Chapter 7 Capital Asset Pricing and Arbitrage Pricing Theory

Chapter Objectives

- Use the implications of capital market theory to compute security risk premiums.
- Construct and use the security market line.
- Specify and use a multifactor security market line.
- Take advantage of an arbitrage opportunity with a portfolio that includes mispriced securities.
- Use arbitrage pricing theory with more than one factor to identify mispriced securities.

7.1 THE CAPITAL ASSET PRICING MODEL

- Historically, the CAPM was developed prior to the index model introduced in the previous chapter (Equation 6.11).
  - The index model was widely adopted as a natural description of the stock market immediately on the heels of the CAPM because the CAPM implications so neatly match the intuition underlying the model.
    - ✓ So it makes sense to use the index model to help understand the lessons of the CAPM.
- The index model describes an empirical relationship between the excess return on an individual stock, R_i, and that of a broad market-index portfolio, R_M: $R_i = \beta_i R_M + \alpha_i + e_i$, where alpha is the expected firm-specific return and e_i is zero-mean "noise," or firm-specific risk.
  - Therefore, the expected excess return on a stock, given (conditional on) the market excess return, R_M, is $E(R_i|R_M) = \beta_i R_M + \alpha_i$.

- What does this mean to portfolio managers?
  - Hunt for positive-alpha stocks, don't invest in negative-alpha stocks, and, better yet, sell short negative-alpha stocks if short sales are not prohibited.
  - Investor demand for a positive-alpha stock will increase its price. As the price of a stock rises, other things being equal, its expected return falls, reducing and ultimately eliminating the very alpha that first created the excess demand.
  - Conversely, the drop in demand for a negative-alpha stock will reduce its price, pushing its alpha back toward zero.
  - In the end, such buying or selling pressure will leave most securities with zero alpha values most of the time. Put another way, unless and until your own analysis of a stock tells you otherwise, you should assume alpha is zero.
- When alpha is zero, there is no reward from bearing firm-specific risk; the only way to earn a higher expected return than the T-bill rate is by bearing systematic risk.

- Recall that Treynor-Black model, in which the position in any active portfolio is zero if the alpha is zero.
  - In that case, the best portfolio is the one that completely eliminates nonsystematic risk, and that portfolio is an indexed portfolio that mimics the broad market.
    - ✓ This is the conclusion of the CAPM.
- But science demands more than a story like this.
  - It requires a carefully set up model with explicit assumptions in which an outcome such as the one we describe will be the only possible result. Here goes.

The Model: Assumptions and Implications

- The **capital asset pricing model,** or **CAPM,** was developed by Treynor, Sharpe, Lintner, and Mossin in the early 1960s, and further refined later.
  - The model predicts the relationship between the risk and equilibrium expected returns on risky assets.

- We begin by laying down the necessary, albeit unrealistic, assumptions that underlie the model.
  - ✓ Thanking about an admittedly unrealistic world allows a relatively easy leap to the solution.
  - ✓ With this accomplished, we can add realism to the environment, one step at a time, and see how the theory must be amended.
  - ✓ This process allows us to develop a reasonable realistic model.

- The conditions that lead to the CAPM ensure competitive security markets, where investors will choose identical efficient portfolios using the mean-variance criterion:
  - 1. Markets for securities are perfectly competitive and equally profitable to all investors.
    - ✓ 1.A. No investor is sufficiently wealthy that his or her actions alone can affect market prices.
    - ✓ 1.B. All information relevant to security analysis is publicly available at no cost.
    - ✓ 1.C. All securities are publicly owned and traded, and investors may trade all of them. Thus, all risky assets are in the investment universe.

- ✓ 1.D. There are no taxes on investment returns. Thus, all investors realize identical returns from securities.
- ✓ 1.E. Investors confront no transaction costs that inhibit their trading.
- ✓ 1.F. Lending and borrowing at a common risk-free rate are unlimited.

◆ 2. Investors are alike in every way except for initial wealth and risk aversion; hence, they all choose investment portfolios in the sample manner.

- ✓ 2.A. Investors plan for the same (single-period) horizon.
- ✓ 2.B. Investors are rational, mean-variance optimizers.
- ✓ 2.C. Investors are efficient users of analytical methods, and by assumption 1.B the have access to all relevant information.
  - ➢ Hence, they use the same inputs and consider identical portfolio opportunity sets.
  - ➢ This assumption is often called *homogeneous expectations*.

■ Obviously, these assumptions ignore many real-world complexities. However, they lead to some powerful insights into the nature of equilibrium in security markets.

- Given these assumptions, we summarize the equilibrium that will prevail in this hypothetical world of securities and investors. We elaborate on these implications in the following sections.
  - 1. All investors will choose to hold the **market portfolio (*M*),** which includes all assets of the security universe.
    - ✓ For simplicity, we shall refer to all assets as stocks.
    - ✓ The proportion of each stock in the market portfolio equals the market value of the stock (price per share times the number of shares outstanding) divided by the total market value of all stocks.
  - 2. The market portfolio will be on the efficient frontier.
    - ✓ Moreover, it will be the optimal risky portfolio, the tangency point of the capital allocation line (CAL) to the efficient frontier.
    - ✓ As a result, the capital market line (CML), the line from the risk-free rate through the market portfolio, *M*, is also the best attainable capital allocation line.
    - ✓ All investors hold *M* as their optimal risky portfolio, differing only in the amount invested in it versus in the risk-free asset.

- 3. The risk premium on the market portfolio will be proportional to the variance of the market portfolio and investors' typical degree of risk aversion. Mathematically

$$E(r_M) - r_f = \overline{A}\sigma_M^2 \quad \Rightarrow \quad E(r_M) = r_f + \overline{A}\sigma_M^2 \tag{7.1}$$

  where σ_M is the standard deviation of the return on the market portfolio and $\overline{A}$ represents the degree of risk aversion of the average investor.

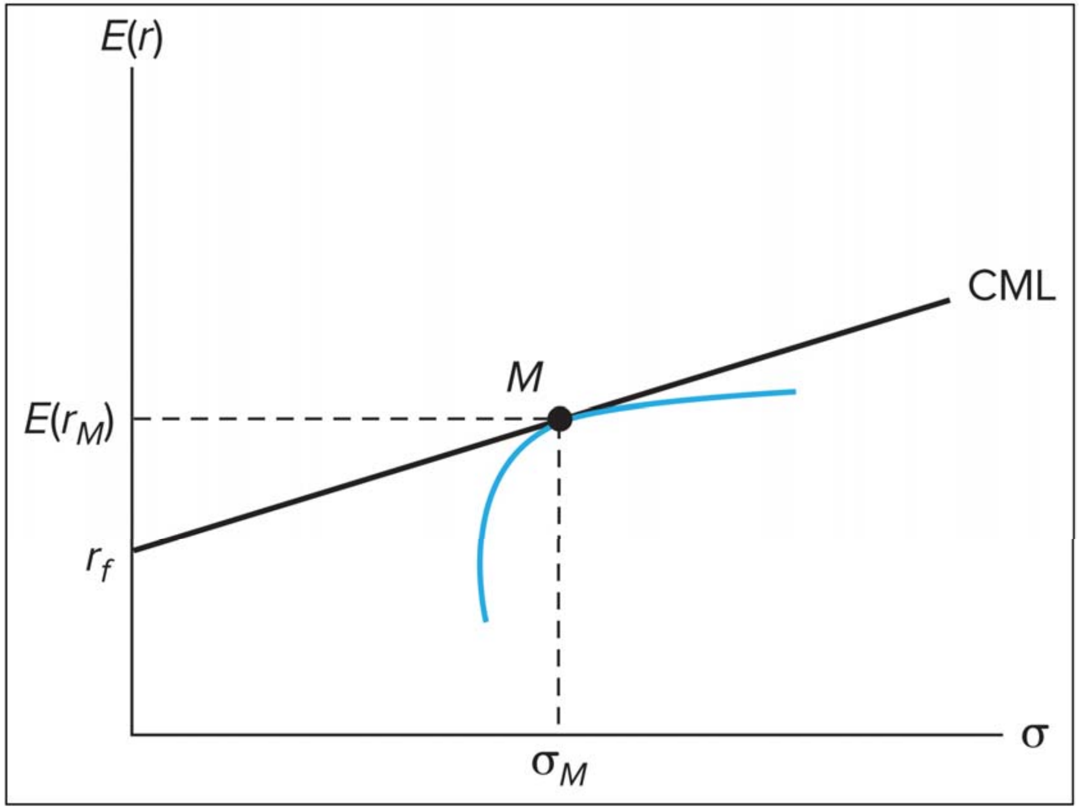
- 4. The risk premium on individual assets will be proportional to the risk premium on the market portfolio (*M*) and to the *beta coefficient* of the security on the market portfolio.
  - ✓ Beta measures the extent to which returns respond to the market portfolio.
  - ✓ Formally, beta is the regression (slope) coefficient of the security return on the market return, representing sensitivity to fluctuations in the overall security market.

Why All Investors Would Hold the Market Portfolio

- Given all our assumptions, it is easy to see why all investors hold identical risky portfolios.

- ◆ If all investors use mean-variance analysis (assumption 2.B), apply it to the same universe of securities (assumptions 1.C and 1.F) with an identical time horizon (assumption 2.A), use the same security analysis (assumption 2.C), and experience identical net returns from the same securities (assumptions 1.A, 1.D, and 1.E), they all must arrive at the same determination of the optimal risky portfolio.

- ■ With everyone choosing to hold the same risky portfolio, stocks will be represented in the aggregate risky portfolio in the same proportion as they are in each investor's (common) risky portfolio.
  - ◆ If Google represents 1% in each common risky portfolio, Google will be 1% of the aggregate risky portfolio.
    - ✓ This in fact is the market portfolio since the market is no more than the aggregate of all individual portfolios.
  - ◆ Because each investor uses the market portfolio for the optimal risky portfolio, the CAL in this case is called the *capital market line*, or CML, as in Figure 7.1.



- Suppose the optimal portfolio of our investors does not include the stock of some company, say, Southwest Air Lines.
  - When no investor is willing to hold Southwest stock, the demand is zero, and the stock price will take a free fall.

- As Southwest stock gets progressively cheaper, it begins to look more attractive, while all other stocks look (relatively) less attractive.
- Ultimately, Southwest will reach a price at which it is desirable to include it in the optimal stock portfolio, and investors will buy.

■ This price adjustment process guarantees that all stocks will be included in the optimal portfolio.

- The only issue is the price.
  - ✓ At a given price level, investors will be willing to buy a stock; at another price, they will not.
- The bottom line is this: If all investors hold an *identical* risky portfolio, this portfolio must be the *market* portfolio.

● The Passive Strategy Is Efficient

■ The CAPM implies that a passive strategy, using the CML as the optimal CAL, is a powerful alternative to an active strategy.

- The market portfolio proportions are a result of profit-oriented "buy" and "sell" orders that cease only when there is no more profit to be made.

- ◆ And in the simple world of the CAPM, all investors use precious resources in security analysis.
  - ✓ A passive investor who takes a free ride by simply investing in the market portfolio benefits from the efficiency of that portfolio.
  - ✓ In fact, an active investor who chooses any other portfolio will end on a CAL that is inferior to the CML used by passive investors.

- ■ We sometimes call this result a **mutual fund theorem** because it implies that only one mutual fund of risky asset—the market index fund—is sufficient to satisfy the investment demands of all investors.
- ■ The mutual fund theorem is another incarnation of the separation property discussed in Chapter 6.
  - ◆ Assuming all investors choose to hold a market index mutual fund, we can separate portfolio selection into two components:
    - ✓ (l) a technical side, in which an efficient mutual fund is created by professional management

- ✓ (2) a personal side, in which an investor's risk aversion determines the allocation of the complete portfolio between the mutual fund and the risk-free asset
- ◆ Here, all investors agree that the mutual fund they would like to hold is invested in the market portfolio.

- ■ While different investment managers do create risky portfolios that differ from the market index, we attribute this in part to the difference in their estimates of risk and expected return (in violation of assumption 2.C).
  - ◆ Nevertheless, a passive investor may view the market index as a reasonable first approximation to an efficient risky portfolio.
- ■ The logical inconsistency of the CAPM is this:
  - ◆ If a passive strategy is costless *and* efficient, why would anyone follow an active strategy?
  - ◆ But if no one does any security analysis, what brings about the efficiency of the market portfolio?

- We have acknowledged from the outset that the CAPM simplifies the real world in its search for a tractable solution.
  - Its applicability to the real world depends on whether its predictions are accurate enough.
  - The model's use is some indication that its predictions are reasonable.
    - ✓ We discuss this issue in Section 7.3 and in greater depth in Chapter 8.

The Risk Premium of the Market Portfolio

- In Chapter 5 we showed how individual investors decide how much to invest in the risky portfolio when they can include a risk-free asset in the investment budget.
- Returning now to the decision of how much to invest in the market portfolio *M* and how much in the risk-free asset, what can we deduce about the equilibrium risk premium of portfolio *M*?
- We asserted earlier that the equilibrium risk premium of the market portfolio, $E(r_M) - r_f$, will be proportional to the degree of risk aversion of the average investor and to the risk of the market portfolio, σ_M^2.

- Now we can explain this result.
  - When investors purchase stocks, their demand drives up prices, thereby lowering expected rates of return and risk premiums.
  - But when risk premiums fall, investors will move some of their funds out from the risky market portfolio into the risk-free asset.
  - In equilibrium, the risk premium on the market portfolio must be just high enough to induce investors to hold the available supply of stocks.
  - If the risk premium is too high, there will be excess demand for securities, and prices will rise; if it is too low, investors will not hold enough stock to absorb the supply, and prices will fall.
  - The *equilibrium* risk premium of the market portfolio is therefore proportional to both the risk of the market, as measured by the variance of its returns, and to the degree of risk aversion of the average investor, denoted by $\overline{A}$ in Equation 7.1.

- Example 7.1: *Market Risk, the Risk Premium, and Risk Aversion*
  - Suppose the risk-free rate is 5%, the average investor has a risk-aversion coefficient of $\bar{A} = 2$, and the standard deviation of the market portfolio is 20%.
    - ✓ Then, from Equation 7.1, we estimate the equilibrium value of the market risk premium as $2 \times .20^2 = .08$.
    - ✓ So the expected rate of return on the market must be

$$E(r_M) = r_f + \text{Equilibrium risk premium} = .05 + .08 = .13 = 13\%$$

  - If investors were more risk averse, it would take a higher risk premium to induce them to hold shares.
    - ✓ For example, if the average degree of risk aversion were 3, the market risk premium would be $3 \times .20^2 = .12$, or 12%, and the expected return would be 17%.
  - Note that to use Equation 7.1, we must express returns in decimal form rather than as percentage.

Expected Returns on Individual Securities

- The CAPM is built on the insight that the appropriate risk premium on an asset will be determined by its contribution to the risk of investors' overall portfolios.
  - Portfolio risk is what matters to investors, and portfolio risk is what governs the risk premiums they demand.
- We know that nonsystematic risk can be reduced to an arbitrarily low level through diversification (Chapter 6).
  - Therefore, investors do not require a risk premium as compensation for bearing nonsystematic risk.
  - They need to be compensated only for bearing systematic risk, which cannot be diversified.
- We know also that the contribution of a single security to the risk of a large diversified portfolio depends only on the systematic risk of the security as measured by its beta, as we saw in Chapter 6, Section 6.5.

- Therefore, it should not be surprising that the risk premium of an asset is proportional to its beta; a security with double the systematic risk of another must pay twice the risk premium.
- Thus, the ratio of risk premium to beta should be the same for any two securities or portfolios.

■ If we equate the ratio of risk premium to systematic risk for the market portfolio, which has a beta of 1, to the corresponding ratio for a particular stocks, for example, Delta Air Lines, we find that

$$\frac{E(r_M) - r_f}{1} = \frac{E(r_D) - r_f}{\beta_D}$$

Rearranging results in the CAPM's **expected return-beta relationship:**

$$E(r_D) = r_f + \beta_D[E(r_M) - r_f] \qquad (7.2)$$

- ◆ In words, an asset's risk premium equals the asset's systematic risk measure (its beta) times the risk premium of the (benchmark) market portfolio.
- ◆ This expected return (or mean return)-beta relationship is the most familiar expression of the CAPM.

■ The mean-beta relationship of the CAPM makes a powerful economic statement.

- ◆ It implies, for example, that a security with a high variance but a relatively low beta of .5 will carry one-third the risk premium of a low-variance security with a beta of 1.5.
- ◆ Equation 7.2 quantifies the conclusion we reached in Chapter 6:
  - ✓ Only systematic risk matters to investors who can diversify, and systematic risk is measured by beta.

■ Example 7.2: *Expected Returns and Risk Premiums*

- ◆ Suppose the risk premium of the market portfolio is 8%, and we estimate the beta of Dell as $\beta_D = 1.2$.
  - ✓ The risk premium predicted for the stock is therefore 1.2 times the market risk premium, or $1.2 \times 8\% = 9.6\%$.

- ✓ The expected rate of return on Delta is the risk-free rate plus the risk premium.
  - ➢ For example, if the T-bill rate were 3%, the expected rate of return would be 3% + 9.6% = 12.6%, or using Equation 7.2 directly,

    $E(r_D) = r_f + \beta_D[\text{Market risk premium}] = 3\% + 1.2 \times 8\% = 12.6\%$

- ✓ If the estimate of the beta of Delta's beta were only 1.1, its required risk premium would fall to 8.8% (= 1.1 × 8%).
- ✓ Similarly, if the market risk premium were only 6% and β_D = 1.2, Delta's risk premium would be only 7.2% (1.2 × 6%).

■ The fact that many investors hold active portfolios that differ from the market portfolio does not necessarily invalidate the CAPM.

- ◆ Recall that reasonably well-diversified portfolios shed almost all firm-specific risk and are subject to only systematic risk.
- ◆ Even if one does not hold the precise market portfolio, a well-diversified portfolio will be so highly correlated with the market that a stock's beta relative to the market still will be a useful risk measure.

- If the mean-beta relationship holds for any individual asset, it must hold for any combination of assets.
  - The beta of a portfolio is simply the weighted average of the betas of the stocks in the portfolio, using as weights the portfolio proportions.
  - Thus, beta also predicts the portfolio's risk premium in accordance with Equation 7.2.
- Example 7.3: *Portfolio Beta and Risk Premium*
  - Consider the following portfolio:

Asset	Beta	Risk Premium	Portfolio Weight
Microsoft	1.2	9.0%	0.5
American Electric Power	0.8	6.0	0.3
Gold	0.0	0.0	0.2
Portfolio	0.84	?	1.0

- ◆ If the market risk premium is 7.5%, the CAPM predicts that the risk premium on each stock is its beta times 7.5%, and the risk premium on the portfolio is .84 × 7.5% = 6.3%.
  - ✓ This is the same result that is obtained by taking the weighted average of the risk premiums of the individual stocks. (Verify this for yourself.)

- ■ A word of caution:
  - ◆ We often hear that well-managed firms will provide high rate of return. We agree this is true if one measures the *firm's* accounting return on investments in plant and equipment.
  - ◆ The CAPM, however, predicts returns on investments in the *securities* of the firm that trade in capital markets.
    - ✓ Say that everyone knows a firm is well run. Its stock price will be bid up, and returns to stockholders at those high prices will not be extreme.
    - ✓ Security *prices* reflect public information about a firm's prospects, but only the risk of the company (as measured by beta) should affect *expected returns*.

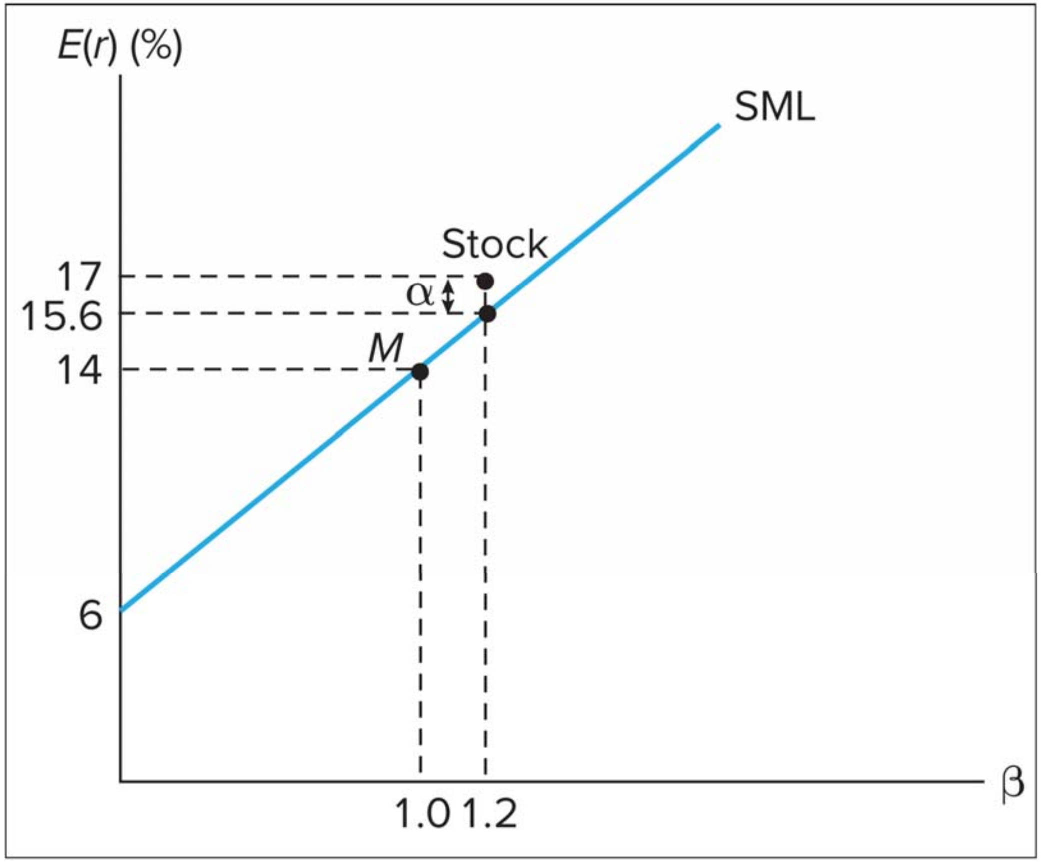
- ✓ In a rational market, investors receive high expected returns only if they bear systematic risk.

● The Security Market Line

- ■ The expected return-beta relationship or equivalently, the mean-beta relationship, is a reward-risk equation.
  - ◆ The beta of a security is the appropriate measure of its risk because beta is proportional to the variance the security contributes to the optimal risky portfolio.
  - ◆ The contribution of a security to portfolio variance equals the variance of the portfolio when the security is included minus the variance when the security is excluded, with the weights of all other securities increased proportionally to bring total weights to 1.
- ■ With approximately normal returns, we measure the stock's contribution to the standard deviation.
  - ◆ Because the beta of a stock measures the stock's contribution to the standard deviation of the market portfolio, we expect the required risk premium to be a function of beta.

- ◆ The CAPM confirms this intuition, stating further that the security's risk premium is directly proportional to both the beta and the risk premium of the market portfolio; that is, the risk premium equals $\beta[E(r_M) - r_f]$.

- ■ The mean-beta relationship is represented as the **security market line (SML)** in Figure 7.2.



- ◆ Its slope is the risk premium of the market portfolio.
- ◆ At the point where $\beta = 1.0$ (the beta of the market portfolio), we can read off the vertical axis the expected return on the market portfolio.

■ It is useful to compare the security market line to the capital market line.

- ◆ The CML graphs the risk premiums of efficient portfolios (made up of the risky market portfolio and the risk-free asset) as a function of portfolio standard deviation.
  - ✓ This is appropriate because standard deviation is a valid measure of risk for portfolios that are candidates for an investor's complete portfolio.
- ◆ The SML, in contrast, graphs *individual asset* risk premiums as a function of asset risk (which we measure by beta).
  - ✓ The relevant measure of risk for an individual asset (which is held as part of a well-diversified portfolio) is not the asset standard deviation but rather the asset beta.
  - ✓ The SML is valid both for individual assets and portfolios.

- The security market line provides a benchmark for evaluation of investment performance.
  - The SML provides the required rate of return that will compensate investors for the beta risk of that investment, as well as for the time value of money.
- Because the SML is the graphical representation of the mean-beta relationship, "fairly priced" assets plot exactly on the SML. The expected returns of such assets are commensurate with their risk.
  - Whenever the CAPM holds, all securities must lie on the SML.
  - Underpriced stocks plot above the SML.
    - ✓ Given beta, their expected returns are greater than is indicated by the CAPM.
  - Overpriced stocks plot below the SML.
  - The difference between the fair and actual expected rates of return on a stock is the **alpha,** denoted α.
  - The expected return on such a mispriced security is given by:

$$E(r_s) = \alpha_s + r_f + \beta_s[E(r_M) - r_f]$$

- Example 7.4: *The Alpha of a Security*
  - Suppose the return on the market is expected to be 14%, a stock has a beta of 1.2, and the T-bill rate is 6%.
    - ✓ The SML would predict an expected return on the stock of

$$E(r) = r_f + \beta[E(r_M) - r_f] = 6\% + 1.2(14\% - 6\%) = 15.6\%$$

    - ✓ If one believes the stock will provide instead a return of 17%, its implied alpha would be 1.4%, as shown in Figure 7.2.
    - ✓ If instead the expected return were only 15%, the stock alpha would be negative, -.6%.

Applications of the CAPM

- One place the CAPM may be used is in the investment management industry.
  - The SML provides a benchmark to assess the *fair* expected return on any risky asset.
    - ✓ Then an analyst calculates the return she actually expects.
    - ✓ Notice that we depart here from the simple CAPM world in that active investors apply their own analysis to derive a private "input list."

- ✓ If a stock is perceived to be a good buy, or underpriced, it will provide a positive alpha, that is, an expected return in excess of the fair return stipulated by the SML.

- ■ The CAPM is also useful in capital budgeting decisions.
  - ◆ When a firm is considering a new project, the SML provides the required return demanded of the project.
    - ✓ This is the cutoff internal rate of return (IRR) or "hurdle rate" for the project.
- ■ Example 7.5: *The CAPM and Capital Budgeting*
  - ◆ Suppose Silverado Springs Inc. is considering a new spring-water bottling plant.
    - ✓ The business plan forecasts an internal rate of return of 14% on the investment.
    - ✓ Research shows the beta of similar products is 1.3.
      - ➢ Thus, if the risk-free rate is 4%, and the market risk premium is estimated at 8%, the hurdle rate for the project should be $4\% + 1.3 \times 8\% = 14.4\%$.
    - ✓ Because the IRR is less than the risk-adjusted discount or hurdle rate, the project has a negative net present value and ought to be rejected.

- Yet another use of the CAPM is in utility rate-making cases.
  - Here the issue is the rate of return a regulated utility should be allowed to earn on its investment in plant and equipment.
- Example 7.6: *The CAPM and Regulation*
  - Suppose shareholder equity invested in a utility is \$100 million, and the beta of the equity is .6.
  - If the T-bill rate is 6%, and the market risk premium is 8%, then a fair annual profit will be $6\% + (.6 \times 8\%) = 10.8\%$ of \$100 million, or \$10.8 million.
  - Because regulators accept the CAPM, they will allow the utility to set prices at a level expected to generate these profits.

7.2 THE CAPM AND INDEX MODELS

- The CAPM has two limitations: It relies on the theoretical market portfolio, which includes *all* assets (such as real estate, foreign stocks, etc.), and it deals with *expected* as opposed to actual returns.
  - To implement the CAPM, we cast it in the form of an *index model* and use realized, not expected, returns.
- An index model replaces the theoretical all-inclusive portfolio with a market index such as the S&P 500.
  - The composition and rate of return of the index are unambiguous and widely published, and therefore provide a clear benchmark for performance evaluation.
- We can start with one central prediction of the CAPM: The market portfolio is mean-variance efficient.
  - An index model can be used to test this hypothesis by verifying that an index designed to represent the full market is mean-variance efficient.

- ◆ To test mean-variance efficiency of an index, we must show that its Sharpe ratio is not surpassed by any other portfolio.
  - ✓ We will examine tests of this equation in the next chapter.

■ The CAPM predicts relationships among *expected* returns. However, all we can observe are realized (historical) holding-period returns, which in a particular holding period seldom, if ever, match initial expectations.

- ◆ For example, the S&P 500 returned -39% in 2008. Could such a large negative return possibly have been expected when investors could have invested in risk-free Treasury bills?
  - ✓ In fact, this logic implies that any stock-index return less than T-bills must entail a negative departure from expectations.
  - ✓ Because expectations must be realized on average, this means that more often than not, positive excess returns exceeded expectations.

The Index Model, Realized Returns, and the Mean-Beta Relationship

- To move from a model cast in expectations to a realized-return framework, we start with a form of the single-index equation in realized excess returns, Equation 6.11:

$$r_{it} - r_{ft} = \alpha_i + \beta_i(r_{Mt} - r_{ft}) + e_{it} \qquad (7.3)$$

where r_{it} is the holding-period return (HPR) on asset i in period t, and α_i and β_i are the intercept and slope of the security characteristic line that relates asset i's realized excess return to the realized excess return of the index.

  - We denote the index return by r_M to emphasize that the index is proxying for the market portfolio.
  - The e_{it} measures firm-specific effects during the holding period t; it is the deviation of security i's realized HPR from the regression line, the forecast of return based on the index's actual HPR.
  - We set the relationship in terms of *excess* returns (over the risk-free rate, r_{ft}), consistent with the CAPM's logic of risk premiums.

- To compare the index model with the CAPM predictions about expected asset returns, we take expectations in Equation 7.3.
- Given that the CAPM is a statement about the expectation of asset returns, we look at the expected return of security *i* predicted by Equation 7.3.
  - Recall that the expectation of e_{it} is zero (the firm-specific surprise is expected to average zero over time), so in terms of expectations, Equation 7.3 becomes

$$E(r_{it}) - r_{ft} = \alpha_i + \beta_i[E(r_{Mt}) - r_{ft}] \qquad (7.\ 4)$$

  - Comparing Equation 7.4 to Equation 7.2 reveals that the CAPM predicts $\alpha_i = 0$.
    - ✓ Thus, we have converted the CAPM prediction about unobserved expectations of security returns relative to an unobserved market portfolio into a prediction about the intercept in a regression of observed variables: realized excess returns of a security relative to those of an observed index.

- Operationalizing the CAPM in the form of an index model has a drawback, however.
  - If intercepts of regressions of returns on an index differ substantially from zero, you will not be able to tell whether it is because you chose a bad index to proxy for the market or because the theory is not useful.

Estimating the Index Model

- To illustrate how to estimate the index model, we will use actual data and apply the model to the stock of Google (G), in a manner similar to that followed by practitioners.
- Let us rewrite Equation 7.3 for Google, denoting Google's excess return as $R_G = r_G - r_f$ and denoting month using the subscript t.

$$R_{Gt} = \alpha_G + \beta_G R_{Mt} + e_{Gt}$$

  - As noted, this relationship may be viewed as a regression equation.
    - ✓ The dependent variable in this case is Google's excess return in each month, explained by the excess return on the market index in that month, R_{Mt}.
    - ✓ The regression coefficients are intercept α_G and slope β_G.

- The alpha of Google is the average of the firm-specific factors during the sample period; the zero-average surprise in each month is captured by the last term in the equation e_{Gt}.
  - ✓ This residual is the difference between Google's actual excess return and the excess return that would be predicted from the regression line:

    Residual = Actual excess return – Predicted excess return for Google

    $$\Rightarrow \quad e_{Gt} = R_{Gt} - (\alpha_G + \beta_G R_{Mt})$$

- We are interested in estimating the intercept α_G and Google's beta as measured by the slope coefficient, β_G.
  - ✓ We estimate Google's firm-specific risk by *residual standard deviation*, which is just the standard deviation of e_{Gt}.

■ We conduct the analysis in three steps:

- Collect and process relevant data.
- Feed the data into a statistical program (here we will use Excel) to estimate and interpret the regression Equation 7.3.

- Use the results to answer these questions about Google's stock.
  - ✓ For example, we will consider the following questions:
    - (*a*) What have we learned about the behavior of Google's returns?
    - (*b*) What required rate of return is appropriate for investment with the same risk as Google's equity?
    - (*c*) How might we assess the performance of a portfolio manager who invested heavily in Google stock during this period?

COLLECTING AND PROCESSING DATA

- We start with the monthly series of Google stock prices and the S&P 500 Index, adjusted for stock splits and dividends over the period January 2006-December 2010.
  - From these series we computed monthly holding-period returns on Google and the market index.
  - For the same period we compiled monthly rates of return on one-month T-bills, which will serve as the risk-free rate.

- With these three series of returns we generate monthly excess return on Google's stock and the market index.
  - Some statistics for these returns are shown in Table 7.1.
  - Notice that the monthly variation in the T-bill return reported in Table 7.1 does not reflect risk, as investors knew the return on bills at the beginning of each month.

TABLE 7.1 Monthly return statistics: T-bills, S&P 500, and Google, January 2006–December 2010

Statistic (%)	T-bills	S&P 500	Google
Average rate of return	0.184	0.239	1.125
Average excess return	–	0.054	0.941
Standard deviation*	0.177	5.11	10.40
Geometric average	0.180	0.107	0.600
Cumulative total 5-year return	11.65	6.60	43.17
Gain Jan 2006–Oct 2007	9.04	27.45	70.42
Gain Nov 2007–May 2009	2.29	−38.87	−40.99
Gain June 2009–Dec 2010	0.10	36.83	42.36

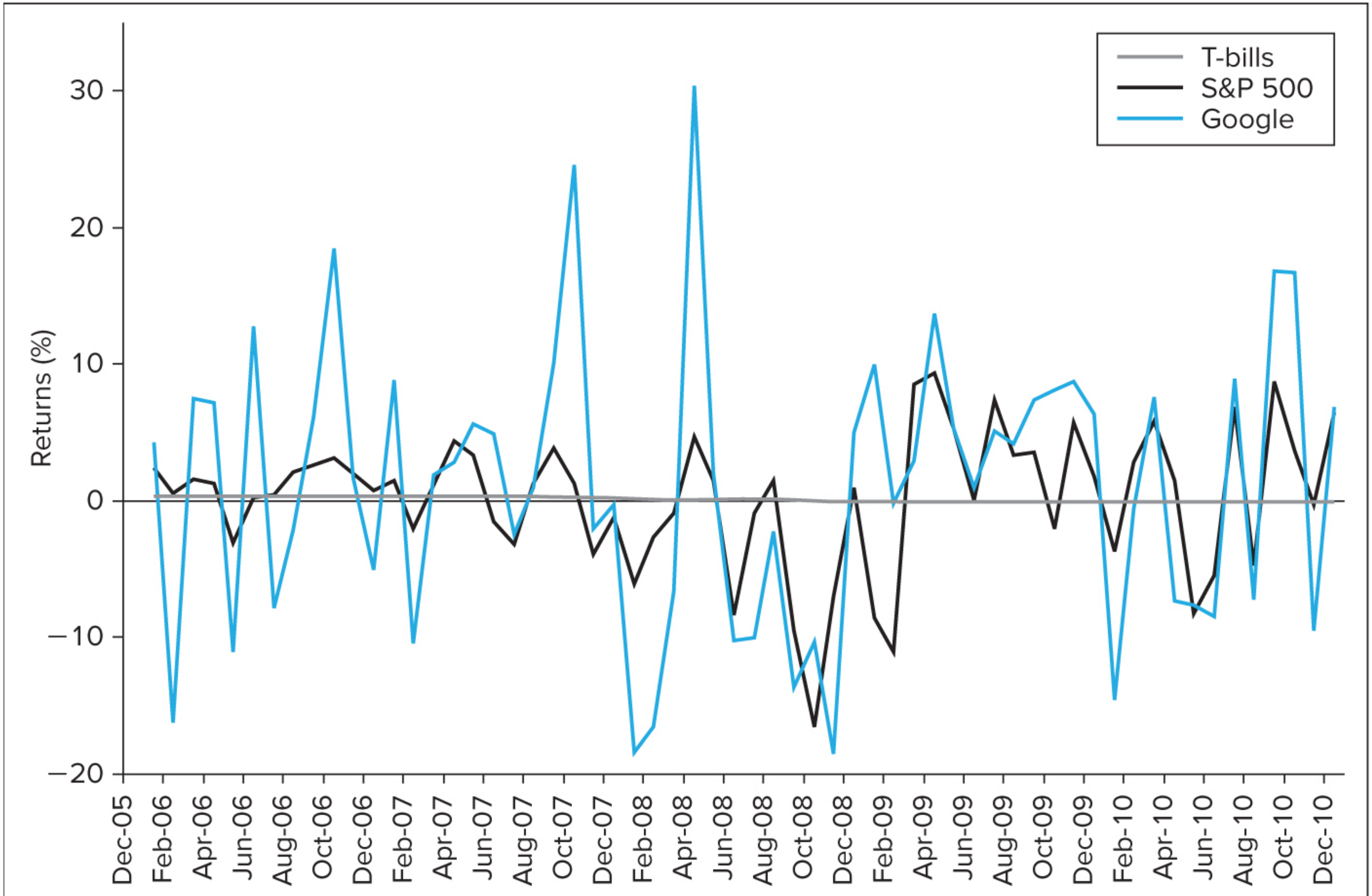
*The rate on T-bills is known in advance; hence SD does not reflect risk.

- The period of January 2006-December 2010 includes the late stage of recovery from the mild 2001 recession, the severe recession that officially began in December 2007, as well as the first stage of the recovery that began in June 2009.

- Table 7.1 shows that the effect of the financial crisis was so severe that the monthly geometric average return of the market index, .107%, was less than that of T-bills, .180%.
  - We noted that in Chapter 5 that arithmetic averages exceed geometric averages, with the difference between them increasing with return volatility.
  - In this period, the monthly SD of the market index, 5.11%, was large enough that despite the market return's lower geometric average, its monthly arithmetic average, .239%, was greater than that of T-bills, .185%, resulting in a positive average excess return of .054% per month.
  - Google had a cumulative five-year return of 43.17%, a lot better than T-bills (11.65%) or the S&P 500 (6.60%). Its monthly standard deviation of 10.40%, about double that of the market, raises the question of how much of that volatility is systematic.

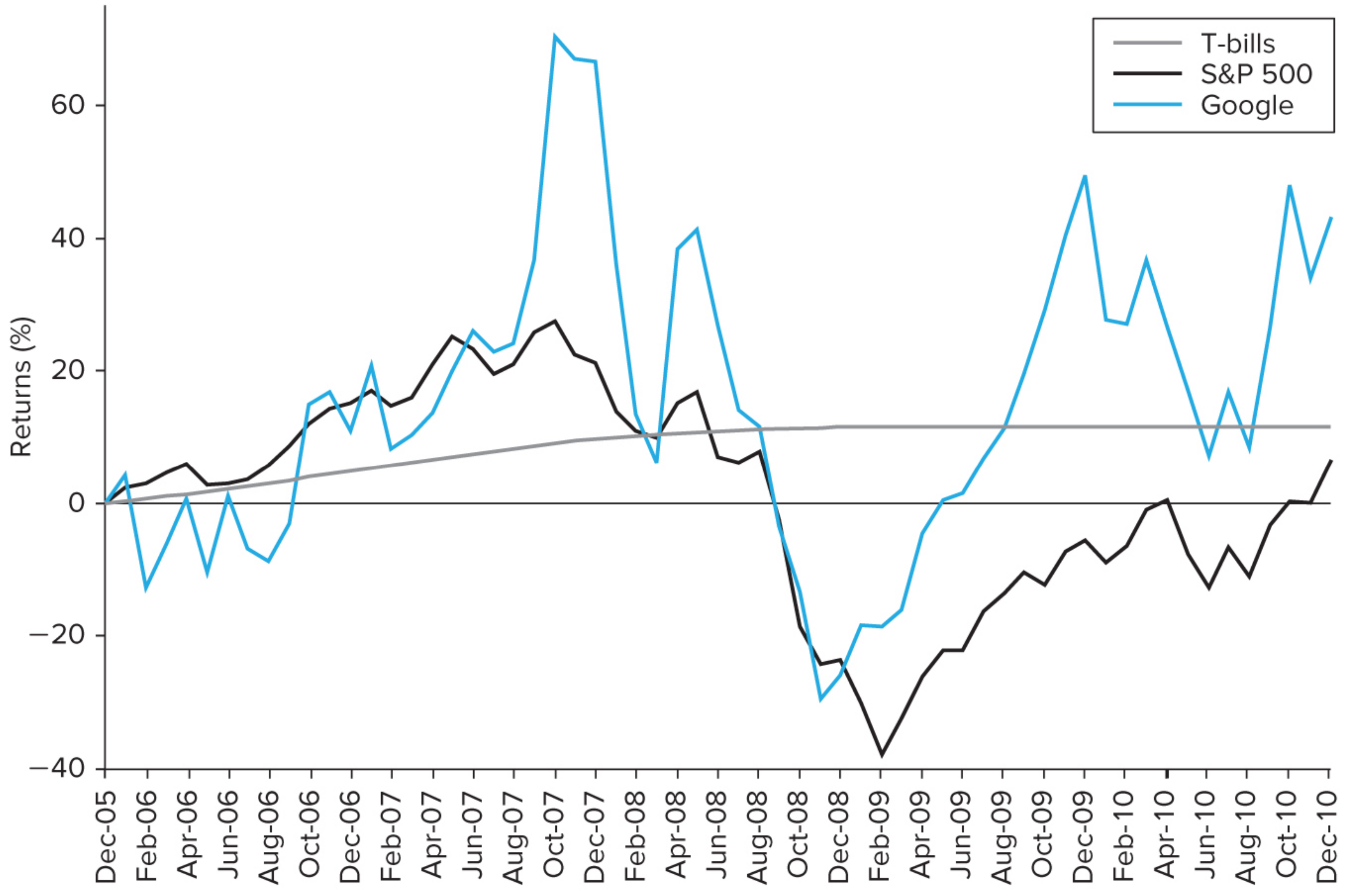
- Google's returns over subperiods within these five years illustrate a common illusion. Observe that Google's precession increase between January 2006 and October 2007 was 70.42%.
  - The subsequent financial crisis decline (November 2007–May 2009) and recovery (June 2009-December 2010) were of similar magnitudes of -40.99% and 42.36%, respectively, and you might think they should have just about canceled out.
  - Yet the total five-year return was "only" 43.17%, around 27% less than the precession gain of 70.42%. Where did that 27% go?
  - It went in the crisis: Then decline and subsequent increase had total impact on cumulative return of $(1 - .4099) \times (1 + .4236) = .8401$, resulting in a loss of about 16%.
  - Apply that loss to the precession value of stock, and you obtain $.8401 \times (1 + .7042) = 1.43$, just equal to the five-year cumulative return.

- Why didn't the 40.99% loss and 42.36% gain roughly cancel out?
    - In general, a large gain following a large loss has a muted impact on cumulative return because it acts on a diminished investment base, while a large loss following a large gain has an amplified impact because it acts on a greater investment base.
    - The greater the fluctuations, the greater the impact on final investment value, which is why the spread between the geometric average (which reflects cumulative return) and the arithmetic average grows with stock volatility.
- Figure 7.3, Panel A shows the monthly return on the securities during the sample period.
    - The significantly higher volatility of Google is evident, and the graph suggests that its beta is greater than 1.0: When the market moves, Google tends to move in the same direction, but by greater amounts.



A: Monthly returns

- Figure 7.3, Panel B shows the evolution of the cumulative returns.

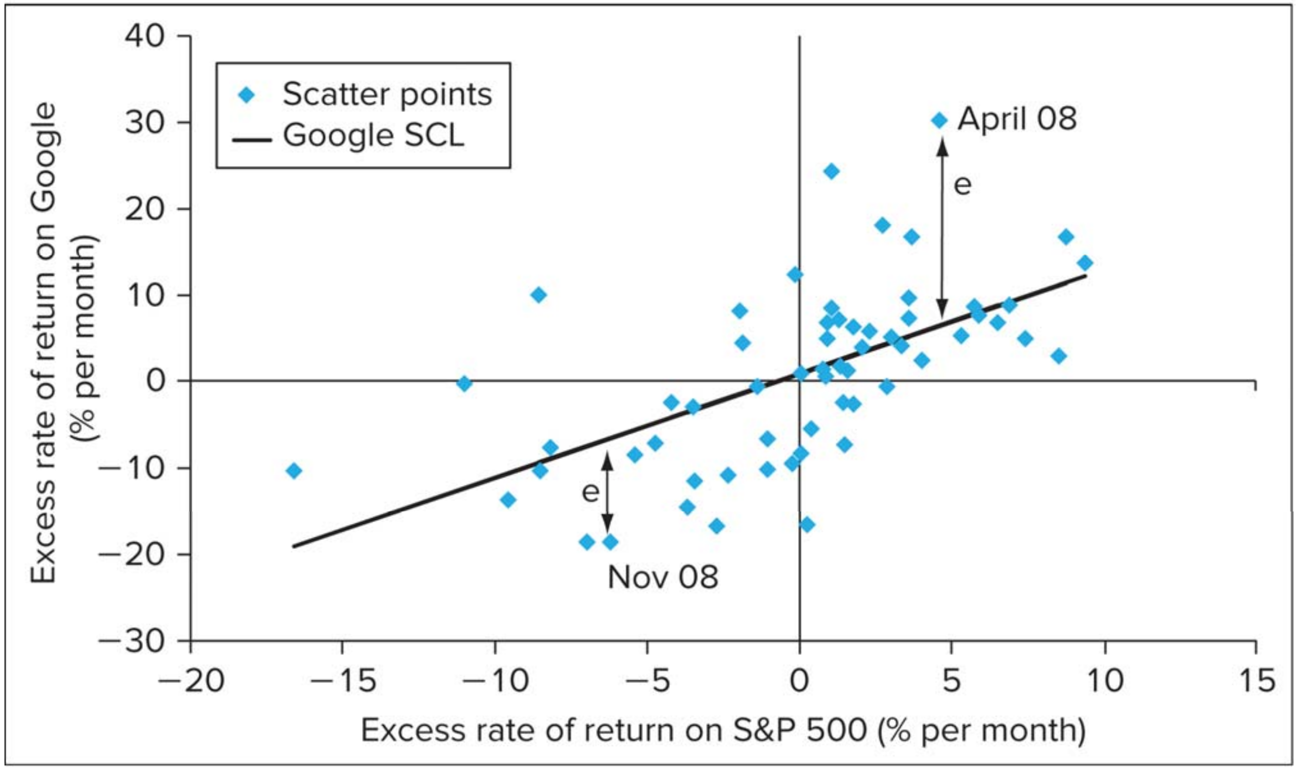


B: Cumulative returns

- ◆ It illustrates the positive index returns in the early years of the sample, the steep decline during the recession, and the significant partial recovery of losses at the end of the sample period.
- ◆ Whereas Google outperforms T-bills, T-bills outperform the market index over the period, highlighting the worse-than-expected realizations in the capital market.

ESTIMATION RESULTS

- ■ We regressed Google's excess returns against those of the index using the Regression command from the Data Analysis menu of Excel.
- ■ The scatter diagram in Figure 7.4 shows the data points for each month as well as the regression line that best fits the data.
  - ◆ As noted in the previous chapter, this is called the **security characteristic line (SCL),** because it describes the relevant characteristics of the stock.
  - ◆ Figure 7.4 allows us to view the residuals, the deviation of Google's return each month from the prediction of the regression equation.
    - ✓ By construction, these residuals *average* to zero, but in any particular month, the residual may be positive or negative.



- The residual for April 2008 (23.81%) and November 2008 (-10.97%) are labeled explicitly.
  - The April 2008 point lies above the regression line, indicating that in this month, Google's return was better than predicted from the market return. The distance between the point and the regression line is Google's firm-specific returns, which is the residual for April.

- The standard deviation of the residuals indicates the accuracy of predictions from the regression line.
  - If there is a lot of firm-specific risk, there will be a wide scatter of points around the line (a high residual standard deviation), indicating that the market return will not enable a precise forecast of Google's return.
- Table 7.2 is the regression output from Excel.
  - The first line shows that the correlation coefficient between the excess returns of Google and the index was .59.
  - The more relevant statistic, however, is the *adjusted R-square* (.3497).
    - ✓ It is the square of the correlation coefficient, adjusted downward for the number of coefficients or "degrees of freedom" used to estimate the regression line.
    - ✓ The adjusted R-square tells us that 34.97% of the variance of Google's excess returns is explained by the variation in the excess returns of the index, and hence the remainder, or 65.03%, of the variance is firm specific, or unexplained by market movements.

✓ The dominant contribution of firm-specific factors to variation in Google's returns is typical of individual stocks, reminding us why diversification can greatly reduce risk.

TABLE 7.2 Security characteristic line for Google (S&P 500 used as market index), January 2006–December 2010

Linear Regression

Regression Statistics

R (correlation)	0.5914
R-square	0.3497
Adjusted *R*-square	0.3385
SE of regression	8.4585
Total number of observations	60

Regression Equation: Google (excess return) = 0.8751 + 1.2031 × S&P 500 (excess return)

	Coefficients	Standard Error	*t*-Statistic	*p*-Value	LCL	UCL
Intercept	0.8751	1.0920	0.8013	0.4262	21.7375	3.4877
S&P 500	1.2031	0.2154	5.5848	0.0000	0.6877	1.7185

LCL—Lower confidence interval (95%)
UCL—Upper confidence interval (95%)

- The standard deviation of the residuals is referred to in the output (below the adjusted R-square) as the "standard error" of the regression (8.46%).
  - ✓ In roughly two-thirds of the months, the firm-specific component of Google's excess return was between ±8.46%.
  - ✓ This wide spread is more evidence of Google's considerable firm-specific volatility.
- Finally, the lower panel of the table shows the estimates of the regression intercept and slope (alpha = .88% and beta = 1.20).
  - ✓ The positive alpha means that, measured by *realized* returns, Google stock was above the security market line (SML) for this period.
  - ✓ But the next column shows considerable imprecision in this estimate as measured by its standard error, 1.09, considerably larger than the estimate itself.
  - ✓ The *t*-statistic (the ratio of the estimate of alpha to its standard error) is only .801, indicating low statistical significance.

- ✓ This is reflected in the large *p*-value in the next column, .426, which indicates that the probability that an estimate of alpha this large could have resulted from pure chance even if the true alpha were zero.
- ✓ The last two columns give the upper and lower bounds of the 95% confidence interval around the coefficient estimate.
  - ➢ This confidence interval tells us that, with a probability of .95, the true alpha lies in the wide interval from -1.74 to 3.49, which includes zero.
  - ➢ Thus, we cannot conclude from this particular sample, with any degree of confidence, that Google's true alpha was not zero, which would be the prediction of the CAPM.

◆ The second line in the panel gives the estimate of Google's beta, which is 1.20.

- ✓ The standard error of this estimate is .215, resulting in a *t*-statistic of 5.58, and a practically zero *p*-value for the hypothesis that the true beta is in fact zero.
  - ➢ In other words, the probability of observing an estimate this large if the true beta were zero is negligible.

- ✓ Another important question is whether Google's beta is significantly different from the average stock beta of 1.
  - ➢ This hypothesis can be tested by computing the *t*-statistic:

$$t = \frac{\text{Estimated value} - \text{Hypothesis value}}{\text{Standard error of estimate}} = \frac{1.2031 - 1}{.2154} = .94$$

    This value is considerably below the conventional threshold for statistical significance; we cannot say with confidence that Google's beta differs from 1. The 95% confidence interval for beta ranges from .69 t0 1.72.

WHAT WE LEARN FROM THIS REGRESSION

- ◆ The regression analysis reveals much about Google, but we must temper our conclusions by acknowledging that the tremendous volatility in stock market returns makes it difficult to derive strong statistical conclusions about the parameters of the index model, at least for individual stocks.
  - ◆ With such noisy variables we must expect imprecise estimates; such is the reality of capital markets.

- Despite these qualifications, we can safely say that Google is a cyclical stock. Its returns vary equally with or more than the overall market, as its beta is higher than the average value of 1, albeit not significantly so.
  - Thus, we would expect Google's excess return to respond, on average, more than one-for-one with the market index.
  - Without additional information, if we had to forecast the volatility of a portfolio that includes Google, we would use the beta estimate of 1.20 to compute the contribution of Google to portfolio variance.
- Moreover, if we had to advise Google's management of the appropriate discount rate for a project that is similar in risk to its equity, we would use this beta estimate in conjunction with the prevailing risk-free rate and our forecast of the expected excess return on the market index.

- Suppose the current T-bill rate is 2.75%, and our forecast for the market excess return is 5.5%. Then the required rate of return for an investment with the same risk as Google's equity would be:

  Required rate = Risk-free rate + β × Expected excess return of index

  $$= r_f + \beta \times (r_M - r_f) = 2.75\% + 1.20 \times 5.5\% = 9.35\%$$

- In light of the imprecision of both the market risk premium and Google's beta estimate, we would try to bring more information to bear on these estimates.
  - ✓ For example, we would compute the betas of other firms in the industry, which ought to be similar to Google's, to sharpen our estimate of Google's systematic risk.
- Finally, suppose we were asked to determine whether, given Google's positive alpha, a portfolio manager was correct in loading up a managed portfolio with Google stock over the period 2006-2010.

- To answer this question, let's find the optimal position in Google that would have been prescribed by the Treynor-Black model of the previous chapter.
  - Let us assume that the manager had an accurate estimate of Google's alpha and beta, as well as its residual standard deviation and correlation with the index (from Tables 7.1 and 7.2).
  - We still need information about the manager's forecast for the index as we know it was *not* the actual return.
    - ✓ Suppose the manager assumed a market-index risk premium of .6%/month (near the historical average) and correctly estimated the index standard deviation of 5.11%/month.
  - Thus, the manager's input list would have included:

Security	**Risk Premium**	**Standard Deviation (%)**	**Correlation**
Index	0.6	5.11	
Google	$0.875 + 1.203 \times 0.6 = 1.60$	10.40	0.59

- ✓ Using Equation 6.10 we calculate for the optimize portfolio (P):

  $w_M = .3911 \quad w_G = .6089 \quad E(R_P) = 1.21\% \quad \sigma_P = 7.69\%$

  - ➢ Thus, it appears the manager would have been quite right to tilt the portfolio heavily toward Google during this period.
  - ➢ This reflects its large positive alpha over the sample period.

- ◆ We can also measure the improvement in portfolio performance.
  - ✓ Using Equation 6.8, the Sharpe ratio of the index and the optimized portfolios based on expected returns are

    $S_M = .12 \quad S_P = .16$

    So the position in Google substantially increased the Sharpe ratio.

- ◆ This exercise would not be complete without the next step, where we observe the performance of the proposed "optimal" portfolio.
  - ✓ After all, analysts commonly use available data to construct portfolios for a future period.

- ✓ We put *optimal* in quotes because everyone in the profession knows that past alpha values do not predict future values.
  - ➢ Hence, a portfolio formed solely, or even primarily, by extrapolating past alpha would never qualify as optimal.
- ✓ However, if we treat this alpha as though it came from security analysis, we can paint a picture of what might go on in the trenches of portfolio management.

◆ Suppose we now observe 10 additional months of returns (January 2011–October 2011) for the S&P 500, Google, and T-bills.

- ✓ We test the proposed portfolio for three future periods following the data collection and analysis period: the next quarter, next semiannual period, and next 10 months. For each of these periods we compare the performance of the proposed portfolio to the passive index portfolio and to T-bills.

✓ The results are as follows:

Cumulative Returns (%) of Three Alternative Strategy Portfolios			
Portfolio	**Proposed Google-S&P 500**	**Passive: S&P 500 Index**	**T-bills**
2011 Q1	1.36	5.42	0.01
2011 First half	-7.35	5.01	0.01
January-October 2011	0.41	-0.35	0.01

- We see that 2011 began as a good year for the market, with a half-year cumulative excess return of 5.01%. Up to this point, however, Google stock dropped and the proposed portfolio stumbled badly.
- Yet the next four months saw a complete reversal of fortune: The market stumbled badly, dragging its cumulative return into negative territory, while Google shined and propelled the proposed "optimal" portfolio into positive territory.

- ➢ It is clear that evaluating performance is fraught with enormous potential estimation error. Even a nonsense portfolio can have its day when volatility is go high.

Predicting Betas

- A single-index may not be fully consistent with the CAPM and may not be a sufficiently accurate predictor of risk premiums. Still the concept of systematic versus diversifiable risk is useful.
  - Systematic risk is approximated well by the regression equation beta and nonsystematic risk by the residual variance of the regression.
- As an empirical rule, it appears that beta exhibit a statistical property called *mean reversion*.
- Often, we estimate betas in order to forecast the rate of return of an asset.
  - This suggests that high-β (that is, $\beta > 1$) securities tend to exhibit a lower β in the future, while low-β (that is, $\beta < 1$) securities exhibit a higher β in future periods.

- Researchers who desire predictions of future betas often adjust beta estimates estimated from historical data to account for regression toward 1.
  - ✓ For this reason, it is necessary to verify whether the estimates are already "adjusted betas."

- A simple way to account for mean reversion is to forecast beta a weighted average of the sample estimate with the value 1.
- Example 7.7: *Forecast of Beta*
  - Suppose that past data yield a beta estimate of .65.
    - ✓ A common weighted scheme is 2/3 on the sample estimate and 1/3 on the value 1. Thus, the adjusted forecast of beta will be

      Adjusted beta = $(2/3) \times .65 + (1/3) \times 1.0 = .77$

      The final forecast of beta is in fact closer to 1 than the sample estimate.

- A more sophisticated technique would base the weight of the sample beta on its statistical quality.
  - A more precise estimate of beta will get a higher weight.
- However, obtaining a precise statistical estimate of beta from past data on individual stocks is a formidable task, because the volatility of rates of return is so large.
  - In particular, there is a lot of “noise” in the data due to firm-specific events.
  - The problem is severe with portfolios because diversification reduces firm-specific variance.
- One might hope that more precise estimates of beta could be obtained by using a long time series of returns.
  - Unfortunately, this is not a solution because betas change over time and old data can provide a misleading guide to current betas.
- Two methods can help improve forecasts of beta.
  - The first is an application of a technique that goes by the name of ARCH models. ARCH models better predict variance and covariance using high-frequency (daily) historical data to identify persistent changes in variance and covariance.

- ✓ *ARCH* stands for "autoregressive conditional heteroscedasticity." The model was developed by Robert F. Engle, who received the 2003 Nobel Prize in Economics.
- ✓ This is a fancy way of saying that the volatility (and covariance) of stock changes over time in ways that can be at least partially predicted from past data.

◆ The second method involves ad additional step where beta estimates from time series regressions are augmented by other information about the firm, for example, P/E ratios.

7.3 THE CAPM AND THE REAL WORLD

- In many ways, portfolio theory and the CAPM have become accepted tools in the practitioner community.
  - ◆ Many investment professionals think about the distinction between firm-specific and systematic risk and are comfortable with the use of beta to measure systematic risk.
    - ✓ One survey found that about three-quarters of financial managers use the CAPM when estimating the cost of capital.
    - ✓ In the money management industry, alpha is regularly computed.
- The CAPM was first published by Sharpe in 1964 and took the world of finance by storm.
  - ◆ Early tests by Black, Jensen, and Scholes (1972) and Fama and MacBeth (1973) were partially supportive of the CAPM.
    - ✓ Average returns were higher for higher beta portfolios, but the reward for beta risk was less than the predictions of the simple version of the CAPM.

- While this sort of evidence against the CAPM remained largely within the ivory towers of academia, Roll's (1977) paper "A Critique of Capital Asset Pricing Tests" shook the practitioner world as well.
  - Roll argued that the true market portfolio can never be observed.
    - ✓ The usual stock market indexes used as proxies for the market portfolio ignore the large majority of investor wealth, for example, real estate, fixed income securities, foreign investments, and not least, the value of human capital.
  - Without a good measure of the return on a broad measure of investor assets, the theory is *necessarily* untestable.
- The publicity given the now classic "Roll's critique" resulted in popular articles such as "Is Beta Dead?" that effectively slowed the permeation of portfolio theory through the world of finance.
  - Although Roll is absolutely correct on theoretical grounds, more recent research suggests that the error introduced by using a broad market index as proxy for the true, unobserved market portfolio is perhaps not even the greatest problem of the CAPM.

- ◆ For example, Fama and French (1992) published a study that dealt the CAPM an even harsher blow.
  - ✓ They found that in contradiction to the CAPM, certain characteristics of the firm, namely, such as size and the ratio of market to book value, were far more useful in predicting future returns than beta.

- ■ Fama and French and several others have published many follow-up studies of this topic.
  - ◆ It seems clear from these studies that beta does not tell the whole story of risk. There seem to be risk factors that affect security returns beyond beta's one-dimensional measurement of market sensitivity.
    - ✓ In the next section, we will introduce a theory of risk premiums that explicitly allows for multiple risk factors.

- ■ Liquidity, a different kind of risk factor, was ignored for a long time.
  - ◆ Although first analyzed by Amihud and Mendelson as early as 1986, it is yet to be accurately measured and incorporated in portfolio management.

- ◆ Measuring liquidity and the premium commensurate with illiquidity is part of a larger field in financial economics, namely, market structure.
  - ✓ We now know that trading mechanisms on stock exchanges can affect the liquidity of assets traded on these exchanges and thus significantly affect their market value.

- ■ Despite all these issues, beta is not dead.
  - ◆ Research shows that when we use a more inclusive proxy for the market portfolio than the S&P 500 (specifically, an index that includes human capital) and allow for the fact that beta changes over time, the performance of beta in explaining security returns is considerably enhanced (Jagannathan and Wang, 1996).
  - ◆ We know that the CAPM is not a perfect model and that it will continue to be refined.
  - ◆ Still, the logic of the model is compelling and capture the two key points made by all of its more sophisticated variants.
    - ✓ First, the crucial distinction between diversifiable risk and systematic risk that cannot be avoided by diversification.

- ✓ Second, the fact that investors will demand a premium for bearing nondiversifiable risk. The CAPM therefore provides a useful framework for thinking rigorously about the relationship between security risk and return.

7.4 MULTIFACTOR MODELS AND THE CAPM

- The index model allows us to decompose stock variance into systematic risk and firm-specific risk that can be diversified in large portfolios.
- In the index model, the return on the market portfolio summarized the aggregate impact of macro factors.
  - In reality, however, systematic risk is not due to one source, but instead derives from uncertainty in a number of economy-wide factors such as business-cycle risk, interest or inflation rate risk, energy price risk, and so on.
- It stands to reason that a more explicit representation of systematic risk, allowing stocks exhibit different sensitivities to its various facets, would constitute a useful refinement of the single-factor model.
  - We can expect that models that allow for several systematic factors—**multifactor models**—can provide better descriptions of security returns.

- Let's illustrate with a two-factor model.
  - Suppose the two most important macroeconomic sources of risk are the state of the business cycle reflected in returns on a broad market index such as the S&P 500 and unanticipated changes in interest rates captured by returns on a Treasury-bond portfolio.
  - The return on any stock will respond both to sources of macro risk as well as to its own firm-specific influences.
  - Therefore, we can expand the single-index model, Equation 7.3, describing the excess rate of return on stock *i* in some time period *t* as follows:

$$R_{it} = \alpha_i + \beta_{iM}R_{Mt} + \beta_{i\mathrm{TB}}R_{\mathrm{TB}t} + e_{it} \tag{7.5}$$

where $\beta_{i\mathrm{TB}}$ is the sensitivity of the stock's excess return to that of the T-bond portfolio and $R_{\mathrm{TB}t}$ is the excess return of the T-bond portfolio in month *t*.

- How will the security market line of the CAPM generalize once we recognize multiple sources of systematic risk?
  - Not surprisingly, a multifactor index model gives rise to a multifactor security market line in which the risk premium is determined by the exposure to *each* systematic risk factor and by a risk premium associated with each of those factors.
    - ✓ Such a multifactor CAPM was first presented by Merton (1973).
- In the two-factor economy of Equation 7.5, the expected rate of return on a security would be the sum of three terms:
  - The risk-free rate of return.
  - The sensitivity to the market index (i.e., the market beta, β_{iM}) times the risk premium of the index, $[E(r_M) - r_f]$.
  - The sensitivity to interest rate risk (i.e., the T-bond beta, $\beta_{i\mathrm{TB}}$) times the risk premium of the T-bond portfolio, $[E(r_{\mathrm{TB}}) - r_f]$.

- This assertion is expressed as a two-factor security market line for security *i*:

$$E(r_i) = r_f + \beta_{iM}[E(r_M) - r_f] + \beta_{i\text{TB}}[E(r_{\text{TB}}) - r_f] \quad (7.6)$$

  - Equation 7.6 is an expansion of the simple security market line.
  - Once we generalize the single-index SML to multiple risk sources, each with its own risk premium, the insights are similar.
- Example 7.8: *A Two-Factor SML*
  - Northeast Airlines has a market beta of 1.2 and a T-bond beta of .7. Suppose the risk premium of the market index is 6%, while that of the T-bond portfolio is 3%.
    - ✓ Then the overall risk premium on Northeast stock is the sum of the risk premiums required as compensation for each source of systematic risk.
      - The portion of the risk premium attributable to market risk is the stock's exposure to that risk, 1.2, multiplied by the corresponding risk premium, 6%, or $1.2 \times 6\% = 7.2\%$.
      - Similarly, the risk premium attributable to interest rate risk is $.7 \times 3\% = 2.1\%$.
      - The total risk premium is $7.2 + 2.1 = 9.3\%$.

- Therefore, if the risk-free rate is 4%, the expected return on the portfolio should be

	4.0%		Risk-free rate
+	7.2%	+	Risk premium for exposure to market risk
+	2.1%	+	Risk premium for exposure to interest-rate risk
	13.3%		Total expected return

- More concisely,

$$E(r) = 4\% + 1.2 \times 6\% + .7 \times 3\% = 13.3\%$$

- The multifactor model clearly gives us a much richer way to think about risk exposures and compensation for those exposures than the single-index model or the CAPM.
- But what are the relevant additional systematic factors?
  - Economic theory suggests factors that affect investor welfare in three possible ways:
    - ✓ Factors that are correlated with prices of important consumption goods, such as housing or health care
    - ✓ Factors that are correlated with future investment opportunities, such as interest rates, return volatility, or risk premiums

- ✓ Factors that correlated with the general state of the economy, such as industrial production and employment.

- ■ Alas, multifactor equations employing likely candidates for these theoretically plausible factors have not produced sufficient improvement over the explanatory power of the single-factor model.
- ■ Instead, pioneering research by Eugene F. Fama and Kenneth French (1992) produced today's state-of-the-art approach to multifactor models.

● The Fama-French Three-Factor Model

- ■ Fama and French (1996) proposed a three-factor model that has become a standard tool for empirical studies of asset returns.
  - ◆ They add to the market-index portfolios formed on the basis of firm size and book-to-market ratio to explain average returns.
  - ◆ These additional factors are motivated by the observations that average returns on stocks of small firms and firms with high ratios of book value of equity to market value of equity have been higher than predicted by the CAPM.

- This observation suggests that size or the book-to-market ratio (B/M) may be *proxies* for exposures to sources of systematic risk not captured by the beta, and thus result in return premiums.

- While a high book-to-market ratio can result from financial distress, which depresses market value relative to book value, for the most part, this group includes relatively mature firms.
  - The latter derive a larger share of their market value from assets already in place, rather than growth opportunities. This group often is called *value stocks*.
  - In contrast, low-B/M companies are viewed as *growth firms* whose market values derive from anticipated future cash flows, rather than from assets already in place.
  - Considerable evidence (which we will review in the following chapter) suggests that value stocks trade at lower prices than growth stocks (or, equivalently, have offered a higher average rate of return); the differential is known as the *value premium*.

- One mechanism by which a high book-to-market ratio may capture aspects of systematic risk is through the impact of large fixed assets.
  - When the economy tanks, these assets cannot be used at full capacity, and a large share of their value may be lost.
  - Therefore, a high B/M ratio, which can result from large investments in fixed assets, may imply higher systematic risk than indicated by historically estimated beta.
- Firm size may also predict some aspects of risk.
  - Shares of large firms may be less risky than those of small firms, other things being equal, because they are covered by more analysts, and there is more accurate information about them.
  - With better-informed investors, prices will better reflect true value and be less susceptible to systematic as well as firm-specific fluctuations.
  - With deeper pockets and greater debt capacity, large firms also can better withstand economic downturns.
  - On both counts, stock in small firms will command higher risk premiums than indicated by beta alone.

- Fama and French also pioneered a method to construct size and B/M factor portfolios to facilitate estimates of each factor's effect on stock returns. Their approach has since been used to construct other factor portfolios.
  - To illustrate, we will follow the same general approach that we applied for Google earlier, but now using the more general model.

COLLECTING AND PROCESSING DATA

- To create portfolios that track the size and B/M factors, one can sort industrial firms by size (market capitalization or market "cap") and by B/M ratio.
  - The size premium is constructed as the difference in returns between small and large firms and is denoted by SMB ("small minus big").
  - Similarly, the B/M premium is calculated as the difference in returns between firms with a high versus low B/M ratio, and is denoted HML ("high minus low" ratio).
- Taking the difference in returns between two portfolios has an economic interpretation.
  - The SMB return, for example, equals the return from a long position in small stocks, financed with a short position in the large stocks.
    - ✓ Note that this is a portfolio that entails no *net* investment.

■ Summary statistics for these portfolios in our sample period are reported in Table 7.3.

TABLE 7.3 Statistics for monthly rates of return (%), January 2006–December 2010

	Excess Return*		Total Return	
Security	**Average**	**Standard Deviation**	**Geometric Average**	**Cumulative Return**
T-bill	0	0	0.18	11.65
Market index†	0.26	5.44	0.30	19.51
SMB	0.34	2.46	0.31	20.70
HML	0.01	2.97	−0.03	−2.06
Google	0.94	10.40	0.60	43.17

*Total return for SMB and HML.

†Includes all NYSE, NASDAQ, and AMEX stocks.

- ◆ We use a broad market index, the value-weighted return on all stocks traded on U.S. national exchanges (NYSE, Amex, and NASDAQ) to compute the excess return on the market portfolio.
- ◆ The "return" of the SMB and HML portfolios require careful interpretation.
  - ✓ As noted above, these portfolios do not by themselves represent investment portfolios, as they entail zero net investment.

- ✓ Rather, they may be interpreted as side bets on whether one type of stock will beat another (e.g., small versus large ones for SMB).

- ■ To apply the FF three-factor portfolio to Google, we need to estimate Google's beta on each factor.
  - ◆ To do so, we generalize the regression Equation 7.3 of the single-index model and fit a multivariate regression:

$$r_{\mathrm{G}} - r_f = \alpha_{\mathrm{G}} + \beta_M(r_M - r_f) + \beta_{\mathrm{HML}} r_{\mathrm{HML}} + \beta_{\mathrm{SMB}} r_{\mathrm{SMB}} + e_{\mathrm{G}} \tag{7.7}$$

  - ◆ To the extent that returns on the size (SMB) and book-to-market (HML) portfolios proxy for risk that is not fully captured by the market index, the beta coefficients on these portfolios represent exposure to systematic risks beyond the market-index beta.
    - ✓ When we estimate Equation 7.7, we must subtract the risk-free return from the market return but not from the returns on the SMB or HML portfolios.
    - ✓ This is because the SMB and HML factors are *already* risk premiums, for size or book-to-market respectively. We subtract the risk-free rate from the market index return to similarly cast it as a risk premium.

ESTIMATION RESULTS

- Both the single-index model (alternatively employing the S&P 500 Index and the broad market index) and the FF three-factor model are summarized in Table 7.4.

TABLE 7.4 Regression statistics for alternative specifications:
1. A Single index with S&P 500 as market proxy
1. B Single index with broad market index (NYSE + NASDAQ + Amex)
2. Fama-French three-factor model (broad market + SMB + HML)

Monthly returns January 2006–December 2010

	Single Index Specification		
Estimate	**S&P 500**	**Broad Market Index**	**FF 3-Factor Specification with Broad Market Index**
Correlation coefficient	0.59	0.61	0.70
Adjusted *R*-Square	0.34	0.36	0.47
Residual SD = Regression SE (%)	8.46	8.33	7.61
Alpha = Intercept (%)	0.88 (1.09)	0.64 (1.08)	0.62 (0.99)
Market beta	1.20 (0.21)	1.16 (0.20)	1.51 (0.21)
SMB (size) beta	–	–	−0.20 (0.44)
HML (book to market) beta	–	–	−1.33 (0.37)

Note: Standard errors in parenthesis.

- The broad market index includes more than 4,000 stocks, while the S&P 500 includes only 500 of the largest U.S. stocks, in which list Google ranked fourteenth in January 2012.
- In this sample, the broad market index tracks Google's returns better than the S&P 500, and the three-factor model is a better specification than the one-factor model.
  - ✓ This is reflected in three aspects of a successful specification: a higher adjusted R-square, a lower residual SD, and a smaller value of alpha.
  - ✓ This outcome turns out to be typical, which makes a broader market index the choice of researchers and the FF model the current first-line empirical model of security returns.
  - ✓ The FF model is often augmented by an additional factor, usually *momentum*, which classifies stocks according to which ones have recently increased or recently decreased in price. Liquidity is also increasingly used as yet another additional factor.

- Google's market beta estimate is very different in the three-factor model (1.51 versus 1.20 or 1.16 in one-factor models).
  - ✓ Moreover, this coefficient value implies high cyclicality and is significantly greater than 1: It exceeds 1 by 2.43 standard errors.
- The SMB beta is negative, (-.20), as you would expect for a firm as large as Google, yet it is not significantly different from zero (standard error = .44).
- Google still exhibits a negative and significant book-to-market beta (coefficient = 1.33, standard error = .37), however, indicating that it is still a growth stock.

WHAT WE LEARN FROM THIS REGRESSION

- While the FF three-factor model offers a richer and more accurate description of asset returns, applying this model requires two more forecasts of future returns, namely, for the SMB HML portfolios.
  - We have so far in this section been using a T-bill rate of 2.75% and a market risk premium of 5.5%.

- If we add to these values a forecast of 2.5% for the SMB premium and 4% for HML, the required rate for an investment with the same risk as Google's equity would be

$$E(r_G) = r_f + \beta_M[E(r_M) - r_f] + \beta_{SMB}E(r_{SMB}) + \beta_{HML}E(r_{HML})$$
$$= 2.75\% + (1.51 \times 5.5\%) + (-.20 \times 2.5\%) + (-1.33 \times 4\%) = 5.24\%$$

  which is considerably lower than the rate derived from cyclical considerations alone (i.e., single-beta models).

- Notice from this example that to obtain expected rates of return, the FF model requires, in addition to a forecast of the market-index return, a forecast of the returns of the SMB and HML portfolios, making the model more difficult to apply. This can be a critical issue.
  - If such forecasts are difficult to devise, the single-factor model may be preferred even if it is less successful in explaining *past* returns.
  - This is a fairly common outcome: Theoretically inferior models with fewer explanatory variables often describe out-of-sample outcomes more accurately than models employing more explanatory variables.

- ✓ This reflects in part the tendency of some researchers to "data mine," that is, to search too aggressively for variables that help describe a sample but have no staying power out of sample.
- ✓ In addition, each explanatory variable of a model must be forecast to make a prediction, and each of those forecasts adds some uncertainty to the prediction.

Multifactor Models and the Validity of the CAPM

- The single-index CAPM fails empirical tests because its empirical representation, the single-index model, inadequately explains returns on too many securities.
  - In short, too many statistically significant values of alpha (which should be zero) show up in single-index regressions.
  - Despite this failure, it is still widely used in the industry.
- Multifactor models such as the FF model may also be tested by the prevalence of significant alpha values.
  - The three-factor model shows a material improvement over the single-index model in that regard.

- But the use of multi-index models comes at a prices: They require forecasts of the additional factor returns.
  - ✓ If forecasts of those additional factors are themselves subject to forecast error, these models will be less accurate than the theoretically inferior single-index model.
- Nevertheless, multifactor models have a definite appeal because it is clear that real-world risk is multifaceted.

■ Merton (1973) first showed that the CAPM could be extended to allow for multiple sources of systematic risk.

- His model results in a multifactor security market line like that of Equation 7.8 but with risk factors that relate to the extra-market sources of risk that investors wish to hedge.
- In this light, a reasonable interpretation of multivariate index models such as FF is that they constitute an application of the multifactor CAPM, rather than a rejection of the underlying logic of the simple model.

- The original FF model has spawned a cottage industry generating new factor portfolios, including, but not limited to, momentum or liquidity factors.
  - One expects that the marginal contributions of these additional factors will be smaller than those of the first two, SMB and HML.
  - Still, the search for the ultimate version of the expected return-risk relationship is far from over.

7. 5 ARBITRAGE PRICING THEORY

- One reason for skepticism about the validity of the CAPM is the unrealistic nature of the assumptions needed to derive it.
  - Most unappealing are assumptions 2.A-C, namely, that all investors are identical in every way but wealth and risk aversion.
  - For this reason, as well as for its economic insights, the arbitrage pricing theory is of great interest.
  - To understand this theory we begin with the concept of *arbitrage*.
- **Arbitrage** is the act of exploiting the mispricing of two or more securities to achieve risk-free profits.
  - As a trivial example, consider a security that is priced differently in two markets.
    - ✓ A long position in the cheaper market financed by a short position in the expensive market will lead to a sure profit.
    - ✓ As investors avidly pursue this strategy, prices are forced back into alignment, so arbitrage opportunities vanish almost as quickly as they materialize.

- The first to apply this concept to equilibrium security returns was Ross (1976), who developed the **arbitrage pricing theory (APT).**
  - The APT depends on the observation that well-functioning capital markets preclude arbitrage opportunities.
    - ✓ A violation of the APT's pricing relationships will cause extremely strong pressure to restore them even if only a limited number of investors become aware of the disequilibrium.
  - Ross's accomplishment is to derive the equilibrium rates of return that would prevail in a market where prices are in aligned to eliminate arbitrage opportunities.
    - ✓ The APT thus avoids the most objectionable assumptions of the CAPM.

Well-Diversified Portfolios and the APT

- To illustrate how the APT works, we begin with a single-index market; generalization to multi-factor markets is straightforward.

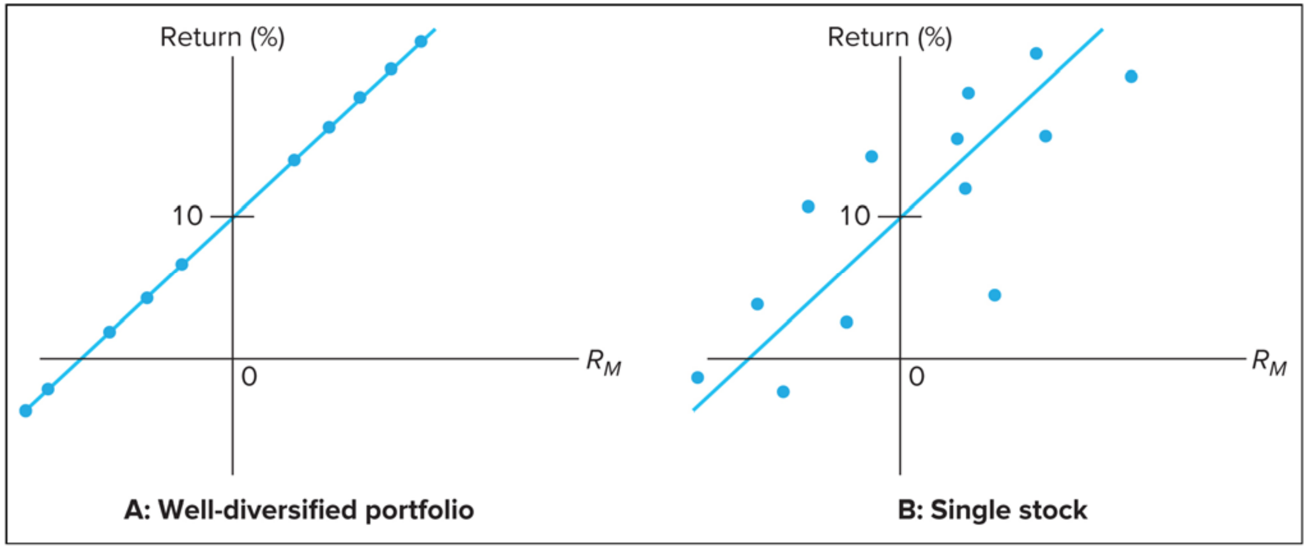
- Suppose a portfolio, *P*, with beta of β_P, is believed to have a positive alpha.
  - If we invest in *P* and add a short position in the broad market benchmark portfolio (with a beta of 1), we can hedge away the systematic risk of *P* and reduce overall beta to zero.
  - We can then go even further and turn the positive-alpha, zero-beta position into a zero-net-investment position by adding an appropriate position in the risk-free asset.
  - In all, we combine the positive-alpha *P* with both the benchmark and T-bills to create a costless, zero-beta "arbitrage portfolio," *A*, with a positive alpha.
- Table 7.5 shows how.

TABLE 7.5 Steps to convert a well-diversified portfolio into an arbitrage portfolio

Portfolio Weight*	In Asset	Contribution to Excess Return
$w_P = 1$	Portfolio *P*	$w_P(\alpha_P + \beta_P R_M + e_P) = \alpha_P + \beta_P R_M + e_P$
$w_M = -\beta_P$	Benchmark	$w_M R_M = -\beta_P R_M$
$w_f = \beta_P - 1$	Risk-free asset	$w_f \times 0 = 0$
$\sum w = 0$	Portfolio *A*	$\alpha_P + e_P$

*When alpha is negative, you would reverse the signs of each portfolio weight to achieve a portfolio *A* with positive alpha and no net investment.

- Table 7.5 shows that portfolio *A* with excess return $\alpha_P + e_P$ is still risky as long as the residual variance, σ_e^2 is positive.
  - ✓ This shows that a zero-investment, zero-beta, positive-alpha portfolio is not necessarily an arbitrage opportunity; true arbitrage implies no risk.
- However, if *P* were highly diversified, its residual risk would be small.
  - ✓ A portfolio with practically negligible residual risk is called a **well-diversified portfolio.**
- The difference in the scatter diagrams of any asset versus that of a well-diversified portfolio with the same beta is shown in Figure 7.5



- Portfolio *A*, when constructed from a *well-diversified portfolio*, is an **arbitrage portfolio.**
  - An arbitrage portfolio is a money machine: It can generate risk-free profits with zero net investment.
  - Therefore, investors who succeed in constructing one will scale it up as much as they can, financing with as much leverage and/or as many short positions as available.
- Example 7.9: *Constructing an Arbitrage Portfolio*
  - Suppose the benchmark, *M*, in Table 7.5 is the observed, broad market index. Image that on December 31, 2015, a portfolio manager possessed the following predictions based on security and macro analyses:
    - ✓ The risk-free rate on T-bills over the next year is estimated at 2.2%.
    - ✓ The return on the market benchmark is estimated at 5.0%.
    - ✓ A highly diversified portfolio *P*, is composed of large-capitalization stocks and is believed to be *underpriced*. Its expected return is forecast at 6.0%.

- ✓ Portfolio P's beta against the benchmark is estimated at .95, which leads to the following calculation for its alpha: 6.0% = 2.2% + α + .95(5.0% – 2.2%); α = 1.14%.
- ◆ Therefore, we set $w_P = 1$, $w_M = -0.95$, and $w_f = -.05$. Arbitrage portfolio A has no residual risk because it is well diversified , and it has no systematic risk because it has zero beta.
- ◆ Yet, its alpha is positive: $\alpha_A = 1.14\%$.

- ■ Example 7.9 shows how to construct an arbitrage portfolio from a mispriced well-diversified portfolio. But it leaves us with an important question: How diversified must a portfolio be to be "well diversified"?
  - ◆ For example, is even the highly diversified S&P 500 a well-diversified portfolio?
  - ◆ Table 7.6 shows the weight of the 10 largest stocks in the S&P 500.
    - ✓ While these firms are only 2% of the firms in the index, they account for almost 20% of the market capitalization, and their weights in the index are far from negligible.

TABLE 7.6 Ten largest capitalization stocks in the S&P 500 and their weights in the index (February 2015)

	Weight (%)		Weight (%)
Apple (AAPL)	4.05	Wells Fargo (WFC)	1.39
ExxonMobil (XOM)	2.04	General Electric (GE)	1.35
Microsoft (MSFT)	1.93	Procter & Gamble (PG)	1.24
Johnson & Johnson (JNJ)	1.52	JP Morgan Chase (JPM)	1.19
Berkshire Hathaway (BRK.B)	1.45	Pfizer (PFE)	1.17
Total for 10 largest firms	17.33		

[18]Obviously, five-year TIPS would carry a practically zero risk to the real rate. But we deal here with active portfolio managers who continuously rebalance their portfolios and must maintain considerable liquidity. For very short holding periods, TIPS would not be practical for these investors.

✓ We will evaluate the S&P relative to a comprehensive market benchmark comprising all stocks traded on the NYSE, NASDAQ, and Amex markets.

- We can estimate the residual risk of the S&P 500 relative to this comprehensive benchmark from a regression of its monthly returns against the benchmark over the prediction period (January 1, 2006, to December 31, 2010).
  - The essential regression output is displayed in Table 7.7.

TABLE 7.7 Regression statistics of the S&P 500 portfolio on the benchmark portfolio, January 2006–December 2010

Linear Regression				
Regression Statistics				
R	0.9933			
R-square	0.9866			
Adjusted *R*-square	0.9864	**Annualized**		
Regression SE	0.5968	2.067		
Total number of observations	60			
S&P 500 = −0.1909 + 0.9337* Benchmark				
	Coefficients	**Standard Error**	***t*-stat**	***p*-level**
Intercept	−0.1909	0.0771	−2.4752	0.0163
Benchmark	0.9337	0.0143	65.3434	0.0000

  - Notice that the (annualized) standard deviation of the regression residuals, called the *standard error of the regression*, was 2.07%. Is this residual SD small enough for us to deem the S&P 500 "well diversified"?

- To answer the question, we recognize that investors who consider an arbitrage strategy must invest their existing wealth *somewhere*. The risk of their alternative portfolio is therefore relevant to the discussion.
  - Obviously, the lowest-risk investment would be to roll over T-bills. A measure of the risk of this strategy is the uncertainty of its real rate of return over the prediction period.
  - Table 7.8 suggests that the annual SD of the real rate from rolling over bills is in the range of .5%–1.5% per year depending on the sample period.

TABLE 7.8 Annual standard deviation of the real, inflation, and nominal rates

Period	Real Rate	Inflation Rate	Nominal Rate
1/1/2006–12/31/2010	1.46	1.46	0.61
1/1/1996–12/31/2000	0.57	0.54	0.17
1/1/1986–12/31/1990	0.86	0.83	0.37

- Our question then comes down to this: What would be the marginal increase in risk from adding an arbitrage portfolio with an SD of about 2% per year to a portfolio with an SD of .5%–1.5% per year?
  - ◆ Because the two rates are uncorrelated, the variance of the portfolio will be the sum of the variances.
    - ✓ The SD of this complete portfolio minus the SD of the T-bill portfolio is the *marginal* risk of the S&P 500 in its use as an arbitrage portfolio.
  - ◆ The following table shows some examples of the marginal risk of the arbitrage portfolio, first treating T-bills as the initial portfolio to which the arbitrage portfolio is added (with three assumptions for the SD of its real return) and then treating the comprehensive benchmark portfolio as the initial position, with an assumed SD = 20%.

SD of Real Rate on Initial Portfolio	SD of Total Portfolio	Marginal Risk
0.5% (T-bills)	$(.005^2 + .0207^2)^{1/2} = 2.13\%$	1.63%
1.0% (T-bills)	2.30%	1.30%
1.5% (T-bills)	2.56%	1.06%
20.0% (benchmark)	20.11%	0.11%

- There is no widely accepted threshold for the acceptable marginal risk of an arbitrage portfolio in a practical application.
  - Nevertheless, the marginal risk in the first three lines of the table, which is just about the same as the SD of the real rate on bills, may well be above the appropriate threshold.
  - Moreover, an alpha of 2% per year in this example is not even decisively statistically significant.
  - We learn from this exercise that well-diversified portfolios are not easy to construct, and arbitrage portfolio is added to the risky benchmark portfolio, the marginal increase in overall standard deviation is minimal.

- We are now ready to derive the APT. Our argument follows from Example 7.9.
  - Investors, however few, will invest large amounts in any arbitrage portfolio they can identify.
    - ✓ This will entail large-scale purchases of positive-alpha portfolios or large-scale shorting of negative-alpha portfolios.
    - ✓ These actions will move the prices of component securities until alpha disappears.
  - In the end, when alphas of all well-diversified portfolios are driven to zero, their return equations become

$$r_P = r_f + \beta_P(r_M - r_f) + e_P \tag{7.8}$$

Taking the expectations in Equation 7.8 results in the familiar CAPM mean-beta equation:

$$E(r_P) = r_f + \beta_P[E(r_M) - r_f] + e_P \tag{7.9}$$

- For portfolios such as the S&P 500 that shed *most* residual risk, we can still expect buying and selling pressure to drive their alpha close to zero.
  - If alphas of portfolios with very small residual risk are near zero, then even less diversified portfolios will tend to have small alpha values.
  - Thus, the APT implies a hierarchy about alphas of portfolios, based on the degree of diversification.

The APT and the CAPM

- Why did we need so many restrictive assumptions to derive the CAPM when the APT seems to arrive at the expected return-beta relationship with seemingly fewer and less objectionable assumptions?
  - The answer is simple: Strictly speaking, the APT applies only to well-diversified portfolios.
    - ✓ Absence of riskless arbitrage alone cannot guarantee that, in equilibrium, the expected return-beta relationship will hold for any and all assets.

- With additional effort, however, one can use the APT to show that the relationship must hold approximately even for individual assets.
  - The essence of the proof is that if the expected return-beta relationship were violated by many individual securities, it would be virtually impossible for all well-diversified portfolios to satisfy the relationship.
  - So the relationship must *almost* surely hold true for individual securities.
- We say "almost" because, according to the APT, there is no guarantee that *all* individual assets will lie on the SML.
  - If only a few securities violated the SML, their effect on well-diversified portfolios could conceivably be negligible.
    - ✓ In this sense, it is possible that the SML relationship is violated for some securities.
  - If many securities violate the expected return-beta relationship, however, the relationship will no longer hold for well-diversified portfolios comprising these securities, and arbitrage opportunities will be available.

- The APT serves many of the same functions as the CAPM.
  - It gives us a benchmark for fair rates of return that can be used for capital budgeting, security evaluation, or performance evaluation of managed portfolios.
  - Moreover, the APT highlights the crucial distinction between nondiversifiable risk (systematic or factor risk) that requires a reward in the form of a risk premium and diversifiable risk that does not.
- The bottom line is that neither of these theories dominates the other.
  - The APT is more general in that it gets us to the expected return-beta relationship without requiring many of the unrealistic assumptions of the CAPM, particularly homogeneous expectations and the reliance on the market portfolio.
    - ✓ The latter improves the prospects for testing the APT.
  - But the CAPM is more general in that it applies to all assets without reservation.
  - The good news is that both theories agree on the expected return-beta relationship.

- It is worth noting that because past tests of the mean-beta relationship examined the rates of return on highly diversified portfolios, they actually came closer to testing the APT than the CAPM.
  - Thus, it appears that econometric concerns, too, favor the APT.

Multifactor Generalization of the APT and CAPM

- So far, we've examined the APT in a one-factor world. In reality, there are several sources of systematic risk such as uncertainty in the business cycle, interest rates, energy prices, and so on.
  - Presumably, exposure to any of these factors will affect a stock's appropriate expected return.
  - We can use a multifactor version of the APT to accommodate these multiple sources of risk.

- Expanding the single-factor model expressed in Equation 7.8 to a two-factor model:

$$R_i = \alpha_i + \beta_{i1} R_{M1} + \beta_{i2} R_{M2} + e_i \tag{7.10}$$

where R_{M1} and R_{M2} are the excess returns on portfolios that represent the two systematic factors.

  - Factor 1 might be, for example, unanticipated changes in industrial production, while factor 2 might represent unanticipated changes in short-term interest rates.
  - We assume again that there are many securities available with any combination of betas.
    - ✓ This implies that we can form well-diversified **factor portfolios** with a beta of 1 on one factor and zero on all others.
    - ✓ Thus, a factor portfolio with a beta of 1 on the first factor will have a rate of return of R_{M1}; a factor portfolio with a beta of 1 on the second factor will have a rate of return of R_{M2}; and so on.
  - Factor portfolios can serve as the benchmark portfolios for a multifactor generalization of the security market line relationship.

- Example 7.10: *Multifactor SML*
  - Suppose the two factor portfolios, here called portfolios 1 and 2, have expected returns $E(r_1) = 10\%$ and $E(r_2) = 12\%$.
  - Suppose further that the risk-free rate is 4%.
    - ✓ The risk premium on the first factor portfolio is therefore 6%, while that on the second factor portfolio is 8%.
  - Now consider an arbitrary well-diversified portfolio (P), with beta on the first factor, $\beta_{P1} = .5$, and on the second factor, $\beta_{P2} = .75$.
    - ✓ The multifactor APT states that the portfolio risk premium must equal the sum of the risk premiums required as compensation to investors for each source of systematic risk.
    - ✓ The risk premium attributable to risk factor 1 is the portfolio's exposure to factor 1, β_{P1}, times the risk premium earned on the first factor portfolio, $E(r_1) - r_f$.
    - ✓ Therefore, the portion of portfolio P's risk premium that is compensation for its exposure to the first risk factor is $\beta_{P1}[E(r_1) - r_f] = .5(10\% - 4\%) = 3\%$, while the risk premium attributable to risk factor 2 is $\beta_{P2}[E(r_2) - r_f] = .75(12\% - 4\%) = 6\%$.

- ✓ The total risk premium on the portfolio, therefore, should be 3% + 6% = 9%, and the total return on the portfolio should be 13%.

4%	Risk-free rate
+ 3%	Risk premium for exposure to factor 1
+ 6%	Risk premium for exposure to factor 2
13%	Total expected return

- ■ Suppose portfolio *P* of Example 7.10 actually has an expected excess return of 11% and therefore a positive alpha of 2%.
  - ◆ We can generalize the methodology of Table 7.5 to construct an arbitrage portfolio for this two-factor problem.
  - ◆ Table 7.9 shows how.
    - ✓ Because *P* is well diversified, e_P must be small, and the excess return on the zero-investment, zero-beta portfolio *A* is just $\alpha_P = 2\%$.

TABLE 7.9 Constructing an arbitrage portfolio with two systematic factors

Portfolio Weight	In Asset	Contribution to Excess Return
1	Portfolio P	$\alpha_P + \beta_{P1}R_1 + \beta_{P2}R_2 + e_P$
$-\beta_{P1} = -0.5$	Factor portfolio 1	$-\beta_{P1}R_1$
$-\beta_{P2} = -0.75$	Factor portfolio 2	$-\beta_{P2}R_2$
$\beta_{P1} + \beta_{P1} - 1 = 0.25$	Risk-free asset	0
Total $= 1$	Portfolio A	$\alpha_P + e_P = 2\% + e_P$

- Here, too, extensive trading by arbitrageurs will eliminate completely alphas of well-diversified portfolios.
- We conclude that, in general, the APT hierarchy of possible alphas values, declining with the extent of portfolio diversification, applies to any multifactor market.
- In the absence of private information from security and macro analyses, investors and corporate officers must use the multifactor SML equation (with zero alpha) to determine the expected rates on securities and the required rates of return on the firm's projects.