Church-Turing Thesis

FOCS, Fall 2020

Turing machines

Model of computation for decision functions $f : \Sigma^* \to \{\text{true}, \text{false}\}$

- A decision function is Turing-computable if there exists a total Turing machine M such that f(u) = true exactly when M accepts string u

Claim: Turing-computability can be taken as the definition of computability

Church-Turing thesis: every "feasible" model of computation can be simulated by a Turing machine

- feasible = no infinite work in finite time, no data with infinite content, ...

Church-Turing thesis

The Church-Turing thesis is a thesis -- it is not a theorem, it cannot be proved

It is an empirical fact, based on evidence

- Every model of computation we have can be simulated by Turing machines

We will see other models of computations later in the course, based on different principles than states + memory, and we will see how to simulate them with Turing machines

- lambda calculus, production grammars, ...

Turing machines as CPUs

Modern computers are driven by a CPU

If we can show how to simulate a CPU using a Turing machine, we get that whatever a modern computer can do can basically be done by a Turing machine

What's a CPU?

- registers holding finite values
- finite memory holding finite values
- simple instructions to transfer values between registers and memory

A simple CPU

Arbitrarily many registers, numbered 0, 1, 2, 3, …

Each register holds an arbitrary natural number ≥ 0

Instructions:

- INC r	increment register *r*
- DEC r, addr	if register *r* > 0, decrement it; else, jump to *addr*
- JMP addr	jump to *addr*
- TRUE	stop and return *true*
- FALSE	stop and return *false*

Example: addition

```
# R0 + R1 =? R2

start:
  DEC 0, compare
  INC 1
  JMP start

compare:
  DEC 1, empty
  DEC 2, reject
  JMP compare
empty:
  DEC 2, accept
reject:
  FALSE
accept:
  TRUE
```

Example: multiplication

```
# R0 * R1 =? R2

clear3:
  # clear R3 = prod
  DEC 3, clear4
  JMP clear3
clear4:
  # clear R4 = temp
  DEC 4, loop
  JMP clear4
```

```
loop0
  DEC 0, compare
loop1
  DEC 1, next
  INC 3
  INC 4
  JMP loop1
next:
  DEC 4, loop0
  INC 1
  JMP next
```

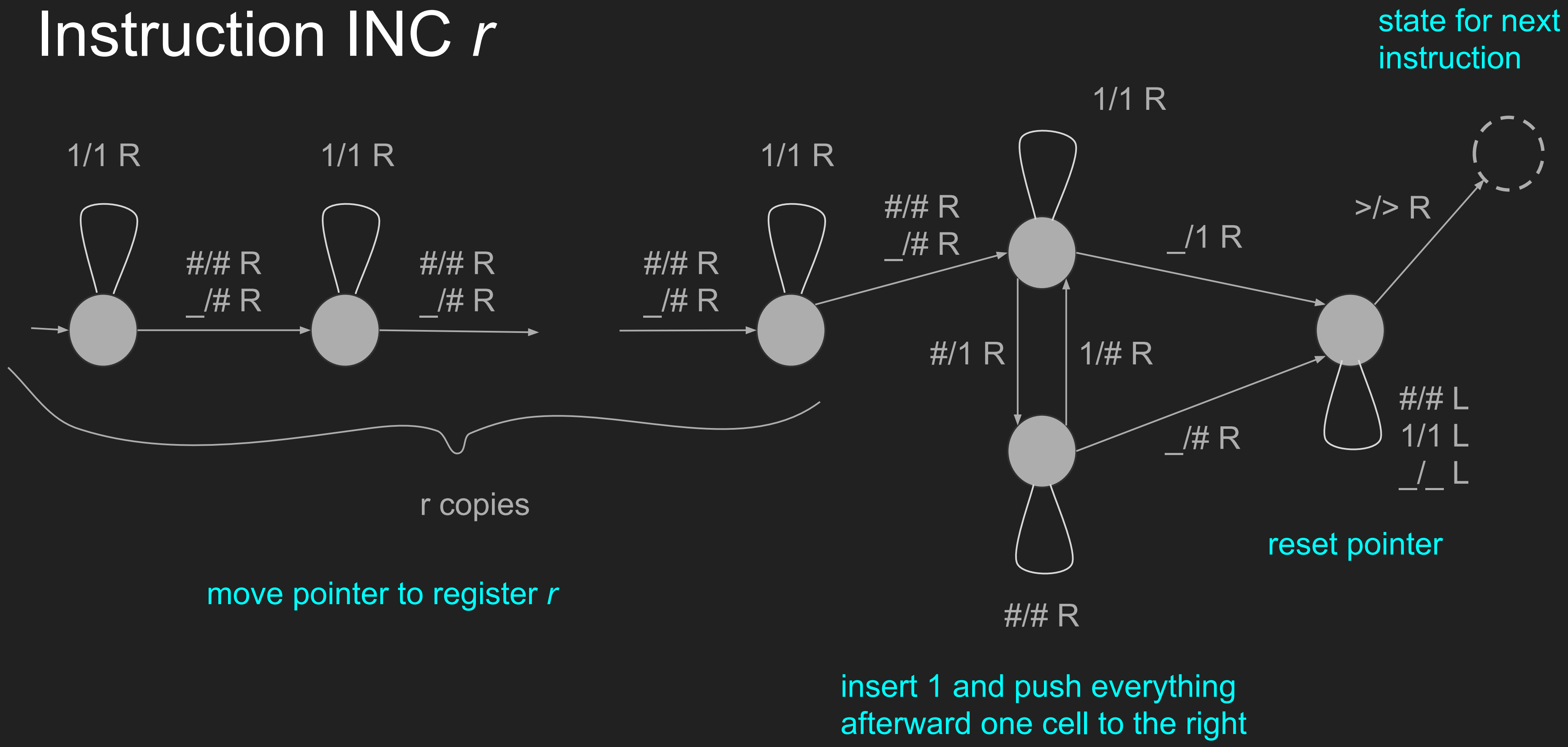
```
compare:
  DEC 3, empty
  DEC 2, reject
  JMP compare
empty:
  DEC 2, accept
reject:
  FALSE
accept:
  TRUE
```

Simulating a program with a Turing machine

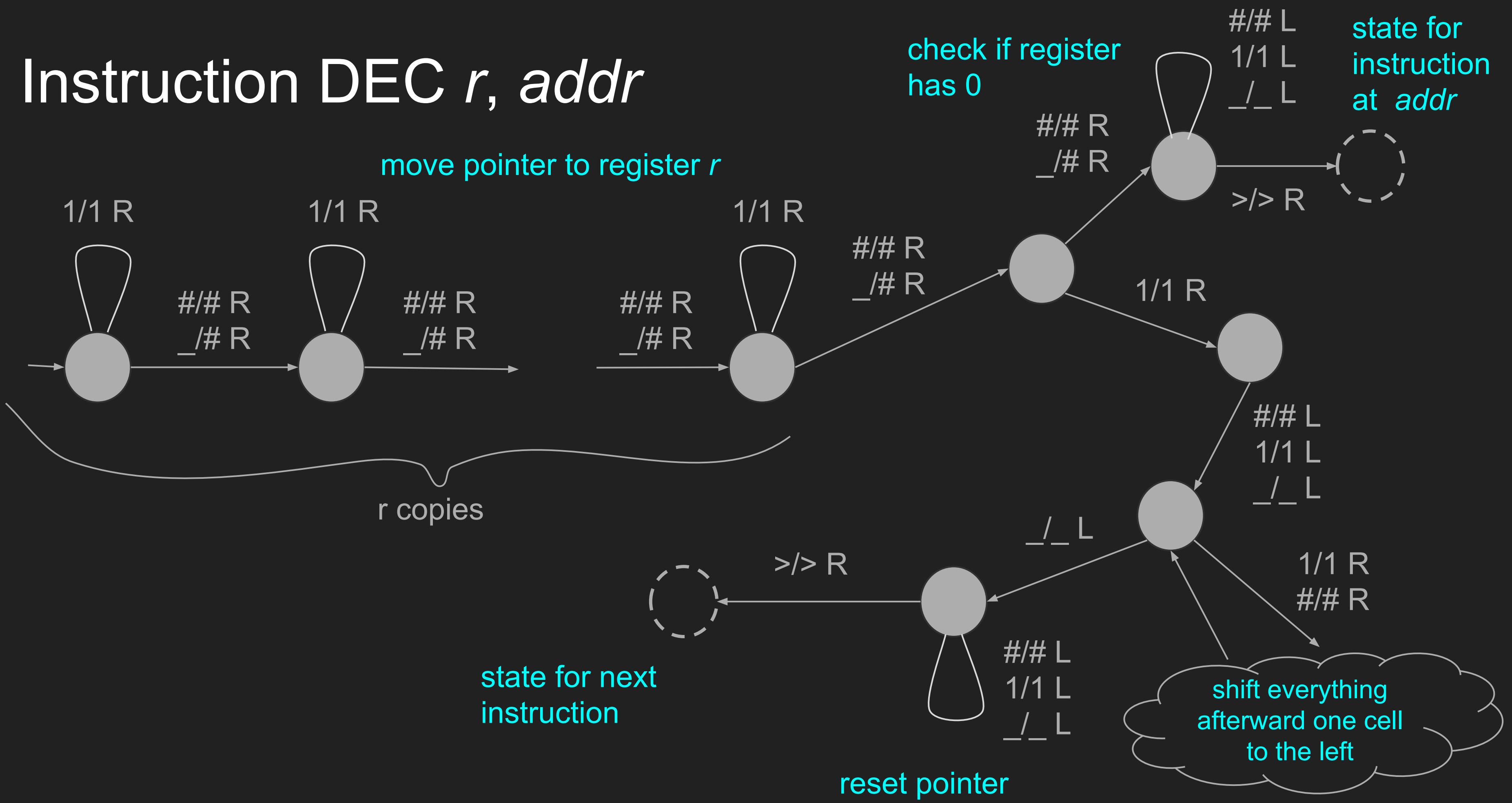
Translate a CPU program into a Turing machine

- Translate each instruction is a set of states in the Turing machine
  - Each set of states for an instruction has an "entry" state
- Jumping to an instruction is jumping to the "entry" state of the corresponding set of states
- The tape holds a value for each register
  - $n_0\#n_1\#n_2\#n_3\#n_4\#...$
  - each stored in *unary* for simplicity 0 = ; 1 = 1; 2 = 11; 3 = 111; 10 = 1111111111; …
- At the beginning of each instruction, tape pointer is on the first register

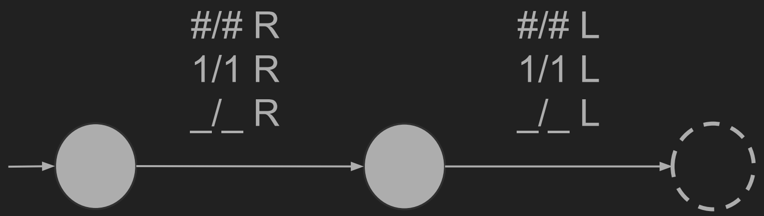
Instruction INC *r*



Instruction DEC *r*, *addr*



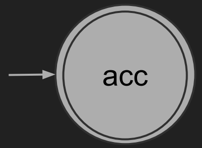
Instruction JMP *addr*



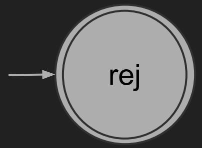
state for instruction at *addr*

Instructions TRUE and FALSE

TRUE

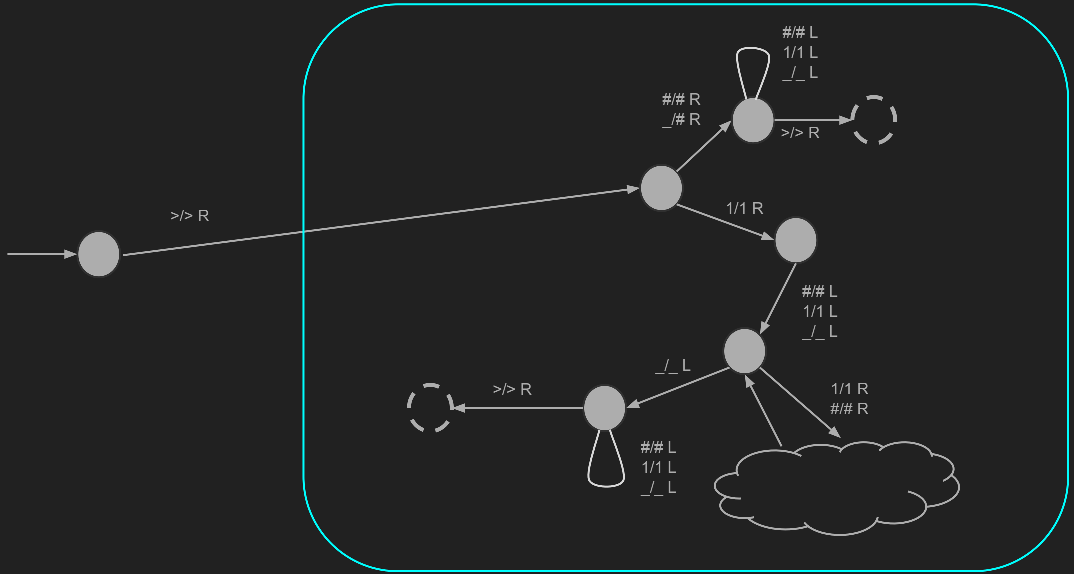


FALSE



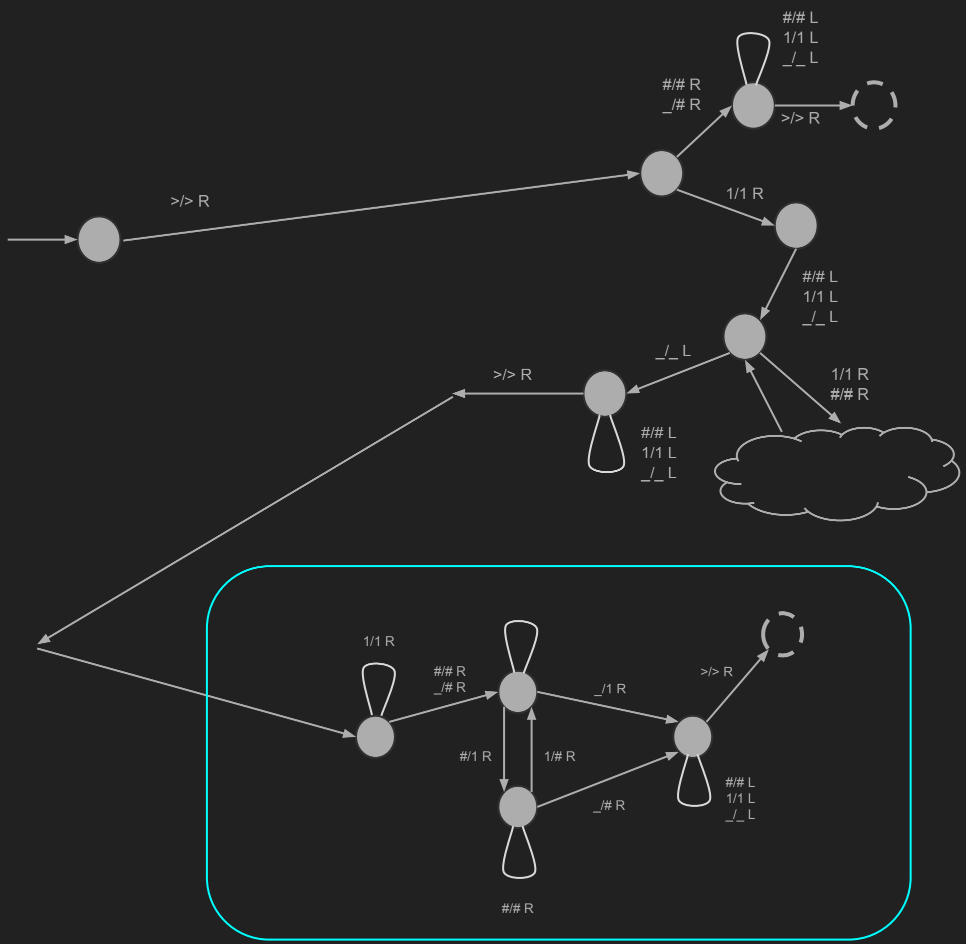
Example

start:
  DEC 0, stop
  INC 1
  JMP start
stop:
  TRUE



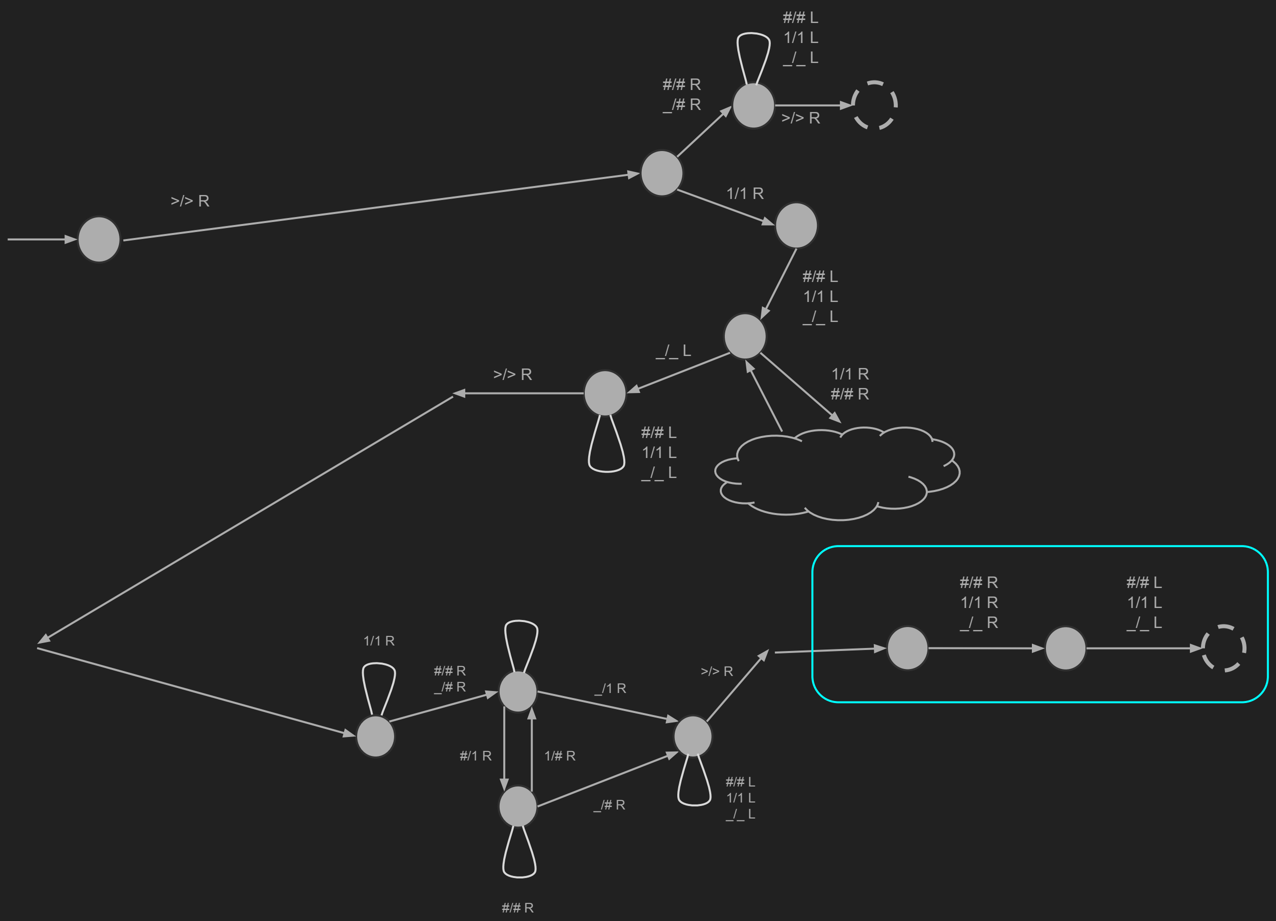
Example

```
start:
  DEC 0, stop
  INC 1
  JMP start
stop:
  TRUE
```



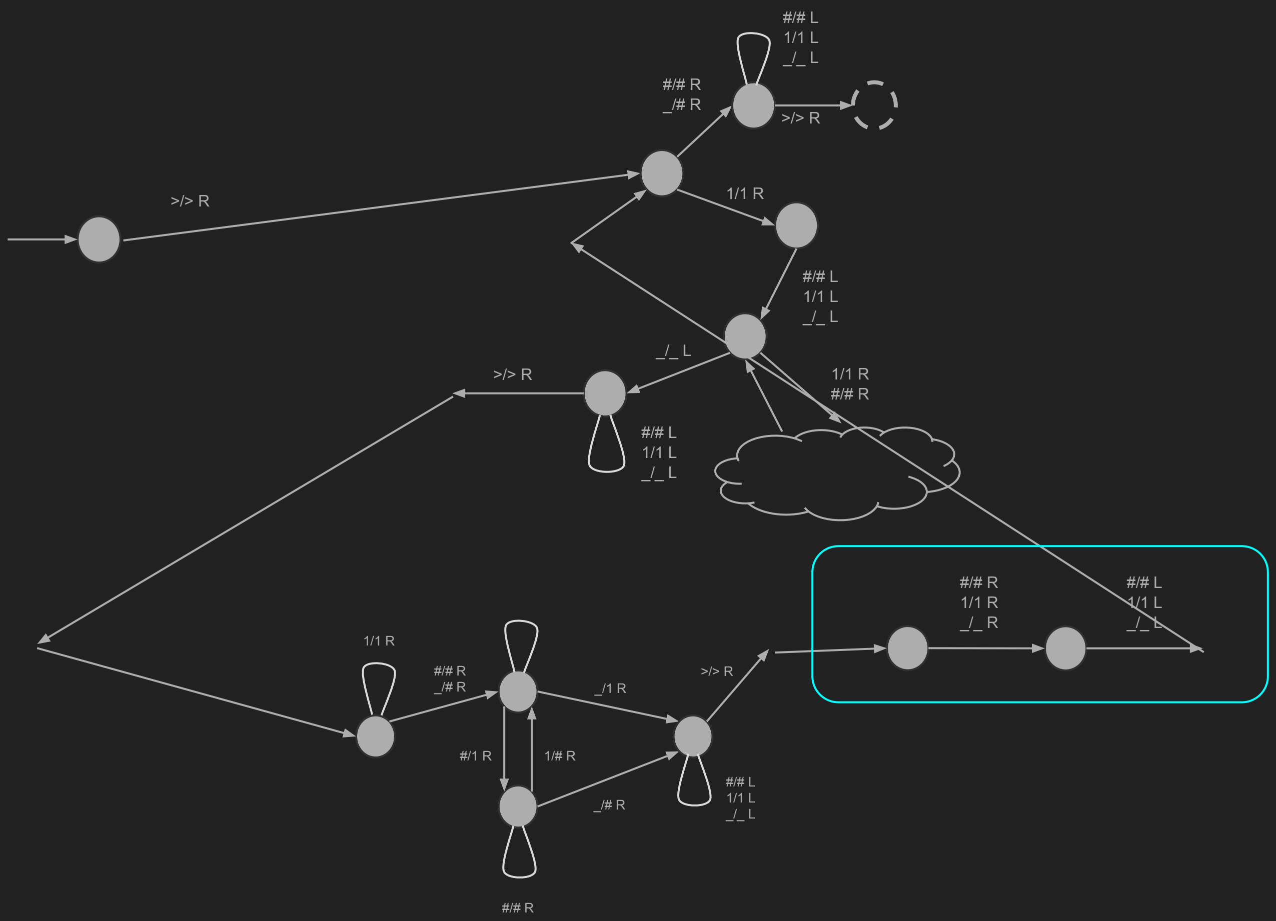
Example

start:
  DEC 0, stop
  INC 1
  JMP start
stop:
  TRUE



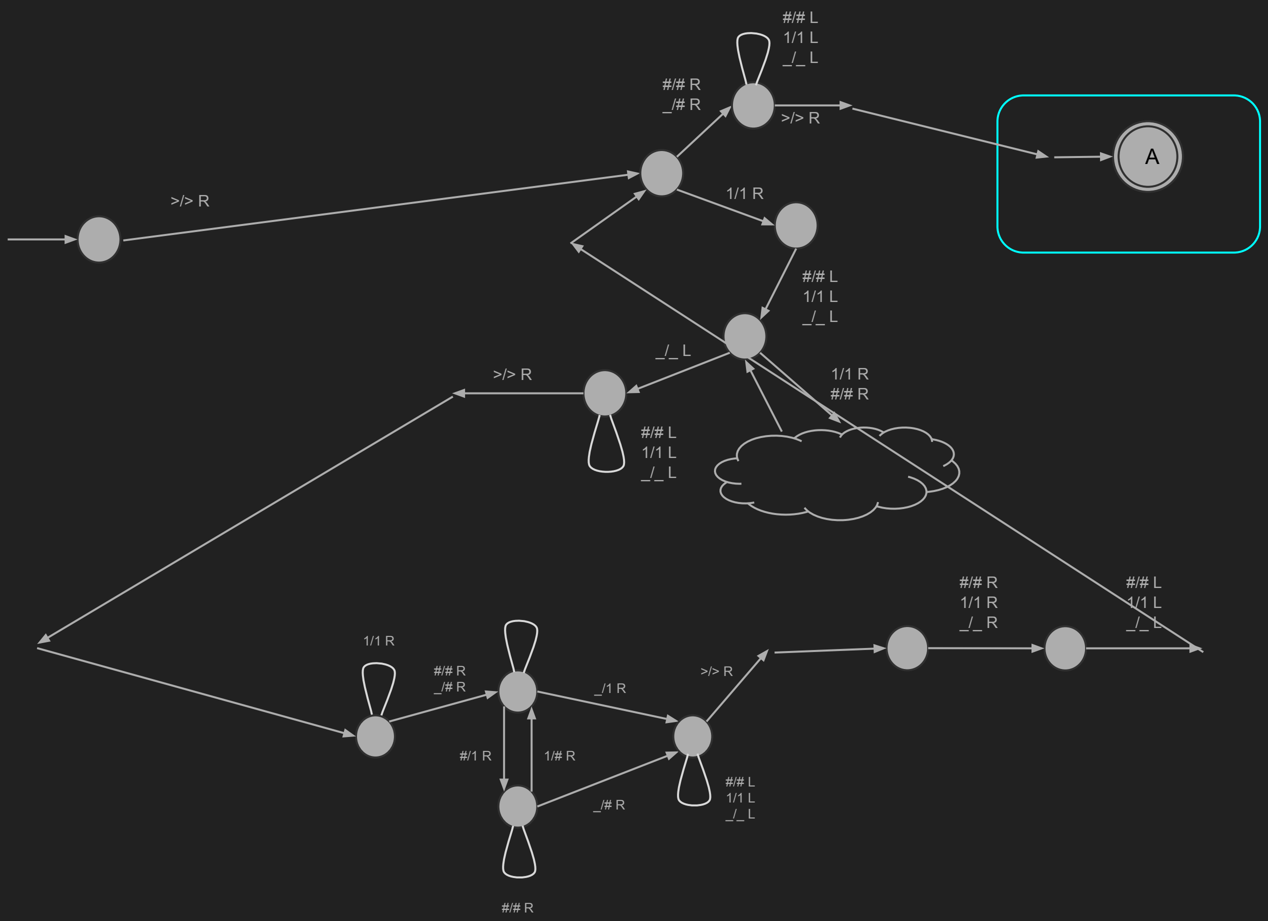
Example

start:
  DEC 0, stop
  INC 1
  JMP start
stop:
  TRUE



Example

start:
  DEC 0, stop
  INC 1
  JMP start
stop:
  TRUE



Higher-level languages

Once you have a small CPU language, you can use it as the target of compilation for higher-level languages

- More on this in the notes

At this point though, this is less about Turing machines, and more about programming languages implementation, with Turing machines as a (very slow) execution model