# Chapter 6 Efficient Diversification

## Chapter Objectives

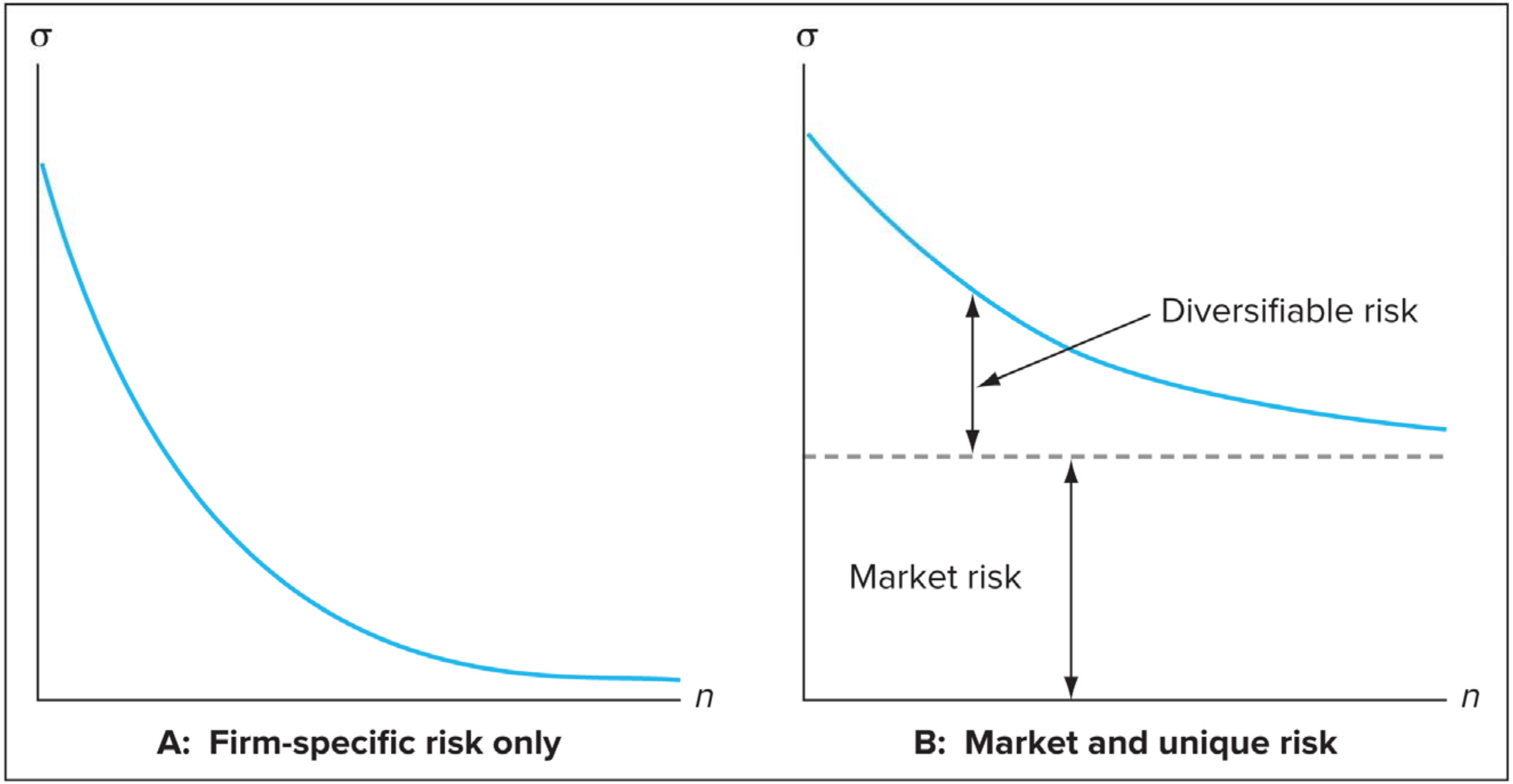
- Show how covariance and correlation affect the power of diversification to reduce portfolio risk.
- Calculate mean, variance, and covariance using either historical data or scenario analysis.
- Construct efficient portfolios and use the Sharpe ratio to evaluate portfolio efficiency.
- Calculate the composition of the optimal risky portfolio.
- Use index models to analyze the risk characteristics of securities and portfolios.
- Understand the effect of investment horizon on portfolio risk.

## 6.1 DIVERSIFICATION AND PORTFOLIO RISK

- Suppose you have in your risky portfolio only one stock, say, Dana Computer Corporation. What are the sources of risk affecting this “portfolio”?
    - We can identify two broad sources of uncertainty.
        - ✓ The first is the risk that has to do with general economic conditions, such as the business cycle, the inflation rate, interest rates, exchange rates, and so forth.
            - None of these macroeconomic factors can be predicted with certainty, and all affect Dana stock.
        - ✓ Then you must firm-specific influences, such as Dana’s success in R&D, its management style and philosophy, and so on.
            - Firm-specific factors are those that affect Dana without noticeably affecting other firms.

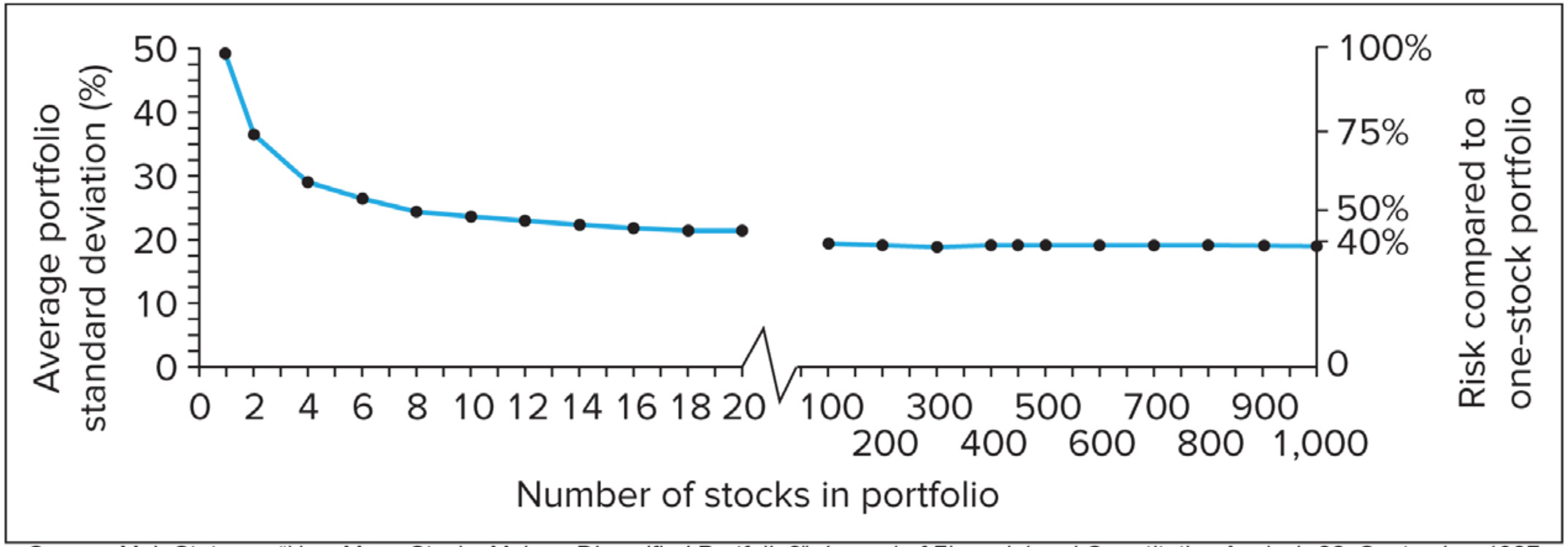
- Now consider a naive diversification strategy, adding another security to the risky portfolio.
  - If you invest half of your risky portfolio in ExxonMobil, leaving the other half in Dana, what happens to portfolio risk?
    - ✓ Because the firm-specific influences on the two stocks differ (statistically speaking, the influences are uncorrelated), this strategy should reduce portfolio risk.
      - For example, when oil prices fall, hurting ExxonMobil, computer prices might rise, helping Dana. The two effects are offsetting, which stabilizes portfolio return.
- But why stop at only two stocks?
  - Diversifying into many more securities continues to reduce exposure to firm-specific factors, so portfolio volatility should continue to fall.
  - Ultimately, however, even with a large number of risky securities in a portfolio, there is no way to avoid all risk.

- To the extent that virtually all securities are affected by common (risky) macroeconomic factors, we cannot eliminate our exposure to general economic risk, no matter how many stocks we hold.
  - Figure 6.1 illustrates these concepts.



- When all risk is firm-specific, as in Figure 6.1A, diversification can reduce risk to low levels.
  - With all risk sources independent, and with investment spread across many securities, exposure to any particular source of risk is negligible.
    - This is just an application of the law of averages.
  - The reduction of risk to very low levels because of independent risk sources is sometimes called the *insurance principle*.
- When common sources of risk affect all firms, however, even extensive diversification cannot eliminate risk.
  - In Figure 6.1B, portfolio standard deviation falls as the number of securities increases, but it is not reduced to zero.
  - The risk that remains even after diversification is called **market risk,** risk that is attributable to marketwide risk sources. Equivalent terms are **systematic risk** or **nondiversifiable risk.**
  - The risk that *can* be eliminated by diversification is called **unique risk, firm-specific risk, nonsystematic risk,** or **diversifiable risk.**

- This analysis is borne out by empirical studies.
  - Figure 6.2 shows the effect of portfolio diversification, using data on NYSE stocks.




Source: Meir Statman, "How Many Stocks Make a Diversified Portfolio?" *Journal of Financial and Quantitative Analysis* 22, September 1987

- ✓ The figure shows the average standard deviations of equally weighted portfolios constructed by selecting stocks at random as a function of the number of stocks in the portfolio.
- ✓ On average, portfolio risk does fall with diversification, but the power of diversification to reduce risk is limited by common sources of risk.

- In light of this discussion, it is worth pointing out that general macroeconomic conditions in the U.S. do not move in lockstep with those in other countries.
  - International diversification may further reduce portfolio risk, but here too, global economic and political factors affecting all countries to various degrees will limit the extent of risk reduction.

# 6.2 ASSET ALLOCATION WITH TWO RISKY ASSETS

## ● Covariance and Correlation

- To optimally construct a portfolio from risky assets, we need to understand how the uncertainties of asset returns interact.
- A key determinant of portfolio risk is the extent to which the returns on the two assets vary either in tandem or in opposition.
  - ◆ Portfolio risk depends on the *correlation* between the returns of the assets in the portfolio.
  - ◆ We can see why using a simple scenario analysis.
- The scenario analysis in Spreadsheet 6.1 posits four possible scenarios for the economy: a severe recession, a mild recession, normal growth, and a boom.

| | A | B | C | D | E | F |
|---|---|---|---|---|---|---|
| 1 | | | **Stock Fund** | | **Bond Fund** | |
| 2 | Scenario | Probability | Rate of Return | Col B × Col C | Rate of Return | Col B × Col E |
| 3 | Severe recession | 0.05 | −37 | −1.9 | −9 | −0.45 |
| 4 | Mild recession | 0.25 | −11 | −2.8 | 15 | 3.8 |
| 5 | Normal growth | 0.40 | 14 | 5.6 | 8 | 3.2 |
| 6 | Boom | 0.30 | 30 | 9.0 | −5 | −1.5 |
| 7 | Expected or Mean Return: | | SUM: | 10.0 | SUM: | 5.0 |

- ◆ The performance of stocks follows the broad economy, returning, respectively, -37%, -11%, 14%, and 30% in the four scenarios.
- ◆ In contrast, bonds perform best in a mild recession, returning 15% (since falling interest rates create capital gains), and in the normal growth scenario, where their return is 8%. They suffer from defaults in severe recession, resulting in a negative return, -9%, and from inflation in the boom scenario, where their return is -5%.
  - ✓ Notice that bonds outperform stocks in both the mild and severe recession scenarios.
  - ✓ In both normal growth and boom scenarios, stocks outperform bonds.
- ◆ The expected return on each fund equals the probability-weighted average of the outcomes in the four scenarios.
  - ✓ The last row of Spreadsheet 6.1 shows that the expected return of the stock fund is 10%, and that of the bond fund is 5%.

■ The variance is the probability-weighted average across all scenarios of the squared deviation between the actual return of the fund and its expected return; the standard deviation is the square root of the variance.

◆ These values are computed in Spreadsheet 6.2.

| | A | B | C | D | E | F | G | H | I | J |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | | | | Stock Fund | | | Bond Fund | | | |
| 2 | | | | Deviation | | | | Deviation | | |
| 3 | | | Rate | from | | Column B | Rate | from | | Column B |
| 4 | | | of | Expected | Squared | × | of | Expected | Squared | × |
| 5 | Scenario | Prob. | Return | Return | Deviation | Column E | Return | Return | Deviation | Column I |
| 6 | Severe recession | 0.05 | −37 | −47 | 2209 | 110.45 | −9 | −14 | 196 | 9.80 |
| 7 | Mild recession | 0.25 | −11 | −21 | 441 | 110.25 | 15 | 10 | 100 | 25.00 |
| 8 | Normal growth | 0.40 | 14 | 4 | 16 | 6.40 | 8 | 3 | 9 | 3.60 |
| 9 | Boom | 0.30 | 30 | 20 | 400 | 120.00 | −5 | −10 | 100 | 30.00 |
| 10 | | | | Variance = SUM | | 347.10 | | | Variance: | 68.40 |
| 11 | | Standard deviation = SQRT(Variance) | | | | 18.63 | | | Std. Dev.: | 8.27 |

■ What about the risk and return characteristics of a portfolio made up from the stock and bond funds?

◆ The portfolio return is the weighted average of the returns on each fund with weights equal to the proportion of the portfolio invested in each fund.

◆ Suppose we form a portfolio with 40% invested in the stock fund and 60% in the bond fund.

✓ Then the portfolio return in each scenario is the weighted average of the returns on the two funds. For example,

Portfolio return in mild recession = 0.40 × (-11%) + 0.60 × 15% = 4.6%

which appears in cell C6 of Spreadsheet 6.3.

| | A | B | C | D | E | F | G |
|---|---|---|---|---|---|---|---|
| 1 | | | Portfolio invested 40% in stock fund and 60% in bond fund | | | | |
| 2 | | | Rate | Column B | Deviation from | | Column B |
| 3 | | | of | × | Expected | Squared | × |
| 4 | Scenario | Probability | Return | Column C | Return | Deviation | Column F |
| 5 | Severe recession | 0.05 | −20.2 | −1.01 | −27.2 | 739.84 | 36.99 |
| 6 | Mild recession | 0.25 | 4.6 | 1.15 | −2.4 | 5.76 | 1.44 |
| 7 | Normal growth | 0.40 | 10.4 | 4.16 | 3.4 | 11.56 | 4.62 |
| 8 | Boom | 0.30 | 9.0 | 2.70 | 2.0 | 4.00 | 1.20 |
| 9 | | Expected return: | | 7.00 | | Variance: | 44.26 |
| 10 | | | | | Standard deviation: | | 6.65 |

➢ Spreadsheet 6.3 shows the rate of return of the portfolio in each scenario.

- Notice that both funds suffer in a severe downturn and, therefore, the portfolio also experiences a substantial loss of 20.2%.
  - This is manifestation of systematic risk affecting a broad spectrum of securities.
  - Declines of more than 25% in the S&P 500 Index have occurred five times in the past 88 years (1930, 1931, 1937, 1974, and 2008), roughly once every 18 years.
  - Avoiding losses in these extreme outcomes would require one to devote a large allocation of the portfolio to risk-free (and correspondingly low return) investments or (expensive) portfolio insurance.
  - Extreme events such as severe recessions make for the large standard deviation of stocks, 18.63%, and even of bonds, 8.27%.
  - Still, the overall standard deviation of the diversified portfolio, 6.65%, is considerably smaller than that of stocks and even smaller than that of bonds.

- The low risk of the portfolio is due to the inverse relationship between the performances of the stock and bond funds.
  - In a mild recession, stocks fare poorly, but this is offset by the large positive return of the bond fund.
  - Conversely, in a boom scenario, bond prices fall, but stocks do well.
  - Notice that while the portfolio's expected return is just the weighted average of the expected return of the two assets, *the portfolio standard deviation is actually less than that of either component fund*.
- Portfolio risk is reduced most when the returns of the two assets most reliably offset each other.
  - The natural question investors should ask, therefore, is how one can measure the tendency of the returns on two assets to vary either in tandem or in opposition to each other.
  - The statistics that provide this measure are the covariance and the correlation coefficient.

- The covariance is calculated in a manner similar to the variance.
  - Instead of multiplying the difference of an asset return from its expected value by itself (i.e., squaring it), we multiply it by the deviation of the *other* asset return from *its* expectation.
  - The sign and magnitude of this product are determined by whether deviations from the mean move together (i.e., are both positive or negative in the same scenarios) and whether they are small or large at the same time.
- We start in Spreadsheet 6.4 with the deviation of the return on each fund from its expected or mean value.

| | A | B | C | D | E | F |
|---|---|---|---|---|---|---|
| 1 | | | Deviation from Mean Return | | Covariance | |
| 2 | Scenario | Probability | Stock Fund | Bond Fund | Product of Dev | Col B × Col E |
| 3 | Severe recession | 0.05 | −47 | −14 | 658 | 32.9 |
| 4 | Mild recession | 0.25 | −21 | 10 | −210 | −52.5 |
| 5 | Normal growth | 0.40 | 4 | 3 | 12 | 4.8 |
| 6 | Boom | 0.30 | 20 | −10 | −200 | −60.0 |
| 7 | | | | Covariance = | SUM: | −74.8 |
| 8 | Correlation coefficient = Covariance/(StdDev(stocks)*StdDev(bonds)) = | | | | | −0.49 |

- For each scenario, we multiply the deviation of the stock fund return from its mean by the deviation of the bond fund return from its mean.
  - ✓ The product will be positive if both asset returns exceed their respective means in that scenario or if both fall short of their respective means, implying that both assets perform well or poorly in the same scenarios.
  - ✓ The product will be negative if one asset exceeds its mean return, while the other falls short of its mean return, implying that one asset performs well when the other is performing poorly.
- Spreadsheet 6.4 shows that the stock fund return in a mild recession falls short of its expected value by 21%, while the bond fund return exceeds its mean by 10%.
  - ✓ Therefore, the product of the two deviations is $-21 \times 10 = -210$, as reported in column E.
- The product of deviations is negative if one asset performs well when the other is performing poorly.
- The product of deviations is positive if both assets perform well or poorly in the same scenarios.

- The probability-weighted average of the products is called *covariance* and measures the *average* tendency of the asset returns to vary in tandem, that is, to co-vary.
  - The formula for the covariance of the returns on the stock and bond portfolios is given in Equation 6.1.

$$\text{Cov}(r_S, r_B) = \sum_{i=1}^{n} p(i)[r_S(i) - E(r_S)][r_B(i) - E(r_B)] \tag{6.1}$$

    - ✓ Each particular scenario in this equation is labeled or "indexed" by *i*.
    - ✓ In general, *i* ranges from scenario 1 to *n* (the total number of scenarios; here, $n = 4$).
    - ✓ The probability of each scenario is denoted $p(i)$.
  - The covariance of the stock and bond funds is computed in the next-to-last line of Spreadsheet 6.4 using Equation 6.1.
  - The negative value for the covariance indicates that the two assets, on average, vary inversely; when one asset performs well, the other tends to perform poorly.

- Like variance, the unit of covariance is percent square, which is why it is difficult to interpret its magnitude.
  - For instance, does the covariance of -74.8 in cell F7 indicate that the inverse relationship between the returns on stock and bond funds is strong? It's hard to say.
- An easier statistic to interpret is the *correlation coefficient*, which is the covariance divided by the product of the standard deviations of the returns on each fund.
  - We denote the correlation coefficient by the Greek letter rho, ρ.

$$\text{Correlation coefficient} = \rho_{SB} = \frac{\text{Cov}(r_S,\ r_B)}{\sigma_S \sigma_B} = \frac{-74.8}{18.63 \times 8.27} = -.49 \qquad (6.2)$$

- Correlations can range from values of -1 to + 1.
  - A correlation of -1 indicates that one asset's return varies perfectly inversely with the other's.
    - ✓ If you were to do a linear regression of one asset's return on the other, the slope coefficient would be negative and the R-square of the regression would be 100%, indicating a perfect fit.

- ✓ The R-square is the square of the correlation coefficient and tells you the fraction of the variance of one return explained by the other.
- ✓ With a correlation of -1, you could predict 100% of the variability of one asset's return if you knew the return on the other asset.

◆ Conversely, a correlation of +1 would indicate perfect positive correlation and also would imply an R-square of 100%.

◆ A correlation of zero indicates that the returns on the two assets are unrelated.

■ The correlation coefficient of $\rho_{SB} = -0.49$ in Equation 6.2 confirms the tendency of the returns on the stock and bond funds to vary inversely.

◆ In fact, a fraction of $(-.49)^2 = .24$ of the variance of stocks can be explained by the returns on bonds.

■ Equation 6.2 shows that whenever the covariance in called for in a calculation we can replace it with the following expression using the correlation coefficient:

$$\text{Cov}(r_S, r_B) = \sigma_{SB} = \sigma_{BS} = \rho_{SB}\sigma_S\sigma_B \qquad (6.3)$$

## ● Using Historical Data

- We've seen that portfolio risk and return depend on the means and variances of the component securities, as well as on the covariance between their returns.
  - One way to obtain these inputs is a scenario analysis as in Spreadsheets 6.1-6.4.
- A common alternative approach to produce these inputs is to make use of historical data.
  - The idea is that variability and covariability change slowly over time.
  - Thus, if we estimate these statistics from recent data, our estimates will provide useful predictions for the near future—perhaps next month or next quarter.
- In this approach, we use realized returns to estimate variance and covariance.
  - Means cannot be as precisely estimated from past returns.
    - ✓ We discuss mean returns in great detail later.
  - The estimate of variance is the average value of the squared deviations around the sample average.
  - The estimate of the covariance is the average value of the cross-product of deviations.

- Instead of using mean returns based on the scenario analysis, we use average returns during the sample period.
- We can illustrate this approach with a simple example.

- Example 6.1: *Using Historical Data to Estimate Means, Standard Deviations, Covariance, and Correlation*
  - Study yourself.
- Two comments on Example 6.1 are in order.
  - First, you may recall from a statistics class and from Chapter 5 that when variance is estimated from a sample of $n$ observed returns, it is common to divide the squared deviations by $n - 1$ rather than by $n$.
    - ✓ This is because we take deviations from an estimated average return rather than the true (but unknown) expected return; this procedure is said to adjust for a "lost of freedom."
    - ✓ In Excel, the function STDEVP computes standard deviation dividing by $n$, while the function STDEV uses $n - 1$.
    - ✓ Excel's covariance and correlation functions both use $n$.

- ✓ In Example 6.1, we ignored this fine point, and divided by *n* throughout.
  - ➢ In any event, the correction for the lost degree of freedom is negligible when there are plentiful observations.
  - ➢ For example, with 60 returns (e.g., five years of monthly data), the difference between dividing by 60 or 59 will affect variance or covariance by a factor of only 1.017.

◆ Second, we repeat the waring about the statistical reliability of historical estimates.

- ✓ Estimates of variance and covariance from past data are generally reliable forecasts (at least for the short term).
- ✓ However, averages of past returns typically provide highly noisy (i.e., imprecise) forecasts of future expected returns.
- ✓ In this example, we use past averages from small samples because our objective is to demonstrate the methodology.
- ✓ In practice, professional investors spend most of their resources on macroeconomic and security analysis to improve their estimates of mean returns.

## The Three Rules of Two-Risky Assets Portfolios

- Suppose a proportion denoted by $w_B$ is invested in the bond fund, and the remainder $1 - w_B$, denoted by $w_S$, is invested in the stock fund. The properties of the portfolio are determined by the following three rules, which apply the rules of statistics governing combinations of random variables:
  - *Rule 1: The rate of return on the portfolio is the weighted average of the returns on the component securities, with the investment proportions as weights.*

$$r_P = w_B r_B + w_S r_S \tag{6.4}$$

  - *Rule 2: The expected rate of return on the portfolio is the weighted average of the expected returns on the component securities, with the portfolio proportions as weights.*

$$E(r_P) = w_B E(r_B) + w_S E(r_S) \tag{6.5}$$

- ✓ Rules 1 and 2 say that a portfolio's actual return and its mean return are linear functions of the component security returns and portfolio weights.
  - ➢ This is not so for portfolio variance, as the third rule shows.

◆ *Rule3 : The variance of the rate of return on a two-risky-asset portfolio is*

$$\begin{aligned}\sigma_P^2 &= \mathrm{Var}(r_P) = \mathrm{Var}(w_B r_B + w_S r_S) = w_B^2 \mathrm{Var}(r_B) + w_S^2 \mathrm{Var}(r_S) + 2 w_B w_S \mathrm{Cov}(r_B,\ r_S) \\ &= w_B^2 \sigma_B^2 + w_S^2 \sigma_S^2 + 2 w_B w_S \sigma_{BS} \\ &= (w_B \sigma_B)^2 + (w_S \sigma_S)^2 + 2 (w_B \sigma_B)(w_S \sigma_S) \rho_{BS} \quad (\text{Note: } \sigma_{BS} = \sigma_B \sigma_S \rho_{BS})\end{aligned} \tag{6.6}$$

*where $\rho_{BS}$ is the correlation coefficient between the returns on the stock and bond funds.*

◆ The variance of the portfolio is the *sum* of the contributions of the component security variances *plus* a term that involves the correlation coefficient (and hence, covariance) between the returns on the component securities.

- ✓ We know from the last section why this last term arises.
  - ➢ When the correlation between the component securities is small or negative, there will be a greater tendency for returns on the two assets to offset each other. This will reduce portfolio risk.
- ✓ Equation 6.6 tells us that portfolio variance is lower when the correlation coefficient is lower.

■ The formula describing portfolio variance is more complicated than that describing portfolio return.

- ◆ This complication has a virtue, however: a tremendous potential for gains from diversification.

## ● The Risk-Return Trade-Off with Two-Risky-Assets Portfolios

■ We can assess the benefit from diversification by using Rules 2 and 3 to compare the risk and expected return of a better-diversified portfolio to a less-diversified benchmark.

- ◆ Suppose an investor estimates the following input data:

  $E(r_B) = 5\%$; $\sigma_B = 8\%$; $E(r_S) = 10\%$; $\sigma_S = 19\%$; $\rho_{BS} = .2$

- ✓ Currently, all funds are invested in the bond fund, but the investor ponders a portfolio invested 40% in stocks and 60% in bonds.

- ◆ Using Rule 2, the expected return of this portfolio is

$E(r_P) = .4 \times 10\% + .6 \times 5\% = 7\%$

which represents a gain of 2% compared to a bond-only investment.

- ◆ Using Rule 3, the portfolio standard deviation is

$$\sigma = \sqrt{(.4\times 19\%)^2 + (.6\times 8\%)^2 + 2(.4\times 19\%)\times(.6\times 8\%)\times .2} = 9.76\%$$

which is less than the weighted average of the component standard deviations: $0.4 \times 19 + 0.6 \times 8 = 12.40\%$.

- ✓ The difference of 2.64% reflects the benefits of diversification.

- ◆ This benefit is cost-free in the sense that diversification allows us to experience the full contribution of the stock's higher expected return, while keeping the portfolio standard deviation below the average of the component standard deviations.

- Example 6.2: *Benefits from Diversification*
  - Suppose we invest 85% in bonds and only 15% in stocks.
    - ✓ We can construct a portfolio with an expected return higher than bonds $(.85 \times 5\%) + (.15 \times 10\%) = 5.75\%$ and, at the same time, a standard deviation less than bonds. Using Equation 6.6 again, we find that the portfolio variance is

      $$(.85 \times 8\%)^2 + (.15 \times 19\%)^2 + 2(.85 \times 8\%)(.15 \times 19\%) \times .2 = 62.1$$

      and, accordingly, the portfolio standard deviation is $\sqrt{62.1} = 7.88\%$, which is less than the standard deviation of either bonds or stocks alone.
  - Taking on a more volatile asset (stocks) actually reduces portfolio risk!
    - ✓ Such is the power of diversification.
- We can find investment proportions that will reduce portfolio risk even further.
  - The risk-minimizing proportions will be 90.7% in bonds and 9.3% in stocks.
    - ✓ With these proportions, the portfolio standard deviation will be 7.80%, and the portfolio's expected return will be 5.47%.

- Is this portfolio preferable to the one considered in Example 6.2, with 15% in the stock fund?
  - ✓ That depends on investor preferences, because the portfolio with the lower variance also has a lower expected return.
- The minimum-variance portfolio is constructed to minimize the variance (and hence standard deviation) of returns, regardless of the expected return.
  - ✓ With a zero correlation coefficient, the variance-minimizing proportion in the bond fund is given by the expression: $\sigma_S^2/(\sigma_B^2+\sigma_S^2)$.
  - ✓ See the following proof:

$$\text{Find } w_B \text{ to minimize } \sigma_P^2 = w_B^2\sigma_B^2 + w_S^2\sigma_S^2 + 2w_B w_S \rho_{BS}\sigma_B\sigma_S$$
$$= w_B^2\sigma_B^2 + (1-w_B)^2\sigma_S^2 + 2w_B(1-w_B)\rho_{BS}\sigma_B\sigma_S$$

The first order condition $=0$ with respect to $w_B$

$$\Rightarrow 2w_B\sigma_B^2 + 2(1-w_B)\sigma_S^2(-1) + (2-4w_B)\rho_{BS}\sigma_B\sigma_S = 0$$

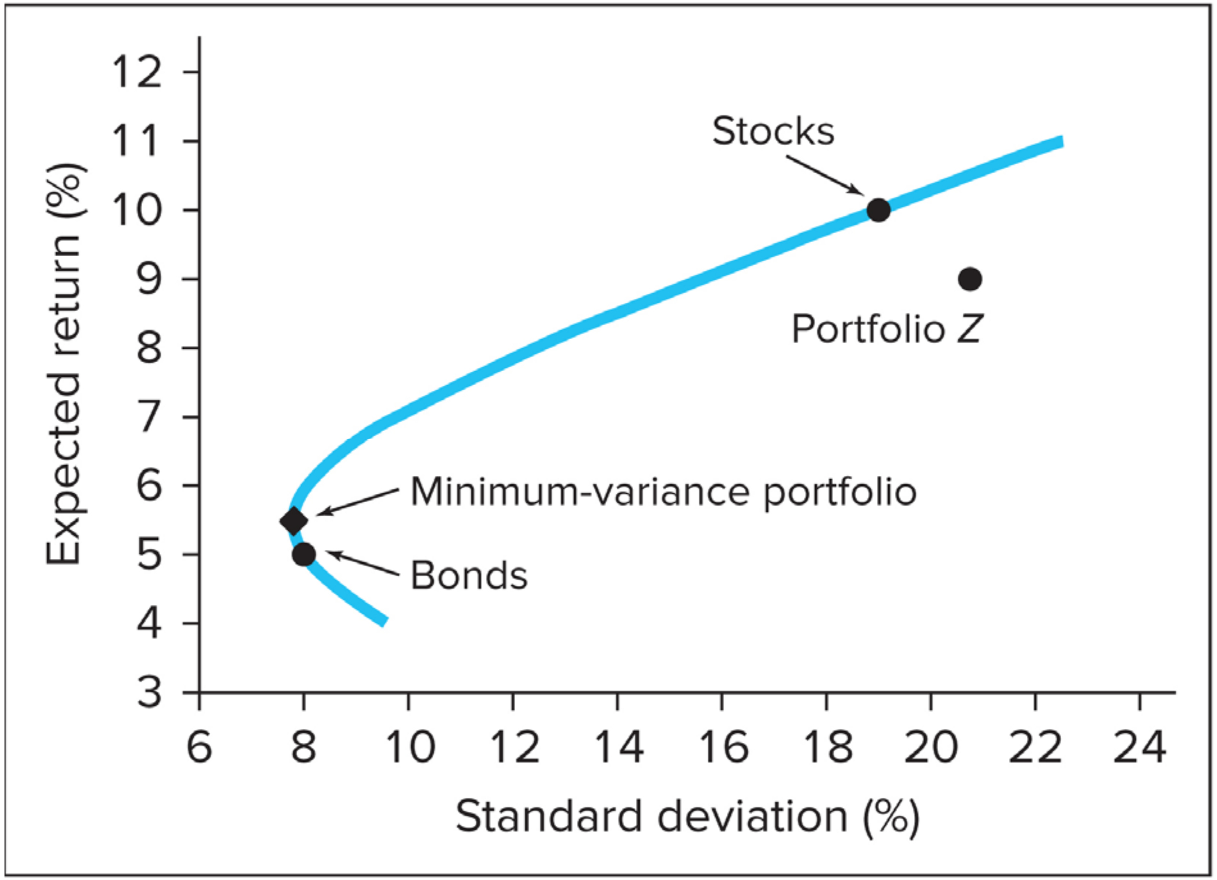
$$\Rightarrow w_B = \frac{\sigma_S^2 - \rho_{BS}\sigma_B\sigma_S}{\sigma_S^2 + \sigma_B^2 - 2\rho_{BS}\sigma_B\sigma_S}$$

$$\Rightarrow w_B = \frac{\sigma_S^2}{\sigma_S^2 + \sigma_B^2} \text{ if } \rho_{BS} = 0$$

- What the analyst can and must do is show investors the entire **investment opportunity set.**
  - This is the set of all attainable combinations of risk and return offered by portfolios formed using the available assets in differing proportions.
  - We find the investment opportunity set using Spreadsheet 6.5.
    - ✓ Columns A and B set out several different proportions for investments in the stock and bond funds.

| | A | B | C | D | E |
|---|---|---|---|---|---|
| 1 | | | **Input Data** | | |
| 2 | $E(r_S)$ | $E(r_B)$ | $\sigma_S$ | $\sigma_B$ | $\rho_{BS}$ |
| 3 | 10 | 5 | 19 | 8 | 0.2 |
| 4 | Portfolio Weights | | Expected Return, $E(r_p)$ | | Std Dev |
| 5 | $w_S = 1 - w_B$ | $w_B$ | Col A*A3 + Col B*B3 | | (Equation 6.6) |
| 6 | −0.2 | 1.2 | 4.0 | | 9.59 |
| 7 | −0.1 | 1.1 | 4.5 | | 8.62 |
| 8 | 0.0 | 1.0 | 5.0 | | 8.00 |
| 9 | 0.0932 | 0.9068 | 5.5 | | 7.804 |
| 10 | 0.1 | 0.9 | 5.5 | | 7.81 |
| 11 | 0.2 | 0.8 | 6.0 | | 8.07 |
| 12 | 0.3 | 0.7 | 6.5 | | 8.75 |
| 13 | 0.4 | 0.6 | 7.0 | | 9.77 |
| 14 | 0.5 | 0.5 | 7.5 | | 11.02 |
| 15 | 0.6 | 0.4 | 8.0 | | 12.44 |
| 16 | 0.7 | 0.3 | 8.5 | | 13.98 |
| 17 | 0.8 | 0.2 | 9.0 | | 15.60 |
| 18 | 0.9 | 0.1 | 9.5 | | 17.28 |
| 19 | 1.0 | 0.0 | 10.0 | | 19.00 |
| 20 | 1.1 | −0.1 | 10.5 | | 20.75 |
| 21 | 1.2 | −0.2 | 11.0 | | 22.53 |
| 22 | Notes: | | | | |
| 23 | 1. Negative weights indicate short positions. | | | | |
| 24 | 2. The weights of the minimum-variance portfolio are computed using the formula in Footnote 1. | | | | |

- ✓ The next columns present the portfolio expected returns and standard deviation corresponding to each allocation.
- ✓ These risk-return combinations are plotted in Figure 6.3.



## The Mean-Variance Criterion

- Investors desire portfolio that lie to the "northwest" in Figure 6.3.
  - These are portfolios with high expected returns (toward the "north" of the figure) and low volatility (to the "west").
  - These preferences mean that we can compare portfolios using a *mean-variance criterion* in the following way.
    - ✓ Portfolio *A* is said to dominate portfolio *B* if all investors prefer *A* over *B*.
      - This will be the case if it has higher mean return and lower variance:

        $E(r_A) \geq E(r_B)$ and $\sigma_A \leq \sigma_B$
- Graphically, when we plot the expected return and standard deviation of each portfolio in Figure 6.3, portfolio *A* will lie to the northwest of *B*.
  - Given a choice between portfolios *A* and *B*, *all* investors would choose *A*.
    - ✓ For example, the stock fund in Figure 6.3 dominates portfolio *Z*; the stock fund has higher expected return and lower volatility.

- Portfolios that lie below the minimum-variance portfolio in the figure can therefore be rejected out of hand as inefficient.
  - Any portfolio on the downward-sloping portion of the curve (including the bond fund) is “dominated” by the portfolio that lies directly above it on the upward-sloping portion of the curve since that portfolio has higher expected return and equal standard deviation.
  - The best choice among the portfolios on the upward sloping portion of the curve is not as obvious, because in this region higher expected return is accompanied by greater risk.
    - ✓ We will discuss the best choice when we introduce the risk-free asset to the portfolio decision.
- So far we have assumed a correlation of .2 between stock and bond returns. We know that low correlations aid diversification and that a higher correlation coefficient results in a reduced effect of diversification.
  - What are the implications of perfect positive correlation between bonds and stocks?

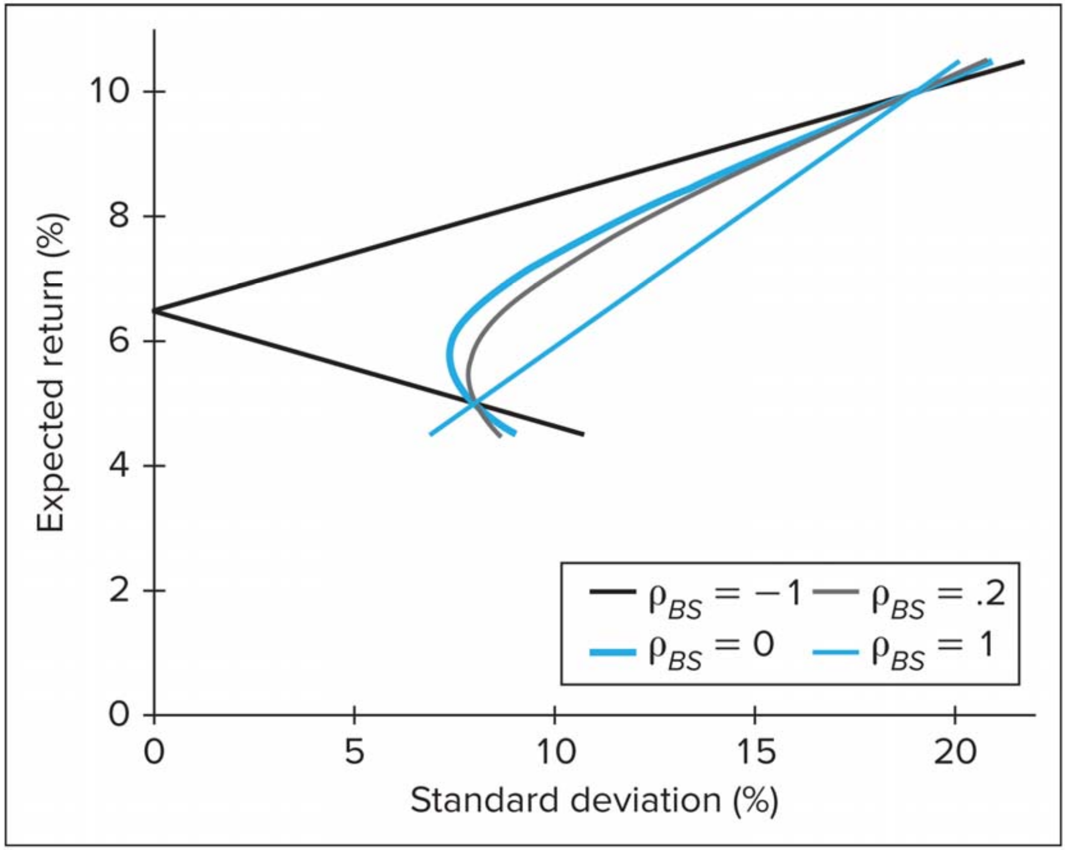
- A correlation coefficient of 1 simplifies Equation 6.6 for portfolio variance.
  - Looking at it again, you will see that substitution of $\rho_{BS} = 1$ in Equation 6.6 allows us to "complete the square" of the quantities $w_B\sigma_B$ and $w_S\sigma_S$ to obtain

$$\sigma_P^2 = (w_B\sigma_B)^2 + (w_S\sigma_S)^2 + 2(w_B\sigma_B)(w_S\sigma_S)\rho_{BS} \qquad (6.6)$$

$$\Rightarrow \quad \sigma_P^2 = w_B^2\sigma_B^2 + w_S^2\sigma_S^2 + 2w_B\sigma_B w_S\sigma_S = (w_B\sigma_B + w_S\sigma_S)^2$$

$$\Rightarrow \quad \sigma_P = w_B\sigma_B + w_S\sigma_S$$

    - ✓ The portfolio standard deviation is a weighted average of the component security standard deviations only in the special case of perfect positive correlation.
    - ✓ In this circumstance, there are no gains to be had from diversification.
    - ✓ Both the portfolio mean and the standard deviation are simple weighted averages.
- Figure 6.4 shows the opportunity set with perfect positive correlation—a straight line through the component securities.




- ◆ No portfolio can be discarded as inefficient in this case, and the choice among portfolios depends only on risk aversion.
- ◆ Diversification in the case of perfect positive correlation is not effective.

■ Perfect positive correlation is the *only* case in which there is no benefit from diversification.

- Whenever $\rho < 1$, the portfolio standard deviation is less than the weighted average of the standard deviations of the component securities.
  - Therefore, *there are benefits to diversification whenever asset returns are less than perfectly correlated.*
- Our analysis has ranged from very attractive diversification benefits ($\rho_{BS} < 0$) to no benefits at all ($\rho_{BS} = 1$).
  - For $\rho_{BS}$ within this range, the benefits will be somewhere in between.
- A realistic correlation coefficient between stocks and bonds based on historical experience is actually around .20.
  - The expected returns and standard deviations that we have so far assumed also reflect historical experience, which is why we include a graph for $\rho_{BS} = .2$ in Figure 6.4.
- Spreadsheet 6.6 enumerates some of the points on the various opportunity sets in Figure 6.4.
  - As the figure illustrates, $\rho_{BS} = .2$ is a lot better for diversification than perfect positive correlation and a bit worse than zero correlation.

| | A | B | C | D | E | F | G |
|---|---|---|---|---|---|---|---|
| 1 | | **Input Data** | | | | | |
| 2 | $E(r_S)$ | $E(r_B)$ | $\sigma_S$ | $\sigma_B$ | | | |
| 3 | 10 | 5 | 19 | 8 | | | |
| 4 | | | | | | | |
| 5 | Weights in Stocks | Portfolio Expected Return | Portfolio Standard Deviation[1] for Given Correlation, $\rho$ | | | | |
| 6 | $w_S$ | $E(r_P)$ = Col A*A3 + (1 − Col A)*B3 | −1 | 0 | 0.2 | 0.5 | 1 |
| 7 | −0.1 | 4.5 | 10.70 | 9.00 | 8.62 | 8.02 | 6.90 |
| 8 | 0.0 | 5.0 | 8.00 | 8.00 | 8.00 | 8.00 | 8.00 |
| 9 | 0.1 | 5.5 | 5.30 | 7.45 | 7.81 | 8.31 | 9.10 |
| 10 | 0.2 | 6.0 | 2.60 | 7.44 | 8.07 | 8.93 | 10.20 |
| 11 | 0.3 | 6.5 | 0.10 | 7.99 | 8.75 | 9.79 | 11.30 |
| 12 | 0.4 | 7.0 | 2.80 | 8.99 | 9.77 | 10.83 | 12.40 |
| 13 | 0.6 | 8.0 | 8.20 | 11.84 | 12.44 | 13.29 | 14.60 |
| 14 | 0.8 | 9.0 | 13.60 | 15.28 | 15.60 | 16.06 | 16.80 |
| 15 | 1.0 | 10.0 | 19.00 | 19.00 | 19.00 | 19.00 | 19.00 |
| 16 | 1.1 | 10.5 | 21.70 | 20.92 | 20.75 | 20.51 | 20.10 |
| 17 | | | | Minimum-Variance Portfolio[2,3,4,5] | | | |
| 18 | $w_S(\min) = (\sigma_B\text{^}2 - \sigma_B\sigma_S\rho)/(\sigma_S\text{^}2 + \sigma_B\text{^}2 - 2^*\sigma_B\sigma_S\rho) =$ | | 0.2963 | 0.1506 | 0.0923 | −0.0440 | −0.7273 |
| 19 | $E(r_P) - w_S(\min)^*A3 + (1 - w_S(\min))^*B3 =$ | | 6.48 | 5.75 | 5.46 | 4.78 | 1.36 |
| 20 | | $\sigma_P =$ | 0.00 | 7.37 | 7.80 | 7.97 | 0.00 |

- ✓ $\sigma_P$ = SQRT[(Col A*C3)^2 + ((1 – Col A)*D3)^2

  + 2*Col A*C3*(1 – Col A)*D3*ρ]
- ✓ The standard deviation is calculated from Equation 6.6 using the weights of the minimum-variance portfolio:

  $\sigma_P$ = SQRT[($w_S$(min)*C3)^2 + ((1 –$w_S$(min))*D3)^2

  + 2* $w_S$(min)*C3*(1 –$w_S$(min))*D3*ρ]
- ✓ As the correlation coefficient grows, the minimum-variance portfolio requires a smaller position in stocks (even a negative position for higher correlation), and the performance of this portfolio become less attractive.
- ✓ Notice that with correlation of .5 or higher, minimum variance is achieved with a short position in stocks. The standard deviation is lower than that of bonds, but the mean is lower as well.
- ✓ With perfect positive correlation (column G), you can drive the standard deviation to zero by taking a large, short position in stocks. The mean return is then as low as 1.36%

- Negative correlation between a pair of assets is also possible.
  - When correlation is negative, there will be even greater diversification benefits.
- Again, let us start with an extreme.
  - With perfect negative correlation, we substitute $\rho_{BS}$ = -1 in Equation 6.6 and simplify it by completing the square.

$$\sigma_P^2 = (w_B\sigma_B)^2 + (w_S\sigma_S)^2 + 2(w_B\sigma_B)(w_S\sigma_S)\rho_{BS}$$

$$\Rightarrow \quad \sigma_P^2 = w_B^2\sigma_B^2 + w_S^2\sigma_S^2 - 2w_B\sigma_B w_S\sigma_S = (w_B\sigma_B - w_S\sigma_S)^2$$

and, therefore,

$$\sigma_P = ABS[w_B\sigma_B - w_S\sigma_S] \tag{6 .7}$$

- ✓ The right-hand side of Equation 6.7 denotes the absolute value of $w_B\sigma_B - w_S\sigma_S$.
- ✓ The solution involves the absolute value because standard deviation cannot be negative.

- With perfect negative correlation, the benefits from diversification stretch to the limit.
  - Equation 6.7 yields the proportions that will reduce the portfolio standard deviation all the way to zero.
  - See the following proof:

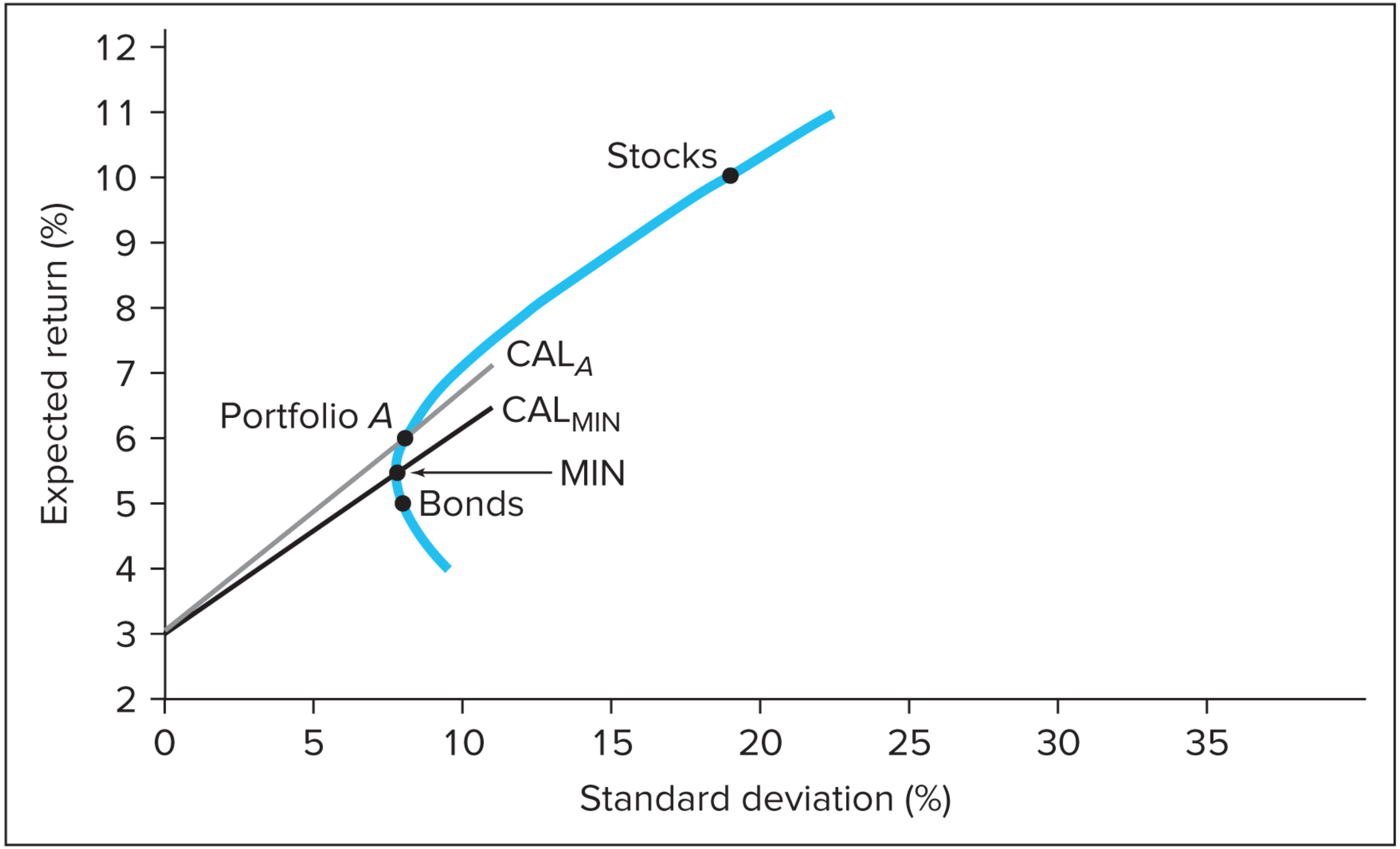
$$\sigma_P = ABS[w_B\sigma_B - w_S\sigma_S] = 0$$
$$\Rightarrow \quad w_B\sigma_B - w_S\sigma_S = 0 \text{ and Note: } w_B + w_S = 1$$
$$\Rightarrow \quad w_B = \frac{\sigma_S}{\sigma_B + \sigma_S} \text{ and } w_S = 1 - w_B = \frac{\sigma_B}{\sigma_B + \sigma_S}$$

  - Compare this formula to the formula for the variance-minimizing proportions when $\rho = 0$. $(w_B = \dfrac{\sigma_S^2}{\sigma_B^2 + \sigma_S^2})$
  - With our data, this will happen when $w_B$ = 70.37% and $w_S$ = 29.63%.
  - While exposing us to zero risk, investing 29.63% in stocks (rather than placing all funds in bonds) will still increase the portfolio expected return from 5% to 6.48%.
  - Of course, we can hardly expect results this attractive in reality.

## 6.3 THE OPTIMAL RISKY PORTFOLIO WITH A RISK-FREE ASSET

- Now we can expand the asset allocation problem to include a risk-free asset.
  - Let us continue to use the input data from Spreadsheet 6.5.
  - Suppose then that we are still confined to the risky bond and stock funds, but now can also invest in risk-free T-bills yielding 3%.
  - When we add the risk-free asset to a stock-plus-bond risky portfolio, the resulting opportunity set is the straight line that we called the CAL (capital allocation line) in Chapter 5.
  - We now consider various CALs constructed from risk-free bills and a variety of possible risky portfolios, each formed by combining the stock and bond funds in alternative proportions.

■ We start in Figure 6.5 with the opportunity set of risky assets constructed only from the bond and stock funds.



- ◆ The lowest-variance risky portfolio is labeled MIN (denoting the *minimum-variance portfolio*).
  - ✓ $CAL_{MIN}$ is drawn through it and shows the risk-return trade-off with various positions in T-bills and portfolio MIN.
- ◆ It is immediately evident from the figure that we could do better (i.e., obtain a higher Sharpe ratio) by using portfolio *A* instead of MIN as the risky portfolio.
  - ✓ $CAL_A$ dominates $CAL_{MIN}$, offering a higher expected return for any level of volatility.
    - ➢ Spreadsheet 6.6 (see bottom panel of column E) shows that portfolio MIN's expected return is 5.46%, and its standard deviation (SD) is 7.80%.
    - ➢ Portfolio *A* (row 10 in Spreadsheet 6.6) offers an expected return of 6% with an SD of 8.07%.

■ The slope of the CAL that uses a risky portfolio, *P*, is the Sharpe ratio of that portfolio, that is, the ratio of its expected excess return to its standard deviation:

$$S_P = \frac{E(r_P) - r_f}{\sigma_P} \tag{6.8}$$

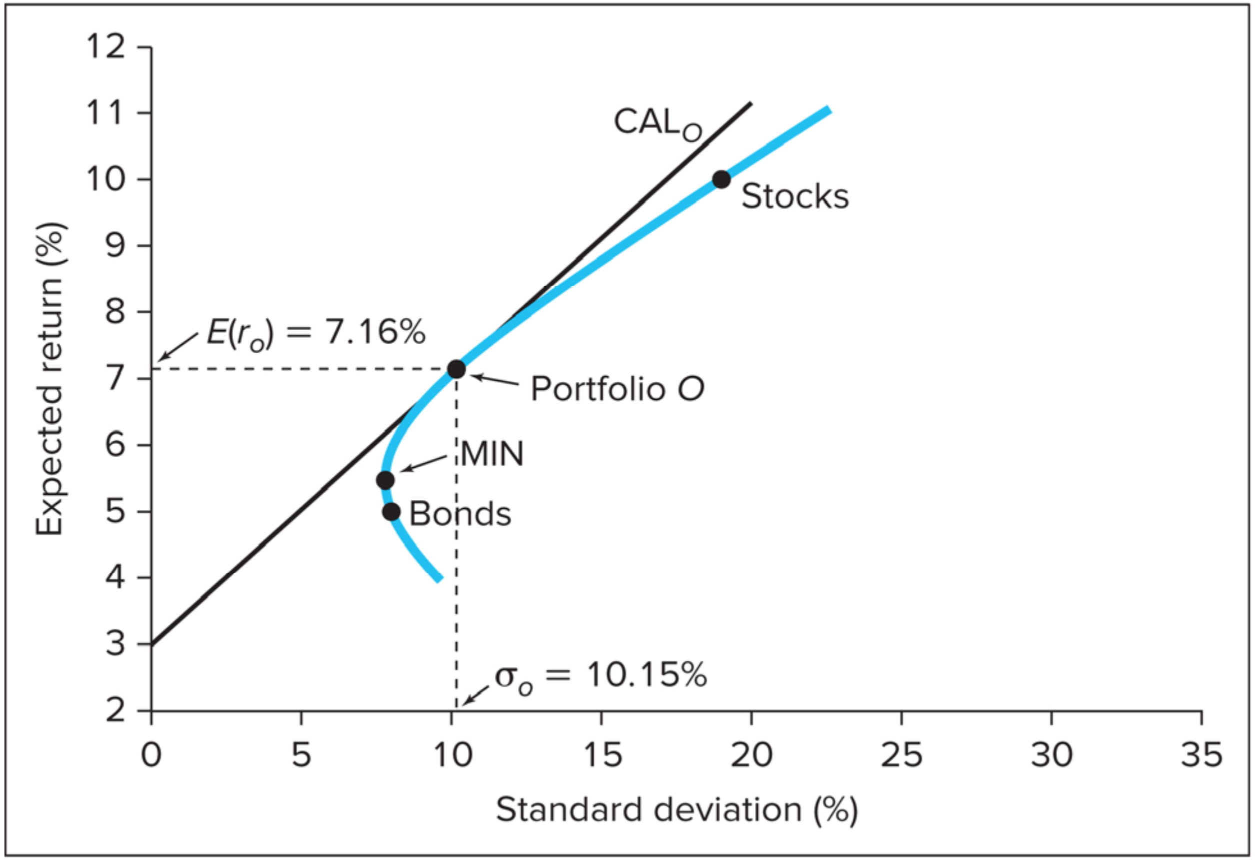
- With T-bill rate of 3% we obtain the Sharpe ratio of the two portfolios:

$$S_{\text{MIN}} = \frac{5.46-3}{7.80} = .32 \qquad S_A = \frac{6-3}{8.07} = .37 \tag{6.9}$$

  - ✓ The higher ratio for portfolio *A* compared to *MIN* measures the improvement it offers in the risk-return trade-off.

- But why stop at portfolio *A*?
  - We can continue to ratchet the CAL upward until it reaches the ultimate point of tangency with the investment opportunity set.
    - ✓ This must yield the CAL with the highest feasible Sharpe ratio.
  - Therefore, the tangency portfolio (*O*) in Figure 6.6 is the **optimal risky portfolio** to mix with T-bills, which may be defined as the risky portfolio that generates the steepest CAL.
    - ✓ Figure 6.6 clearly show the improvement in the risk-return trade-off obtained with $\text{CAL}_O$.

- ✓ With the exception of the optimal portfolio $O$, any portfolio formed by combining just the stock and bond funds has lower expected return than a portfolio on $CAL_O$ with the same standard deviation.



- To find the composition of the optimal risky portfolio, *O*, we search for weights in the stock and bond funds that maximize the portfolio's Sharpe ratio.
  - With only two risky assets, we can solve for the optimal portfolio weights using the following formula:

$$w_B = \frac{[E(r_B)-r_f]\sigma_S^2 - [E(r_S)-r_f]\sigma_B\sigma_S\rho_{BS}}{[E(r_B)-r_f]\sigma_S^2 + [E(r_S)-r_f]\sigma_B^2 - [E(r_B)-r_f+E(r_S)-r_f]\sigma_B\sigma_S\rho_{BS}}$$

$$w_S = 1 - w_B \qquad (6.10)$$

    - ✓ The above formula is obtained by maximizing the objective function, $S_P$, as follows:

$$\max_{w_B} S_P = \frac{E(r_P)-r_f}{\sigma_P} = \frac{w_B E(r_B) + w_S E(r_S) - r_f}{w_B^2\sigma_B^2 + w_S^2\sigma_S^2 + 2w_B w_S \sigma_B \sigma_S \rho_{BS}}$$

subject to $w_B + w_S = 1$

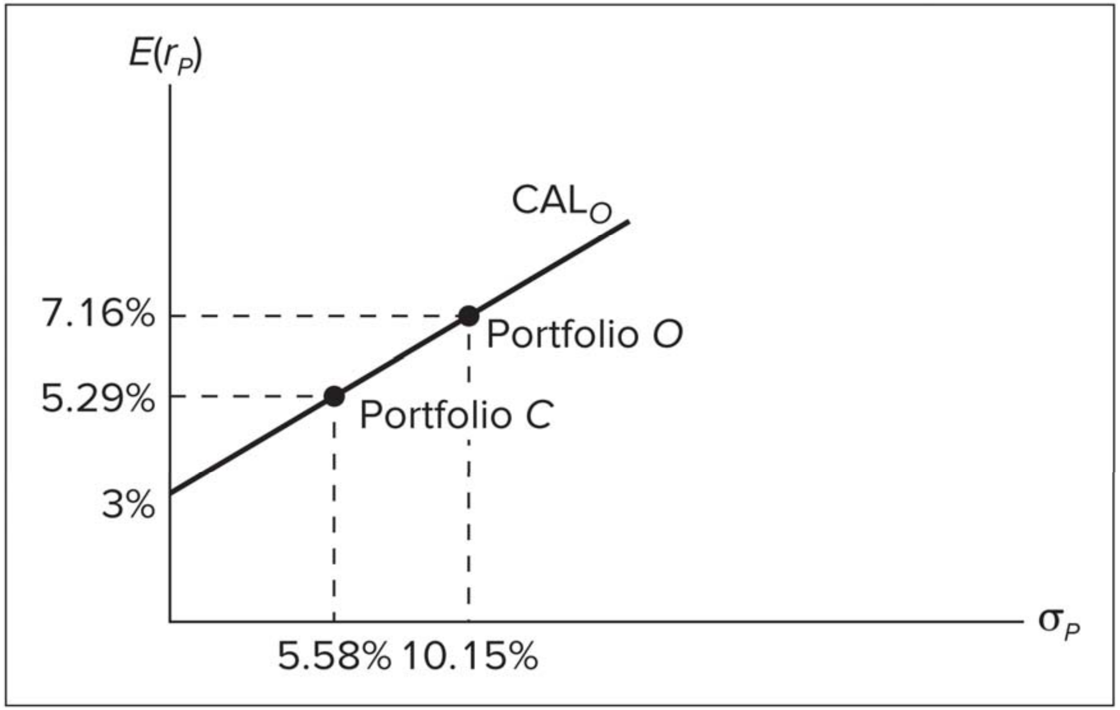
- Using the risk premiums (expected excess return over the risk-free rate) of the stock and bond funds, their standard deviations, and the correlation between their returns in Equation 6.10, we find that the weights of the optimal portfolio are $w_B(O) = .568$ and $w_S(O) = .432$.
  - Using these weights, Equations 6.5, 6.6, and 6.8 imply that $E(r_O) = 7.16\%$, $\sigma_O = 10.15\%$, and therefore the Sharpe ratio of the optimal portfolio (the slope of its CAL) is

$$S_O = \frac{E(r_O) - r_f}{\sigma_O} = \frac{7.16 - 3}{10.15} = .41$$

    - ✓ This Sharpe ratio is significantly higher than those provided by either the bond or stock portfolios alone.
- In the last chapter we saw that the preferred *complete* portfolio formed from a risky portfolio and a risk-free asset depends on the investor's risk aversion.
  - More risk-averse investors will prefer low-risk portfolios despite the lower expected return, while more risk-tolerant investors will choose higher-risk, higher-return portfolios.

- Both investors, however, will choose portfolio *O* as their risky portfolio since it provides highest return per unit of risk, that is, the steepest capital allocation line.
  - ✓ Investors will differ only in their allocation of investment funds between portfolio *O* and the risk-free asset.

- Figure 6.7 shows one possible choice for the preferred complete portfolio, *C*.



- The investor places 55% of wealth in portfolio *O* and 45% in Treasury bills.
  - ✓ The rate of return and volatility of the portfolio are

$$E(r_C) - r_f = y[E(r_P) - r_f] \quad \Rightarrow \quad E(r_C) = r_f + y[E(r_P) - r_f] \tag{5.19}$$

$$E(r_C) = 3\% + .55 \times (7.16\% - 3\%) = 5.29\%$$

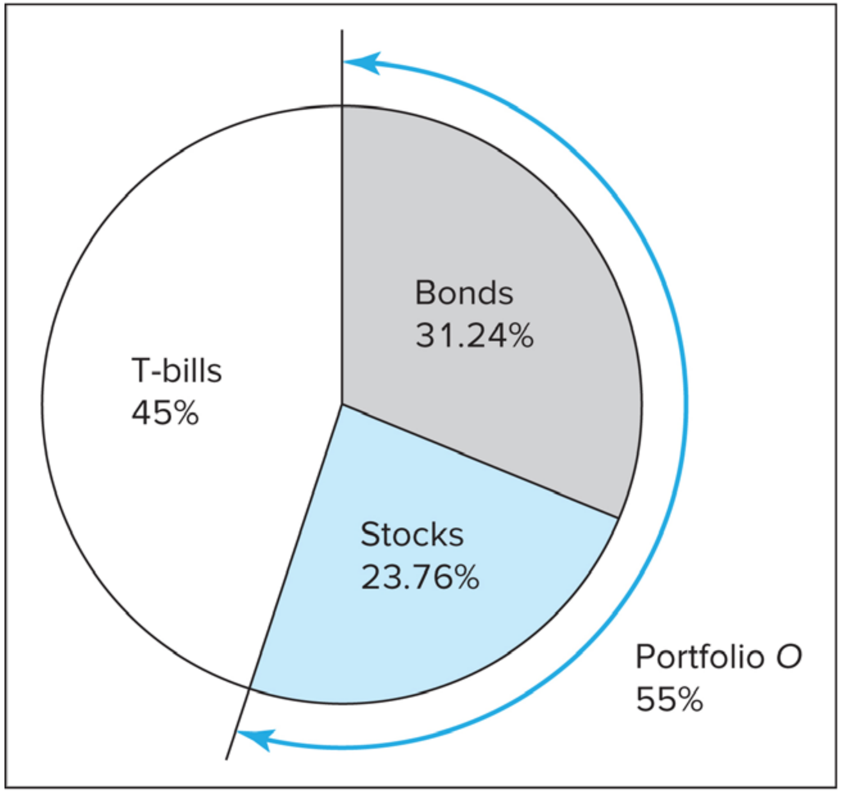
$$\sigma_C = y\sigma_P \tag{5.20}$$

$$\sigma_C = .55 \times 10.15\% = 5.58\%$$

- Because portfolio *O* is a mix of the bond fund and stock fund with weights of 56.8% and 43.2%, the overall asset allocation of the *complete* portfolio is as follows:

| | | |
|---|---|---|
| Weight in risk-free asset | | 45.00% |
| Weight in bond fund | .568 × 55% = | 31.24% |
| Weight in stock fund | .432 × 55% = | 23.76% |
| Total | | 100.00% |

- Figure 6.8 depicts the overall asset allocation.



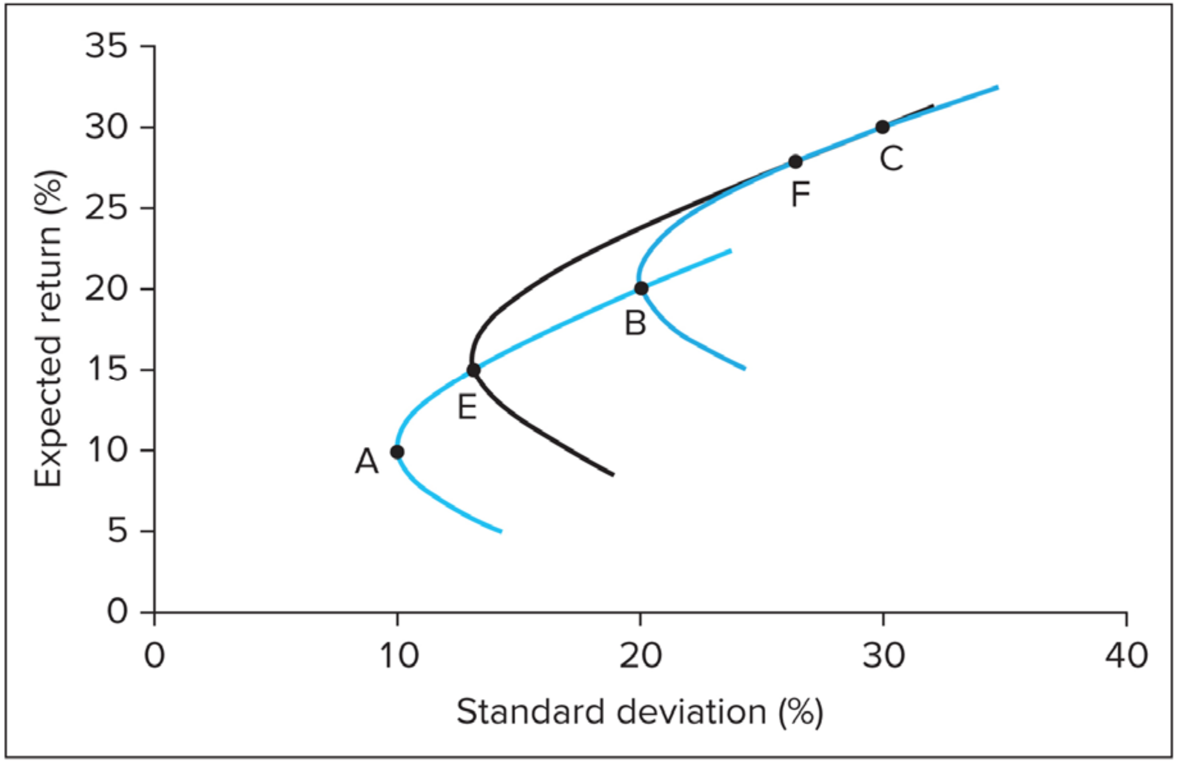

- The allocation reflects considerations of both efficient diversification (the construction of the optimal risky portfolio, *O*) and risk aversion (the allocation of funds between the risk-free asset and the risky portfolio *O* to form the complete portfolio, *C*).

## 6.4 EFFICIENT DIVERSIFICATION WITH MANY RISKY ASSETS

- We can extend the two-risky-assets portfolio methodology to many risky assets in three steps.
  - First, we generalize the two-risky-asset opportunity set to allow for many assets.
  - Next we determine the optimal risky portfolio that supports the steepest CAL, that is, maximizes the Sharpe ratio.
  - Finally, we choose a complete portfolio on $\text{CAL}_O$ based on risk aversion by mixing the risk-free asset with the optimal risky portfolio.

### The Efficient Frontier of Risky Assets

- To get a sense of how additional risky assets can improve investment opportunities, look at Figure 6.9.
  - Points *A*, *B*, and *C* represent the expected returns and standard deviations of three stocks.
    - ✓ The curve passing through *A* and *B* shows the risk-return combinations of portfolios formed from those two stocks. Similarly, the curve passing through *B* and *C* shows portfolios formed from these two stocks.

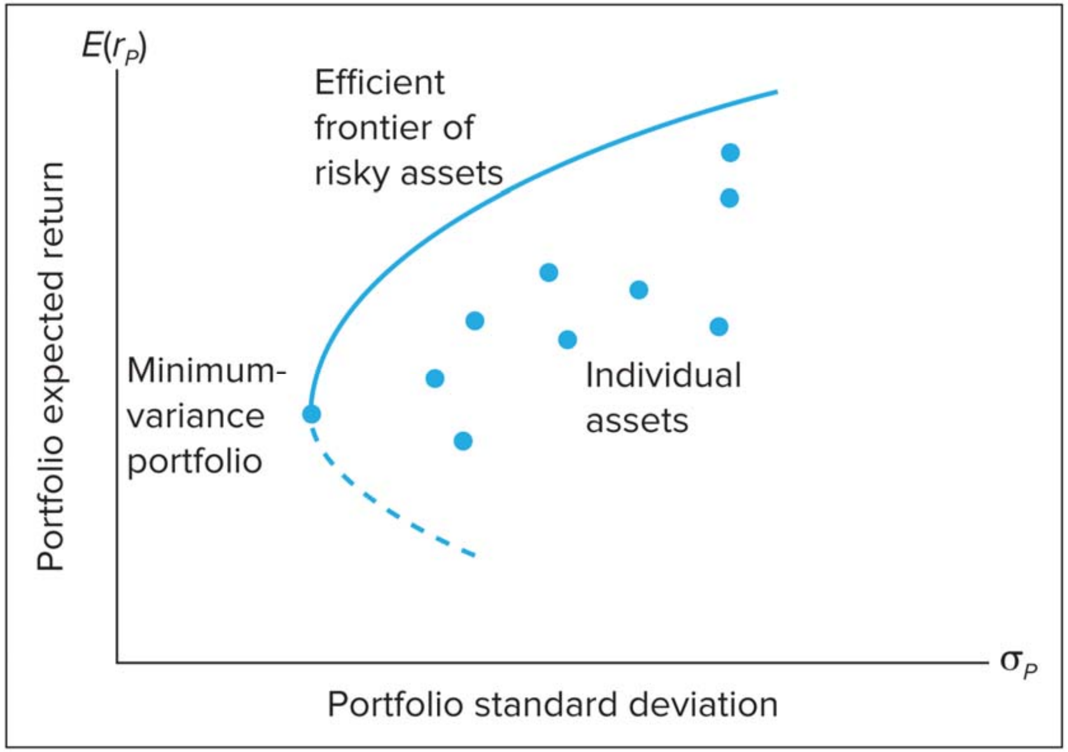



◆ Now observe point *E* on the *AB* curve and point *F* on the *BC* curve.

- ✓ These points represent two portfolios chosen from the set of *AB* and *BC* combinations.
  - ➢ The curve that passes through *E* and *F* in turn represents portfolios constructed from portfolios *E* and *F*.

- Since *E* and *F* are themselves constructed from *A*, *B*, and *C*, this curve shows some of the portfolio constructed from these *three* securities.
- Notice that curve *EF* extends the investment opportunity set to the northwest, which is the desired direction.

- Now we can continue to take other points (each representing portfolios) from these three curves and further combine them into new portfolios, thus shifting the opportunity set even farther to the northwest.
  - You can see that this process would work even better with more stocks.
  - Moreover, the boundary or "envelope" of all the curves thus developed will lie quite away from the individual stocks in the northwesterly direction, as shown in Figure 6.10.
- The analytical technique to derive the efficient frontier of risky assets was developed by Harry Markowitz at the University of Chicago in 1951 and ultimately earned him the Nobel Prize in economics. We will sketch his approach here.




- First, we determine the risk-return opportunity set.
  - ✓ The aim is to construct the northwestern-most portfolios in terms of expected return and standard deviation from the universe of securities.
    - ➢ The inputs are the expected returns and standard deviations of each asset in the universe, along with the correlation coefficients between each pair of assets.

- ✓ The graph that connects all the northwestern-most portfolios is called the **efficient frontier** of risky assets.
  - ➢ It represents the set of portfolios that offers the highest possible expected rate of return for each level of portfolio standard deviation.
  - ➢ These portfolios may be viewed as efficiently diversified.
  - ➢ One such frontier is shown in Figure 6.10.

- ■ There are three equivalent ways to produce the efficient frontier.
  - ◆ We will sketch each in a way that allows you to participate and gain insight into the logic and mechanics of the efficient frontier: Take a pencil and paper and draw the graph as you follow along with our discussion.
  - ◆ For each method, first draw the horizontal axis for portfolio standard deviation and the vertical axis for risk premium.
  - ◆ We focus on the risk premium (expected excess returns), $R$, rather than total returns, $r$, so that the risk-free asset will lie at the origin (with zero SD and zero risk premium).

- ◆ We begin with the minimum-variance portfolio—make it as point *G* (for *global* minimum variance).
  - ✓ Imagine that *G*'s coordinates are .10 (SD = 10%) and .03 (risk premium = 3%); that is your first point on the efficient frontier.
  - ✓ Later, we add detail about how to find these coordinates.
- ■ The three ways to draw the efficient frontier are (1) maximize the risk premium for any level of SD; (2) minimize the SD for any level of risk premium; and (3) maximize the Sharpe ratio for any level of SD (or risk premium).
- ■ For the first method, maximizing the risk premium for any level of SD, draw a few vertical lines to the right of *G* (there can be no portfolio with SD less than *G*'s).
  - ◆ Choose the vertical line drawn at SD = 12%; we therefore search for the portfolio with the highest possible expected return consistent with an SD of 12%.
  - ◆ So we give the computer an assignment to maximize the risk premium subject to two constraints: (i) The portfolio weights sum to 1 (this is called the *feasibility* constraint, since any legitimate portfolio must have weights that sum to 1), and (ii) the portfolio SD must match the constraint value, $\sigma = .12$.

- ◆ The optimization software searches over all portfolios with $\sigma$ = .12 and finds the highest risk premium. Assume that for this portfolio $R$ = .04.
- ◆ You now have your second point on the efficient frontier. Do the same for other vertical lines to the right of .12, and when you "connect the dots," you will have drawn a frontier like that in Figure 6.10.

■ The second method is minimize the SD for any level of risk premium.

- ◆ Here, you need to draw a few horizontal lines above $G$ (portfolios lying below $G$ are inefficient because they offer a *lower* risk premium and *higher* variance than $G$). Draw the first horizontal line at $R$ = .04. Now the computer's assignment is to minimize the SD subject to the usual feasibility constraint.
- ◆ But in this method, we replace the constraint on SD by one on the portfolio's risk premium ($R$ = .04). Now the computer seeks the portfolio that is farthest to the left along the horizontal line—this is the portfolio with the lowest SD consistent with a risk premium of 4%.
  - ✓ You already know that this portfolio must be at $\sigma$ = .12, since the first point on the efficient frontier that you found using method 1 was ($\sigma$, $R$) = (.12, .04).

- Repeat this approach using other risk premiums, and you will find other points along the efficient frontier. Again, connect the dots and you will have the frontier of Figure 6.10.

- The third approach to forming the efficient frontier, maximizing the Sharpe ratio for any SD or risk premium, is easiest to visualize by revisiting Figure 6.6.
  - Observe that each portfolio on the efficient frontier provides the highest Sharpe ratio, the slope of a ray from the risk-free rate, for any choice of SD or expected return.
  - Let's start by specifying the SD constraint, achieved by using the vertical lines to the right of *G*. To each line, we draw rays from the origin at ever-increasing slopes, and we assign the computer to find the *feasible* portfolio with the highest slope.
    - ✓ This is similar to sliding up the vertical line to find the highest risk premium.
    - ✓ We must find the same frontier as that found with either of the first two methods.
  - Similarly, we could instead specify a risk-premium constraint and construct rays from the origin to horizontal lines. We assign the computer to find the *feasible* portfolio with the highest slope to the given horizontal line.
    - ✓ This is similar to sliding to the left on horizontal lines in method 2.

- We started the efficient frontier from the minimum-variance portfolio, *G*.
  - *G* is found with a program that minimizes SD subject *only* to the feasibility constraint.
  - This portfolio has the lowest SD for *any* risk premium, which is why it is called "global" minimum-variance portfolio.
- By the same principle, the optimal portfolio, *O*, will maximize the Sharpe ratio globally, subject only to the feasibility constraint.
  - Any *individual* asset ends up inside the efficient frontier, because single-asset portfolios are inefficient—they are not efficiently diversified.
- Various constraints may preclude a particular investor from choosing portfolios on the efficient frontier, however.
  - If an institution is prohibited by law from taking short positions in any asset, the portfolio manager must add constraints to the computer-optimization program that rule out negative (short) positions.

- Some clients may want to assure a minimum level of expected dividend yield.
  - In this case, input data must include a set of expected dividend yields.
  - The optimization program is made to include a constraint to ensure that the expected *portfolio* dividend yield will equal or exceed the desired level.
- Another common constraint forbids investments in companies engaged in "undesirable social activity."
  - This constraint implies that portfolio weights in these companies must equal zero.
- In principle, portfolio managers can tailor an efficient frontier to meet any particular objective.
  - Of course, satisfying constraints carries a price tag.
    - ✓ An efficient frontier subject to additional constraints will offer a lower Sharpe ratio.
    - ✓ Clients should be aware of this cost and may want to think twice about constraints that are not mandated by law.

- Deriving the efficient frontier and graphing it with any number of assets and any set of constraints is quite straightforward.
  - For a not-too-large number of assets, the efficient frontier can be computed and graphed even with a spreadsheet program.
- Imposing the restriction against short sales requires that each weight in the optimal portfolio be greater than or equal to zero.
  - One way to see whether the short-sale constraint actually matters is to find the efficient portfolio without it.
    - ✓ If one or more of the weights in the optimal portfolio turn out negative, we know the short-sale restrictions will result in a different efficient frontier with a less attractive risk-return trade-off.

## Choosing the Optimal Risky Portfolio

- The second step of the optimization plan involves the risk-free asset.
  - Using the current risk-free rate, we search for the capital allocation line with the highest Sharpe ratio (the steepest slope), as shown in Figures 6.5 and 6.6.

- The CAL formed from the optimal risky portfolio (*O*) will be tangent to the efficient frontier of risky assets discussed above.
  - This CAL dominates all alternative feasible CALs. Portfolio *O*, therefore, is the optimal risky portfolio.
- Because we know that an investor will choose a point on the CAL that mixes the *optimal* risky portfolio with T-bills, there is actually no need to either provide access to or derive the entire efficient frontier.
  - Therefore, rather than solving for the entire efficient frontier, we can proceed directly to determining the optimal portfolio.
  - This requires maximizing the Sharpe ratio subject only to the feasibility constraint.
  - The "global" maximize-Sharpe-ratio portfolio is the optimal portfolio *O*.
  - The ray from the origin to *O* and beyond is the optimal CAL.

## The Preferred Complete Portfolio and the Separation Property

- Finally, in the third step, the investor chooses the appropriate mix between the optimal risky portfolio (*O*) and T-bills, exactly as in Figure 6.7.

- A portfolio manager will offer the same risky portfolio (*O*) to all clients, no matter what their degrees of risk aversion.
  - Risk aversion comes into play only when clients select their desired point on the CAL.
    - ✓ Regardless of risk aversion, all clients will use portfolio *O* as the optimal risky investment vehicle.
- This result is called a **separation property,** introduced by James Tobin (1958), the 1983 Nobel Laureate for economics: Its name reflects the fact that portfolio choice can be separated into two independent tasks.
  - The first task, to determine the optimal risky portfolio (*O*), is surely technical.
    - ✓ Given the input data, the best risky portfolio is the same for all clients regardless of risk aversion.
  - The second task, construction of the complete portfolio from bills and portfolio *O*, is personal and depends on risk aversion.
    - ✓ Here the client is the decision maker.

- In practice, optimal risky portfolios for different clients may vary because of constraints on short sales, dividend yield, tax considerations, or other client preferences.
  - Our analysis, though, suggests that a few portfolios may be sufficient to serve the demands of a wide range of investors.
  - We see here the theoretical basis of the mutual fund industry.
    - ✓ If the optimal portfolio is the same for all clients, professional management is more efficient and less costly.
    - ✓ One management firm can serve many clients with relatively small incremental administrative costs.
- The (computerized) optimization technique is the easiest part of portfolio construction.
  - When different managers use different input data, they will develop different efficient frontiers and offer different "optimal" portfolios.
  - Therefore, the real arena of the competition among portfolio managers is in the sophisticated security analysis that produces the input estimates.

- The rule of GIGO (garbage in-garbage out) applies fully to portfolio selection.
  - ✓ If the quality of security analysis is poor, a passive portfolio such as a market-index fund will yield better results than an active portfolio tilted toward *seemingly* favorable securities.

## 6.5 A SINGLE-INDEX ASSET MARMET

- We started this chapter with the distinction between systematic and firm-specific risk.
  - Systematic risk is largely macroeconomic, affecting all securities.
  - Firm-specific risk factors affect only one particular firm or, at most, a cluster of firms.
- **Index models** are statistical models designed to estimate these two components of risk for a particular security or portfolio.
  - The first to use an index model to explain the benefits of diversification was another Nobel Prize winner, William F. Sharpe (1963).
    - ✓ We will introduce his major work (the capital asset pricing model) in the next chapter.
- The popularity of factor modes is due to their practicality.
  - To construct the efficient frontier from a universe of 100 securities, we would need to estimate 100 expected returns, 100 variances, and $100 \times 99/2 = 4{,}950$ covariances. And a universe of 100 securities is actually quite small.

- A universe of 1,000 securities would require estimates of $1{,}000 \times 999/2 = 499{,}500$ covariances, as well as 1,000 expected returns and variances.
- Assuming that one common factor is responsible for all the covariability of stock returns, with all other variability due to firm-specific factors, dramatically simplifies the analysis.

- Let us use $R_i$ to denote the **excess return** on a security, that is, the rate of return in excess of the risk-free rate: $R_i = r_i - r_f$.
  - Then we can express the distinction between macroeconomic and firm-specific factors by decomposing this excess return in some holding period into three components:

$$R_i = \beta_i R_M + e_i + \alpha_i \tag{6.11}$$

- The first two terms on the right-hand side of Equation 6.11 reflect the impact of two sources of uncertainty.
  - $R_M$ is the excess return on a broad market index (the S&P 500 is commonly used for this purpose), so variation in this term reflects the influence of economywide or macroeconomic events that generally affect all stocks to greater or lesser degrees.
  - The security's **beta,** $\beta_i$, is the typical response of that particular stock's excess return to changes in the market index's excess return.
    - ✓ As such, beta measures a stock's comparative sensitivity to macroeconomic news.
      - A value greater than 1 would indicate a stock with greater sensitivity to the economy than the average stock in the market. These are known as *cyclical stocks*.
      - Betas less than 1 indicate below-average sensitivity and therefore are known as *defensive stocks*.
  - Recall that the risk attributable to the stock's exposure to uncertain market returns is called market or *systematic* risk, because it relates to the uncertainty that pervades the whole economic system.

- The term $e_i$ in Equation 6.11 represents the impact of **firm-specific** or **residual risk.**
  - ✓ The expected value of $e_i$ is zero, as the impact of unanticipated events must average out to zero.
- Both residual risk and systematic risk contribute to the total volatility of returns.

■ The term $\alpha_i$ in Equation 6.11 is not a risk measure.

- Instead, $\alpha_i$ represents the expected return on the stock *beyond* any return induced by movements in the market index. This term is called the security **alpha.**
  - ✓ A positive alpha is attractive to investors and suggests an underpriced security.
    - ➢ Among securities with identical sensitivity (beta) to the market index, securities with higher alpha values will offer higher expected returns.
  - ✓ Conversely, stocks with negative alphas are apparently overpriced.
    - ➢ For any value of beta, they offer lower expected returns.

- In sum, the index model separates the realized rate of return on a security into macro (systematic) and micro (firm-specific) components. The excess rate of return on each security is the sum of three components:

| | **Symbol** |
|---|---|
| 1. The component of return due to movements in the overall market (as represented by the index $R_M$); $\beta_i$ is the security's responsiveness to the market. | $\beta_i R_M$ |
| 2. The component attributable to unexpected events that are relevant only to this security (firm-specific). | $e_i$ |
| 3. The stock's expected excess return if the market factor is neutral, that is, if the market-index excess return is zero. | $\alpha_i$ |

- Because the firm-specific component of the firm's return is uncorrelated with the market return, we can write the variance of the excess return of the stock as

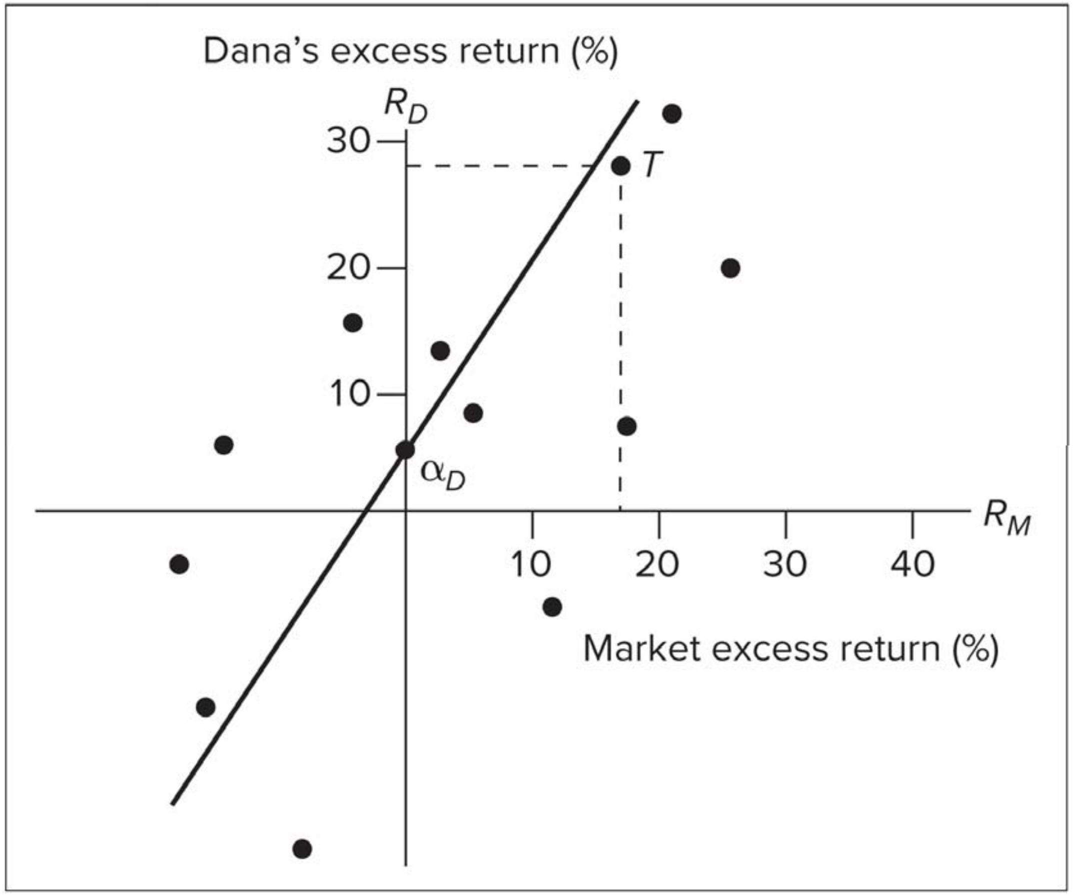
$$\begin{aligned} \text{Variance}(R_i) &= \text{Variance}(\alpha_i + \beta_i R_M + e_i) \\ &= \text{Variance}(\beta_i R_M) + \text{Variance}(e_i) \\ &= \beta_i^2 \sigma_M^2 + \sigma^2(e_i) \\ &= \text{Systematic risk} + \text{Firm-specific risk} \end{aligned} \tag{6.12}$$

  - In Equation 6.12, the covariance between $R_M$ and $e_i$ is zero.
  - Notice that because $\alpha_i$ is a constant, it has no bearing on the variance of $R_i$.
- Therefore, the total variability of the rate of return of each security is a sum of two components:
  - 1. The variance attributable to the uncertainty common to the entire market.
    - ✓ This variance depends on both the variance of $R_M$, denoted $\sigma_M^2$, *and* the beta of the stock on $R_M$.
  - 2. The variance of firm-specific returns, $e_i$, which is independent of market performance.

- This single-index model is convenient.
  - It relates security returns to a market index that investors follow.
  - Moreover, as we soon shall see, its usefulness goes beyond mere convenience.

## Statistical and Graphical Representation of the Single-Index Model

- Equation 6.11, $R_i = \alpha_i + \beta_i R_M + e_i$, may be interpreted as a single-variable *regression equation* of $R_i$ on the market excess return $R_M$.
  - The excess return on the security ($R_i$) is the dependent variable that is to be explained by the regression.
  - On the right-hand side of the equation are the intercept $\alpha_i$; the regression (or slope) coefficient beta, $\beta_i$, multiplying the independent (or explanatory) variable $R_M$; and the security residual (unexplained) return, $e_i$.
- We can plot this regression relationship as in Figure 6.12, which shows a possible scatter diagram of 12 observations of Dana's excess return paired against the excess return of the market index.




- The horizontal axis of the scatter diagram measures the explanatory variable, here the market excess return $R_M$.
- The vertical axis measures the dependent variable, here Dana's excess return, $R_D$.

- Each point on the scatter diagram represents a sample pair of returns ($R_M$, $R_D$) observed over a particular holding period.
  - ✓ Point *T*, for instance, describes a holding period when the excess return was 17% for the market index and 27% for Dell.

- Regression analysis uses a sample of historical returns to estimate the coefficients (alpha and beta) of the index model.
  - The algorithm finds the regression line, shown in Figure 6.12, that minimizes the sum of the squared deviations around it.
    - ✓ Hence, we say the regression line "best fits" the data in the scatter diagram.
    - ✓ The line is called the **security characteristic line (SCL).**

◆ The regression line is obtained by minimizing the "sum of squares" function as follows:

$$\text{Min } \; S(\alpha_i,\ \beta_i) = \textstyle\sum_{t=1}^{N}(R_{it} - \alpha_i - \beta_i R_{Mt})^2$$

$$\Rightarrow \quad \frac{\partial S}{\partial \alpha_i} = 2N\alpha_i - 2\textstyle\sum_{t=1}^{N} R_{it} + 2(\sum_{t=1}^{N} R_{Mt})\beta_i = 0$$

$$\Rightarrow \quad \frac{\partial S}{\partial \beta_i} = 2(\textstyle\sum_{t=1}^{N} R_{Mt}^2)\beta_i - 2\sum_{t=1}^{N} R_{it}R_{Mt} + 2(\sum_{t=1}^{N} R_{Mt})\alpha_i = 0$$

Then solve the above two equations for $\alpha_i$ and $\beta_i$, we can get the following formulas:

$$\beta_i = \frac{N\sum_{t=1}^{N} R_{Mt}R_{it} - \sum_{t=1}^{N} R_{Mt}\sum_{t=1}^{N} R_{it}}{N\sum_{t=1}^{N} R_{Mt}^2 - (\sum_{t=1}^{N} R_{Mt})^2} = \frac{\mathrm{Cov}(R_{it},\ R_{Mt})}{\mathrm{Var}(R_{Mt})}$$

$$\alpha_i = \frac{\sum_{t=1}^{N} R_{it}}{N} - \beta_i \frac{\sum_{t=1}^{N} R_{Mt}}{N} = \bar{R}_{it} - \beta_i \bar{R}_{Mt}$$

- The regression intercept ($\alpha_D$) is measured from the origin to the intersection of the regression line with the vertical axis.
  - Any point on the vertical axis represents zero market excess return, so the intercept gives us the *expected excess* return on Dana when market return is "neutral," that is, equal to the T-bill return.
    - ✓ The intercept in Figure 6.12 is 4.5%.
- The slope of the regression line, the ratio of rise to run, is called the *regression coefficient* or simply the beta.
  - In Figure 6.12, Dell's beta is 1.4.
  - A stock beta measures systematic risk since it predicts the response of the security to each extra 1% return on the market index.
- The regression line does not represent the *actual* returns.
  - The points on the scatter diagram almost never lie exactly on the regression line.
  - Rather, the line represents average tendencies; it shows the *expectation* of $R_D$ given the market excess return, $R_M$.

- The algebraic representation of the regression line is

  $$E(R_D|R_M) = \alpha_D + \beta_D R_M \quad (6.13)$$

  which reads: The expectation of $R_D$ given a value of $R_M$ equals the intercept plus the slope coefficient times the given value of $R_M$.

- Because the regression line represents expectations and these expectations may not be realized (as the scatter diagram shows), the *actual* returns also include a residual, $e_i$, reflecting the firm-specific component of return.
  - This surprise (at point $T$, for example) is measured by the vertical distance between the point of the scatter diagram and the regression line.
    - ✓ The expected return on Dana, given a market return of 17%, would have been 4.5% + 1.4 × 17% = 28.3%. The actual return was only 27%, so point $T$ falls below the regression line by 1.3%.

- Equation 6.12 shows that the greater the beta of the security, that is, the greater the slope of the regression, the greater the systematic risk and total variance.
  - Because the market index comprises all securities, the typical response to a market movement must be one for one.
  - An "aggressive" investment will have a beta higher than 1; that is, the security has above-average market risk.
  - Conversely, securities with betas lower than 1 are called defensive.
- A security may have a negative beta.
  - Its regression line will then slope downward, meaning that, for more favorable macro events (higher $R_M$), we would expect a *lower* return, and vice versa.
    - ✓ The latter means that when the macro economy goes bad (negative $R_M$) and securities with positive beta are expected to have negative excess returns, the negative-beta security will shine.
    - ✓ The result is that a negative-beta security provides a hedge against systematic risk.

- The dispersion of the scatter of actual returns about the regression line is determined by the residual variance $\sigma^2(e_D)$.
  - The magnitude of firm-specific risk varies across securities.
  - One way to measure the relative importance of systematic risk is to measure the ratio of systematic variance to total variance.

$$\rho^2 = \frac{\text{Systematic (or explained) variance}}{\text{Total variance}}$$

$$= \frac{\beta_D^2 \sigma_M^2}{\sigma_D^2} = \frac{\beta_D^2 \sigma_M^2}{\beta_D^2 \sigma_M^2 + \sigma^2(e_D)} = \left( \frac{\text{Cov}(R_D, R_M)}{\text{Var}(R_M)} \right)^2 \frac{\sigma_M^2}{\sigma_D^2} = \frac{\text{Cov}(R_D, R_M)^2}{\sigma_M^2 \sigma_D^2} \qquad (6.14)$$

where $\rho$ is the correlation coefficient between $R_D$ and $R_M$.

    - ✓ Its square measures the ratio of explained variance to total variance, that is, the proportion of total variance that can be attributed to market fluctuations.
  - But if beta is negative, so is the correlation coefficient, an indication that the explanatory and dependent variables are expected to move in opposite directions.

- At the extreme, when the correlation coefficient is either 1 or -1, the security return is fully explained by the market return and there are no firm-specific effects.
    - All the points of the scatter diagram will lie exactly on the line.
        - ✓ This is called *perfect correlation* (either positive or negative); the return on the security is perfectly predictable from the market return.
    - A large correlation coefficient (in absolute value terms) means systematic variance dominates the total variance; that is, firm-specific variance is relatively unimportant.
    - When the correlation coefficient is small (in absolute value terms), the market factor plays a relatively unimportant part in explaining the variance of the asset, and firm-specific factors dominate.
- Example 6.3 illustrates how you can use a spreadsheet to estimate the single-index model from historical data.

- Example 6.3: *Estimating the Index Model Using Historical Data*
  - A direct way to calculate the slope and intercept of the characteristic lines for ABC and XYZ is from the variances and covariances.
    - ✓ Here, we use the Data Analysis menu of Excel to obtain the covariance matrix in the following spreadsheet.
  - The slope coefficient for ABC is given by the formula

$$\beta_{ABC} = \frac{\mathrm{Cov}(R_{ABC}, R_{Market})}{\mathrm{Var}(R_{Market})} = \frac{773.31}{669.01} = 1.156$$

  - The intercept for ABC is

$$\alpha_{ABC} = \mathrm{Average}(R_{ABC}) - \beta_{ABC} \times \mathrm{Average}(R_{Market}) = 15.20 - 1.156 \times 9.40 = 4.33$$

  - Therefore, the security characteristic line of ABC is given by

$$R_{ABC} = 4.33 + 1.156 R_{Market}$$

  - This result also can be obtained by using the "Regression" command from Excel's Data Analysis menu, as we show at the bottom of the spreadsheet.

| | A | B | C | D | E | F | G | H | I |
|---|---|---|---|---|---|---|---|---|---|
| 2 | | | **Annualized Rates of Return** | | | | | **Excess Returns** | |
| 3 | Week | ABC | XYZ | Mkt. Index | Risk free | | ABC | XYZ | Market |
| 4 | 1 | 65.13 | −22.55 | 64.40 | 5.23 | | 59.90 | −27.78 | 59.17 |
| 5 | 2 | 51.84 | 31.44 | 24.00 | 4.76 | | 47.08 | 26.68 | 19.24 |
| 6 | 3 | −30.82 | −6.45 | 9.15 | 6.22 | | −37.04 | −12.67 | 2.93 |
| 7 | 4 | −15.13 | −51.14 | −35.57 | 3.78 | | −18.91 | −54.92 | −39.35 |
| 8 | 5 | 70.63 | 33.78 | 11.59 | 4.43 | | 66.20 | 29.35 | 7.16 |
| 9 | 6 | 107.82 | 32.95 | 23.13 | 3.78 | | 104.04 | 29.17 | 19.35 |
| 10 | 7 | −25.16 | 70.19 | 8.54 | 3.87 | | −29.03 | 66.32 | 4.67 |
| 11 | 8 | 50.48 | 27.63 | 25.87 | 4.15 | | 46.33 | 23.48 | 21.72 |
| 12 | 9 | −36.41 | −48.79 | −13.15 | 3.99 | | −40.40 | −52.78 | −17.14 |
| 13 | 10 | −42.20 | 52.63 | 20.21 | 4.01 | | −46.21 | 48.62 | 16.20 |
| 14 | **Average:** | | | | | | **15.20** | **7.55** | **9.40** |
| 15 | | | | | | | | | |
| 16 | **COVARIANCE MATRIX** | | | | | | | | |
| 17 | | ABC | XYZ | Market | | | | | |
| 18 | ABC | 3020.933 | | | | | | | |
| 19 | XYZ | 442.114 | 1766.923 | | | | | | |
| 20 | Market | 773.306 | 396.789 | 669.010 | | | | | |
| 21 | | | | | | | | | |
| 22 | **SUMMARY OUTPUT OF EXCEL REGRESSION** | | | | | | | | |
| 23 | | | | | | | | | |
| 24 | Regression Statistics | | | | | | | | |
| 25 | Multiple R | 0.544 | | | | | | | |
| 26 | R-Square | 0.296 | | | | | | | |
| 27 | Adj. R-Square | 0.208 | | | | | | | |
| 28 | Standard Error | 48.918 | | | | | | | |
| 29 | Observations | 10.000 | | | | | | | |
| 30 | | | | | | | | | |
| 31 | | | | | | | | | |
| 32 | | Coefficients | Std. Error | t-Stat | p-value | | | | |
| 33 | Intercept | 4.336 | 16.564 | 0.262 | 0.800 | | | | |
| 34 | Market return | 1.156 | 0.630 | 1.834 | 0.104 | | | | |
| 35 | | | | | | | | | |

- ✓ The minor differences between the direct regression output and our calculation above are due to rounding error.
- ✓ Note: This is the output provided by the Data Analysis tool in Excel.
  - ➢ As a technical aside, we should point out that the covariance matrix produced by Excel does not adjust for degrees of freedom.
  - ➢ In other words, it divides total squared deviations from the sample average (for variance) or total cross product of deviations from sample averages (for covariance) by total observations, despite the fact that sample averages are estimated parameters.
  - ➢ This procedure does not affect regression coefficients, however, because in the formula for beta, both the numerator (i.e., the covariance) and denominator (i.e., the variance) are affected equally.

## Diversification in a Single-Factor Security Market

- Imagine a portfolio that is divided equally among securities whose returns follow the single-index model of Equation 6.11. What are the systematic and nonsystematic variances of this portfolio?
  - The beta of the portfolio is the simple average of the individual security betas; hence, the systematic variance equals $\beta_P^2\sigma_M^2$.
    - ✓ This is the level of market risk in Figure 6.1B.
  - The market variance $(\sigma_M^2)$ and the beta of the portfolio determine its market risk.
    - ✓ See the following proof:

$$
\begin{aligned}
& R_i=\alpha_i+\beta_i R_M+e_i \Rightarrow \sum_{i=1}^{n} w_i R_i=\sum_{i=1}^{n} w_i(\alpha_i+\beta_i R_M+e_i) \\
& \Rightarrow R_P=\alpha_P+\beta_P R_M+e_P \\
& \Rightarrow \operatorname{Var}(R_P)=\operatorname{Var}(\alpha_P+\beta_P R_M+e_P) \\
& \qquad =\operatorname{Var}(\beta_P R_M)+\operatorname{Var}(e_P)+2\beta_P \operatorname{Cov}(R_M, e_P) \\
& \qquad =\beta_P^2\sigma_M^2+\sigma^2(e_P) \quad (\text{Note: } \operatorname{Cov}(R_M, e_P)=0)
\end{aligned}
$$

Total firm-specific risk

$$= \mathrm{Var}(e_P) = \mathrm{Var}(w_1 e_1 + w_2 e_2 + \ldots + w_n e_n)$$

$$= \mathrm{Var}(\frac{1}{n}e_1 + \frac{1}{n}e_2 + \ldots + \frac{1}{n}e_n) + 2(\frac{1}{n})(\frac{1}{n})\sum_{\substack{i=1 \\ i \neq j}}^{n}\sum_{\substack{j=1 \\ i > j}}^{n}\mathrm{Cov}(e_i,\ e_j) \quad (\text{Note: } \mathrm{Cov}(e_i,\ e_j) = 0)$$

$$= \frac{1}{n^2}\sum_{i=1}^{n}\sigma_i^2 = \frac{1}{n}\sum_{i=1}^{n}\frac{1}{n}\sigma_i^2 = \frac{1}{n}\bar{\sigma}^2 \xrightarrow{n \to \infty} 0$$

- The systematic component of each security return, $\beta_i R_M$, is driven by the market factor and therefore is perfectly correlated with the systematic part of any other security's return.
  - ✓ Hence, there are no diversification effects on systematic risk no matter how many securities are involved.
- As far as *market risk* goes, a single security has the same systematic risk as a diversified portfolio with the same beta.
  - ✓ The number of securities makes no difference.

- It is quite different with firm-specific risk.
  - Consider a portfolio of $n$ securities with weights, $w_i$ (where $\sum_{i=1}^{n} w_i = 1$), in securities with nonsystematic risk, $\sigma_{e_i}^2$. The nonsystematic portion of the portfolio return is

$$e_p = \sum_{i=1}^{n} w_i e_i$$

  - Because the firm-specific terms, $e_i$, are uncorrelated, the portfolio nonsystematic variance is the weighted sum of the individual firm-specific variance:

$$\sigma_{e_p}^2 = \sum_{i=1}^{n} w_i^2 \sigma_{e_i}^2 \qquad (6.15)$$

    - ✓ Each individual nonsystematic variance is multiplied by the *square* of the portfolio weight.
  - With diversified portfolios, the squared weights are very small.
    - ✓ For example, if $w_i = .01$ (think of a portfolio with 100 securities), then $w_i^2 = .0001$. The sum in Equation 6.15 is far less than the average firm-specific variance of the stocks in the portfolio.

- ◆ We conclude that the impact of nonsystematic risk becomes negligible as the number of securities grows and the portfolio becomes ever more diversified.
  - ✓ This is why the number of securities counts more than the size of their nonsystematic variance.
- ■ In sum, when we control the systematic risk of the portfolio by manipulating the average beta of the component securities, the number of securities is of no consequence.
- ■ But for *nonsystematic* risk, the number of securities involved is more important than the firm-specific variance of the securities.
  - ◆ Sufficient diversification can virtually eliminate firm-specific risk.
    - ✓ Understanding this distinction is essential to understanding the role of diversification in portfolio construction.
- ■ We have just seen that when forming highly diversified portfolios, firm-specific risk becomes *irrelevant*. Only systematic risk remains.
  - ◆ This means that for diversified investors, the relevant risk measure for a security will be the security's beta, $\beta$, since firms with higher $\beta$ have greater sensitivity to market risk.

- As Equation 6.12 makes clear, systematic risk will be determined both by market volatility, $\sigma_M^2$, and the firm's sensitivity to the market, β.