Discrete Local Volatility

Likelihood Ratio Risk, Affine Dividends, Multi-Asset Pricing, Quanto, Stochastic Vol, Jumps

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Do not get diffused

Motivation

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Discrete Local Volatility:

– Given a potentially sparse set of strikes and maturities we construct the transition matrices of a discrete state martingale, which has the following properties:

1. **Fixes Arbitrage:**
   If the input data is arbitragable - for example during Stress calculations -, we find efficiently a *globally* L^1-closest fit to the input data, with higher weights for observed market prices vs. interpolated data or points with large bid/ask.
   This method is useful independently in order to manage arbitrage violations.

2. **Large Steps:**
   Allows taking large steps, fully consistently between forward (MC) and backward (FD) schemes.

3. **Small Steps:**
   Allows taking small steps, fully consistent with the large step transition operators.

4. **Risk by Strike:**
   Our approach allows for a clear definition and implementation of Vega risk by strike/maturity.

Dupire's Local Volatility

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Setup:

– Assume we are given an equity S with
  - **Discount Factors** DF_t for all $t \in [0,\infty)$.
  - **Forwards** F_t for all $t \in [0,\infty)$.
  - A continuous volatility surface, or equivalently, a surface of **European Call** prices $Call(t,K)$ for all $t \in [0,\infty)$ and cash strikes $K \in (0,\infty)$

Objective:

– Define also "pure" call prices $C(t,k) := Call(t,k\,F_t)/DF_t$.
We aim to derive an arbitrage-free pricing model $S_t = F_t X_t$ for a diffusion X_t which "fits" the market in the sense that

$$\mathrm{DF}_t \, \mathrm{E}\left[(S_t - K)^+\right] = Call(t,K)$$

or, equivalently, that

$$\mathrm{E}\left[(X_t - k)^+\right] = C(t,k)$$

Dupire's Local Volatility

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Classic Dupire Local Volatility:

– There is a unique continuous Markov "local volatility" process X of the form

$$dX_t = X_t \sigma_t (X_t) dW_t$$

where

$$\sigma_t(k)^2 = \frac{f\Theta(t,k)}{\frac{1}{2}k^2 dt\,\Gamma(t,k)}$$

with

- **Forward-Theta** $f\Theta(t,k) := C(t+dt,k) - C(t,k)$; and
- **Gamma** $\Gamma(t,k) := \partial_{kk} C(t,k)$.

This definition of **Gamma** represents the second order derivative of the option price in *strike*, not spot. It only coincides with the latter under the assumption of a sticky strike implied volatility surface – which is not compatible with any known dynamic martingale model.

Absence of Arbitrage:

– We call the option price surface C or its implied volatility surface Dupire-arbitrage-free if σ is non-negative, real and bounded, i.e. if

- Both $f\Theta$ and Γ are non-negative, and
- $f\Theta$ is zero whenever Γ is.

There are a few additional technical conditions to strictly ensure existence of a solution to $dX_t = X_t \sigma_t(X_t) dW_t$ but those are not really relevant in practice and not pertinent to the discussion here.

Practical Usage and Limitations

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Discrete, sparse market data

The Volatility Surface is only observed at discrete strikes for listed maturities, typically at least 1M, mostly 3M wide.

Interpolate using Implied Volatility Scheme which may exhibit arbitrage

First fit an “Implied Volatility” scheme to those discrete volatilities to generate a continuous surface. However, no such is scheme is known which is truly arbitrage-free and fits well to most observed market data.

Fit Dupire’s Local Volatility

Bootstrap with FD if arbitrage is found e.g. under Stress or if dividend assumptions are changed.

Dupire’s Local Volatilty calibration is not robust vs. errors in arbitrage.

A solution is to “bootstrap” the local volatility function with FD to be able to recover from local errors in the implied volatility surface.

Price product with daily time steps even if only large steps are required

Local Volatility requires us to simulate the resulting process with small time steps in order to be able to “fit” the market – even if the actual product does not require dense time steps for valuation.

Computing strike wise risk very noisy

Computing strike-wise risk is noisy due to impact of local arbitrage during shifts.

Constructing Arbitrage-Free Surfaces

Fast

Discrete Local Volatility

Assumptions: discrete world

- **Maturities**
  Assume we are given listed maturities $0=t_0<\ldots<t_m$.
  Set $dt_j^+:=t_{j+1}-t_j$ and $dt_j^-:=t_j-t_{j-1}$.

- **Strikes**
  For each maturity t_j, we are given n_j strikes $k_j^{-1}<\ldots 1\ldots<k_j^{nj}$.
  We will drop the subscript j wherever possible, e.g. we define
  $dk_+^i:=k^{i+1}-k^i$ and $dk_-^i:=k^i-k^{i-1}$.
  We also add arbitrary **ghost strikes** $k^{-2}<k^{-1}$ (which might be negative) and $k^{n+1}>k^n$.

- **Market Prices**
  For each strike and maturity, we are given input market call prices
  $C_j^i := C(t_j,k_j^i)$.

Definitions:

- **Model Prices**
  We will use generally $c_j^i := c(t_j,k_j^i)$ to refer to model prices.
  We impose that all model prices are intrinsic at the **boundary strikes** k^{-2},k^{-1} and k^n,k^{n+1}.

- **Quality of Fit**
  Using positive weights w_j^i which sum up to 1, we define the norm

$$\| c \| := \sum_{i,j} w_j^i \, \| c_j^i - C_j^i \|$$

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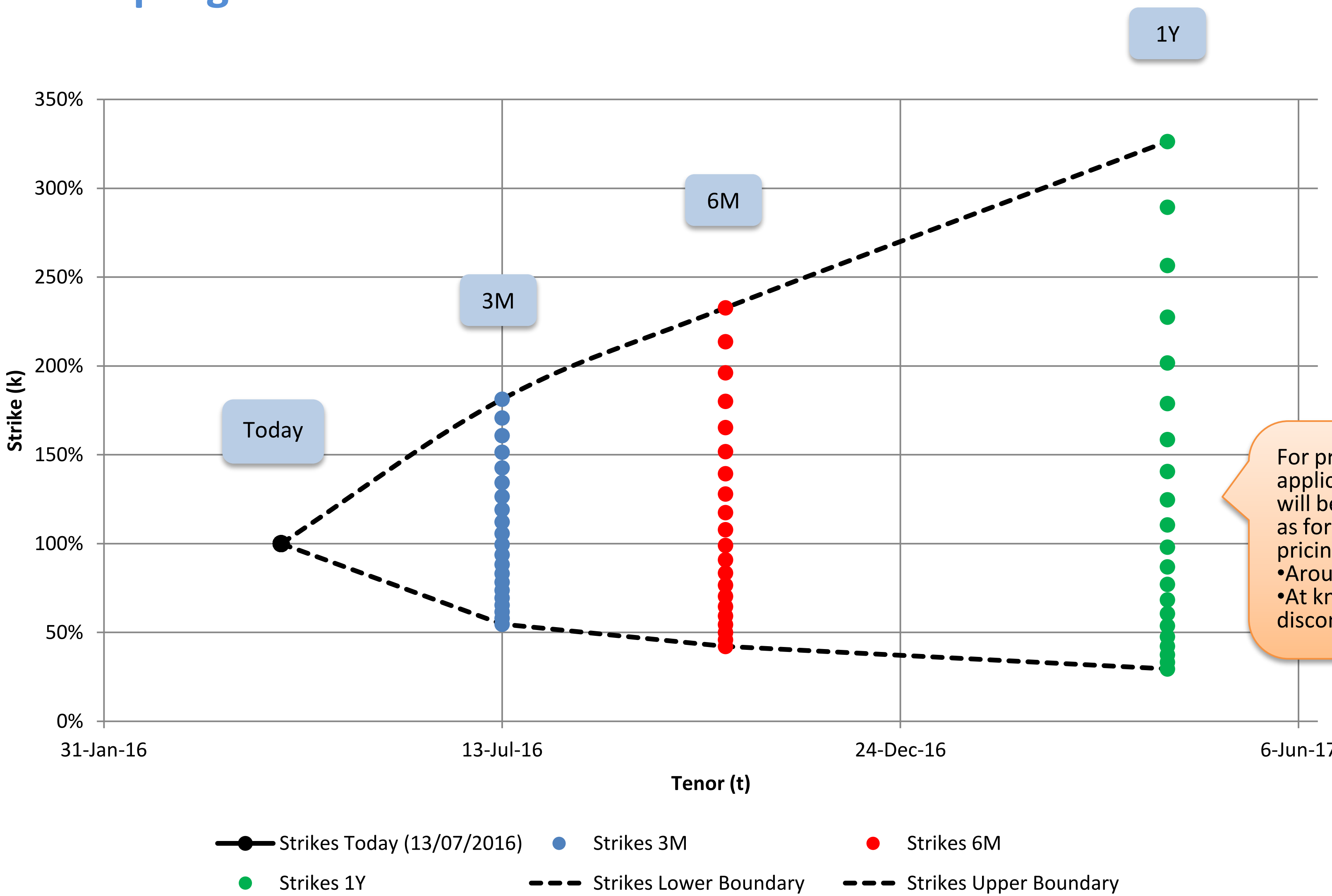
Example grid with a fixed number of normalized strikes



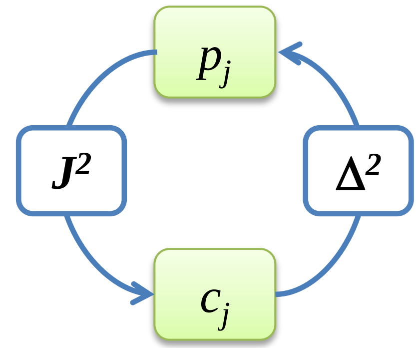
For practical applications, grids will be expanded as for classic FD pricing:
- Around barriers
- At known discontinuities

Discrete Local Volatility

Algebra for Discrete Martingales in Strikes:

- Assume $p=(p^i)$ is a discrete density over strikes $k=(k^i)$
  Its call prices on the given strikes are given in terms of the linear integral-type operator J^2 as

$$c^i := (J^2 p)^i := \sum_{u=i+1}^{n} p^u (k^i - k^u)$$



- Its inverse operator over call prices c is given as by applying the operator Δ^2 given as:

$$p^i = (\Delta^2 c)^i := \left(\frac{c^{i+1} - c^i}{dk_+^i} - \frac{c^i - c^{i-1}}{dk_-^i} \right)$$

  The operator Δ^2 is related to the classic second order difference operator D^2 by

$$(\Delta^2 c)^i = \tfrac{1}{2}(dk_+^i + dk_-^i)(D^2 c)^i$$

- **Gamma** is as usual defined as

$$\Gamma^i := (D^2 c)^i$$

Theorem (Absence of Arbitrage for one Maturity [BR15])

- Let c_j be candidate call price function which is intrinsic at the boundary strikes as defined before. If $\Gamma_j \geq 0$, then c_j is arbitrage-free in the sense that

$$p_j^i := \tfrac{1}{2}(dk_+^i + dk_-^i)\Gamma_j^i$$

  is a density.

Absence of Arbitrage

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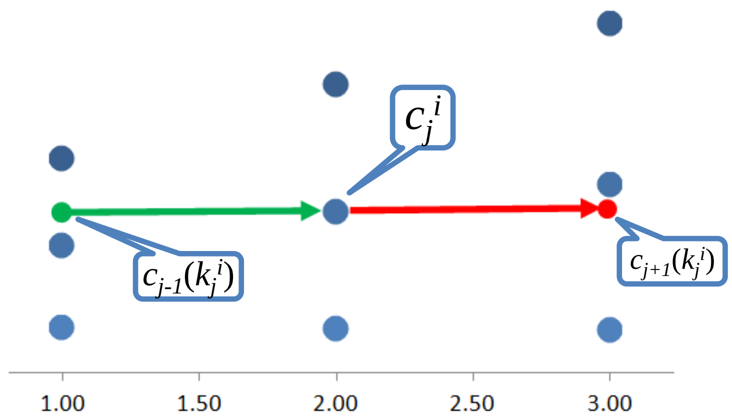
Algebra for Discrete Martingales in Time:

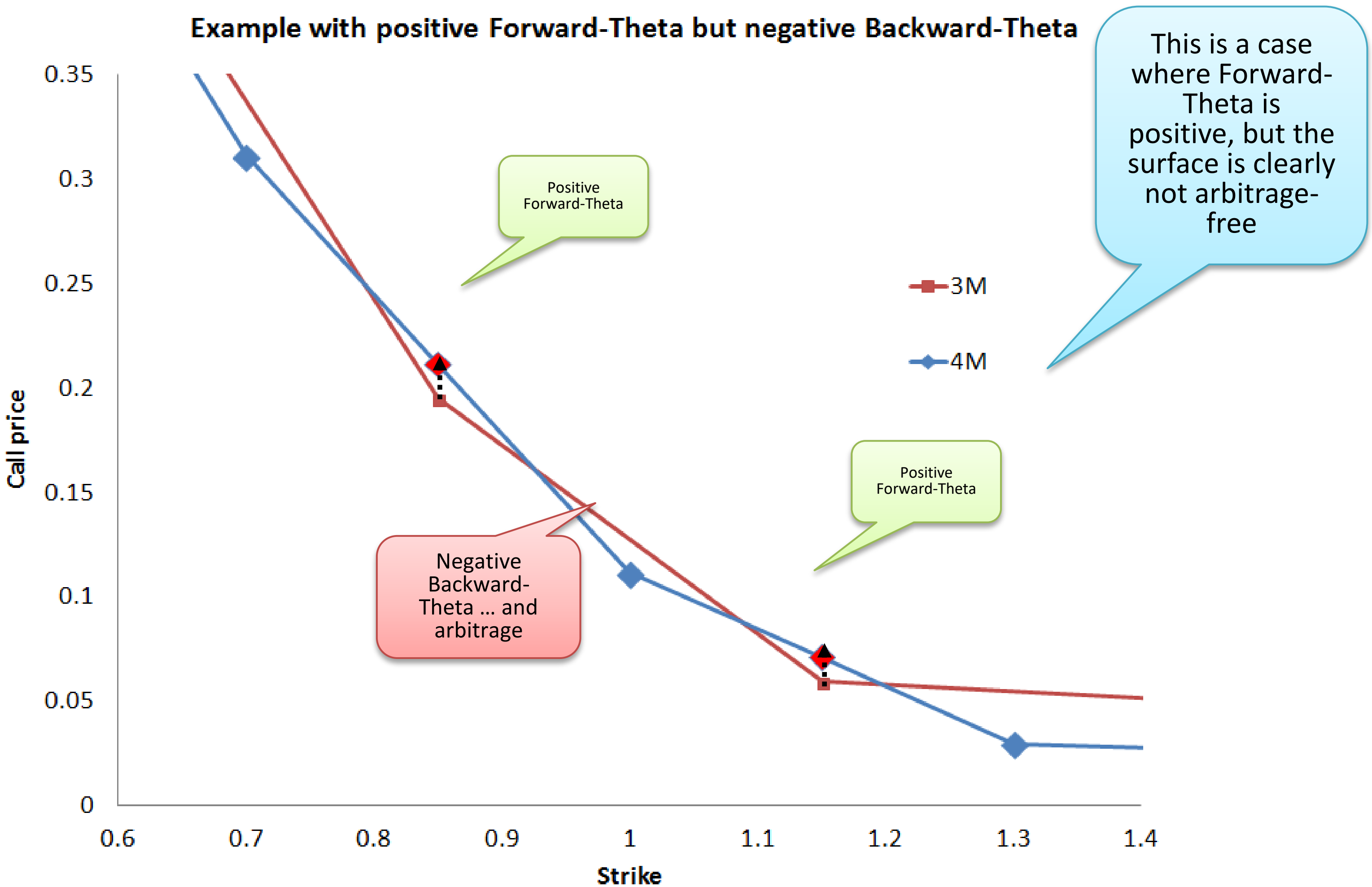
- Assume $p=(p_j^i)$ is a discrete density over strikes $k=(k_j^i)$ with call prices $c=(c_j^i)$. Recall that we allowed for different strikes per maturity.
  We denote by
  $$c_j(x) := \sum p_j^u (k_j^u - x)^+$$
  the call prices for off-grid strikes. We note that this is equivalent to *linear interpolation* in call prices.
- **Forward-Theta** is defined as
  $$f\Theta^i := c_{j+1}(k_j^i) - c_j^i$$
- **Backward-Theta** is defined as
  $$b\Theta^i := c_j^i - c_{j-1}(k_j^i)$$



Theorem (Absence of Arbitrage [BR15])

- Assume that for each maturity j, c_j is arbitrage-free with density p_j. Then, the surface c is arbitrage-free in the sense that there is a discrete martingale X with marginal densities p_j if and only if $b\Theta \geq 0$.
- The conclusion does *not* hold for $f\Theta \geq 0$.

Absence of Arbitrage



Absence of Arbitrage

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Finding the Closest Arbitrage-Free Surface [BR15] - *fast*

– A call price surface is arbitrage-free in the sense that there exist a martingale which fits c if and only if the two linear conditions on c hold:
  1. $\Gamma^j \geq 0$
  2. $b\Theta \geq 0$

– Assume that C are given market prices with weights w.
Then, we may find a closest arbitrage-free surface by solving the *linear program*

$$c^* := \arg\min \left\{ \sum_{i,j} w_j^i \, \| c_j^i - C_j^i \| \, \middle| \, c : \Gamma \geq 0, b\Theta \geq 0 \right\}$$

over the set c which have intrinsic value at the boundary strikes.
  - It is straight forward to impose bounds on implied volatility.
  - Other norms than L^1 can easily be used.

– Note that the conditions 1. and 2. above do not imply that Dupire's local volatility exists.
In particular, we do not exclude the case where $\Gamma_j^i = 0$ while $b\Theta_j^i > 0$.

Stress Test

– The LP form means that this problem can be solved practically on-the-fly.
This means that we have a clearly defined, robust method of finding "closest" abritrage-free surfaces, e.g. under ill-defined stress scenarios.

Example of Fixing Arbitrage

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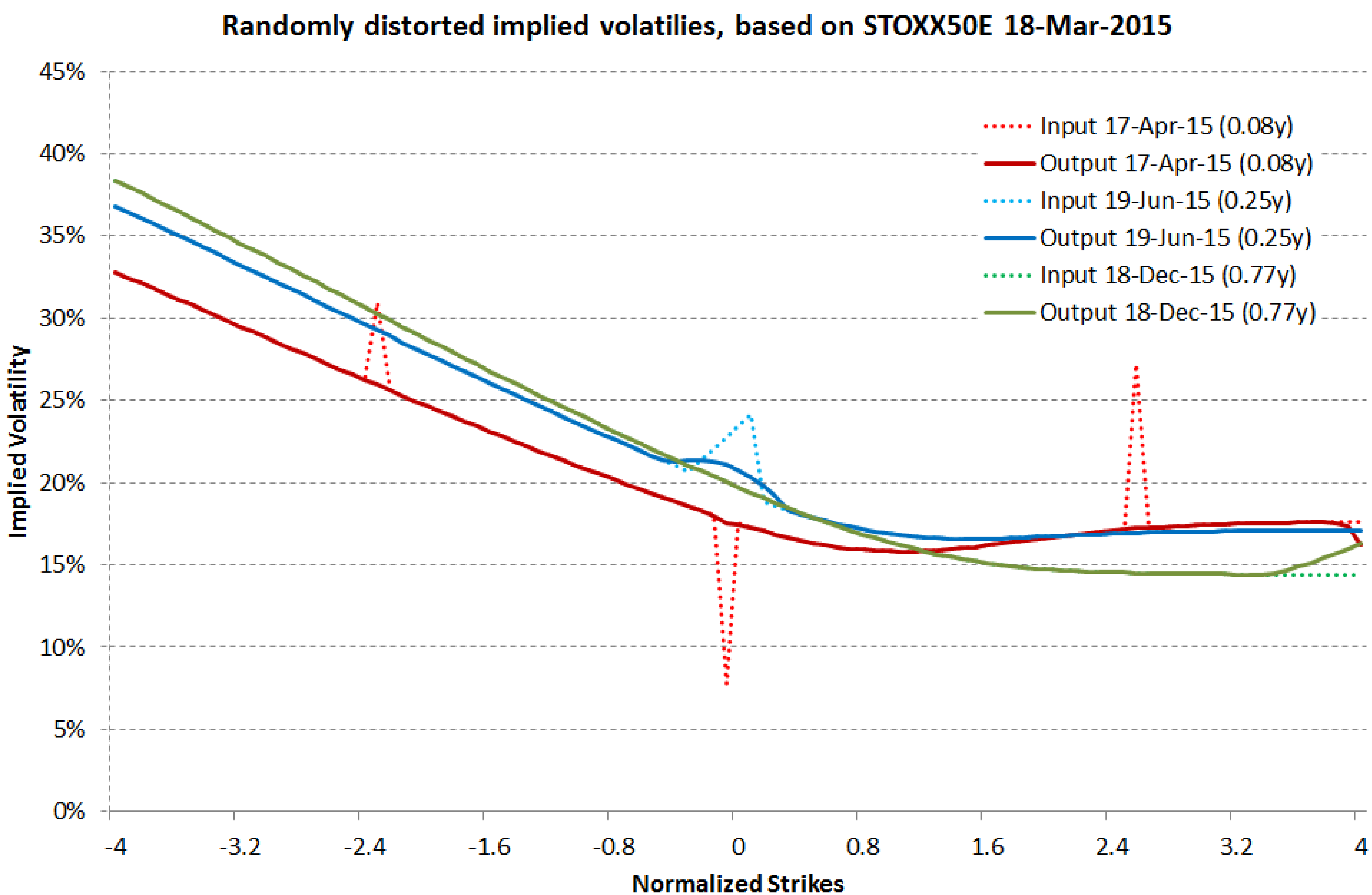


Discrete Local Volatility
Construction of Discrete Martingales

Step 1: Time Interpolation

Step 1: Interpolation in time for an expanding grid

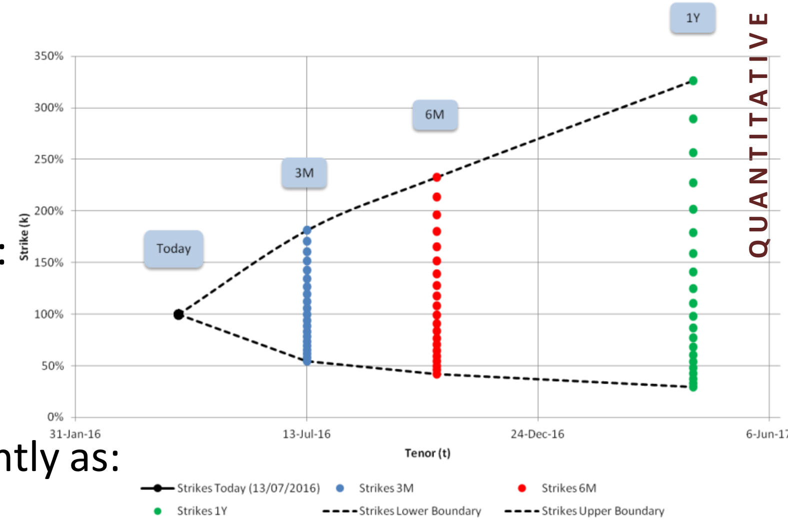
– Fix $j\text{-}1$ and consider the call prices c_{j-1} defined over k_{j-1}

1. Compute call prices cc_{j-1} using the current density p_{j-1} for new strikes k_j as:

$$cc^i_{j-1} := c_{j-1}(k^i_j)$$

2. Define the associated interpolated density q_{j-1} again for strikes k_j consistently as:

$$q^i_{j-1} := (\Delta^2 cc_{j-1})^i$$



– Both operations are *linear* and jointly define a linear operator which maps the density p_{j-1} defined over strikes k_{j-1} into the density q_{j-1} defined over k_j:

$$\Xi_j : p_{j-1} \mapsto q_{j-1}$$

Obviously, if $k_j=k_{j-1}$, then $p_{j-1}=q_{j-1}$.

Theorem (Interpolation using a martingale kernel) [BR15]

– Ξ is a *transition kernel*, i.e.

- Ξ is a probability matrix: $1\Xi_j=1$ and $\Xi_j \geq 0$.
- It is a transition matrix $q_{j-1}=\Xi_j p_{j-1}$.
- It is a martingale kernel $k_{j+1}\Xi_j=k_j$.

All of these calculations are simple algebra and can virtually be done on a spread sheet.

Step 1: Time Interpolation

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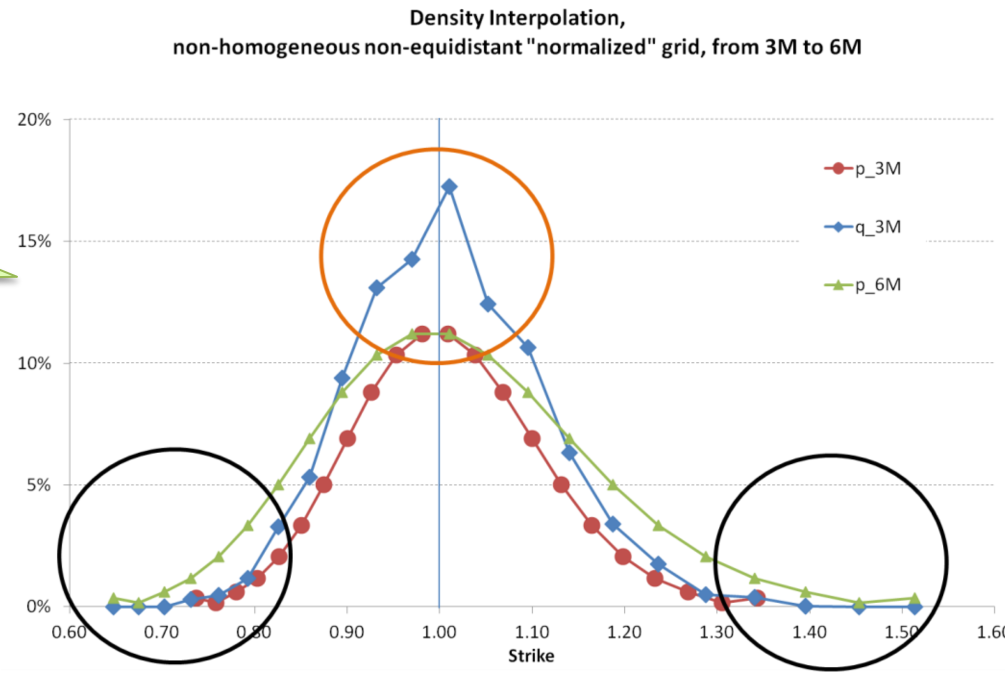
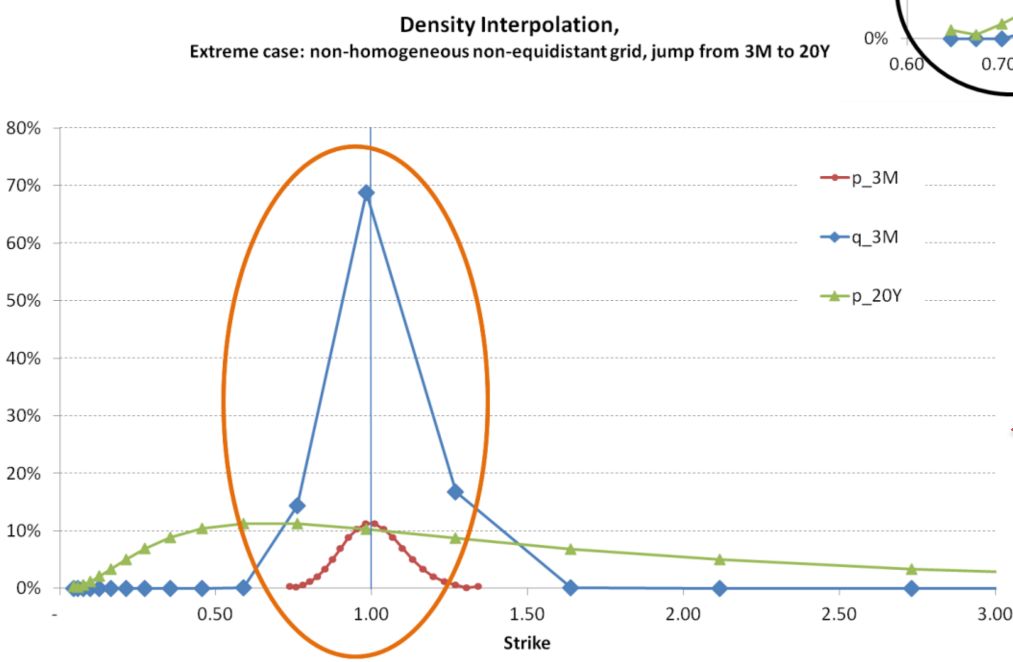
Interpolated density, for reasonable time steps





Interpolated density, for large time steps

Step 2: Transition Operators

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Step 2: Transition operators from Implicit FDs

– Assume now that strikes are *homogeneous* between t_{j-1} and t_j.

- **Define prior model:** recall the equation $dX_t = X_t \sigma_t(X_t) dW_t$. Its density $\pi(t,x) := \mathrm{P}[X_t = x]$ satisfies the forward-PDE

$$d\pi(t,x) = \frac{1}{2} \partial^2_{xx} \left\{ x^2 \sigma_t(x)^2 \pi(t,x) \right\} dt$$

- **Implicit FD:** discretize in time using an *implicit* scheme for $\pi_j(x) := \pi(t_j, x)$:

$$\pi_j(x) - \pi_{j-1}(x) = \frac{1}{2} \partial^2_{xx} \left\{ x^2 \sigma_j(x)^2 \pi_j(x) \right\} dt\_$$

$$\pi_{j-1} = I_j^{-1} \pi_j \qquad I^{-1} := 1 - \frac{1}{2} \partial^2_{xx} \left\{ x^2 \sigma_j(x)^2 \bullet \right\} dt\_$$

Standard FD discretization in space yields the tridiagonal matrix

$$I^{-1} := 1 - \frac{1}{2} D^2_{xx} \left\{ x^2 \sigma_j(x)^2 \bullet \right\} dt\_$$

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Discretization of Forward-PDE matrices:

- Note that when discretizing

$$I_j^{-1} := 1 - \frac{1}{2} D_{xx}^2 \left\{ x^2 \sigma_j(x)^2 \bullet \right\}$$

  we do <u>not</u> expand the second order derivative in separate derivatives of $x^2\sigma(x)^2$ and $\bullet$ as was proposed in Andreasen-Huge [AH11], but we discretize it as is.
- In this form, it is worth noting that I is actually just the transpose of the *backward* FD operator BI defined on the same grid via

$$BI_j^{-1} := 1 - \frac{1}{2} x^2 \sigma_j(x)^2 D_{xx}^2 \bullet$$

- In other words, this discretization scheme is *consistent for forward and backward operators*.
  We more generally have:

Theorem (consistent forward and backward operators) [BR15]

- The backward operator of a diffusion with unattainable boundaries is the adjoint (transpose) of its forward operator.
  - The same is true for a finite state Markov chain, i.e. forward and backward operators are consistent if the density has a Neumann-boundary condition.

Step 2: Transition Operators

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Theorem (Z-Matrix) see also Andreasen-Huge [AH11]

– Assume that M is a square matrix whose columns [rows] add up to 1, and where all off-diagonal elements are non-positive.
  Then, its inverse exists, is non-negative, and its columns [rows] add up to 1; in other words M^{-1} is a transition matrix.
  (see [BR15] for a brief proof)

$$\begin{bmatrix} 1 & -a & & & \\ & 1+a+b & -a & & \\ & -b & 1+a+b & -a & \\ & & -b & 1+a+b & \\ & & & -b & 1 \end{bmatrix}$$

Illustration

– Our tridiagonal matrix

$$I_j^{-1} := 1 - \frac{1}{2} D_{xx}^2 \left\{ x^2 \sigma_j(x)^2 \bullet \right\}$$

does indeed fit this description, hence I is a transition kernel for π.

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Backward Local Volatility

– How does that help?
  Most likely the discretized π is not even a density.
  We now aim to find a local volatility σ such that $p_j = I_j p_{j-1}$ for the given model densities (recall that we currently assume homogeneous strikes).
– To this end, we write the FD out, which gives:

$$p_j - p_{j-1} \stackrel{!}{=} \frac{1}{2} D_{xx}^2 \left\{ x^2 \sigma_j(x)^2 p_j(x) \right\} = \frac{1}{2} \Delta_{xx}^2 \left\{ x^2 \sigma_j(x)^2 \Gamma_j(x) \right\}$$

We now apply the inverse integral operator $J_{xx}{}^2$ such that

$$C_j^i - C_{j-1}^i \stackrel{!}{=} \frac{1}{2} k_j^{i\,2} \sigma_j^{i\,2} \Gamma_j^i dt_-$$

which gives rise to the definition of **backward local volatility** as:

$$\sigma_j^{i\,2} := \frac{b\Theta_j^i}{\frac{1}{2} k_j^{i\,2} \Gamma_j^i dt_-}$$

Step 2: Transition Operators

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Theorem (Bounded Discrete Local Volatility) [BR15]

- Let c be a call price surface which is intrinsic at the boundaries, and which satisfies for $0 \le \sigma_{min} < \sigma_{max}$ the linear constraints
  1. $\Gamma \geq 0$ and
  2. $\frac{1}{2}\Gamma\, k^2\, dt\, \sigma_{min}{}^2 \le b\Theta \le \frac{1}{2}\Gamma\, k^2\, dt\, \sigma_{max}{}^2$
- Then, c is arbitrage free, and the transition matrix from $p_{j\text{-}1}$ to p_j is given by

$$\Pi_j := I_j \Xi_j$$

  where
  - Ξ is given by the interpolation operator defined before; and
  - I is the well-defined inverse of the tridiagonal matrix I^{-1} given as

$$I_j^{-1} := 1 - \frac{1}{2} D_{xx}^2 \left\{ x^2 \sigma_j(x)^2 \bullet \right\} \quad \text{with} \quad \sigma_j^{i\,2} := \frac{b\Theta_j^i}{\frac{1}{2} k_j^{i\,2} \Gamma_j^i dt_-}$$

  with **bounded** "backward local volatility" σ.

- Moreover, conditions 1. and 2. above are linear, hence for a given market surface C we may find a closest arbitrage-free surface with bounded backward local volatility by solving the appropriate linear program.

Step 3: Small Steps

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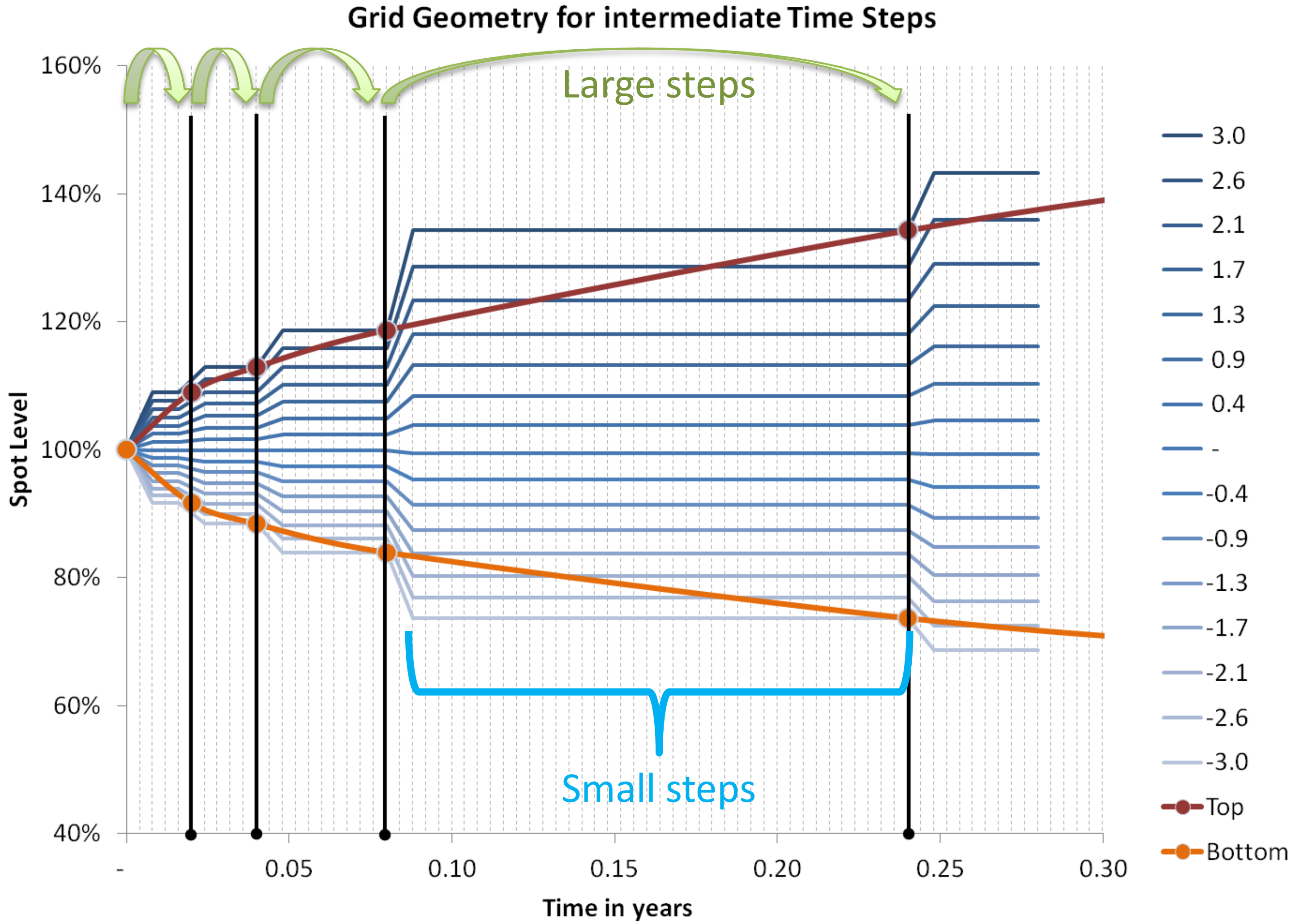

Step 3: Small Steps

- We have constructed a discrete martingale for our reference time steps, for example listed maturities.
  - How do we price options which require more frequent or non-standard observations?
  - Recall that our transition operator is given as

$$\Pi_j := I_j \Xi_j$$

  - Our matrix admits an eigen-decomposition [BR15], hence we may write it in terms of a unitary matrix X and a diagonal matrix D as

$$I_j := X_j' D_j X_j$$

Quick, since I_j^{-1} is tridiagonal.

Therefore, for any positive α we may write

$$I_j^\alpha := X_j' D_j^\alpha X_j$$

  - For any $t_{j-1}<t<t_j$, let $\alpha:=(t-t_{j-1})/(t_j-t_{j-1})$ and define the two transition matrices

$$II_{j-1}^t := X_j' I_j^a X_j \qquad II_t^j := X_j' I_j^{1-a} X_j$$

whose product, obviously, is again I.

Result

- In other words, we have constructed transition operators from t_{j-1} to t, and from t to t_j, which are consistent with the overall operator from t_{j-1} to t_j.

Summary of Approach

1. Use interpolation operator Ξ to reduce to the homogeneous strike case.

2. For homogeneous strikes:
   a. Define "prior model" with associated forward PDE for $\Sigma_t(x) := x\, \sigma_t(x)$:
   $$d\pi_t(x) = \tfrac{1}{2}\partial_{xx}^2 \left\{\Sigma_t^2 \pi_t\right\} dt$$
   b. Use implicit FD operator discretization which gives us a transition matrix for a given Σ:
   $$I_j^{-1} = 1 - \tfrac{1}{2}\Delta_{xx}^2 \left\{\Sigma_j^2 \bullet\right\} dt$$
   c. The transition property for p imposes the following equation for Σ:
   $$p_j - p_{j-1} \overset{!}{=} \tfrac{1}{2} D_{xx}^2 \left\{\Sigma_j^2 \Gamma_j\right\} dt$$
   d. Solve for Σ by applying the inverse J^2 of the operator D^2:
   $$C_j - C_{j-1} =: b\Theta_j \overset{!}{=} \tfrac{1}{2}\Sigma_j^2 \Gamma_j dt_j^-$$
   e. Bounds on Σ yield **linear no-arbitrage conditions** for the option price surface:
   $\Gamma \geq 0$ and $\tfrac{1}{2}\,\Gamma\, dt\, \Sigma_{min}^2 \leq b\Theta \leq \tfrac{1}{2}\,\Gamma\, k^2\, dt\, \Sigma_{max}^2$.

All of these calculations are simple algebra and can virtually be done on a spread sheet.

3. Interpolate to intermediate time steps by eigen-decomposition
$$I_j := X_j' D_j X_j$$

Spot vs. forward discretization

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Our discretization focuses on getting the spot-distributions interpolated most efficiently.

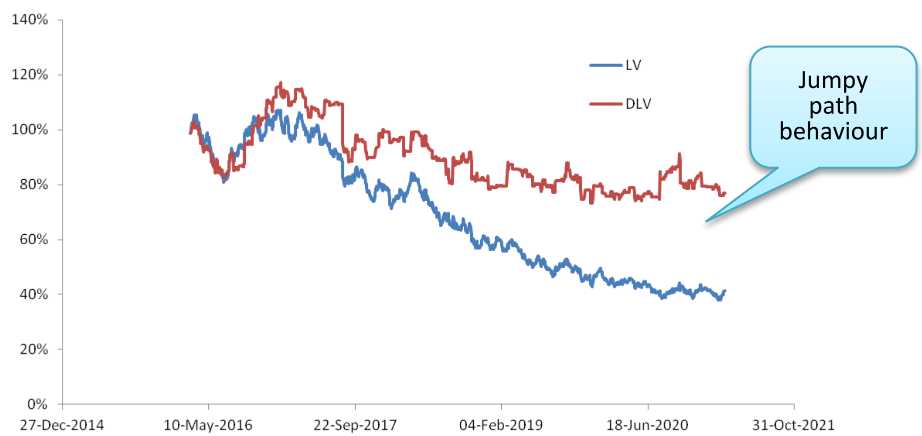
- The prior process was:

$$dX_t = X_t \sigma_t(X_t) dW_t$$

- By using increasing strikes, we are essentially modelling the coordinate-transformed process

$$Y_t = \frac{X_t}{\sqrt{\mathrm{var}(X_t)}} \approx \int \frac{\sigma_t(X_t)}{\sqrt{t}} dW_t - \ldots \rightarrow \int \frac{1}{\sqrt{t}} dW_t$$

  which means that the effective forward-distribution degenerates with time.
  In the case of our discrete process, it means that our forward increments become more "jumpy" – the process is less likely to change state, but if it does it jumps (relatively) far.



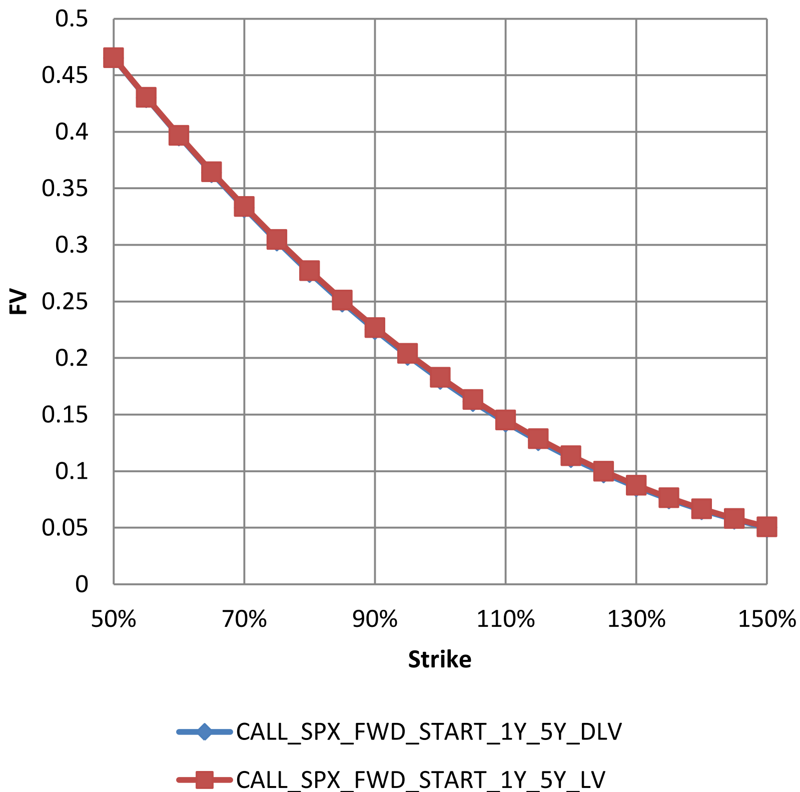
- A partial solution is to add homogeneous strikes around ATM.
- This does not affect standard structured products, but needs to be taken into account for variance products (which we won't price on DLV anyway).

DLV in action

Test instrument:

1Y forward started Vanilla on SPX

Maturity: 5Y

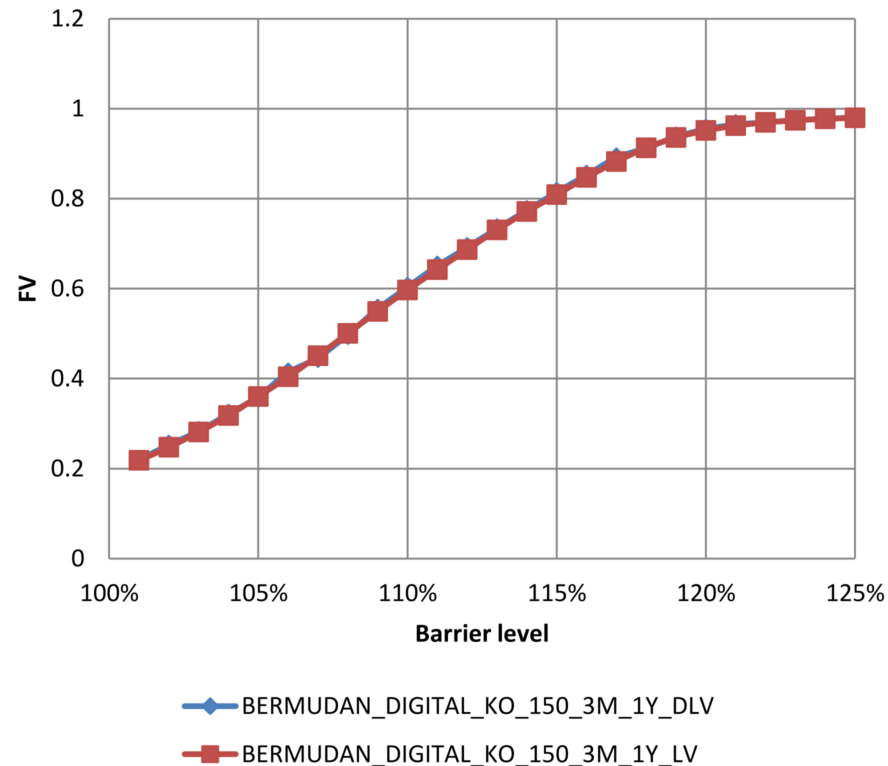


Test instrument:

Bermudan digital Knock Out option with an upper barrier

– Maturity: 1Y
– Barrier observation: quarterly

Notice that extra strikes were added to the grid around the barrier level.



Efficient Risk

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Risk in Monte-Carlo

- We have constructed a discrete Markov martingale with transition kernels Π. The joint density for a sample path $(x_0, \ldots, x_m)$ is

$$P(x_1, \ldots, x_m \mid x_{0,}) := \Pi_1(x_1 \mid x_0) \cdots \Pi_m(x_m \mid x_{m-1})$$

- Classic approach for risk: re-sample paths with same random numbers.
- However, if we know all transition densities, we may use **Likelihood Ratio** greeks:
  - Assume that under perturbation, our "perturbed" kernels are Π^*, such that the joint path density P^* is given as

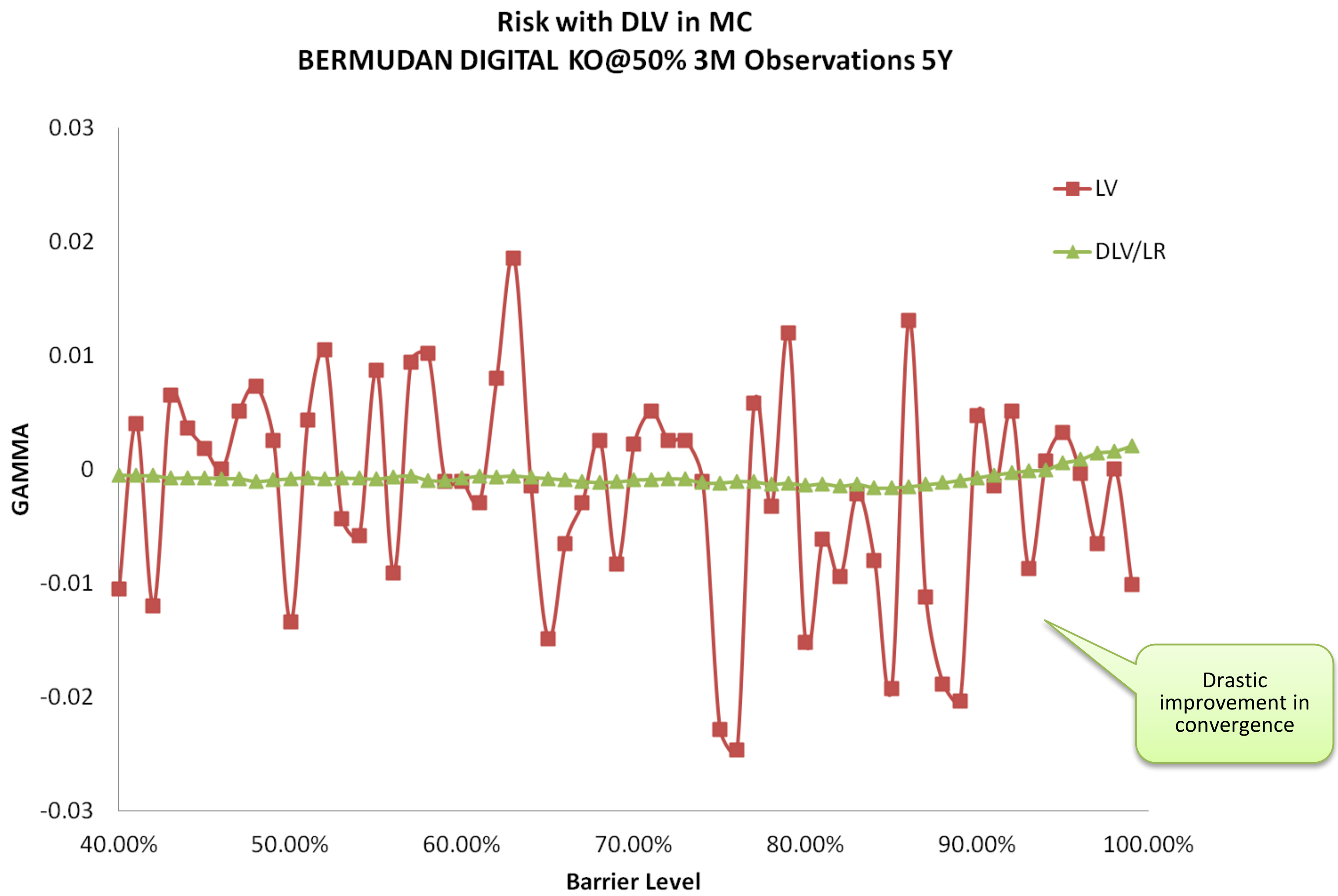
$$P^*(x_1, \ldots, x_m \mid x_{0,}) := \Pi_1^*(x_1 \mid x_0) \cdots \Pi_m^*(x_m \mid x_{m-1})$$

  - Then, we may compute the perturbed price of a payoff F using

$$\begin{aligned} \mathrm{E}^*\left[F(X_{0,} X_1, \ldots, X_m)\right] &= \int F(x_{0,} x_1, \ldots, x_m) P^*(x_1, \ldots, x_m \mid x_{0,}) dx \\ &= \int F(x_{0,} x_1, \ldots, x_m) \frac{P^*(x_1, \ldots, x_m \mid x_{0,})}{P(x_1, \ldots, x_m \mid x_{0,})} P(x_1, \ldots, x_m \mid x_{0,}) dx \\ &= \mathrm{E}\left[F(X_{0,} X_1, \ldots, X_m) \frac{P^*(X_1, \ldots, X_m \mid x_{0,})}{P(X_1, \ldots, X_m \mid x_{0,})}\right] \end{aligned}$$

- This is computationally much more efficient than re-sampling paths.
- The noise of this method is generally lower than with perturbation, except for short-end exposures for products which have short-end fixings (e.g. Delta or Front-Term-Vega for daily barriers which are close to ATM).

Examples


Risk with DLV in MC
BERMUDAN DIGITAL KO@50% 3M Observations 5Y
0.03
0.02
0.01
0
-0.01
-0.02
-0.03
GAMMA
40.00%
50.00%
60.00%
70.00%
80.00%
90.00%
100.00%
Barrier Level
LV
DLV/LR
Drastic improvement in convergence

Pointwise Vega

- It is generally difficult to compute strike/maturity vega risk for Dupire Local Volatility.
- In DLV, we can mostly compute *analytically* the sensitivity of the two surrounding transition operators to changes in option prices.
- Still, by nature of a small change: in Monte-Carlo:
  - Sensitivity per *every* strike to low, better to group them
  - Add control variates to reduce noise
  - Need to increase number of paths

DLV in action

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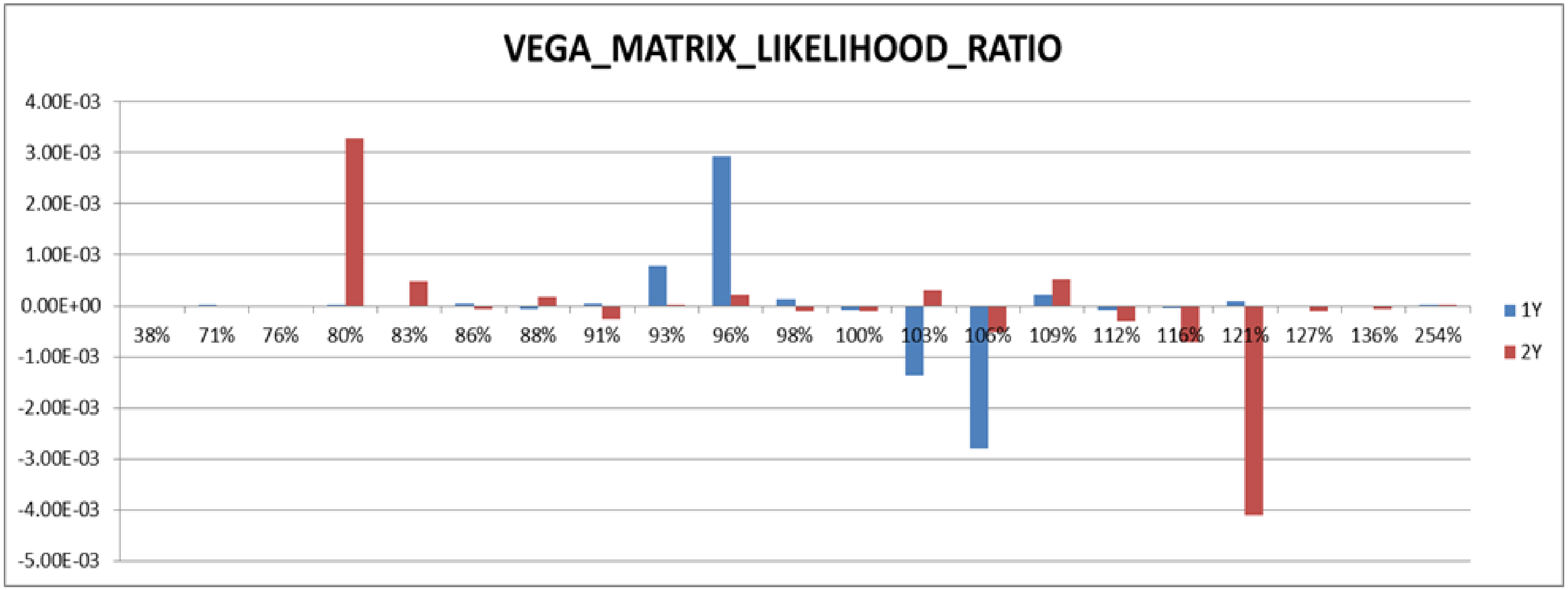
QUANTITATIVE RESEARCH

Test instrument: 1Y and 2Y Call spreads on NKY

$$payoff = \left(\frac{S_{t_1}}{S_0} - 0.95\right)^+ - \left(\frac{S_{t_1}}{S_0} - 1.05\right)^+ + \left(\frac{S_{t_2}}{S_0} - 0.8\right)^+ - \left(\frac{S_{t_2}}{S_0} - 1.2\right)^+ \text{ where } t_1 = 1Y,\, t_2 = 2Y$$

True vega is at (1Y, 95%), (1Y, 105%) and (2Y, 80%), (2Y, 120%)

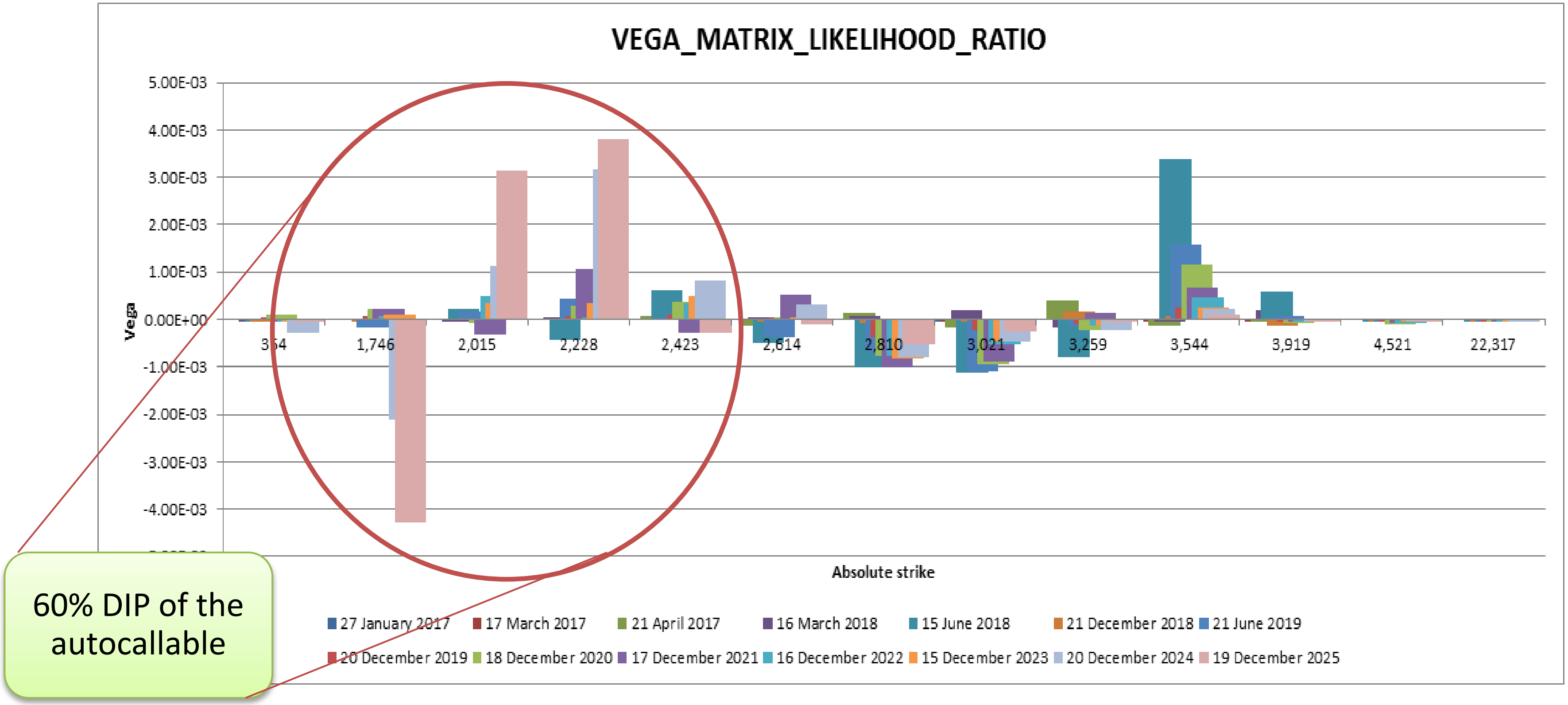


Decomposing Vega down to individual DLVGrid point turned out very noisy. To improve convergence we

1. Bucket vega into 10-15 buckets around ATMF
2. Use Variate Control Estimator (improves convergence by the factor of 4-10)
3. Use large number of iterations (500K or even 1M)

DLV in action

Test instrument: 8Y Auto callable (from prod) with 2M forward start



Affine Dividends

Affine Dividends

- Assume an equity price process pays at ex-dividend dates $\tau=(\tau_k)_k$ dividends which have a proportional component d and a cash component δ, i.e.

$$S_{\tau_k} = S_{\tau_k -} - S_{\tau_k -} e^{d_k} - \delta_k$$

Define the proportional drift:

$$R_t = \exp\left\{ \int_0^t \mu_s ds + \sum_{k:\tau_k > t} d_k \right\}$$

and the *discounted future dividends*,

$$D_t := R_t \left(\sum_{k:\tau_k > t} \frac{\delta_k}{R_{\tau_k}} \right)$$

- It was then shown in Buehler [B10] that *every (*)* arbitrage-free representation of S has the form

$$S_t = (F_t - D_t) X_t + D_t$$

where X is a (local) martingale.

(*) that is true for all processes which do not have certain jumps at dividend dates.

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QUANTITATIVE RESEARCH

Discrete Local Volatility for Affine Dividends

- Assume we are given call prices C on S.
  As shown in [B10], we derive "pure" call prices CC for X by the simple transformation

$$CC(t,k) := \mathrm{E}[(X_t - k)^+] := \frac{C(t, k(F_t - D_t) + D_t)}{\mathrm{DF}_t}$$

Observation:

- If an implied volatility surface is marked with "Black-Scholes" proportional dividends only, then there is no arbitrage-free process with (truly) affine dividends which fits the continuous option price surface.

Result:

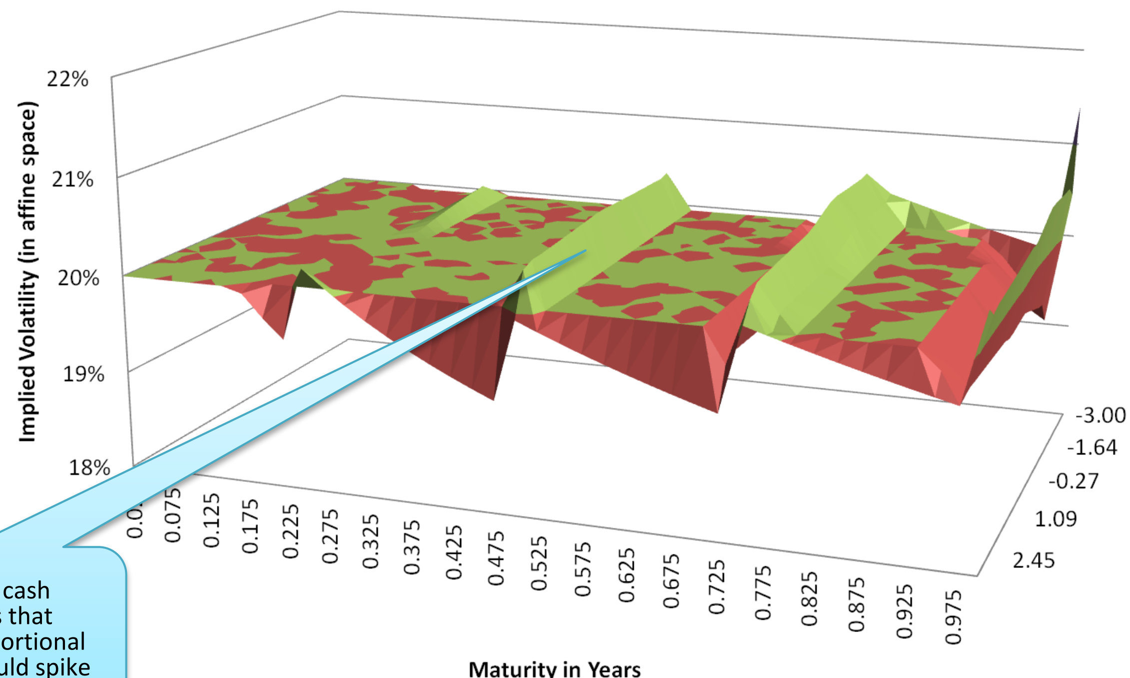
- Our Approach provides in a well-defined way the "closest" affine dividend arbitrage-free option price surface given to an input surface market with proportional dividends.
- Extension of this statement to simple credit risk with deterministic default intensity is trivial along the arguments presented in [B10].

Examples

– Synthetic asset, quarterly cash dividends of 5% for 1.5Y
– All options priced @ Black-Scholes with vol set at 20% → inconsistent with cash dividend assumption

1. Convert into "pure" implied volatilities for X
2. Find new "closest" no-arbitrage volatilities for X and re-price options in cash-space
3. Compute again Black-Scholes implied volatilities for S itself

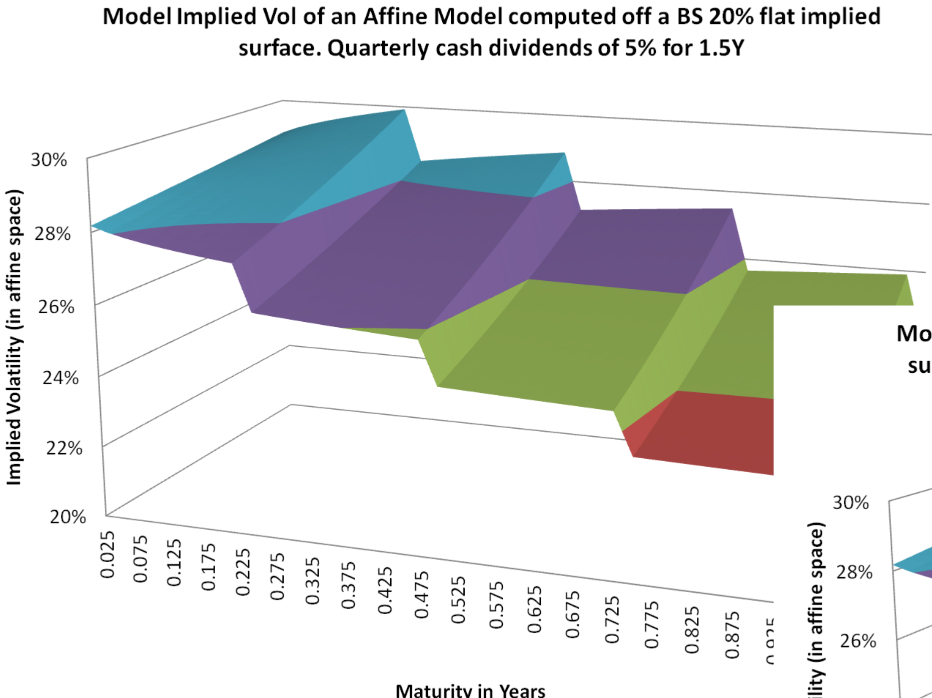
BS Implied Vol of an Arbitrage-Free fit under an Affine Model fitted to a BS 20% flat implied surface Quarterly cash dividends of 5% for 1.5Y. Local Vol bounds 1% and 200%

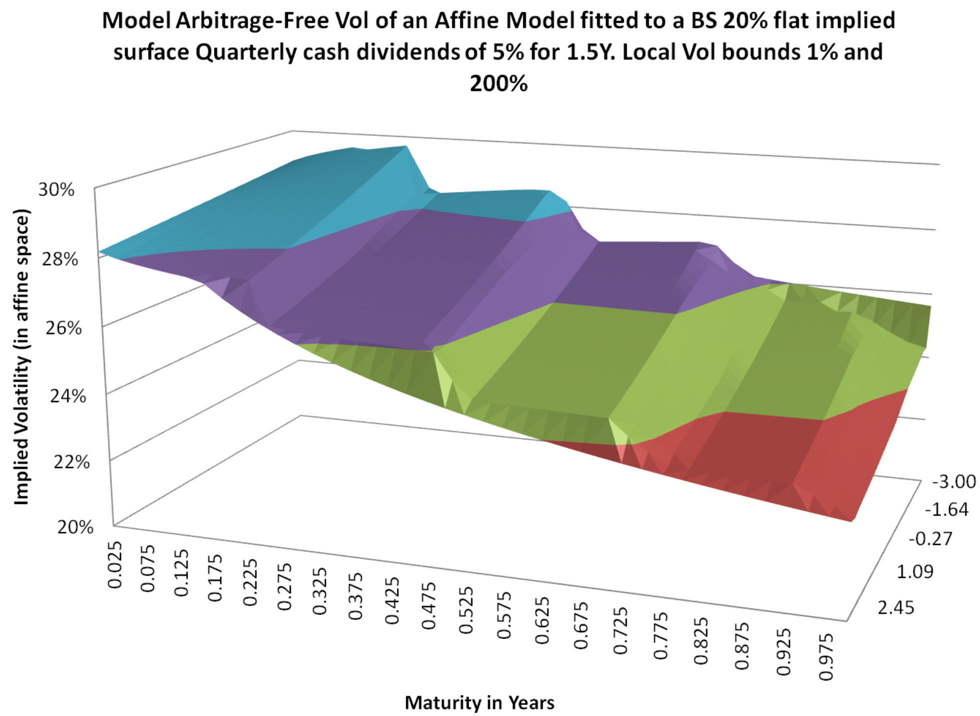


The presence of cash dividends means that volatilities in a proportional dividend model should spike

Examples

FYI corresponding “model” volatilities for X under a cash dividend model





Dividends: AT&T Volatility Smile

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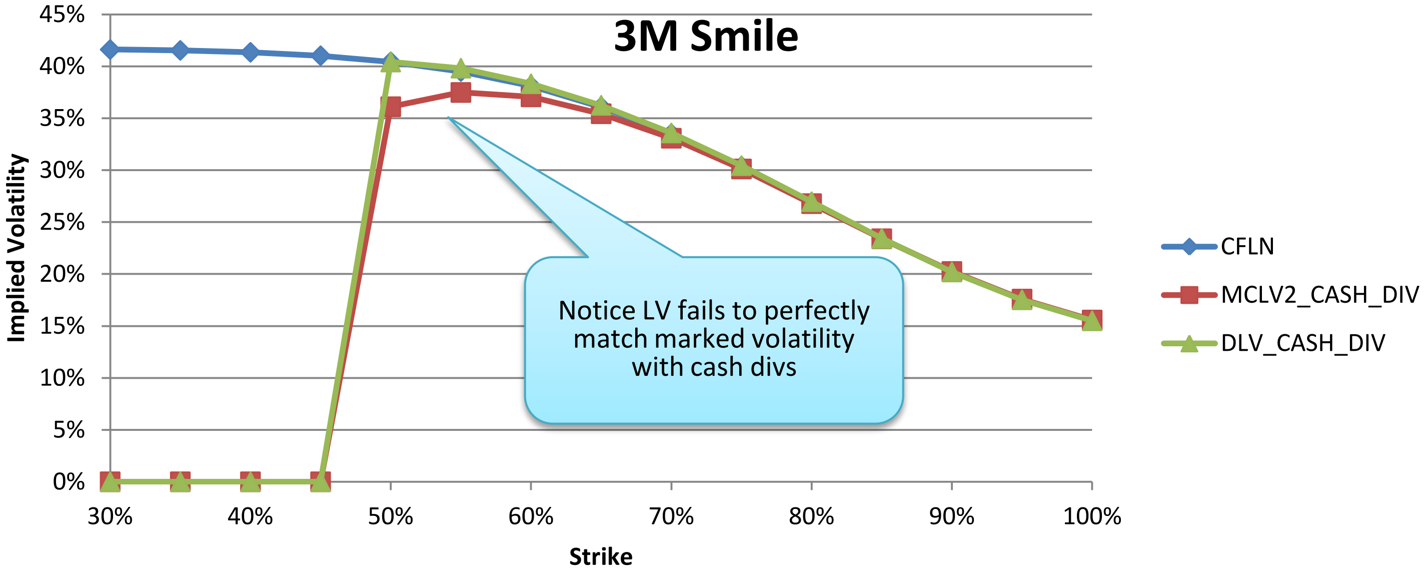
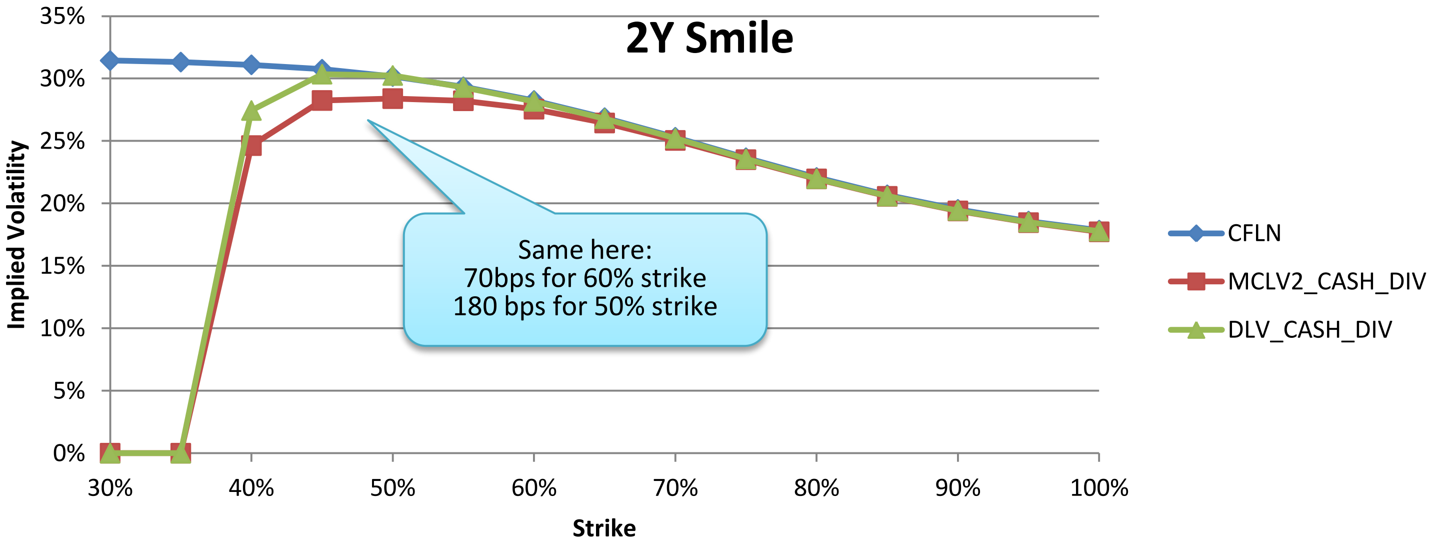
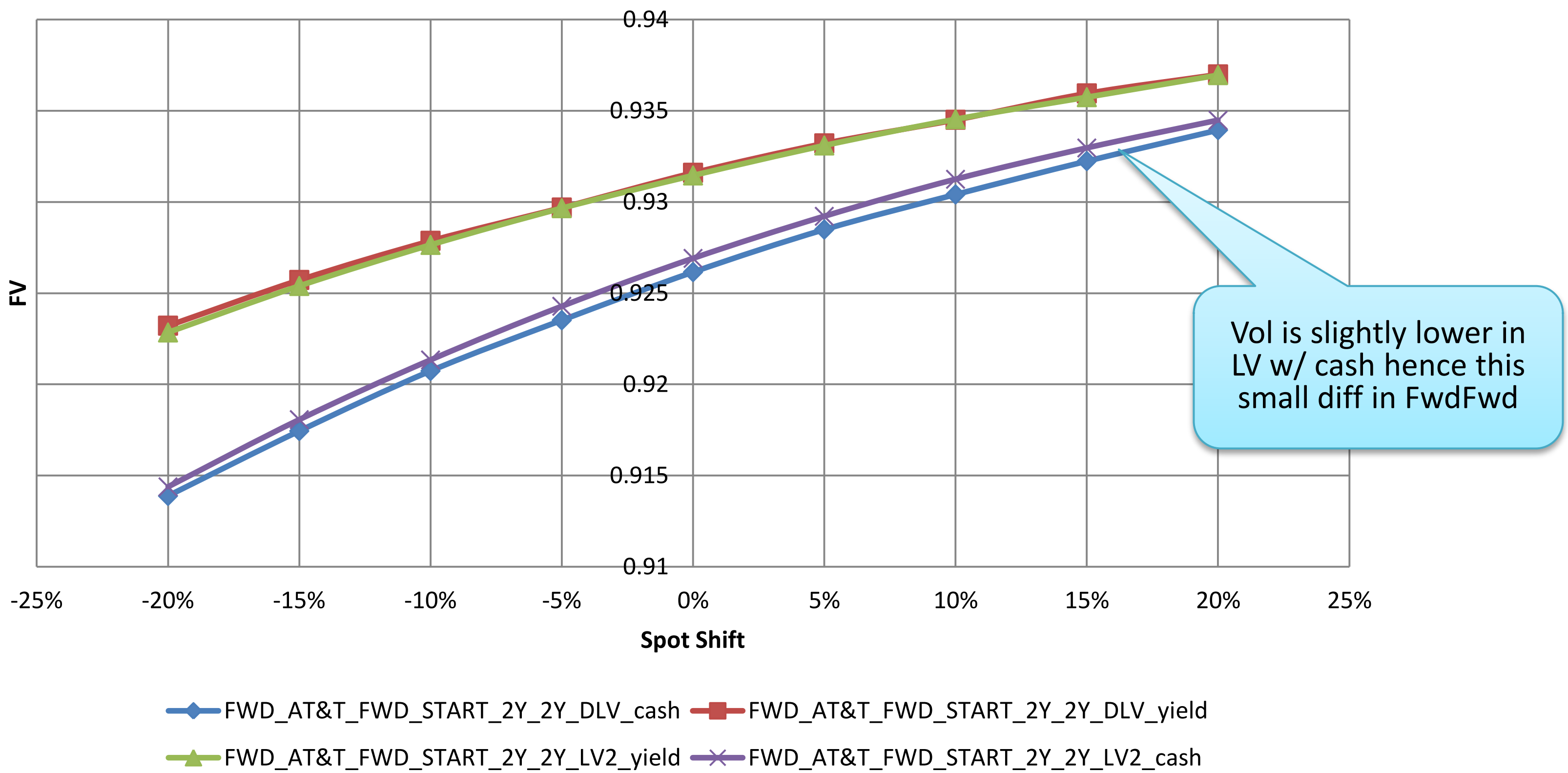




Dividends: ForwardForward on AT&T 2Y into 2Y

ForwardForward in proportional vs. cash dividends
Notice DLV matches LV with both div modelling assumptions.



DLV allows to see the impact of div modelling assumptions without introducing too many vol differences.

Multiple Assets and Quanto DLV

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QUANTITATIVE RESEARCH

Multi-Asset Monte-Carlo

– Naïve approach: when sampling from the transition kernels from two assets, correlate the driving random noise via Gaussian copula
→ equivalent to joint transition operator constructed with conditional Gaussian copulas.

Discrete Gaussian Copula

– Background
  - Let $p^0,\dots,p^{n-1}$ and $q^0,\dots,q^{N-1}$ be two densities on strikes $x^0,\dots,x^{n-1}$ and $y^0,\dots,y^{N-1}$ with distributions $f^0,\dots,f^{n-1}$ and $g^0,\dots,g^{N-1}$, i.e. $f^i = p^0+\dots+p^i$. Set $f^{-1}=g^{-1}:=0$.
  - Define the discrete 2D Gaussian density with correlation ρ in terms of the Gaussian continuous distribution N as

$$G_\rho(i,u) := N_\rho(f^i,g^u) - N_\rho(f^{i-1},g^u) - N_\rho(f^i,g^{u-1}) + N_\rho(f^{i-1},g^{u-1})$$

  - Compute the scaling factor

$$\Psi^\rho(i,u) := \frac{D_\rho(i,u)}{D_0(i,u)}$$

– Application: construct Ψ conditionally for each starting point (l,r) and set

$$P[X_j = k^i, Y_j = y^u \mid X_{j-1}^l = k^l, Y_{j-1} = y^r] := \Pi_j^{i,l}\Pi_j^{u,r}\Psi_j^\rho(i,u \mid l,r)$$

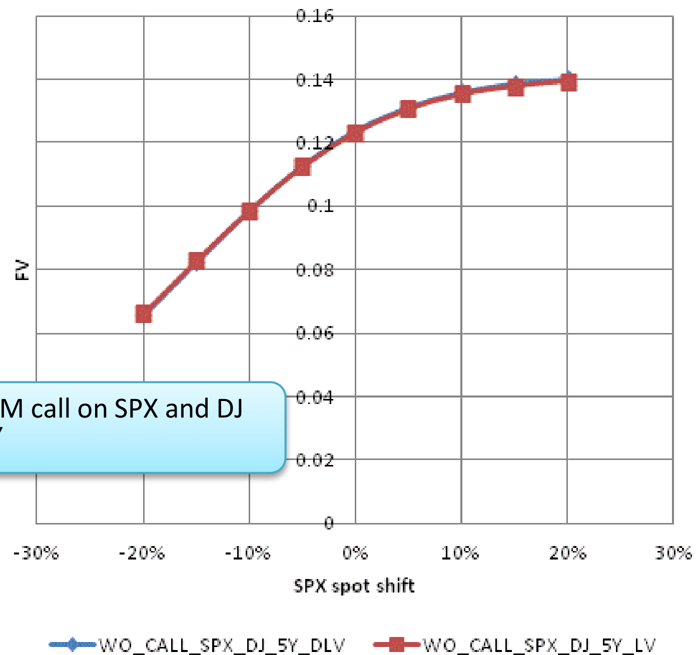
– Leads to fully implicit <u>dense</u> FD scheme which can handle *any* copula structure;
c.f. the discussion in Andreasen/Huge [AH11] and various papers on 7-point approximations for 2F FD schemes.

– Practical application via very efficient monotone approximation of Ψ.

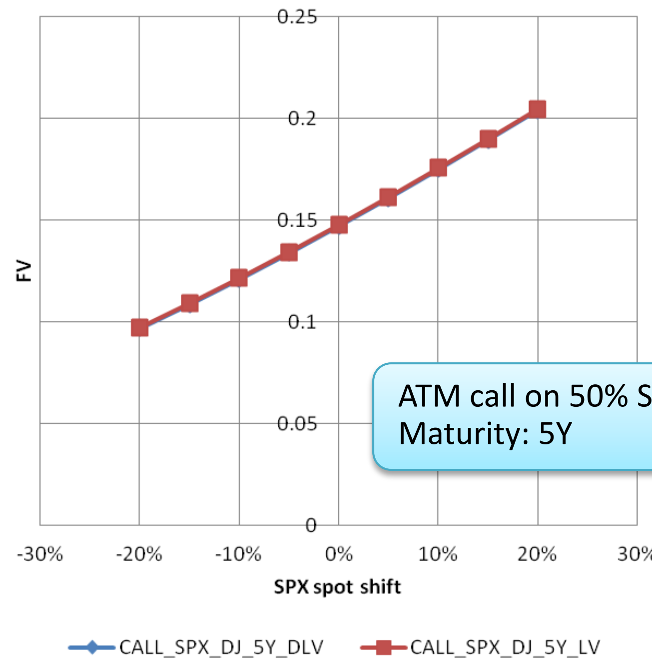
Multi-Asset DLV in action

Multi-Asset Monte-Carlo

- Naïve approach: when sampling from the transition kernels from two assets, correlate the driving random noise via Gaussian copula
- Converges in $dt{\downarrow}0$ to Dupire Local Volatility simulated with classic Gaussian driver.
- Surprisingly works out of the box (we did not manage to prove this theoretically yet)



Worst-of ATM call on SPX and DJ
Maturity: 5Y



ATM call on 50% SPX + 50% DJ
Maturity: 5Y

Difference in DLV and LV prices is smaller that MC noise.
Pricing is up to 20x-50x faster in DLV as large steps are taken.

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QUANTITATIVE RESEARCH

Quanto Discrete Algebra

- Denote by P the EUR measure and by Q the USD measure
- Assume X is a EUR asset, F is the EUR/USD FX rate such $\mathrm{E}^{\mathrm{Q}}F = \mathrm{E}^{\mathrm{P}}$.
- Joint density under P specified using Discrete Gaussian Copula.
- Notation

$$P_j^{i,u|l,r} := P[X_j = k^i, F_j = f^u \mid X_{j-1} = k^l, F_{j-1} = f^r]$$

- The relation $\mathrm{E}^{\mathrm{Q}}F = \mathrm{E}^{\mathrm{P}}$ yields trivially the joint density under Q:

$$Q_j^{i,u|l,r} = \mathrm{E}^P\left[\frac{F_{j-1}}{F_j}\{X_j = k^i, F_j = f^u\} \mid k^l, f^r\right] = \frac{f^r}{f^u} P_j^{i,u|l,r}$$

Quanto DLV in action

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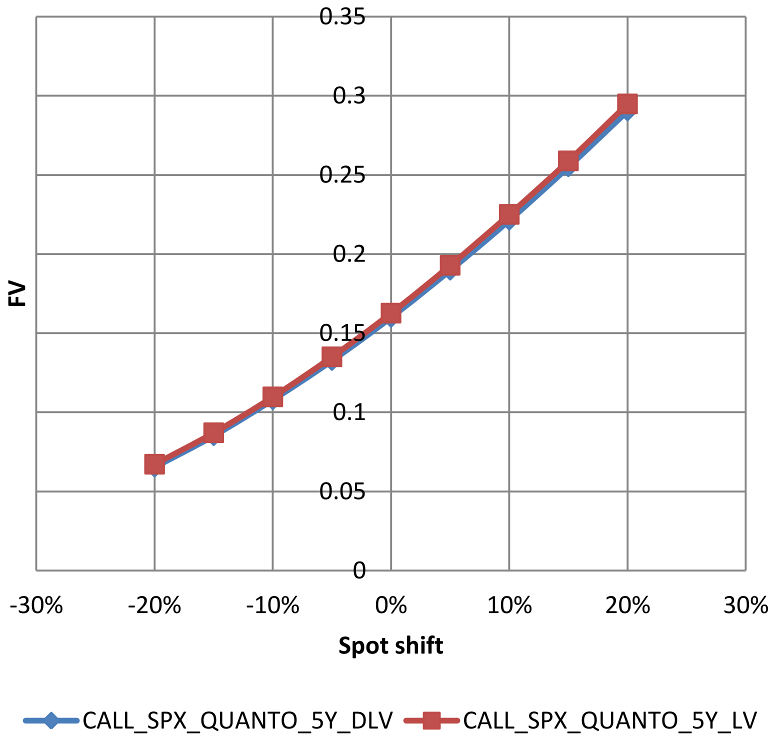
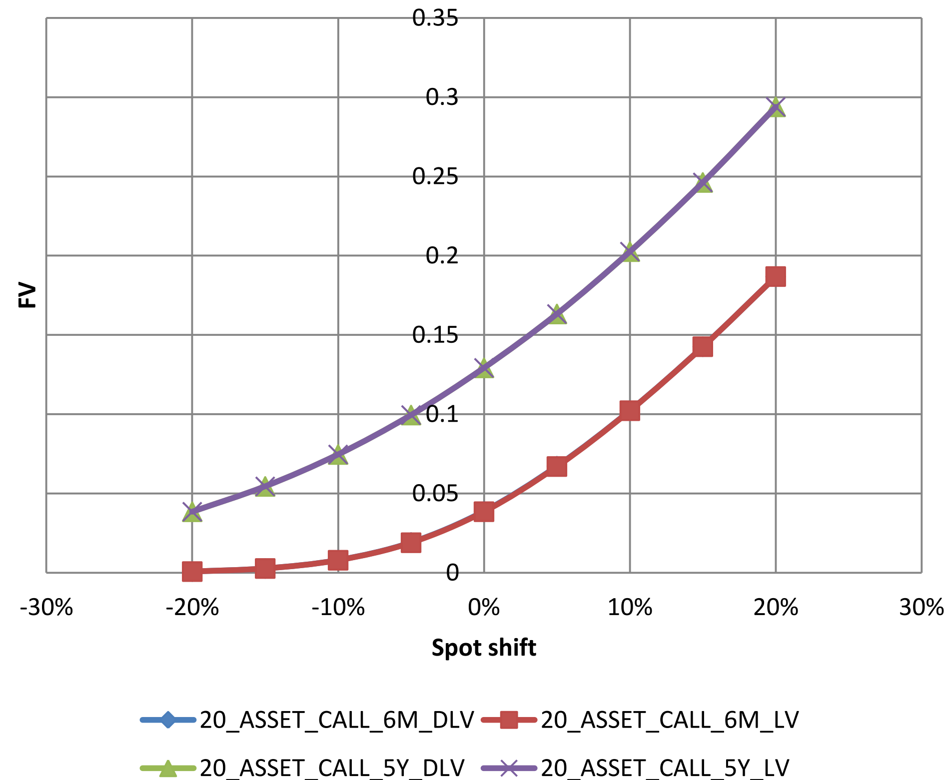
QUANTITATIVE RESEARCH

Test instrument:
Quanto Vanilla on SPX
Maturity: 5Y



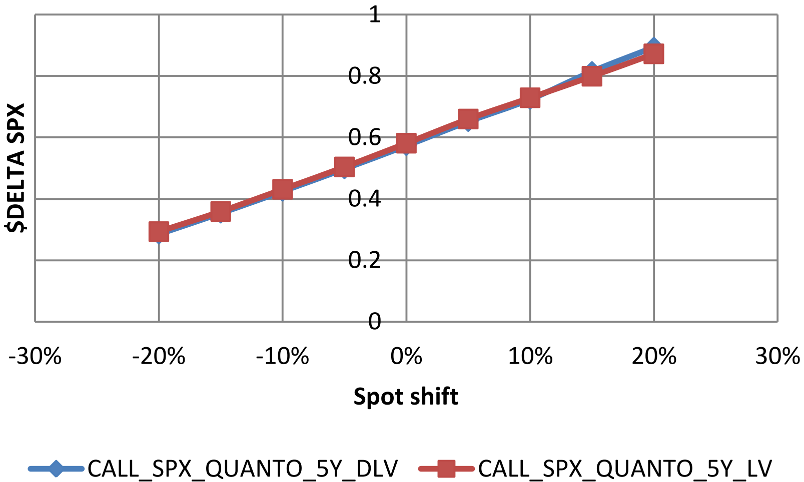
Test instrument:
Basket Call on 20 single stocks with 10 quanto
Maturity: 6M, 5Y

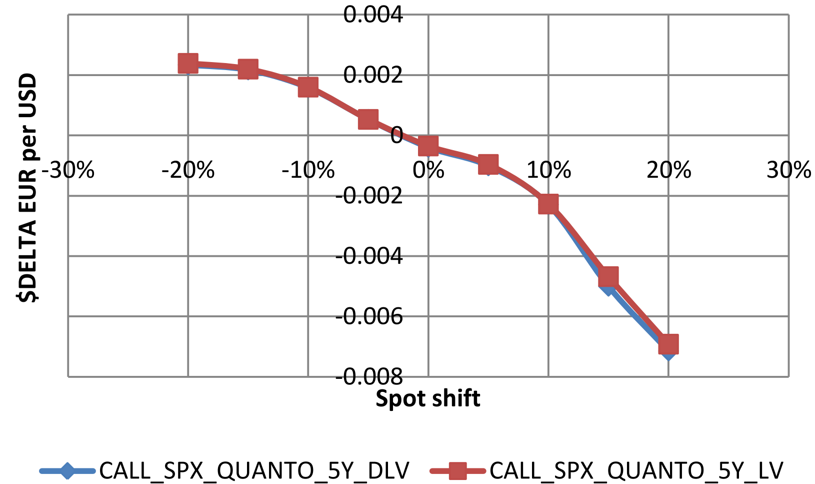


LV engine is using Dupire for Equities and FX and Gaussian driver.

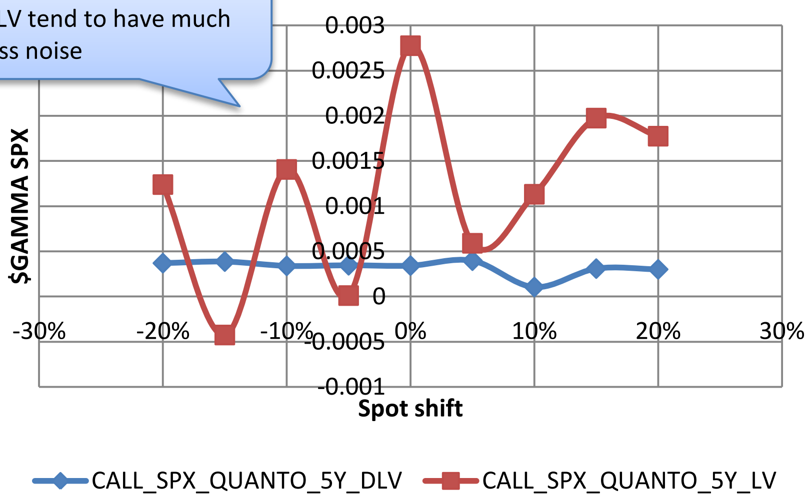
Difference in DLV and LV prices is smaller that MC noise. Pricing is up to 20x-50x faster in DLV as daily levels are not diffused.

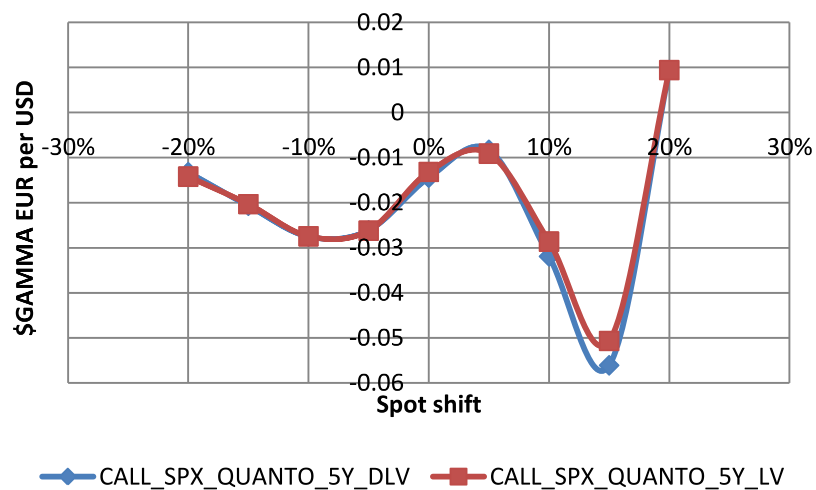
Quanto DLV Greeks





Second order Greeks in DLV tend to have much less noise





Random Grids
Discrete Stochastic Volatility

Discrete Stochastic Volatility

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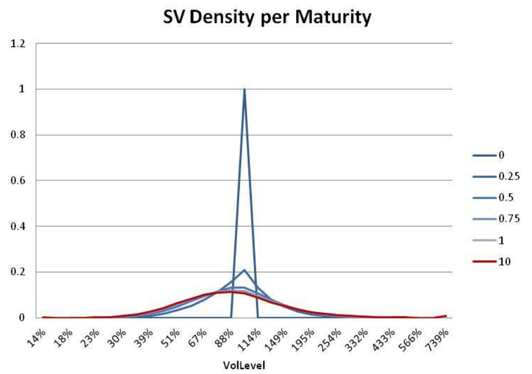
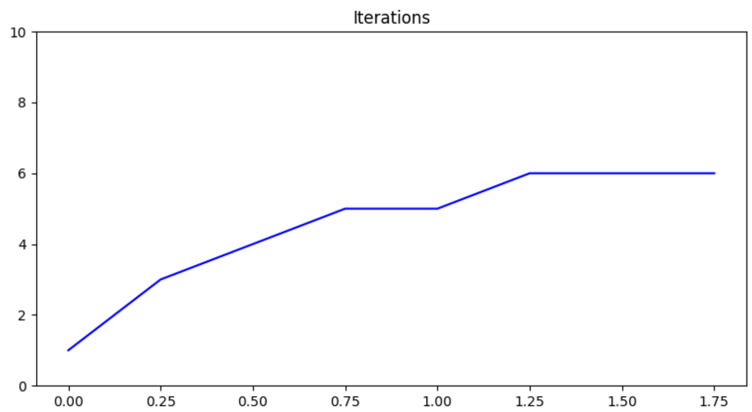
Discrete Stochastic Volatility a'la Andreasen-Huge [AH11]

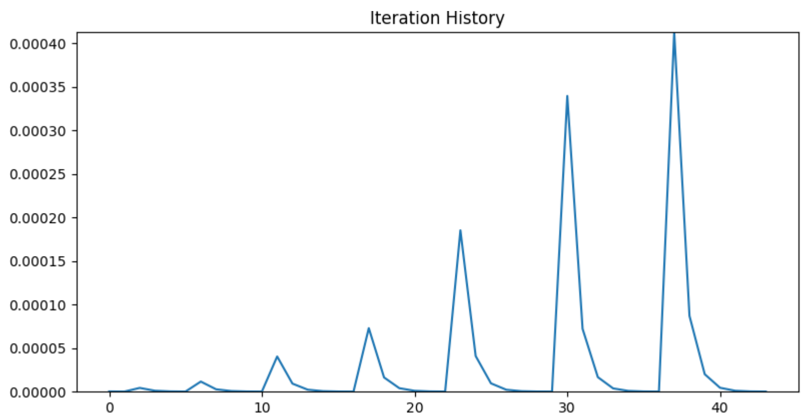
- Find arbitrage-free surface using our 1F method
- Write 2F implicit FD for the 2F process

$$dv_t = k(1 - v_t)dt + \Sigma v_t dB_t$$

- As in [AH11]
  - Use upwind scheme for the drift
  - Use operator splitting to speed up calibration
- For each iteration
  - Use uncorrelated operator first.
  - Calibrate local volatility overlay, sadly, iteratively (usually only few steps needed, six in below example)
  - Impose full correlation on the transition operator using our *discrete copula.* This eliminates the restriction on correlations, c.f. [AH11]



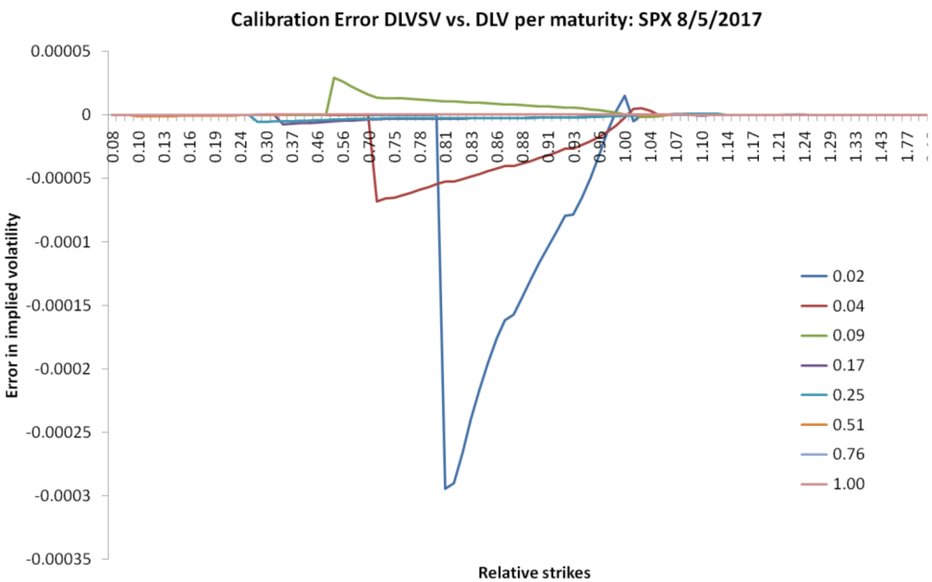


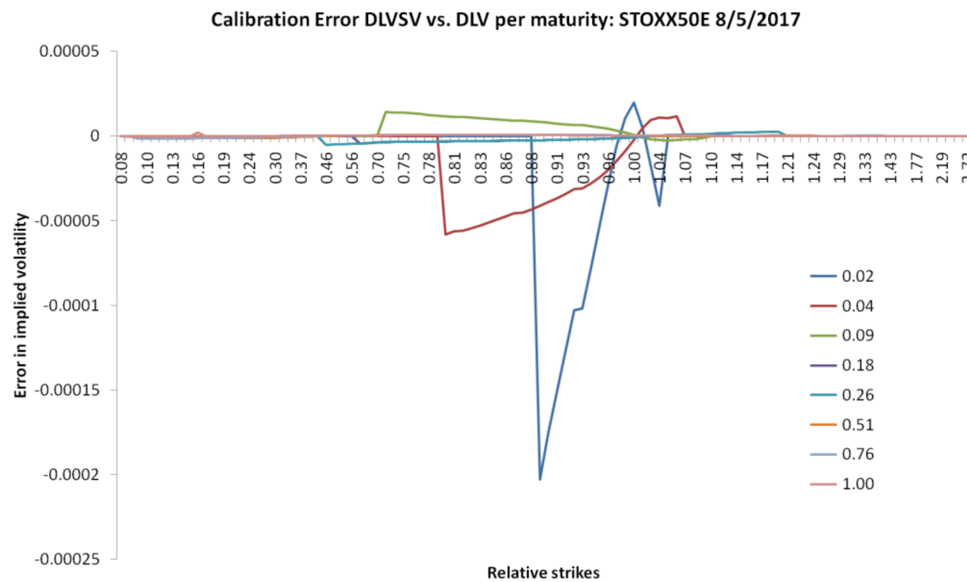


Discrete Stochastic Volatility

Discrete Stochastic Volatility a'la Andreasen-Huge [AH11]

- Works very well
- Example uses $\kappa=1$, $\Sigma=1$, Correlation -70%, 100 equity strikes and 31 strikes in vol space





- Small steps: use matrix logarithm instead of eigen-decomposition.

Jumps

Jumps in DLV

Approach 1: jumps via convolution

- Assume we wish to a model a martingale of the “prior” form

$$X_t = Y_t U_t$$

  where Y is a local volatility process and where U is a given jump process whose jumps occurrences are independent of Y

- From option prices on X_t extract option prices on Y_t
- Build DLV for Y_t

- In practice we discretize U_t on the DLV Grid so that transition kernel is decomposed into jump part and diffusion part

Pure diffusion process

Pure jump process

$$\boldsymbol{\Pi}_j = \boldsymbol{I}_j \times \boldsymbol{J}_j \Xi_j$$

Interpolation stays here as before

- This kernel allows to separate jumps and local vol diffusion

Jumps in DLV

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Approach 1: jumps via convolution

- This kernel allows to separate jumps and local vol diffusion

Pure diffusion process

Pure jump process

$$\boldsymbol{\Pi}_j = \boldsymbol{I}_j \times \boldsymbol{J}_j \boldsymbol{\Xi}_j$$

Interpolation stays here as before

- For each simulation date:
  - Draw a random number and 'jump' according to $\boldsymbol{J}_j \boldsymbol{\Xi}_j$
  - Then draw one more random number and 'diffuse' according to $\boldsymbol{I}_j$

- **Pros:** flexible model e.g. in quanto you can have different eq/fx jump correlation and eq/fx diffusion correlation

- **Cons:** *Small steps* do not work. If you want daily simulation you have to calibrate daily DLV grid -> doable but slow.

Jumps in DLV

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Approach 2: Via discretization of PDE with jumps

– Take an 'underlying' model with jumps:

$$\frac{dX_t}{X_t} = \sigma_t(X_t)dW_t + \lambda dt - jdN$$

$$\partial_t \pi_t(k) = (1 - \lambda dt)\frac{1}{2}\partial_{kk}^2\{\sigma_t^2(k)k^2\pi_t(k)\} + \lambda dt\pi_t\big(k(1-j)\big)$$

– Discretize making sure you get a transition matrix, you'll get

Diffusion and jumps mixed together

$$\widetilde{\boldsymbol{\Pi}}_j = \big(\tilde{\boldsymbol{I}}_j + (\mathbf{J}_j - \mathbf{1})\big) \times \boldsymbol{\Xi}_j$$

Interpolation stays here as before

– **Pros:** *Small steps* work now, $\big(\tilde{I}_j + (J_j - 1)\big)$ has eigenvalue decomposition.

– **Cons:** Not easy to separate jumps from local vol diffusion: in multi-asset case eq-eq jump correl would be equal to eq-eq diffusion correl.

Back to basics
Discrete Quantitative Finance

Summary

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QUANTITATIVE RESEARCH

Discrete Local Volatility:

– Given a potentially sparse set of strikes and maturities we constructed the transition matrices of a discrete state martingale, which has the following properties:

1. **Fixes Arbitrage:**
   If the input data is arbitragable - for example during Stress calculations -, we find efficiently a *globally* L^1-closest fit to the input data, with higher weights for observed market prices vs. interpolated data or points with large bid/ask.
   This method is useful independently in order to manage arbitrage violations.

2. **Large Steps:**
   Allows taking large steps, fully consistently between forward (MC) and backward (FD) schemes.

3. **Small Steps:**
   Allows taking small steps, fully consistent with the large step transition operators.

4. **Risk by Strike:**
   Our approach allows for a clear definition and implementation of Vega risk by strike/maturity.

– We applied our approach to pricing under affine dividends and we commented on introducing skew with jumps

Back to Basics

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QUANTITATIVE RESEARCH

Discrete Pricing Scheme is back to basics:

- *Everything is easier*
- Models
  - Mean-reverting assets a'la VIX: see paper [BR15]
  - Jumps:
    - Convolution – see [BR15]
    - Replace Dupire Prior with joint jump process – same methods apply here (but: pure jump process diffusion matrix does not admit eigen-decomp)
  - Stochastic Vol:
    - Still iterative a'la Andreasen/Huge [AH11] but our Discrete Gaussia Copula solves their 'zero correlation' issue
  - Stochastic / Local Dividends, Credit, …
- Quantitative Finance
  - Numerical sampling methods, control variates, …
  - Non-linear pricing methods:
    - Uncertain "Discrete Local Vol" project with Prof. Schied
    - HJB methods
  - VaR, Statistical Modelling, ….

Thank you very much for your attention

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