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MA 521
Homework #10
4/25/2019

Chapter 7: 4.3, 4.6, 4.7
Chapter 8: 1.5, 1.11, 1.12, 1.17

Exercise 4.3 Let $B = \{0\} \cup \{\frac{-1}{n^2}\}_{n\in\mathbb{N}}$ and $E = \mathbb{R} \setminus B$. Consider the series

$$\sum_{n=1}^{\infty} \frac{1}{1+n^2x}$$

on the set E.

1. Prove that the series converges absolutely for all $x \in E$; therefore it converges pointwise to a function $f : E \to \mathbb{R}$.
2. Prove that the series converges uniformly to f on $(-\infty, -\delta) \cup (\delta, \infty) \setminus B$ for any $\delta > 0$, but that it does not converge uniformly to f on E.
3. Prove that f is continous.
4. Prove that $f(0+) = +\infty$, that therefore f is not a bounded function.

1. Consider the series, $A = \sum_{n=1}^{\infty} \frac{1}{xn^2}$. For the case that $x > 0$, we know that $\sum_{n=1}^{\infty} |\frac{1}{1+n^2x}| = \sum_{n=1}^{\infty} \frac{1}{1+n^2x} \leq A$. A converges so for this case this series absolutely converges. The other case is $x < 0$. For this case, $1 + n^2x < -n^2$ after some value N. This value can be selected in this fashion: $-n^2 - n^2x \geq 1 \to -n^2 \geq \frac{1}{1+x} \to n^2 \geq |\frac{1}{1+x}|$. So from (N, ∞) this series absolutely converges by Comparison Test.
2. From part 1, select a series that converges that is larger than f, such as $\sum_{n=1}^{\infty} \frac{1}{1-\delta n^2} = M_n$. This is the largest choice since δ is the smallest positive and largest negative so it will cover both cases in terms of x. By the Weierstrass M-Test, this converges uniformly on the given interval. This is not true for E because $\exists\epsilon > 0$ s.t. $x = \delta - \epsilon$ s.t. $M_n < \frac{1}{|1-(\delta-\epsilon)n^2|}$.
3. By part 2, we know that f converges uniformly on $(-\infty, -\delta) \cup (\delta, \infty)$ so pick an arbitrary nbd $(a, b) \in (\delta, \infty)$. Uniform Convergence guarantees that $|f(x_1) - f(x_2)| < \epsilon$ for $|x_1 - x_2| < \delta$ which translates directly to the $\epsilon - \delta$ of continutiy.
4. Choose the $\sum_{n=1}^{\infty} \frac{1}{n}$. For $x \leq \frac{1}{4}$, choose $n \in \mathbb{N}$ s.t. that $n \geq 2$ then $\frac{1}{1+xn^2} > \frac{1}{n}$ and we know that $\frac{1}{n}$ diverges by p-test so by CST this series diverges as $x \to 0$

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Exercise 4.6 Find the radius of convergence for each of the following power series:

$$\sum_{n=0}^{\infty} n^n z^n \quad \sum_{n=0}^{\infty} \frac{z^n}{n!} \quad \sum_{n=0}^{\infty} z^n \quad \sum_{n=0}^{\infty} \frac{z^n}{n} \quad \sum_{n=0}^{\infty} \frac{z^n}{n^2}.$$

1. $\lim_{n\to\infty} \sup(n^n)^{\frac{1}{n}} \to \lim_{n\to\infty} \sup n \to +\infty$. So, radius of convergence is zero.

2. Apply ratio test so $|\frac{(n+1)!}{(n)!}| = |n+1|$ thus $\lim_{n\to\infty} \sup|n+1| \to \infty$ so the radius of convergence is zero.

3. $\lim_{n\to\infty} \sup \sqrt[n]{1|z|}$ thus the radius of convergence is (-1,1).

4. $\lim_{n\to\infty} \sup \sqrt[n]{\frac{1}{n}} = 1|z|$ thus the radius of convergence is (-1,1).

5. $\lim_{n\to\infty} \sup \sqrt[n]{\frac{1}{n^2}} = 1|z|$ thus the radius of convergence is (-1,1).

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Exercise 4.7 Consider the power series $\sum_{n=0}^{\infty} c_n z^n$. Let R be the radius of convergence of the power series, and assume $R > 0$. Let $f : (-R, R) \to \mathbb{R}$ be the function defined by $f(z) = \sum_{n=0}^{\infty} c_n z^n$. Prove the following statements, which refine Thm 4.5.

1. For any $r \in (0, R)$, the series $\sum_{n=0}^{\infty} c_n z^n$ converges uniformly on $(-r, r)$ to f.

2. f is continuous on all of $(-R, R)$.

1. For any $\epsilon > 0$, take the Weierstrauss-M test for any r, we have $\sum_{n=0}^{\infty} c_n r^n < \sum_{n=0}^{\infty}|c_n r^n| < \sum_{n=0}^{\infty} c_n(r+\epsilon)^n$. We can ensure convergence of $\sum_{n=0}^{\infty} c_n(r+\epsilon)^n$ s.t. $r+\epsilon \in (0, R)$. So, by Weierstrauss-M test, $\sum_{n=0}^{\infty} c_n r^n$ converges uniformly for all $r \in (0, R)$.

2. For any $r, \epsilon > 0$ and $r < R$, $\sum_{n=0}^{\infty} c_n(r-\epsilon)^n < \sum_{n=0}^{\infty} c_n r^n < \sum_{n=0}^{\infty} c_n(r+\epsilon)^n$. So, $-\sum_{n=0}^{\infty} c_n(r+\epsilon)^n < \sum_{n=0}^{\infty} c_n r^n < \sum_{n=0}^{\infty} c_n(r+\epsilon)^n$. So, $|\sum_{n=0}^{\infty} c_n r^n| \le \epsilon$ which implies $\sum_{n=0}^{\infty} c_n r^n \in B(\sum_{n=0}^{\infty} c_n r^n, \epsilon)$ for any δ so f is continuous.

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Exercise 1.5 Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that $|f(x) - f(y)| \le (x-y)^2$ for all $x, y \in \mathbb{R}$. Prove that f is constant.

$\lim_{t\to x} \frac{|f(t)-f(x)|}{t-x} \le |f(t) - f(x)| \le (t-x)^2$. $|f(t) - f(x)| \le (t-x)^2 \to 0$. So then, $\lim_{t\to x} \frac{|f(t)-f(x)|}{t-x} \to 0$. This shows that $\frac{f(t)-f(x)}{t-x} \to 0$ showing that f is constant. ■

Exercise 1.11 Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable, and assume $\lim_{x\to+\infty} x|f'(x)| = 0$. Define a sequence (a_n) in $\mathbb{R}$ by $a_n = f(2n) - f(n)$ for each $n \in \mathbb{N}$. Prove that $a_n \to$ as $n \to \infty$.

If $\lim_{x\to+\infty}|f'(x)| = 0$ thus $f'(x) \to$ a constant function as $x \to \infty$ $\frac{f(2n)-f(n)}{2n-n} \to 0$ as $n \to \infty$. $2n - n \forall n \in \mathbb{N}$, then $f(2n) - f(n) \to 0$ as $n \to \infty$. Thus, $a_n \to 0$ as $n \to \infty$. ∎

Exercise 1.12 Let $f : (a,b) \to \mathbb{R}$ be differentiable with $f'(x) > 0$ for all $x \in (a,b)$.

1. Prove that f is injective.
2. By part (1), there exists a function $g : f((a,b)) \to (a,b)$ such that $g(f(x)) = x$ for all $x \in (a,b)$. Prove that g is continuous.
3. Prove that g is differentiable, and that $g'(f(x)) = \frac{1}{f'(x)}$, for all $x \in (a,b)$.

1. Because f is strictly increasing by Cor. 1.10, there are only unique elements that are in the codomain, that is $\mathbb{R}$. This implies that no two elements in (a,b) can map to the same element in $\mathbb{R}$ proving f is injective.
2. g implies that f is surjective as well so it is bijective. Because f is strictly increasing we can say that $x \in [x-\epsilon, x+\epsilon] \subset (a,b)$ so that $f(x) \in [f(x-\epsilon), f(x+\epsilon))]$. So, by Theorem 2.24, g is continuous at x and since ϵ was arbitrary, g is continuous for all x on (a,b).
3. $\frac{g(f(x_n))-g(f(x))}{f(x_n)-f(x)} = \frac{x_n-x}{f(x_n)-f(x)} = \frac{1}{f'(x)}$ which is greater than 0 so the limit exists and g is differerentiable.

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Exercise 1.17 Use Taylor's Theorem with remainder to estimate $e^{\frac{1}{2}}$ to an accuracy of within 10^{-3}. Prove your answer has the desired accuracy.

We need $\frac{|f^{n+1}(x^*)|}{2^{n+1}(n+1)!} < .001$. So this means that we need to pick n=4 for the series so that $(n+1)!2^{n+1} > 1000$. This translates into the Taylor approximation formula, $e^x = \sum_{n=1}^{\infty} \frac{x^k}{k!} = 1 + \frac{1}{2} + \frac{1}{8} + \frac{1}{48} + \frac{1}{384} = \frac{211}{128} = 1.64864375$ compared to the calculator's answer 1.6489 which is within the bounds. ∎