1:- Uncertainty U should be a function which continuous in all P_i, where $P_i \equiv P(x_i)$.

$$U(\mathbb{X}) = U(P_1, P_2, \cdots, P_N) = -\sum_{i=1}^{N} P_i \log_n(P_i) \quad P_i \in [0,1]$$

$$\frac{\partial U(P_1, \cdots, P_N)}{\partial P_i} = -\left[P_i \cdot \frac{1}{P_i \log_e(n)} + \log_n(P_i)\right]$$

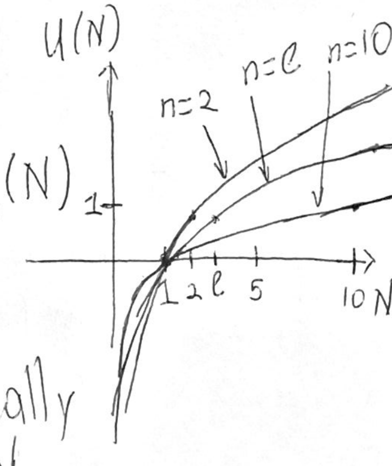
$$= -\log_n(P_i) - \frac{1}{\log_e(n)}$$

which means U is differentiable on P_i ($\frac{\partial U}{\partial P_i}$ is defined) which then implies U is continuous on P_i.

2:- If all events equally possible then U should be monotonically increasing with N (number of events).

$$P_i = P_1 = \cdots = P_N = 1/N = P(X_i)$$

$$U(\mathbb{X}) = -\sum_{i=1}^{N} \frac{1}{N} \log_n\left(\frac{1}{N}\right) = \log_n(N)$$



$$\frac{dU}{dN} = \frac{1}{N \log_e(n)} > 0 \quad \forall N \in \mathbb{N}$$

Therefore monotonically increasing with N.

3: If an event is broken down to successive events the original $\mathcal{U}$ should be the weighted sum of the individual values of $\mathcal{U}$ in the succession.

Example: A coin and a dice. We decide to categorize the events as:

1) Coin being heads. (P_1)
2) Coin being tails and dice a prime number. (P_2)
3) All other possible events. (P_3)

Suppose we throw coin and dice at the same time, then: $P_1 = 1/2$, $P_2 = \frac{1}{2} \cdot \frac{3}{6} = 1/4$, $P_3 = 1/4$

$$\mathcal{U}\left(\frac{1}{2}, \frac{1}{4}, \frac{1}{4}\right) = -\frac{1}{2}\log_2\left(\frac{1}{2}\right) - \frac{1}{4}\log_2\left(\frac{1}{4}\right) - \frac{1}{4}\log_2\left(\frac{1}{4}\right)$$

$\boxed{\text{We choose } n=2 \text{ for convenience}}$

$$= \frac{1}{2} + \frac{1}{2} + \frac{1}{2} = 3/2$$

Now we throw the coin first, the the dice. On throwing the coin we have: $P_1 = \frac{1}{2}$, $P_2 = 0$, $P_3 = \frac{1}{2}$. Now, throwing the dice is only necessary if we get tails in the coin which has a probability of 1/2. This 1/2 is the weight of our second event (throwing the dice) then in this second event we have: $P_1 = 0$ (it's already tails), $P_2 = \frac{1}{2}$, $P_3 = \frac{1}{2}$.

Then the total uncertainty is $\mathcal{U}\left(\frac{1}{2}, 0, \frac{1}{2}\right) + \frac{1}{2}\mathcal{U}\left(0, \frac{1}{2}, \frac{1}{2}\right)$

$$= \frac{1}{2} + \frac{1}{2} + \frac{1}{2}\left(\frac{1}{2} + \frac{1}{2}\right) = 3/2$$

Uncertainty should be independent on the "way" events happen.