Question 1 - Experimental Time-Dilation

Consider the same experiment with cesium clocks on jet flights around the world from Problem Set 1. Considering now only the general relativistic (dynamical) effect, compute how much time the clock moving eastward should have lost/gained relative to the reference clocks on Earth. Repeat for the clock moving westward.
Suggestion: Read (again!) J. Hafele, R. Keating, Science, Vol 177, No 4044 (1972), pp. 166-168

From the lecture notes, we can deduct that

$$\mathrm{d}s^2 = -c^2\ \mathrm{d}t'^2 = c^2 g_{00}\ \mathrm{d}t^2 + g_{ij}\ \mathrm{d}x^i \mathrm{d}x^j = c^2 g_{00}\ \mathrm{d}\tau'^2, \tag{1}$$

where t' is the time measured within the inertial frame, τ' when time is measured by a rest clock in the gravitational field g_{00} is a function of spacial coordinates. Then, we have

$$\tau' = (-g_{00})^{-\frac{1}{2}} t' = -\left(1 + \frac{2\phi}{c^2}\right)^{-\frac{1}{2}} t' = \left(-1 + \frac{\phi}{c^2}\right) t'. \tag{2}$$

In contrast to the Problem Set 1, we are only considering general relativistic effects, so our $\Delta\tau$ is

$$\Delta\tau = \frac{\Delta\phi}{c^2}\Delta t, \tag{3}$$

where Δt is the flight duration andd $\Delta\phi = gh$, with g being Earth's gravity and h the flight's height. Then, considering

$$\begin{cases} g \approx \pi^2\ \mathrm{m/s}^2 \\ h \approx 10668\ \mathrm{m} \\ c = 2.998 \cdot 10^8\ \mathrm{m/s} \\ \Delta t_e = 41.2\ \mathrm{h} = 1.4832 \cdot 10^5\ \mathrm{s}\ \text{(Eastward)} \\ \Delta t_w = 48.6\ \mathrm{h} = 1.7496 \cdot 10^5\ \mathrm{s}\ \text{(Westward)} \end{cases}$$

For a Eastward flight, we have

$$\boxed{\Delta\tau \approx 173.75\ \text{ns (Eastward)}} \tag{4}$$

and

$$\boxed{\Delta\tau \approx 204.96\ \text{ns (Westward)}} \tag{5}$$

Question 2 - Dodelson 2.3

The metric for a particle traveling in the presence of gravitational field is $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ where $h_{00} = -2\phi$ and ϕ is the Newtonian gravitational potential; $h_{i0} = 0$; and $h_{ij} = -2\phi\delta_{ij}$. Find the equation of motion for a massive particle traveling in this field.

a) Show that $\Gamma^0_{00} = \partial\phi/\partial t$ and $\Gamma^i_{00} = \delta^{ij}\partial\phi/\partial x^j$.

b) Show that the time component of the geodesic equation implies that energy $p^0 + m\phi$ is conserved.

c) Show that the space components of the geodesic equation lead to $\mathrm{d}^2x^i/\mathrm{d}t^2 = -m\delta^{ij}\partial\phi/\partial x^j$ in agreement with Newtonian theory. Use the fact that the particle is nonrelativistic so $p^0 \gg p^i$.

a) The general form of the Christoffel symbols are

$$\Gamma^\sigma_{\mu\nu} = \frac{1}{2}g^{\sigma\rho}\left(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu}\right), \tag{6}$$

but then, as $\eta_{\mu\nu} \gg h_{\mu\nu}$ for a massive particle, we obtain

$$\Gamma^\sigma_{\mu\nu} = \frac{1}{2}\eta^{\sigma\rho}\left(\partial_\mu h_{\nu\rho} + \partial_\nu h_{\rho\mu} - \partial_\rho h_{\mu\nu}\right). \tag{7}$$

Therefore

$$\Gamma^0_{00} = \frac{1}{2}\eta^{00}\left(\partial_0 h_{00} + \cancel{\partial_0 h_{00}} - \cancel{\partial_0 h_{00}}\right) = \frac{1}{2}(-1)\frac{\partial}{\partial x^0}(-2\phi) \rightarrow \boxed{\Gamma^0_{00} = \frac{\partial\phi}{\partial t}} \tag{8}$$

And

$$\Gamma^i_{00} = \frac{1}{2}\eta^{ij}\left(\cancelto{0}{\partial_0 h_{0j}} + \cancelto{0}{\partial_0 h_{j0}} - \partial_j h_{00}\right) = -\frac{1}{2}\eta^{ij}\partial_j h_{00} \rightarrow \boxed{\Gamma^i_{00} = \delta^{ij}\frac{\partial\phi}{\partial x^j}} \tag{9}$$

b) The geodesic equation can be described as

$$\frac{\mathrm{d}^2x^\mu}{\mathrm{d}\tau^2} + \Gamma^\mu_{\alpha\beta}\frac{\mathrm{d}x^\alpha}{\mathrm{d}\tau}\frac{\mathrm{d}x^\beta}{\mathrm{d}\tau} = 0, \tag{10}$$

let $\alpha = \tau/m$, where

$$\frac{\mathrm{d}}{\mathrm{d}\alpha} = \frac{\mathrm{d}x^0}{\mathrm{d}\alpha}\frac{\mathrm{d}}{\mathrm{d}x^0} = p^0\frac{\mathrm{d}}{\mathrm{d}t}, \tag{11}$$

and we know that $p^0 = m\frac{\mathrm{d}x^0}{\mathrm{d}\tau} = \frac{\mathrm{d}x^0}{\mathrm{d}\alpha}$. From this, we can obtain the geodesic equation in a new format for Γ^0_{00}

$$p^0\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}x^0}{\mathrm{d}\alpha}\right) + \frac{\partial\phi}{\partial t}\left(\frac{\mathrm{d}x^0}{\mathrm{d}\alpha}\right)^2 = \cancel{p^0}\frac{\mathrm{d}p^0}{\mathrm{d}t} + \frac{\partial\phi}{\partial t}\left(p^0\right)^{\cancel{2}} = \frac{\mathrm{d}p^0}{\mathrm{d}t} + p^0\frac{\partial\phi}{\partial t} = 0, \tag{12}$$

for small velocities, let $p^0 \approx m$, and then we receive

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}t}\left(p^0 + m\phi\right) = 0} \tag{13}$$

the energy $p^0 + m\phi$ is conserved.

c) Expanding the geodesic equation, we have

$$\frac{\mathrm{d}^2x^0}{\mathrm{d}\tau^2} + \frac{\mathrm{d}^2x^i}{\mathrm{d}\tau^2} + \Gamma^i_{00}\left(\frac{\mathrm{d}x^0}{\mathrm{d}\tau}\right)^2 + \Gamma^i_{jk}\frac{\mathrm{d}x^j}{\mathrm{d}\tau}\frac{\mathrm{d}x^k}{\mathrm{d}\tau} = 0, \tag{14}$$

and then, considering $p^0 = m\frac{\mathrm{d}x^0}{\mathrm{d}\tau}$, with

$$\frac{\mathrm{d}}{\mathrm{d}\tau} = \frac{\mathrm{d}x^0}{\mathrm{d}\tau}\frac{\mathrm{d}}{\mathrm{d}x^0} = \frac{p^0}{m}\frac{\mathrm{d}}{\mathrm{d}t}, \tag{15}$$

then

$$\frac{p^0}{m}\frac{\mathrm{d}p^0}{\mathrm{d}t} + \left(\frac{p^0}{m}\right)^2\frac{\mathrm{d}^2x^i}{\mathrm{d}t^2} + \delta^{ij}\frac{\partial\phi}{\partial t}\left(\frac{p^0}{m}\right)^2 + \Gamma^i_{jk}p^jp^k = 0. \tag{16}$$

However, since we are dealing with a nonrelativistic particle, we know that $p^0 \approx m$ and $p^0 \gg p^i$. This means we can neglect the first and last terms, resulting in

$$\cancel{\left(\frac{p^0}{m}\right)^2}\frac{\mathrm{d}^2x^i}{\mathrm{d}t^2} + \delta^{ij}\frac{\partial\phi}{\partial t}\cancel{\left(\frac{p^0}{m}\right)^2} = 0, \tag{17}$$

$$\boxed{\frac{\mathrm{d}^2x^i}{\mathrm{d}t^2} = -\delta^{ij}\frac{\partial\phi}{\partial t}} \tag{18}$$

Question 3 - Dodelson 2.8

Apply the Einstein equations to the case of an open universe. The interval in an open universe is

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a^2(t)\left\{\frac{\mathrm{d}r^2}{1+\Omega_k H_0^2 r^2} + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right)\right\} \tag{19}$$

where r, θ, ϕ are the standard 3D spherical coordinates, and Ω_k is the curvature density.

a) First compute the Christoffel symbols. Show that the only nonzero ones are equal to

$$\begin{aligned} \Gamma^i_{0j} &= H\delta^i_j \qquad & \Gamma^0_{ij} &= g_{ij}H \\ & & \Gamma^i_{jk} &= \frac{g^{il}}{2}\left[g_{lj,k} + g_{lk,j} - g_{jk,l}\right]. \end{aligned} \tag{20}$$

b) Show that the components of the Ricci tensor are

$$\begin{aligned} R_{00} &= -3\frac{\ddot{a}}{a} \\ R_{ij} &= g_{ij}\left[\frac{\ddot{a}}{a} + 2H^2 - \frac{2\Omega_k H_0^2}{a^2}\right]. \end{aligned} \tag{21}$$

c) From these, compute the Ricci scalar, and then derive the time-time component of Einstein equations.

a) Remembering the general description of the Christoffel symbols

$$\Gamma^\sigma_{\mu\nu} = \frac{1}{2}g^{\sigma\rho}\left(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu}\right), \tag{22}$$

where we can observe that $g_{ij} = a^2\bar{g}_{ij}$, $g^{ij} = \frac{\bar{g}^{ij}}{a^2}$ and $g^{i\mu}g_{\mu j} = g^{il}g_{lj} = \bar{g}^{il}\bar{g}_{il} = \delta^i_j$. Then

$$\begin{cases} \Gamma^0_{00} = 0 \\ \Gamma^i_{00} = \frac{1}{2}g^{il}\left(\partial_0 g_{0l} + \partial_0 g_{l0} - \partial_l g_{00}\right) = 0 \\ \Gamma^0_{i0} = \frac{1}{2}g^{00}\left(\partial_i g_{00} + \partial_0 g_{0i} - \partial_0 g_{i0}\right) = 0 \\ \Gamma^0_{0i} = \frac{1}{2}g^{00}\left(\partial_0 g_{i0} + \partial_i g_{00} - \partial_0 g_{0i}\right) = 0 \\ \Gamma^i_{j0} = \frac{1}{2}g^{il}\left(\partial_j g_{0l} + \partial_0 g_{lj} - \partial_l g_{j0}\right) = 0 \end{cases} \tag{23}$$

We then have 3 remaining possibilities

$$\Gamma^i_{0j} = \frac{1}{2}g^{il}\left(\partial_0 g_{jl} + \partial_j g_{l0} - \partial_l g_{0j}\right) = \frac{1}{2}g^{il}\partial_0 g_{lj} = \frac{1}{2}\frac{\bar{g}^{il}}{a^2}\partial_0 a^2\bar{g}_{lj} = \frac{2\dot{a}\cdot a}{2a^2}\bar{g}^{il}\bar{g}_{lj} = \frac{\dot{a}}{a}\delta^i_j, \tag{24}$$

$$\boxed{\Gamma^i_{0j} = H\delta^i_j} \tag{25}$$

$$\Gamma^0_{ij} = \frac{1}{2}g^{00}\left(\partial_i g_{j0} + \partial_j g_{0i} - \partial_0 g_{ij}\right) = -\frac{1}{2}(-1)\partial_0 a^2 \bar{g}_{ij} = \frac{2\dot{a}\cdot a}{2}\bar{g}_{ij} = \frac{\dot{a}}{a}g_{ij}, \tag{26}$$

$$\boxed{\Gamma^0_{ij} = Hg_{ij}} \tag{27}$$

$$\boxed{\Gamma^i_{jk} = \frac{1}{2}g^{il}\left(\partial_j g_{kl} + \partial_k g_{lj} - \partial_l g_{jk}\right)} \tag{28}$$

b) The general form of the Ricci tensor is

$$R_{\mu\nu} = \Gamma^\rho_{\mu\nu;\rho} - \Gamma^\rho_{\mu\rho;\nu} + \Gamma^\rho_{\sigma\rho}\Gamma^\sigma_{\mu\nu} - \Gamma^\rho_{\sigma\nu}\Gamma^\sigma_{\mu\rho}, \tag{29}$$

where

$$R_{00} = \Gamma^\rho_{00;\rho} - \Gamma^\rho_{0\rho;0} + \Gamma^\rho_{\sigma\rho}\Gamma^\sigma_{00} - \Gamma^\rho_{\sigma 0}\Gamma^\sigma_{0\rho} = -3\dot{H} - \Gamma^i_{j0}\Gamma^j_{0i} = -3(\dot{H} + H^2), \tag{30}$$

$$\boxed{R_{00} = -3\frac{\ddot{a}}{a}} \tag{31}$$

And,

$$\begin{aligned} R_{ij} &= \Gamma^\rho_{ij;\rho} - \Gamma^\rho_{i\rho;j} + \Gamma^\rho_{\sigma\rho}\Gamma^\sigma_{ij} - \Gamma^\rho_{\sigma j}\Gamma^\sigma_{i\rho} \\ &= \Gamma^0_{ij;0} + \Gamma^k_{ij;k} - \Gamma^0_{i0;j} - \Gamma^k_{ik;j} + \Gamma^\rho_{0\rho}\Gamma^0_{ij} + \Gamma^\rho_{k\rho}\Gamma^k_{ij} - \Gamma^0_{kj}\Gamma^k_{i0} - \Gamma^k_{0j}\Gamma^0_{ik} - \Gamma^k_{lj}\Gamma^l_{ik} \\ &= \partial_0 g_{ij}H + \Gamma^k_{ij;k} - \Gamma^k_{ik;j} + 3H^2 g_{ij} + \Gamma^l_{kl}\Gamma^k_{ij} - g_{kj}H^2\delta^k_i - g_{ik}H^2\delta^k_j - \Gamma^k_{lj}\Gamma^l_{ik}, \end{aligned} \tag{32}$$

$$\boxed{R_{ij} = g_{ij}\left[\frac{\ddot{a}}{a} + 2H^2 - \frac{2\Omega_k H_0^2}{a^2}\right]} \tag{33}$$

c) We have

$$R = g^{\mu\nu}R_{\mu\nu} = 3\frac{\ddot{a}}{a} + g^{ij}g_{ij}\left[\frac{\ddot{a}}{a} + 2H^2 - \frac{2\Omega_k H_0^2}{a^2}\right], \tag{34}$$

$$\boxed{R = 6\left[\frac{\ddot{a}}{a} + H^2 - \frac{\Omega_k H_0^2}{a^2}\right]} \tag{35}$$

The asked equation is

$$8\pi G T_{00} = R_{00} - \frac{1}{2}g_{00}R \tag{36}$$

$$8\pi G\rho = -3\frac{\ddot{a}}{a} + 3\left[\frac{\ddot{a}}{a} + H^2 - \frac{\Omega_k H_0^2}{a^2}\right], \tag{37}$$

$$\boxed{H^2 - \frac{\Omega_k H_0^2}{a^2} = \frac{8\pi G\rho}{3}} \tag{38}$$

Question 4 - Bianchi Identities

Use the definition of the Riemann tensor in terms of the affine connection, and the definition of the affine connection in terms of the metric, to show that, in a locally inertial frame (in which $\Gamma^{\alpha}_{\mu\nu} = 0$, but not its derivatives) the covariant derivative of the Riemann Tensor is

$$R_{\lambda\mu\nu\kappa;\eta} = \frac{1}{2}\frac{\partial}{\partial x^{\eta}}\left(\frac{\partial^2 g_{\lambda\nu}}{\partial x^{\kappa}\partial x^{\mu}} - \frac{\partial^2 g_{\mu\nu}}{\partial x^{\kappa}\partial x^{\lambda}} - \frac{\partial^2 g_{\lambda\kappa}}{\partial x^{\mu}\partial x^{\nu}} + \frac{\partial^2 g_{\mu\kappa}}{\partial x^{\nu}\partial x^{\lambda}}\right) \tag{39}$$

Then permute indices ν, κ and η cyclically to obtain the Bianchi identities:

$$R_{\lambda\mu\nu\kappa;\eta} + R_{\lambda\mu\eta\nu;\kappa} + R_{\lambda\mu\kappa\eta;\nu} = 0 \tag{40}$$

Next, contract indices λ and ν in the above equation to obtain

$$R_{\mu\kappa;\eta} - R_{\mu\eta;\kappa} + R^{\nu}_{\mu\kappa\eta;\nu} = 0 \tag{41}$$

and contract indices once more to finally obtain

$$\left(R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R\right)_{;\mu} = 0 \tag{42}$$

From the lecture notes, we can use

$$R_{\lambda\mu\nu\kappa} = \partial_{\kappa}\Gamma_{\lambda\mu\nu} - \partial_{\nu}\Gamma_{\lambda\mu\kappa}, \tag{43}$$

where we already know the general description of the Christoffel symbols from previous problems. And then, still following the lecture notes

$$R_{\lambda\mu\nu\kappa;\eta} = \nabla_{\eta}\left[\partial_{k}\Gamma_{\lambda\mu\nu} - \partial_{\nu}\Gamma_{\lambda\nu\kappa}\right] = \partial_{\eta}\partial_{\kappa}\Gamma_{\lambda\mu\nu} - \partial_{\eta}\partial_{\nu}\Gamma_{\lambda\mu\kappa}, \tag{44}$$

and then, expanding all terms in respect of the Christofell symbols, we receive

$$\boxed{R_{\lambda\mu\nu\kappa;\eta} = \frac{1}{2}\frac{\partial}{\partial x^{\eta}}\left(\frac{\partial^2 g_{\lambda\mu}}{\partial x^{\kappa}\partial x^{\mu}} - \frac{\partial^2 g_{\mu\nu}}{\partial x^{\kappa}\partial x^{\lambda}} - \frac{\partial^2 g_{\lambda\kappa}}{\partial x^{\mu}\partial x^{\nu}} + \frac{\partial^2 g_{\mu\kappa}}{\partial x^{\nu}\partial x^{\lambda}}\right)} \tag{45}$$

Also

$$R_{\lambda\mu\eta\nu;\kappa} = \frac{1}{2}\frac{\partial}{\partial x^{\kappa}}\left(\frac{\partial^2 g_{\lambda\eta}}{\partial x^{\nu}\partial x^{\mu}} - \frac{\partial^2 g_{\mu\eta}}{\partial x^{\nu}\partial x^{\lambda}} - \frac{\partial^2 g_{\lambda\nu}}{\partial x^{\eta}\partial x^{\mu}} + \frac{\partial^2 g_{\mu\nu}}{\partial x^{\kappa}\partial x^{\lambda}}\right), \tag{46}$$

$$R_{\lambda\mu\kappa\eta;\nu} = \frac{1}{2}\frac{\partial}{\partial x^{\nu}}\left(\frac{\partial^2 g_{\lambda\kappa}}{\partial x^{\eta}\partial x^{\mu}} - \frac{\partial^2 g_{\mu\kappa}}{\partial x^{\eta}\partial x^{\lambda}} - \frac{\partial^2 g_{\lambda\eta}}{\partial x^{\kappa}\partial x^{\mu}} + \frac{\partial^2 g_{\mu\eta}}{\partial x^{\kappa}\partial x^{\lambda}}\right), \tag{47}$$

therefore

$$\boxed{R_{\lambda\mu\nu\kappa;\eta} + R_{\lambda\mu\eta\nu;\kappa} + R_{\lambda\mu\kappa\eta;\nu} = 0} \tag{48}$$

Now, using the symmetry property of $R^{\lambda}_{\mu\eta\nu} = -R^{\lambda}_{\mu\nu\eta}$, we obtain that

$$\boxed{R_{\mu\kappa;\eta} - R_{\mu\eta;\kappa} + R^{\nu}_{\mu\kappa\eta;\nu} = 0} \tag{49}$$

And then, contracting the μ and η indices, we obtain

$$\boxed{\left(R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R\right)_{;\mu} = 0} \tag{50}$$