Design and Analysis of Algorithms

L36: Bellman-Ford algorithm

Dynamic Programming

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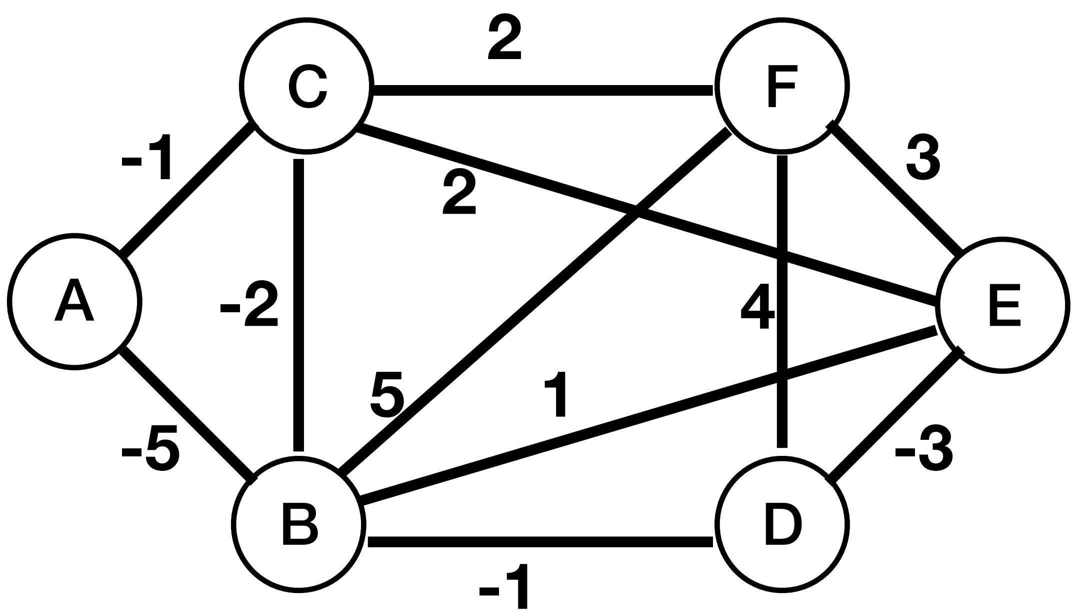
Resources

- Text book 2: Horowitz
  – Sec 5.1, 5.2, **5.4**, 5.8, 5.9
- Text book 1: Levitin
  – Sec 8.2-8.4
- R1: Introduction to Algorithms
  - Cormen et al.

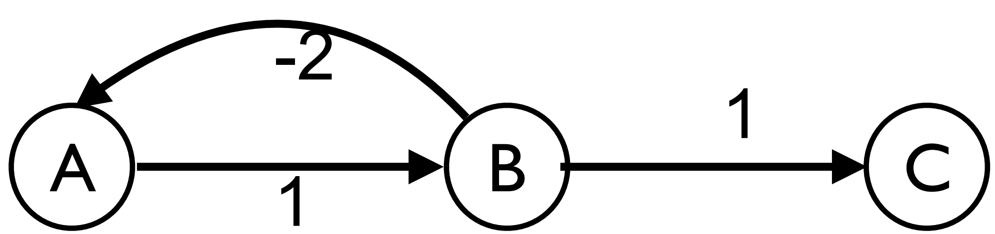
Single Source Shorest Paths

- Issues with Dijkstra's algo
  - Does not work with negative edge weights
  - Consider a graph below
    - Shortest distance `d(F,B)=-4`, but algo gives `0`.



Graph with Negative edges

- Key requirement:
  - the shortest path must consist of finite number of edges.
  - Essentially, there should be no negative cycles.
  - Consider the graph below.
    - Length of shortest path from A to C is $-\infty$
    - Length of path is A, B, A, B, …, A, B, C



Single Source Shortest Path

- Consider a graph G with `n` nodes.
  - If a path has more than `n-1` edges,
    - it must contain a cycle (at least `1` vertex is repeated)
  - Elimination of path results in a shorter path
    - With same source and destination
  - When there are no negative cycles
  - Then there can be at most `n-1` edges in the shortest path from the source to the farthest node
    - this path will include all the nodes of the graph
  - Minimum path length will correspond to `1` edge
  - Use the path length as the dynamic programming approach

Single Source Shortest Path

- Consider source vertex as `s`.
- Let $\mathtt{dist}^k(\mathtt{v})$ denote the length of shortest path from vertex `v` from `s` containing at most `k` edges.
- Thus, $\mathtt{dist}^k(\mathtt{v})=\mathtt{cost[s][v]},\ 1 \le \mathtt{v} \le \mathtt{n}$.
- When there are no negative cycles, then
  - restrict the search for shortest paths to length `n−1`
  - Such a length would be $\mathtt{dist}^{n-1}(\mathtt{v})$.

Dynamic Programming Approach

- If the shortest path from `s` to `u` with at most `k` edges, has no more than `k−1` edges, then
  - $\mathtt{dist}^{k}(u) = \mathtt{dist}^{k-1}(u)$
- If the shortest path from `s` to `u` with at most `k` edges, has exactly `k` edges, then
  – it must have shortest path from `s` to some vertex `j` followed by edge `(j,u)`.
  – The shortest path from `s` to `j` will have `k−1` edges
    - with its length as $\mathtt{dist}^{k-1}(j)$
    - All vertices `i` such that edge `(i,v)` is in the graph will be candidate for vertex `j`.

  – Vertex `i` that minimizes ($\mathtt{dist}^{k-1}(i)$+cost[i][u]) is the right vertex for shortest path from `s` to `u`

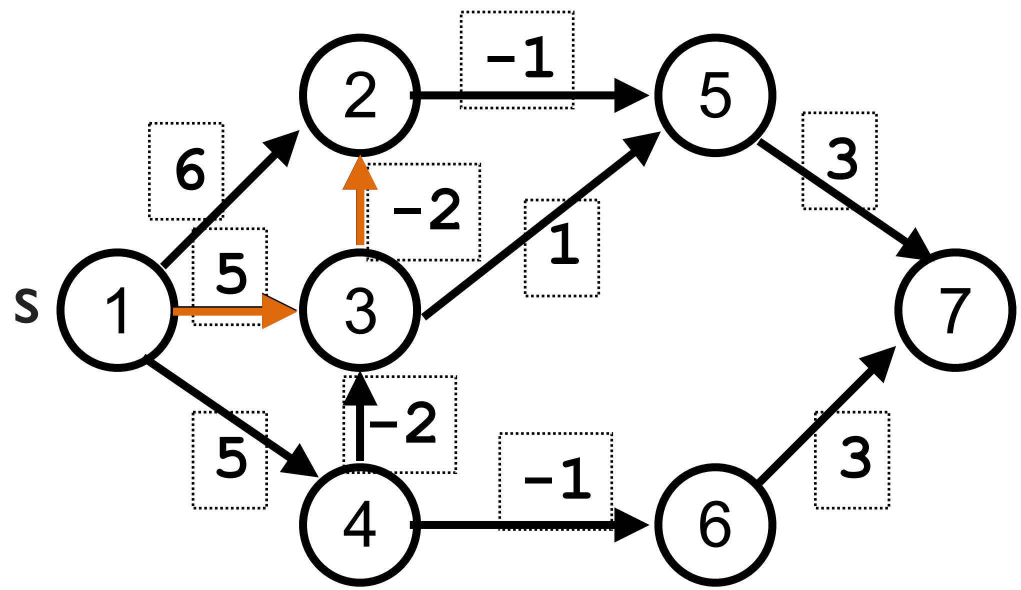
Dynamic Programming Approach

- The recurrence equation for shortest path

$$dist^{k}(u) = \min\{dist^{k-1}(u), \min_i\{dist^{k-1}(i) + cost[i][u]\}\}$$

- Use the recurrence equation to compute shortest path $dist^{k}(u)$ for $k=2,3,\ldots,n-1$
- Approach involves
  – use adjacent matrix for cost
  – First compute all shortest paths of lenth $k=2$
  – Then compute shortest paths of length $k=3$
  :
  – Finally, compute shortest paths of length $k=n-1$

Example: DP Approach



dist^k[1..7]

k↓	1	2	3	4	5	6	7
1	0	6	5	5	∞	∞	∞
2	0	3	3	5	5	4	∞
3	0	1	3	5	2	4	7
4	0	1	3	5	0	4	5
5	0	1	3	4	0	4	3
6	0	1	3	4	0	4	3

Algo: Bellman Ford

```
Algo BellmanFord(v, n, cost[][],dist[])
//computes single source all dstn shortest with -ve edge costs
// i/p: source v, number of nodes: n
for i=1 to n
   dist[i]=cost[v][i]  //default to ∞ when no edge
for k=2 to n-1
   for (each u such that u≠v, and u has incoming edge)
      for each edge <i,u> in the graph
         if dist[u] > dist[i]+cost[i][u] then
            dist[u]=dist[i]+cost[i][u]
         fi //end if
      // end for each edge
   // end for each u≠v
// end for k
```

Time Complexity: Bellman Ford

- Using adjancency matrix
  - 3 nested for loops
    - first `n-1` times (`k=2` to `n-1`)
    - other could potentially run n times.
  - Time complexity: $O(n^3)$
- Using adjancy matrix
  - outer most nest loop: `n` times (`k=2` to `n-1`)
  - Innermost two for loop together at most `e` times
  - Time complexity: `O(ne)`
- Note: if for one iteration of `k`. none of `dist[]` changes
  - Then it won't change for next iteration as well
  - Loop can be terminated with iterating till `n-1`

Summary

- Graph with negative edges
  - Dijkstra's algo does not work.
- Graph with negative cycles.
- Dynamic programming approach to graphs with negative cycles
  - Bellman Ford algorithm.