THEOREM: AN AVL TREE OF HEIGHT H HAS AT LEAST $2^{H/2-1}$ NODES

PROOF:

LET MIN(H) BE THE MINIMUM NUMBER OF NODES IN AN AVL TREE OF HEIGHT H.

WE WANT TO SHOW $MIN(H) \geq 2^{H/2-1}$ $\forall H \geq 1$.

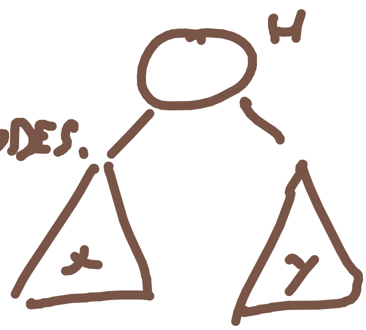
WE PROCEED BY (STRONG) INDUCTION ON H.

FOR H = 1, 2, NOTE THAT

$$MIN(1) = 1 \geq 2^{1/2-1} = 1/\sqrt{2}$$
$$MIN(2) = 2 \geq 2^{2/2-1} = 2^0 = 1$$

FOR $H \geq 3$, ASSUME BY (STRONG) INDUCTION THAT THE RESULT IS TRUE FOR ALL $n < H$. WE SHOW THE RESULT FOR H:

CONSIDER AN AVL TREE OF HEIGHT H WITH MIN NODES. THEN X OR Y MUST HAVE HEIGHT H-1, AND THE OTHER MUST HAVE HEIGHT H-2



$$MIN(H) \geq 1 + MIN(H-1) + MIN(H-2)$$
$$> 2\,MIN(H-2)$$

AND BY INDUCTION, $MIN(H-2) \geq 2^{\frac{H-2}{2}-1}$

SO

$$MIN(H) > 2 \cdot 2^{\frac{H-2}{2}-1} = 2^{\frac{H-2}{2}} = 2^{H/2-1}$$

$\square$

COROLARY: THE HEIGHT OF AN AVL TREE WITH N ELEMENTS IS $O(\log_2 N)$

PROOF: FROM THE THEOREM, WE KNOW THAT

$$N \geq 2^{H/2-1} \quad \text{WHERE } H \text{ IS THE HEIGHT}$$

TAKING LOGS: $\log_2 N \geq H/2 - 1$

$$\Rightarrow H \leq 2\log_2 N + 2$$

$$\Rightarrow H \text{ IS } O(\log_2 N)$$