Geometry:

$\overline{r}(x)$ - continuously differentiable regular curve.

$\forall t : \overline{r}(t) \perp \overline{r}'(t)$

Prove, that $\overline{r}([a,b])$ - part of circle.

Solution:

$r = (x, y)$
$\forall t : \overline{r}(t) \perp \overline{r}'(t) \Leftrightarrow (x(t), y(t)) \perp (x'(t), y'(t)) \Leftrightarrow x(t)x'(t) + y(t)y'(t) = 0$
$(x(t), y(t)) \perp (x'(t), y'(t)) \Rightarrow (x'(t), y'(t)) \parallel (-y(t), x(t)) \Rightarrow$
$\begin{cases} x'(t) = -\lambda y(t) \\ y'(t) = \lambda x(t) \end{cases}$
$xx' + yy' = xx' + \lambda yx = 0$
$y = \frac{-1}{\lambda} x'$
$r = \left(x, \frac{-1}{\lambda} x'\right), r' = \left(x', \frac{-1}{\lambda} x''\right)$
$r \perp r' \Leftrightarrow xx' + \frac{1}{\lambda^2} x'x'' = 0$
$if\ x' \neq 0 :$
$x + \frac{1}{\lambda^2} x'' = 0 \Rightarrow x'' = -\lambda^2 x \Rightarrow x = sin(\lambda t), y = -cos(\lambda t)$

Algebra:

Check group structure:

1) $G = R \backslash \{-1\}, a * b = ab + a + b$

Solution:

$\{a \neq -1 \Leftrightarrow (a+1) \neq 0\}$
$\{a*b = ab+a+b \neq -1\ since\ otherwise\ ab+a+b+1 = 0 = (a+1)(b+1) \Leftrightarrow a = -1\ or\ b = -1\}$
$[a : x \to (a+1)x]$
$[a \circ b = (x \to (a+1)x)(x \to (b+1)x) = (x \to (a+1)(b+1)x) =$
$= (x \to ((ab+a+b)+1)x) = ab+a+b]$
$(a*b)*c = (ab+a+b)*c = abc+ac+bc+c+ab+a+b+c =$
$= abc+ac+bc+ab+a+b+c$
$a*(b*c) = a*(bc+b+c) = abc+ab+ac+a+bc+b+c =$
$= abc+ac+bc+ab+a+b+c = (a*b)*c$
$a*e = e*a = a = ae+a+e \Rightarrow ae = -e \Rightarrow e = 0$
$[e : x \to x = x \to (e+1)x \Rightarrow e+1 = 1 \Leftrightarrow e = 0]$
$a*a^{-1} = 0 = aa^{-1} + a + a^{-1} = a^{-1}(a+1) + a \Rightarrow a^{-1} = \frac{-a}{a+1}$
$\left[a^{-1} : (a+1)x \to x = x \to \frac{x}{a+1} = x \to \left(\frac{-a}{a+1} + 1\right)x\right]$

2) $G=\left\{\begin{pmatrix}0&\alpha&\beta\\0&0&\gamma\\0&0&0\end{pmatrix}, A*B=A+B+\frac{1}{2}(AB-BA)\right\}$

Solution:

$G=\left\{\begin{pmatrix}0&\alpha&\beta\\0&0&\gamma\\0&0&0\end{pmatrix}, A*B=A+B+\frac{1}{2}(AB-BA)\right\}$

Function is: (a,b,c):(x,y,z)$\rightarrow (a+x, b+y+\frac{1}{2}(az-cx), c+z)$

A*B=$\begin{pmatrix}0&a_1&a_2\\0&0&a_3\\0&0&0\end{pmatrix}*\begin{pmatrix}0&b_1&b_2\\0&0&b_3\\0&0&0\end{pmatrix}=\begin{pmatrix}0&a_1+b_1&a_2+b_2+\frac{1}{2}(a_1b_3-a_3b_1)\\0&0&a_3+b_3\\0&0&0\end{pmatrix}$

Will write $\begin{pmatrix}0&\alpha&\beta\\0&0&\gamma\\0&0&0\end{pmatrix}$ as $(\alpha,\beta,\gamma), so$

$(a_1,a_2,a_3)(b_1,b_2,b_3)=(a_1+b_1,a_2+b_2+\frac{1}{2}(a_1b_3-a_3b_1),a_3+b_3)$

C=$(c_1,c_2,c_3)$

(AB)C=$(a_1+b_1,a_2+b_2+\frac{1}{2}(a_1b_3-a_3b_1),a_3+b_3)(c_1,c_2,c_3)=$

$=(a_1+b_1+c_1,a_2+b_2+\frac{1}{2}(a_1b_3-a_3b_1)+c_2+\frac{1}{2}((a_1+b_1)c_3-(a_3+b_3)c_1),a_3+b_3+c_3)=$

$=(a_1+b_1+c_1,a_2+b_2+c_2+\frac{1}{2}(a_1b_3+a_1c_3+b_1c_3-a_3b_1-a_3c_1-b_3c_1),a_3+b_3+c_3)$

A(BC)=$(a_1,a_2,a_3)(b_1+c_1,b_2+c_2+\frac{1}{2}(b_1c_3-b_3c_1),b_3+c_3)=$

$=(a_1+b_1+c_1,a_2+b_2+c_2+\frac{1}{2}(b_1c_3-b_3c_1)+\frac{1}{2}(a_1(b_3+c_3)-a_3(b_1+c_1)),a_3+b_3+c_3)=$

$=(a_1+b_1+c_1,a_2+b_2+c_2+\frac{1}{2}(b_1c_3+a_1b_3+a_1c_3-b_3c_1-a_3b_1-a_3c_1),a_3+b_3+c_3)=(AB)C$

e=$(e_1,e_2,e_3)$

Ae=eA=A

$(a_1+e_1,a_2+e_2+\frac{1}{2}(a_1e_3-a_3e_1),a_3+e_3)=$

$=(a_1+e_1,a_2+e_2+\frac{1}{2}(e_1a_3-e_3a_1),a_3+e_3)=$

$=(a_1,a_2,a_3)\Rightarrow e_1=e_3=0$

$(a_1,a_2+e_2,a_3)=(a_1,a_2,a_3)\Rightarrow e_2=0\Rightarrow e=(0,0,0)$

$A^{-1}=(a_1^{-1},a_2^{-1},a_3^{-1})$

$AA^{-1}=(a_1+a_1^{-1},a_2+a_2^{-1}+\frac{1}{2}(a_1a_3^{-1}-a_3a_1^{-1}),a_3+a_3^{-1})=(0,0,0)=e$

$\Rightarrow a_1^{-1}=-a_1, a_3^{-1}=-a_3AA^{-1}=(0,a_2+a_2^{-1},0)=(0,0,0)\Rightarrow a_2^{-1}=-a_2$

$A^{-1}=(-a_1,-a_2,-a_3)$

$A^{-1}A=(0,\frac{1}{2}(a_1(-a_3)-a_3(-a_1)),0)=(0,0,0)=e$

Differential equations:

58. $y'-xy^2=2xy$

Solution:

$y' - xy^2 = 2xy = \frac{dy}{dx} - xy^2$

$xy(2+y)dx = dy$

$xdx = \frac{dy}{y(y+2)}$

$\int xdx = \int \frac{dy}{y(y+2)} = \frac{1}{2}\int \left(\frac{1}{y} - \frac{1}{y+2}\right) dy = \frac{1}{2}\int \frac{dy}{y} - \frac{1}{2}\int \frac{dy}{y+2}$

$\frac{1}{2}x^2 = \frac{1}{2}(ln(y) - ln(y+2)) + C$

59. $e^{-s}(1 + ds/dt) = 1$

Solution:

$e^{-s}(1 + \frac{ds}{dt}) = 1$

$e^{-s} + e^{-s}\frac{ds}{dt} = 1$

$e^{-s}dt + e^{-s}ds = dt$

$(1 - e^{-s})dt = e^{-s}ds$

$dt = \frac{e^{-s}}{1-e^{-s}}ds$

$t = \int \frac{e^{-s}}{1-e^{-s}}ds$

$z = e^{-s}, s = -ln(z), ds = \frac{-dz}{z}$

$\int \frac{e^{-s}}{1-e^{-s}}ds = \int \frac{1}{1-z}dz = ln(1-z) + C = ln(1-e^{-s}) + C$

$t = ln(1-e^{-s}) + C$

60. $z' = 10^{x+z}$

Solution:

$z' = 10^{x+z} = \frac{dz}{dx} = 10^x 10^z$

$10^{-z}dz = 10^x dx$

$\int 10^{-z}dz = \int 10^x dx$

$\int e^{-ln(10)z}dz = \int e^{ln(10)x}dx$

$\frac{-1}{ln(10)}10^{-z} = \frac{1}{ln(10)}10^x + C$

$10^x + 10^{-z} + C = 0$

62. $y' = cos(y-x)$:

Solution:

$y' = cos(y-x)$

$z = y - x, y = z + x, y' = z' + 1$

$z' + 1 = cos(z)$

$\frac{dz}{dx} = cos(z) - 1$

$\int \frac{dz}{cos(z)-1} = \int dx = x + C$

$-\int \frac{d\frac{z}{2}}{sin^2(z)} = ctg\left(\frac{z}{2}\right) + C$

$ctg\left(\frac{y-x}{2}\right) = x + C$