

Chapter 7: 4.3, 4.6, 4.7
Chapter 8: 1.5, 1.11, 1.12, 1.17

Exercise 4.3 Let $B = \{0\} \cup \{\frac{-1}{n^2}\}_{n \in \mathbb{N}}$ and $E = \mathbb{R} \setminus B$. Consider the series

$$\sum_{n=1}^{\infty} \frac{1}{1+n^2x}$$

on the set E .

1. Prove that the series converges absolutely for all $x \in E$; therefore it converges pointwise to a function $f : E \rightarrow \mathbb{R}$.
2. Prove that the series converges uniformly to f on $(-\infty, -\delta) \cup (\delta, \infty) \setminus B$ for any $\delta > 0$, but that it does not converge uniformly to f on E .
3. Prove that f is continuous.
4. Prove that $f(0+) = +\infty$, that therefore f is not a bounded function.

1. Consider the series, $A = \sum_{n=1}^{\infty} \frac{1}{xn^2}$. For the case that $x > 0$, we know that $\sum_{n=1}^{\infty} |\frac{1}{1+n^2x}| = \sum_{n=1}^{\infty} \frac{1}{1+n^2x} \leq A$. A converges so for this case this series absolutely converges. The other case is $x < 0$. For this case, $1+n^2x < -n^2$ after some value N . This value can be selected in this fashion: $-n^2 - n^2x \geq 1 \rightarrow -n^2 \geq \frac{1}{1+x} \rightarrow n^2 \geq |\frac{1}{1+x}|$. So from (N, ∞) this series absolutely converges by Comparison Test.
2. From part 1, select a series that converges that is larger than f , such as $\sum_{n=1}^{\infty} \frac{1}{1-\delta n^2} = M_n$. This is the largest choice since δ is the smallest positive and largest negative so it will cover both cases in terms of x . By the Weierstrass M -Test, this converges uniformly on the given interval. This is not true for E because $\exists \epsilon > 0$ s.t. $x = \delta - \epsilon$ s.t. $M_n < \frac{1}{|1-(\delta-\epsilon)n^2|}$.
3. By part 2, we know that f converges uniformly on $(-\infty, -\delta) \cup (\delta, \infty)$ so pick an arbitrary nbd $(a, b) \in (\delta, \infty)$. Uniform Convergence guarantees that $|f(x_1) - f(x_2)| < \epsilon$ for $|x_1 - x_2| < \delta$ which translates directly to the $\epsilon - \delta$ of continuity.
4. Choose the $\sum_{n=1}^{\infty} \frac{1}{n}$. For $x \leq \frac{1}{4}$, choose $n \in \mathbb{N}$ s.t. that $n \geq 2$ then $\frac{1}{1+xn^2} > \frac{1}{n}$ and we know that $\frac{1}{n}$ diverges by p-test so by CST this series diverges as $x \rightarrow 0$

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Exercise 4.6 Find the radius of convergence for each of the following power series:

$$\sum_{n=0}^{\infty} n^n z^n \quad \sum_{n=0}^{\infty} \frac{z^n}{n!} \quad \sum_{n=0}^{\infty} z^n \quad \sum_{n=0}^{\infty} \frac{z^n}{n} \quad \sum_{n=0}^{\infty} \frac{z^n}{n^2}.$$

1. $\lim_{n \rightarrow \infty} \sup(n^n)^{\frac{1}{n}} \rightarrow \lim_{n \rightarrow \infty} \sup n \rightarrow +\infty$. So, radius of convergence is zero.
- 2.
- 3.
- 4.

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Exercise 4.7 Consider the power series $\sum_{n=0}^{\infty} c_n z^n$. Let R be the radius of convergence of the power series, and assume $R > 0$. Let $f : (-R, R) \rightarrow \mathbb{R}$ be the function defined by $f(z) = \sum_{n=0}^{\infty} c_n z^n$. Prove the following statements, which refine Thm 4.5.

1. For any $r \in (0, R)$, the series $\sum_{n=0}^{\infty} c_n z^n$ converges uniformly on $(-r, r)$ to f .
2. f is continuous on all of $(-R, R)$.

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Exercise 1.5 Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a function such that $|f(x) - f(y)| \leq (x - y)^2$ for all $x, y \in \mathbb{R}$. Prove that f is constant.

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Exercise 1.11 Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be differentiable, and assume $\lim_{x \rightarrow +\infty} x|f'(x)| = 0$. Define a sequence (a_n) in \mathbb{R} by $a_n = f(2n) - f(n)$ for each $n \in \mathbb{N}$. Prove that $a_n \rightarrow 0$ as $n \rightarrow \infty$.

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Exercise 1.12 Let $f : (a, b) \rightarrow \mathbb{R}$ be differentiable with $f'(x) > 0$ for all $x \in (a, b)$.

1. Prove that f is injective.
2. By part (1), there exists a function $g : f((a, b)) \rightarrow (a, b)$ such that $g(f(x)) = x$ for all $x \in (a, b)$. Prove that g is continuous.
3. Prove that g is differentiable, and that $g'(f(x)) = \frac{1}{f'(x)}$, for all $x \in (a, b)$.

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Exercise 1.17 Use Taylor's Theorem with remainder to estimate $e^{\frac{1}{2}}$ to an accuracy of within 10^{-3} . Prove your answer has the desired accuracy.

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