Theory of Distributions and Linear Differential Equations

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Outline

Introduction

- Laurent Schwartz and Sergei Sobolev;

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- a more perspicuous language for PDE analysts;

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- a generalised approach to solve linear PDEs.

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- a more perspicuous language for PDE analysts;
- a generalised approach to solve linear PDEs.
- treat functions as functionals acting on some ‘good’ functions

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for $x > 0$.

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Does it admit an extension from $\mathbb{R}_+$ to $\mathbb{R}$? No classical solution, but there are distributional solutions!

Test functions

Definition (Test functions.)

We write the test functions over region Ω to be all smooth (C^∞, infinitely many times differentiable) functions over Ω with compact supports (in short: all non-zero value of which lie in a compact set).

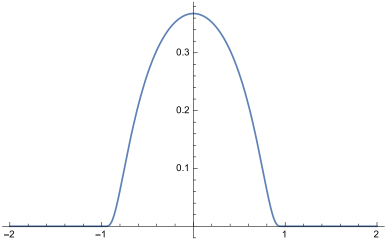
Test functions

Example

The function

$$\Psi(x) = \begin{cases} \exp(-\frac{1}{1-x^2}), & |x| < 1 \\ 0, & \text{otherwise} \end{cases}$$

is a test function on $\mathbb{R}$.



Test functions

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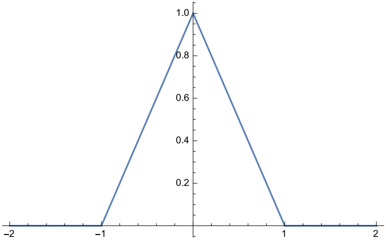
The followings are not test functions on $\mathbb{R}$:



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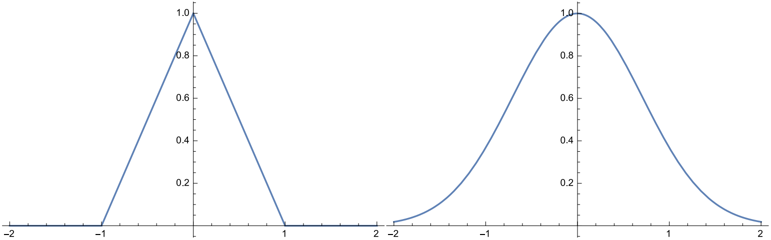


the latter of which is function $exp(-x^2)$.

Distributions

Definition (Distributions.)

We then define the distributions, i.e. generalised functions, to be defined as continuous linear functionals over all test functions on Ω.

Example (Regular distributions.)

All L^1_{loc}-functions f are automatically defined as distributions:

$$\langle f, \varphi \rangle = \int f(x) \cdot \varphi(x) \, \mathrm{d}x$$

for any test function φ.

Distributions

Example (Heaviside function.)

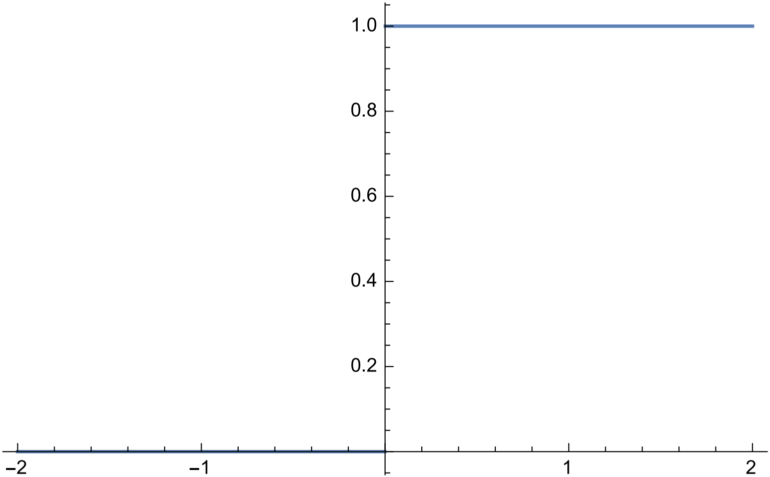
The Heaviside step function is defined as

$$\theta(x) = \begin{cases} 0, & x < 0 \\ \frac{1}{2}, & x = 0 \\ 1, & x > 0 \end{cases} .$$

It admits a regular distribution

$$\langle \theta(x), \varphi(x) \rangle = \int_{\mathbb{R}} \theta(x)\varphi(x)\ dx = \int_{\mathbb{R}_+} \varphi(x)\ dx.$$

Distributions



Distributions

Example (Dirac distribution.)

The Dirac distribution $\delta(x)$ is defined as

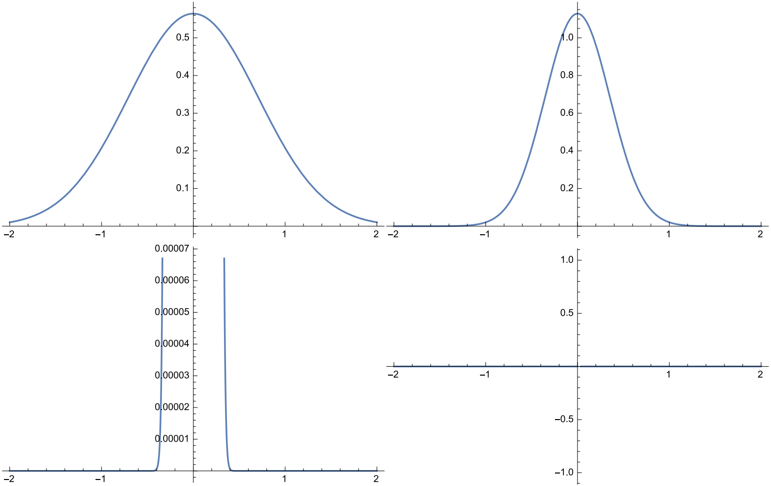
$$\langle \delta(x), \varphi(x) \rangle = \varphi(0).$$

It is an irregular distribution. It can be seen as an approximation of

$$\delta_\varepsilon(x) = \frac{1}{\varepsilon\sqrt{\pi}} e^{-x^2/\varepsilon^2}$$

as $\varepsilon \to 0$.

Distributions



Multiplication & differentiation

Definition (Multiplication by C^∞-functions.)

Given a distribution g on Ω and a C^∞-function f on Ω, we define the distribution fg on Ω to be

$$\langle fg, \varphi\rangle = \langle g, f \cdot \varphi\rangle$$

for any test function φ on Ω.

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Definition (Differentiation.)

Given a distribution f on $\mathbb{R}^n$, for $1 \le k \le n$ we define the distribution $\frac{\partial}{\partial x^k} g$ on $\mathbb{R}^n$ to be

$$\langle \frac{\partial}{\partial x^k} g, \varphi \rangle = -\langle g, \frac{\partial}{\partial x^k} \varphi \rangle$$

for any test function φ on $\mathbb{R}^n$.

Multiplication & differentiation

Example

We want to find the derivative of Heaviside step function:

$$\langle \theta'(x), \varphi(x) \rangle = -\langle \theta(x), \varphi'(x) \rangle = -\int_{\mathbb{R}_+} \varphi'(x)\ dx = -\left[\varphi(x)\right]_0^\infty$$
$$= \varphi(0) = \langle \delta(x), \varphi(x) \rangle,$$

for φ is a test function with compact support (it equals 0 when x is large enough). Hence $\theta'(x) = \delta(x)$.

Multiplication & differentiation

Example

$$x \cdot \delta(x) = 0$$

as of

$$\langle x \cdot \delta(x), \varphi(x) \rangle = \langle \delta(x), x\varphi(x) \rangle = 0.$$

Multiplication & differentiation

Definition (Principal value distributions.)

We define the principal value distribution of $1/x$ on $\mathbb{R}$ to be

$$\langle \mathcal{P}(\frac{1}{x}), \varphi(x) \rangle = \lim_{\varepsilon \to 0^+} \int_{\mathbb{R} \setminus [-\varepsilon, \varepsilon]} \frac{1}{x} \varphi(x) \; dx.$$

Example

$$x \cdot \mathcal{P}(\frac{1}{x}) = 1.$$

Tensor product & convolution

Definition (Tensor product.)

If distributions $f(x)$ is defined on Ω_1 and $g(y)$ is defined on Ω_2, then we define

$$\langle f(x) \otimes g(y), \varphi(x;y)\rangle = \langle f(x), \langle g(y), \varphi(x;y)\rangle\rangle$$

to be a distribution over $\Omega_1 \times \Omega_2$.

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Definition (Convolution.)

We for a moment assume that distributions $f(x), g(y)$ are defined on the same region Ω, and f is a distribution of a compact support. Then

$$\langle f * g, \varphi\rangle = \langle f(x) \otimes g(y), \varphi(x + y)\rangle.$$

Tensor product & convolution

Example (Convolution with Dirac distribution.)

$$\begin{aligned}\langle f(x) * \delta(x), \varphi(x)\rangle = \langle f(x) \otimes \delta(y), \varphi(x+y)\rangle = \langle f(x), \langle \delta(y), \varphi(x+y)\rangle\rangle \\ = \langle f(x), \varphi(x+0)\rangle = \langle f(x), \varphi(x)\rangle.\end{aligned}$$

Hence $f(x) * \delta(x) = f(x)$.

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$$= \langle f(x), \varphi(x+0) \rangle = \langle f(x), \varphi(x) \rangle.$$

Hence $f(x) * \delta(x) = f(x)$.

Proposition

If distributions f and g are defined on $\mathbb{R}^n$:

$$\frac{\partial}{\partial x^k} f * g = (\frac{\partial}{\partial x^k} f) * g = f * (\frac{\partial}{\partial x^k} g).$$

Linear ODE with singularity

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for constants α, β.

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for constants α, β. Then

$$y(x) = \frac{1}{x}\alpha\theta(x) + \beta\mathcal{P}(\frac{1}{x}) + \gamma\delta(x)$$

Linear ODE with singularity

one can show that

$$x \cdot (\frac{1}{2}\mathcal{P}\frac{1}{|x|} + \frac{1}{2}\mathcal{P}(\frac{1}{x})) = \theta(x)$$

where $\mathcal{P}\frac{1}{|x|}$ is the regularisation of $\frac{1}{|x|}$ defined by

$$\langle \mathcal{P}\frac{1}{|x|}, \varphi(x)\rangle = \int_{|x|<1} \frac{\varphi(x) - \varphi(0)}{|x|}\mathrm{d}x + \int_{|x|>1} \frac{\varphi(x)}{|x|}\mathrm{d}x.$$

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After reseting all coefficients we obtain solutions in the form

$$y(x) = \alpha\mathcal{P}\frac{1}{|x|} + \beta\mathcal{P}\frac{1}{x} + \gamma\delta(x).$$

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When we restrain the domain from $\mathbb{R}$ to $\mathbb{R}_+$, $\mathcal{P}\frac{1}{|x|}$ and $\mathcal{P}\frac{1}{x}$ are exactly the distributions induced by $\frac{1}{|x|}$ and $\frac{1}{x}$.

Fundamental solution

Definition (Fundamental solution.)

We define the fundamental solutions to a constant coefficient differential operator $L(\partial)$ to be the distributional solution E which satisfies

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Let f be a distribution on Ω s.t. the convolution $E * f$ exists as a distribution, then the solution of equation $L(\partial)E = f(x)$ exists in the space of distributions and is given by the formula

$$u = E * f.$$

This solution is unique in the class of distributions.

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Hint:

$$L(\partial)(E * f) = (L(\partial)E) * f = \delta(x) * f(x) = f(x)$$

Laplace operator

We aim to solve

$$\triangle u(x_1, x_2, x_3) = f(x_1, x_2, x_3) \Leftrightarrow \frac{\partial^2}{\partial x_1^2} u + \frac{\partial^2}{\partial x_2^2} u + \frac{\partial^2}{\partial x_3^2} u = f.$$

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Consider the equation where $x \in \mathbb{R}^3$

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It is a known fact that

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By the previous proposition, assume $E_3(x) * f(x)$ exists, then

$$u(x) = E_3(x) * f(x) = -\frac{1}{4\pi} |x|^{-1} * f$$

is a unique distributional solution to the equation in the class of distributions.

Distributional solutions to Navier-Stokes equations

Definition (Navier-Stokes equations.)

The incompressibe Navier-Stokes equations in $\mathbb{R}^3$ is defined by the system:

$$v_t + v \cdot \nabla v - \mu \triangle v + \nabla p = 0, \quad \nabla \cdot v = 0.$$

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Theorem (Leray (1934))

For $v_0 \in L_2(\mathbb{R}^3)$ with $\nabla \cdot v_0 = 0$, there exists a solution v of the Navier-Stokes equations in the sense of distributions.

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Theorem (Leray (1934))

For $v_0 \in L_2(\mathbb{R}^3)$ with $\nabla \cdot v_0 = 0$, there exists a solution v of the Navier-Stokes equations in the sense of distributions.

No one knows whether there are any classical solutions to the equations.

Flaw of the theory

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- a L_2-function can be multiplied by another L_2-function to get a L_1-function, automatically a distribution.

The End

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