Question 1 - Galaxy Cluster Abundance

In PS 6 you computed the variance σ^2 of linear fluctuations on a scale R

$$\sigma^2(z,R) = D^2(z)\int \frac{k^2 \mathrm{d}k}{2\pi^2}|W(kR)|^2 P_L(k) = D^2(z)\sigma^2(z=0,R) \tag{1}$$

where $P_L(k)$ is the linear matter power spectrum at redshift $z = 0$ (e.g. from CAMB) and

$$W(kR) = \frac{3}{k^2R^2}\left[\frac{\sin(kR)}{kR} - \cos(kR)\right] \tag{2}$$

is the Fourier Transform of a spherical top-hat window of radius R.
a) Use your previous results to compute $\sigma(z,M) = D(z)\sigma(z=0,M)$ at a scale R that encloses mass M at the background density $\bar{\rho}_{m0}$ today. You just need to convert from radius to mass using $M = \bar{\rho}_{m0}4\pi R^3/3$. Plot $\sigma(z,M)$ versus M for $z = 0$ and $z = 1$ in log scale, for the range $M = [10^{12}, 10^{16}]M_\odot/h$ and choose an appropriate range in the y-axis. What value of M corresponds to $\sigma(z=0,M) = \delta_c = 1.686$?
It will be useful for the next items if you compute $\sigma(z=0,M)$ for certain values of M and define an interpolating function (e.g. spline) that gives you $\sigma(z=0,M)$ for any values of M (check that you have a sufficient number of points for the interpolation to work well). Then $\sigma(z,M) = D(z)\sigma(z=0,M)$ gives you σ for any z and M.
b) Compute $\mathrm{d}\sigma/\mathrm{d}M$ by finite difference of the previous result, and use this to compute

$$\frac{\mathrm{d}\ln\sigma^{-1}}{\mathrm{d}\ln M} = -\frac{M}{\sigma}\frac{\mathrm{d}\sigma}{\mathrm{d}M} \tag{3}$$

Plot $\mathrm{d}\ln\sigma^{-1}/\mathrm{d}\ln M$ versus M in the same mass range as in a). Again, define an interpolating function that gives you this function at any value of M.
c) Use the results from a) and b) to compute the halo mass function as

$$\frac{\mathrm{d}n(z,M)}{\mathrm{d}\ln M} = f(\sigma)\frac{\bar{\rho}_m}{M}\frac{\mathrm{d}\ln\sigma^{-1}}{\mathrm{d}\ln M} \tag{4}$$

for the fit from Tinker et al. 2008 (https://arxiv.org/abs/0803.2706), i.e.

$$f(\sigma) = A\left[\left(\frac{\sigma}{b}\right)^{-a} + 1\right]\exp\left[-\frac{c}{\sigma^2}\right] \tag{5}$$

and choose values for A, a, b, c that are appropriate for $\Delta = 200$ (see https://arxiv.org/abs/0803.2706Tinker Eq. 3 and Table 2.). Plot $\mathrm{d}n/\mathrm{d}\ln M$ versus M in the same range as in a), for $z = 0$ and $z = 1$. Is $f(\sigma)$ properly normalized? What about $g(\sigma)$ from Eq. C2 and Table C4 in Appendix C?
d) Integrate $\mathrm{d}n/\mathrm{d}\ln M$ in mass M for masses above $M_{lim} = 10^{14}M_\odot/h$ for various values of z and interpolate to finally obtain the number density $n(z)$ at any z:

$$n(z) = \int_{M_{lim}}^{\infty}\mathrm{d}\ln M\frac{\mathrm{d}n(z,M)}{\mathrm{d}\ln M} \tag{6}$$

Plot $n(z)$ versus z, for $z = [0, 2]$.

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e) Finally, integrate $n(z)$ in comoving volume $\mathrm{d}V = \Delta\Omega \mathrm{d}z D_A^2(z)/H(z)$, for $\Delta\Omega = 5000\mathrm{deg}^2$ (convert $\mathrm{deg}^2 \to \mathrm{rad}^2$) to find the number $N(z_i)$ of halos in redshift bins of width $\Delta z = 0.1$:

$$N(z_i)\Delta\Omega \int_{z_i}^{z_i+\Delta z} \mathrm{d}z \frac{D_A^2(z)}{H(z)} n(z) \tag{7}$$

Plot $N(z_i)$ versus z_i for 20 bins in z_i, i.e. from 0 to 2.

Item a)

Computing $\sigma(z, M)$ for $z = 0$ and $z = 1$, we receive Figure 1 with $M = 2.57 \cdot 10^{12}\ M_\odot/h$ for $\sigma(M) = \delta_c$.

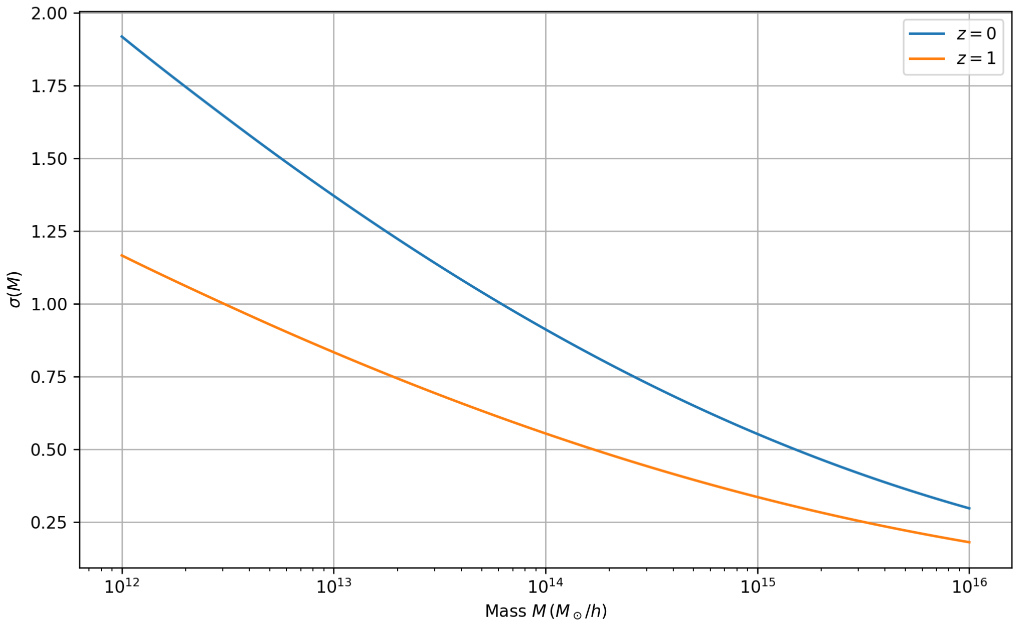

Figure 1: $\sigma(M)$ in function of the mass in units of $M_\odot/h$.

Item b)

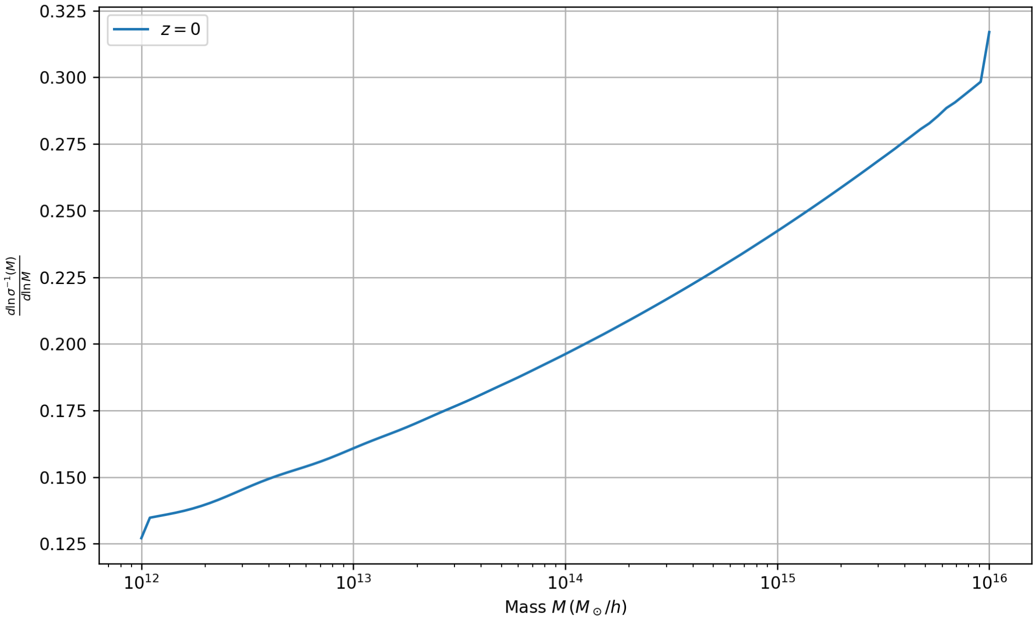

Figure 2: $\frac{\mathrm{d} \ln \sigma^{-1}}{\mathrm{d} \ln M}$ in function of the mass in units of $M_\odot/h$.

Item c)

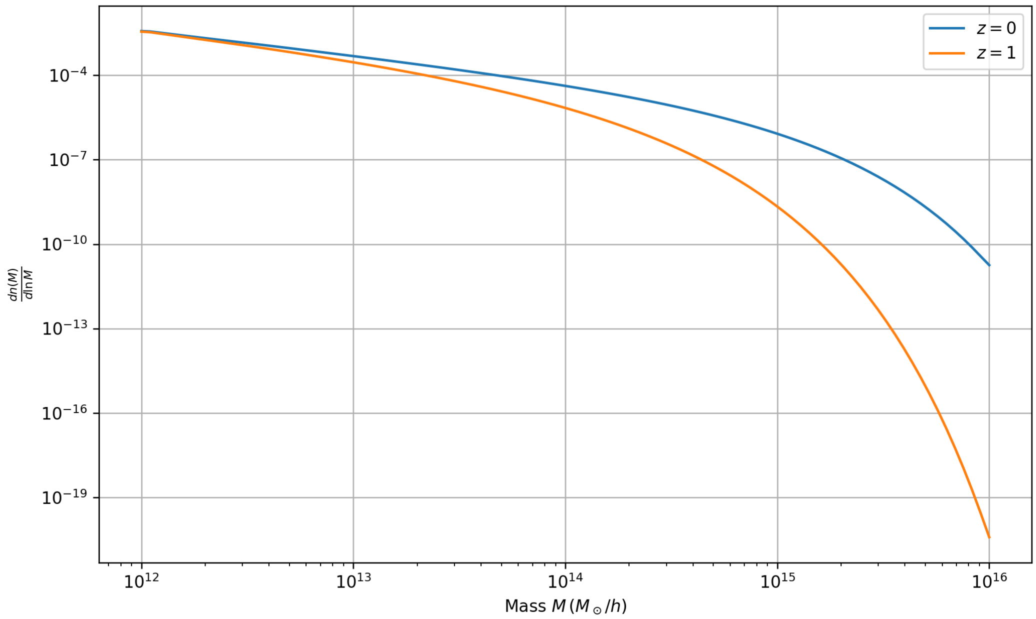

Figure 3: $\frac{\mathrm{d}n}{\mathrm{d}\ln M}$ in function of the mass in units of $M_\odot/h$.

From the paper, we observe that $f(\sigma)$ is not normalized, but $g(\sigma)$ (Equation 8) is.

$$g(\sigma) = B\left[\left(\frac{\sigma}{e}\right)^{-d} + \sigma^{-f}\right] e^{-g/\sigma^2}. \tag{8}$$

This happens because $f(\sigma)$ is arbitrary for lower masses, but $g(\sigma)$ is better behaved at $z = 0$.

Item d)

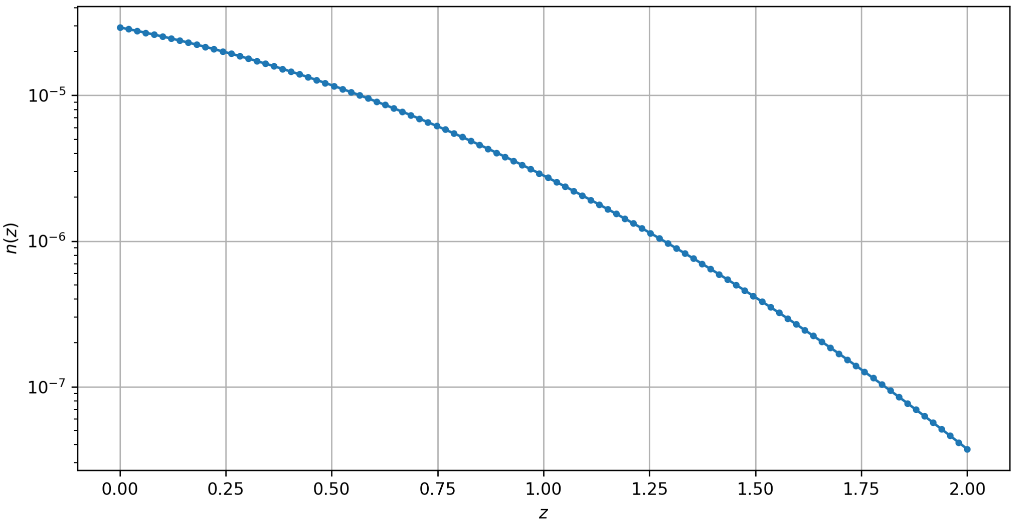

Figure 4: $n(z)$ in function of redshift.

Item e)

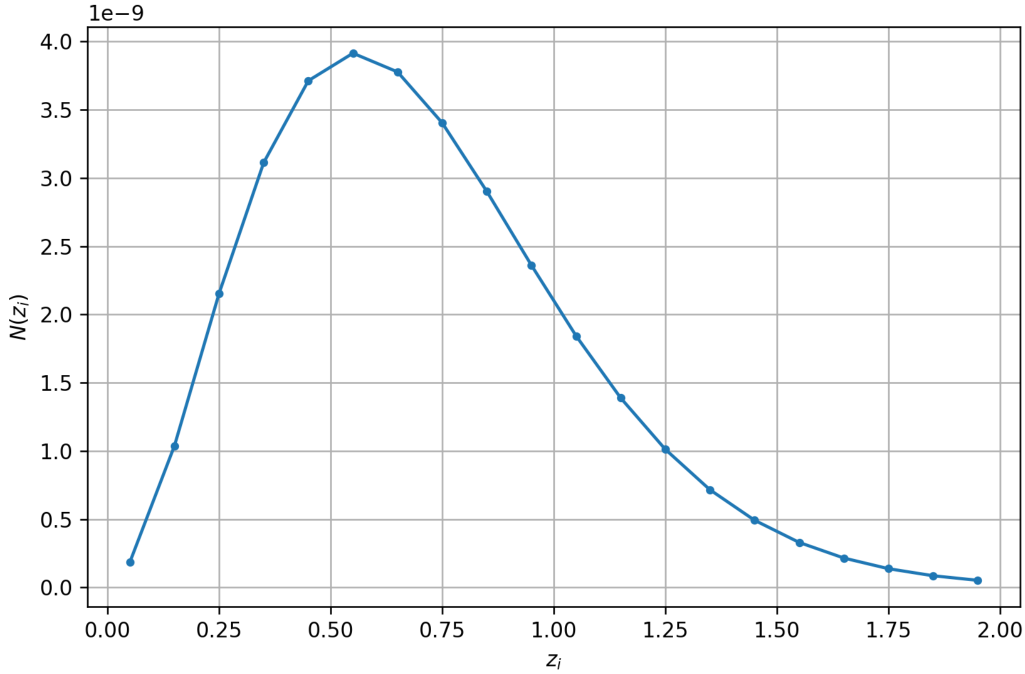

Figure 5: $N(z_i)$ in function of redshift bins z_i.