Design and Analysis of Algorithms

L44: LC Branch and Bound

0-1 Knapsack Problem

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Resources

- Text book 2: Horowitz
  – Sec 8.2
- Text book 1: Levitin
  – Sec 12.1, 12.2
- R1: Introduction to Algorithms
  - Cormen et al.
- Youtube link for lecture recording
  – https://www.youtube.com/watch?v=j556E7LgvbI
- Youtube (other)
  – https://www.youtube.com/watch?v=yV1d-b_NeK8

BB Search: State Space Tree

```
Algo BBSearch(node t) // search tree with root at t.
if t is an answer node
    output t and return
E←t // make t an E-node
Initialize the list L of live nodes to empty list.
do
    for each child x of E
        if x is an answer node
            output the path from x to E and return
        Add(x)  to list L of live nodes
        parent(x)←E
    if L is empty // there are no more live nodes
        output "No answer nodes" and return
    E←Next(L) // take the next live node from to search
while True
```

BB Search: State Space Tree

- Three possible implementation of search space
  - Depends upon how the list `L` is implemented
  - and how the `Next(L)` is taken out
- L is Queue i.e. FIFO (First In First Out)
  - E-nodes are removed in the order they are added
  - Also called BFS (Breadth First search)
- L is Stack i.e. LIFO (Last in First Out)
  - E-nodes are removed in the reverse order it is added
  - Also called D-search (Depth First search)
- L is Heap (can be min or max heap)
  - E-nodes are removed as min (or max) value
  - Called Least Cost (LC) Search

$0-1$ Knapsack Problem

- Knapsack problem:
  – Given n items of known weights $w_1, \ldots, w_n$, and
  – Values $v_1, \ldots, v_n$ and knapsack capacity m
  – Find the most valuable subset of items that fit into the knapsack.
    - i.e.maximize the value of knapsack
    - An item has to be included in full
      ($0-1$ knapsack problem)
  – Note: All the weights w_i's and knapsack capacity m are integers, but values v_i's can be real numbers.
- $0-1$ knapsack is a maximization problem
  – Branch and Bound solves minimization problem.
  – So convert knapsack to minimization problem

0–1 Knapsack Problem

- 0–1 Knapsack problem (maximization problem)
  - maximize $\sum_{1 \le i \le n} v_i x_i$,
  - subject to $\sum_{1 \le i \le n} w_i x_i \le m$
  - x_i is 0 or 1, and $1 \le i \le n$
- Problems of TSP and Job Assignment were minimization problem solved using Branch-n-Bound
- Convert knapsack maximization to minimization

  minimize $-\sum_{1 \le i \le n} v_i x_i$, (call it cost)
  - it maximizes $\sum_{1 \le i \le n} v_i x_i$ (values)

  subject to $\sum_{1 \le i \le n} w_i x_i \le m$ (knapsack constraint)
- State space tree formation
  - Using fixed tuple size, one variable for each weight
  - Using variable tuple size, uses the index of weight

0−1 Knapsack Problem

- State space tree formation
  - Using fixed tuple size, one variable for each weight
    - Each variable has two values 0 or 1
    - Thus, Each node has two children
  - Using variable tuple size, uses the index of weight
    - Can be easily built from fixed tuple size case
- Implementation: define two terms:
    - cost per node (what can be reached theoreically)
    - upper bound per node (what can be achieved)
  - Define $c(x) = -\Sigma_{1 \le i \le n} v_i x_i$ for each answer node x
    - $c(x) = \infty$ for infeasible leaf nodes
  - For non-leaf nodes, define $c(x)$ recursively as
    - `min{c(Lchild(x), Rchild(x)}`
    - Thus, computation recursively becomes exponential

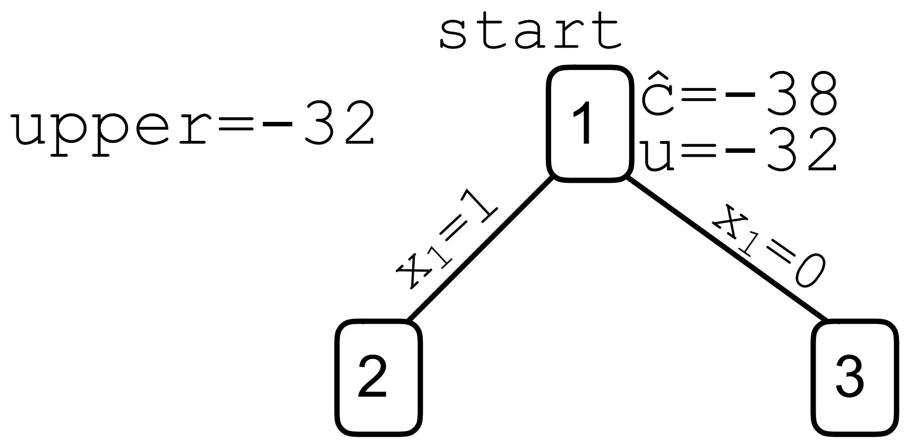
0-1 Knapsack Implementation

- Define $\hat{c}(x)$: a heuristic value for $c(x)$
  - cost till the first node which doesn't fit the knapsack
    - Thus, include its partial value to max the knapsack
- Define $u(x)$: an upper bound for node x.
  - the cost till the first node which doesn't fit the knapsack, but without including the partial value.
- Thus, two functions follows the constraints for node x
  $\hat{c}(x) \leq c(x) \leq u(x)$
- Maintain single `upper` variable.
  - This indicates the best value i.e. minimum cost solution achieved so far.
- Thus, for any node when $\hat{c}(x) > upper$
  - Discard that path (i.e. kill that node), prune the tree

Example: LCBB $0-1$ Knapsack

- Consider knapsack instance with $n=4, m=15$, and
  - values $(v_1, v_2, v_3, v_4)=(10, 10, 12, 18)$, and
  - weights $(w_1, w_2, w_3, w_4)=(2, 4, 6, 9)$
- Using fixed tuple implementation, trace LCBB
  - Fixed implementation implies 4 tuple varaibles
    - x_1, x_2, x_3, x_4 and each can take value 0 or 1.
- We need to compute following values for each node
  $\hat{c}(x), u(x)$, upper
- Consider root node i.e. start node at level 1.
  - Least Cost (LC) approach
    - Among all live nodes, choose the node with lowest cost to explore (i.e. it becomes E-node)
    - List L of live nodes is implemented as Heap

LCBB: 0-1 Knapsack



	1	2	3	4
v	10	10	12	18
w	2	4	6	9

n=4,m=15

- **start node** 1:

  $\hat{c}(x)$: w_1, w_2, and w_3 contributes fully, w_4 exceeds knapsack

  $\hat{c}=-(10+10+12+((15-12)/9)*18))=-38$

  $u(x)$: w_1, w_2, and w_3 contributes fully, w_4 exceeds knapsack

  $u=-(10+10+12+0)=-32]$

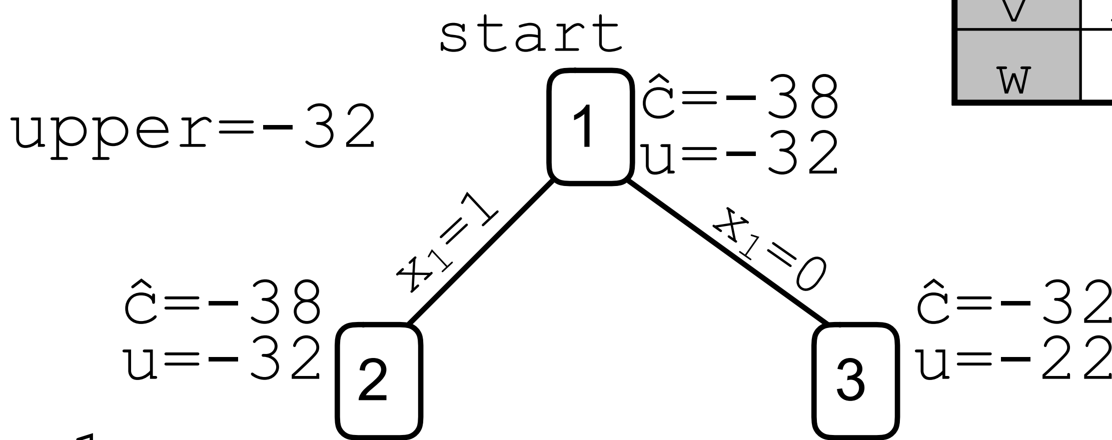
  upper=-32

  - This node is live node ($\hat{c} \leq$ upper) and only node so far,
  - Explore this node, two children
    - $x_1=1$ (include w_1), $x_1=0$ (exclude w_1)

LCBB: 0-1 Knapsack

	1	2	3	4
v	10	10	12	18
w	2	4	6	9

n=4, m=15

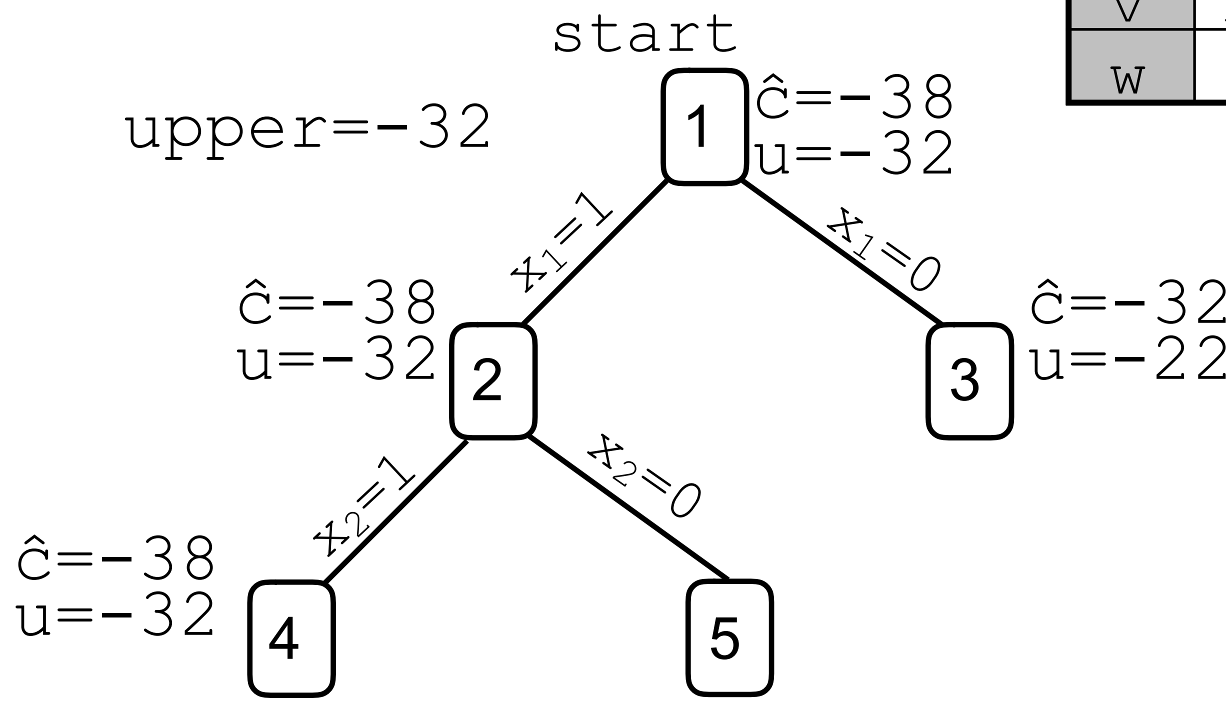


- **node** 2: $x_1=1$
  - $\hat{c}(x)$: w_1, w_2, **and** w_3 **contributes fully,** w_4 **exceeds knapsack**
    - $\hat{c}=-(10+10+12+((15-12)/9)*18))=-38$
  - $u(x)$: w_1, w_2, **and** w_3 **contributes fully,** w_4 **exceeds knapsack**
    - $u=-(10+10+12+0)=-32$
    - $\hat{c}(x)$, $u(x)$, upper **don't change**
- **node** 3: $x_1=0$ **(partial weight of** w_4 **becomes** 5**)**
  - $\hat{c}=-(0+10+12+((15-10)/9)*18))=-32$
  - $u=-(0+10+12+0)=-22$
  - upper **remains** -32 **(doesn't change)**
- **Alives nodes are:** 2 **and** 3 ($\hat{c}(x) \leq$ upper)

LCBB: 0-1 Knapsack

	1	2	3	4
v	10	10	12	18
w	2	4	6	9

n=4,m=15



- Least Cost (−38) among live nodes is for node 2.
- Explore node 2.
  - x_2=1 (node 4), and x_2=0(node 5)
- Node 4:

  $\hat{c}=-(10+10+12+((15-12)/9)*18))=-38$

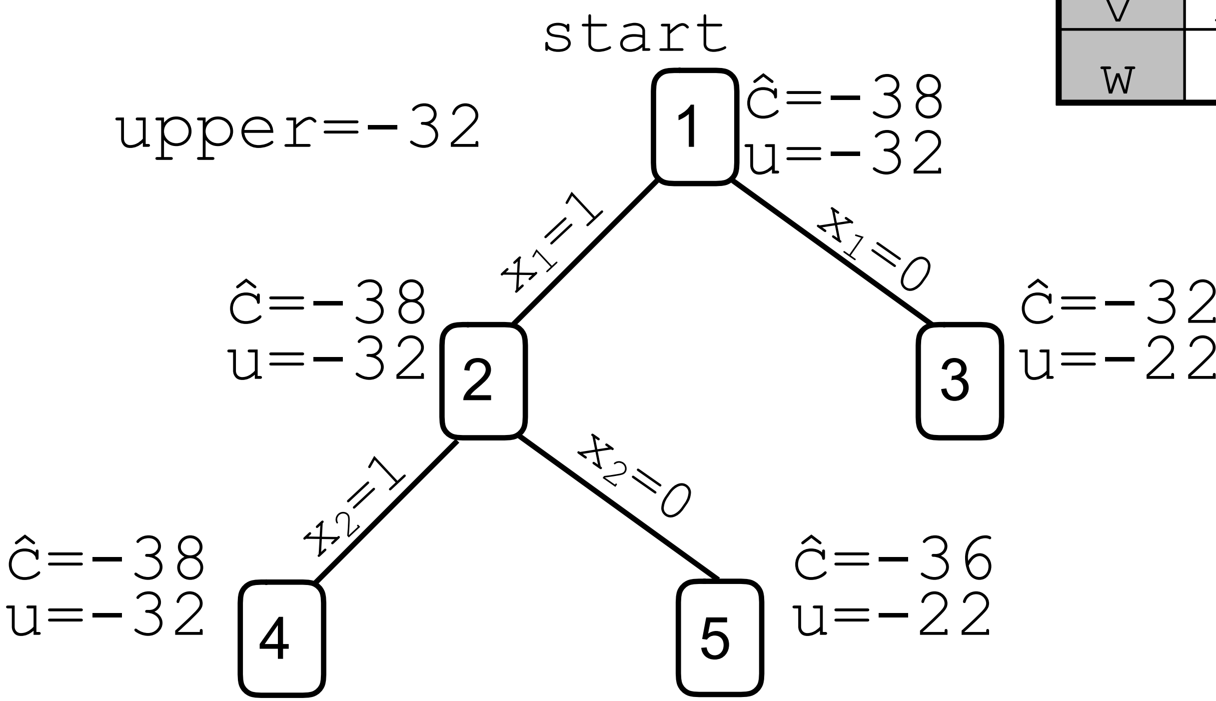
  $u=-(10+10+12+0)=-32$

  upper remains same and doesn't change

LCBB: 0-1 Knapsack

	1	2	3	4
v	10	10	12	18
w	2	4	6	9

n=4,m=15

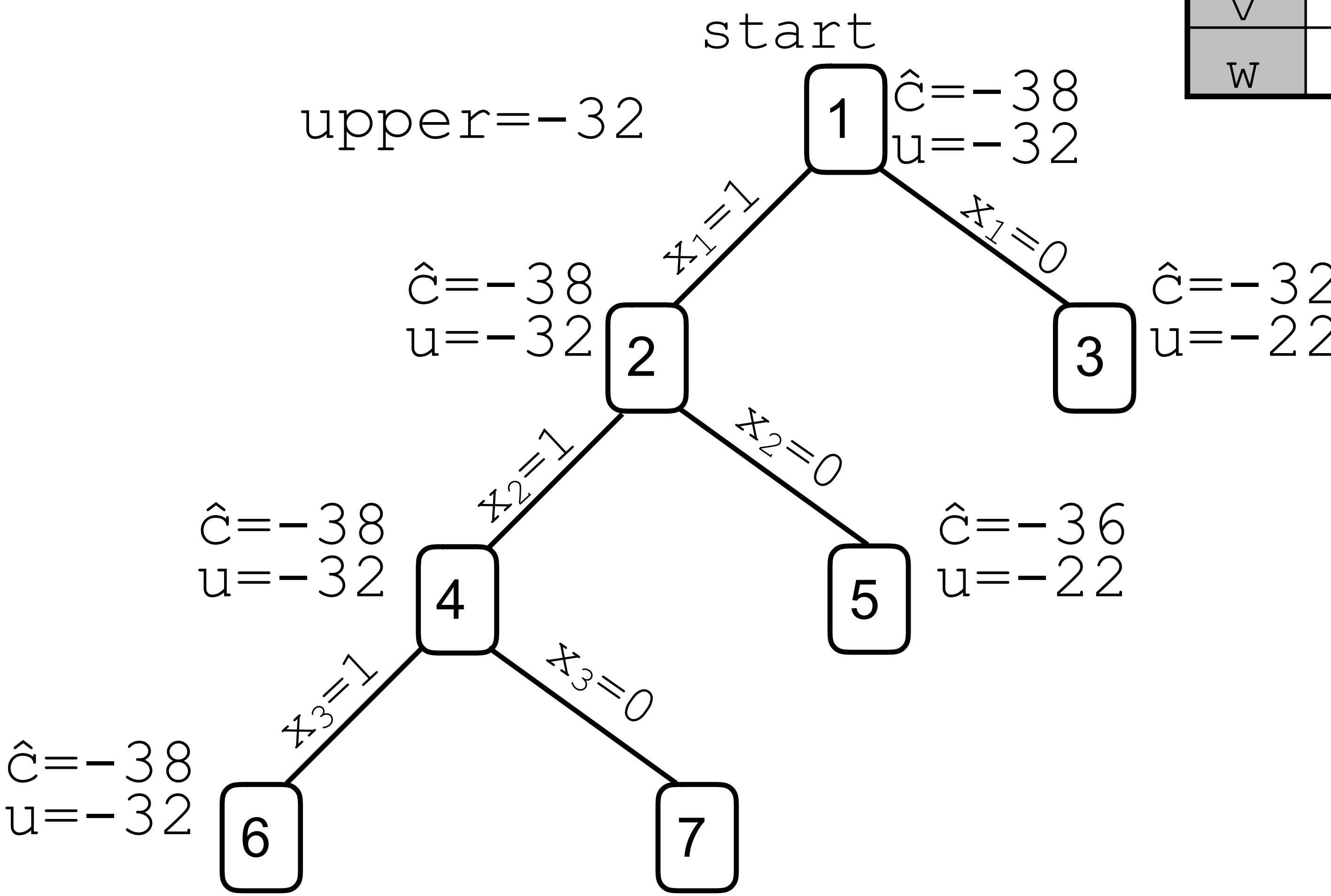


- **Node** 5 **(partial weight of** w_4 **changes)**
  ĉ=-(10+0+12+((15-8)/9)*18))=-36
  u=-(10+0+12+0)=-22
  upper **remains same and doesn't change**
- **Lives nodes now:** 3, 4, 5 (c(x)≤upper)
- **Least Cost node is** 4. **Explore it**
  - x_3=1 **(node** 6**), and** x_3=0 **(node** 7**)**

LCBB: 0-1 Knapsack

	1	2	3	4
v	10	10	12	18
w	2	4	6	9

n=4,m=15



Node 6 (x_3=1)

$\hat{c}=-(10+10+12+((15-12)/9)*18))=-38$

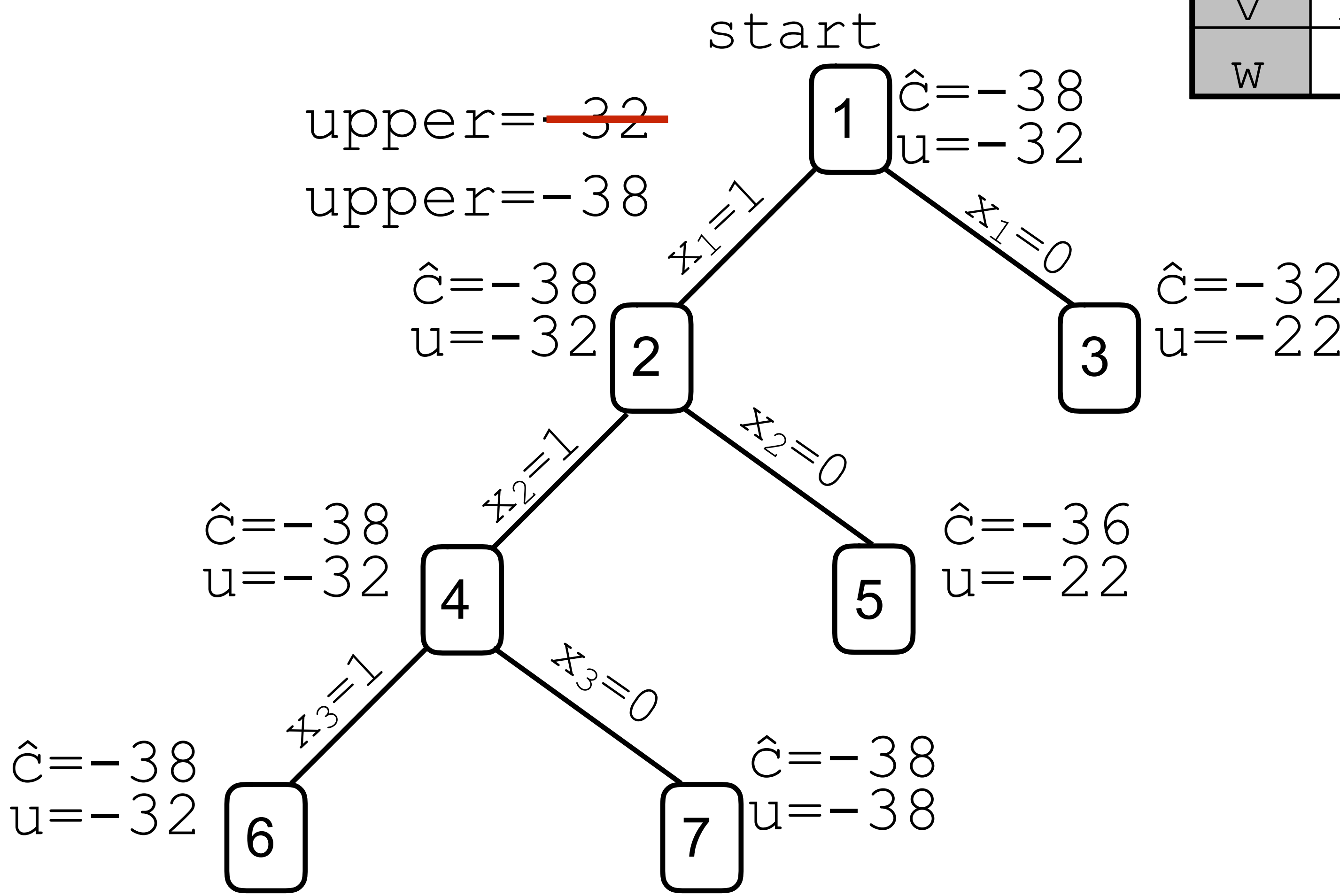
$u=-(10+10+12+0)=-32$

upper **remains same and doesn't change**

LCBB: 0-1 Knapsack

	1	2	3	4
v	10	10	12	18
w	2	4	6	9

n=4,m=15



Node 7 (x_3=0)

$\hat{c}$=-(10+10+0+18))=-38

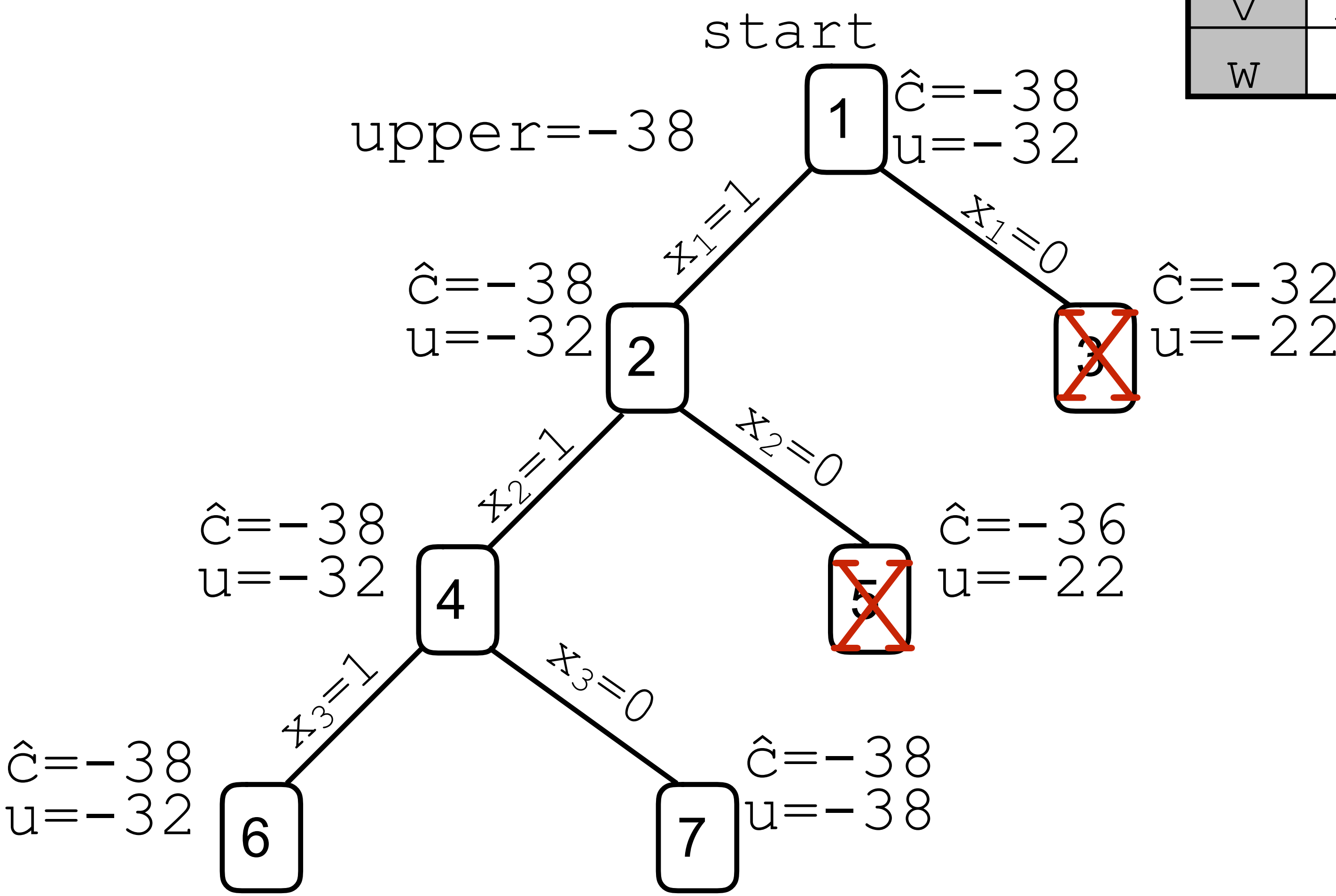
u=-(10+10+0+18)=-38

upper **becomes less and hence changes to** −38

LCBB: 0-1 Knapsack

	1	2	3	4
v	10	10	12	18
w	2	4	6	9

n=4, m=15

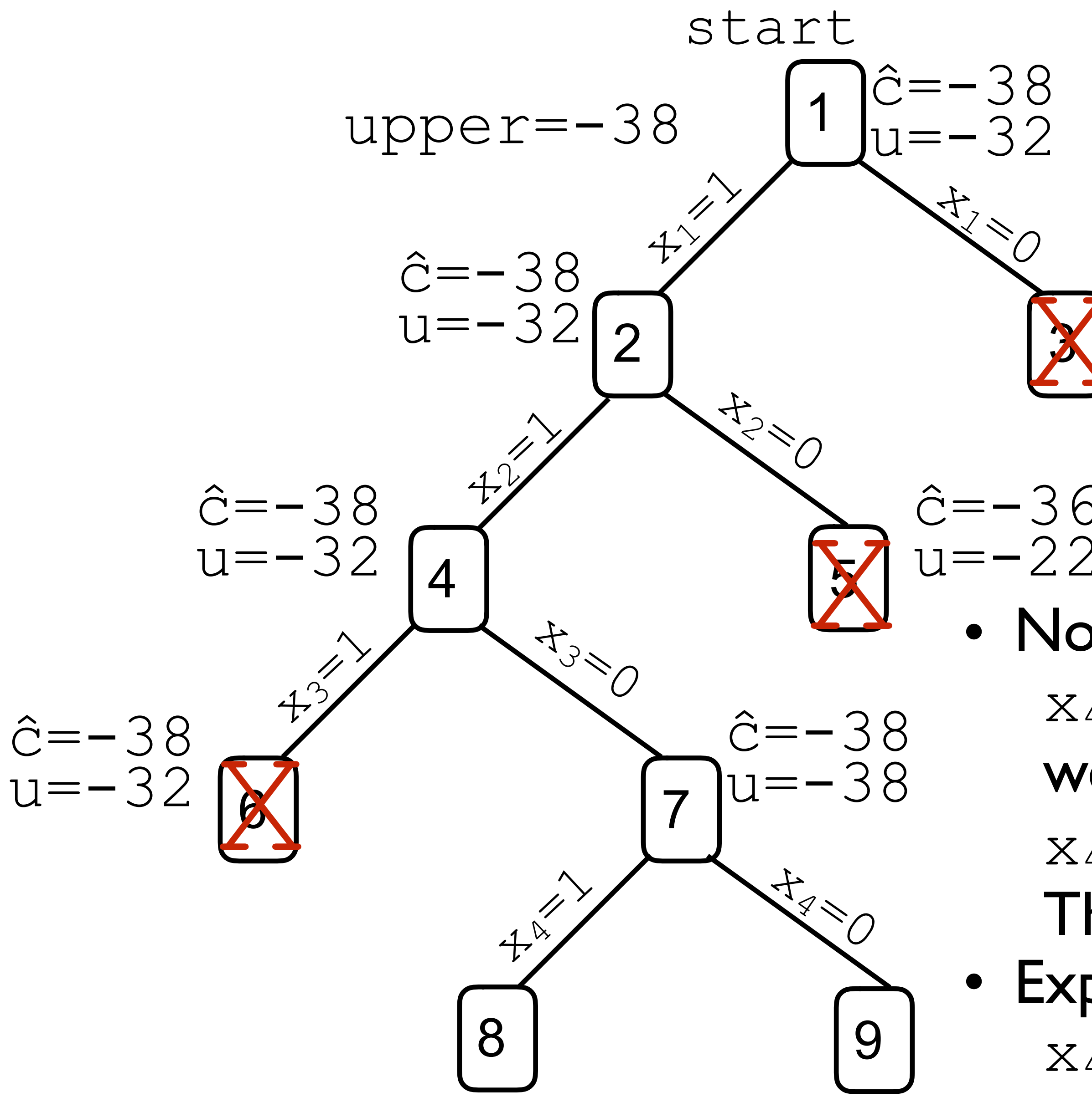


- Live node are 6 and 7. (ĉ(6)≤-38, ĉ(7)≤-38)
- Nodes 3 and 5 are killed, ĉ(3)>upper, ĉ(5)>upper
- Least Cost live node: can be taken either 6 or 7, both are equal
- Take 6 as least cost node.

LCBB: 0-1 Knapsack

	1	2	3	4
v	10	10	12	18
w	2	4	6	9

n=4, m=15

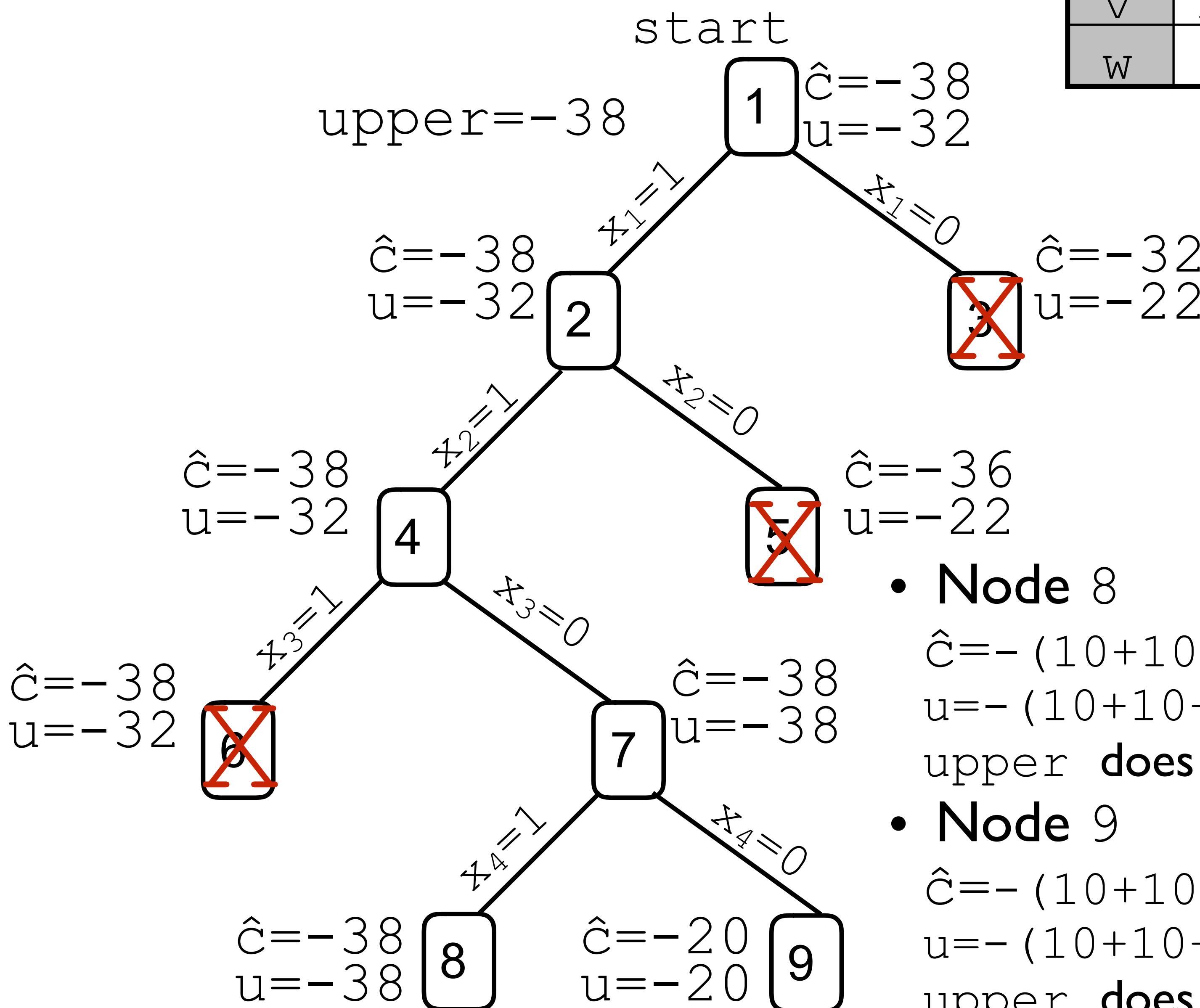


- Nodes 6
  x_4=1 makes knapsack weight 21, can't consider
  x_4=0 decreases $\hat{c}$ to −32
  Thus, kill the node 6.
- Explore node 7
  x_4=1 (node 8),
  x_4=0 (node 9)

LCBB: 0-1 Knapsack

	1	2	3	4
v	10	10	12	18
w	2	4	6	9

n=4, m=15

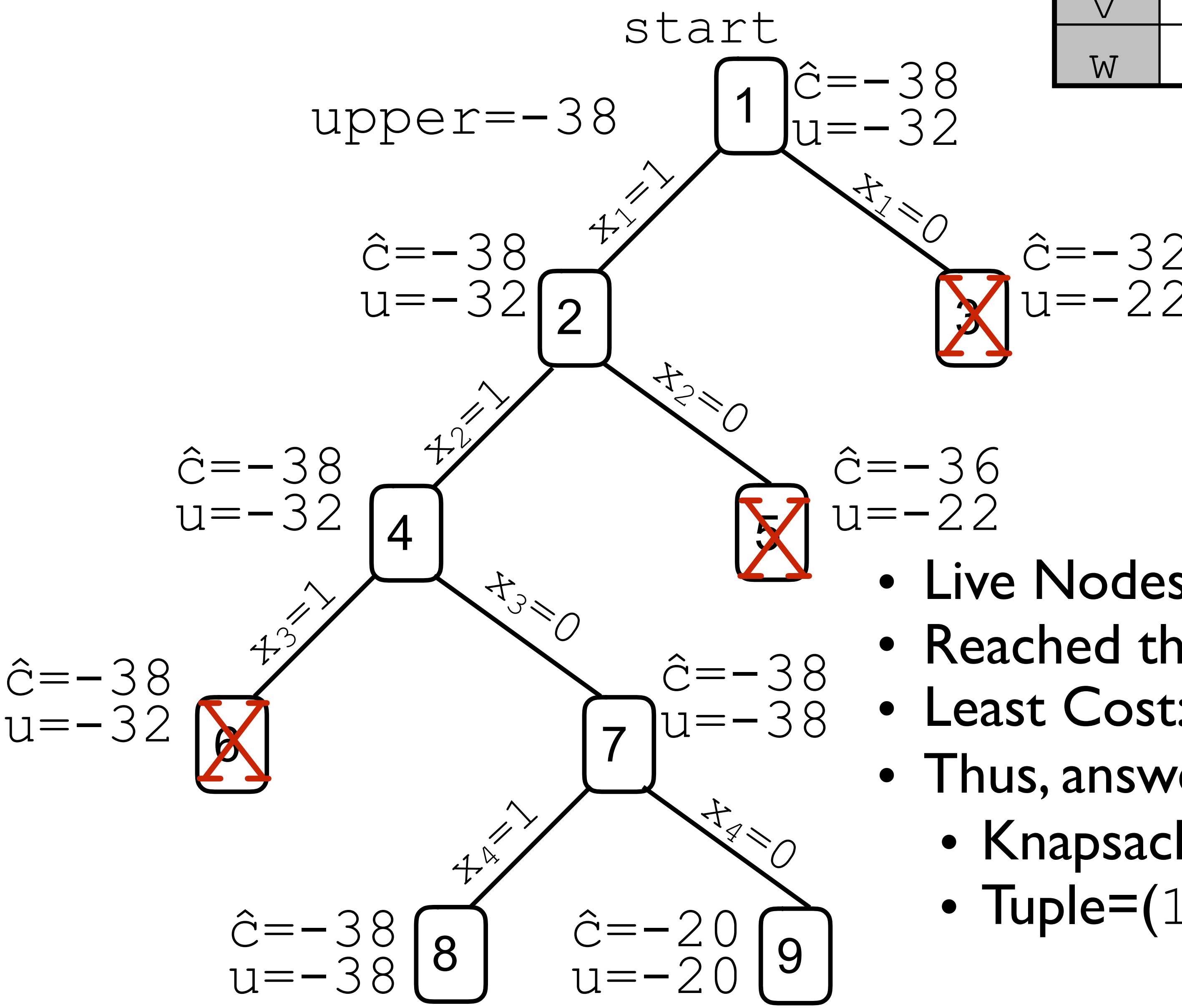


- **Node** 8
  ĉ=-(10+10+0+18)=-38
  u=-(10+10+0+18)=-38
  upper **does not change**
- **Node** 9
  ĉ=-(10+10+0+0)=-20
  u=-(10+10+0+0)=-20
  upper **does not change**

LCBB: 0-1 Knapsack

	1	2	3	4
v	10	10	12	18
w	2	4	6	9

n=4,m=15



- Live Nodes 8, 9
- Reached the leaf nodes
- Least Cost: node 8
- Thus, answer node is 8
  - Knapsack value=38
  - Tuple=(1,1,0,1)

`0-1` Knapsack Implementation

- State space tree is a binary tree with depth `n+1`
  - Define two functions `Bound`(), `UBound`() as shown in next slide,
    - `Bound()` is used to compute cost
    - `Bound()` is used to compute upper value

$$u(x)=UBound(-\Sigma_{1\le i<j} v_i x_i, \Sigma_{1\le i<j} w_i x_i, j-1, m)$$
$$c(x) \ge Bound(\Sigma_{1\le i<j} v_i x_i, \Sigma_{1\le i<j} w_i x_i, j-1)$$

Bound()

`Proc Bound(float cv, float cw, int k)`

// provides an upper bound (partial knapsack) on best solution obtainable (by expanding any node `Z` at level `k+1`)

// includes the partial value of node which exceeds knapsack

//`cp`: current total value, `cw`: current total weight

// `k` is the index of last removed item of knapsack

```
    float b=cp;  float c=cw;
    for i←k+1 to n do ..............................................B1
        if (c+wi<m) then  ..........................................B2
            c=c+wi          ........................................B2
            b = b - vi .............................................B4
        else
            return (b-(m-c)/wi)*vi  .................................B5
    return b          ..............................................B6
```

UBound()

```
Proc UBound(float cv, float cw, int k,float m)
```

// provides an upper bound (0−1 knapsack) on best solution obtainable by expanding any node Z at level k+1

// does not include the cost last node that exceeds knapsack

//cp: current total value, cw: current total weight

// k is the index of last removed item of knapsack

float b=cp;

float c=cw;

for $i \leftarrow k+1$ **to** n **do** U1

  if $(c+w_i \le m)$ **then** U2

   $c = c+w_i$ U3

   $b = b-v_i$ U4

return b U5

Summary:

- Least Cost Branch and Bound for
  - `0-1` Knapsack problem
- Next to explore
  - FIFO Branch and Bound