Question 1 - Dodelson 3.4

Solve the rate equation 1 numerically to determine the neutron fraction as a function of temperature. Ignore decays. There are (at least) two ways to perform this computation. The first is to treat it as a simple ordinary differential equation and solve it numerically. The second is to proceed analytically and reduce the problem to an evaluation of a single numerical integral. This second method, which I'll lead you through here, is based on a numerical coincidence noted by Bernstein, Brown, and Feinberg (1988).

$$\frac{\mathrm{d}X_n}{\mathrm{d}x} = \frac{x\lambda_{np}}{H(x=1)}\{e^{-x} - X_n(1+e^{-x})\} \tag{1}$$

where

$$H(x=1) = \sqrt{\frac{4\pi^3 GQ^4}{45}} \cdot \sqrt{10.75} \approx 1.13\,\mathrm{sec}^{-1} \tag{2}$$

(a) Using standard differential equation techniques, show that a formal solution to Eq. 1 is:

$$X_n(x) = \int_{x_i}^{x} dx' \frac{\lambda_{np}(x')e^{-x'}}{x'H(x')} e^{\mu(x')-\mu(x)} \tag{3}$$

where x_i is some initial, very high temperature, and

$$\mu(x) \equiv \int_{x_i}^{x} \frac{dx'}{x'H(x')} \lambda_{np}(x') \left[1 + e^{-x'}\right]. \tag{4}$$

(b) Use Eqs. 5 and 6

$$\begin{aligned}
\lambda_{np} &= \frac{255}{\tau_n x^5}\left(12 + 6x + x^2\right) &(5)\\
\rho &= \frac{\pi^2}{30}T^4 \left[\sum_{\text{i = bosons}} g_i + \frac{7}{8}\sum_{\text{i = fermions}} g_i\right] \quad (i \text{ relativistic})\\
&\equiv g_* \frac{\pi^2}{30}T^4 &(6)
\end{aligned}$$

to compute the integrating factor μ analytically. Show that it is equal to

$$\mu = -\frac{255}{\tau_n \mathcal{Q}}\left[\frac{4\pi^3 G\mathcal{Q}^2 g_*}{45}\right]^{-1/2} \cdot \left[\left(\frac{4}{x^3} + \frac{3}{x^2} + \frac{1}{x}\right) + \left(\frac{4}{x^3} + \frac{1}{x^2}\right)e^{-x}\right]\Bigg|_{x_i}^{x}. \tag{7}$$

The simple form for μ is the result of numerical coincidence alone.
(c) With the results of part **(b)**, do the single numerical integral in **(a)** numerically. Compare the asymptotic result at $x = \infty$ with the result in the text, $X_n(x=\infty) = 0.15$.

Item a)

We can write the differential equation as

$$\frac{\mathrm{d}X_n}{\mathrm{d}x}+\frac{x\lambda_{np}}{H(x=1)}(1+e^{-x})X_n=\frac{x\lambda_{np}}{H(x=1)}e^{-x}\rightarrow\frac{\mathrm{d}X_n}{\mathrm{d}x}+\frac{\lambda_{np}}{xH(x)}(1+e^{-x})X_n=\frac{\lambda_{np}}{xH(x)}e^{-x},$$

which can be solved by integrating $\mu(x)$:

$$\mu(x)=\int_{x_i}^{x}\frac{\mathrm{d}x'}{x'H(x')}\lambda_{np}(x')(1+e^{-x'}),$$

then X_n is

$$X_n(x)=\frac{1}{e^{\mu(x)}}\left(\int_{x_i}^{x}e^{\mu(x')}\frac{\lambda_{np}(x')e^{-x'}}{x'H(x')}\,\mathrm{d}x'\right),$$

then we receive

$$\boxed{X_n(x)=\int_{x_i}^{x}\mathrm{d}x'\frac{\lambda_{np}(x')e^{-x'}}{x'H(x')}e^{\mu(x')-\mu(x)}}$$

Item b)

We have that

$$H(x)=\sqrt{\frac{8\pi G\rho}{3}}=\left[\frac{8\pi G}{3}\frac{g_*\pi^2T^4}{30}\right]^{-1/2},$$

but we know from the lecture notes that $T=Q/x$, then

$$H(x)=\frac{Q}{x}\left[\frac{4\pi^3GQ^2g_*}{45}\right]^{-1/2},$$

so, substituting $H(x')$ and $\lambda_{np}(x')$ in $\mu(x)$, we have

$$\begin{aligned}\mu(x)&=\frac{255}{\tau_nQ}\left[\frac{4\pi^3GQ^2g_*}{45}\right]^{-1/2}\int_{x_i}^{x}\frac{x'^2}{x'^5}\frac{1}{x'}(12+6x'+x'^2)(1+e^{-x'})\,\mathrm{d}x'\\&=\frac{255}{\tau_nQ}\left[\frac{4\pi^3GQ^2g_*}{45}\right]^{-1/2}\left[\int_{x_i}^{x}\left(\frac{12}{x'^4}+\frac{6}{x'^3}+\frac{1}{x'^2}\right)\,\mathrm{d}x'+\int_{x_i}^{x}\left(\frac{12}{x'^4}+\frac{6}{x'^3}+\frac{1}{x'^2}\right)e^{-x'}\,\mathrm{d}x'\right],\end{aligned}$$

and then

$$\boxed{\mu(x)=-\frac{255}{\tau_nQ}\left[\frac{4\pi^3GQ^2g_*}{45}\right]^{-1/2}\cdot\left[\left(\frac{4}{x^3}+\frac{3}{x^2}+\frac{1}{x}\right)+\left(\frac{4}{x^3}+\frac{1}{x^2}\right)e^{-x}\right]\Bigg|_{x_i}^{x}}$$

Item c)

Solving numerically (`Python`) for $x=10000$, $\tau_n=886.7$ s, $G=1.545\cdot10^{-2}$ MeV^{-4} s^{-2}, $g_*=10.75$ and $Q=1.293$ MeV, we receive

$$\boxed{X_n=0.1481}$$

which is a close answer for $X_n(x\rightarrow\infty)\rightarrow0.15$.

Question 2 - Dodelson 3.8

Solve for the evolution of the free electron fraction. Do not compare your results with Figure 3.4 until you finish part **(d)**. Throughout, take parameters $\Omega_m = 1, \Omega_b = 0.06, h = 0.5$.

(a) Use as an evolution variable $x \equiv \epsilon_0/T$ instead of time in Eq. (3.39). Rewrite the equation in terms of x and the Hubble rate at $T = \epsilon_0$.

(b) Using the methods of Section 3.4, find the final freeze-out abundance of the free electron fraction, $X_e(x = \infty)$.

(c) Numerically integrate the equation from **(a)** from $x = 1$ down to $x = 1000$. What is the final frozen-out X_e?

(d) Peebles (1968) argued that even captures to excited states would not be important except for the fraction of times that the $n = 2$ state decays into two photons or expansion redshifts the Lyman alpha photon so that it cannot pump up a ground-state atom. Quantitatively, he multiplied the right-hand side of Eq. 8

$$\frac{\mathrm{d}X_e}{\mathrm{d}t} = \{(1 - X_e)\beta - X_e^2 n_b \alpha^{(2)}\} \tag{8}$$

by the correction factor,

$$C = \frac{\Lambda_\alpha + \Lambda_{2\gamma}}{\Lambda_\alpha + \Lambda_{2\gamma} + \beta^{(2)}} \tag{9}$$

where the two-photon decay rate is $\Lambda_{2\gamma} = 8.227\,\mathrm{sec}^{-1}$; Lyman alpha production is $\beta^{(2)} = \beta^{3\epsilon_0/4T}$; and

$$\Lambda_\alpha = \frac{H(3\epsilon_0)^2}{(8\pi)^2}. \tag{10}$$

Do this and show how it changes your final answer. Now compare the freezeout abundance with the result of **(c)** and the evolution with Figure 3.4.

Item a)

The Equation (3.39) can be written as

$$\frac{\mathrm{d}X_e}{\mathrm{d}t} = (1 - X_e)\beta - n_b \alpha^{(2)} X_e^2,$$

where X_e is the free electron fraction, n_b is the baryon number density, $\alpha^{(2)}$ is the recombination coefficient (or the rate of recombination) and β is the ionization rate.

We can rewrite the Hubble parameter in terms of T considering the early universe, as it was matter-dominated (we're asked to consider $\Omega_m = 1$), and then

$$H(T) = H_0 \sqrt{\Omega_m} \left(\frac{T}{T_0}\right)^{3/2},$$

where H_0 is the Hubble constant, Ω_m is the matter density and T_0 is the present-day temperature of the universe.

Now, to set $x = \epsilon_0/T$, we have

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -\frac{\epsilon_0}{T}\frac{\mathrm{d}T}{\mathrm{d}t},$$

but we know that
$$H(T) = -\frac{1}{T}\frac{\mathrm{d}T}{\mathrm{d}t},$$
then
$$\frac{\mathrm{d}x}{\mathrm{d}t} = xH(T) \to \mathrm{d}t = \frac{\mathrm{d}x}{xH(T)}.$$

We can now do
$$\begin{aligned} \frac{\mathrm{d}X_e}{\mathrm{d}t} \to \frac{\mathrm{d}X_e}{\mathrm{d}x}\frac{\mathrm{d}x}{\mathrm{d}t} &= (1-X_e)\beta - n_b\alpha^{(2)}X_e^2 \\ \frac{\mathrm{d}X_e}{\mathrm{d}x}xH(T) &= (1-X_e)\beta - n_b\alpha^{(2)}X_e^2, \end{aligned}$$
and we receive
$$\boxed{\frac{\mathrm{d}X_e}{\mathrm{d}x} = \frac{(1-X_e)\beta - n_b\alpha^{(2)}X_e^2}{xH(T=\epsilon_0)}}$$

Item b)

Considering the same (3.39) Equation from Dodelson, we understand that at freeze-out, the rate change of X_e is effectively zero, so we set
$$\frac{\mathrm{d}X_e}{\mathrm{d}t} = 0 \to 0 = (1-X_e)\beta - n_b\alpha^{(2)}X_e^2,$$
and when expanding all terms, we receive
$$n_b\alpha^{(2)}X_e^2 + \beta X_e - \beta = 0,$$
which we can solve with the quadratic equation by applying
$$X_e = \frac{-\beta \pm \sqrt{\beta^2 + 4n_b\alpha^{(2)}\beta}}{2n_b\alpha^{(2)}},$$
and X_e must be positive, so we choose the positive root, which results in
$$X_e = \frac{-\beta + \sqrt{\beta^2 + 4n_b\alpha^{(2)}\beta}}{2n_b\alpha^{(2)}}.$$

β can be written in function of x by
$$\beta = \beta^{(0)}e^{-x},$$
where $\beta^{(0)}$ is a constant. And we understand that when $x \to \infty$, $\beta \approx 0$. In these terms, we receive that
$$\boxed{X_e \approx 0}$$

In the freeze-out condition, the value of X_e is expected to be small, so our result is close to what we are looking.

Item c)

Item d)