Question 1 - Experimental Time-Dilation

On October 1971, cesium beam clocks were flown on jet flights around the world twice (eastward and westward) and then compared with reference clocks at the US Naval Observatory. From the flight paths of each trip, and considering only the special relativistic (kinematic) effect, compute how much time the clock moving eastward should have lost/gained relative to the reference clocks. Repeat the computation for the clock moving westward.

Note: On top of the kinematical effect, there is also a larger time dilation due to a gravitational effect from General Relativity (see Problem Set 2). Both the kinematical and gravitational effects are comparable and necessary to explain the observed time gain/loss.

Suggestion: Read the original paper *J. Hafele, R. Keating, Science, Vol 177, No 4044 (1972), pp. 166-168*

From Chapter 1, section 1.1.8 of the lecture notes, we can represent the events in two reference frames:

$$\begin{cases} K : (x_1, t_1) \to (x_2, t_2) \\ K' : (x'_1, t'_1) \to (x'_2, t'_2) \end{cases}$$

where $\Delta x' = x'_2 - x'_1 = 0$ and $\Delta t' \equiv \Delta\tau = t'_2 - t'_1$, where the time measured in the K' reference coincides with the proper time.

In the K frame, we have $\Delta t = t_2 - t_1$, with the events not happening in the same position, $\Delta x = x_2 - x_1 \neq 0$.

From Chapter 1, section 1.1.5, we have the Lorentz transformations:

$$x'^{\mu} = \Lambda^{\mu}_{\nu} x^{\nu}, \tag{1}$$

where

$$\Lambda^{\mu}_{\nu} = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{2}$$

with

$$\beta = \frac{u}{c}, \tag{3}$$

$$\gamma = \frac{1}{\sqrt{1-\beta^2}}, \tag{4}$$

and then

$$x'^0 = \gamma(x^0 - \beta x^1) \to x'_i = \gamma(x_i - \beta c t_i) = \gamma(x_i - v t_i), \tag{5}$$

and

$$x'^1 = \gamma(x^1 - \beta x^0) \to c t'_i = \gamma(c t_i - \beta x_i). \tag{6}$$

Moving forward, we have

$$\begin{aligned} \Delta x' = x'_2 - x'_1 &= \gamma(x_2 - v t_2) - \gamma(x_1 - v t_1) \\ &= \gamma(x_2 - x_1) - \gamma u(t_2 - t_1) \\ &= \gamma\Delta t - \gamma u \Delta t = 0, \end{aligned}$$

$$\Delta x = u\Delta t. \tag{7}$$

The proper time can be computed as

$$\begin{aligned}
c\Delta\tau = ct'_2 - ct'_1 = \gamma(ct_2 - \beta x_2) - \gamma(ct_1 - \beta x_1) &= \gamma c(t_2 - t_1) - \gamma\beta(x_2 - x_1) \\
&= \gamma c\Delta t - \gamma\beta\Delta x \\
&= \gamma(c\Delta t - \beta\Delta x) \\
&= \gamma(c\Delta t - \beta u\Delta t),\ u = \beta c \\
&= \gamma c(\Delta t - \beta^2\Delta t) \\
&= \gamma c\Delta t(1 - \beta^2),\ (1 - \beta^2) = \frac{1}{\gamma^2}, \\
&= c\frac{\Delta t}{\gamma},
\end{aligned}$$

$$\Delta\tau = \frac{\Delta t}{\gamma} \to \Delta t = \gamma\Delta\tau. \tag{8}$$

Now,

$$\Delta t - \Delta\tau = \gamma\Delta\tau - \Delta\tau = \Delta\tau(\gamma - 1), \tag{9}$$

where, for $u^2 \ll c^2$, we have that

$$\gamma \approx 1 - \frac{u^2}{2c^2}, \tag{10}$$

then

$$\Delta t - \Delta\tau = -\frac{\Delta\tau^2 u^2}{2c^2}. \tag{11}$$

According to JC Hafele (1972), we can approximate Equation 11 to

$$\Delta t - \Delta\tau = -\frac{(2R\Omega v + v^2)\Delta\tau}{2c^2}, \tag{12}$$

where R is Earth's radius and Ω its angular speed; v is the airplane ground speed with $v > 0$ for a eastward circumnavigation and $v < 0$ for a westward circumnavigation.

Considering

$$\begin{cases}
R = 6.371 \cdot 10^6 \text{ m} \\
\Omega = 7.27 \cdot 10^{-5} \text{ rad/s} \\
v = 250 \text{ m/s} \\
\Delta\tau_e = 41.2 \text{ h} = 1.4832 \cdot 10^5 \text{ s (Eastward)} \\
\Delta\tau_w = 48.6 \text{ h} = 1.7496 \cdot 10^5 \text{ s (Westward)}
\end{cases} \tag{13}$$

Substituting in Equation 12, we receive

$$\Delta t - \Delta\tau_e \approx -485.30 \text{ ns (Eastward flight)}, \tag{14}$$

$$\Delta t - \Delta\tau_w \approx 329.14 \text{ ns (Westward flight)}. \tag{15}$$

Question 2 - E & B Fields

a) Write out the $\mu = 0$ component of the covariant force equation $f^\mu = qU_\nu F^{\mu\nu}$ in terms of the particle energy, velocity and the **E** & **B** fields, and provide an interpretation for it.

b) Using the Lorentz transformations on $F_{\mu\nu}$, show how **E** and **B** transform under a boost along the x-axis.

c) Show that for a general tensor $B_{\mu\nu}$, the contraction $B^{\mu\nu}B_{\mu\nu}$ is a scalar. Apply this result to the electromagnetic field tensor $F_{\mu\nu}$ to obtain the scalar in this case.

d) The energy-momentum tensor for electromagnetism is

$$T^{\mu\nu}_{(\text{EM})} = F^{\mu\lambda}F^{\nu}{}_{\lambda} - \frac{1}{4}\eta^{\mu\nu}F^{\lambda\sigma}F_{\lambda\sigma},$$

compute $T^{00}_{(\text{EM})}$ and $T^{0i}_{(\text{EM})}$ in terms of **E** and **B**.

a) From Chapter 1, section 1.1.11 of the lecture notes, we know that

$$U^\mu = \gamma(c, \mathbf{v}) \to U_\mu = \gamma \begin{pmatrix} -c \\ v_x \\ v_y \\ v_z \end{pmatrix}, \tag{16}$$

also, from the $F^{\mu\nu}$ tensor, we know that

$$F^{0\nu} = \begin{pmatrix} 0 & \frac{E_x}{c} & \frac{E_y}{c} & \frac{E_z}{c} \end{pmatrix}. \tag{17}$$

It's then easier to write the $\mu = 0$ component of f^μ in its covariant form using the $\eta_{\mu\alpha}$ metric. The $\mu = 0$ component of the metric is

$$\eta_{0\alpha} = \begin{pmatrix} -1 & 0 & 0 & 0 \end{pmatrix}, \tag{18}$$

where the only component that should matter is $\eta_{00} = -1$. Then we have

$$\eta_{\mu\alpha} f^\mu = q\eta_{\mu\alpha}U_\nu F^{\mu\nu} \to f_\mu = q\eta_{\mu\alpha}U_\nu F^{\mu\nu}, \tag{19}$$

$$\begin{aligned} f_0 = q\eta_{00}U_\nu F^{0\nu} &= -q\left(U_0F^{00} + U_1F^{01} + U_2F^{02} + U_0FF^{03}\right) \\ &= -q\gamma\left(-c\cdot 0 + v_x\frac{E_x}{c} + v_y\frac{E_y}{c} + v_z\frac{E_z}{c}\right), \end{aligned}$$

$$\therefore f_0 = -\gamma\frac{\mathbf{v}\cdot(q\mathbf{E})}{c}. \tag{20}$$

The 4-force tensor can be arranged as

$$\mathbf{F} = \left(\gamma\frac{\mathbf{v}\cdot\mathbf{f}}{c}, \gamma\mathbf{f}\right), \tag{21}$$

where $\mathbf{v}\cdot\mathbf{f}$ is the power of the 4-force. Also the electric field force is given by $q\mathbf{E}$, so the result of Equation 20 is represented as the power of the 4-force tensor for an electric field.

b) From Chapter 1, section 1.1.13, we have that

$$F'_{\mu\nu} = \Lambda^\alpha_\mu \Lambda^\beta_\nu F_{\alpha\beta} = \Lambda^\alpha_\mu F_{\alpha\beta} \Lambda^\beta_\nu = (\Lambda F \Lambda^T)_{\mu\nu}, \tag{22}$$

where

$$F_{\mu\nu} = \begin{pmatrix} 0 & -E_x/c & -E_y/c & -E_z/c \\ E_x/c & 0 & B_z & -B_y \\ E_y/c & -B_z & 0 & B_x \\ E_z/c & B_y & -B_x & 0 \end{pmatrix}, \tag{23}$$

$$\Lambda^\nu_\mu = \begin{pmatrix} \gamma & \beta\gamma & 0 & 0 \\ \beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{24}$$

This product will result in

$$F'_{\mu\nu} = \begin{pmatrix} 0 & -\frac{E_x}{c} & -\gamma\left(\frac{E_y}{c} - \beta B_z\right) & -\gamma\left(\frac{E_z}{c} + \beta B_y\right) \\ \frac{E_x}{c} & 0 & \gamma\left(B_z - \beta\frac{E_y}{c}\right) & -\gamma\left(B_y + \beta\frac{E_z}{c}\right) \\ \gamma\left(\frac{E_y}{c} - \beta B_z\right) & -\gamma\left(B_z - \beta\frac{E_y}{c}\right) & 0 & B_x \\ \gamma\left(\frac{E_z}{c} + \beta B_y\right) & \gamma\left(B_y + \beta\frac{E_z}{c}\right) & -B_x & 0 \end{pmatrix}, \tag{25}$$

we then receive

$$\begin{cases} \dfrac{E'_x}{c} = \dfrac{E_x}{c} \to E'_x = E_x \\ \dfrac{E'_y}{c} = \dfrac{\gamma}{c}\left(E_y - vB_z\right) \to E'_y = \gamma\left(E_y - vB_z\right) \\ \dfrac{E'_z}{c} = \dfrac{\gamma}{c}\left(E_z + vB_y\right) \to E'_z = \gamma\left(E_z + vB_y\right) \end{cases} \tag{26}$$

$$\begin{cases} B'_x = B_x \\ B'_y = \gamma\left(B_y + v\dfrac{E_z}{c^2}\right) \\ B'_z = \gamma\left(B_z - v\dfrac{E_y}{c^2}\right) \end{cases} \tag{27}$$

c) We have that

$$B^{\mu\nu}B_{\mu\nu} = \sum_{\nu,\mu}^{n} = B^{00}B_{00} + B^{01}B_{01} + \ldots + B^{0n}B_{0n} + B^{10}B_{10} + \ldots + B^{nn}B_{nn}, \tag{28}$$

so $B^{\mu\nu}B_{\mu\nu} = B_{00}B^{00} + \ldots + B^{nn}B_{nn}$, which is a scalar. Then, we have

$$\begin{aligned} F^{\mu\nu}F_{\mu\nu} &= F^{00}F_{00} + F^{01}F_{01} + F^{02}F_{02} + F^{03}F_{03} + F^{10}F_{10} + F^{11}F_{11} + F^{12}F_{12} + F^{13}F_{13} \\ &\quad + F^{20}F_{20} + F^{21}F_{21} + F^{22}F_{22} + F^{23}F_{23} + F^{30}F_{30} + F^{31}F_{31} + F^{32}F_{32} + F^{33}F_{33} \\ &= \sum_{i=x}^{z} \frac{E_i}{c}\cdot\left(-\frac{E_i}{c}\right) + \sum_{i=x}^{z}\left(-\frac{E_i}{c}\right)\cdot\frac{E_i}{c} + B_z^2 + B_y^2 + B_z^2 + B_x^2 + B_y^2 + B_x^2 \\ &= -2\left(\frac{E_x^2}{c^2} + \frac{E_y^2}{c^2} + \frac{E_z^2}{c^2}\right) + 2\left(B_x^2 + B_y^2 + B_z^2\right), \end{aligned}$$

$$\therefore F^{\mu\nu}F_{\mu\nu} = 2|\mathbf{B}|^2 - \frac{2}{c^2}|\mathbf{E}|^2. \tag{29}$$

d) For the energy-momentum tensor for electromagnetism, we have that

$$G^{\nu}_{\lambda} = \eta^{\mu\alpha} F_{\alpha\lambda}, \tag{30}$$

so

$$T^{\mu\nu}_{(\mathrm{EM})} = F^{\mu\lambda}\eta^{\mu\alpha}F_{\alpha\lambda} - \frac{1}{4}\eta^{\mu\nu}F^{\lambda\sigma}F_{\lambda\sigma}. \tag{31}$$

For $\mu = \nu = 0$, we have that $\eta^{0\alpha}$ is only significant when $\alpha = 0$ because when $\alpha \neq 0$, $\eta^{0i} = 0$. Furthermore, from the previous item, we know that $F^{\alpha\lambda}F_{\alpha\lambda} = 2|\mathbf{B}|^2 - \frac{2}{c^2}|\mathbf{E}|^2$. Then we get:

$$\begin{aligned}
T^{00}_{(\mathrm{EM})} &= F^{0\lambda}\eta^{00}F_{0\lambda} - \eta^{00}\left(\frac{1}{2}|\mathbf{B}|^2 - \frac{1}{2c^2}|\mathbf{E}|^2\right) \\
&= -\left(F^{00}F_{00} + F^{01}F_{01} + F^{02}F_{02} + F^{03}F_{03}\right) + \frac{1}{2}|\mathbf{B}|^2 - \frac{1}{2c^2}|\mathbf{E}|^2 \\
&= -\sum_{i=x}^{z}\frac{E_i}{x}\left(-\frac{E_i}{x}\right) + \frac{1}{2}|\mathbf{B}|^2 - \frac{1}{2c^2}|\mathbf{E}|^2 \\
&= \frac{1}{c^2}|\mathbf{E}|^2 + \frac{1}{2}|\mathbf{B}|^2 - \frac{1}{2c^2}|\mathbf{E}|^2,
\end{aligned}$$

$$\therefore T^{00}_{(\mathrm{EM})} = \frac{1}{2}|\mathbf{B}|^2 + \frac{1}{2c^2}|\mathbf{E}|^2. \tag{32}$$

For $T^{0i}_{(\mathrm{EM})}$, with $i = 1, 2, 3$, we have that $\eta^{i\alpha} \neq 0$ only for $\alpha = i$ and as $\eta^{0i} = 0$, the second term of the equation goes to 0 altogether. Then, we are left with

$$T^{0i}_{(\mathrm{EM})} = F^{0\lambda}\cancelto{1}{\eta^{ii}}F_{i\lambda} = F^{0\lambda}F_{i\lambda}. \tag{33}$$

1. for $i = 1$, we have

$$F^{0\lambda}F_{1\lambda} = F^{00}F_{10} + F^{01}F_{11} + F^{02}F_{12} + F^{03}F_{13} = \frac{E_y}{c}B_z - \frac{E_z}{c}B_y. \tag{34}$$

2. for $i = 2$, we have

$$F^{0\lambda}F_{2\lambda} = F^{00}F_{20} + F^{01}F_{21} + F^{02}F_{22} + F^{03}F_{23} = \frac{E_z}{c}B_x - \frac{E_x}{c}B_z. \tag{35}$$

3. for $i = 3$, we have

$$F^{0\lambda}F_{3\lambda} = F^{00}F_{30} + F^{01}F_{31} + F^{02}F_{32} + F^{03}F_{33} = \frac{E_x}{c}B_y - \frac{E_y}{c}B_x. \tag{36}$$

Finally, our result is

$$T^{0i}_{(\mathrm{EM})} = \frac{1}{c}\left(\mathbf{E}\times\mathbf{B}\right)_i. \tag{37}$$

Question 3 - Dodelson 2.1

Convert the following quantities by inserting the appropriate factors of c, $\hbar$ and k_B:

- $T_0 = 2.725$ K $\to$ eV
- $\rho_\gamma = \pi^2 T_0^4/15 \to$ eV^4 and g cm^{-3}
- $1/H_0 \to$ cm
- $m_{Pl} \equiv 1.2 \cdot 10^{19}$ GeV $\to$ K, cm^{-1}, s^{-1}

In addition to this problem, express:

a) The critical density today $\rho_c = 3H_0^2/8\pi G$ in units of $h^2 M_\odot \text{Mpc}^{-3}$;

b) c/H_0 in units of h^{-1}Mpc.

We can convert T_0 from Kelvin to eV with the equation $T' \equiv E = k_B T_0$, where $k_B = 8.62 \cdot 10^{-5}$ eV/K and $T_0 = 2.725$ K,

$$\therefore T_0 = 8.62 \cdot 10^{-5} \cdot 2.725 = 2.35 \cdot 10^{-4} \text{ eV}. \tag{38}$$

The convertion of ρ_γ to eV^4 is simple, we only need to express T_0 in terms of eV, as done before,

$$\therefore \rho_\gamma = \pi^2 \frac{(2.35 \cdot 10^{-4})^4}{15} = 2 \cdot 10^{-15} \text{ eV}^4. \tag{39}$$

To convert ρ_γ to g cm^{-1}, we need to use energy identities:

$$E = mc^2 \to m = \frac{E}{c^2}, \tag{40}$$

$$E = \frac{\hbar c}{k} \to k^3 = \frac{\hbar^3 c^3}{E^3}, \tag{41}$$

and then

$$\frac{m}{k^3} = \frac{E^4}{\hbar^3 c^5} \to \left[\frac{m}{k^3}\right] = \frac{[M]}{[L^3]} = \frac{kg}{m^3} = \frac{(J/K)^4}{(J \cdot s)^3 (m/s)^5}. \tag{42}$$

We already have the convertion parameters, we only need to consider $E = k_B T_0$ eV, as previously used and then we receive:

$$\rho_\gamma = \frac{\pi^2}{15}\frac{(k_B T_0)^4}{\hbar^3 c^5} = \frac{\pi^2}{15}\frac{(2.725 \cdot 1.380649 \cdot 10^{-23})^4}{(1.054571817 \cdot 10^{-34})^3 \cdot (2.998 \cdot 10^8)^5} = 4.64 \cdot 10^{-31} \text{ kg/m}^3, \tag{43}$$

where $k_B = 1.380649 \cdot 10^{-23}$ J/K, $\hbar = 1.054571817 \cdot 10^{-34}$ J·s and $c = 2.998 \cdot 10^8$ m/s,

$$\therefore \rho_\gamma = 4.64 \cdot 10^{-31} \frac{10^3}{10^6} \text{ g/cm}^3 = 4.64 \cdot 10^{-34} \text{ g/cm}^3. \tag{44}$$

To convert $1/H_0$ we can use the latest Planck 2018 Data Release for $H_0 \approx 67.4$ km s^{-1} Mpc^{-1}, where $1/H_0 = 1/67.4$ km^{-1} s Mpc. To convert this value, we can use the light speed in cm/s with $c = 2.998 \cdot 10^{10}$ cm/s. We can also convert Mpc to $3.0857 \cdot 10^{19}$ km. Now,

$$\frac{1}{H_0} = \frac{1}{67.4} \cdot 2.998 \cdot 10^{10} \cdot 3.0857 \cdot 10^{19} \approx 1.37 \cdot 10^{28} \text{ cm}. \tag{45}$$

To convert m_{Pl} from GeV to K, we only need to divide m_{Pl} by the Boltzmann Constant in GeV/K:

$$m_{Pl} = \frac{1.2 \cdot 10^{19}}{8.62 \cdot 10^{-9} 10^{-5}} = 1.4 \cdot 10^{32} \text{ K}. \tag{46}$$

To convert m_{Pl} from GeV to cm^{-1}, we can use the previously used identity $k = \hbar c / k$:

$$m_{Pl} = \frac{1.2 \cdot 10^{19}}{\hbar \cdot c} = \frac{1.2 \cdot 10^{19}}{6.58211957 \cdot 10^{-25} \cdot 2.998 \cdot 10^{10}} \approx 6.1 \cdot 10^{32} \text{ cm}^{-1}, \tag{47}$$

where $\hbar = 6.58211957 \cdot 10^{-25}$ GeV·s and c is in cm/s.

To convert m_{Pl} from GeV to s^{-1}, we only need to divide m_{Pl} by $\hbar$ in GeV·s:

$$m_{Pl} = \frac{1.2 \cdot 10^{19}}{6.58211957 \cdot 10^{-25}} = 1.8 \cdot 10^{43} \text{ s}^{-1}. \tag{48}$$

a) Let's set some quantities:

$$\begin{cases} H_0 = 100h \text{ km s}^{-1} \text{ Mpc}^{-1} \text{ (Hubble's constant in function of its reduced form)} \\ \text{kg} = 5 \cdot 10^{-31}\ M_\odot \text{ (kg to solar mass)} \\ \text{m} = 3.241 \cdot 10^{-23} \text{ Mpc (m to Mpc)} \\ \text{km} = 3.241 \cdot 10^{-20} \text{ Mpc (km to Mpc)} \end{cases} \tag{49}$$

The gravitational constant can be quantified as

$$G = 6.67 \cdot 10^{-11} \text{ N m}^2 \text{ kg}^{-2} = 6.67 \cdot 10^{-11} \text{ kg m s}^{-2} \text{ m}^2 \text{ kg}^{-2} = 6.67 \cdot 10^{-11} \text{ m}^3 \text{ s}^{-2} \text{ kg}^{-1}, \tag{50}$$

and using our convertion for m to Mpc, we then have

$$G = 6.67 \cdot 10^{-11} \cdot (3.241 \cdot 10^{-23})^3 \approx 2.27 \cdot 10^{-78} \text{ Mpc}^3 \text{ s}^{-2} \text{ kg}^{-1}, \tag{51}$$

and lastly, we can convert kg to solar mass

$$G = \frac{2.27 \cdot 10^{-78}}{5 \cdot 10^{-31}} \approx 4.54 \cdot 10^{-48} \text{ Mpc}^3 \text{ s}^{-2}\ M_\odot^{-1}. \tag{52}$$

And then

$$\rho_c = \frac{3H_0^2}{8\pi G} = \frac{3 \cdot 10^4}{8\pi G}\ h^2 \text{ km}^2 \text{ s}^{-2} \text{ Mpc}^{-2}, \tag{53}$$

converting km to Mpc

$$\rho_c = \frac{3 \cdot 10^4}{8\pi G}(3.241 \cdot 10^{-20})^2 \approx \frac{3.94}{\pi G} \cdot 10^{-36}\ h^2 \text{ s}^{-2}, \tag{54}$$

now, substituting G, we have

$$\rho_c = \frac{1}{\pi} \cdot \frac{3.94 \cdot 10^{-36}}{4.54 \cdot 10^{-48}} \approx 2.76 \cdot 10^{11}\ h^2\ M_\odot \text{ Mpc}^{-3}. \tag{55}$$

b) We can set the speed of light in terms of Mpc/s, where $c = 9.71561 \cdot 10^{-15}$ Mpc/s. Now,

$$\frac{c}{H_0} = \frac{9.71561 \cdot 10^{-15}}{100} = 9.71561 \cdot 10^{-17}\ h^{-1}\ \text{km}^{-1}\ \text{Mpc}^2, \tag{56}$$

converting km to Mpc, we have

$$\frac{c}{H_0} = \frac{9.71561 \cdot 10^{-17}}{3.241 \cdot 10^{-20}} \approx 3 \cdot 10^3\ h^{-1}\ \text{Mpc}. \tag{57}$$