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MA 521
Homework #7
3/28/2018

1.2, 2.2, 2.9, 2.10, 2.11, 2.23, 2.32, 2.36, 2.37.

Exercise 1.2 Let (X, d_X) and (Y, d_Y) be metric spaces, and let E be a subset of X. Let $f : E \to Y$ be a function, and let p be a limit point of E in X. Prove that $f(x) \to q$ as $x \to p$ if and only if for every $\epsilon > 0$, there exists $\delta > 0$ such that $x \in E$ and $0 < d_X(x, p) < \delta$ imply together that $d_Y(f(x), q) < \epsilon$.

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Exercise 2.2 Let (X, d_X) and (Y, d_Y) be metric spaces; let $f : X \to Y$ be a function. Prove that f is continuous at $p \in X$ if and only if for every ϵ ¿ 0, there exists $\delta > 0$ such that $x \in B_X(p, \delta)$ implies $f(x) \in B_Y(f(p), \epsilon)$.

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Exercise 2.9 Assume $f : \mathbb{R} \to \mathbb{R}$ is a function satisfying $\lim_{h\to 0}[f(x+h) - f(x-h)] = 0$, for all $x \in \mathbb{R}$. Does it follow that f must be continuous? If so, give a proof; if not, give a counterexample.

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Exercise 2.10 Let (X, d_X) and (Y, d_Y) be metric spaces and $f : X \to Y$ a function.

1. Show that f is continuous if and only if $f^{-1}(C)$ is closed on X whenever C is closed in Y.
2. Show that $f : X \to Y$ is continuous if and only if $f(\overline{A}) \subset \overline{f(A)}$ for every subset A of X.
3. Consider the (continuous) function $g : \mathbb{R} \to \mathbb{R}$ given by $g(x) = \frac{1}{1+x^2}$. Give an example of a subset A of $\mathbb{R}$ such that $g(\overline{A}) \neq \overline{g(A)}$.

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Exercise 2.11 Let (X, d_X) and (Y, d_Y) be metric spaces and let f and g be continuous functions from X to Y. Assume E is a dense subset of X.

1. Prove that $f(E)$ is dense in $f(X)$. (Hint: Use Exercise 1.30) in Chapter 4 and Exercise 2.10 above.)
2. Prove that if $f(x) = g(x)$ for all $x \in E$, then $f(x) = g(x)$ for all $x \in X$.

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Exercise 2.23

1. Find a closed subset of E of $\mathbb{R}$ and a continuous function $f : \mathbb{R} \to \mathbb{R}$ is continuous such that $f(E)$ is not closed.
2. Find a bounded subset E of $\mathbb{R}$ and a continuous function $f : E \to \mathbb{R}$ such that $f(E)$ is not bounded.
3. Show that if E is a bounded subset of $\mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$ is continuous, then $f(E)$ is bounded.

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Exercise 2.32 Prove that the set $R^2 \setminus \{0, 0\}$ is path-connected, and therefore connected. Then, use the function $x \setminus |x|$ to show that $S = \{x \in \mathbb{R}^2 : |x| = 1\}$ is connected.

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Exercise 2.36 Assume $f : X \to Y$ and $g : Y \to Z$ are uniformly continuous functions, where $(X, d_X), (Y, d_Y)$, and (Z, d_Z) are metric spaces. Prove that $g \circ f$ is uniformly continuous.

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Exercise 2.37 Let E be a bounded subset of $\mathbb{R}^k$ and let $f : E \to \mathbb{R}$ be a uniformly continuous function. Show that f is bounded. (Hint: You will need to use compactness of $\overline{E}$ at some point.)

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