

Chapter 2: 5.4, 5.5, 6.4  
Chapter 3: 1.7, 2.3, 2.4, 2.6, 2.8, 2.11, 2.12, 2.16, 2.20

**Exercise 5.4.** Let  $a$  and  $b$  be real numbers. Show that the following three equalities hold:

$$\bigcap_{x>b} (a, x) = (a, b), \quad \bigcup_{n=1}^{\infty} [a + \frac{1}{n}, b - \frac{1}{n}) = (a, b), \quad \bigcap_{n=1}^{\infty} (a + n, +\infty) = \emptyset.$$

1.  $\bigcap_{x>b} (a, x) = \{y \in \mathbb{R} : y \in (a, x) \forall x > b\}$ .  $\mathbb{R}$  has a least upper bound so call this bound,  $d$  so  $y \leq d \forall y \in \bigcap_{x>b} (a, x)$ . Assume  $d < b$ , then  $a < y \leq d < b < x \exists p \in \bigcap_{x>b} (a, x)$  s.t.  $a < b < p < x$ . So,  $d$  is not the least upper bound. So,  $b \leq d$  and  $a < y \leq b \leq d \forall \bigcap_{x>b} (a, x)$ .  $\bigcap_{x>b} (a, x) = \{y \in \mathbb{R} : a < y \leq b\} = (a, b]$ .
2.  $\bigcup_{n=1}^{\infty} [a + \frac{1}{n-1}, b - \frac{1}{n-1}) \cup [a + \frac{1}{n}, b - \frac{1}{n}) = \{y \in \mathbb{R} : a + \frac{1}{n} \leq y \leq b - \frac{1}{n}\} = B$ .
  - $a - \frac{1}{2}$  is a lower bound since  $\nexists y \in B$  s.t.  $y < a + \frac{1}{n}$ . Assume  $\exists$  a lower bound called  $\beta$  s.t.  $\beta > a + \frac{1}{n}$ . If so then,  $B = \{a + \frac{1}{n} < y < b - \frac{1}{n}\}$ . But,  $\exists y \in B$  s.t.  $y = a + \frac{1}{n}$ . So,  $\nexists$  any  $\beta$  so  $\inf B = a + \frac{1}{n}$  and  $a < a + \frac{1}{n} = \inf B$ .
  - $b - \frac{1}{n}$  is an upper bound since  $\nexists y \in B$  s.t.  $y > b - \frac{1}{n}$ . Assume  $\exists$  some upper bound  $\alpha$  s.t.  $\alpha < b - \frac{1}{n}$ . So,  $b - \frac{1}{n} - \alpha > 0$ . Choose  $\gamma \in \mathbb{N}$  so  $\frac{1}{\gamma} < b - \frac{1}{n} - \alpha$ .  $(b - \frac{1}{n} - \frac{1}{\gamma}) > b - \frac{1}{n} - (b - \frac{1}{n} - \alpha)$ . So,  $\alpha$  is not an upper bound. So,  $\sup B = b - \frac{1}{n} < b$ .

Thus,  $B = \{a < \inf B \leq y < \sup B < b \forall y \in B\} = (a, b)$ .

3. Let's do proof by contradiction. Assume  $\bigcap_{n=1}^{\infty} (a + n, +\infty) = X$  s.t.  $X = \{\beta\}, \beta \in \mathbb{R}, \beta < a + n$ . Enumerate  $\bigcap_{n=1}^{\infty} (a + n, +\infty) = (a + 1, \infty) \cap (a + 2, \infty) \cap \dots \cap (a + n, \infty) \cap \dots$ . Take a look at set,  $(a + n, \infty) = \{x \in \mathbb{R} : a + n < x < \infty\} = B$ . Clearly  $a + n$  is a lower bound for  $B$  since  $a + n < x \forall x \in B$ . Assume that  $k$  is a lower bound s.t.  $k > a + n$ . So,  $B = \{x \in \mathbb{R} : a + n < k < x < \infty\}$ . So,  $k - a - n > 0$ . Choose  $\phi$  so  $\phi > k - a - n$ .  $a + n + \phi > a + n + k - a - n$ . So,  $k$  is not a lower bound. So,  $\inf B = a + n \rightarrow \beta < a + n = \inf B$ .  $\beta \notin B$  so  $\beta \notin \bigcap_{n=1}^{\infty} (a + n, +\infty)$ . This establishes a contradiction so  $\bigcap_{n=1}^{\infty} (a + n, +\infty) = \emptyset$ . ■

**Exercise 5.5.** Let  $a_1, a_2, \dots$  be any enumeration of the negative rational numbers; let  $b_1, b_2, \dots$  be any enumeration of the positive rational numbers. Show that the following two equalities hold:

$$\bigcap_{j=1}^{\infty} (a_j, b_j) = \{0\}, \quad \bigcup_{j=1}^{\infty} (a_j, b_j) = \mathbb{R}.$$

1. Let's analyze one set of this general intersection,  $(a_n, b_n)$  and call it  $A$ .  $\sup A$  exists by the least upper bound property since  $A \subset \mathbb{R}$  is bounded above and nonempty by least upper bound property. Similar argument for existence of  $\inf A$  but it is bounded below. Let us prove what  $\sup A, \inf A$  are supposed to be.

- $a_n$  is a lower bound  $\rightarrow x \notin A$  s.t.  $x < a_n \forall x \in A$ . Assume that  $l$  is a lower bound s.t.  $l > a_n$ . So,  $A = \{x \in \mathbb{R} : a_n < x < b_n\}$ . So,  $l - a_n > 0$ . Choose  $\alpha \in \mathbb{N}$  so  $\alpha > l - a_n$ .  $\alpha + a_n > a_n + l - a_n$ . Thus,  $l$  is not a lower bound. So,  $\inf A = a_n$ .
- $b_n$  is an upper bound  $\rightarrow x \notin A$  s.t.  $x > b_n \forall x \in A$ . Assume that  $u$  is an upper bound s.t.  $u < b_n$ . So,  $A = \{x \in \mathbb{R} : a_n < x < u < b_n\}$ . So,  $u - b_n > 0$ . Choose  $n \in \mathbb{N}$  so  $u - b_n > \frac{1}{n}$ . So,  $b_n - \frac{1}{n} > u$ . Thus,  $u$  is not an upper bound. So,  $\sup A = b_n$ .

Since  $a_n \in -\mathbb{Q}$  and  $b_n \in +\mathbb{Q} \forall n \in \mathbb{N}, 0 \in (a_n, b_n) \forall n$  so  $\bigcap_{j=1}^{\infty} (a_n, b_n) = \{0\}$ .

2. Let's analyze one set of this general union,  $(a_n, b_n)$  and call it  $A$ .  $\sup A$  exists by the least upper bound property since  $A \subset \mathbb{R}$  is bounded above and nonempty by least upper bound property. Similar argument for existence of  $\inf A$  but it is bounded below. Let us prove what  $\sup A, \inf A$  are supposed to be.

- $a_n$  is a lower bound  $\rightarrow x \notin A$  s.t.  $x < a_n \forall x \in A$ . Assume that  $l$  is a lower bound s.t.  $l > a_n$ . So,  $A = \{x \in \mathbb{R} : a_n < x < b_n\}$ . So,  $l - a_n > 0$ . Choose  $\alpha \in \mathbb{N}$  so  $\alpha > l - a_n$ .  $\alpha + a_n > a_n + l - a_n$ . Thus,  $l$  is not a lower bound. So,  $\inf A = a_n$ .
- $b_n$  is an upper bound  $\rightarrow x \notin A$  s.t.  $x > b_n \forall x \in A$ . Assume that  $u$  is an upper bound s.t.  $u < b_n$ . So,  $A = \{x \in \mathbb{R} : a_n < x < u < b_n\}$ . So,  $u - b_n > 0$ . Choose  $n \in \mathbb{N}$  so  $u - b_n > \frac{1}{n}$ . So,  $b_n - \frac{1}{n} > u$ . Thus,  $u$  is not an upper bound. So,  $\sup A = b_n$ .

Assume that there is some  $\beta \in \mathbb{R}$ . If  $\beta \notin A$ ,  $\exists$  some set in  $\bigcup_{j=1}^{\infty} (a_j, b_j)$  s.t.  $\beta \in \{a_n - \epsilon < x < b_n + \epsilon\}$  for any  $\epsilon > 0$ . So, if true for any  $\beta$ ,  $\bigcup_{j=1}^{\infty} (a_j, b_j) = \mathbb{R}$ . ■

**Exercise 6.4.** Prove there exists no order  $\leq$  that makes  $(\mathbb{C}, +, \cdot, \leq)$  into an ordered field.

Let us argue by contradiction. Assume that  $\mathbb{C}$  is an ordered field. Let us analyze the case  $(0, 1) = i$ .

- If  $i > 0$ ,  $x, y \in \mathbb{C}$  and  $x = i, y = i, i \cdot i > 0 \implies -1 > 0$ , which is a contradiction.
- If  $i < 0$ ,  $x, y \in \mathbb{C}$  and  $x = -i, y = -i$  (since  $x, y > 0$  for the second condition to hold),  $-i \cdot -i < 0 \rightarrow i^2 > 0 \implies -1 > 0$ , which is a contradiction. ■

**Exercise 1.7.** Let  $\|\cdot\|$  be a norm on a real vector space  $V$ . Prove the *reverse triangle inequality*:

$$|||x| - |y|| \leq \|x - y\|$$

$$\begin{aligned} \|x\| &= \|(x - y) + y\| \leq \|x - y\| + \|y\|. \quad |||x| - |y|| \leq |||(x - y) + \|y\| - \|y\|| = \\ &= |||(x - y)||| = \|x - y\|. \\ \|y\| &= \|(y - x) + x\| = \|y - x\| + \|x\|. \quad |||x| - |y|| \leq |||x| - (\|y - x\| + \|x\|)| \leq \|x - y\|. \end{aligned}$$

■

**Exercise 2.3.** Let  $X$  be any set. Prove that the discrete metric  $d : X \times X \rightarrow \mathbb{R}$  (defined by  $d(x, y) = 1$  if  $x \neq y$  and  $d(x, x) = 0$  for  $x \in X$ ) satisfies the triangle inequality and is therefore a metric on  $X$ .

Let's argue via proof by contrapositive. Assume  $d(x, y) > d(x, z) + d(z, y) \forall x, y, z \in X$  so  $d$  is not a metric on  $X$ . Let  $x \neq y$  and enumerate what values  $z$  can take on.

- If  $z \neq y \neq x$ , then  $1 > 2$ .
- If  $z = x$  and  $z \neq y$ , vice versa, then  $1 > 1$ .
- If  $z = y = x$ , then  $1 > 0$ . However, this contradicts our initial assumption that  $x \neq y$ .

Clearly, all these pose contradictions, so it must be that  $d$  satisfies the triangle inequality and is a metric on  $X$ .

■

**Exercise 2.4.** Determine which of the following functions are metrics on  $\mathbb{R}$ . Prove your answer in each case.

- $d_1(x, y) = \sqrt{|x - y|}$ .
- $d_2(x, y) = |x - 2y|$ .
- $d_3(x, y) = \frac{|x - y|}{1 + |x - y|}$ .

■

**Exercise 2.6.** Consider the function  $d : \mathbb{R}^2 \times \mathbb{R}^2 \rightarrow \mathbb{R}$ , defined by

$$d(x, y) = |x_1 - y_1| + |x_2 - y_2|, \quad (x = (x_1, x_2), y = (y_1, y_2)).$$

1. Prove that  $d$  is a metric on  $\mathbb{R}^2$ .
2. On a sheet of graph paper, draw the set  $B_d((5, 1), 3)$ . Use dotted lines to indicate the boundary, which is not included in the set you are drawing. (Hint: it may be easier to figure out what the set looks like if you first consider  $B_d((0, 0), 3)$ .)
3. On the same graph as in the previous part, draw  $B_{d_u}((-3, 2), 1)$ , where  $d_u$  denotes the square metric.

**Exercise 2.8.** Let  $(X, d)$  be a metric space, and let  $E$  be a subset of  $X$ . The *diameter* of  $E$  in  $(X, d)$  is defined by the formula

$$\text{diam}_d(E) = \sup\{d(x, y) : x, y \in E\}.$$

1. Prove that for any  $r > 0$  and  $x \in X$ , we have  $\text{diam}(B(x, r)) \leq 2r$ .
2. If  $X$  is any set and  $d$  is the discrete metric, show  $\text{diam}(B(x, r)) = 0$  for any  $r \leq 1$ , while  $\text{diam}(B(x, r)) = 1$  for any  $r > 1$ .
3. If  $X = \mathbb{R}^n$  for some  $n \in \mathbb{N}$  and  $d$  is the Euclidean metric, prove that  $\text{diam}(B(x, r)) = 2r$ .

**Exercise 2.11.** As in Example 2.7, let  $X = \mathbb{R}^2$ ,  $Y = [-1, 3] \times [2, 4]$ , and let  $d$  denote the Euclidean metric on  $X = \mathbb{R}^2$ . Let  $p = (3, 4)$  and let  $q = (2, 4)$ . Arguing *directly from the definition of an interior point* (i.e., without using Exercise 2.12), show that  $q$  is an interior point  $B_Y(p, 2)$  with respect to  $Y$ , but  $q$  is not an interior point  $B_Y(p, 2)$  with respect to  $X$ . In addition, draw a picture on a piece of graph paper that illustrates the idea of your proof.

**Exercise 2.12.** Let  $(X, d)$  be a metric space, and let  $Y$  be a subset of  $X$ . Prove that for any subset  $U$  of  $Y$ , we have

$$(*) \text{Int}_X(U) = \text{Int}_Y(U) \cap \text{Int}_X(Y).$$

In the notation of Exercise 2.11, the equality  $(*)$  gives an alternate explanation of why  $q$  is not an interior point of  $B_Y(p, 2)$  with respect to  $X$ : It is because  $q \notin \text{Int}_X(Y)$ , as can be seen from the picture you drew in that Exercise.

**Exercise 2.16.** Let  $(X, d)$  be a metric space, and let  $U$  be a subset of  $X$ . Use Proposition 2.15 to prove that  $\text{Int}_X(U)$  is open in  $X$ .

**Exercise 2.20.** Let  $(X, d)$  be a metric space. Assume that  $U \subset Y \subset X$ , and additionally that  $Y$  is open in  $X$ . Prove that  $U$  is open in  $Y$  if and only if  $U$  is open in  $X$ . (Note: There are at least two possible solutions; one uses Theorem 2.19, the other uses Exercise 2.12.)