another estimation of $D(a)$ by performing the integral in Eq. 7.77 and plot these results in the same plot as the previous calculation. Do both types of calculations for $w = -1$ and two flat models with $\Omega_m, \Omega_\Lambda = (1.0, 0.0), (0.3, 0.7)$ and an open model with $\Omega_m, \Omega_\Lambda = (0.3, 0.0)$. Show all cases in the same plot for comparison, using different colors for the differential equation solver (blue) and integration method (red) and different line types (solid, dashed, dotted) to differentiate the models.
How do both your results compare to each other? And to Dodelson's Figure?

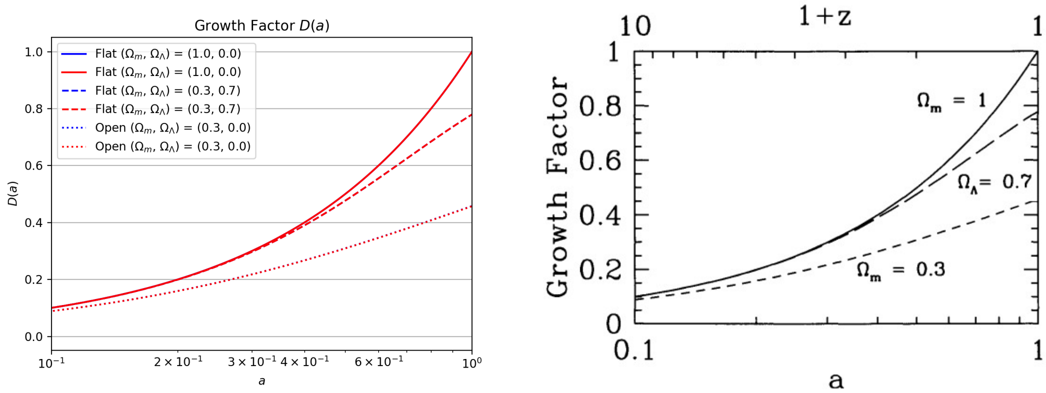

Figure 1: Left: Differential equation solution in blue and integration method in red. Right: Figure 7.12 from Dodelson.

From Figure 1, we can see that both differential equation solver and integration method are compatible with each other. Furthermore, it's clear that my plot is also compatible with Dodelson's plot.