Statement of Knowledge Structure

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1 Overview

The document explicates my mathematical curriculum, up to merger, split and concision. The purpose of this composition is to grant specific readers a quick but decent check of my fundamental knowledge about mathematics. It does not contain any extracurricular readings.

2 Specialisation

2.1 Pure and applied analysis

- *Measure theory*: Families of sets, σ-algebras, Borel sets, measurability of functions, measures, null sets, completeness, Borel measures, Lebesgue measure in $\mathbb{R}^n$. Integral with respect to a measure, Fatou's lemma, monotone convergence theorem, dominated convergence theorem. Caratheodory construction, product measures, Fubini's theorem, Lebesgue integral in $\mathbb{R}^n$. Absolute continuity of measures, Radon-Nikodym theorem. k-dimensional measures in $\mathbb{R}^n$.

- *Linear functional analysis*: Young's, Holder and Minkowski inequalities. Normed linear spaces and Banach spaces. ℓ^p spaces, c_0, c_{00}; $\mathcal{L}^p$ spaces. $C(X)$ and $C_p(X)$ spaces. Completion. Finite-dimensional normed linear spaces. Hilbert spaces. Orthogonal systems and the orthogonalisation. Riesz representation theorem. Tietze extension theorem. Dini, Stone-Weierstrass and Arzelà-Ascoli theorems. Zorn's lemma and the Hahn-Banach theorem. Linear functionals and duality. Second dual and reflexive spaces. Baire category theorem. Uniform boundedness theorem, open mapping theorem, closed graph theorem.

- *Analysis on manifolds*: Functions on $\mathbb{R}^n$, differentiation, chain rule, partial derivatives. Inverse function theorem. Implicit function theorem. Multiple Riemann integrals. Tensors, alternating tensors, tensor product and

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wedge product. Differential forms, closed and exact forms, pullback and pushforward, differentials. Simplexes and chains, integration on chains. Manifolds, integration on manifolds, Stoke's Theorem. The volume element, classical Stoke's theorem, Green's theorem and divergence theorem.

- *Distribution theory*: Test functions. Distributions, differentiation, multiplication by smooth functions. Distributions with compact support, distributions supported at a point. Convolution of distributions. Fourier transform on distributions.

- *PDE analysis*: Linear differential operators. Elliptic/parabolic/hyperbolic classification. Convolution and fundamental solutions of linear PDEs; the classical integral representation formulae. Schwartz kernel theorem. Fourier transform, convolution, Poisson's summation formula, Sobolev spaces, elliptic regularity. Maximum principles and energy. The Dirichlet problem for harmonic functions.

- *Measure-theoretic probability*: Probability space, events, random variables. Probability laws, distribution function, densities. Joint laws. Expectation, variance. Chebyshev's inequality. Independence. The Borel-Cantelli Lemmas. Bernstein's inequality. Tail σ-algebras and Kolmogorov's $0-1$ law. Weak convergence. Characteristic functions and Fourier transforms. The Parseval-Plancherel Theorem. The Central Limit Theorem. Conditional expectations, filtrations, and martingales. Stopping times. The Optional Stopping Theorem. Doob's inequality. Kolmogorov's inequality. Kolmogorov's Theorem. Strong Law of Large Numbers. Previsible processes and the martingale transform. Martingale Convergence Theorem. Convergence of uniformly integrable martingales.

2.2 Differential geometry and its application

- *Differential geometry of curves and surfaces*: Curves. Curvature and torsion of curves and Frenet-Serret formulae, isoperimetric inequality, total curvature and the global geometry of curves. Surfaces in $\mathbb{R}^3$. First fundamental form, length and area, covariant derivative, normal and geodesic curvature, geodesics, second fundamental form, the Gauss map, principal, Gaussian, mean, geodesic and normal curvatures, Theorema Egregium, Gauss-Bonnet Formula, Euler characteristic, minimal surfaces, applications. Manifolds. Topology, topological manifolds.

- *Mathematics for general relativity*: Minkowski spacetime, Lorentzian transformations. Contraction of length and time. Vector fields and covector fields, contraction of tensors and tensor product. Lorentzian manifolds. Geodesics. Christoffel symbols, Riemann curvature tensors and Ricci curvature tensors. Einstein field equations. Schwartzschild metrics, Eddington-Finkelstein coordinates and Kruskal coordinates.

3 Fundamentals

3.1 Analysis

- *Analysis on* $\mathbb{R}$: Basic properties of $\mathbb{R}$. Sequences and convergence. Bolzano-Weierstrass theorem. Series and convergence tests. Boundedness and continuity, the intermediate value theorem and inverses. Differentiation on $\mathbb{R}$. Rolle's Theorem. Mean Value Theorem. Riemann integration. The fundamental theorem of calculus. Improper integrals. Cauchy sequences. Uniform continuity. L'Hôpital's Rule. Taylor's Theorem.

- *Complex analysis*: Basic properties of $\mathbb{C}$. Convergence of sequences. Continuous, holomorphic functions, Cauchy-Riemann equations, power series. Harmonic functions. Conformal mapping. Linear fractional transformations. Integration along curves. Goursat's theorem, Cauchy's theorem, Cauchy's integral formulas. Taylor's theorem, Laurent's theorem, Liouville's theorem, Morera's theorem. The Fundamental theorem of algebra. Meromorphic functions, zeros, poles, residue theorem, Cauchy's argument principle, Rouché's theorem, Casorati-Weierstrass' theorem, the identity theorem. Maximum modulus principle.

- *Real analysis*: Normed and metric spaces. Openness and closeness. Continuity. Compactness. Weierstrass function. Pointwise and uniform convergence. Weierstrass approximation theorem. Banach contraction mapping theorem. Orthonormal systems and Fourier Series. Bessel inequalities. Riemann's lemma. Dirichlet's theorem.

3.2 Algebra

- *Linear Algebra*: Matrices, elementary operations. Row echelon form. Solution of system. Linear independence. Basis and dimension. Permutations. Kernel-rank theorem. Determinants. Adjoint and inverse. Eigenvalues and eigenvectors, characteristic equation, diagonalisation. The Cayley-Hamilton Theorem. Direct sums. Invariant subspaces. Minimum polynomial. Primary decomposition theorem. Jordan normal form. Linear forms, dual spaces. Symmetric bilinear forms. Quadratic forms. Congruence of matrices. Real and complex canonical forms , signatures. Polarization. Sylvester's Law of Inertia. Definite and semi-definite forms. Cauchy-Schwarz inequality, orthogonal basis and Gram-Schmidt process. Orthogonal complements. Adjoint mapping. Orthogonal and unitary maps. Spectral theorem.

- *Group theory*: Groups and order. Subgroups, cyclic groups. Lagrange's Theorem. Symmetric group. Families of finite groups: C_n; D_{2n}; A_n; S_n; GL_n. Homomorphisms, isomorphisms and automorphisms. Automorphism group. Generators. Semi-direct product. Group action. Stabiliser, the class equation. Sylow's Theorem. Groups of order pq^m where $q < p$.

- *Ring theory*: Ring, field, unit, reducible and irreducible elements. Division rings. Subrings, ideals, quotient rings, integral domains. Principal ideals. Unit group and Euler's totient function. Extension fields as quotients. Eisenstein's Criterion. Gauss's Lemma. Cyclotomic polynomials.

- *Number theory*: Degree function. Highest common factor. Primes. Euclids algorithm. The h, k-lemma. Unique factorization. Division algorithm, Euclid's theorem. Remainder theorem. Fermat's Little Theorem and the Chinese Remainder Theorem.

3.3 Methods

- *Multivariable calculus*: Partial differentiation. Chain rule. Taylor series. Maxima and minima of functions of more than one independent variable. Line integrals. Potential energy. Gradient. Integration over plane areas. Volumes. Change of variables. Jacobians. Integration over volumes. Flux, Gauss theorem, Greens theorem, Stokes theorem, divergence, curl, standard identities and manipulations, Einstein summation convention.

- *Basic PDE methods*: Fourier series. Parseval's theorem. Euler-Lagrange equations with and without constraints. First-order linear partial differential equations. Characteristics. Quasilinear equations. One-dimensional wave equation. D'Alembert's solution. Initial/boundary value problems involving the use of Fourier series. Normal modes of vibration. Solution of the diffusion and Laplace equations by separation of variables.

- *Probability and statistics*: Kolmogorov's axioms. Bernoulli, binomial, geometric, negative binomial, hypergeometric, Poisson, uniform, exponential, gamma, beta, normal distributions. Joint and conditional distributions. Iterated expectation; multinomial and multivariate normal distributions. Moment and probability generating functions. Independence. Weak law of large numbers. Central Limit Theorem. Data, estimation and hypothesis testing. Normal probability models. χ^2, t and F distributions. Confidence intervals. Regression and correlation. Least-squares.

- *Numerical analysis*: Theoretical foundations of computational mathematics: conditioning, stability, consistency, numerical stability, convergence, the equivalence theorem. Numerical solution of nonlinear equations: the bisection algorithm, fixed point iteration, Newtons method and its relatives. Numerical solution of large linear systems: iterative methods, Gauss'/Jacobi's methods, Richards' method and the gradient method, preconditioning, the conjugate gradient method. Methods for ordinary differential equations: the Cauchy problem, the basic single step methods, explicit and implicit methods, zero-stability and absolute stability, truncation error, convergence, extensions to system of equations. Introduction to boundary value problems using finite differences.