Question 1 - Scale Factor Evolution

In this problem, you will solve the Friedmann equation **numerically**.

$$\left(\frac{\dot{a}}{a}\right)^2 = H_0^2 \left[\underbrace{\Omega_k a^{-2}}_{\text{Curvature}} + \underbrace{\Omega_m a^{-3}}_{\text{Matter}} + \underbrace{\Omega_r a^{-4}}_{\text{Radiation}} + \underbrace{\Omega_{DE} a^{-3(1+w)}}_{\text{Dark Energy}} \right] \tag{1}$$

$$\text{where} \quad \Omega_k = 1 - (\Omega_m + \Omega_r + \Omega_{DE}) \tag{2}$$

For that, you must write a numerical program that uses a differential equation solver (e.g. Runge-Kutta or better).
In each case below, set up appropriate **initial conditions** using the dominant component at early times. For instance, for a universe with matter and radiation, at early times radiation dominates and you may use the analytical solution for $a(t)$ to set the correct value of $a_0 = a(t_0)$ at the initial time t_0.
Note that the only quantity with units here is H_0 (units of time^{-1}). Use H_0 such that you present your results with time in Gigayears (Gyr). For all cases, fix $h = 0.72$.
a) First do the single-component cases. Leave one $\Omega_i = 1$ at a time, and set all others equal to zero (notice the constraint in Eq. 2 above). For the dark energy, choose a cosmological constant, i.e. $w = -1$. For each case, plot the numerically derived scale factor as a function of time and compare your numerical solution to the analytical solution, plotting also the analytical solution.
b) Now do intermediate **two-component** cases, containing i) matter + cosmological constant and ii) matter + curvature (choose closed universes ($k > 0$), i.e. $\Omega_k < 0$ and $\Omega_m > 1.0$). In these cases you also have analytical solutions to compare.
c) Now do the **complete** case with all terms. Use **fiducial** values: $\Omega_k = 0$ (flat universe), $\Omega_m = 0.25$, $\Omega_{DE} = 0.75$, $w = -1$, $\Omega_r = 8.2 \cdot 10^{-5}$. Compute the **age of the Universe**, by finding the time t_0 that corresponds to today, i.e. the time when $a(t_0) = 1$. Change the parameter values one at a time (Ω's and w for cases below) and **plot** the corresponding $a(t)$ for each variation set. See also the impact on the age of the Universe

i) $\Omega_m = 0.1, 0.25, 0.5, 1.0$ (with all other parameters equal to fiducial)

ii) $\Omega_{DE} = 0.5, 0.75, 1.0$ (with all other parameters equal to fiducial)

iii) $w = -0.8, -1.0, -1.2$ (with all other parameters equal to fiducial)

iv) $\Omega_r = (6, 8, 10) \times 10^{-5}$ (with all other parameters equal to fiducial)

v) Flat cases: $(\Omega_m, \Omega_{DE}) = (0.0, 1.0), (0.1, 0.9), (0.25, 0.75), (0.75, 0.25), (1.0, 0.0)$.

Indicate clearly in the plots the cases you are showing.

Item a)

The single-component analytical solutions for the Friedmann equation can be obtained by zeroing every unwanted component, leaving only the desired one.

All numerical solutions will be obtained with `Python` package `scypy` (`solve_ivp`).

Matter

For the matter dominated Universe, we have

$$\begin{aligned}\left(\frac{\dot{a}}{a}\right)^2 &= H_0^2\Omega_m a^{-3} \\ \frac{\mathrm{d}a}{\mathrm{d}t} &= H_0 a\sqrt{\Omega_m a^{-3}} \\ \frac{\mathrm{d}a}{\mathrm{d}t} &= H_0\sqrt{\Omega_m a^{-1}} \\ \sqrt{a}\ \mathrm{d}a &= H_0\sqrt{\Omega_m}\ \mathrm{d}t, \end{aligned} \tag{3}$$

integrating both sides from $0 \to a$ and $0 \to t$, we receive

$$\frac{2}{3}a^{3/2} = H_0\sqrt{\Omega_m}t, \tag{4}$$

and then

$$a(t) = \left(\frac{3}{2}H_0 t\sqrt{\Omega_m}\right)^{2/3}. \tag{5}$$

The comparison with the numerical solution can be observed in Figure 1

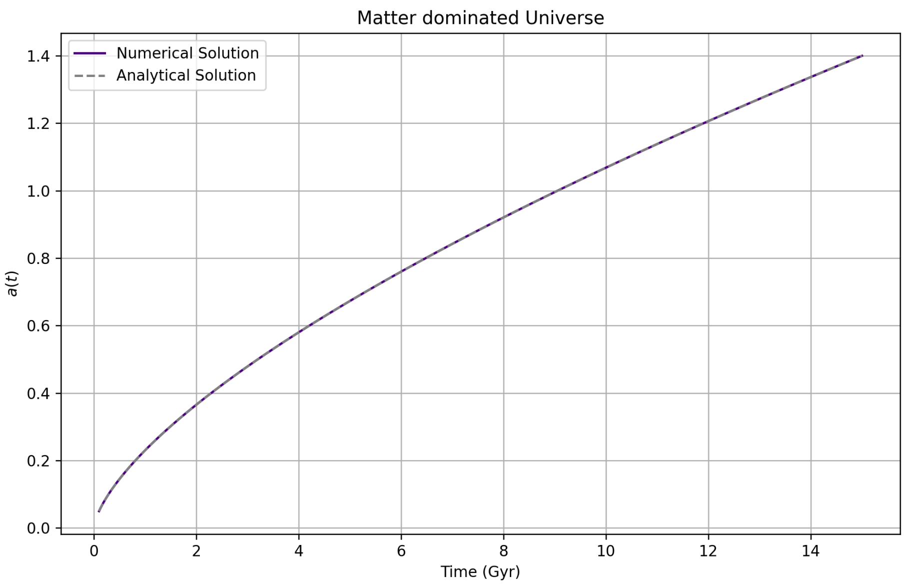

Figure 1: Matter dominated Universe, comparison between numerical and analytical solutions.

Radiation

For a Radiation dominated Universe, we have

$$\left(\frac{\dot{a}}{a}\right)^2 = H_0^2 \Omega_r a^{-4}, \tag{6}$$

in the same way as the Matter dominated Universe, we receive

$$a(t) = \left(2H_0 t\sqrt{\Omega_r}\right)^{1/2}. \tag{7}$$

Then, comparing with the numerical solution (Figure 2).

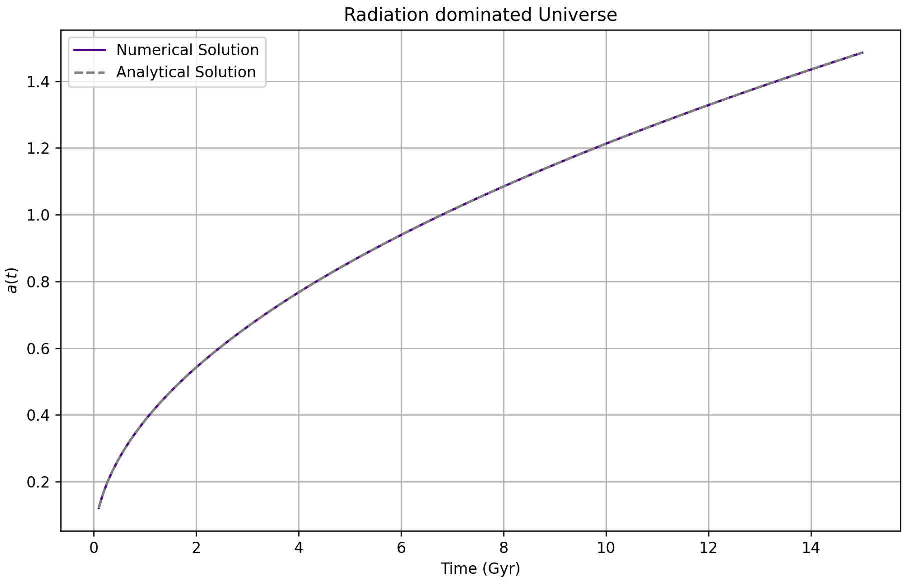

Figure 2: Radiation dominated Universe, comparison between numerical and analytical solutions.

Dark Energy

For a Dark Energy dominated Universe, we have

$$\left(\frac{\dot{a}}{a}\right)^2 = H_0^2\Omega_{DE}, \tag{8}$$

in the same way as the Matter dominated Universe, we receive

$$\frac{\mathrm{d}a}{a} = H_0\sqrt{\Omega_{DE}}t \tag{9}$$

integrating both sides from $0 \to a$ and $0 \to t$, we receive

$$\ln(a) = H_0 t\sqrt{\Omega_{DE}}, \tag{10}$$

and then

$$a(t) = \exp\Big(H_0 t\sqrt{\Omega_{DE}}\Big). \tag{11}$$

Then, comparing with the numerical solution (Figure 3).

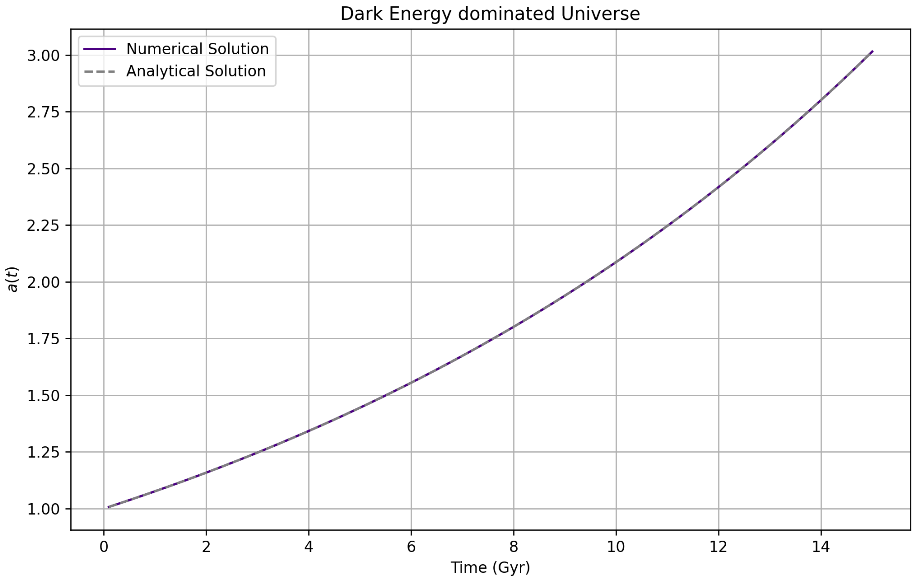

Figure 3: Dark Energy dominated Universe, comparison between numerical and analytical solutions.

Curvature

For a Curvature dominated Universe, we have

$$\left(\frac{\dot{a}}{a}\right)^2 = H_0^2 \Omega_k a^{-2}, \tag{12}$$

in the same way as the Matter dominated Universe, we receive

$$a(t) = tH_0\sqrt{\Omega_k}. \tag{13}$$

Then, comparing with the numerical solution (Figure 4).

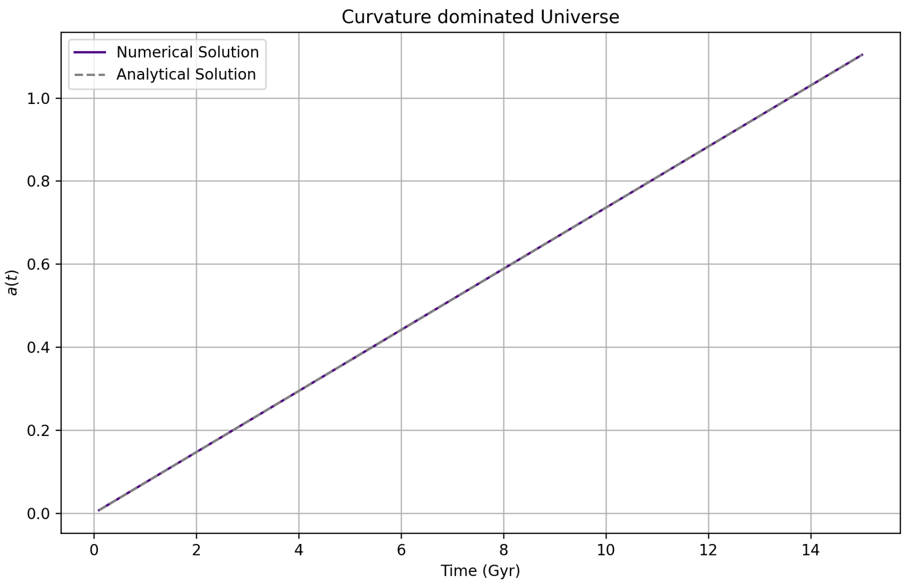

Figure 4: Curvature dominated Universe, comparison between numerical and analytical solutions.

Item b)

Matter + Curvature

For a Matter + Curvature model, we have

$$\frac{\mathrm{d}a}{\mathrm{d}t} = H_0\sqrt{\Omega_m a^{-1} + \Omega_k} \rightarrow \frac{\mathrm{d}a}{\sqrt{\Omega_m a^{-1} + \Omega_k}} = H_0 \ \mathrm{d}t, \tag{14}$$

from the lecture notes, we receive two equations

$$a(\theta) = -\frac{\Omega_m}{2\Omega_k}\left[1 - \cos\theta\right], \tag{15}$$

and

$$t(\theta) = \frac{\Omega_m}{12H_0(-\Omega_k)^{3/2}}\left[\theta - \sin\theta\right], \tag{16}$$

which can be interpolated to produce the analytical solution (Figure 5).

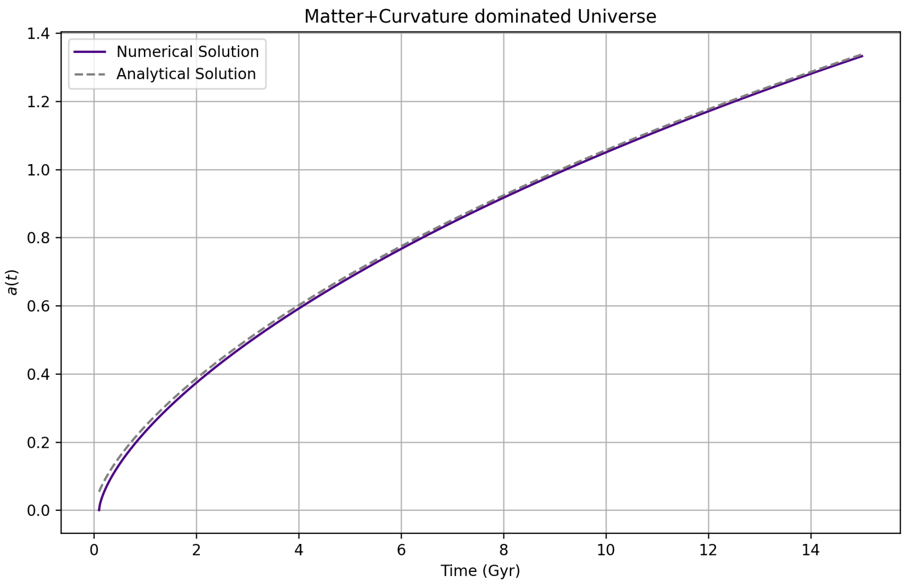

Figure 5: Matter + Curvature dominated Universe, comparison between numerical and analytical solutions. We have $\Omega_k = -0.5$ and $\Omega_m = 1.3$ (all else is zero).

Matter + Dark Energy

For the Matter + Dark Energy (Cosmological constant) model, we have

$$\frac{\mathrm{d}a}{\mathrm{d}t} = H_0\sqrt{\Omega_m a^{-1} + \omega_{DE}a^2} \rightarrow \frac{\mathrm{d}a}{\sqrt{\Omega_m a^{-1} + \Omega_{DE}a^2}} = H_0 \ \mathrm{d}t, \tag{17}$$

and again, from the lecture notes, we receive

$$a(t) = \left(\frac{\Omega_m}{\Omega_{DE}}\right)^{1/3} \sinh^{2/3}\left(\frac{3H_0 t\sqrt{\Omega_{DE}}}{2}\right). \tag{18}$$

And then, we can compare both analytical and numerical solutions in Figure 6.

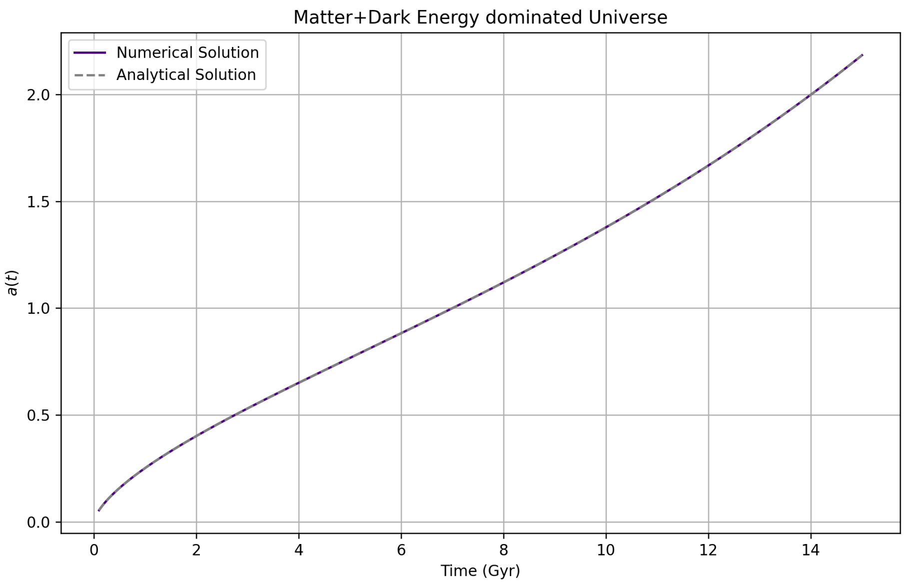

Figure 6: Matter + Dark Energy dominated Universe, comparison between numerical and analytical solutions. We have $\Omega_m = 1.3$, $\Omega_{DE} = 1.3$ and $w = -1$ (all else is zero).

Item c)

Now we can produce the numerical solutions with all components (with $\Omega_k = 0$ for the Flat Universe cases). The sampled Universe age can be found in each plot.

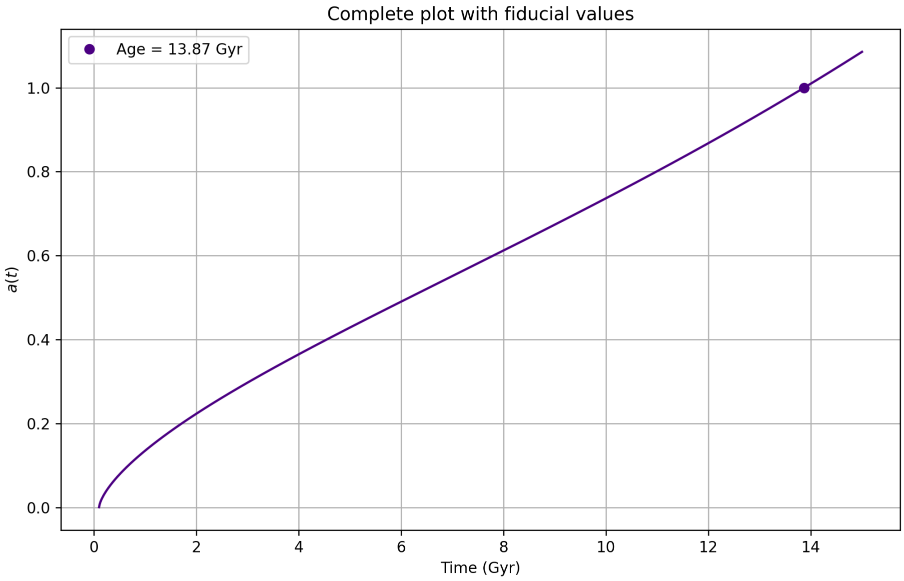

Figure 7: Plot with all fiducial values.

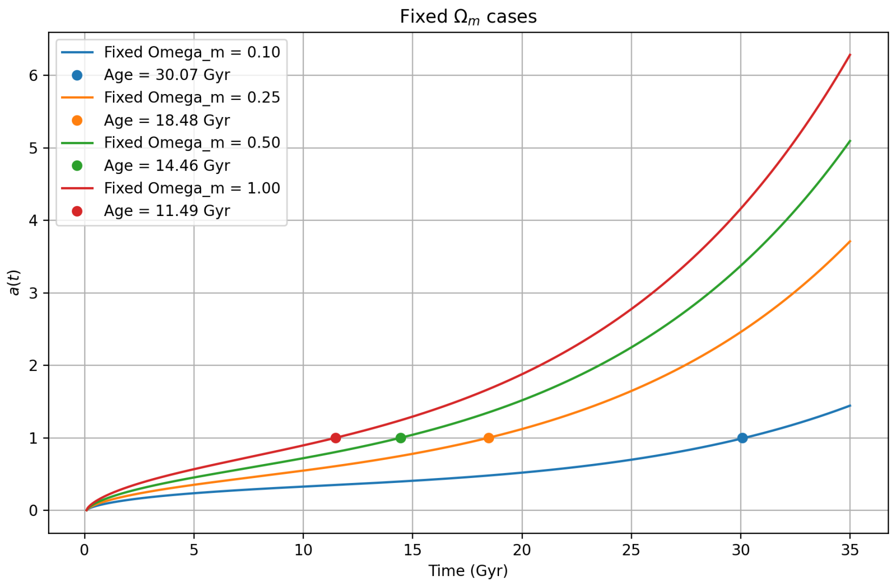

Figure 8: Plot with Ω_m fixed.

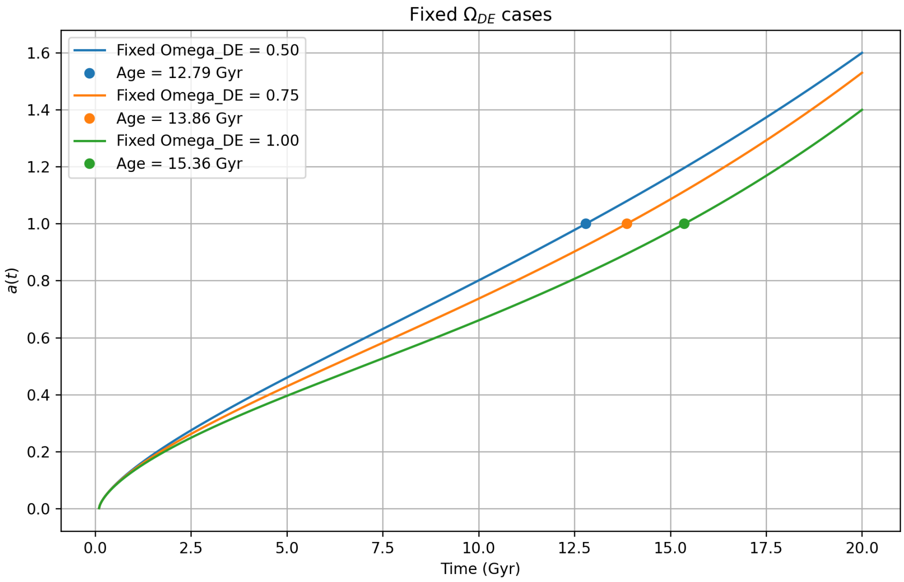

Figure 9: Plot with Ω_{DE} fixed.

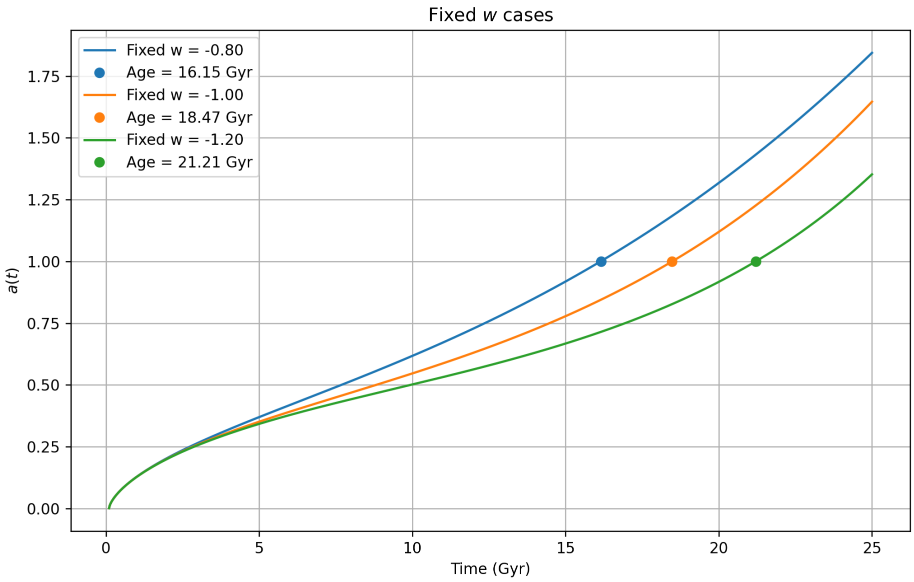

Figure 10: Plot with w fixed.

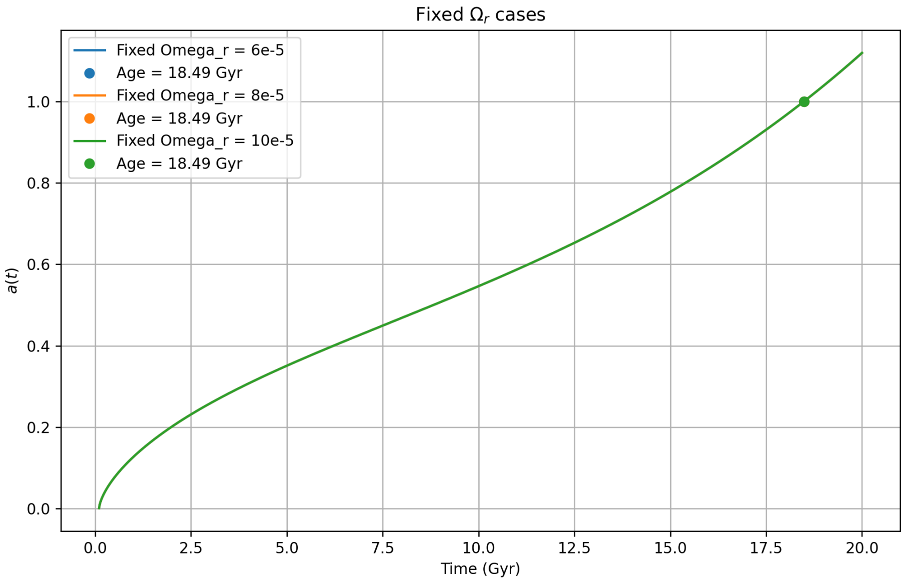

Figure 11: Plot with Ω_r fixed.

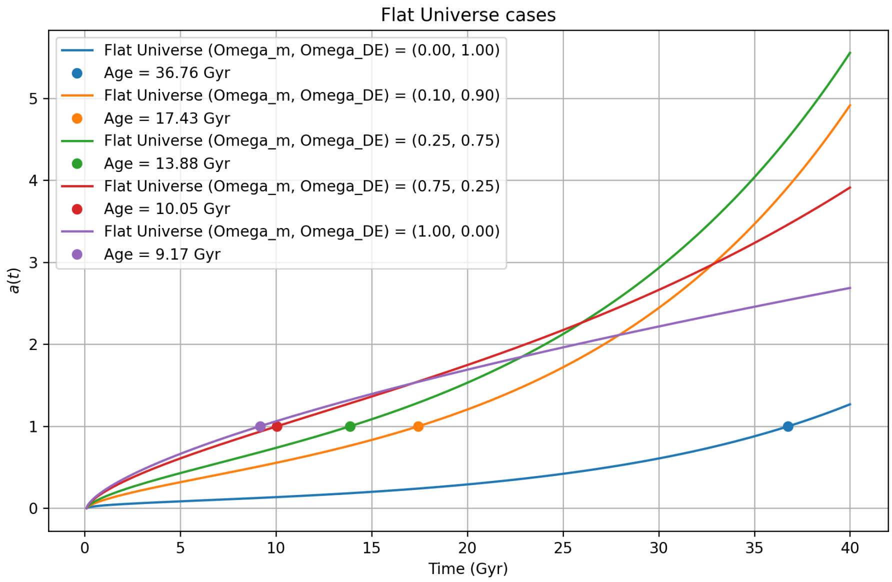

Figure 12: Plot with Flat Universe cases.