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MA 521
Homework #5
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1.25, 1.26, 1.28, 1.30, 1.31, 2.7, 2.9, 2.16, 2.18.

Exercise 1.25 Let (X, d) be a metric space. Let E and Y be subsets of X such that $E \subset Y$. Prove that

$$\mathrm{Cl}_Y(E) = \mathrm{Cl}_X(E) \cap Y.$$

For $\subset$, let $x \in \mathrm{Cl}_Y(E)$. $\mathrm{Cl}_Y(E) = E \cup \mathrm{Lim}_Y(E) = E \cup (\mathrm{Lim}_X(E) \cap Y)$ (by Exercise 1.10) $= (E \cup \mathrm{Lim}_X(E)) \cap Y = \mathrm{Cl}_X(E) \cap Y$. So, $x \in \mathrm{Cl}_X(E) \cap Y$. For $\supset$, let $x \in \mathrm{Cl}_X(E), Y$. So, $\mathrm{Cl}_X(E) = (E \cup \mathrm{Lim}_X(E)) \cap Y = E \cup (\mathrm{Lim}_X(E) \cap Y) = E \cup \mathrm{Lim}_Y(E) = Cl_Y(E)$. ∎

Exercise 1.26 Let (X, d) be a metric space.

1. Prove that for any collection $\mathbb{E}$ of subsets of X, we have

$$\bigcup_{E \in \mathbb{E}} \overline{E} \subset \overline{\bigcup_{E \in \mathbb{E}} E}$$

and equality holds if $\mathbb{E}$ is finite.

2. Prove that for any collection $\mathbb{E}$ of subsets of X, we have

$$\bigcap_{E \in \mathbb{E}} \overline{E} \supset \overline{\bigcap_{E \in \mathbb{E}} E}$$

and equality holds if $\mathbb{E}$ is finite.

3. Give examples that demonstrate that equality might fail in part (1) is $\mathbb{E}$ is not finite, and equality might fail in part (2) even if $\mathbb{E}$ is finite.

1. Let $x \in \bigcup_{E \in \mathbb{E}} \overline{E}$. So, for some E, $x = E \cup \mathrm{Lim}_X(E)$. $\overline{\bigcup_{E \in \mathbb{E}}(E)} = [\bigcup_{E \in \mathbb{E}} E] \cup [\mathrm{Lim}_X(\bigcup_{E \in \mathbb{E}})]$. So, $E \subset \bigcup_{E \in \mathbb{E}} E$ and $\mathrm{Lim}_X(E) \subset \mathrm{Lim}_X(\bigcup_{E \in \mathbb{E}})$ by Exercise 1.9. So, $x \in \overline{\bigcup_{E \in \mathbb{E}} E}$. For $\supset$, let $K = \bigcup_{E \in \mathbb{E}} \overline{E}$. Since K is union of finite number of closed sets, K is a closed set. All $x \in \bigcap_{E \in \mathbb{E}}(E) \to x \in \mathrm{Cl}_X(E)$ and so $x \subset K$. Thus, $\bigcap_{E \in \mathbb{E}}(E) \subset K$.
2.
3.

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Exercise 1.28 Let (X, d) be a metric space.

1. Prove that $x \in X$ and $r > 0$, we have $\overline{B_X(x,r)} \subset \{y \in X : d(x,y) \leq r\}$. (Hint: Take complements and draw a picture.) Note the inclusion $\overline{B_X(x,r)} \subset B_X(x, r+\epsilon)$ follows for any $\epsilon > 0$.
2. Give an example using the discrete metric that demonstrates that equality need not hold in the inclusion $\overline{B_X(x,r)} \subset \{y \in X : d(x,y) \leq r\}$ that you proved in (1).
3. Prove that in $\mathbb{R}^n$ under the Euclidean metric $d(x,y) = \|x - y\|$, we have $\overline{B_{\mathbb{R}^n}(x,r)} = \{y \in \mathbb{R}^n : \|x-y\| \leq r\}$. (Again a picture might be useful)
4. Using part (1), prove if A is bounded in (X, d) then $\overline{A}$ is also bounded in (X, d).

1. $\overline{B_X(x,r)} = B_X(x,r) \cup \mathrm{Lim}_X(B_X(x,r))$. Let's analyze some $z \in B_X(x,r)$. For any such z, $d(x,z) < r \to z \in \{y \in X : d(x,y) \leq r\}$. Now assume that $y \in \mathrm{Lim}_X(B_X(x,r))$. $\mathrm{Lim}_X(B_X(x,r)) = \{x : U \cap \{B_X(x,r) \setminus \{x\} \neq \emptyset\}\}$ where U is any neighborhood of x. The set of limit points will contain those points with $d(x,y) \leq r$. Any y that has $d(x,y) \geq r + \epsilon$ will violate definition of limit point. So, $\mathrm{Lim}_X(B_X(x,r)) \subset \{y \in X : d(x,y) \leq r\}$. This completes the inclusion, $\subset$.
2. Let $r = 1$ and $y \in \{y \in X : d(x,y) \leq r\}. y \notin B_X(x,1) = \{x\}$ and we need to show $y \notin \mathrm{Lim}_X(B_X(x,1)) = U \cap (\{x\} \setminus y)$. Choose U to be $B_X(y,1)$ so $y \notin \mathrm{Lim}_X(B_X(x,1))$.
3. Let $D = \{y \in \mathbb{R}^n : \|x-y\| \leq r\}$. For $\subset$, $\overline{B_{\mathbb{R}^n}(x,r)} = B_{\mathbb{R}^n}(x,r) \cup \mathrm{Lim}_X(B_{\mathbb{R}^n}(x,r))$. If $y \in B_{\mathbb{R}^n}(x,r)$, $d(x,y) < r$ so $y \in D$. If $y \in \mathrm{Lim}_X(B_{\mathbb{R}^n}(x,r)) = U \cap (B_{\mathbb{R}^n}(x,r) \setminus \{y\})$ for any neighborhood around y. This includes y when $d(x,y) = r$ so $y \in D$. For $\supset$, choose $y \in D$ s.t. $d(x,y) < r$ so $y \in B_{\mathbb{R}^n}(x,r)$. Let $z \in D$ be s.t. $d(x,z) = r$. So, for any $\epsilon > 0$, $\nexists B_X(z,\epsilon)$ s.t. $B_X(z,\epsilon) \cap (B_{\mathbb{R}^n}(x,r) \setminus \{z\})$. Thus, $z \in \mathrm{Lim}_X(B_{\mathbb{R}^n}(x,r))$.
4. Let $a \in \mathrm{Lim}_X(A) = B_X(x,R) \cup A \subset \{a\} = \{\dots, b, \dots\}$. $d(a,x) \leq d(a,b) + d(b,x) \leq R$. So, $\mathrm{Lim}_X(A) \in B_X(x,R)$ so $\overline{A}$ is bounded.

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