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MA 521
Homework #2
2/6/2019

Chapter 1: 4.18, 4.19, 4.22;
Chapter 2: 1.6, 1.7, 2.3, 2.4, 2.5, 3.3, 4.6, 4.8, 4.9, 4.10

**Exercise 4.18.** Let $A$ and $B$ be sets. Assume $A$ is infinite, $B$ is countable, and $A$ and $B$ are disjoint. Prove $A \sim A \cup B$. Hint: The strategy of Theorem 4.16 may be useful.

If $A$ is infinite, we have $C \subset A$, a countably infinite set. By Proposition 4.21, the union of two countable sets is still countable, $B \cup C$, which is countably infinite. Since $((A \cup B) \setminus B \cup C) \cap C$ and $((A \cup B) \setminus B \cup C) \cap (B \cup C)$ are both empty, $A \cup B = ((A \cup B) \setminus B \cup C) \cup (B \cup C) \sim ((A \cup B) \setminus B \cup C) \cup C = A$. ■

**Exercise 4.19.** Let $X$ and $Y$ be sets. Assume $Y$ is countable and $X \setminus Y$ is infinite. Prove that $X \sim X \cup Y \sim X \setminus Y$. Hint: Each of the equivalences can be done extremely quickly if you use the previous exercise and some set manipulations.

If $X \setminus Y$ is infinite, $X \setminus Y$ must have a countably infinite subset. This means that $X$ must be infinite. We can use Exercise 4.18 but we need to prove that $X$ and $Y$ are disjoint sets. Let's solve by contradiction.

Assume $a_1 \in X, Y$ s.t. $X \cap Y = \{a_1\}$. This means that $X \setminus Y$ will be a proper subset of $X$. We can apply Theorem 4.16 to say $X \sim X \setminus Y$. But then, $X \cap Y \sim (X \cap Y) \cap Y = \emptyset$, which is a contradicts our assumption. This suggest that $X$ and $Y$ are disjoint and apply Exercise 4.18 directly to say $X \sim X \cup Y \sim X \setminus Y$. ■

**Exercise 4.22.** Let $X$ be a countable set.

1. Prove inductively that $X^n \sim X^{n-1} \times X$ for any $n \in \mathbb{N}$.
2. Prove inductively that $X^n$ is countable for any $n \in \mathbb{N}$.

1.
2.

■

**Exercise 1.6.** Let $E, F$, and $G$ be nonempty subsets of an ordered set $(S, \leq)$. Prove the following statements.

1. If $\alpha \in S$ is a lower bound for $E$ and $\beta \in S$ is an upper bound for $E$, then $\alpha \leq \beta$.
2. $\sup E \leq \inf F$ if and only if $x \leq y$ for any $x \in E, y \in F$.
3. If $E \subset G$, then $\sup E \leq \sup G$.

1.
2.
3.

■

**Exercise 1.7.** Let $(S, \leq)$ be an ordered set, let $f$ and $g$ be functions from $X$ to $S$ and let $A$ be a subset of $X$. Assume that $f(x) \leq g(x)$ for all $x \in A$, and that furthermore $\sup_A f$ and $\sup_A g$ exist in $S$. Prove that $\sup_A f \leq \sup_A g$.

■

**Exercise 2.3.** Let $A$ be a nonempty subset of an ordered field $(F, +, \cdot, \leq)$. Assume that $\sup A$ and $\inf A$ exist in $F$, and let $c$ be any element of $F$. Define the set $cA := \{ca : a \in A\}$.

1. Prove that $c \leq 0$, then $\sup(cA) = c \sup A$.
2. What is $\sup(cA)$ if $c \leq 0$? Prove your answer is correct.

1.
2.

■

**Exercise 2.4** Let $A$ be a nonempty subset of an ordered field $(F, +, \cdot, \leq)$. Assume that $\sup A$ and $\inf A$ exist in $F$. Define $A + B := \{a + b : a \in A, b \in B\}$. Prove that $\sup(A + B) = \sup A + \sup B$ by filling in the details of the following outline:

- Denote $s = \sup A, t = \sup B$. Then $s + t$ is an upper bound for $A + B$.
- Let $u$ be any upper bound for $A + B$, and let $a$ be any element of $A$. Then $t \leq u - a$.
- We have $s + t \leq u$. Consequently, $\sup(A + B)$ exists in $F$ and is equal to $s + t = \sup A + \sup B$.

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**Exercise 2.5.** Let $f$ and $g$ be functions from a set $X$ to an ordered field $(F, +, \cdot, \leq)$. Let $A$ be a subset of $X$.

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**Exercise 3.3.** Using the strategies similar to those proofs in this section, prove the following statements.

1. There is no rational whose square is 20.
2. The set $A := \{r \in \mathbb{Q} : r^2 \leq 20\}$ has no least upper bound in $\mathbb{Q}$.

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**Exercise 4.6.** Elements of $\mathbb{R} \setminus \mathbb{Q}$ are called *irrational numbers*.

1. Assume $r$ is rational and $x$ is irrational. Show that $r + x$ and $rx$ are irrational.
2. Use the Archimedean property of $\mathbb{R}$ to prove that the set of irrational numbers is dense in $\mathbb{R}$. (Hint: First prove if $x$ and $y$ are real numbers with $y - x > \sqrt{2}$, then there exists an integer $m$ such that $x < m\sqrt{2} < y$.)

1.
2.

■

**Exercise 4.8.** Assume $a, b \in \mathbb{R}$. Prove that $a \leq b$ if and only if $a \leq b + \epsilon$ for every $\epsilon > 0$.

■

**Exercise 4.9.** Let $E$ be a set of real numbers, let $s$ be an upper bound for $E$. Prove that $s = \sup E$ if and only if for every $\epsilon > 0$ there exists $x \in E$ such that $x > s - \epsilon$.

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**Exercise 4.10.** Let $A$ and $B$ be nonempty sets of real numbers. Decide whether the following statements are true or false. If true, give a proof; if false, give a counterexample.

1. If $\sup A < \inf B$, then there exists a $c \in \mathbb{R}$ satisfying $a < c < b$ for all $a \in A$ and $b \in B$.
2. If there exists a $c \in \mathbb{R}$ satisfying $a < c < b$ for all $a \in A$ and $b \in B$, then $\sup A < \inf B$.

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