Spectral Problems on 3-tori

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Abstract

It is known that the standard curl operator on a 3-torus endowed with trivial Euclidean metric obtains spectrum with explicit eigenfunctions. General non-constant metrics usually lead to no explicit solutions of eigenfunctions, but we investigate the curl operator on two 3-tori endowed with distinct special non-constant metric tensors under which explicit axisymmetric solutions are found. We also find reasonable approximations of eigenvalues and corresponding eigenfunctions under non-axisymmetric cases.

Contents

1 Spectrum of Standard Curl Operator

Let M be a 3-torus of periodicity 2π, the vector x on which satisfies

$$x = (x^1, x^2, x^3) = (x^1 + 2n_1\pi, x^2 + 2n_2\pi, x^3 + 2n_3\pi) \tag{1.1}$$

for all $n_1, n_2, n_3 \in \mathbb{Z}$. Write 3-covectors v on M^* as

$$v = (v_1, v_2, v_3) \tag{1.2}$$

and consider the M is endowed with standard Riemannian metric

$$g_{\alpha\beta} = \delta^{\alpha}{}_{\beta}, \quad \alpha, \beta = 1, 2, 3. \tag{1.3}$$

We then have the standard curl operator

$$\operatorname{curl} = \begin{pmatrix} 0 & -\partial_3 & \partial_2 \\ \partial_3 & 0 & -\partial_1 \\ -\partial_2 & \partial_1 & 0 \end{pmatrix} \tag{1.4}$$

which is a mapping from M^* to M^*.

Now we consider the spectral problem with respect to the standard curl operator (1.4):

$$\operatorname{curl} v = \lambda v \tag{1.5}$$

for some $\lambda \in \mathbb{C}$. Separate the variables by

$$v = u \exp(ip_\alpha x^\alpha), \quad \alpha = 1, 2, 3, \quad u \in \mathbb{C}^3 \tag{1.6}$$

to obtain a linear system of equations

$$i \begin{pmatrix} 0 & -p_3 & p_2 \\ p_3 & 0 & -p_1 \\ -p_2 & p_1 & 0 \end{pmatrix} u = \lambda u. \tag{1.7}$$

By solving it we obtain the spectrum of the standard curl operator (1.4):

- 0 is an eigenvalue of multiplicity two.
- For any $p \in \mathbb{Z}^3$, we have an eigenvalue $\|p\|$ unique up to rescaling with eigenfunction of form $u \exp(ip_\alpha x^\alpha)$.
- For any $p \in \mathbb{Z}^3$, we have an eigenvalue $-\|p\|$ unique up to rescaling with eigenfunction of form $u \exp(ip_\alpha x^\alpha)$.

2 Curl Operator under Special Riemannian Metric A

Now we introduce the special axisymmetric Riemannian metric tensor A depending only on x^1 with perturbation ϵ:

$$g_{\alpha\beta}(x^1;\epsilon)dx^\alpha dx^\beta = \left[dx^1\right]^2 + \left[1+\epsilon\cos x^1 dx^2 + \epsilon\sin x^1 dx^3\right]^2 + \left[\epsilon\sin x^1 dx^2 + (1-\epsilon\cos x^1)dx^3\right]^2. \quad (2.1)$$

This can also be written as

$$g_{\alpha\beta}(x^1;\epsilon) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1+\epsilon^2+2\epsilon\cos x^1 & 2\epsilon\sin x^1 \\ 0 & 2\epsilon\sin x^1 & 1+\epsilon^2-2\epsilon\cos x^1 \end{pmatrix}. \quad (2.2)$$

The corresponding contravariant metric tensor is

$$g^{\alpha\beta}(x^1;\epsilon) = (1-\epsilon^2)^{-2}\begin{pmatrix} (1-\epsilon^2)^2 & 0 & 0 \\ 0 & 1+\epsilon^2-2\epsilon\cos x^1 & -2\epsilon\sin x^1 \\ 0 & -2\epsilon\sin x^1 & 1+\epsilon^2+2\epsilon\cos x^1 \end{pmatrix}. \quad (2.3)$$

The curl operator is defined on Riemannian manifolds by

$$(\operatorname{curl} v)_\gamma = \sqrt{\det g_{\mu\nu}}\ g^{\alpha'\alpha}g^{\beta'\beta}(\partial_\alpha v_\beta)\varepsilon_{\alpha'\beta'\gamma}. \quad (2.4)$$

On a 3-torus M endowed with special metric A (2.1), we have our curl operator in the form

$$\begin{aligned}\operatorname{curl} = (1-\epsilon^2)^{-1}&\begin{pmatrix} 0 & 0 & 0 \\ 0 & 2\epsilon\sin x^1 & -(1+\epsilon^2+2\epsilon\cos x^1) \\ 0 & 1+\epsilon^2-2\epsilon\cos x^1 & -2\epsilon\sin x^1 \end{pmatrix}\partial_1 \\ &+ (1-\epsilon^2)^{-1}\begin{pmatrix} 0 & 0 & 1 \\ -2\epsilon\sin x^1 & 0 & 0 \\ -(1+\epsilon^2-2\epsilon\cos x^1) & 0 & 0 \end{pmatrix}\partial_2 \\ &+ (1-\epsilon^2)^{-1}\begin{pmatrix} 0 & -1 & 0 \\ 1+\epsilon^2+2\epsilon\cos x^1 & 0 & 0 \\ 2\epsilon\sin x^1 & 0 & 0 \end{pmatrix}\partial_3. \end{aligned} \quad (2.5)$$

It is easy to check that $\operatorname{curl}(\nabla) = 0$, $\operatorname{div}(\operatorname{curl}) = 0$ and $\langle \operatorname{curl} u, v\rangle = \langle u, \operatorname{curl} v\rangle$ under

$$\operatorname{div} u := \frac{\partial g^{\alpha\beta}u_\beta}{\partial x^\alpha} \quad (2.6)$$

and

$$\langle u, v\rangle := \int_{\mathbb{T}^3} g^{\alpha\beta}v_\alpha u_\beta \sqrt{\det g_{\mu\nu}}\ dx^1dx^2dx^3, \quad (2.7)$$

the latter of which implies that curl operator (2.5) is self-adjoint.

We then consider the curl operator that works on the half-density:

$$
\begin{aligned}
&\operatorname{curl}_{1/2} := \sqrt{g^{\alpha\beta}}\operatorname{curl}\sqrt{g_{\alpha\beta}}\\
&= \begin{pmatrix}0&0&0\\0&0&-1\\0&1&0\end{pmatrix}\partial_1 + (1-\epsilon^2)^{-1}\begin{pmatrix}0&\epsilon\sin x^1&1-\epsilon\cos x^1\\-\epsilon\sin x^1&0&0\\-(1-\epsilon\cos x^1)&0&0\end{pmatrix}\partial_2\\
&\quad + (1-\epsilon^2)^{-1}\begin{pmatrix}0&-(1+\epsilon\cos x^1)&-\epsilon\sin x^1\\1+\epsilon\cos x^1&0&0\\\epsilon\sin x^1&0&0\end{pmatrix}\partial_3\\
&\quad - (1-\epsilon^2)^{-1}\epsilon\begin{pmatrix}0&0&0\\0&\epsilon+\cos x^1&\sin x^1\\0&\sin x^1&\epsilon-\cos x^1\end{pmatrix}. \qquad (2.8)
\end{aligned}
$$

It is easy to check $\langle \operatorname{curl} u, v\rangle' = \langle u, \operatorname{curl} v\rangle'$ under the standard Euclidean inner product on 3-covectors

$$\langle u, v\rangle' := \int_{\mathbb{T}^3} v_\alpha u_\beta \; dx^1 dx^2 dx^3. \qquad (2.9)$$

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3 Spectrum of axisymmetric Curl under Metric A

Consider the spectral problem of half-density curl operator under special metric A (2.8) where the eigenfunctions may only depend on x^1. Then the half-density curl operator is simplified to its axisymmetric version

$$\operatorname{curl}_{1/2} = \begin{pmatrix}0&0&0\\0&0&-1\\0&1&0\end{pmatrix}\partial_1 - (1-\epsilon^2)^{-1}\epsilon\begin{pmatrix}0&0&0\\0&\epsilon+\cos x^1&\sin x^1\\0&\sin x^1&\epsilon-\cos x^1\end{pmatrix}. \qquad (3.1)$$

Since the operator does not involve the first entry of the covector operated, we only concentrate on its effect on v_2 and v_3. We modify the operator:

$$\widetilde{\operatorname{curl}_{1/2}} = R\operatorname{curl}_{1/2} R^*, \qquad (3.2)$$

in which R is the unitary operator

$$R = \frac{1}{\sqrt{2}}\begin{pmatrix}1&i\\i&1\end{pmatrix} \qquad (3.3)$$

and R^* is the conjugate transpose of R. It is known that $\widetilde{\operatorname{curl}_{1/2}}$ and $\operatorname{curl}_{1/2}$ share same pairs of eigenvalue and eigenfunction. We write down the modified axisymmetric half-density operator explicitly:

$$\widetilde{\operatorname{curl}_{1/2}} = \begin{pmatrix}i&0\\0&-i\end{pmatrix}\partial_1 - \frac{\epsilon^2}{1-\epsilon^2}Id + \frac{\epsilon}{1-\epsilon^2}\begin{pmatrix}0&ie^{ix^1}\\-ie^{-ix^1}&0\end{pmatrix}. \qquad (3.4)$$

By solving the spectral problem

$$\widetilde{\operatorname{curl}_{1/2}} v = \lambda v \tag{3.5}$$

we obtain the explicit spectrum of the axisymmetric half-density curl operator (3.1):

- For each $n \in \mathbb{Z}$, we have an eigenvalue

$$\lambda_n^+ = -\frac{1}{2} - \frac{\epsilon^2}{1-\epsilon^2} + \sqrt{n^2 - n + \frac{1}{4} + \frac{\epsilon^2}{(1-\epsilon^2)^2}} \tag{3.6}$$

with the corresponding eigenfunction

$$v_n^+ = \begin{pmatrix} 0 \\ \epsilon i e^{inx^1} \\ (1-\epsilon^2)(-\frac{1}{2} + n + \sqrt{n^2 - n + \frac{1}{4} + \frac{\epsilon^2}{(1-\epsilon^2)^2}}) e^{i(n-1)x^1} \end{pmatrix}. \tag{3.7}$$

- For each $n \in \mathbb{Z}$, we have an eigenvalue

$$\lambda_n^- = -\frac{1}{2} - \frac{\epsilon^2}{1-\epsilon^2} - \sqrt{n^2 - n + \frac{1}{4} + \frac{\epsilon^2}{(1-\epsilon^2)^2}} \tag{3.8}$$

with the corresponding eigenfunction

$$v_n^- = \begin{pmatrix} 0 \\ \epsilon i e^{inx^1} \\ (1-\epsilon^2)(-\frac{1}{2} + n - \sqrt{n^2 - n + \frac{1}{4} + \frac{\epsilon^2}{(1-\epsilon^2)^2}}) e^{i(n-1)x^1} \end{pmatrix}. \tag{3.9}$$

It is clear that the eigenvalues do not split:

$$\lambda_n^+ = \lambda_{1-n}^+, \quad \lambda_n^- = \lambda_{1-n}^- \tag{3.10}$$

for $n \in \mathbb{Z}$. We can also observe spectral asymmetry.

4 Non-axisymmetric Curl under Metric A

Now we consider the spectral problem of the half-density curl operator under special metric A (2.8) where the solutions involve x^1, x^2 and x^3. We modify the operator (2.8) by

$$\widetilde{\mathrm{curl}} = R\,\mathrm{curl}_{1/2}\,R^* = \begin{pmatrix} 0 & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & -i \end{pmatrix} \partial_1 + \frac{1}{1-\epsilon^2} \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\epsilon^2 & i\epsilon e^{ix^1} \\ 0 & -i\epsilon e^{-ix^1} & -\epsilon^2 \end{pmatrix} + \frac{1}{1-\epsilon^2}$$
$$\begin{pmatrix} 0 & -\frac{i}{\sqrt{2}}(1-\epsilon\cos x^1) + \frac{1}{\sqrt{2}}\epsilon\sin x^1 & \frac{1}{\sqrt{2}}(1-\epsilon\cos x^1) - \frac{i}{\sqrt{2}}\epsilon\sin x^1 \\ -\frac{i}{\sqrt{2}}(1-\epsilon\cos x^1) - \frac{1}{\sqrt{2}}\epsilon\sin x^1 & 0 & 0 \\ -\frac{1}{\sqrt{2}}(1-\epsilon\cos x^1) - \frac{i}{\sqrt{2}}\epsilon\sin x^1 & 0 & 0 \end{pmatrix} \partial_2$$
$$+ \frac{1}{1-\epsilon^2}$$
$$\begin{pmatrix} 0 & -\frac{1}{\sqrt{2}}(1+\epsilon\cos x^1) + \frac{i}{\sqrt{2}}\epsilon\sin x^1 & \frac{i}{\sqrt{2}}(1+\epsilon\cos x^1) - \frac{1}{\sqrt{2}}\epsilon\sin x^1 \\ \frac{1}{\sqrt{2}}(1+\epsilon\cos x^1) + \frac{i}{\sqrt{2}}\epsilon\sin x^1 & 0 & 0 \\ \frac{i}{\sqrt{2}}(1+\epsilon\cos x^1) + \frac{1}{\sqrt{2}}\epsilon\sin x^1 & 0 & 0 \end{pmatrix} \partial_3, \tag{4.1}$$

in which

$$R = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \\ 0 & \frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}. \tag{4.2}$$

To solve the spectral problem

$$\widetilde{\mathrm{curl}} v = \lambda v \tag{4.3}$$

we separate the variables by

$$v(x^1, x^2, x^3) = \begin{pmatrix} u_1(x^1) \\ u_2(x^1) \\ u_3(x^1) \end{pmatrix} \exp(im_\alpha x^\alpha), \quad \alpha = 2, 3. \tag{4.4}$$

We then obtain a system of differential equations concerning u only:

$$\frac{1}{\sqrt{2}(1-\epsilon^2)}$$
$$\begin{pmatrix} 0 & m_2 - im_3 - (m_2+im_3)\epsilon e^{-ix^1} & im_2 - m_3 - (im_2+m_3)\epsilon e^{ix^1} \\ m_2 + im_3 + (-m_2+im_3)\epsilon e^{ix^1} & -\sqrt{2}\epsilon^2 & i\sqrt{2}\epsilon e^{ix^1} \\ -im_2 - m_3 + (im_2 - m_3)\epsilon e^{-ix^1} & -i\sqrt{2}\epsilon e^{-ix^1} & -\sqrt{2}\epsilon^2 \end{pmatrix}$$
$$u + \begin{pmatrix} 0 & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & -i \end{pmatrix} \frac{du}{dx^1} = \lambda u. \tag{4.5}$$

Then the spectral problem about v is converted to another spectral problem about u:

$$\mathrm{curl_u}\, u = \lambda u \tag{4.6}$$

in which

$$\mathrm{curl_u} = \frac{1}{2}\left[\begin{pmatrix}0&0&0\\0&i&0\\0&0&-i\end{pmatrix}\frac{du}{dx^1}+\frac{du}{dx^1}\begin{pmatrix}0&0&0\\0&i&0\\0&0&-i\end{pmatrix}\right]+\frac{1}{\sqrt{2}(1-\epsilon^2)}$$
$$\begin{pmatrix}0 & m_2-im_3-(m_2+im_3)\epsilon e^{-ix^1} & im_2-m_3-(im_2+m_3)\epsilon e^{ix^1}\\ m_2+im_3+(-m_2+im_3)\epsilon e^{ix^1} & -\sqrt{2}\epsilon^2 & i\sqrt{2}\epsilon e^{ix^1}\\ -im_2-m_3+(im_2-m_3)\epsilon e^{-ix^1} & -i\sqrt{2}\epsilon e^{-ix^1} & -\sqrt{2}\epsilon^2\end{pmatrix} \tag{4.7}$$

is self-adjoint.

We now consider the simplest case in which $m_2 = 1$ and $m_3 = 0$ and our spectral problem becomes:

$$\begin{pmatrix}0&0&0\\0&i&0\\0&0&-i\end{pmatrix}\frac{du}{dx^1}+\frac{1}{\sqrt{2}(1-\epsilon^2)}\begin{pmatrix}0 & 1-\epsilon e^{-ix^1} & i-i\epsilon e^{ix^1}\\ 1-\epsilon e^{ix^1} & -\sqrt{2}\epsilon^2 & i\sqrt{2}\epsilon e^{ix^1}\\ -i+i\epsilon e^{-ix^1} & -i\sqrt{2}\epsilon e^{-ix^1} & -\sqrt{2}\epsilon^2\end{pmatrix}u=\lambda u. \tag{4.8}$$

We consider the unperturbed zero-order term with $\epsilon = 0$:

$$\begin{pmatrix}0&1&i\\1&0&0\\-i&0&0\end{pmatrix} \tag{4.9}$$

the eigenvalues of which are

$$\lambda = 0, \pm 1. \tag{4.10}$$

We are going to investigate more details about the spectrum around 1 specifically. Since explicit solutions are not found in our non-axisymmetric case, we want to have at least an approximation to one pair of eigenvalue and eigenfunction. We set a small perturbation $\epsilon = 0.1$. Consider the inner product on 3-covectors:

$$\langle u, v\rangle = \int_0^{2\pi} u^* v \; dx; \tag{4.11}$$

Consider the following orthonormal basis $\{\phi_{n,k}\}$ for the space of 3-covectors, in which $n \in \mathbb{Z}$ and $k = 1, 2, 3$:

$$(\phi_{n,k})_\mu = \frac{1}{\sqrt{2\pi}} e^{inx^1} \delta^k{}_\mu, \quad \mu = 1, 2, 3. \tag{4.12}$$

We restrain the basis to $n \in [-5, 5]$, a basis of thirty-three elements and apply Galerkin method with the abovementioned inner product. We obtain thirty-three eigenvalues in this constrained search, the one closest to our earlier chosen eigenvalue 1 of which is

$$\tilde{\lambda} = 1.00501. \tag{4.13}$$

We also find its corresponding eigenfunction:

$$
\begin{aligned}
\tilde{u} = &\begin{pmatrix} 8.50917\times 10^{-10}i \\ -1.52518\times 10^{-10}i \\ -2.38349\times 10^{-9} \end{pmatrix} e^{-5ix^1} + \begin{pmatrix} 1.5644\times 10^{-7}i \\ -3.73333\times 10^{-8}i \\ -3.2823\times 10^{-7} \end{pmatrix} e^{-4ix^1} + \begin{pmatrix} 0.0000163316i \\ -5.8545\times 10^{-6}i \\ -0.0000228409 \end{pmatrix} e^{-3ix^1} \\
&+ \begin{pmatrix} 0.00071697i \\ -0.000516426i \\ -0.000501673 \end{pmatrix} e^{-2ix^1} + \begin{pmatrix} -0.000285756i \\ -0.0202466i \\ -0.0000511699 \end{pmatrix} e^{-ix^1} + \begin{pmatrix} -0.282071i \\ -0.198455i \\ -0.198455 \end{pmatrix} + \begin{pmatrix} 0.000285756i \\ -0.0000511699i \\ -0.0202466 \end{pmatrix} e^{ix^1} \\
&+ \begin{pmatrix} 0.00071697i \\ -0.000501673i \\ -0.000516426 \end{pmatrix} e^{2ix^1} + \begin{pmatrix} 0.0000163316i \\ -0.0000228409i \\ -5.8545\times 10^{-6} \end{pmatrix} e^{3ix^1} + \begin{pmatrix} 1.5644\times 10^{-7}i \\ -3.2823\times 10^{-7}i \\ -3.73333\times 10^{-8} \end{pmatrix} e^{4ix^1} \\
&+ \begin{pmatrix} 8.50917\times 10^{-10}i \\ -2.38349\times 10^{-9}i \\ 1.52518\times 10^{-10}i \end{pmatrix} e^{5ix^1}.
\end{aligned}
\tag{4.14}
$$

Then we consider how close our approximation is to the actual eigenvalue. Consider the discrepancy function of non-axisymmetric half-density curl operator under special metric A:

$$
\varepsilon(u,\lambda) = \left[\begin{pmatrix} 0 & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & -i \end{pmatrix} \frac{d}{dx^1} + \frac{1}{\sqrt{2}(1-\epsilon^2)} \begin{pmatrix} 0 & 1-\epsilon e^{-ix^1} & i - i\epsilon e^{ix^1} \\ 1-\epsilon e^{ix^1} & -\sqrt{2}\epsilon^2 & i\sqrt{2}\epsilon e^{ix^1} \\ -i + i\epsilon e^{-ix^1} & -i\sqrt{2}\epsilon e^{-ix^1} & -\sqrt{2}\epsilon^2 \end{pmatrix} - \lambda Id \right] u.
\tag{4.15}
$$

Evaluate the L^2-norm of the discrepancy of our approximation:

$$
\|\varepsilon(\tilde{u},\tilde{\lambda})\| \approx 2.72807\times 10^{-10}. \tag{4.16}
$$

The spectral-theoretic interpretation of this norm is that we have an exact eigenvalue λ lies in 2.72807×10^{-10}-neighbourhood of our approximated eigenvalue $\tilde{\lambda}$ when $\epsilon = 0.1$, that is:

$$
\lambda \in [1.00501 - 2.72807\times 10^{-10}, 1.00501 + 2.72807\times 10^{-10}]. \tag{4.17}
$$

5 Curl Operator under Special Riemannian Metric B

We now move to 3-torus endowed with another Riemmannian metric B. Consider

$$
g_{\alpha\beta}(x^1;\epsilon)dx^\alpha dx^\beta = \left[dx^1 + \epsilon\cos x^1 dx^2 + \epsilon \sin x^1 dx^3\right]^2 + \left[dx^2\right]^2 + \left[dx^3\right]^2. \tag{5.1}
$$

We write it as the matrix form

$$
g_{\alpha\beta}(x^1;\epsilon) = \begin{pmatrix} 1 & \epsilon\cos x^1 & \epsilon\sin x^1 \\ \epsilon\cos x^1 & 1+\epsilon^2\cos^2 x^1 & \epsilon^2\cos x^1 \sin x^1 \\ \epsilon \sin x^1 & \epsilon^2 \cos x^1 \sin x^1 & 1+\epsilon^2\sin^2 x^1 \end{pmatrix}, \tag{5.2}
$$

the contravariant metric tensor corresponding to which is

$$g^{\alpha\beta}(x^1;\epsilon) = \begin{pmatrix} 1+\epsilon^2 & -\epsilon\cos x^1 & -\epsilon\sin x^1 \\ -\epsilon\cos x^1 & 1 & 0 \\ -\epsilon\sin x^1 & 0 & 1 \end{pmatrix}. \tag{5.3}$$

It is noted that the covariant metric tensor (5.2) can also be written as product $M^T M$ in which

$$M = \begin{pmatrix} 1 & \epsilon\cos x^1 & \epsilon\sin x^1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{5.4}$$

is its coframe and M^{-1} is its frame.

With curl operator defined in (2.4) we have the one written under special metric B:

$$\begin{aligned} &\mathrm{curl} = \\ &\begin{pmatrix} -\epsilon\sin x^1\partial_2 + \epsilon\cos x^1\partial_3 & \epsilon\sin x^1\partial_1 - \partial_3 & -\epsilon\cos x^1\partial_1 + \partial_2 \\ -\epsilon^2\cos x^1\sin x^1\partial_2 + (1+\epsilon^2\cos^2 x^1)\partial_3 & \epsilon^2\cos x^1\sin x^1\partial_1 - \epsilon\cos x^1\partial_3 & -(1+\epsilon^2\cos^2 x^1)\partial_1 + \epsilon\cos x^1\partial_2 \\ -(1+\epsilon^2\sin^2 x^1)\partial_2 + \epsilon^2\cos x^1\sin x^1\partial_3 & (1+\epsilon^2\sin^2 x^1)\partial_1 - \epsilon\sin x^1\partial_3 & -\epsilon^2\cos x^1\sin x^1\partial_1 + \epsilon\sin x^1\partial_2 \end{pmatrix} \\ &= \begin{pmatrix} 0 & \epsilon\sin x^1 & -\epsilon\cos x^1 \\ 0 & \epsilon^2\cos x^1\sin x^1 & -(1+\epsilon^2\cos^2 x^1) \\ 0 & 1+\epsilon^2\sin^2 x^1 & -\epsilon^2\cos x^1\sin x^1 \end{pmatrix}\partial_1 + \begin{pmatrix} -\epsilon\sin x^1 & 0 & 1 \\ -\epsilon^2\cos x^1\sin x^1 & 0 & \epsilon\cos x^1 \\ -(1+\epsilon^2\sin^2 x^1) & 0 & \epsilon\sin x^1 \end{pmatrix}\partial_2 \\ &\quad + \begin{pmatrix} \epsilon\cos x^1 & -1 & 0 \\ 1+\epsilon^2\cos^2 x^1 & -\epsilon\cos x^1 & 0 \\ \epsilon^2\cos x^1\sin x^1 & -\epsilon\sin x^1 & 0 \end{pmatrix}\partial_3. \end{aligned} \tag{5.5}$$

It is easy to check that $\mathrm{curl}(\nabla) = 0$, $\mathrm{div}(\mathrm{curl}) = 0$ and $\langle \mathrm{curl}\, u, v\rangle = \langle u, \mathrm{curl}\, v\rangle$ under (2.6) and (2.7).

We then consider the curl operator working on half-density.

$$\begin{aligned} &\mathrm{curl}_{1/2} = (M^T)^{-1}\,\mathrm{curl}\, M^T \\ &\begin{pmatrix} 0 & \epsilon\sin x^1 & -\epsilon\cos x^1 \\ -\epsilon\sin x^1 & 0 & -1 \\ \epsilon\cos x^1 & 1 & 0 \end{pmatrix}\partial_1 + \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}\partial_2 + \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\partial_3 - \begin{pmatrix} \epsilon^2 & 0 & 0 \\ \epsilon\cos x^1 & 0 & 0 \\ \epsilon\sin x^1 & 0 & 0 \end{pmatrix}. \end{aligned} \tag{5.6}$$

It is still easy to check the half-density curl operator is self-adjoint under the standard Euclidean inner product on 3-covectors.

6 Spectrum of axisymmetric Curl under Metric B

Consider the spectral problem of half-density curl operator under special metric B (5.6) where the eigenfunctions may only depend on x^1. The half-density curl operator is simplified to its axisymmetric version

$$\operatorname{curl}_{1/2} = \begin{pmatrix} 0 & \epsilon \sin x^1 & -\epsilon \cos x^1 \\ -\epsilon \sin x^1 & 0 & -1 \\ \epsilon \cos x^1 & 1 & 0 \end{pmatrix} \partial_1 - \begin{pmatrix} \epsilon^2 & 0 & 0 \\ \epsilon \cos x^1 & 0 & 0 \\ \epsilon \sin x^1 & 0 & 0 \end{pmatrix}. \tag{6.1}$$

We modify the operator:

$$\widetilde{\operatorname{curl}_{1/2}} = R \operatorname{curl}_{1/2} R^*, \tag{6.2}$$

in which R is unitary

$$R = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \\ 0 & \frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \tag{6.3}$$

and R^* is its conjugate transpose. We write down the modified curl operator explicitly:

$$\widetilde{\operatorname{curl}_{1/2}} = \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} i\epsilon e^{-ix^1} & -\frac{1}{\sqrt{2}} \epsilon e^{ix^1} \\ \frac{1}{\sqrt{2}} i\epsilon e^{ix^1} & i & 0 \\ \frac{1}{\sqrt{2}} \epsilon e^{-ix^1} & 0 & -i \end{pmatrix} \frac{d}{dx^1} - \begin{pmatrix} \epsilon^2 & 0 & 0 \\ \frac{1}{\sqrt{2}} \epsilon e^{ix^1} & 0 & 0 \\ \frac{1}{\sqrt{2}} i\epsilon e^{-ix^1} & 0 & 0 \end{pmatrix}. \tag{6.4}$$

By solving the spectral problem

$$\operatorname{curl}_{1/2} v = \lambda v, \tag{6.5}$$

we explicate the spectrum of the axisymmetric half-density curl operator (6.1):

- For each $n \in \mathbb{Z}$, we have an eigenvalue

$$\lambda_n^+ = -\frac{\epsilon^2}{2} - 1 + \sqrt{n^2\epsilon^2 + n^2 + \frac{\epsilon^4}{4}} \tag{6.6}$$

  with the corresponding eigenfunction

$$v_n^+ = \begin{pmatrix} \left(-\frac{\epsilon^2}{2} + n + \sqrt{n^2\epsilon^2 + n^2 + \frac{\epsilon^4}{4}}\right)\left(-\frac{\epsilon^2}{2} - n + \sqrt{n^2\epsilon^2 + n^2 + \frac{\epsilon^4}{4}}\right) e^{inx} \\ -\frac{1}{\sqrt{2}}\epsilon \left(-\frac{\epsilon^2}{2} - n + \sqrt{n^2\epsilon^2 + n^2 + \frac{\epsilon^4}{4}}\right)(n+1)e^{i(n+1)x} \\ i\frac{1}{\sqrt{2}}\epsilon \left(-\frac{\epsilon^2}{2} + n + \sqrt{n^2\epsilon^2 + n^2 + \frac{\epsilon^4}{4}}\right)(n-1)e^{i(n-1)x} \end{pmatrix}. \tag{6.7}$$

- For each $n \in \mathbb{Z}$, we have an eigenvalue

$$\lambda_n^- = -\frac{\epsilon^2}{2} - 1 - \sqrt{n^2\epsilon^2 + n^2 + \frac{\epsilon^4}{4}} \tag{6.8}$$

with the corresponding eigenfunction

$$v_n^- = \begin{pmatrix} \left(-\frac{\epsilon^2}{2} + n - \sqrt{n^2\epsilon^2 + n^2 + \frac{\epsilon^4}{4}}\right)\left(-\frac{\epsilon^2}{2} - n - \sqrt{n^2\epsilon^2 + n^2 + \frac{\epsilon^4}{4}}\right) e^{inx} \\ -\frac{1}{\sqrt{2}}\epsilon\left(-\frac{\epsilon^2}{2} - n - \sqrt{n^2\epsilon^2 + n^2 + \frac{\epsilon^4}{4}}\right)(n+1)e^{i(n+1)x} \\ i\frac{1}{\sqrt{2}}\epsilon\left(-\frac{\epsilon^2}{2} + n - \sqrt{n^2\epsilon^2 + n^2 + \frac{\epsilon^4}{4}}\right)(n-1)e^{i(n-1)x} \end{pmatrix}. \tag{6.9}$$

- 0 is an eigenvalue of multiplicity at least two. We have found explicitly two eigenvalues equal to zero, namely

$$\lambda_1^+(\epsilon) = \lambda_{-1}^+(\epsilon) = 0. \tag{6.10}$$

It is clear that the eigenvalues do not split:

$$\lambda_n^+ = \lambda_{-n}^+, \quad \lambda_n^- = \lambda_{-n}^- \tag{6.11}$$

for $n \in \mathbb{Z}$. Spectral asymmetry is observed.

7 Non-axisymmetric Curl under Metric B

Now we consider the spectral problem of the half-density curl operator under special metric B (5.6) where the solutions involve x^1, x^2 and x^3. We modify the operator (5.6) by

$$\begin{aligned}\widetilde{\operatorname{curl}} = R\operatorname{curl}_{1/2}R^* &= \begin{pmatrix} 0 & \frac{1}{\sqrt{2}}i\epsilon e^{-ix^1} & -\frac{1}{\sqrt{2}}\epsilon e^{ix^1} \\ \frac{1}{\sqrt{2}}i\epsilon e^{ix^1} & i & 0 \\ \frac{1}{\sqrt{2}}\epsilon e^{-ix^1} & 0 & -i \end{pmatrix}\partial_1 \\ &+ \begin{pmatrix} 0 & -\frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{i}{\sqrt{2}} & 0 & 0 \\ -\frac{1}{\sqrt{2}} & 0 & 0 \end{pmatrix}\partial_2 + \begin{pmatrix} 0 & -\frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & 0 \\ \frac{i}{\sqrt{2}} & 0 & 0 \end{pmatrix}\partial_3 - \begin{pmatrix} \epsilon^2 & 0 & 0 \\ \frac{1}{\sqrt{2}}\epsilon e^{ix^1} & 0 & 0 \\ \frac{1}{\sqrt{2}}i\epsilon e^{-ix^1} & 0 & 0 \end{pmatrix}.\end{aligned} \tag{7.1}$$

in which

$$R = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \\ 0 & \frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}. \tag{7.2}$$

Let

$$v = u\exp(ip_\alpha x^\alpha), \quad \alpha = 1,2,3, \quad u \in \mathbb{C}^3 \tag{7.3}$$

be a separable solution. We simplify the eigenvalue problem and obtain a system of differential equations concerning u only:

$$
\begin{pmatrix} 0 & \frac{i}{\sqrt{2}}\epsilon e^{-ix^1} & -\frac{1}{\sqrt{2}}\epsilon e^{ix^1} \\ \frac{i}{\sqrt{2}}\epsilon e^{ix^1} & i & 0 \\ \frac{1}{\sqrt{2}}\epsilon e^{-ix^1} & 0 & -i \end{pmatrix} \frac{du}{dx^1}
+ \begin{pmatrix} -\epsilon^2 & \frac{1}{\sqrt{2}}m_2 - \frac{i}{\sqrt{2}}m_3 & \frac{i}{\sqrt{2}}m_2 - \frac{1}{\sqrt{2}}m_3 \\ \frac{1}{\sqrt{2}}m_2 + \frac{i}{\sqrt{2}}m_3 - \frac{1}{\sqrt{2}}\epsilon e^{ix^1} & 0 & 0 \\ -\frac{i}{\sqrt{2}}m_2 - \frac{1}{\sqrt{2}}m_3 - \frac{i}{\sqrt{2}}\epsilon e^{-ix^1} & 0 & 0 \end{pmatrix} u = \lambda u. \tag{7.4}
$$

The spectral problem about v is converted to a spectral problem about u:

$$
\mathrm{curl_u}\, u = \lambda u \tag{7.5}
$$

in which

$$
\mathrm{curl_u} = \frac{1}{2}\left[\begin{pmatrix} 0 & \frac{i}{\sqrt{2}}\epsilon e^{-ix^1} & -\frac{1}{\sqrt{2}}\epsilon e^{ix^1} \\ \frac{i}{\sqrt{2}}\epsilon e^{ix^1} & i & 0 \\ \frac{1}{\sqrt{2}}\epsilon e^{-ix^1} & 0 & -i \end{pmatrix} \frac{d}{dx^1} + \frac{d}{dx^1} \begin{pmatrix} 0 & \frac{i}{\sqrt{2}}\epsilon e^{-ix^1} & -\frac{1}{\sqrt{2}}\epsilon e^{ix^1} \\ \frac{i}{\sqrt{2}}\epsilon e^{ix^1} & i & 0 \\ \frac{1}{\sqrt{2}}\epsilon e^{-ix^1} & 0 & -i \end{pmatrix} \right]
+ \begin{pmatrix} -\epsilon^2 & \frac{1}{\sqrt{2}}m_2 - \frac{i}{\sqrt{2}}m_3 - \frac{1}{2\sqrt{2}}\epsilon e^{-ix^1} & \frac{i}{\sqrt{2}}m_2 - \frac{1}{\sqrt{2}}m_3 + \frac{i}{2\sqrt{2}}\epsilon e^{ix^1} \\ \frac{1}{\sqrt{2}}m_2 + \frac{i}{\sqrt{2}}m_3 - \frac{1}{2\sqrt{2}}\epsilon e^{ix^1} & 0 & 0 \\ -\frac{i}{\sqrt{2}}m_2 - \frac{1}{\sqrt{2}}m_3 - \frac{i}{2\sqrt{2}}\epsilon e^{-ix^1} & 0 & 0 \end{pmatrix} \tag{7.6}
$$

is self-adjoint.

Consider the simplest case in which $m_2 = 1$ and $m_3 = 0$ and our spectral problem turns to be:

$$
\begin{pmatrix} 0 & \frac{i}{\sqrt{2}}\epsilon e^{-ix^1} & -\frac{1}{\sqrt{2}}\epsilon e^{ix^1} \\ \frac{i}{\sqrt{2}}\epsilon e^{ix^1} & i & 0 \\ \frac{1}{\sqrt{2}}\epsilon e^{-ix^1} & 0 & -i \end{pmatrix} \frac{du}{dx^1} + \begin{pmatrix} -\epsilon^2 & \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\epsilon e^{ix^1} & 0 & 0 \\ -\frac{i}{\sqrt{2}} - \frac{i}{\sqrt{2}}\epsilon e^{-ix^1} & 0 & 0 \end{pmatrix} u = \lambda u. \tag{7.7}
$$

Consider the unperturbed zero-order term with $\epsilon = 0$:

$$
\begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & 0 \\ -\frac{i}{\sqrt{2}} & 0 & 0 \end{pmatrix}, \tag{7.8}
$$

the eigenvalues of which are

$$
\lambda = 0, \pm 1. \tag{7.9}
$$

We want to learn more about the eigenvalues around 1 when perturbed by ϵ. Although no explicit solutions are found, we can find an approximation to the exact eigenvalue near 1. Apply the Galerkin method with respect to inner product (4.11) and Hilbert basis (4.12) with constraints $n \in [-5, 5]$, a basis of thirty-three elements. We obtain thirty-three eigenvalues in the constrained search, the one closest to our chosen unperturbed eigenvalue 1 of which is

$$\tilde{\lambda} = 0.997514. \tag{7.10}$$

Its corresponding eigenfunction is

$$\begin{aligned}\tilde{u} = &\begin{pmatrix} -2.15745 \times 10^{-8} i \\ 3.8115 \times 10^{-9} i \\ 5.23399 \times 10^{-8} \end{pmatrix} e^{-5ix^1} + \begin{pmatrix} 9.31018 \times 10^{-7} i \\ -2.17229 \times 10^{-7} i \\ -1.49199 \times 10^{-6} \end{pmatrix} e^{-4ix^1} + \begin{pmatrix} -0.0000286894 i \\ 0.000010032 i \\ 0.0000226916 \end{pmatrix} e^{-3ix^1} \\ &+ \begin{pmatrix} 0.000523242 i \\ -0.000365023 i \\ 0.00011933 \end{pmatrix} e^{-2ix^1} + \begin{pmatrix} -0.0000869346 i \\ 0.00984421 i \\ -0.00999194 \end{pmatrix} e^{-ix^1} + \begin{pmatrix} -0.281394 i \\ -0.199472 i \\ -0.199472 \end{pmatrix} + \begin{pmatrix} 0.0000869346 i \\ 0.00999194 i \\ -0.00984421 \end{pmatrix} e^{ix^1} \\ &+ \begin{pmatrix} 0.000523242 i \\ 0.00011933 i \\ -0.000365023 \end{pmatrix} e^{2ix^1} + \begin{pmatrix} 0.0000286894 i \\ -0.0000226916 i \\ -0.000010032 \end{pmatrix} e^{3ix^1} + \begin{pmatrix} 9.31018 \times 10^{-7} i \\ -1.49199 \times 10^{-6} i \\ -2.17229 \times 10^{-7} \end{pmatrix} e^{4ix^1} \\ &+ \begin{pmatrix} 2.15745 \times 10^{-8} i \\ -5.23399 \times 10^{-8} i \\ -3.8115 \times 10^{-9} \end{pmatrix} e^{5ix^1}. \end{aligned} \tag{7.11}$$

To examine how accurate our approximation is, we write down the discrepancy function of non-axisymmetric half-density curl operator under special metric B:

$$\varepsilon(u, \lambda) = \left[\begin{pmatrix} 0 & \frac{i}{\sqrt{2}} \epsilon e^{-ix^1} & -\frac{1}{\sqrt{2}} \epsilon e^{ix^1} \\ \frac{i}{\sqrt{2}} \epsilon e^{ix^1} & i & 0 \\ \frac{1}{\sqrt{2}} \epsilon e^{-ix^1} & 0 & -i \end{pmatrix} \frac{d}{dx^1} + \begin{pmatrix} -\epsilon^2 & \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} \epsilon e^{ix^1} & 0 & 0 \\ -\frac{i}{\sqrt{2}} - \frac{i}{\sqrt{2}} \epsilon e^{-ix^1} & 0 & 0 \end{pmatrix} - \lambda Id \right] u. \tag{7.12}$$

We evaluate the L^2-norm of the discrepancy of our approximation:

$$\|\varepsilon(\tilde{u}, \tilde{\lambda})\| \approx 3.27973 \times 10^{-8}. \tag{7.13}$$

The spectral-theoretic interpretation of this result is that we have an exact eigenvalue λ lies in 3.27973×10^{-8}-neighbourhood of our eigenvalue of approximation $\tilde{\lambda}$ when $\epsilon = 0.1$:

$$\lambda \in [0.997514 - 3.27973 \times 10^{-8}, 0.997514 + 3.27973 \times 10^{-8}]. \tag{7.14}$$