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MA 521
Homework #10
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Chapter 7: 4.3, 4.6, 4.7
Chapter 8: 1.5, 1.11, 1.12, 1.17

Exercise 4.3 Let $B = \{0\} \cup \{\frac{-1}{n^2}\}_{n\in\mathbb{N}}$ and $E = \mathbb{R} \setminus B$. Consider the series

$$\sum_{n=1}^{\infty} \frac{1}{1+n^2x}$$

on the set E.

1. Prove that the series converges absolutely for all $x \in E$; therefore it converges pointwise to a function $f : E \to \mathbb{R}$.
2. Prove that the series converges uniformly to f on $(-\infty, -\delta) \cup (\delta, \infty) \setminus B$ for any $\delta > 0$, but that it does not converge uniformly to f on E.
3. Prove that f is continous.
4. Prove that $f(0+) = +\infty$, that therefore f is not a bounded function.

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Exercise 4.6 Find the radius of convergence for each of the following power series:

$$\sum_{n=0}^{\infty} n^n z^n \quad \sum_{n=0}^{\infty} \frac{z^n}{n!} \quad \sum_{n=0}^{\infty} z^n \quad \sum_{n=0}^{\infty} \frac{z^n}{n} \quad \sum_{n=0}^{\infty} \frac{z^n}{n^2}.$$

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Exercise 4.7 Consider the power series $\sum_{n=0}^{\infty} c_n z^n$. Let R be the radius of convergence of the power series, and assume $R > 0$. Let $f : (-R, R) \to \mathbb{R}$ be the function defined by $f(z) = \sum_{n=0}^{\infty} c_n z^n$. Prove the following statements, which refine Thm 4.5.

1. For any $r \in (0, R)$, the series $\sum_{n=0}^{\infty}$ converges uniformly on $(-r, r)$ to f.
2. f is continuous on all of $(-R, R)$.

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Exercise 1.5 Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that $|f(x) - f(y)| \leq (x-y)^2$ for all $x, y \in \mathbb{R}$. Prove that f is constant.

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Exercise 1.11 Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable, and assume $\lim_{x \to +\infty} x|f'(x)| = 0$. Define a sequence (a_n) in $\mathbb{R}$ by $a_n = f(2n) - f(n)$ for each $n \in \mathbb{N}$. Prove that $a_n \to$ as $n \to \infty$.

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Exercise 1.12 Let $f : (a,b) \to \mathbb{R}$ be differentiable with $f'(x) > 0$ for all $x \in (a,b)$.

1. Prove that f is injectibe.
2. By part (1), there exists a function $g : f((a,b)) \to (a,b)$ such that $g(f(x)) = x$ for all $x \in (a,b)$. Prove that g is continuous.
3. Prove that g is differentiable, and that $g'(f(x)) = \frac{1}{f'(x)}$, for all $x \in (a,b)$.

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Exercise 1.17 Use Taylor's Theorem with remainder to estimate $e^{\frac{1}{2}}$ to an accuracy of within 10^{-3}. Prove your answer has the desired accuracy.

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