Proof that the maximum uncertainty for N events is obtain when all events ~~hax~~ are equally possible.

- Goal is to min $U = -\sum_{i=1}^{N} P_i \log P_i$ where $P_i \equiv P(X_i)$ (1)

under the constraint $1 = \sum_{i=1}^{N} P_i$ (2).

From (2) we have $P_N = 1 - \sum_{i=1}^{N-1} P_i$, substituting into (1)

$$U = -\sum_{i=1}^{N-1} P_i \log P_i - \left(1 - \sum_{i=1}^{N-1} P_i\right) \log\left(1 - \sum_{j=1}^{N-1} P_j\right)$$

Useful
$\frac{\partial}{\partial x}(-x\log x) = -1 - \log x$

then

$$\frac{\partial U}{\partial P_j} = -1 - \log P_j + \left(-1 - \log\left(1 - \sum_{i=1}^{N-1} P_i\right)\right) \frac{\partial\left(1 - \sum_{i=1}^{N-1} P_i\right)}{\partial P_j}$$

$$= -\log P_j + \log\left(1 - \sum_{i=1}^{N-1} P_i\right) = \log\left(\frac{1 - \sum_{i=1}^{N-1} P_i}{P_j}\right)$$

then I need to solve

$$(3) \quad 1 - \sum_{i=1}^{N-1} P_i = P_j$$ as this will yield $\frac{\partial U}{\partial P_j} = \log(1) = 0$

from (3) we have $N-1$ linear equations which we write a single one for j as $2P_j + \sum_{\substack{i=1 \\ \neq j}}^{N-1} P_i = 1$

which we can easily represent as

$$\underbrace{\begin{bmatrix} 2 & 1 & \cdots & 1 \\ 1 & \ddots & & \vdots \\ \vdots & & \ddots & 1 \\ 1 & \cdots & 1 & 2 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} P_1 \\ \vdots \\ P_{N-1} \end{bmatrix}}_{P} = \underbrace{\begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}}_{C}.$$

$A^{-1}A$
if multiplying row i from A^{-1} by column j from A with $i \neq j$ (off-diagonal elements or $A^{-1}A$), ~~the su~~ notice that the sum of all rows of j except row i is constant and equal to $2 + (N-3) = N-1$, the remaining elemen (i,j) of A is 1 which means that if the element (i,i) of A^{-1} is $N-1$ and remaining colums of the same row in A^{-1} are -1, off diagonal elements will always be zero in $A^{-1}A$. Furthermore the digonal will be $2(N-1) - (N-2) = N \Rightarrow A^{-1} = \frac{1}{N}\begin{bmatrix} N-1 & & -1 \\ -1 & \ddots & \\ & & N-1 \end{bmatrix}$

$$A^{-1} = \frac{1}{N}\begin{bmatrix} N-1 & -1 & \cdots & -1 \\ -1 & \ddots & & \vdots \\ \vdots & & \ddots & -1 \\ -1 & \cdots & -1 & N-1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} p_1 \\ \vdots \\ p_{N-1} \end{bmatrix} = \frac{1}{N}\begin{bmatrix} N-1 & & -1 \\ & \ddots & \\ -1 & & N-1 \end{bmatrix}\begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} \cancel{= \frac{1}{N}[N-1-(N-2)]}$$

$$\Rightarrow p_j = \frac{1}{N}[N-1-(N-2)] = \frac{1}{N} \quad j \in \{1, \dots, N-1\}$$

and $p_N = 1 - \sum_{i=1}^{N-1} p_i = 1 - \frac{1}{N}(N-1) = \frac{1}{N}$

$\Rightarrow p_i = \frac{1}{N}$ $i \in \{1, \dots, N\}$ maximizes the uncertainty $u = \sum_{i=1}^{N} p_i \log p_i$