Implicit and Explicit Finite Difference Method

Presented By

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Finite Difference Method

Taylor Series

$$f(x \pm dx) = f(x) \pm \mathrm{dx} f^{'}(x) + \frac{dx^2}{2!} f^{''}(x) \pm \frac{dx^3}{3!} f^{'''}(x) + \frac{dx^4}{4!} f^{''''}(x) \pm \ldots$$

Forward Difference

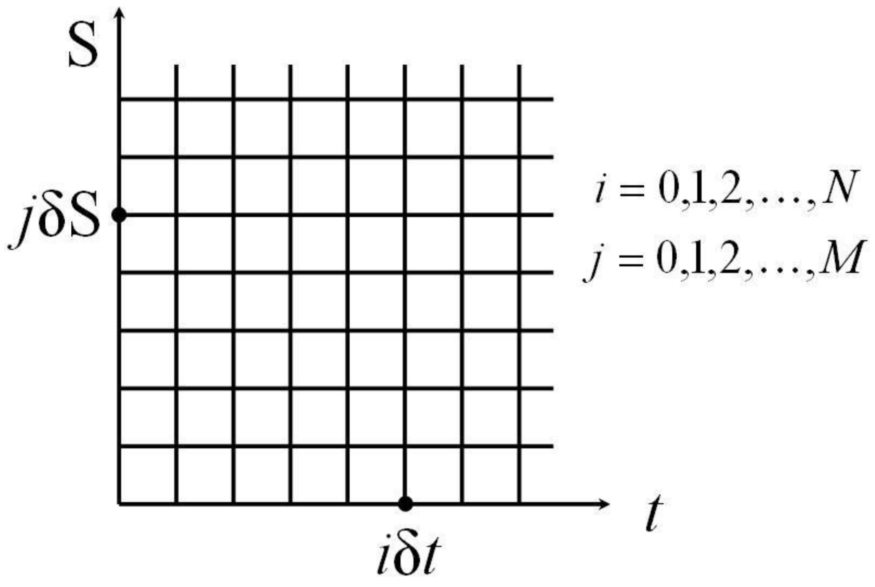
$$\frac{df(x)}{dx} = \frac{f(x+h) - f(x)}{h} + O(h)$$

Backward Difference

$$\frac{df(x)}{dx} = \frac{f(x) - f(x-h)}{h} + O(h)$$

Central Difference

$$\frac{df(x)}{dx} = \frac{f(x+h) - f(x-h)}{2h} + O(h^2)$$

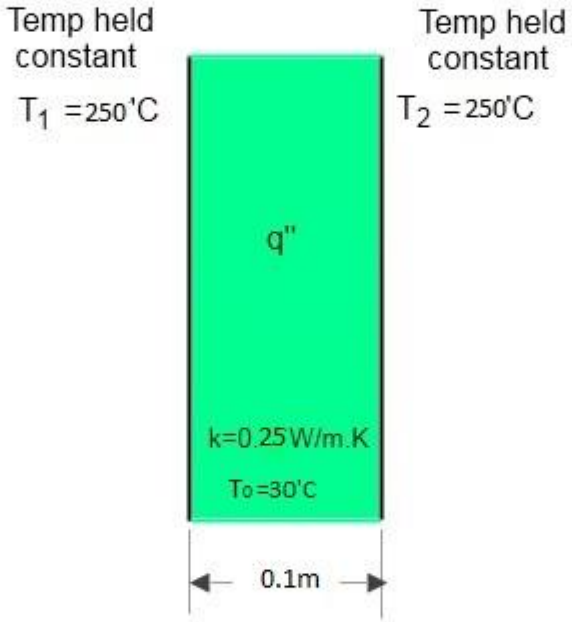


Objectives

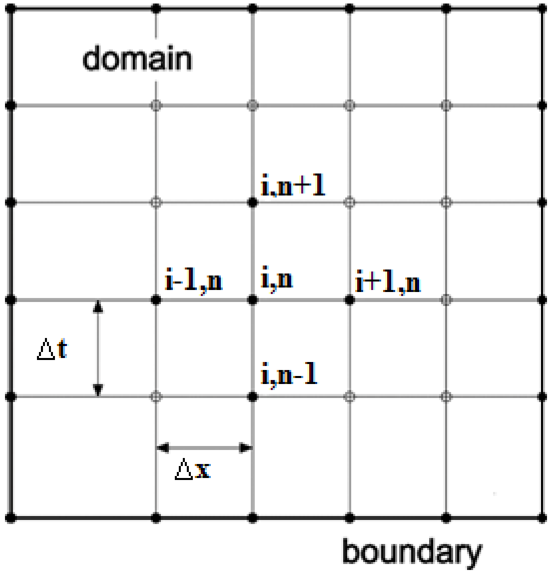
- To understand the problem statement
- Explicit Approach
- Implicit Approach
- Stability Criterion
- Boundary Nodes

Problem Statement and comparison

Let a plate of thickness 10cm of uniform temperature 30^0C is placed in an environment where ambient temperature is 250^0C. Here temperature variation only along the thickness is considered which made this problem one dimensional unsteady heat conduction problem.



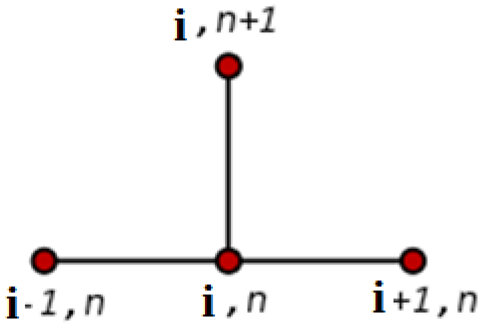
$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2}$$



In the grid diagram
j subscript denotes the node variable in space
n denotes time

Explicit Formulation

The 1-D heat equation is to be solved to arrive at the temperature distribution



$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2}$$

After Discretization using Taylor series

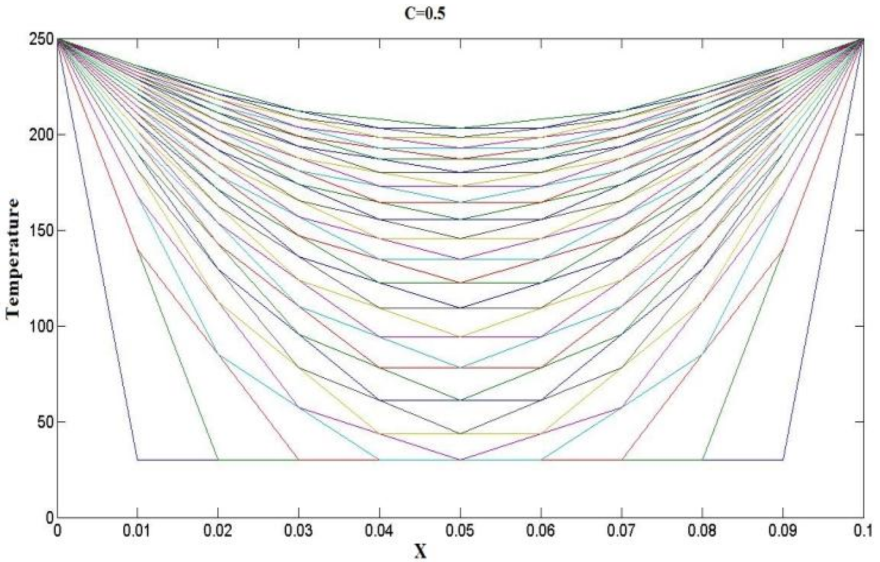
$$\frac{T_i^{n+1} - T_i^n}{\Delta t} = \alpha \frac{(T_{i+1}^n - 2T_i^n + T_{i-1}^n)}{(\Delta x)^2}$$

Rearrangement of the equation gives:

$$T_i^{n+1} = T_i^n + \alpha \frac{\Delta t}{(\Delta x)^2} (T_{i+1}^n - 2T_i^n + T_{i-1}^n \Big)$$

Numerical errors are proportional to the time step and the square of the space step:

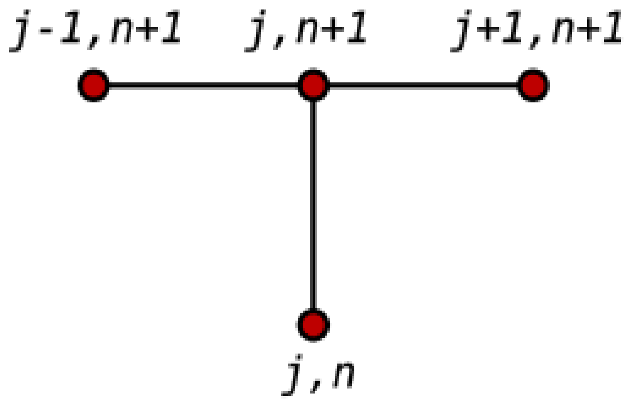
$\Delta u = O(\Delta t) + O(\Delta x^2)$



Implicit Formulation

The right hand side values between time levels n & n+1, the equation will be

$$\frac{T_i^{n+1} - T_i^n}{\Delta t} = \frac{T_{i+1}^{n+1} - 2T_i^{n+1} + T_{i-1}^{n+1}}{(\Delta x)^2}$$

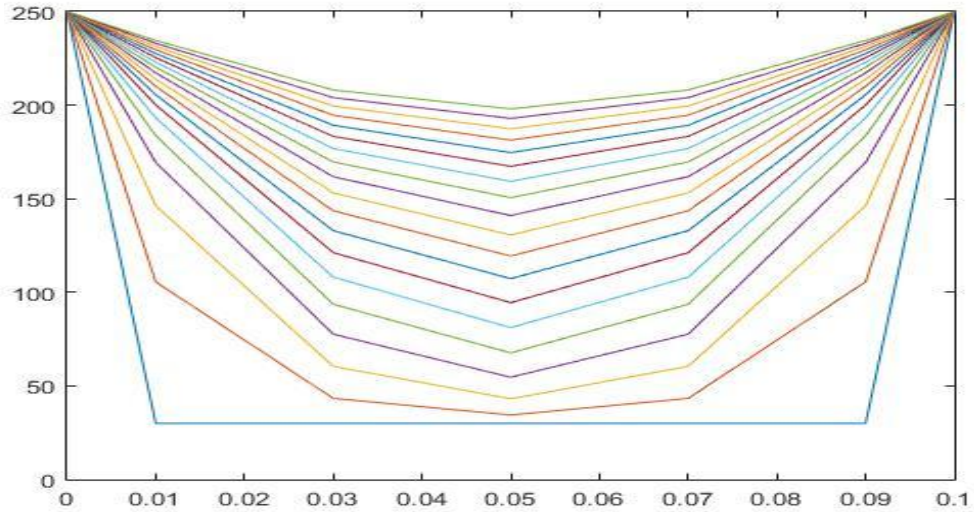


After rearranging the equation

$$-rT_{i-1}^{n+1} + (1+2r)T_i^{n+1} - rT_{i+1}^{n+1} = T_i^n$$

$$A = -r\,;\; B = -2r - 1;\; K_i = T_i^n \qquad ; \qquad r = \frac{\alpha \Delta t}{(\Delta x)^2}$$

$$\begin{bmatrix} -B & A & 0 & 0 & 0 \\ A & -B & A & 0 & 0 \\ 0 & A & -B & A & 0 \\ 0 & 0 & A & -B & A \\ 0 & 0 & 0 & A & -B \end{bmatrix} \begin{bmatrix} T_2 \\ T_3 \\ T_4 \\ T_5 \\ T_6 \end{bmatrix} = \begin{bmatrix} K'_2 \\ K_3 \\ K_4 \\ K_5 \\ K'_6 \end{bmatrix}$$



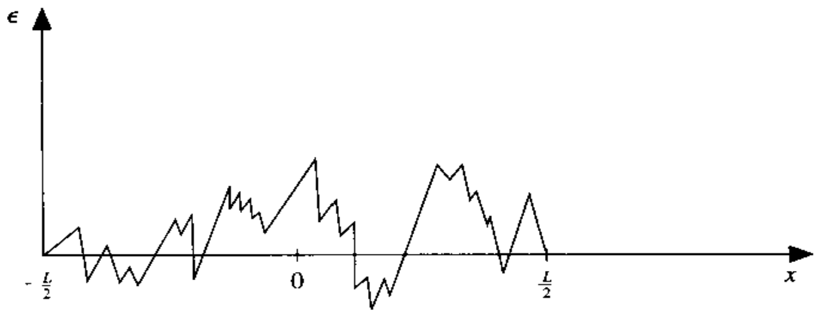
Numerical errors are proportional to the time step and the square of the space step:

$$\Delta u = O(\Delta t) + O(\Delta x^2)$$

Stability Criteria

- Stability of a numerical method means in every marching step the numerical error should decrease.
- N = D + ε , N=Numerical solution from a real computer with finite accuracy and D=Exact solution of difference equation, ε is the round-off error.
- The error must satisfy this difference equation

$$\frac{\varepsilon_i^{n+1} - \varepsilon_i^n}{\Delta t} = \alpha \frac{(\varepsilon_{i+1}^n - 2\varepsilon_i^n + \varepsilon_{i-1}^n)}{(\Delta x)^2}$$



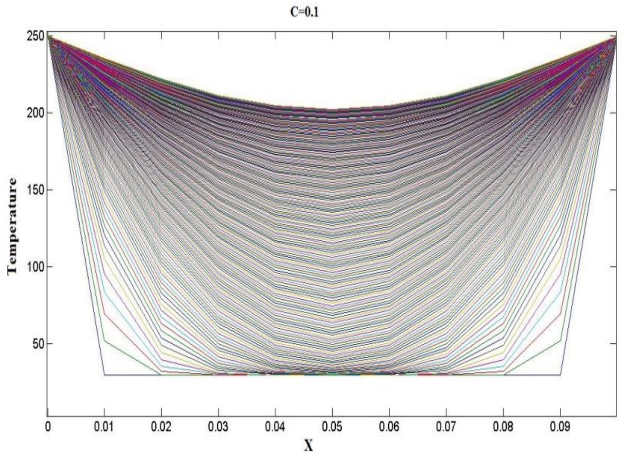
- For the solution to be stable $\left|\frac{\varepsilon_i^{n+1}}{\varepsilon_i^n}\right| \le 1$
- Round off error can be written as $\varepsilon(x,t) = \sum_{m=1}^{N/2} e^{at} e^{ik_m x}$

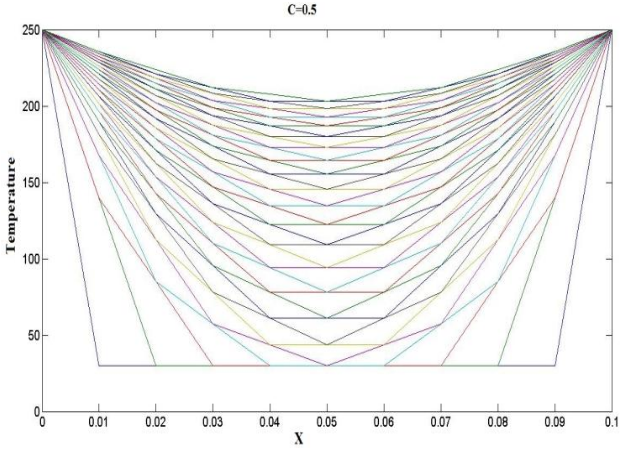
$$e^{a\Delta t} = 1 - \frac{4\alpha\Delta t}{(\Delta x)^2} sin^2 \frac{k_m x}{2}$$

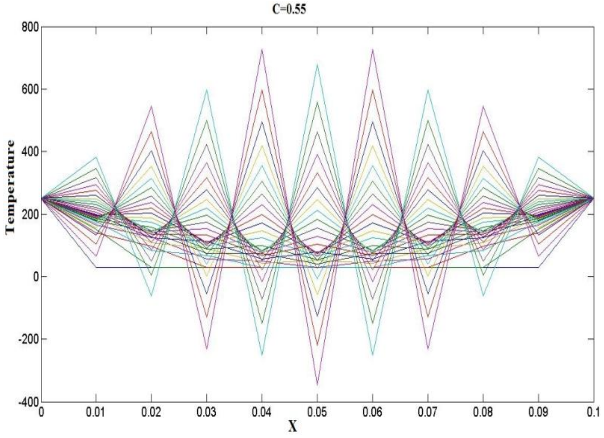
- Condition for stability in explicit method is $\frac{\alpha \Delta t}{(\Delta x)^2} \le \frac{1}{2}$
- Implicit method is unconditionally stable.

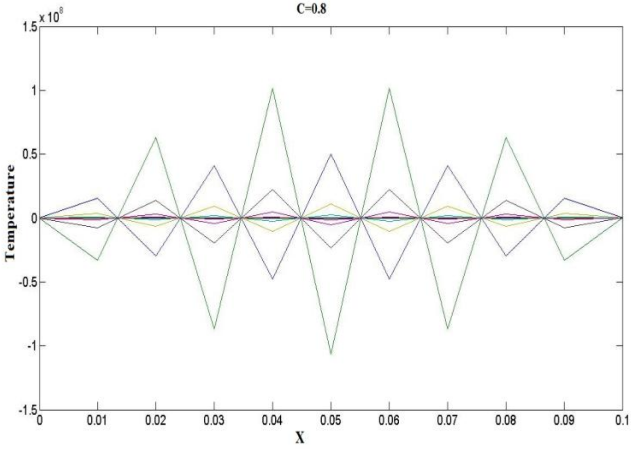
Stability of Two Formulation

- Result obtained by using Explicit Method

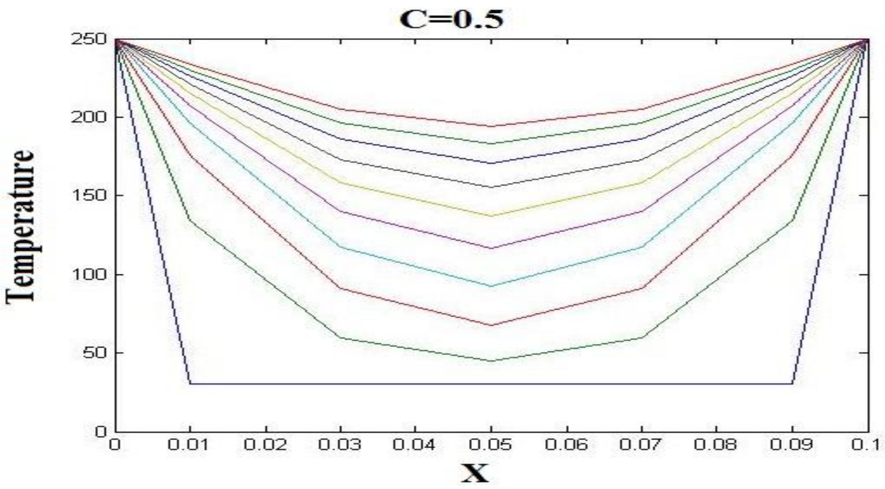


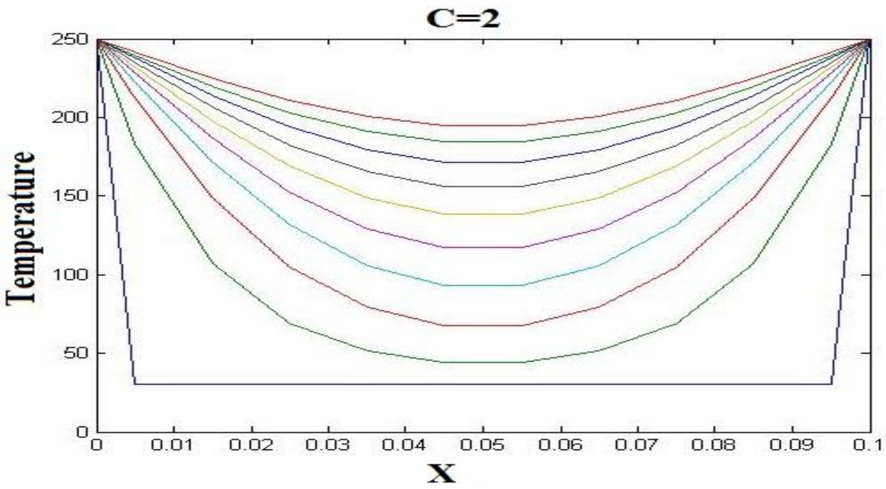




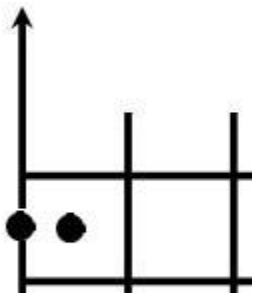


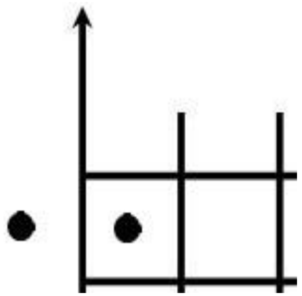
- Result obtained by using Implicit Method





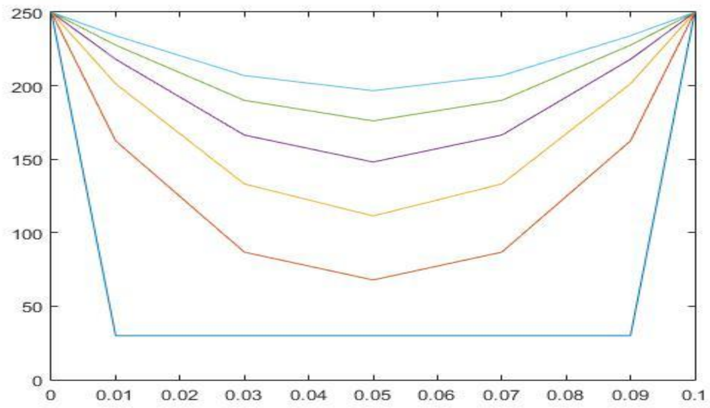
Boundary node problem

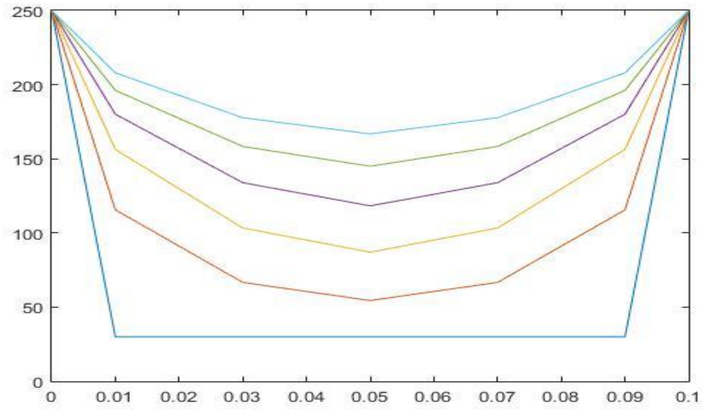




	1	2	3	4	5	
250	30	30	30	30	30	250
250	162.4731	86.7742	67.8495	86.7742	162.4731	250
250	201.34	133.1703	111.3967	133.1703	201.34	250
250	217.976	166.4051	148.069	166.4051	217.976	250
250	227.5982	190.0113	176.0305	190.0113	227.5982	250
250	234.0196	206.8747	196.5933	206.8747	234.0196	250

	1	2	3	4	5	
250	30	30	30	30	30	250
250	115.5556	66.6667	54.4444	66.6667	115.5556	250
250	156.2963	103.3333	87.037	103.3333	156.2963	250
250	180.0617	133.8889	118.2716	133.8889	180.0617	250
250	196.1317	158.3333	144.9794	158.3333	196.1317	250
250	207.939	177.6852	166.7833	177.6852	207.939	250





Conclusion

- Since the discretization schemes normally involve some sort of round-off error, the error will accumulate over each step.
- Explicit Schemes, allows the calculation of a flow field at an appropriate step from calculations, obtained from previous steps.
- Explicit method are known to be stable for a limited step size.
- Implicit Scheme uses the iterative method to obtain the temperature distribution after a time interval.
- Implicit Method are unconditionally stable, i.e. Stability can be maintained over much large value of Δt than for a corresponding explicit method.
- The node selection at the boundary can affect the temperature distribution considerably.

Reference

- Computational Fluid Dynamics- The Basics with Application - J.D. Anderson
- Numerische Methoden 1 – B.J.P. Kaus
- Finite Difference Method – Mark Davis, Imperial College of London