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MA 521
Homework #5
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1.25, 1.26, 1.28, 1.30, 1.31, 2.7, 2.9, 2.16, 2.18.

Exercise 1.25 Let (X, d) be a metric space. Let E and Y be subsets of X such that $E \subset Y$. Prove that
$$\mathrm{Cl}_Y(E) = \mathrm{Cl}_X(E) \cap Y.$$

For $\subset$, let $x \in \mathrm{Cl}_Y(E)$. $\mathrm{Cl}_Y(E) = E \cup \mathrm{Lim}_Y(E) = E \cup (\mathrm{Lim}_X(E) \cap Y)$ (by Exercise 1.10) $= (E \cup \mathrm{Lim}_X(E)) \cap Y = \mathrm{Cl}_X(E) \cap Y$. So, $x \in \mathrm{Cl}_X(E) \cap Y$. For $\supset$, let $x \in \mathrm{Cl}_X(E), Y$. So, $\mathrm{Cl}_X(E) = (E \cup \mathrm{Lim}_X(E)) \cap Y = E \cup (\mathrm{Lim}_X(E) \cap Y) = E \cup \mathrm{Lim}_Y(E) = Cl_Y(E)$. ∎

Exercise 1.26 Let (X, d) be a metric space.

1. Prove that for any collection $\mathcal{E}$ of subsets of X, we have
$$\bigcup_{E\in\mathcal{E}} \overline{E} \subset \overline{\bigcup_{E\in\mathcal{E}} E}$$
and equality holds if $\mathcal{E}$ is finite.
2. Prove that for any collection $\mathcal{E}$ of subsets of X, we have
$$\bigcap_{E\in\mathcal{E}} \overline{E} \supset \overline{\bigcap_{E\in\mathcal{E}} E}$$
3. Give examples that demonstrate that equality might fail in part (1) is $\mathcal{E}$ is not finite, and equality might fail in part (2) even if $\mathcal{E}$ is finite.

1. Let $x \in \bigcup_{E\in\mathcal{E}} \overline{E}$. So, for some E, $x = E \cup \mathrm{Lim}_X(E)$. $\overline{\bigcup_{E\in\mathcal{E}}(E)} = [\bigcup_{E\in\mathcal{E}} E] \cup [\mathrm{Lim}_X(\bigcup_{E\in\mathcal{E}})]$. So, $E \subset \bigcup_{E\in\mathcal{E}} E$ and $\mathrm{Lim}_X(E) \subset \mathrm{Lim}_X(\bigcup_{E\in\mathcal{E}})$ by Exercise 1.9. So, $x \in \overline{\bigcup_{E\in\mathcal{E}} E}$. For $\supset$, let $K = \bigcup_{E\in\mathcal{E}} \overline{E}$. Since K is union of finite number of closed sets, K is a closed set. All $x \in \bigcup_{E\in\mathcal{E}}(E) \to x \in \mathrm{Cl}_X(E)$ and so $x \subset K$. Thus, $\overline{\bigcup_{E\in\mathcal{E}}(E)} \subset K$.
2. $\bigcap_{E\in\mathcal{E}} \overline{E}$ is an intersection of closed sets so it is also closed. It also contains $\bigcap_{E\in\mathcal{E}} E$ so it must contain $\overline{\bigcap_{E\in\mathcal{E}} E}$ by definition of closure.
3. For part 1, let $\mathcal{E}$ be the collection of $\mathbb{Q}$. So then, $\bigcup_{E\in\mathcal{E}} \overline{E} = \mathbb{Q} \neq \overline{\bigcup_{E\in\mathcal{E}}(E)} = \mathbb{R}$. For part 2, let $E_1 = (1, 2)$ and $E_2 = (2, 3)$ so clearly $E_1 \cap E_2 = \emptyset$ and $\overline{E_1 \cap E_2} = \emptyset$. $\bigcap_{E\in\mathcal{E}} \overline{E} = \{2\}$ and clearly $2 \not\subset \emptyset$.

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Exercise 1.28 Let (X, d) be a metric space.

1. Prove that $x \in X$ and $r > 0$, we have $\overline{B_X(x,r)} \subset \{y \in X : d(x,y) \leq r\}$. (Hint: Take complements and draw a picture.) Note the inclusion $\overline{B_X(x,r)} \subset B_X(x, r+\epsilon)$ follows for any $\epsilon > 0$.
2. Give an example using the discrete metric that demonstrates that equality need not hold in the inclusion $\overline{B_X(x,r)} \subset \{y \in X : d(x,y) \leq r\}$ that you proved in (1).
3. Prove that in $\mathbb{R}^n$ under the Euclidean metric $d(x,y) = \|x - y\|$, we have $\overline{B_{\mathbb{R}^n}(x,r)} = \{y \in \mathbb{R}^n : \|x - y\| \leq r\}$. (Again a picture might be useful)
4. Using part (1), prove if A is bounded in (X, d) then $\overline{A}$ is also bounded in (X, d).

1. $\overline{B_X(x,r)} = B_X(x,r) \cup \mathrm{Lim}_X(B_X(x,r))$. Let's analyze some $z \in B_X(x,r)$. For any such z, $d(x,z) < r \to z \in \{y \in X : d(x,y) \leq r\}$. Now assume that $y \in \mathrm{Lim}_X(B_X(x,r))$. $\mathrm{Lim}_X(B_X(x,r)) = \{x : U \cap \{B_X(x,r) \setminus \{x\} \neq \emptyset\}\}$ where U is any neighborhood of x. The set of limit points will contain those points with $d(x,y) \leq r$. Any y that has $d(x,y) \geq r + \epsilon$ will violate definition of limit point. So, $\mathrm{Lim}_X(B_X(x,r)) \subset \{y \in X : d(x,y) \leq r\}$. This completes the inclusion, $\subset$.
2. Let $r = 1$ and $y \in \{y \in X : d(x,y) \leq r\}. y \notin B_X(x,1) = \{x\}$ and we need to show $y \notin \mathrm{Lim}_X(B_X(x,1)) = U \cap (\{x\} \setminus y)$. Choose U to be $B_X(y,1)$ so $y \notin \mathrm{Lim}_X(B_X(x,1))$.
3. Let $D = \{y \in \mathbb{R}^n : \|x - y\| \leq r\}$. For $\subset$, $\overline{B_{\mathbb{R}^n}(x,r)} = B_{\mathbb{R}^n}(x,r) \cup \mathrm{Lim}_X(B_{\mathbb{R}^n}(x,r))$. If $y \in B_{\mathbb{R}^n}(x,r)$, $d(x,y) < r$ so $y \in D$. If $y \in \mathrm{Lim}_X(B_{\mathbb{R}^n}(x,r)) = U \cap (B_{\mathbb{R}^n}(x,r) \setminus \{y\})$ for any neighborhood around y. This includes y when $d(x,y) = r$ so $y \in D$. For $\supset$, choose $y \in D$ s.t. $d(x,y) < r$ so $y \in B_{\mathbb{R}^n}(x,r)$. Let $z \in D$ be s.t. $d(x,z) = r$. So, for any $\epsilon > 0$, $\nexists B_X(z,\epsilon)$ s.t. $B_X(z,\epsilon) \cap (B_{\mathbb{R}^n}(x,r) \setminus \{z\})$. Thus, $z \in \mathrm{Lim}_X(B_{\mathbb{R}^n}(x,r))$.
4. Let $a \in \mathrm{Lim}_X(A) = B_X(x,R) \cup A \subset \{a\} = \{\ldots, b, \ldots\}$. $d(a,x) \leq d(a,b) + d(b,x) \leq R$. So, $\mathrm{Lim}_X(A) \in B_X(x,R)$ so $\overline{A}$ is bounded.

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Exercise 1.30 Let (X, d) be a metric space, and let E be a subset of X.

1. Show that E is dense in X if and only if any nonempty open subset of X contains a point of E.
2. Suppose $E \subset Y \subset X$. Prove that E is dense in Y if and only if $\mathrm{Cl}_X(E) \supset Y$.

1. For $\to$, let $x \in \mathrm{Lim}_X(E)$. Clearly, for any open set U, $U \cap E \setminus \{x\} \neq \emptyset$ so U contains another point than x. For $\leftarrow$, let's argue via proof by contrapositive. Assume that no nonempty open subsets contain a point of E. So, $x \in U$ but $x \notin E$ and $x \notin \mathrm{Lim}_X(E)$ since $U \cap (E \setminus \{x\}) = \emptyset$. So, $x \in X$ but $x \notin \mathrm{Cl}_X(E)$. So, E is not dense in X for this case.
2. E is dense in Y if and only if $\mathrm{Cl}_Y(E) = Y$. So, $\mathrm{Cl}_X(E) \cap Y$ (apply properties from Exer 1.10) $= Y$ if and only if $\mathrm{Cl}_X(E) = Y$.

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Exercise 1.31 Previously, we said that a subset E of $\mathbb{R}$ was dense in $\mathbb{R}$ if for any real numbers a and b, there exists a number $c \in E$ which lies between a and b. Show that in $\mathbb{R}$, the new, more general definition of dense agrees with the old one. That is, show that a subset E of $\mathbb{R}$ is dense in $\mathbb{R}$ according to the new definition if and only if it is dense according to the old one. (Hint: Use Exercise 1.30(1).)

Applying the first part of 1.30 yields $\text{Cl}_{\mathbb{R}}(E) = \mathbb{R}$ if and only if any nonempty open subset of $\mathbb{R}$ contains a point of E. If $a, b \in \mathbb{R}$ are taken as end points of an open interval, there will be another point $c \in E$ which directly leads to the old definition. This proves both directions. ∎

Exercise 2.7 Let $S = (p_n)_{n=1}^{\infty}$ be a sequence in $\mathbb{R}$ whose image is $(\mathbb{Q} \cap (0,1)) \cup \{5\}$. What are the two possibilities for S^*? Justify your answers.

Set $\operatorname{Im} S = (\mathbb{Q} \cap (0,1)) \cup \{5\}$. So, $(\operatorname{Im} S)' = [0,1]$ and $S_\infty = \{5\}$. $S^* = \{[0,1] \cup \{5\}, [0,1]\}$ because 5 might appear a finite number of times in the sequence so $5 \notin S_\infty$. ∎

Exercise 2.9 Prove Proposition 2.8.

Prove (1) $\to$ (2). $p_n \to p$ in X means $p_n \in B_X(p,r)$ for $r > 0$ s.t. $N \in \mathbb{N}$ and $n \geq N$. Suppose $\exists$ some subsequence p_n for $n \geq N$ s.t. $p_n \notin B_X(p,r)$. Then, p_n does not converge to p as it violates the definition of $p_n \to p$ in X as $n \to \infty$. Prove (2) $\to$ (3). $S^* = (\operatorname{Im} S)' \cup S_\infty$, This is true by definition since S^* is the set of all subsequential limits and all subsequences converges to p in X. So, $S^* = p$. Since all subsequences converge to a point p in X, they converge in X. Prove (3) $\to$ (1). If $p \in S_\infty$ then $\exists\, p \in B_X(x,r)$ for all but finitely many points of p_n for $n \geq N, N \in \mathbb{N}$. If $p \in (\operatorname{Im} S)'$, then $\text{Lim}_X(\text{Im}(p_n)) = \{p\}$. Thus, $\exists$ a neighbourhood, U, of p s.t. $(U) \cap \text{Im}(S) \setminus \{p\} \neq \emptyset$. Also, every sybsequence of $p_n \to p$ as $n \to \infty$. Thus, $\exists n \geq N$ s.t. $p_n \in U \forall n$, which shows that $p_n \to p$ as $n \to \infty$. ∎

Exercise 2.16 Let (X,d) be a metric space, and let $(x_n)_{n=1}^{\infty}$ be a sequence in X. Prove the following statements.

1. If $(x_n)_{n=1}^{\infty}$ converges in X, then it is Cauchy in X.
2. If $(x_n)_{n=1}^{\infty}$ is Cauchy in X, then it is bounded in X.

1. By definition of convergence, we have any neighbourhood U s.t. $n \geq N$ implies $x_n \in U$. Let $U = B_X(x,r)$. By the triangle inequality. $d(x_m, x_n) \leq d(x_m, x) + d(x_n, x) \leq 2r$ so Cauchy in X.
2. If (x_n) is Cauchy then $m \geq n \geq N$ implies $d(x_m, x_n) < \epsilon$. Thus, all but finitely many points are within $B_X(x,\epsilon)$. Let $r = \max(d(x_j, x))$ when $j \leq N$. Construct $B_X(x, r+\epsilon)$ where $\epsilon > 0$. Thus all points of $\text{Im}(x_n)$ are contained within $B_X(x, r+\epsilon)$ so $\text{Im}(x_n)$ is bounded so (x_n) is bounded.

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Exercise 2.18 Let (X, d) be a metric space, and let Y be a subset of X. Prove the following statements.

1. If Y is complete, then Y is closed in X.
2. If X is complete and Y is closed in X, then Y is complete.

1. Every Cauchy sequence in Y converges in Y. So, $\exists$ a sequence $(x_n)_{n=1}^{\infty}$ in $Y\{x\}$ that converges to x in X. So $x \in \mathrm{Lim}_X(Y)$ and $\mathrm{Lim}_X(Y) \subset Y$ so Y is closed.
2. Since Y is closed, $\mathrm{Lim}_X(Y) \subset Y$. From Exer 1.13, every Cauchy sequence in X s.t. $x_n \to X$ and $x \in Y$ menas every Cauchy sequence also converges to x in Y. This means it is a limit point of Y which are all contained in Y. Thus. Y is complete.

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