Discrete Local Volatility

Pricing with a Discrete Smile

Global Derivatives Conference 2016, Budapest
May 9-13, 2016

Hans Buehler Global Head of Equities and Investor Services Quantitative Research

Evgeny Ryskin Equity Derivatives Quantitative Research London

J.P.Morgan
QUANTITATIVE RESEARCH

Do not get diffused

Motivation

Discrete Local Volatility:

– Given a potentially sparse set of strikes and maturities we construct the transition matrices of a discrete state martingale, which has the following properties:

1. **Fixes Arbitrage:**
   If the input data is arbitragable - for example during Stress calculations -, we find efficiently a *globally* L^1-closest fit to the input data, with higher weights for observed market prices vs. interpolated data or points with large bid/ask.
   This method is useful independently in order to manage arbitrage violations.

2. **Large Steps:**
   Allows taking large steps, fully consistently between forward (MC) and backward (FD) schemes.

3. **Small Steps:**
   Allows taking small steps, fully consistent with the large step transition operators.

4. **Risk by Strike:**
   Our approach allows for a clear definition and implementation of Vega risk by strike/maturity.

– We apply our approach to pricing under affine dividends and we comment on introducing skew with jumps

Dupire's Local Volatility

J.P.Morgan

Setup:

– Assume we are given an equity S with
  - **Discount Factors** DF_t for all $t \in [0,\infty)$.
  - **Forwards** F_t for all $t \in [0,\infty)$.
  - A continuous volatility surface, or equivalently, a surface of **European Call** prices $Call(t,K)$ for all $t \in [0,\infty)$ and cash strikes $K \in (0,\infty)$

Objective:

– Define also "pure" call prices $C(t,k) := Call(t,k\,F_t)/DF_t$.
  We aim to derive an arbitrage-free pricing model $S_t=F_tX_t$ for a diffusion X_t which "fits" the market in the sense that

$$\mathrm{DF}_t\,\mathrm{E}\left[(S_t-K)^+\right]=Call(t,K)$$

or, equivalently, that

$$\mathrm{E}\left[(X_t-k)^+\right]=C(t,k)$$

Dupire's Classic Local Volatility:

– There is a unique continuous Markov "local volatility" process X of the form

$$dX_t = X_t \sigma_t(X_t) dW_t$$

where W is a driving Brownian motion.
We now use Ito inside the expectation operator to show

$$\begin{aligned} \mathrm{E}[d(X_t - k)^+] &= \mathrm{E}[1_{X_t > k} dX_t] + \tfrac{1}{2}\mathrm{E}[\delta_{X_t = k} d\langle X_t \rangle^2] \\ &= 0 + \tfrac{1}{2} dt\, \mathrm{E}[\delta_{X_t = k} X_t^2 \sigma_t(X_t)^2] \\ &= \tfrac{1}{2} k^2 \sigma_t(k)^2 dt\, \mathrm{E}[\delta_{X_t = k}]. \end{aligned}$$

Hence our local volatility σ is given by Dupire's famous '96 formula [D96]

$$\sigma_t(k)^2 = \frac{f\Theta(t,k)}{\tfrac{1}{2} k^2 dt\, \Gamma(t,k)}$$

with

- **Forward-Theta** $f\Theta(t,k) := C(t+dt,k) - C(t,k)$; and
- **Gamma** $\Gamma(t,k) := \partial_{kk} C(t,k)$.

This definition of **Gamma** represents the second order derivative of the option price in *strike*, not spot. It only coincides with the latter under the assumption of a sticky strike implied volatility surface – which is not compatible with any known dynamic martingale model.

Absence of Arbitrage:

– Recall the formula

$$\sigma_t(k)^2 = \frac{f\Theta(t,k)}{\frac{1}{2}k^2 dt\,\Gamma(t,k)}$$

with

- **Forward-Theta** $f\Theta(t,k) := C(t+dt,k) - C(t,k)$; and
- **Gamma** $\Gamma(t,k) := \partial_{kk}C(t,k)$.

– We call the option price surface C or its implied volatility surface Dupire-arbitrage-free if σ is non-negative, real and bounded, i.e. if

- Both $f\Theta$ and Γ are non-negative, and
- $f\Theta$ is zero whenever Γ is.

There are a few additional technical conditions to strictly ensure existence of a solution to $dX_t = X_t\sigma_t(X_t)dW_t$ but those are not really relevant in practice and not pertinent to the discussion here.

Practical Usage and Limitations

Summary of Steps when using Local Volatility:

Discrete, sparse market data

The Volatility Surface is only observed at discrete strikes for listed maturities, typically at least 1M, mostly 3M wide.

Interpolate using Implied Volatility Scheme which may exhibit arbitrage

First fit an "Implied Volatility" scheme to those discrete volatilities to generate a continuous surface. However, no such is scheme is known which is truly arbitrage-free and fits well to most observed market data.

Fit Dupire's Local Volatility

Bootstrap with FD if arbitrage is found e.g. under Stress or if dividend assumptions are changed.

Dupire's Local Volatilty calibration is not robust vs. errors in arbitrage.

A solution is to "bootstrap" the local volatility function with FD to be able to recover from local errors in the implied volatility surface.

Price product with daily time steps even if only large steps are required

Local Volatility requires us to simulate the resulting process with small time steps in order to be able to "fit" the market – even if the actual product does not require dense time steps for valuation.

Computing strike wise risk very noisy

Computing strike-wise risk is noisy due to impact of local arbitrage during shifts.

Absence of Arbitrage

Discrete Local Volatility

J.P.Morgan

QUANTITATIVE RESEARCH

Assumptions:

- **Maturities**
  Assume we are given listed maturities $0=t_0<\ldots<t_m$.
  Set $dt_j^+:=t_{j+1}-t_j$ and $dt_j^-:=t_j-t_{j-1}$.

- **Strikes**
  For each maturity t_j, we are given n_j strikes $k_j^{-1}<\ldots 1\ldots<k_j^{nj}$.
  We will drop the subscript j wherever possible, e.g. we define
  $dk_+^i:=k^{i+1}-k^i$ and $dk_-^i:=k^i-k^{i-1}$.
  We also add arbitrary **ghost strikes** $k^{-2}<k^{-1}$ (which might be negative) and $k^{n+1}>k^n$.

- **Market Prices**
  For each strike and maturity, we are given input market call prices
  $C_j^i := C(t_j,k_j^i)$.

Definitions:

- **Model Prices**
  We will use generally $c_j^i := c(t_j,k_j^i)$ to refer to model prices.
  We impose that all model prices are intrinsic at the **boundary strikes** k^{-2},k^{-1} and k^n,k^{n+1}.

- **Quality of Fit**
  Using positive weights w_j^i which sum up to 1, we define the norm

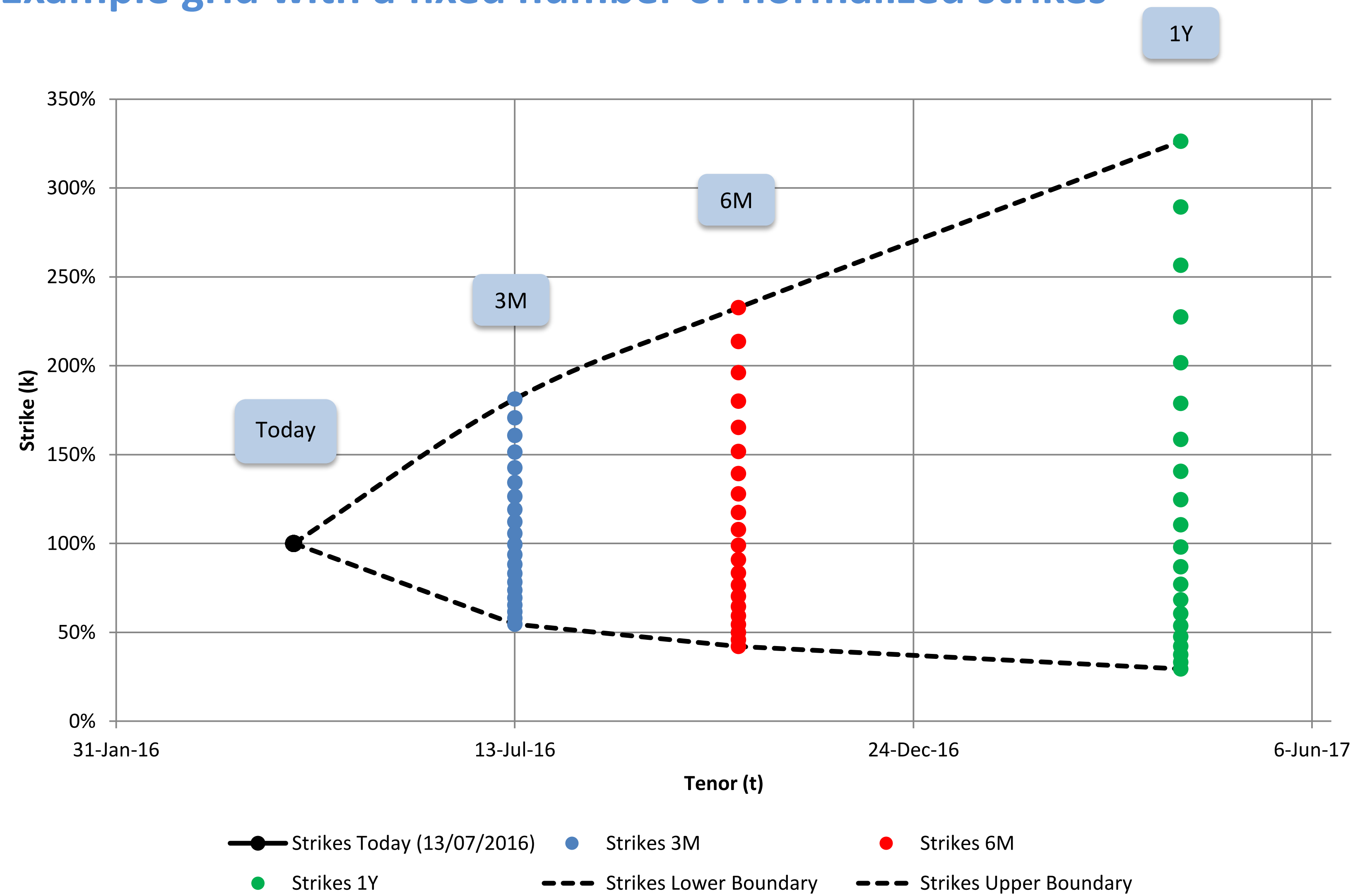
$$\| c \| := \sum_{i,j} w_j^i \, \| c_j^i - C_j^i \|$$

Discrete Local Volatility

J.P.Morgan

QUANTITATIVE RESEARCH

Example grid with a fixed number of normalized strikes

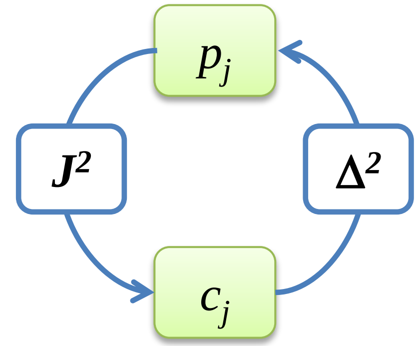


Absence of Arbitrage

Algebra for Discrete Martingales in Strikes:

– Assume $p=(p^i)$ is a discrete density over strikes $k=(k^i)$
  Its call prices on the given strikes are given in terms of the linear integral-type operator J^2 as

$$c^i := (J^2 p)^i := \sum_{u=i+1}^{n} p^u (k^i - k^u)$$



– Its inverse operator over call prices c is given as by applying the operator Δ^2 given as:

$$p^i = (\Delta^2 c)^i := \left(\frac{c^{i+1} - c^i}{dk_+^i} - \frac{c^i - c^{i-1}}{dk_-^i} \right)$$

  The operator Δ^2 is related to the classic second order difference operator D^2 by

$$(\Delta^2 c)^i = \tfrac{1}{2}(dk_+^i + dk_-^i)(D^2 c)^i$$

– **Gamma** is as usual defined as

$$\Gamma^i := (D^2 c)^i$$

Theorem (Absence of Arbitrage for one Maturity [BR15])

– Let c_j be candidate call price function which is intrinsic at the boundary strikes as defined before. If $\Gamma_j \geq 0$, then c_j is arbitrage-free in the sense that

$$p_j^i := \tfrac{1}{2}(dk_+^i + dk_-^i)\Gamma_j^i$$

  is a density.

Absence of Arbitrage

J.P.Morgan

QUANTITATIVE RESEARCH

Algebra for Discrete Martingales in Time:

– Assume $p=(p_j^i)$ is a discrete density over strikes $k=(k_j^i)$ with call prices $c=(c_j^i)$. Recall that we allowed for different strikes per maturity. We denote by

$$c_j(x) := \sum p_j^u (k_j^u - x)^+$$

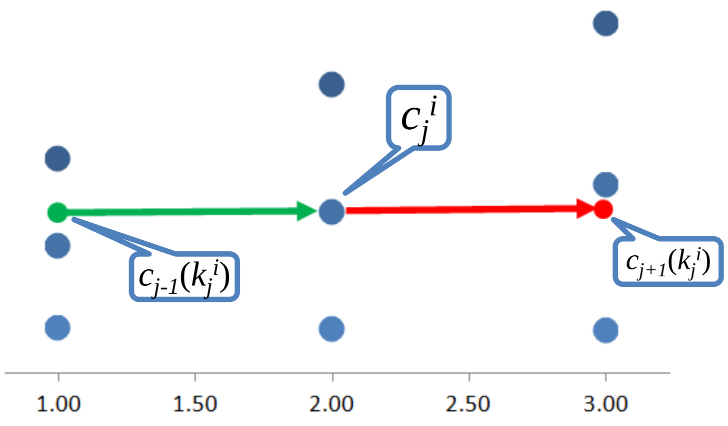
the call prices for off-grid strikes. We note that this is equivalent to *linear interpolation* in call prices.

– **Forward-Theta** is defined as

$$f\Theta^i := c_{j+1}(k_j^i) - c_j^i$$

– **Backward-Theta** is defined as

$$b\Theta^i := c_j^i - c_{j-1}(k_j^i)$$



Theorem (Absence of Arbitrage [BR15])

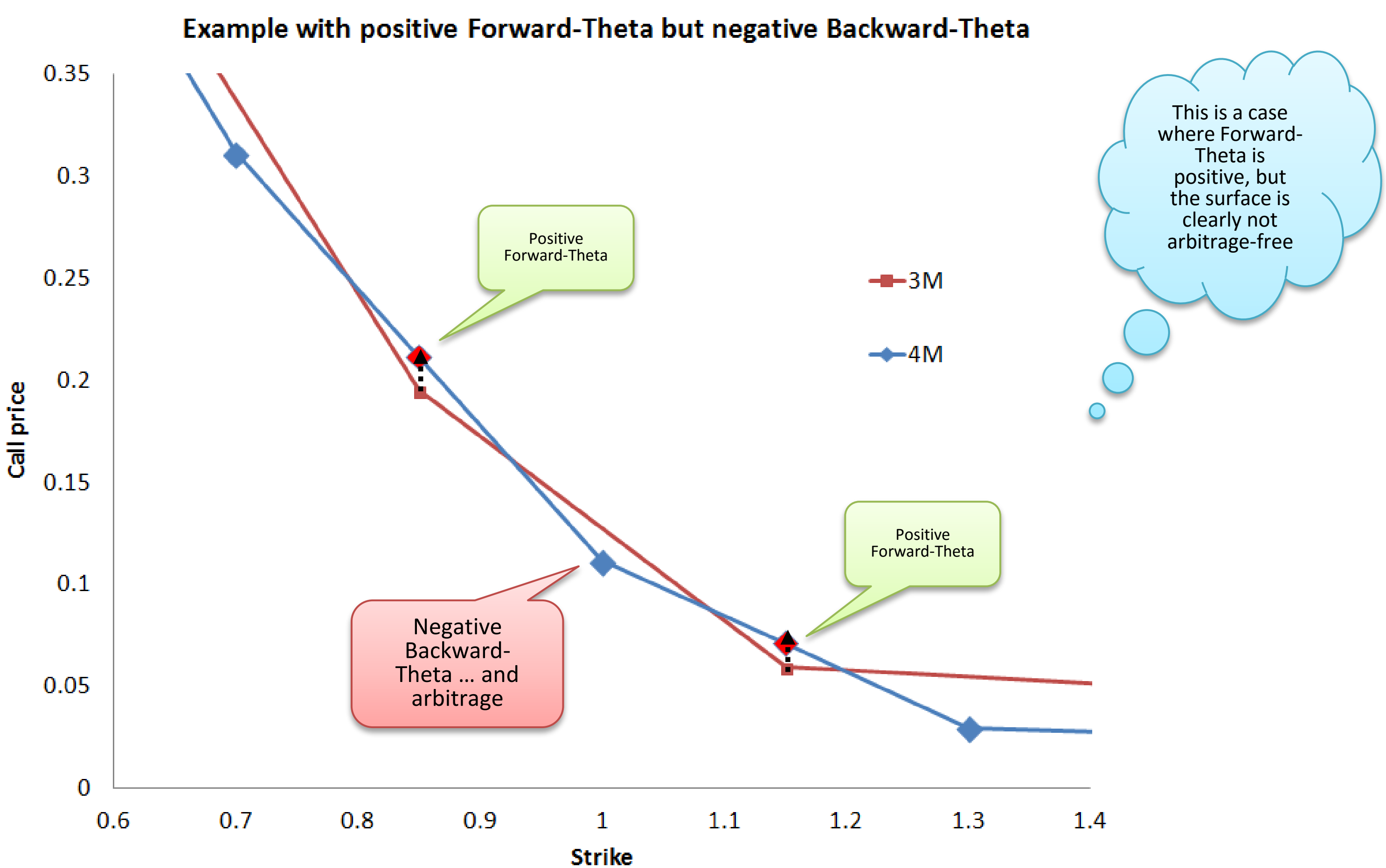
– Assume that for each maturity j, c_j is arbitrage-free with density p_j. Then, the surface c is arbitrage-free in the sense that there is a discrete martingale X with marginal densities p_j if and only if $b\Theta \geq 0$.

– The conclusion does *not* hold for $f\Theta \geq 0$.

Absence of Arbitrage

J.P.Morgan

QUANTITATIVE RESEARCH



Absence of Arbitrage

Find the Closest Arbitrage-Free Surface [BR15]

- A call price surface is arbitrage-free in the sense that there exist a martingale which fits c if and only if the two linear conditions on c hold:
  1. $\Gamma^j \geq 0$
  2. $b\Theta \geq 0$

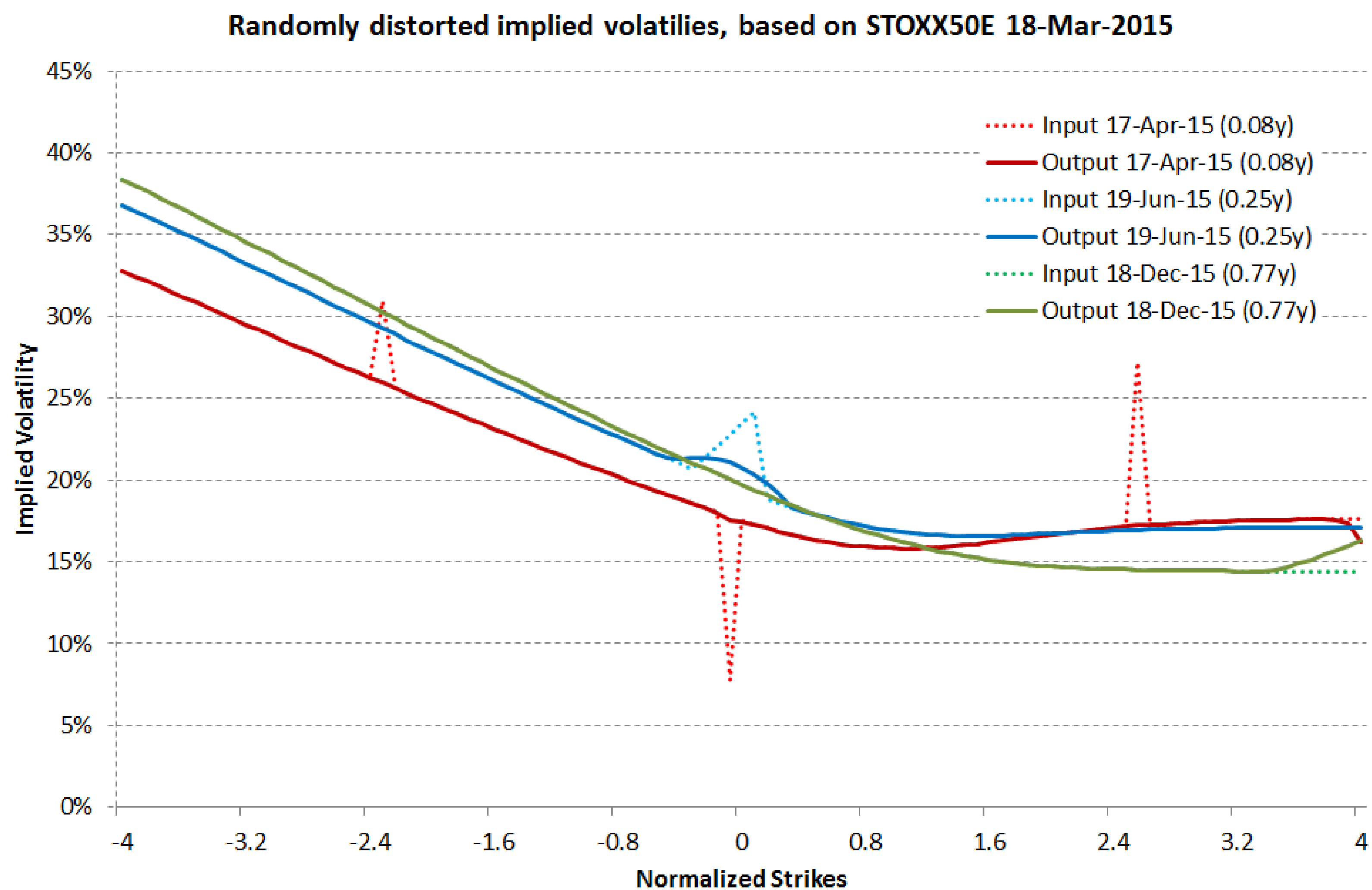
- Assume that C are given market prices with weights w.
  Then, we may find a closest arbitrage-free surface by solving the linear program

$$c^* := \arg\min \left\{ \sum_{i,j} w_j^i \, \| c_j^i - C_j^i \| \,\middle|\, c : \Gamma \geq 0, b\Theta \geq 0 \right\}$$

  over the set c which have intrinsic value at the boundary strikes.
  - It is straight forward to impose bounds on implied volatility.
  - Other norms than L^1 can easily be used.

- Note that the conditions 1. and 2. above do not imply that Dupire's local volatility exists.
  In particular, we do not exclude the case where $\Gamma_j^i = 0$ while $b\Theta_j^i > 0$.

Example of Fixing Arbitrage



Discrete Local Volatility
Construction of Discrete Martingales

Step 1: Time Interpolation

J.P.Morgan

QUANTITATIVE RESEARCH

Step 1: Interpolation in time

– Fix j-1 and consider the call prices $c_{j\text{-}1}$ defined over $k_{j\text{-}1}$

1. Compute call prices $cc_{j\text{-}1}$ using the current density $p_{j\text{-}1}$ for new strikes k_j as:

$$cc^i_{j-1} := c_{j-1}(k^i_j)$$

2. Define the associated interpolated density $q_{j\text{-}1}$ again for strikes k_j consistently as:

$$q^i_{j-1} := (\Delta^2 cc_{j-1})^i$$

– Both operations are *linear* and jointly define a linear operator which maps the density $p_{j\text{-}1}$ defined over strikes $k_{j\text{-}1}$ into the density $q_{j\text{-}1}$ defined over k_j:

$$\Xi_j : p_{j-1} \mapsto q_{j-1}$$

Obviously, if $k_j = k_{j\text{-}1}$, then $p_{j\text{-}1} = q_{j\text{-}1}$.

All of these calculations are simple algebra and can virtually be done on a spread sheet.

Theorem (Interpolation using a martingale kernel) [BR15]

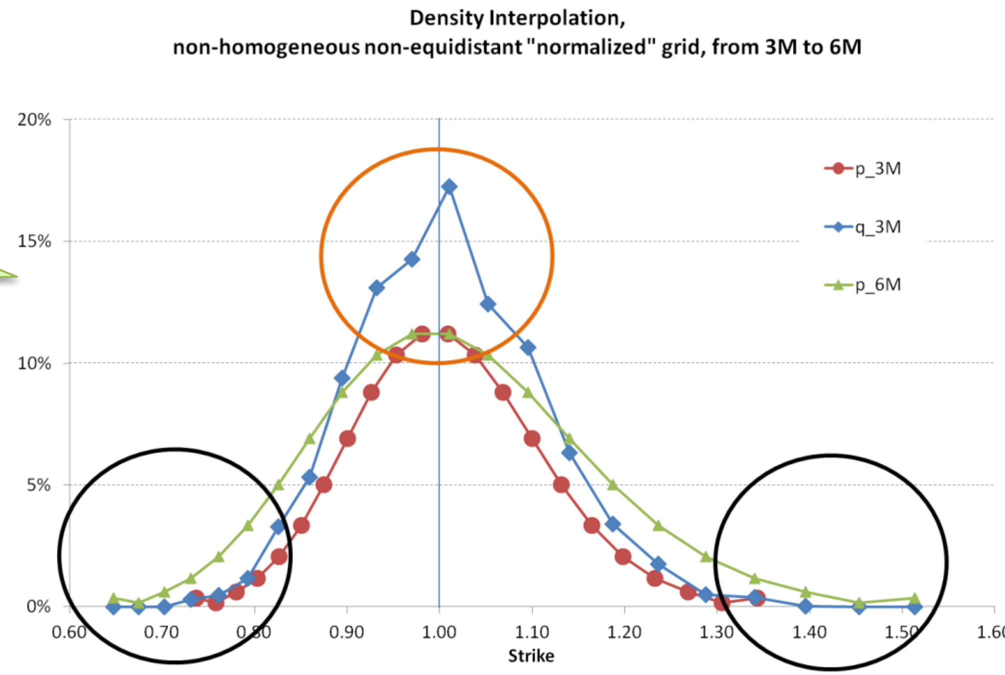
– Ξ is a *transition kernel*, i.e.
  - Ξ is a probability matrix: $1\Xi_j = 1$ and $\Xi_j \geq 0$.
  - It is a transition matrix $q_{j\text{-}1} = \Xi_j p_{j\text{-}1}$.
  - It is a martingale kernel $k_{j+1}\Xi_j = k_j$.

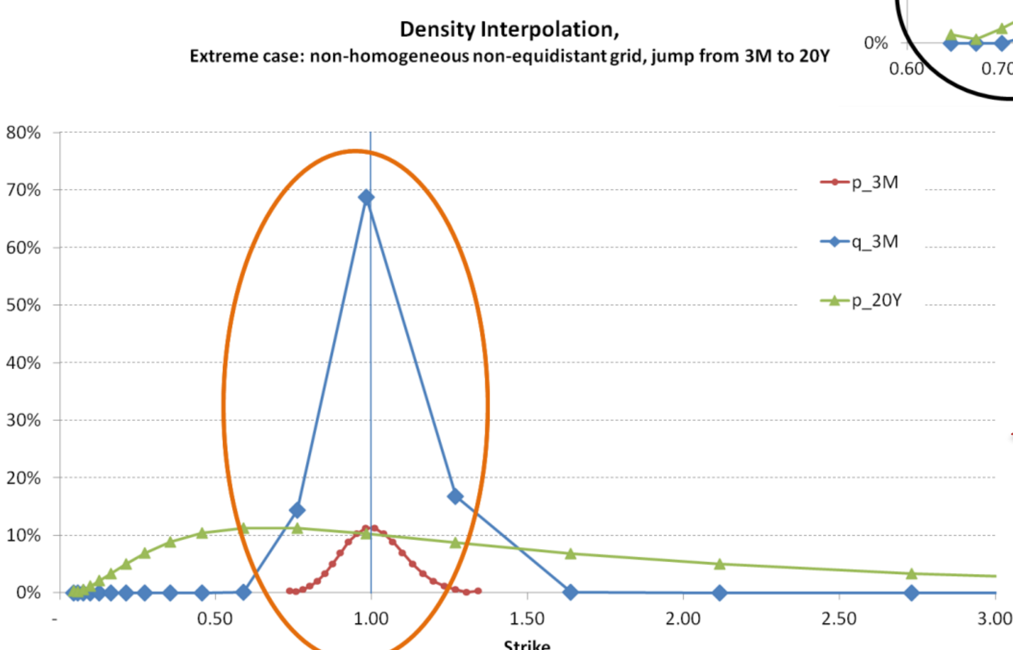
Step 1: Time Interpolation

J.P.Morgan

RESEARCH

Interpolated density, for reasonable time steps





Interpolated density, for large time steps

Step 2: Transition operators from Implicit FDs

– Assume now that strikes are *homogeneous* between t_{j-1} and t_j.

- **Define prior model:** recall the equation $dX_t=X_t\sigma_t(X_t)dW_t$.
  Its density $\pi(t,x):=P[X_t=x]$ satisfies the forward-PDE

$$d\pi(t,x)=\frac{1}{2}\partial_{xx}^2\left\{x^2\sigma_t(x)^2\pi(t,x)\right\}dt$$

- **Implicit FD:** discretize in time using an *implicit* scheme for $\pi_j(x):=\pi(t_j,x)$:

$$\pi_j(x)-\pi_{j-1}(x)=\frac{1}{2}\partial_{xx}^2\left\{x^2\sigma_j(x)^2\pi_j(x)\right\}dt\_$$

$$\pi_{j-1}=I_j^{-1}\pi_j \qquad I^{-1}:=1-\frac{1}{2}\partial_{xx}^2\left\{x^2\sigma_j(x)^2\bullet\right\}dt\_$$

  Standard FD discretization in space yields the tridiagonal matrix

$$I^{-1}:=1-\frac{1}{2}D_{xx}^2\left\{x^2\sigma_j(x)^2\bullet\right\}dt\_$$

Step 2: Transition Operators

Discretization of Forward-PDE matrices:

- Note that when discretizing

$$I_j^{-1} := 1 - \frac{1}{2} D_{xx}^2 \left\{ x^2 \sigma_j(x)^2 \bullet \right\}$$

we do <u>not</u> expand the second order derivative in separate derivatives of $x^2\sigma(x)^2$ and $\bullet$ as was proposed in [AH11], but we discretize it as is.

- In this form, it is worth noting that I is actually just the transpose of the *backward* FD operator BI defined on the same grid via

$$BI_j^{-1} := 1 - \frac{1}{2} x^2 \sigma_j(x)^2 D_{xx}^2 \bullet$$

- In other words, this discretization scheme is *consistent for forward and backward operators*.
  We more generally have:

Theorem (consistent forward and backward operators) [BR15]

- The backward operator of a diffusion with unattainable boundaries is the adjoint (transpose) of its forward operator.
  - The same is true for a finite state Markov chain, i.e. forward and backward operators are consistent if the density has a Neumann-boundary condition.

J.P.Morgan

QUANTITATIVE RESEARCH

Theorem (Z-Matrix) see also Andreasen-Huge [AH11]

- Assume that M is a square matrix whose columns [rows] add up to 1, and where all off-diagonal elements are non-positive.
  Then, its inverse exists, is non-negative, and its columns [rows] add up to 1; in other words M^{-1} is a transition matrix.
  (see [BR15] for a brief proof)

$$\begin{bmatrix} 1 & -a & & & \\ & 1+a+b & -a & & \\ & -b & 1+a+b & -a & \\ & & -b & 1+a+b & \\ & & & -b & 1 \end{bmatrix}$$

Illustration

- Our tridiagonal matrix

$$I_j^{-1} := 1 - \frac{1}{2} D_{xx}^2 \left\{ x^2 \sigma_j(x)^2 \bullet \right\}$$

  does indeed fit this description, hence I is a transition kernel for π.

Step 2: Transition Operators

J.P.Morgan

QUANTITATIVE RESEARCH

Backward Local Volatility

- How does that help?
  Most likely the discretized π is not even a density.
  We now aim to find a local volatility σ such that $p_j = I_j p_{j-1}$ for the given model densities (recall that we currently assume homogeneous strikes).
- To this end, we write the FD out, which gives:

$$p_j - p_{j-1} \stackrel{!}{=} \frac{1}{2} D_{xx}^2 \left\{ x^2 \sigma_j(x)^2 p_j(x) \right\} = \frac{1}{2} \Delta_{xx}^2 \left\{ x^2 \sigma_j(x)^2 \Gamma_j(x) \right\}$$

We now apply the inverse integral operator $J_{xx}{}^2$ such that

$$C_j^i - C_{j-1}^i \stackrel{!}{=} \frac{1}{2} k_j^{i\,2} \sigma_j^{i\,2} \Gamma_j^i dt_-$$

which gives rise to the definition of **backward local volatility** as:

$$\sigma_j^{i\,2} := \frac{b\Theta_j^i}{\frac{1}{2} k_j^{i\,2} \Gamma_j^i dt_-}$$

Step 2: Transition Operators

Theorem (Bounded Discrete Local Volatility) [BR15]

- Let c be a call price surface which is intrinsic at the boundaries, and which satisfies for $0 \leq \sigma_{min} < \sigma_{max}$ the linear constraints
  1. $\Gamma \geq 0$ and
  2. $\frac{1}{2}\Gamma k^2\, dt\, \sigma_{min}{}^2 \leq b\Theta \leq \frac{1}{2}\Gamma k^2\, dt\, \sigma_{max}{}^2$
- Then, c is arbitrage free, and the transition matrix from p_{j-1} to p_j is given by

$$\Pi_j := I_j \Xi_j$$

  where
  - Ξ is given by the interpolation operator defined before; and
  - I is the well-defined inverse of the tridiagonal matrix I^{-1} given as

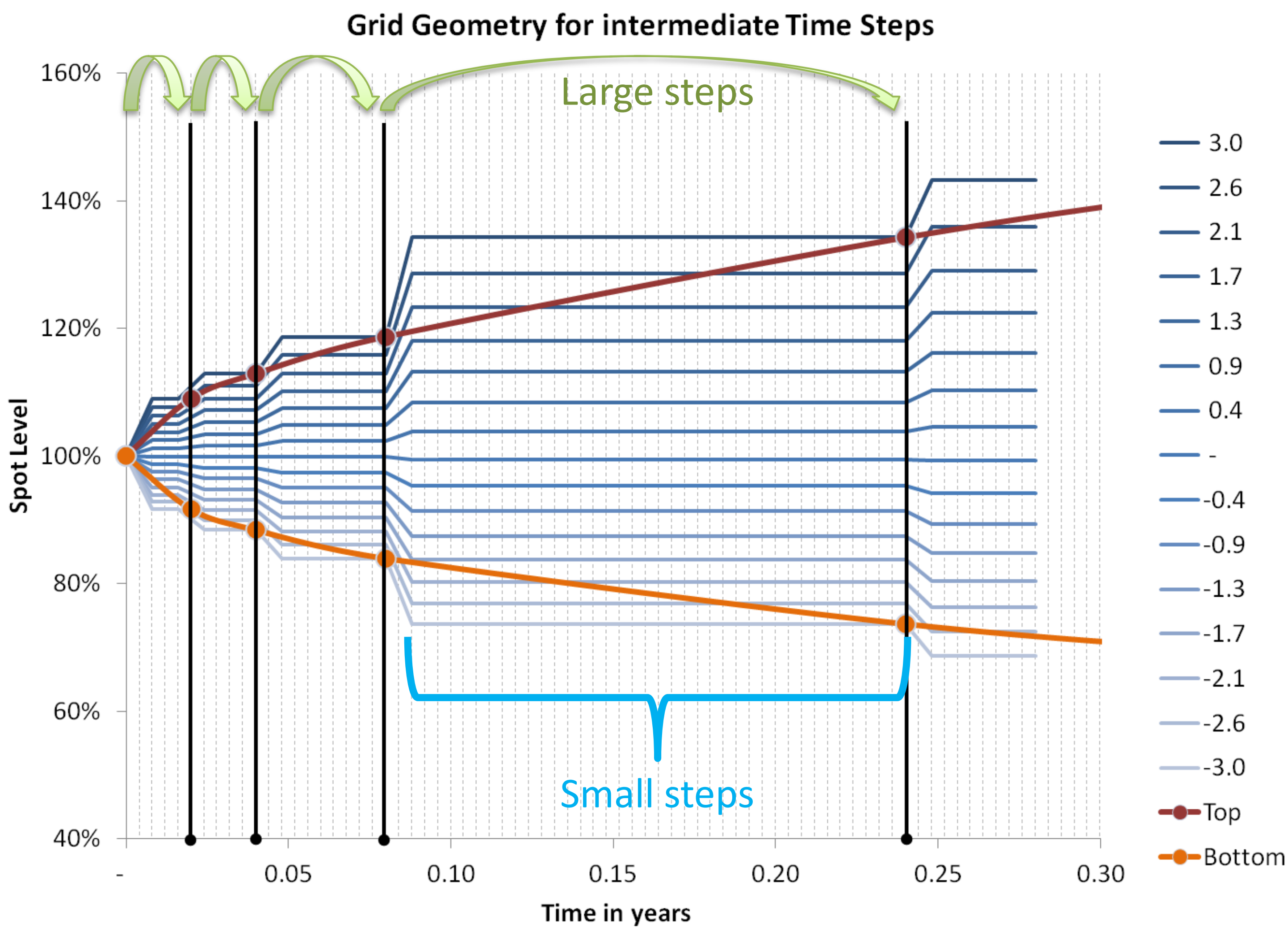
$$I_j^{-1} := 1 - \frac{1}{2} D_{xx}^2 \left\{ x^2 \sigma_j(x)^2 \bullet \right\} \quad \text{with} \quad \sigma_j^{i\,2} := \frac{b\Theta_j^i}{\frac{1}{2} k_j^{i\,2} \Gamma_j^i dt\_}$$

  with **bounded** "backward local volatility" σ.
- Moreover, conditions 1. and 2. above are linear, hence for a given market surface C we may find a closest arbitrage-free surface with bounded backward local volatility by solving the appropriate linear program.

Step 3: Small Steps

J.P.Morgan

QUANTITATIVE RESEARCH



Step 3: Small Steps

J.P.Morgan

QUANTITATIVE RESEARCH

Step 3: Small Steps

- We have constructed a discrete martingale for our reference time steps, for example listed maturities.
  - How do we price options which require more frequent or non-standard observations?
  - Recall that our transition operator is given as

$$\Pi_j := I_j \Xi_j$$

  - Since I is positive definite, we may write it in terms of a unitary matrix X and a diagonal matrix D as

$$I_j := X_j' D_j X_j$$

Quick, since I_j^{-1} is tridiagonal.

Hence, for any positive α we may write

$$I_j^{\alpha} := X_j' D_j^{\alpha} X_j$$

  - For any $t_{j-1}<t<t_j$, let $\alpha:=(t-t_{j-1})/(t_j-t_{j-1})$ and define the two transition matrices

$$II_{j-1}^{t} := X_j' I_j^{a} X_j \qquad II_t^{j} := X_j' I_j^{1-a} X_j$$

whose product, obviously, is again I.

Result

- In other words, we have constructed transition operators from t_{j-1} to t, and from t to t_j, which are consistent with the overall operator from t_{j-1} to t_j.

Discrete Local Volatility

Summary of Approach

1. Use interpolation operator Ξ to reduce to the homogeneous strike case.

2. For homogeneous strikes:
   a. Define "prior model" with associated forward PDE for $\Sigma_t(x) := x\,\sigma_t(x)$:
   $$d\pi_t(x) = \tfrac{1}{2}\partial^2_{xx}\left\{\Sigma_t^2 \pi_t\right\}dt$$
   b. Use implicit FD operator discretization which gives us a transition matrix for a given Σ:
   $$I_j^{-1} = 1 - \tfrac{1}{2}\Delta^2_{xx}\left\{\Sigma_j^2 \bullet\right\}dt$$
   c. The transition property for p imposes the following equation for Σ:
   $$p_j - p_{j-1} \overset{!}{=} \tfrac{1}{2} D^2_{xx}\left\{\Sigma_j^2 \Gamma_j\right\}dt$$
   d. Solve for Σ by applying the inverse J^2 of the operator D^2:
   $$C_j - C_{j-1} =: b\Theta_j \overset{!}{=} \tfrac{1}{2}\Sigma_j^2 \Gamma_j dt_j^-$$
   e. Bounds on Σ yield **linear no-arbitrage conditions** for the option price surface: $\Gamma \geq 0$ and $\tfrac{1}{2}\Gamma\, dt\, \Sigma_{min}^2 \leq b\Theta \leq \tfrac{1}{2}\Gamma\, k^2\, dt\, \Sigma_{max}^2$.

All of these calculations are simple algebra and can virtually be done on a spread sheet.

3. Interpolate to intermediate time steps by decomposing
$$I_j := X_j' D_j X_j$$

Application: Affine Dividends

Discrete Local Volatility

J.P.Morgan

Affine Dividends

- Assume an equity price process pays at ex-dividend dates $\tau=(\tau_k)_k$ dividends which have a proportional component d and a cash component δ, i.e.

  $$S_{\tau_k} = S_{\tau_k -} - S_{\tau_k -} e^{d_k} - \delta_k$$

  Define the proportional drift:

  $$R_t = \exp\left\{ \int_0^t \mu_s ds + \sum_{k:\tau_k > t} d_k \right\}$$

  and the *discounted future dividends*,

  $$D_t := R_t \left(\sum_{k:\tau_k > t} \frac{\delta_k}{R_{\tau_k}} \right)$$

- It was then shown in Buehler [B10] that *every (*)* arbitrage-free representation of S has the form

  $$S_t = (F_t - D_t) X_t + D_t$$

  where X is a (local) martingale.

(*) that is true for all processes which do not have certain jumps at dividend dates.

Discrete Local Volatility

Discrete Local Volatility for Affine Dividends

– Assume we are given call prices C on S.
As shown in [B10], we derive "pure" call prices CC for X by the simple transformation

$$CC(t,k) := \mathrm{E}[(X_t - k)^+] := \frac{C(t, k(F_t - D_t) + D_t)}{\mathrm{DF}_t}$$

Observation:

– If an implied volatility surface is marked with "Black-Scholes" proportional dividends only, then there is no arbitrage-free process with (truly) affine dividends which fits the continuous option price surface.

Result:

– Our Approach provides in a well-defined way the "closest" affine dividend arbitrage-free option price surface given to an input surface market with proportional dividends.
– Extension of this statement to simple credit risk with deterministic default intensity is trivial along the arguments presented in [B10].

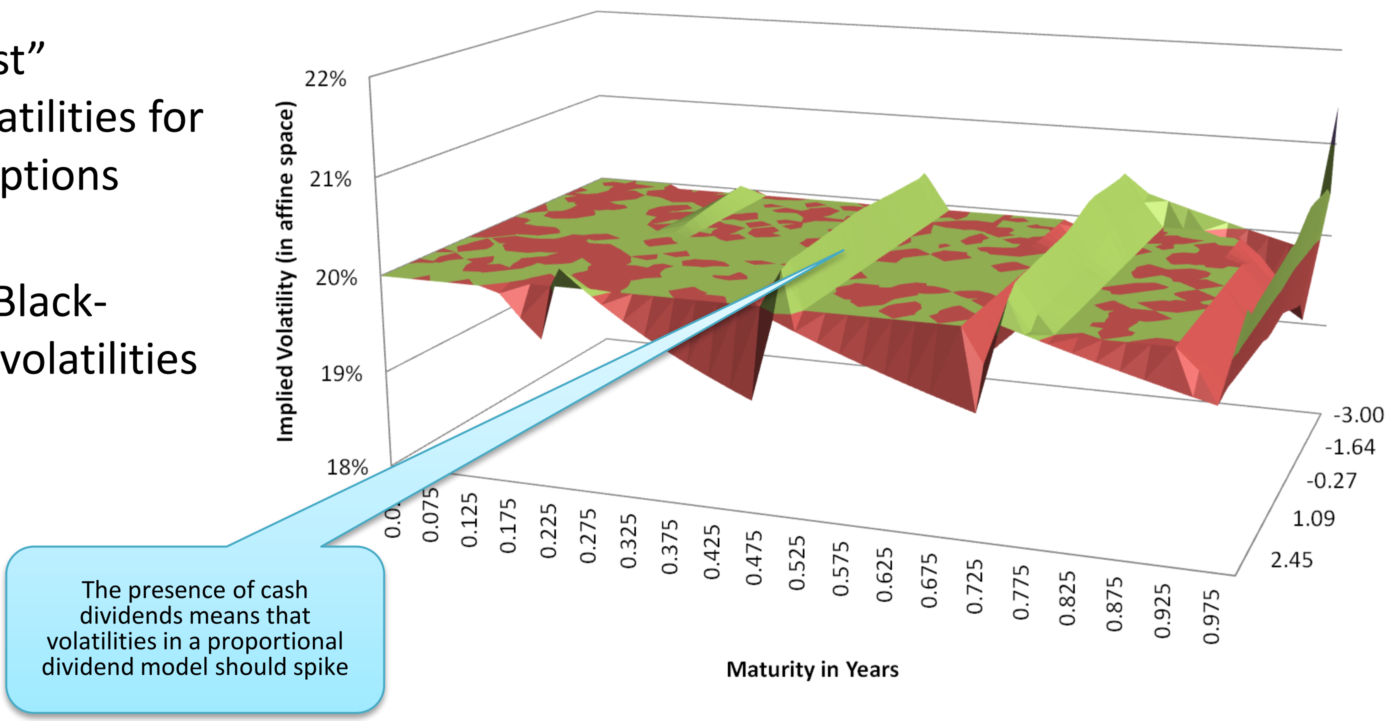
Examples

J.P.Morgan

– Synthetic asset, quarterly cash dividends of 5% for 1.5Y
– All options priced @ Black-Scholes with vol set at 20% → inconsistent with cash dividend assumption

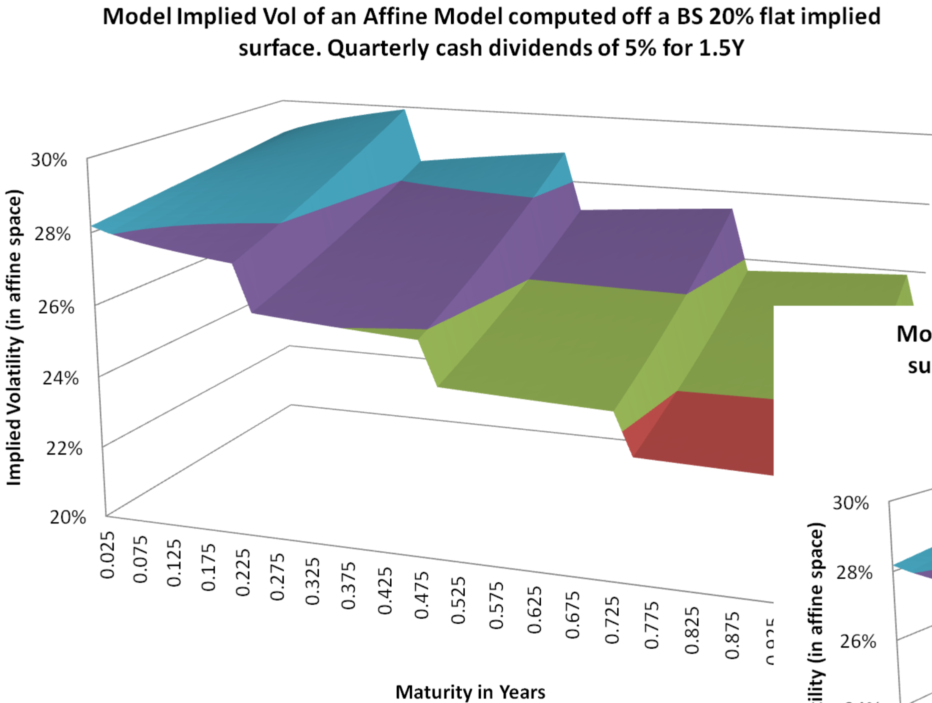
1. Convert into "pure" implied volatilities for X
2. Find new "closest" no-arbitrage volatilities for X and re-price options in cash-space
3. Compute again Black-Scholes implied volatilities for S itself

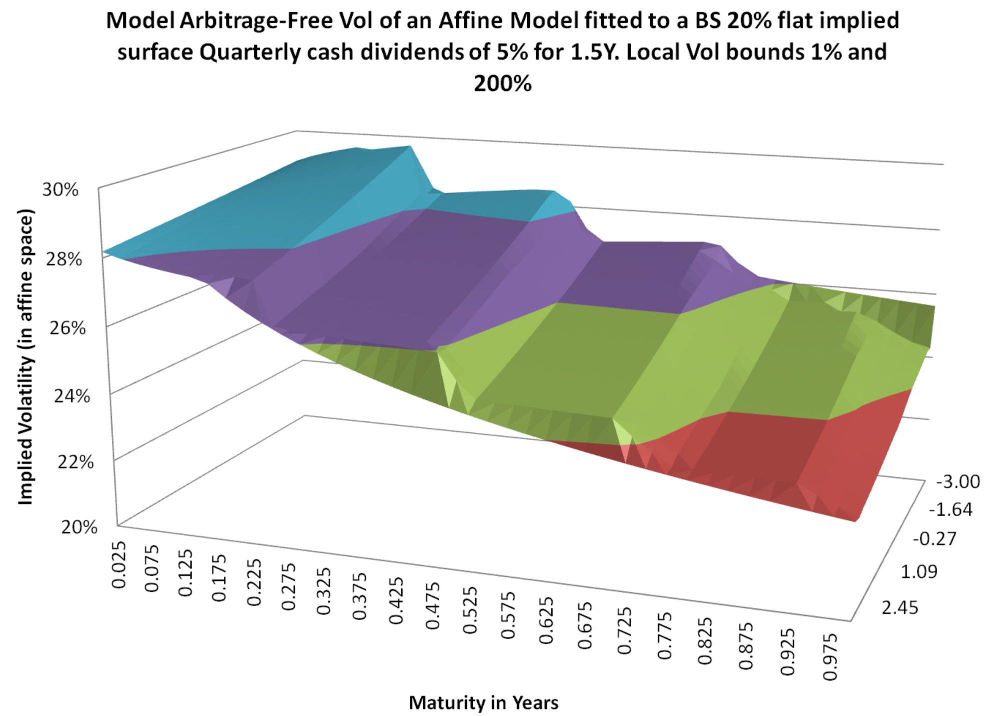
BS Implied Vol of an Arbitrage-Free fit under an Affine Model fitted to a BS 20% flat implied surface Quarterly cash dividends of 5% for 1.5Y. Local Vol bounds 1% and 200%



Examples

FYI corresponding “model” volatilities for X under a cash dividend model





Practical Implementation

Risk in Monte-Carlo

- We have constructed a discrete Markov martingale with transition kernels Π. The joint density for a sample path $(x_0,\dots,x_m)$ is

$$P(x_1,\dots,x_m \mid x_{0,}) := \Pi_1(x_1 \mid x_0)\cdots\Pi_m(x_m \mid x_{m-1})$$

- Classic approach for risk: re-sample paths with same random numbers.
- However, if we know all transition densities, we may use **Likelihood Ratio** greeks:
  - Assume that under perturbation, our "perturbed" kernels are Π^*, such that the joint path density P^* is given as

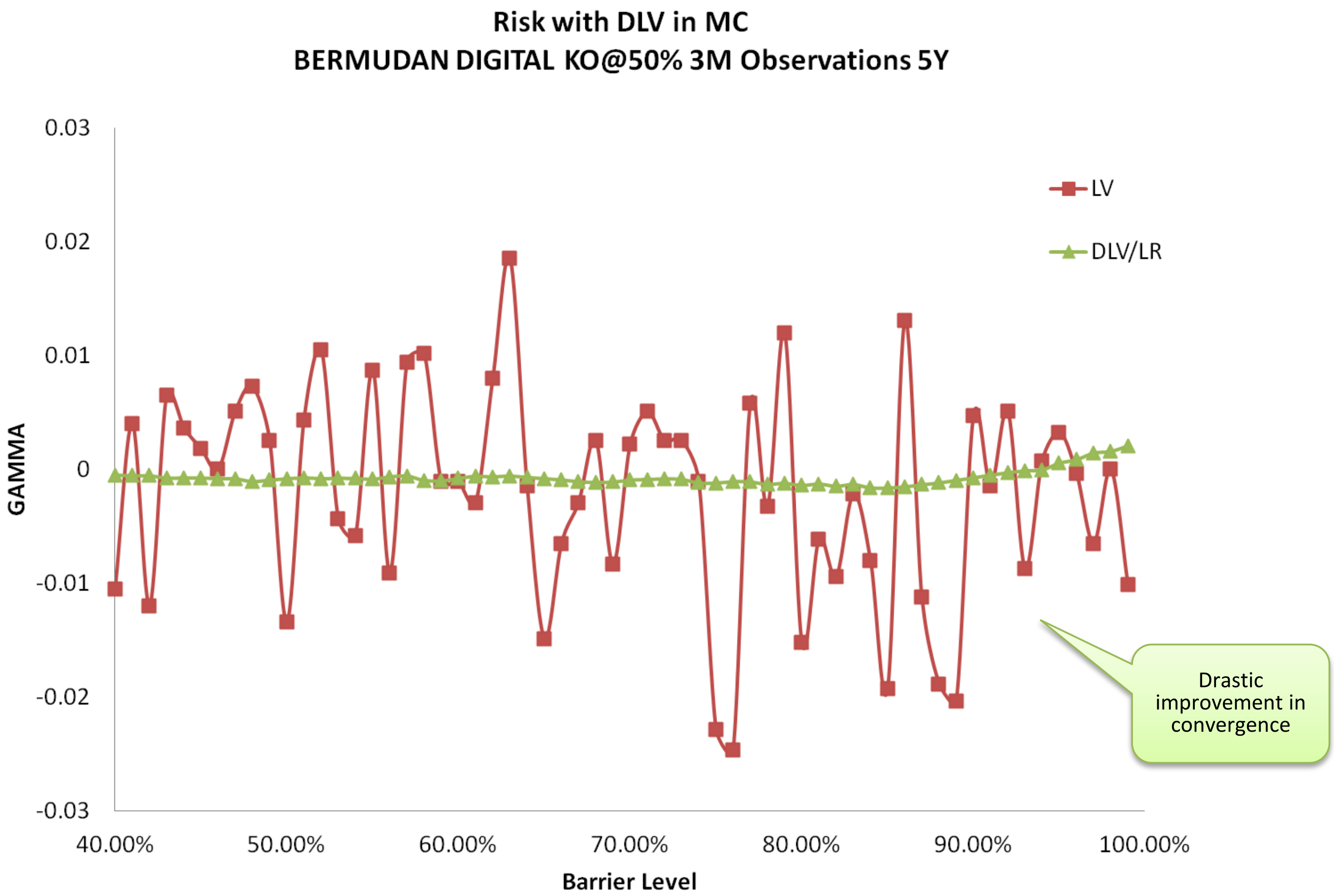
$$P^*(x_1,\dots,x_m \mid x_{0,}) := \Pi_1^*(x_1 \mid x_0)\cdots\Pi_m^*(x_m \mid x_{m-1})$$

  - Then, we may compute the perturbed price of a payoff F using

$$\begin{aligned}
\mathrm{E}^*\left[F(X_{0,}X_1,\dots,X_m)\right] &= \int F(x_{0,}x_1,\dots,x_m)P^*(x_1,\dots,x_m \mid x_{0,})dx \\
&= \int F(x_{0,}x_1,\dots,x_m)\frac{P^*(x_1,\dots,x_m \mid x_{0,})}{P(x_1,\dots,x_m \mid x_{0,})}P(x_1,\dots,x_m \mid x_{0,})dx \\
&= \mathrm{E}\left[F(X_{0,}X_1,\dots,X_m)\frac{P^*(X_1,\dots,X_m \mid x_{0,})}{P(X_1,\dots,X_m \mid x_{0,})}\right]
\end{aligned}$$

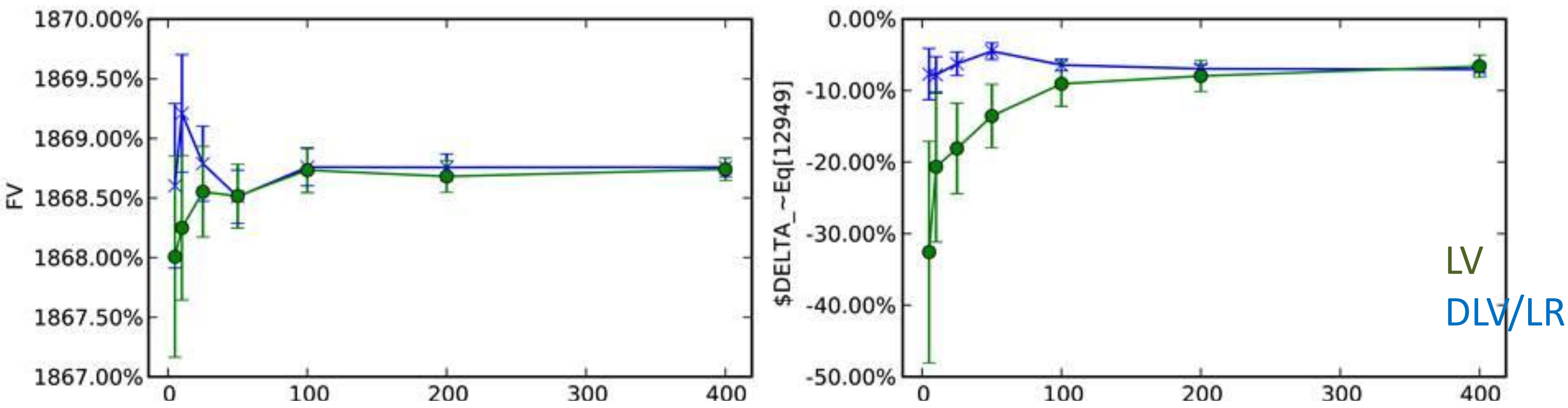
- This is computationally much more efficient than re-sampling paths.
- The noise of this method is generally lower than with perturbation, except for short-end exposures for products which have short-end fixings (e.g. Delta or Front-Term-Vega for daily barriers which are close to ATM).

Examples

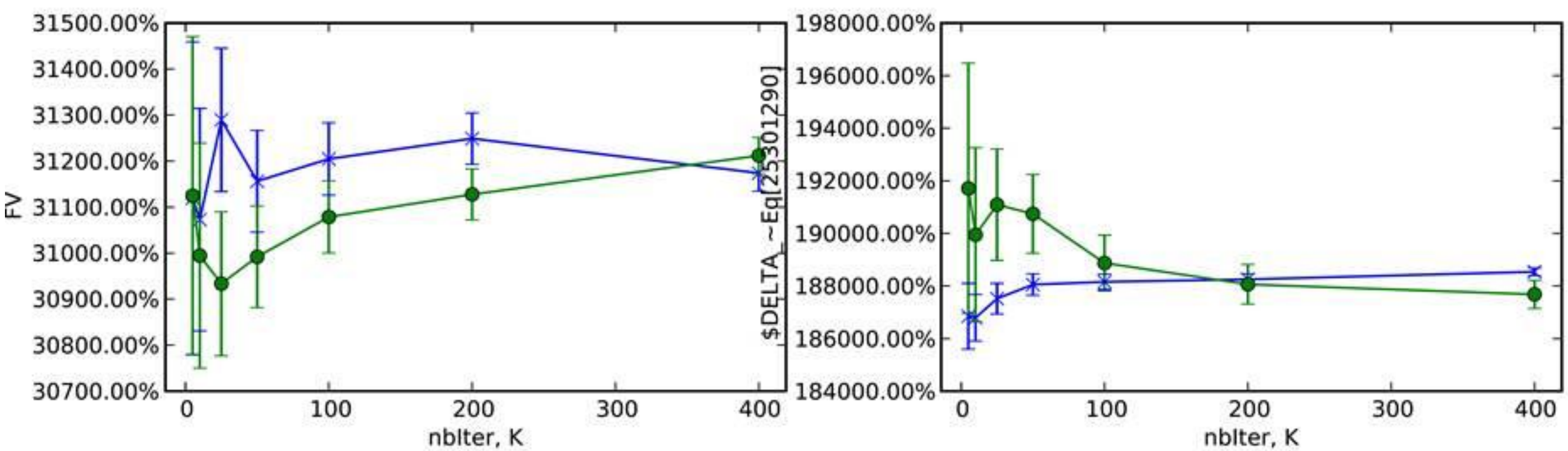

Risk with DLV in MC
BERMUDAN DIGITAL KO@50% 3M Observations 5Y
0.03
0.02
0.01
0
-0.01
-0.02
-0.03
GAMMA
40.00%
50.00%
60.00%
70.00%
80.00%
90.00%
100.00%
Barrier Level
LV
DLV/LR
Drastic improvement in convergence

Examples

1. Convergence test FLEX_1D (*)



2. Convergence test VAN

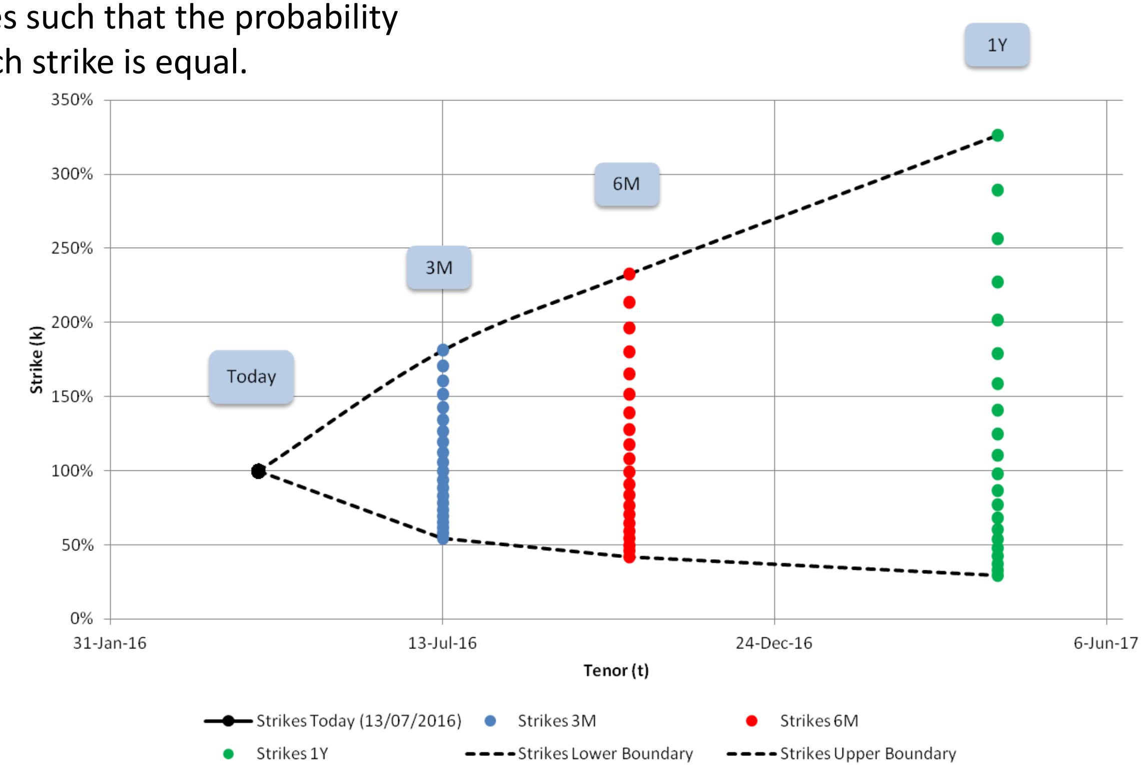


Exotic example found in production: call on a systematic rebalancing strategy

Strike Selection

Strike placement

- Place strikes on listed or otherwise important points
- Place strikes around barriers to avoid noise during greek calculations with perturbation
- Add additional strikes which "scale" with maturity
  Mini algorithm:
  - Calibrate model for ATM-normalized strikes for wide stddev.
  - Find new strikes such that the probability of reaching each strike is equal.



Strike Selection

J.P.Morgan

Path of the scaled process

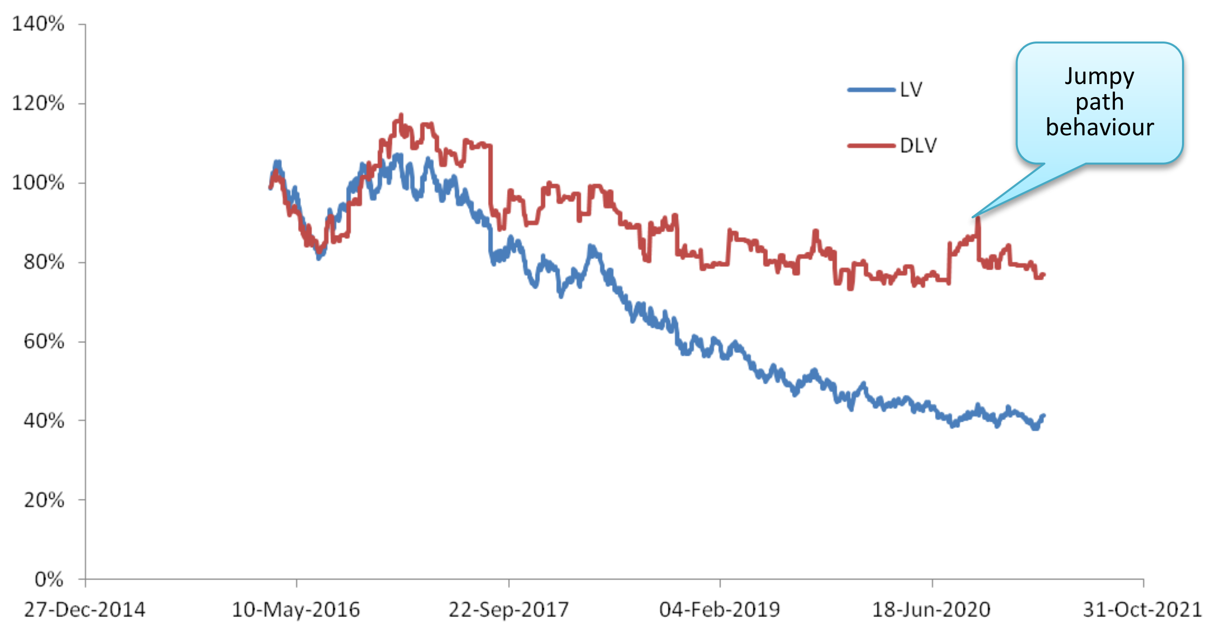
– The original aim was to approximate a process of the form

$$d\log X_t = \sigma_t(X_t)dW_t - \tfrac{1}{2}\sigma_t(X_t)^2 dt$$

– By using increasing strikes, we are essentially modelling the equivalent of the continuous process

$$Y_t = \frac{X_t}{\sqrt{\mathrm{var}(X_t)}} \approx \int \frac{\sigma_t(X_t)}{\sqrt{t}}dW_t - \ldots \to \int \frac{1}{\sqrt{t}}dW_t$$

which means that the effective forward-distribution degenerates with time.
In the case of our discrete process, it means that our forward increments become more "jumpy" – the process is less likely to change state, but if it does it jumps (relatively) far.



Skew via Jumps

Jumps

J.P.Morgan

QUANTITATIVE RESEARCH

Recall the Construction of our Time Interpolation Scheme:

1. Assume we are at t_{j-1}.
2. Using c_{j-1}, interpolate call prices cc_{j-1} for the strikes k_j of the next maturity → this gives us valid option prices over k_j.
3. Use the Δ operator to compute the respective density $q_{j-1} := \Delta^2 \, cc_{j-1}$.

J.P.Morgan

QUANTITATIVE RESEARCH

Jumps

– Assume we wish to a model a martingale of the "prior" form

$$X_t = Y_t U_t$$

where Y is a local volatility process and where U is a jump process whose jumps occurrences are independent of Y, e.g. the classic "Merton"-case:

$$U_t = \exp\left\{\int_0^t \xi_t dN_t - \Lambda dt\right\}$$

- ξ is an iid sequence of jumps independent of N and W,
- N is a Poisson-process with intensity λ, and $\Lambda := \mathrm{E}[1-e^{\xi}]\lambda$.

– For the period $[t_{j-1}, t_j]$ let

$$u_{j-1} := \exp\left\{\xi dN_{t_j} - \Lambda dt_j^-\right\}$$

– Define then the convolution

$$cc_{j-1}^i := \mathrm{E}\left[(X_{j-1} u_{j-1} - k_j^i)^+\right]$$

– Proceed as before to define $q_{j-1} := \Delta^2\, cc_{j-1}$.

– The resulting linear operator $\mathrm{u}\Xi_{j-1} : p_{j-1} \to q_{j-1}$ is again a transition matrix.

Summary

Discrete Local Volatility:

– Given a potentially sparse set of strikes and maturities we constructed the transition matrices of a discrete state martingale, which has the following properties:

1. **Fixes Arbitrage:**
   If the input data is arbitragable - for example during Stress calculations -, we find efficiently a *globally* L^1-closest fit to the input data, with higher weights for observed market prices vs. interpolated data or points with large bid/ask.
   This method is useful independently in order to manage arbitrage violations.

2. **Large Steps:**
   Allows taking large steps, fully consistently between forward (MC) and backward (FD) schemes.

3. **Small Steps:**
   Allows taking small steps, fully consistent with the large step transition operators.

4. **Risk by Strike:**
   Our approach allows for a clear definition and implementation of Vega risk by strike/maturity.

– We applied our approach to pricing under affine dividends and we commented on introducing skew with jumps

Thank you very much for your attention

hans.buehler@jpmorgan.com
evgeny.ryskin@jpmorgan.com

References

[AH11] Andreasen, Huge: ”Random Grids", Risk 24.7 (Jul 2011): 62-67.

[B10] Buehler, “Volatility and Dividends - Volatility Modelling with Cash Dividends and Simple Credit Risk", WP February 2010

[BR15] Buehler, Ryskin, “Discrete Local Volatility for Large Time Steps”, WP November 2015

[D96] Dupire, “Pricing with a Smile", Risk, 7 (1), pp. 18-20, 1996