Spectral Problems on 3-tori

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Abstract

Consider first the standard curl operator on a 3-torus endowed with Euclidean metric. It is known that in this case the eigenvalues and eigenfunctions can be calculated explicitly. For general non-constant metrics one usually cannot evaluate the eigenvalues and eigenfunctions explicitly, but we investigate two special families of non-constant metrics for which explicit axisymmetric solutions (i.e. axisymmetric eigenfunctions and corresponding eigenvalues) can be found. For these special families of metrics we also find good numerical approximations for non-axisymmetric solutions.

Introduction

A function from a vector space to another is called an **operator**. An operator L is **linear** if

$$L(u + av) = Lu + aLv \tag{1}$$

for any vector u, v and any scalar a. A function $f \neq 0$ is said to be an **eigenfunction** of a linear operator L acting between function spaces if there is a scalar λ such that

$$Lf = \lambda f. \tag{2}$$

The number λ is called the **eigenvalue** corresponding to the eigenfunction f. We will be studying an operator whose spectrum is discrete, and in this case the **spectrum** is simply the set of all possible eigenvalues.

An n-**manifold** is a geometric object that resembles n-dimensional Euclidean space near each point. A 3-**torus** is a 3-manifold defined with 2π-periodicity in each argument: for any function f on the manifold, we have

$$f(x^1, x^2, x^3) = f(x^1 + 2n_1\pi, x^2 + 2n_2\pi, x^3 + 2n_3\pi) \tag{3}$$

for any integers n_1, n_2, n_3. A **Riemannian metric** is used to describe how curved the space is. In our case Riemannian metric can be written as a 3×3 matrix-function $g_{\alpha\beta}(x)$. A **perturbation** is used to describe a small variation under which we examine how the quantity we investigate changes.

The partial differentiation is simply written as

$$\partial_\alpha = \frac{\partial}{\partial x^\alpha} \tag{4}$$

The **curl** operator on covector fields is defined by the following formula, where $\alpha, \beta, \alpha', \beta'$ run from 1 to 3 and where we are using Einstein's summation convention:

$$(\operatorname{curl} v)_\gamma = \sqrt{\det g_{\mu\nu}}\, g^{\alpha'\alpha} g^{\beta'\beta} (\partial_\alpha v_\beta)\varepsilon_{\alpha'\beta'\gamma}. \tag{5}$$

In particular, if the metric is Euclidean, then curl takes its familiar form:

$$\operatorname{curl} = \begin{pmatrix} 0 & -\partial_3 & \partial_2 \\ \partial_3 & 0 & -\partial_1 \\ -\partial_2 & \partial_1 & 0 \end{pmatrix}. \tag{6}$$

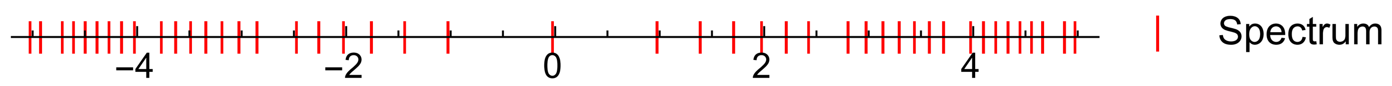
This is the **standard curl operator**.

Spectrum of standard curl operator

It is known the standard curl operator has the following spectrum:

- 0 is an eigenvalue of multiplicity two.
- For any $p \in \mathbb{Z}^3$, we have an eigenvalue $\|p\| = \sqrt{(p_1)^2 + (p_2)^2 + (p_3)^2}$.
- For any $p \in \mathbb{Z}^3$, we have an eigenvalue $-\|p\| = -\sqrt{(p_1)^2 + (p_2)^2 + (p_3)^2}$.

We plot all eigenvalues within $[-5, 5]$:



Spectrum of curl under metric A

We introduce a 3-torus endowed with trigonometric Riemannian metric A with perturbation ε:

$$g_{\alpha\beta}(x^1;\varepsilon)dx^\alpha dx^\beta = \left[dx^1\right]^2 + \left[(1+\varepsilon\cos x^1)dx^2 + \varepsilon\sin x^1 dx^3\right]^2 + \left[\varepsilon\sin x^1 dx^2 + (1-\varepsilon\cos x^1)dx^3\right]^2. \tag{7}$$

We write down the half-density curl operator corresponding to the metric A:

$$\begin{aligned}\operatorname{curl}_{1/2} := {} & \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}\partial_1 + (1-\varepsilon^2)^{-1}\begin{pmatrix} 0 & \varepsilon\sin x^1 & 1-\varepsilon\cos x^1 \\ -\varepsilon\sin x^1 & 0 & 0 \\ -(1-\varepsilon\cos x^1) & 0 & 0 \end{pmatrix}\partial_2 \\ & + (1-\varepsilon^2)^{-1}\begin{pmatrix} 0 & -(1+\varepsilon\cos x^1) & -\varepsilon\sin x^1 \\ 1+\varepsilon\cos x^1 & 0 & 0 \\ \varepsilon\sin x^1 & 0 & 0 \end{pmatrix}\partial_3 \\ & - (1-\varepsilon^2)^{-1}\varepsilon\begin{pmatrix} 0 & 0 & 0 \\ 0 & \varepsilon+\cos x^1 & \sin x^1 \\ 0 & \sin x^1 & \varepsilon-\cos x^1 \end{pmatrix}.\end{aligned} \tag{8}$$

Consider the spectral problem of half-density curl operator under special metric A where the eigenfunctions may only depend on x^1. Then the half-density curl operator is simplified to its axisymmetric version

$$\operatorname{curl}_{1/2} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}\partial_1 - (1-\varepsilon^2)^{-1}\varepsilon\begin{pmatrix} 0 & 0 & 0 \\ 0 & \varepsilon+\cos x^1 & \sin x^1 \\ 0 & \sin x^1 & \varepsilon-\cos x^1 \end{pmatrix}. \tag{9}$$

Spectrum of the axisymmetric half-density curl operator (9) can be found in explicit forms:

- For each $n \in \mathbb{Z}$, we have an eigenvalue

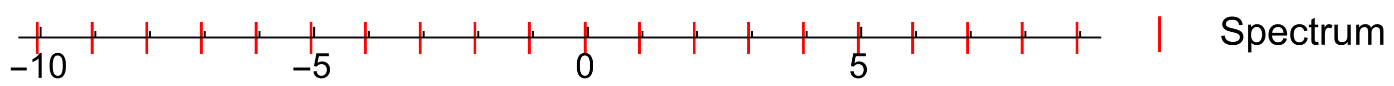
$$\lambda_n^+ = -\frac{1}{2} - \frac{\varepsilon^2}{1-\varepsilon^2} + \sqrt{n^2 - n + \frac{1}{4} + \frac{\varepsilon^2}{(1-\varepsilon^2)^2}}. \tag{10}$$

- For each $n \in \mathbb{Z}$, we have an eigenvalue

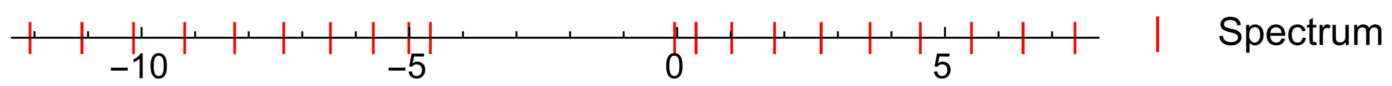
$$\lambda_n^- = -\frac{1}{2} - \frac{\varepsilon^2}{1-\varepsilon^2} - \sqrt{n^2 - n + \frac{1}{4} + \frac{\varepsilon^2}{(1-\varepsilon^2)^2}}. \tag{11}$$

We plot the distribution of eigenvalues with respect to different degrees of perturbation:

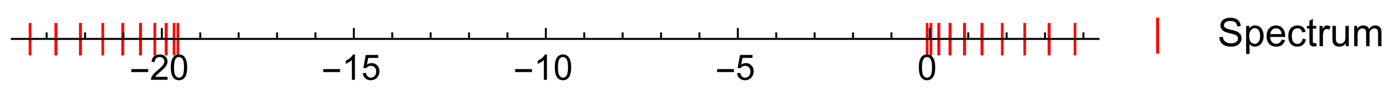
ε=0.1



ε=0.8



ε=0.95



Note that the spectrum does not split:

$$\lambda_n^+ = \lambda_{1-n}^+, \quad \lambda_n^- = \lambda_{1-n}^-. \tag{12}$$

The spectrum is asymmetric with $\lambda_n^+ \neq \lambda_{-n}^+$ and $\lambda_n^- \neq \lambda_{-n}^-$ generally.

In the non-axisymmetric case the spectrum is not explicitly computable, though by Galerkin method a nice approximation is found.

There is an exact eigenvalue λ to the non-axisymmetric problem (8) such that

$$\lambda \in [1.00501 - 2.72807\times 10^{-10}, 1.00501 + 2.72807\times 10^{-10}]. \tag{13}$$

Spectrum of curl under metric B

We introduce another 3-torus endowed with trigonometric Riemannian metric B with perturbation ε:

$$g_{\alpha\beta}(x^1;\varepsilon)dx^\alpha dx^\beta = \left[dx^1 + \varepsilon\cos x^1 dx^2 + \varepsilon\sin x^1 dx^3\right]^2 + \left[dx^2\right]^2 + \left[dx^3\right]^2. \tag{14}$$

Similarly write down the half-density curl operator under metric B, and concentrate on the axisymmetric case:

We obtain the explicit spectrum of the axisymmetric operator:

- For each $n \in \mathbb{Z}$, we have an eigenvalue

$$\lambda_n^+ = -\frac{\varepsilon^2}{2} - 1 + \sqrt{n^2\varepsilon^2 + n^2 + \frac{\varepsilon^4}{4}}. \tag{15}$$

- For each $n \in \mathbb{Z}$, we have an eigenvalue

$$\lambda_n^- = -\frac{\varepsilon^2}{2} - 1 - \sqrt{n^2\varepsilon^2 + n^2 + \frac{\varepsilon^4}{4}}. \tag{16}$$

- 0 is an eigenvalue of multiplicity at least two. We have found explicitly two eigenvalues equal to zero, namely

$$\lambda_1^+(\varepsilon) = \lambda_{-1}^+(\varepsilon) = 0. \tag{17}$$

The spectrum does not split and spectral asymmetry is also observed. Similarly an approximation to the non-axisymmetric case is found.

There is an exact eigenvalue λ such that

$$\lambda \in [0.997514 - 3.27973\times 10^{-8}, 0.997514 + 3.27973\times 10^{-8}]. \tag{18}$$

Further Work

One could possibly formally classify all metrics for which reasonable operators (Laplacian, Dirac, curl) have explicitly computable axisymmetric spectra. In some sense our analysis might be related to integrable systems.

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