

Chapter 2: 5.4, 5.5, 6.4
Chapter 3: 1.7, 2.3, 2.4, 2.6, 2.8, 2.11, 2.12, 2.16, 2.20

Exercise 5.4. Let a and b be real numbers. Show that the following three equalities hold:

$$\bigcap_{x>b} (a, x) = (a, b], \quad \bigcup_{n=1}^{\infty} \left[a + \frac{1}{n}, b - \frac{1}{n}\right] = (a, b), \quad \bigcap_{n=1}^{\infty} (a + n, +\infty) = \emptyset.$$

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Exercise 5.5. Let a_1, a_2, \dots be any enumeration of the negative rational numbers; let b_1, b_2, \dots be any enumeration of the positive rational numbers. Show that the following two equalities hold:

$$\bigcap_{j=1}^{\infty} (a_j, b_j) = \{0\}, \quad \bigcup_{j=1}^{\infty} (a_j, b_j) = \mathbb{R},$$

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Exercise 6.4. Prove there exists no order \leq that makes $(\mathbb{C}, +, \cdot, \leq)$ into an ordered field.

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Exercise 1.7. Let $\|\cdot\|$ be a norm on a real vector space V . Prove the *reverse triangle inequality*:

$$|||x| - |y|| \leq \|x - y\|$$

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Exercise 2.3. Let X be any set. Prove that the discrete metric $d : X \times X \rightarrow \mathbb{R}$ (defined by $d(x, y) = 1$ if $x \neq y$ and $d(x, x) = 0$ for $x \in X$) satisfies the triangle inequality and is therefore a metric on X .

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Exercise 2.4. Determine which of the following functions are metrics on \mathbb{R} . Prove your answer in each case.

- $d_1(x, y) = \sqrt{|x - y|}$.
- $d_2(x, y) = |x - 2y|$.
- $d_3(x, y) = \frac{|x - y|}{1 + |x - y|}$.

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Exercise 2.6. Let $\|\cdot\|$ denote the Euclidean norm \mathbb{R}^2 , i.e. $\|(x_1, x_2)\| = \sqrt{x_1^2 + x_2^2}$. Consider the function from $\mathbb{R}^2 \times \mathbb{R}^2 \rightarrow \mathbb{R}$, defined by

$$d(x, y) = \|x_1 - y_1\| + \|x_2 - y_2\|, \quad (x = (x_1, x_2), y = (y_1, y_2)).$$

1. Prove that d is a metric on \mathbb{R}^2 .
2. On a sheet of graph paper, draw the set $B_d((5, 1), 3)$. Use dotted lines to indicate the boundary, which is not included in the set you are drawing. (Hint: it may be easier to figure out what the set looks like if you first consider $B_d((0, 0), 3)$).
3. On the same graph as in the previous part, draw $B_{d_u}((-3, 2), 1)$, where d_u denotes the square metric.

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Exercise 2.7. For example, let $X = \mathbb{R}^2$, $Y = [-1, 3] \times [2, 4]$, and d denote the Euclidean metric on $X = \mathbb{R}^2$. Let p denote the point $(3, 4)$, the upper right corner of Y . Then $B_{(Y, d)}(p, 2)$ looks like a quarter of the ball $B_{(X, d)}(p, 2)$. (Draw a picture to see this)

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Exercise 2.8. Let (X, d) be a metric space, and let E be a subset of X . The *diameter* of E in (X, d) is defined by the formula

$$\text{diam}_d(E) = \sup\{d(x, y) : x, y \in E\}.$$

1. Prove that for any $r > 0$ and $x \in X$, we have $\text{diam}(B(x, r)) \leq 2r$.
2. If X is any set and d is the discrete metric, show $\text{diam}(B(x, r)) = 0$.
3. If $X = \mathbb{R}^n$ for some $n \in \mathbb{N}$ and d is the Euclidean metric, prove that $\text{diam}(B(x, r)) = 2r$.

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Exercise 2.11. As in Example 2.7, let $X = \mathbb{R}^2$, $Y = [-1, 3] \times [2, 4]$, and let d denote the Euclidean metric on $X = \mathbb{R}^2$. Let $p = (3, 4)$ and let $q = (2, 4)$. Arguing *directly from the definition of an interior point* (i.e., without using Exercise 2.12), show that q is an interior point $B_Y(p, 2)$ with respect to Y , but q is not an interior point $B_Y(p, 2)$ with respect to X . In addition, draw a picture on a piece of graph paper that illustrates the idea of your proof.

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Exercise 2.12. Let (X, d) be a metric space, and let Y be a subset of X . Prove that

$$(*) \operatorname{Int}_X(U) = \operatorname{Int}_Y(U) \cap \operatorname{Int}_X(Y).$$

In the notation of Exercise 2.11, the equality $(*)$ gives an alternate explanation of why q is not an interior point of $B_Y(p, 2)$ with respect to X : It is because $q \notin \operatorname{Int}_X(Y)$, as can be seen from the picture you drew in that Exercise.

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Exercise 2.16. Let (X, d) be a metric space, and let U be a subset of X . Use Proposition 2.15 to prove that $\operatorname{Int}_X(U)$ is open in X .

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Exercise 2.20. Let (X, d) be a metric space. Assume that $U \subset Y \subset X$, and additionally that Y is open in X . Prove that U is open in Y if and only if U is open in X . (Note: There at least two possible solutions; one uses Theorem 2.19, the other uses Exercise 2.12.)

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