Design and Analysis of Algorithms

L44: Branch and Bound

Assignment Problem
TSP Problem

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Resources

- Text book 1: Levitin
  - Sec 12.1, ***12.2***
- Text book 2: Horowitz
  - Sec 7.1, 7.2, 7.3, 7.4, 7.5, 8.2, 11.1=
- R1: Introduction to Algorithms
  - Cormen et al.

Example use cases

- Assignment problem
- Traveling salesperson problem
- `0-1` knapsack problem

Exact Solution Strategies

- Exhaustive search (brute force)
  - Search for all possible combinations (exponential time)
  - Useful only for small instances
- *Dynamic programming*
  - applicable to some problems
    - Where problem can be recursively mapped to smaller other problems
- *Backtracking*
  - Build a state space tree
  - Eliminates unnecessary cases from consideration
  - Solutions may still take exponential time
- *Branch-and-bound*
  - Further refines backtracking for optimization problems

Branch and Bound

- Additional mechanisms compared to backtracking
  - Provide a bound on the best value of objective function for every node of the state-space tree
  - The value of best solution so far
    - at the current node of state space tree
- Approach
  - Compare the node's bound value with the value of best solution seen so far.
    - If the bound is not better, terminate the search (prune the solution)
      - not smaller than the best sol^{n} in a minimization problem
      - not greater than the best sol^{n} in a maximization problem

Branch and Bound

- Termination of the search path in state space tree using branch-n-bound algo, when
  - The value of node's bound is not better than the value of best solution seen so far
  - The node represents no feasible solution because of the constraints of the problem are already violated
  - The subset of feasible solutions represented by the node consists of a single point
    - i.e. reached the end of solution and no more choices
    - Compare the value of objective function with that the best solution seen so far
      - Update the latter if former is better than latter.

Assignment Problem

- Consider a problem of assigning n jobs to n people so that cost is minimised.
  - The cost of each job done by each person is given
  - Represented in a matrix.
- Consider an example of 4 person job assignment cost

	J_1	J_2	J_3	J_4
P_a	9	2	7	8
P_b	6	4	3	7
P_c	5	8	1	8
P_d	7	6	9	4

Assignment Problem

	J_1	J_2	J_3	J_4
P_a	9	2	7	8
P_b	6	4	3	7
P_c	5	8	1	8
P_d	7	6	9	4

- Problem can be stated as follows
  - Select one element in each row
    - so that two selected elements are in same column
    - such that their sum is smallest possible.
- Solution with branch and bound
  - Consider the lowest possible sum
  - Take the lowest element in each row.
  - This may not be optimal but can act as lower bound
    - Two elements may belong to same column
  - Smallest possible values for above example
    - $P_a(J_2)=2+P_b(J_3)=3+P_c(J_3)=1+P_d(J_4)=4$
    - $=2+3+1+4=10$
    - It is not legitimate though (P_b and P_c have same job J_3)

Assignment Problem

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- Backtracking:
  - generate a child of last promising node
- Branch and Bound approach
  - Generate the child of most promising node
    - among non-terminated live leaves in current tree
    - Achieved by comparing lower bounds of all live nodes
  - Intuitively, it is better to consider a node with best bound as most promising
    - It is possible that optimal solution may lie in other branch of the tree.

Assignment Problem

	J_1	J_2	J_3	J_4
P_a	9	2	7	8
P_b	6	4	3	7
P_c	5	8	1	8
P_d	7	6	9	4

- Start at root
  - with no elements selected from matrix
  - Lower bound lb=10

Start
lb=2+3+1+4=10

- It has 4 live nodes.

  $J_1=9$, $J_2=2$, $J_3=7$, $J_4=8$,
  - Lower bounds for each of these are
    - LB for $P_a(J_1=9)=9+3+1+4=17$

  $P_a(J_1=9)+P_b(J_3=3)+P_c(J_3=1)+P_d(J_4=4)$
    - LB for $P_a(J_1=2)=2+3+1+4=10$
    - LB for $P_a(J_1=7)=7+4+5+4=20$
    - LB for $P_a(J_1=8)=8+3+1+6=18$
- Most promising node for P_a is 2.
  - Explore this node further.

Assignment Problem

	J_1	J_2	J_3	J_4
P_a	9	2	7	8
P_b	6	4	3	7
P_c	5	8	1	8
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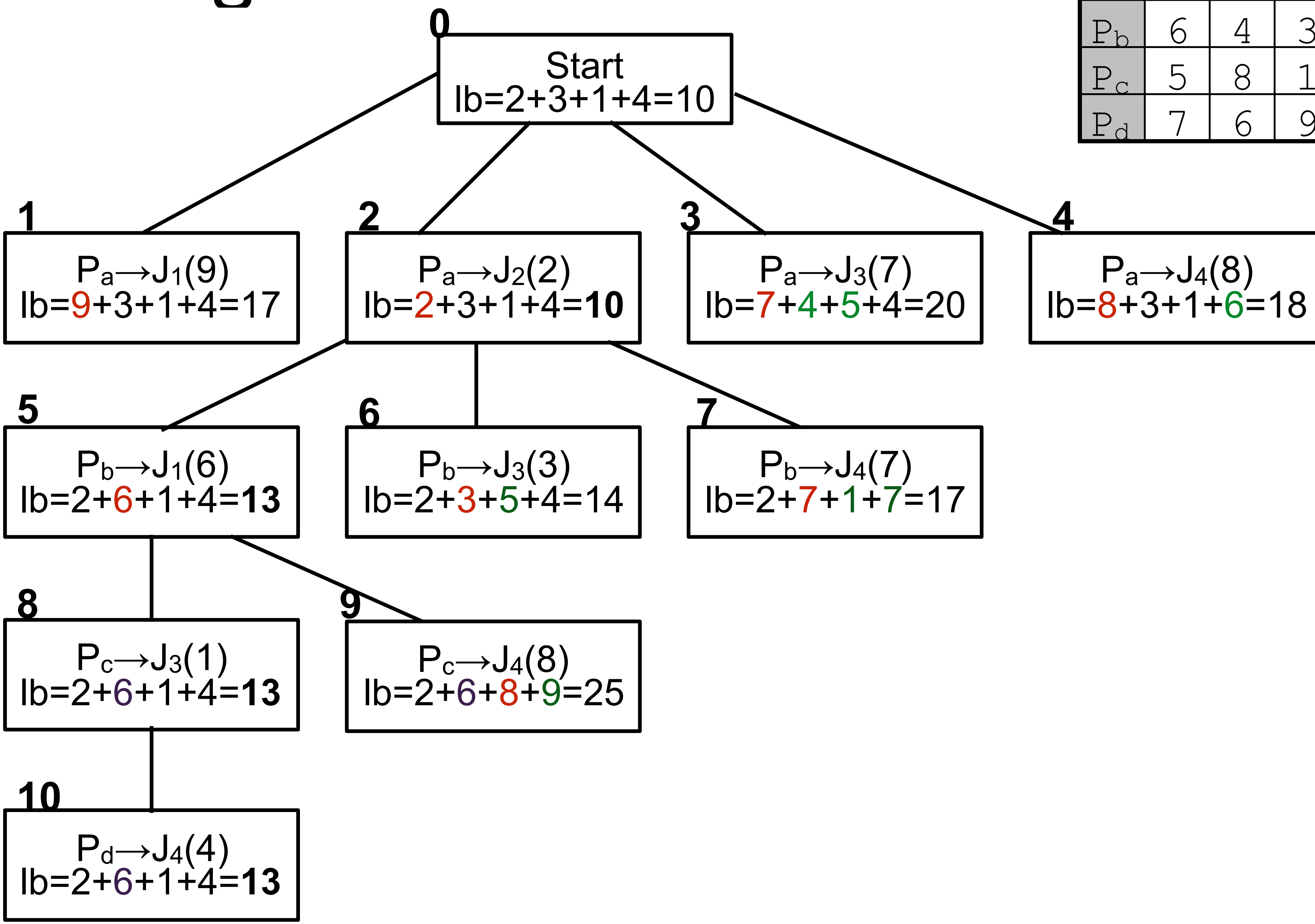
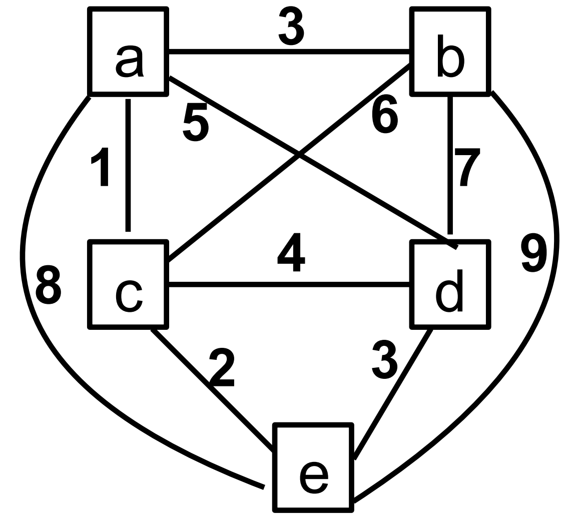


Traveling Salesperson Problem



- Branch and Bound approach
- Define a lower bound
- Simple approach:
  - Take the lowest edge cost
  - Multiply it by number of nodes
    - `LB=1*5=5`
- More information but less obvious
  - Does not require much computation too
  - For each node, find two nearest nodes
    - Find the average of two
  - Sum this average (ceiling) for all nodes

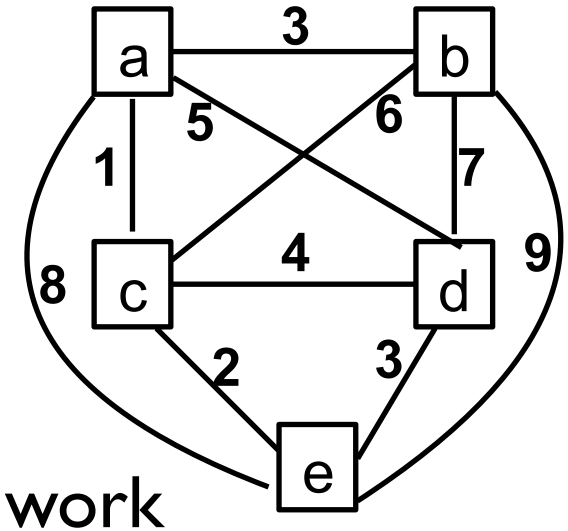
`LB=((1+3)+(3+6)+(1+2)+(3+4)+(2+3))/2=14`

- When any tour includes a particular edge, compute the lower bound accordingly.

Traveling Salesperson Problem

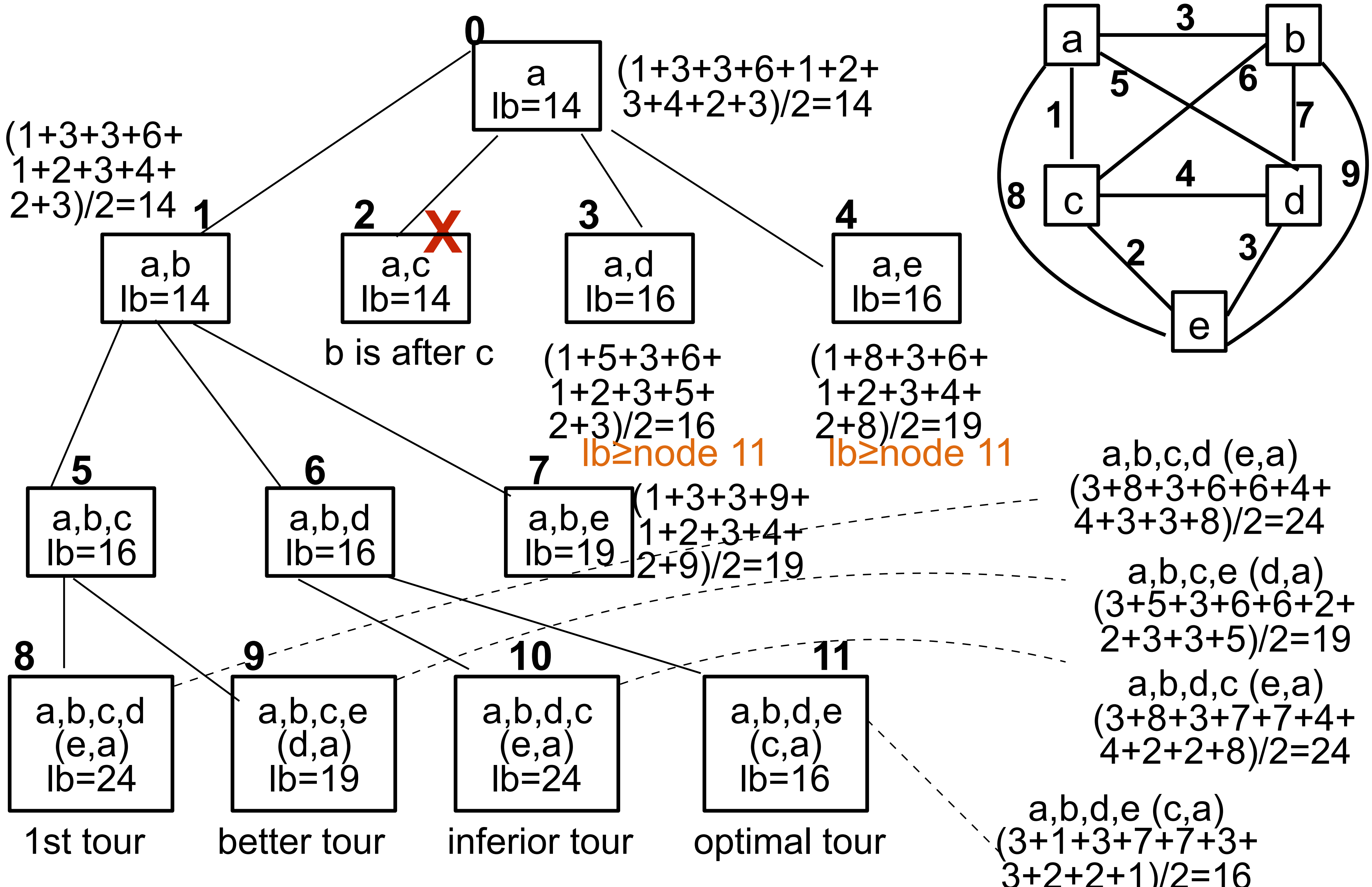
- When any tour includes a particular edge,
  - compute the lower bound accordingly.
- example: Consider `(a,d)` is included.

$$LB=\lceil((1+5)+(3+6)+(1+2)+(5+3)+(2+3))/2\rceil=\lceil 31/2\rceil=16$$



- Further, to reduce the amount of potential work
  - We can consider that tour starts at node `a`, and
  - Since graph is undirected,
    - Generate tours in which `b` appears before `c`.
  - After visiting `n-1=4` nodes,
    - Tour has to visit the last unvisited node, and
    - return to the starting node.

Traveling Salesperson Problem



Branch and Bound

- Finding a good bound function a challenging task
  – May not be always easy to find one
- Bounding function should be easy to compute
- It should not be too simple
  – it may fail to prune the many branches of state space tree as soon as possible
- Finding the balance between two requirements (easy to compute, and not too simplistic) may require intensive experimentation
  – with a wide variety of problem in question

Summary

- Assignment problem
- Traveling Salesperson Problem