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MA 521
Homework #3
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Chapter 2: 5.4, 5.5, 6.4
Chapter 3: 1.7, 2.3, 2.4, 2.6, 2.8, 2.11, 2.12, 2.16, 2.20

Exercise 5.4. Let a and b be real numbers. Show that the following three equalities hold:

$$\bigcap_{x>b}(a,x) = (a,b], \qquad \bigcup_{n=1}^{\infty}[a+\frac{1}{n}, b-\frac{1}{n}) = (a,b), \qquad \bigcap_{n=1}^{\infty}(a+n,+\infty) = \emptyset.$$

1. $\bigcap_{x>b}(a,x) = \{y \in \overline{\mathbb{R}} : y \in (a,x) \forall x > b\}$. $\overline{\mathbb{R}}$ has a least upper bound so call this bound, d so $y \leq d \forall y \in \bigcap_{x>b}(a,x)$. Assume $d < b$, then $a < y \leq d < b < x \exists p \in \bigcap_{x>b}(a,x)$ s.t. $a < b < p < x$. So, d is not the least upper bound. So, $b \leq d$ and $a < y \leq b \leq d \forall \bigcap_{x>b}(a,x)$. $\bigcap_{x>b}(a,x) = \{y \in \overline{\mathbb{R}} : a < y \leq b\} = (a,b]$.
2. $\bigcup_{n=1}^{\infty} = [a+\frac{1}{n-1}, b-\frac{1}{n-1}) \cup [a+\frac{1}{n}, b-\frac{1}{n}) = \{y \in \overline{\mathbb{R}} : a+\frac{1}{n} \leq y \leq b-\frac{1}{n}\} = B$.
   - $a - \frac{1}{2}$ is a lower bound since $\nexists y \in B$ s.t. $y < a + \frac{1}{n}$. Assume $\exists$ a lower bound called β s.t. $\beta > a + \frac{1}{n}$. If so then, $B = \{a + \frac{1}{n} < y < b - \frac{1}{n}\}$. But, $\exists y \in B$ s.t. $y = a + \frac{1}{n}$. So, $\nexists$ any β so $\inf B = a + \frac{1}{n}$ and $a < a + \frac{1}{n} = \inf B$.
   - $b - \frac{1}{n}$ is a upper bound since $\nexists y \in B$ s.t. $y > b - \frac{1}{n}$. Assume $\exists$ some upper bound α s.t. $\alpha < b - \frac{1}{n}$. So, $b - \frac{1}{n} - \alpha > 0$. Choose $\gamma \in \mathbb{N}$ so $\frac{1}{\gamma} < b - \frac{1}{n} - \alpha.(b - \frac{1}{n} - \frac{1}{\gamma}) > b - \frac{1}{n} - (b - \frac{1}{n} - \alpha)$. So, α is not an upper bound. So, $\sup B = b - \frac{1}{n} < b$.

   Thus, $B = \{a < \inf B \leq y < \sup B < b \forall y \in B\} = (a,b)$.
3. Let's do proof by contradiction. Assume $\bigcap_{n=1}^{\infty}(a+n,+\infty) = X$ s.t. $X = \{\beta\}, \beta \in \mathbb{R}, \beta < a+n$. Enumerate $\bigcap_{n=1}^{\infty}(a+n,+\infty) = (a+1,\infty) \cap (a+2,\infty) \cap \cdots \cap (a+n,\infty) \cap \ldots$. Take a look at set, $(a+n,\infty) = \{x \in \overline{\mathbb{R}} : a+n < x < \infty\} = B$. Clearly $a+n$ is a lower bound for B since $a+n < x \, \forall x \in B$. Assume that k is a lower bound s.t. $k > a+n$. So, $B = \{x \in \overline{\mathbb{R}} : a+n < k < x < \infty\}$. So, $k - a - n > 0$. Choose ϕ so $\phi > k - a - n . a + n + \phi > a + n + k - a - n$. So, k is not a lower bound. So, $\inf B = a + n \to \beta < a + n = \inf B . \beta \nexists B$ so $\beta \nexists \bigcap_{n=1}^{\infty}(a+n,+\infty)$. This establishes a contradiction so $\bigcap_{n=1}^{\infty}(a+n,+\infty) = \emptyset$.

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Exercise 5.5. Let $a_1, a_2, \ldots$ be any enumeration of the negative rational numbers; let $b_1, b_2, \ldots$ be any enumeration of the positive rational numbers. Show that the following two equalities hold:

$$\bigcap_{j=1}(a_j, b_j) = \{0\}, \qquad \bigcup_{j=1}^{\infty}(a_j, b_j) = \mathbb{R}.$$

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Exercise 6.4. Prove there exists no order $\leq$ that makes $(\mathbb{C}, +, \cdot, \leq)$ into an ordered field.

Let us argue by contradiction. Assume that $\mathbb{C}$ is a ordered field. Let us analyze the case $(0,1) = i$.

- If $i > 0$, $x, y \in \mathbb{C}$ and $x = i, y = i, i \cdot i > 0 \implies -1 > 0$, which is a contradiction.
- If $i < 0$, $x, y \in \mathbb{C}$ and $x = -i, y = -i$ (since $x, y > 0$ for the second condition to hold), $-i \cdot -i < 0 \to i^2 > 0 \implies -1 > 0$, which is a contradiction.

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Exercise 1.7. Let $\|\cdot\|$ be a norm on a real vector space V. Prove the *reverse triangle inequality*:

$$|\|x\| - \|y\|| \leq \|x - y\|$$

$\|x\| = \|(x-y) + y\| \leq |\|x-y\| + \|y\||$. $|\|x\| - \|y\|| \leq |\|(x-y) + \|y\| - \|y\|\|| = |\|(x-y)\|| = \|x-y\|$.
$\|y\| = \|(y-x) + x\| = \|y-x\| + \|x\|.|\|x\| - \|y\|| \leq |\|x\| - (\|y-x\| + \|x\|)| \leq \|x-y\|$.
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Exercise 2.3. Let X be any set. Prove that the discrete metric $d : X \times X \to \mathbb{R}$(defined by $d(x,y) = 1$ if $x \neq y$ and $d(x,x) = 0$ for $x \in X$) satisfies the triangle inequality and is therefore a metric on X.

Let's argue via proof by contrapositive. Assume $d(x,y) > d(x,z) + d(z,y) \, \forall x, y, z \in X$ so d is not a metric on X. Let $x \neq y$ and enumerate what values z can take on.

- If $z \neq y \neq$ z, then $1 > 2$.
- If $z = x$ and $z \neq y$, vice versa, then $1 > 1$.
- If $z = y = x$, then $1 > 0$. However, this contradicts our initial assumption that $x \neq y$.

Clearly, all these pose contradictions, so it must be that d satisfies the triangle inequality and is a metric on X.

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Exercise 2.4. Determine which of the following functions are metrics on $\mathbb{R}$. Prove your answer in each case.

- $d_1(x,y) = \sqrt{|x-y|}$.
- $d_2(x,y) = |x - 2y|$.
- $d_3(x,y) = \frac{|x-y|}{1+|x-y|}$.

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Exercise 2.6. Let $\|\cdot\|$ denote the Euclidean norm $\mathbb{R}^2$, i.e. $\|(x_1, x_2)\| = \sqrt{x_1^2 + x_2^2}$. Consider the function from $\mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$, defined by

$$d(x, y) = \|x_1 - y_1\| + \|x_2 - y_2\|, \qquad (x = (x_1, x_2), y = (y_1, y_2)).$$

1. Prove that d is a metric on $\mathbb{R}^2$.
2. On a sheet of graph paper, draw the set $B_d((5, 1), 3)$. Use dotted lines to indicate the boundary, which is not included in the set you are drawing. (Hint: it may be easier to figure out what the set looks like if you first consider $B_d((0, 0), 3)$.
3. On the same graph as in the previous part, draw $B_{d_u}((-3, 2), 1)$, where d_u denotes the square metric.

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Exercise 2.8. Let (X, d) be a metric space, and let E be a subset of X. The *diameter* of E in (X, d) is defined by the formula

$$\mathrm{diam}_d(E) = \sup\{d(x, y) : x, y \in E\}.$$

1. Prove that for any $r > 0$ and $x \in X$, we have $\mathrm{diam}(B(x, r)) \leq 2r$.
2. If X is any set and d is the discrete metric, show $\mathrm{diam}(B(x, r)) = 0$.
3. If $X = \mathbb{R}^n$ for some $n \in \mathbb{N}$ and d is the Euclidean metric, prove that $\mathrm{diam}(B(x, r)) = 2r$.

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Exercise 2.11. As in Example 2.7, let $X = \mathbb{R}^2, Y = [-1, 3] \times [2, 4]$, and let d denote the Euclidean metric on $X = \mathbb{R}^2$. Let $p = (3, 4)$ and let $q = (2, 4)$. Arguing *directly from the definition of an interior point* (i.e., without using Exercise 2.12), show that q is an interior point $B_Y(p, 2)$ with respect to Y, but q is not an interior point $B_Y(p, 2)$ with respect to X. In addition, draw a picture on a piece of graph paper that illustrates the idea of your proof.

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Exercise 2.12. Let (X, d) be a metric space, and let Y be a subset of X. Prove that

$$(*)\ \mathrm{Int}_X(U) = \mathrm{Int}_Y(U) \cap \mathrm{Int}_X(Y).$$

In the notation of Exercise 2.11, the equality $(*)$ gives an alternate explanation of why q is not an interior point of $B_Y(p, 2)$ with respect to X: It is because $q \notin \mathrm{Int}_X(Y)$, as can be seen from the picture you drew in that Exercise.

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Exercise 2.16. Let (X, d) be a metric space, and let U be a subset of X. Use Proposition 2.15 to prove that $\mathrm{Int}_X(U)$ is open in X.

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Exercise 2.20. Let (X, d) be a metric space. Assume that $U \subset Y \subset X$, and additionally that Y is open X. Prove that U is open in Y if and only if U is open in X. (Note: There at least two possible solutions; one uses Theorem 2.19, the other uses Exercise 2.12.)

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