

Chapter 5: 3.9, 3.10
Chapter 6: 1.9, 4.2, 4.5

Exercise 3.9 A collection \mathcal{A} of real-valued functions on a set E is said to be *uniformly bounded* on E if there exists $M > 0$ such that $|f(x)| \leq M$ for all $x \in E$, for all $f \in \mathcal{A}$. (So each function is bounded, and the same bound works for all functions in \mathcal{A} .) Let (f_n) be a sequence of bounded functions which converges uniformly to a limit function f . Prove that $\{f_n\}$ is uniformly bounded.

(f_n) contains sequence of all bounded functions. By Prop 3.8, $f_n \rightarrow f$ uniformly iff $d_u(f, f_n) = \sup |f_n(x) - f(x)|$ as $n \rightarrow \infty$. So, choose $n \in \mathbb{N}$ s.t. $\max(|f_1(x) - f(x)|, \dots, |f_n(x) - f(x)|, \dots) \forall x \in E$. Take this value s.t. $M = \sup |f_n(x) - f(x)| + 1$. By Prop 3.8, this is the largest deviation possible and all other functions will lie in $B(E)$ so they will also be bounded by M . So, $\{f_n\}$ is uniformly bounded. ■

Exercise 3.10 Let (f_n) and (g_n) be sequences of real-valued functions on a set E , which converge uniformly on E to limit functions f and g , respectively.

1. Prove that $(f_n + g_n)$ converges to $f + g$, uniformly on E .
2. If each f_n and each g_n is bounded, show that $(f_n g_n)$ converges uniformly to fg on E .

1. So, for $(f_n + g_n)$ to converge uniformly, we need to show that $|f_n(x) + g_n(x) - f(x) - g(x)| < \epsilon \forall \epsilon > 0$. Apply the triangle inequality so $|f_n(x) + g_n(x) - f(x) - g(x)| \leq |f_n(x) - f(x)| + |g_n(x) - g(x)| < \epsilon_1 + \epsilon_2$, where ϵ_1 is the $\sup |f_n - f|$ and ϵ_2 is the $\sup |g_n - g|$. Since f, g both converge uniformly on E , $f_n + g_n$ is also uniformly converges on E .
2. $(f_n) \leq M, (g_n) \leq L$, where $L, M \in \mathbb{R}$, $|g_n(x) - g(x)| < \epsilon_1$, and $|f_n(x) - f(x)| < \epsilon_2$. We need to prove that $|f_n(x)g_n(x) - f(x)g(x)| < \epsilon \forall \epsilon > 0$. So, $|f_n(x)g_n(x) - f(x)g(x)| \leq |f_n(x)||g_n(x) - g(x)| + |g(x)||f_n(x) - f(x)| = M\epsilon_1 + L\epsilon_2$. So, $f_n + g_n$ uniformly converges to fg on E . ■

Exercise 1.9 Prove the second and third points in Prop 1.8.

1. For \rightarrow , by definition of limits, for all nbd V of q , \exists a nbd U of $+\infty$ s.t. $x \in U \cap B \setminus \{+\infty\} \neq \emptyset \rightarrow g(x) \in V$. So pick M s.t. $(M, +\infty) \subset B$. So, pick $V = B_{\overline{\mathbb{R}}}(q, \epsilon)$ for some ϵ in V . So, $g(x) \in B_{\overline{\mathbb{R}}}(q, \epsilon) \rightarrow |g(x) - q| < \epsilon$. For the other direction, for every $\epsilon > 0$, $\exists M \in \mathbb{R}$ s.t. $x > M$ and $x \in B$ together imply that $|g(x) - q| < \epsilon$. $|g(x) - q| < \epsilon \rightarrow g(x) \in B_{\overline{\mathbb{R}}}(q, \epsilon)$ for every $\epsilon > 0$. Pick an M in \mathbb{R} and set U in \mathbb{R} as $(M, +\infty)$. So $x \in (M, +\infty)$. So $x \in (M, +\infty)$ and $x \in B$ and $(M, +\infty) \cap B \setminus \{+\infty\} \neq \emptyset$.
2. For every neighborhood V of $+\infty$, \exists a nbd U of $+\infty$ s.t. $x \in U \cap C \setminus \{+\infty\} \neq \emptyset$. Let U be the neighborhood of $+\infty$ for some $P \in \mathbb{R}$, $(P, +\infty)$. So, let $x \in C$ and $x > P$ to be in $(P, +\infty)$. So, let $h(x) \in V$ around $+\infty$. So, for any choice of N , $h(x)$ will always be in V . For \leftarrow , let U be a neighborhood of $+\infty$ s.t. $x \in (P, +\infty)$ and $x \in C$. $h(x) > N$ implies that V can be chosen as $(N, +\infty)$. To see if $+\infty$ is limit point, $U \cap C \setminus \{+\infty\} \neq \emptyset$.

■

Exercise 4.2 Let $(s_n)_{n=1}^{\infty}$ and $(t_n)_{n=1}^{\infty}$ be sequences in $\overline{\mathbb{R}}$ and let $(u_n)_{n=1}^{\infty}$ be a sequence in \mathbb{R} . Prove the following statements.

1. If $s_n \leq t_n$ for each $n \in \mathbb{N}$ and $\lim_{n \rightarrow \infty} s_n = +\infty$, then $\lim_{n \rightarrow \infty} t_n = +\infty$ as well.
2. If (s_n) and (t_n) converge in $\overline{\mathbb{R}}$ to s and t , respectively, and if $s_n \leq t_n$ for each $n \in \mathbb{N}$, then $s \leq t$.

- 1.
- 2.

■

Exercise 4.2 Let $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ be sequences in of real numbers. Prove that

$$\limsup_{n \rightarrow \infty} (a_n + b_n) \leq \limsup_{n \rightarrow \infty} (a_n) + \limsup_{n \rightarrow \infty} (b_n),$$

provided that the RHS isn't of the form $\infty - \infty$.

■