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MA 521
Homework #8
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Chapter 5: 3.9, 3.10
Chapter 6: 1.9, 4.2, 4.5

Exercise 3.9 A collection $\mathcal{A}$ of real-valued functions on a set E is said to be *uniformly bounded* on E if there exists $M > 0$ such that $|f(x)| \leq M$ for all $x \in E$, for all $f \in \mathcal{A}$. (So each function is bounded, and the same bound works for all functions in $\mathcal{A}$.) Let (f_n) be a sequence of bounded functions which converges uniformly to a limit function f. Prove that $\{f_n\}$ is uniformly bounded.

(f_n) contains sequence of all bounded functions. By Prop 3.8, $f_n \to f$ uniformly iff $d_u(f, f_n) = \sup|f_n(x) - f(x)|$ as $n \to \infty$. So, choose $n \in \mathbb{N}$ s.t. $\max(|f_1(x) - f(x)|, \ldots, |f_n(x) - f(x)|, \ldots) \forall x \in E$. Take this value s.t. $M = |f_n(x) - f(x)| + 1$. By Prop 3.8, this is the largest deviation possible and all other functions will lie in $B(E)$ so they will also be bounded by M. So, $\{f_n\}$ is uniformly bounded. ∎

Exercise 3.10 Let (f_n) and (g_n) be sequences of real-valued functions on a set E, which converge uniformly on E to limit functions f and g, respectively.

1. Prove that $(f_n + g_n)$ converges to $f + g$, uniformly on E.
2. If each f_n and each g_n is bounded, show that $(f_n g_n)$ converges uniformly to fg on E.

1. So, for $(f_n + g_n)$ to converge uniformly, we need to show that $|f_n(x) + g_n(x) - f(x) + g(x)| < \epsilon \, \forall \epsilon > 0$. Apply the trianle inequality so $|f_n(x) + g_n(x) - f(x) + g(x)| \leq |f_n(x) - f(x)| + |g_n(x) - g(x)| < \epsilon_1 + \epsilon_2$, where ϵ_1 is the $\sup|f_n - f|$ and ϵ_2 is the $\sup|g_n - g|$. Since f, g both converge uniformly on E, $f_n + g_n$ is also uniformly converges on E.
2. $(f_n) \leq M, (g_n) \leq L$, $|g_n(x) - g(x)| < \epsilon_1$, and $|f_n(x) - f(x)| < \epsilon_2$. We need to prove that $|f_n(x)g_n(x) - f(x)g(x)| < \epsilon \forall \epsilon > 0$. So, $|f_n(x)g_n(x) - f(x)g(x)| \leq |f_n(x)||g_n(x) - g(x)| + |g(x)||f_n(x) - f(x)| = M\epsilon_1 + L\epsilon_2$. So, $f_n + g_n$ uniformly converges to fg on E.

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Exercise 1.9 Prove the second and third points in Prop 1.8.

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Exercise 4.2 Let $(s_n)_{n=1}^{\infty}$ and $(t_n)_{n=1}^{\infty}$ be sequences in $\overline{\mathbb{R}}$ and let $(u_n)_{n=1}^{\infty}$ be a sequence in $\mathbb{R}$. Prove the following statements.

1. If $s_n \leq t_n$ for each $n \in \mathbb{N}$ and $\lim_{n\to\infty} s_n = +\infty$, then $\lim_{n\to\infty} t_n = +\infty$ as well.
2. If (s_n) and (t_n) converge in $\overline{\mathbb{R}}$ to s and t, respectively, and if $s_n \leq t_n$ for each $n \in \mathbb{N}$, then $s \leq t$.

1.
2.

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Exercise 4.2 Let $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ be sequences in of real numbers. Prove that

$$\lim_{n\to} \sup(a_n + b_n) \leq \lim_{n\to\infty} \sup(a_n) + \lim_{n\to\infty} \sup(b_n),$$

provided that the RHS isn't of the form $\infty - \infty$.

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