Halo Model: From Mass-function to the Nonlinear Power Spectrum

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January 31, 2025

ABSTRACT

The Halo Model provides a framework for describing the distribution of dark matter and its connection to galaxy formation and large-scale structure. In this work, we compute key components of the Halo Model: the Halo Mass-function, Halo Bias, Halo Profile, and the Nonlinear Power Spectrum. Using `Python` libraries such as `CAMB`, `numpy`, and `scipy`, we calculate the variance of linear fluctuations, σ^2, the mass-function, and the bias describing halo clustering. We model the halo density profile with the Navarro-Frenk-White (NFW) profile and its Fourier Transform. For the power spectrum, we compute both the 1-halo and 2-halo terms, which dominate at small and large scales, respectively, and compare results with those derived using linear and nonlinear power spectra from `CAMB`. Our results show that the mass-function declines with increasing mass, the bias increases in denser regions, and the halo profile matches the expected NFW behavior. The Fourier Transform highlights the transition between scales dominated by individual halos and large-scale clustering.

Key words: Cosmology – Halo Model – Power Spectrum

1 INTRODUCTION

As we understand, galaxy formation is heavily connected to gravitational phenomena, which trigger star formation in our universe. In this context, we often look at the impacts of dark matter, which has unknown origins. It is widely accepted that all galaxies are embedded within dark matter halos, which are gravitationally bound structures that extend beyond the visible boundaries of the galaxy.

A dark matter halo is the consequence of matter assembly, and studying its properties leads us to better understand the distribution and evolution of matter in the Universe. In this work, we adopt the Halo Model, which assumes that all matter is contained within the dark matter halos.

This model is often used as a foundation for connecting observational data from problems such as galaxy clustering and weak lensing. The project's objective is to explore the Halo Model properties, and is organized as follows:

- In Section 2.1, we compute the Halo Mass-function, which quantifies the number density of halos as a function of their mass;
- In Section 2.2, we calculate the Halo Bias, which describes the halo clustering relative to matter distribution;
- In Section 2.3, we derive the Halo Profile and its Fourier Transform;
- And finally, in Section 2.4, we compute the 1 and 2-halo components of the Nonlinear Power Spectrum, along with the linear and nonlinear derived by CAMB.

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2 METHODS

All computations in this work are done with `Python`, using libraries such as `CAMB`, `numpy`, `scipy` (for the integrations) and `matplotlib`.

2.1 Halo Mass-function

The first step in computing the Halo Mass-function is to write the variance σ^2 of linear fluctuations on a scale of R, which is the radius of a spherical region in comoving space that corresponds to a certain mass M

$$M = \frac{4}{3}\pi R^3 \bar{\rho}_m(0), \tag{1}$$

with $\bar{\rho}_m(0) = 2.775 \cdot 10^{11}\ h^2 M_\odot \mathrm{Mpc}^{-3}$. Therefore, σ^2 can be described as

$$\sigma^2(z, R) = G^2(z) \int \frac{k^2 \mathrm{d}k}{2\pi^2} |W(kR)|^2 P_L(k), \tag{2}$$

where $P_L(k)$ is the linear matter power spectrum at redshift $z = 0$ (which is computed by `CAMB` with fiducial cosmology), $G(z) = D(z)/D(z=0)$ and

$$W(kR) = \frac{3}{k^2R^2}\left[\frac{\sin(kR)}{kR} - \cos(kR)\right], \tag{3}$$

is the Fourier Transform of a spherical top-hat window of radius R. Now for the derivative of the inverse of σ, we have

$$\frac{\mathrm{d}\ln\sigma^{-1}}{\mathrm{d}\ln M} = -\frac{M}{\sigma}\frac{\mathrm{d}\sigma}{\mathrm{d}M}. \tag{4}$$

And finally, we can compute the Halo Mass-function

$$\frac{\mathrm{d}z(z, M)}{\mathrm{d}\ln M} = g(\sigma)\frac{\bar{\rho}_m}{M}\frac{\mathrm{d}\ln\sigma^{-1}}{\mathrm{d}\ln M}, \tag{5}$$

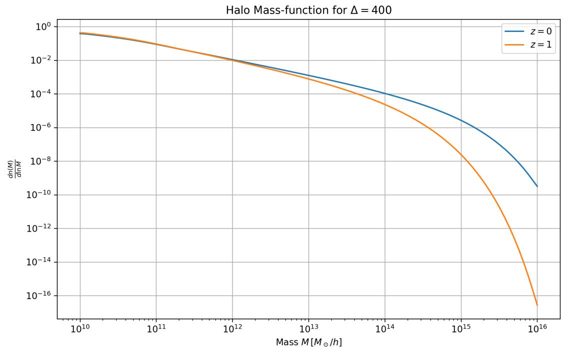

Figure 1. Halo Mass-function for $\Delta = 400$. Blue curve is for $z = 0$ and orange is for $z = 1$.

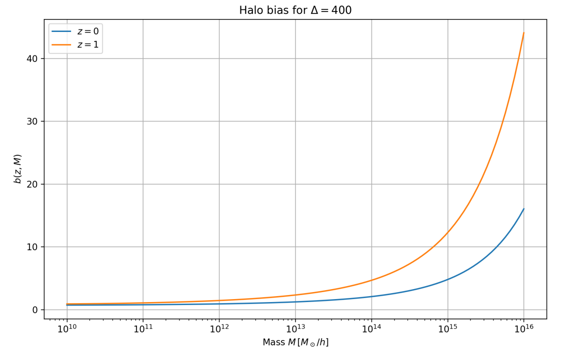

Figure 2. Halo Bias function for $\Delta = 400$. Blue curve is for $z = 0$ and orange is for $z = 1$.

Parameter	$f(\Delta)$
A	$1.0 + 0.24y\exp\left[-(4/y)^4\right]$
a	$0.44y - 0.88$
B	0.183
b	1.5
C	$0.019 + 0.107y + 0.19\exp\left[-(4/y)^4\right]$
c	2.4

Table 1. Parameters of the halo bias function (Tinker et al. 2010). Note that $y \equiv \log_{10}\Delta$.

where, $\bar{\rho}_m = \bar{\rho}_m(0)\cdot\Omega_m$ (Here, we are considering $\Omega_m = 0.23$ and $\Omega_\Lambda = 0.77$), and $g(\sigma)$ (Tinker et al. 2008) is

$$g(\sigma) = B\left[\left(\frac{\sigma}{e}\right)^{-d} + \sigma^{-f}\right]\exp\left[-\frac{g}{\sigma^2}\right], \tag{6}$$

where for $\Delta = \Delta_c/\Omega_m = 400$, $B = 0.494$, $d = 2.30$, $e = 0.93$, $f = 0.48$ and $g = 1.403$. The function $g(\sigma)$ is properly normalized[1], as

$$\int \frac{g(\sigma)}{\sigma}\,\mathrm{d}\sigma = 1. \tag{7}$$

In Figure 1, it is possible to observe the behavior of the Halo Mass-function, where it drops as the mass grows. This probably happens because the universe's expansion eventually halts the accretion matter onto these halos.

2.2 Halo Bias

In essence, the Halo Bias can describe how likely is to halos be found in dense regions, compared to other random positions in the universe. According to Tinker et al. (2010), the halo bias function can be written as

$$b(z, M) = 1 - A\frac{\nu^a}{\nu^a + \delta_c^a} + B\nu^b + C\nu^c, \tag{8}$$

where $\nu = \delta_c/\sigma$, $\delta_c = 1.686$ (check Table 1 for halo bias parameters).

[1] Integrating with `scipy`, we receive that the integral $I = 0.99 \pm 2.56\cdot 10^{-10} \approx 1$.

The halo bias function is normalized[2] as it is confirmed that

$$\int \frac{g(\sigma)b(\sigma)}{\sigma}\,\mathrm{d}\sigma = 1. \tag{9}$$

In Figure 2, we can check that in fact, we tend to see dark matter halos in higher dense regions of the universe, as expected.

2.3 Halo Profile

For the purpose of computing the Halo Profile, we need to write the following equations:

$$E^2(z) = \Omega_m(1+z)^3 + \Omega_\Lambda(1+z)^{3(1+w)}, \tag{10}$$

where $E^2(z)$ is the squared Hubble parameter normalized to H_0 and w is the dark energy equation-of-state parameter.

$$\rho_{\rm crit} = \bar{\rho}_m(0)\cdot E^2(z), \tag{11}$$

where $\rho_{\rm crit}$ is the critical density of the universe at redshift z.

$$w_m(z) = \frac{\Omega_m(1+z)^3}{E^2(z)}, \tag{12}$$

where $w_m(z)$ is the fractional matter density as a function of redshift.

$$\Delta_c(z) = 18\pi^2 + 82x - 39x^2 \text{ (Bryan \& Norman 1998)}, \tag{13}$$

where Δ_c is the overdensity parameter (it is used to define the virial overdensity of collapsed dark matter halos), and $x = w_m(z) - 1$.

$$r_{\rm vir} = \left(\frac{3M_{\rm vir}}{4\pi\rho_{\rm crit}(z)\Delta_c(z)}\right)^{1/3}, \tag{14}$$

which computes the virial radius $r_{\rm vir}$ of a dark matter halo of virial mass $M_{\rm vir}$.

The concetration parameter c, which uses a characteristic stellar mass $M_\star$ is computed as

$$c(z, M_{\rm vir}) = \frac{9}{1+z}\left(\frac{M_{\rm vir}}{M_\star}\right)^{-0.13} \text{ (Bullock et al. 2001)}, \tag{15}$$

[2] Integrating with `scipy`, we receive that the integral $I = 0.99 \pm 2.01\cdot 10^{-9} \approx 1$.

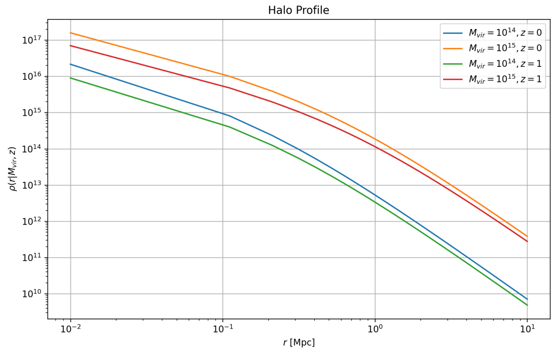

Figure 3. Halo Profile function for different virial masses at $z = 0$ and $z = 1$.

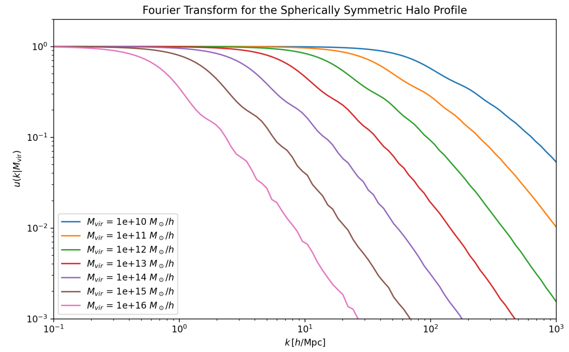

Figure 4. Fourier Transform for the spherically symmetric Halo Profile function at redshift $z = 0$.

and we can find $M_\star$ for $z = 0$:

$$\frac{\delta_c}{\sigma(M, z=0)} - 1 = 0 \rightarrow M_\star(z=0) \approx 2.06 \cdot 10^{11}\ M_\odot/h, \tag{16}$$

and for $z = 1$

$$M_\star(z=1) = M_\star(z=0)\frac{D(z=1)}{D(z=0)} \approx 1.32 \cdot 10^{11}\ M_\odot/h. \tag{17}$$

The scale density for an Navarro-Frenk-White (NFW) halo profile can be computed as

$$\rho_s = \frac{M_{\text{vir}} c^3}{4\pi r_{\text{vir}}^3 \left(\ln(1+c) - \frac{c}{1+c}\right)}. \tag{18}$$

The density $\rho(r)$ at a radial distance r from the center of an NFW halo (Halo Profile function) can be written as

$$\rho(r|M_{\text{vir}}, z) = \frac{\rho_s}{cr/r_{\text{vir}}(1 + cr/r_{\text{vir}})^2}. \tag{19}$$

The NFW halo profile follows the ρ_r function as

$$\rho(r) = \frac{\rho_s}{\frac{r}{r_s}\left(1 + \frac{r}{r_s}\right)^2}, \tag{20}$$

for $r_s = r_{\text{vir}}/c$, which behaves as follows:

$$\begin{cases} r \ll r_s : \ \rho(r) \propto r^{-1} \\ r \gg r_s : \ \rho(r) \propto r^{-3} \end{cases} \tag{21}$$

This means that the density decreases slowly near the center and rapidly in outer regions of the halo, which is seen in Figure 3.

And finally, we can compute the Fourier Transform for the spherically symmetric Halo Profile function as

$$u(k|M_{\text{vir}}) = \frac{1}{M_{\text{vir}}} \int_0^{r_{\text{vir}}} 4\pi r^2 \rho(r) \frac{\sin(kr)}{kr}\, \mathrm{d}r. \tag{22}$$

As expected, we can see in Figure 4 that $u(k|M_{\text{vir}}) \rightarrow 1$ as $k \rightarrow 0$ for all values of M_{vir} and is similar to what Cooray & Sheth (2002) found.

2.4 Halo Model: Nonlinear Power Spectrum

For $z = 0$, we can compute the 1 and 2-halo terms of the Nonlinear Power Spectrum as follows

$$P^{1h}(k) = \int \mathrm{d}\ln M \frac{M^2}{\bar{\rho}_m^2(0)} \frac{\mathrm{d}n}{\mathrm{d}\ln M} |u(k|M)|^2, \tag{23}$$

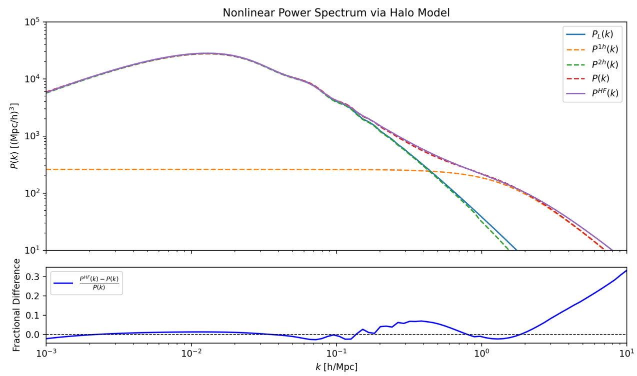

Figure 5. Halo Model: Nonlinear Power Spectrum for $z = 0$. $P_L(k)$ (computed by `CAMB`) is the linear power spectrum, $P^{1h}(k)$ is the 1-halo term, $P^{2h}(k)$ is the 2-halo term, $P(k)$ is the total halo model spectrum and $P^{HF}(k)$ is the non-linear spectrum computed by `CAMB`.

$$P^{2h}(k) = \left[\int \mathrm{d}\ln M \frac{M}{\bar{\rho}_m(0)} \frac{\mathrm{d}n}{\mathrm{d}\ln M} b(M) u(k|M)\right]^2 P_L(k), \tag{24}$$

$$P(k) = P^{1h}(k) + P^{2h}(k), \tag{25}$$

and note that we can transform $\mathrm{d}\ln M \rightarrow \frac{\mathrm{d}M}{M}$.

From Figure 5, we can observe that for small k, $P^{2h}(k) \rightarrow P_L(k)$, as expected, and for large k, they diverge. Furthermore, $P(k)$ is very similar to the $P^{HF}(k)$ curve.

3 CONCLUSIONS

In conclusion, it was possible to study the Halo Model and compute functions that are similar to what is expected in the literature. We explored the Halo Model framework to compute the Halo Mass-function, Halo Bias, Halo Profile, and the Nonlinear Power Spectrum, which are components that have been proven helpful in providing insights into the distribution and clustering of matter in the universe.

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