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MA 521
Homework #3
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Chapter 2: 5.4, 5.5, 6.4
Chapter 3: 1.7, 2.3, 2.4, 2.6, 2.8, 2.11, 2.12, 2.16, 2.20

**Exercise 5.4.** Let $a$ and $b$ be real numbers. Show that the following three equalities hold:

$$\bigcap_{x>b}(a,x) = (a,b], \qquad \bigcup_{n=1}^{\infty}[a+\frac{1}{n}, b-\frac{1}{n}) = (a,b), \qquad \bigcap_{n=1}^{\infty}(a+n,+\infty) = \emptyset.$$

1. $\bigcap_{x>b}(a,x) = \{y \in \overline{\mathbb{R}} : y \in (a,x) \forall x > b\}$. Clearly, $(a,b] \subset \bigcap_{x>b}(a,x)$ since it is contained in every set of the general intersection. For the other inclusion, let $z$ be an upper bound of $\bigcap_{x>b}(a,x)$ where $z < x \, \forall x > b$. For any $\epsilon > 0$, $z = b + \epsilon$. By Exer. 4.8, $b \leq z \to b \leq b$. So, $b$ is upper bound on $\bigcap_{x>b}(a,x)$ so $\bigcap_{x>b}(a,x) \subset (a,b]$.
2. $\bigcup_{n=1}^{\infty} = [a+\frac{1}{n-1}, b-\frac{1}{n-1}) \cup [a+\frac{1}{n}, b-\frac{1}{n}) = \{y \in \overline{\mathbb{R}} : a+\frac{1}{n} \leq y \leq b-\frac{1}{n}\} = B$.
   - $a - \frac{1}{2}$ is a lower bound since $\nexists y \in B$ s.t. $y < a + \frac{1}{n}$. Assume $\exists$ a lower bound called $\beta$ s.t. $\beta > a + \frac{1}{n}$. If so then, $B = \{a + \frac{1}{n} < y < b - \frac{1}{n}\}$. But, $\exists y \in B$ s.t. $y = a + \frac{1}{n}$. So, $\nexists$ any $\beta$ so $\inf B = a + \frac{1}{n}$ and $a < a + \frac{1}{n} = \inf B$.
   - $b - \frac{1}{n}$ is a upper bound since $\nexists y \in B$ s.t. $y > b - \frac{1}{n}$. Assume $\exists$ some upper bound $\alpha$ s.t. $\alpha < b - \frac{1}{n}$. So, $b - \frac{1}{n} - \alpha > 0$. Choose $\gamma \in \mathbb{N}$ so $\frac{1}{\gamma} < b - \frac{1}{n} - \alpha.(b - \frac{1}{n} - \frac{1}{\gamma}) > b - \frac{1}{n} - (b - \frac{1}{n} - \alpha)$. So, $\alpha$ is not an upper bound. So, $\sup B = b - \frac{1}{n} < b$.

   Thus, $B = \{a < \inf B \leq y < \sup B < b \forall y \in B\} = (a,b)$.
3. Let's do proof by contradiction. Assume $\bigcap_{n=1}^{\infty}(a+n,+\infty) = X$ s.t. $X = \{\beta\}, \beta \in \mathbb{R}, \beta < a+n$. Enumerate $\bigcap_{n=1}^{\infty}(a+n,+\infty) = (a+1,\infty) \cap (a+2,\infty) \cap \cdots \cap (a+n,\infty) \cap \ldots$. Take a look at set, $(a+n,\infty) = \{x \in \overline{\mathbb{R}} : a+n < x < \infty\} = B$. Clearly $a+n$ is a lower bound for $B$ since $a+n < x \, \forall x \in B$. Assume that $k$ is a lower bound s.t. $k > a+n$. So, $B = \{x \in \overline{\mathbb{R}} : a+n < k < x < \infty\}$. So, $k - a - n > 0$. Choose $\phi$ so $\phi > k - a - n.a + n + \phi > a + n + k - a - n$. So, $k$ is not a lower bound. So, $\inf B = a + n \to \beta < a + n = \inf B.\beta \notin B$ so $\beta \notin \bigcap_{n=1}^{\infty}(a+n,+\infty)$. This establishes a contradiction so $\bigcap_{n=1}^{\infty}(a+n,+\infty) = \emptyset$.

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**Exercise 5.5.** Let $a_1, a_2, \ldots$ be any enumeration of the negative rational numbers; let $b_1, b_2, \ldots$ be any enumeration of the positive rational numbers. Show that the following two equalities hold:

$$\bigcap_{j=1}^{\infty}(a_j, b_j) = \{0\}, \qquad \bigcup_{j=1}^{\infty}(a_j, b_j) = \mathbb{R}.$$

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1. Let's analyze one set of this general intersection, $(a_n, b_n)$ and call it $A$. $\sup A$ exists by the least upper bound property since $A \subset \mathbb{R}$ is bounded above and nonempty by least upper bound property. Similar argument for existence of $\inf A$ but it is bounded below. Let us prove what $\sup A, \inf A$ are supposed to be.
   - $a_n$ is a lower bound $\rightarrow x \notin A$ s.t. $x < a_n \, \forall x \in A$. Assume that $l$ is a lower bound s.t. $l > a_n$. So, $A = \{x \in \overline{\mathbb{R}} : a_n < k < x < b_n\}$. So, $l - a_n > 0$. Choose $\alpha \in \mathbb{N}$ so $\alpha > l - a_n . \alpha + a_n > a_n + l - a_n$. Thus, $l$ is not a lower bound. So, $\inf A = a_n$.
   - $b_n$ is a upper bound $\rightarrow x \notin A$ s.t. $x > b_n \, \forall x \in A$. Assume that $u$ is an upper bound s.t. $u < b_n$. So, $A = \{x \in \overline{\mathbb{R}} : a_n < x < u < b_n\}$. So, $u - b_n > 0$. Choose $n \in \mathbb{N}$ so $u - b_n > \frac{1}{n}$. So, $b_n - \frac{1}{n} > u$. Thus, $u$ is not an upper bound. So, $\sup A = b_n$.

   Since $a_n \in -\mathbb{Q}$ and $b_n \in +\mathbb{Q} \; \forall n \in \mathbb{N}, 0 \in (a_n, b_n) \, \forall n$ so $\bigcap_{j=1}^{\infty} (a_n, b_n) = \{0\}$.

2. Let's analyze one set of this general union, $(a_n, b_n)$ and call it $A$. $\sup A$ exists by the least upper bound property since $A \subset \mathbb{R}$ is bounded above and nonempty by least upper bound property. Similar argument for existence of $\inf A$ but it is bounded below. Let us prove what $\sup A, \inf A$ are supposed to be.
   - $a_n$ is a lower bound $\rightarrow x \notin A$ s.t. $x < a_n \, \forall x \in A$. Assume that $l$ is a lower bound s.t. $l > a_n$. So, $A = \{x \in \overline{\mathbb{R}} : a_n < k < x < b_n\}$. So, $l - a_n > 0$. Choose $\alpha \in \mathbb{N}$ so $\alpha > l - a_n . \alpha + a_n > a_n + l - a_n$. Thus, $l$ is not a lower bound. So, $\inf A = a_n$.
   - $b_n$ is a upper bound $\rightarrow x \notin A$ s.t. $x > b_n \, \forall x \in A$. Assume that $u$ is an upper bound s.t. $u < b_n$. So, $A = \{x \in \overline{\mathbb{R}} : a_n < x < u < b_n\}$. So, $u - b_n > 0$. Choose $n \in \mathbb{N}$ so $u - b_n > \frac{1}{n}$. So, $b_n - \frac{1}{n} > u$. Thus, $u$ is not an upper bound. So, $\sup A = b_n$.

   Assume that there is some $\beta \in \mathbb{R}$. If $\beta \notin A$, $\exists$ some set in $\bigcup_{j=1}^{\infty} (a_j, b_j)$ s.t. $\beta \in \{a_n - \epsilon < x < b_n + \epsilon\}$ for any $\epsilon > 0$. So, if true for any $\beta$, $\bigcup_{j=1}^{\infty} (a_j, b_j) = \mathbb{R}$.

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**Exercise 6.4.** Prove there exists no order $\leq$ that makes $(\mathbb{C}, +, \cdot, \leq)$ into an ordered field.

Let us argue by contradiction. Assume that $\mathbb{C}$ is a ordered field. Let us analyze the case $(0, 1) = i$.

- If $i > 0$, $x, y \in \mathbb{C}$ and $x = i, y = i, i \cdot i > 0 \implies -1 > 0$, which is a contradiction.
- If $i < 0$, $x, y \in \mathbb{C}$ and $x = -i, y = -i$ (since $x, y > 0$ for the second condition to hold), $-i \cdot -i < 0 \rightarrow i^2 > 0 \implies -1 > 0$, which is a contradiction.

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**Exercise 1.7.** Let $\|\cdot\|$ be a norm on a real vector space $V$. Prove the *reverse triangle inequality*:

$$|\|x\| - \|y\|| \leq \|x - y\|$$

$\|x\| = \|(x-y)+y\| \leq |\|x-y\| + \|y\||$. $|\|x\| - \|y\|| \leq ||\|(x-y) + \|y\| - \|y\|\|| = ||\|(x-y)\|| = \|x-y\|$.
$\|y\| = \|(y-x)+x\| = \|y-x\| + \|x\|.|\|x\| - \|y\|| \leq |\|x\| - (\|y-x\| + \|x\|)| \leq \|x-y\|$.

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**Exercise 2.3.** Let $X$ be any set. Prove that the discrete metric $d : X \times X \to \mathbb{R}$(defined by $d(x,y) = 1$ if $x \neq y$ and $d(x,x) = 0$ for $x \in X$) satisfies the triangle inequality and is therefore a metric on $X$.

Let's argue via proof by contrapositive. Assume $d(x,y) > d(x,z) + d(z,y)\,\forall x,y,z \in X$ so $d$ is not a metric on $X$. Let $x \neq y$ and enumerate what values $z$ can take on.

- If $z \neq y \neq$ z, then $1 > 2$.
- If $z = x$ and $z \neq y$, vice versa, then $1 > 1$.
- If $z = y = x$, then $1 > 0$. However, this contradicts our initial assumption that $x \neq y$.

Clearly, all these pose contradictions, so it must be that $d$ satisfies the triangle inequality and is a metric on $X$.

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**Exercise 2.4.** Determine which of the following functions are metrics on $\mathbb{R}$. Prove your answer in each case.

- $d_1(x,y) = \sqrt{|x-y|}$.
- $d_2(x,y) = |x - 2y|$.
- $d_3(x,y) = \frac{|x-y|}{1+|x-y|}$.

1. For any $x, y \in \mathbb{R}$, $|x-y| \geq 0$ so $\sqrt{|x-y|} \geq 0$. $d_1(x,y) = \sqrt{|x-y|} = \sqrt{|(-1)y-x|} = \sqrt{|y-x|} = d_1(y,x)$. $\sqrt{|x-y|} \leq \sqrt{|x-z|} + \sqrt{|z-y|} \to |x-y| \leq |x-z| + c + |z-y|$ where $c \leq 0$. So, $-(|x-z| + c + |z-y|) \leq x - y \leq |x-z| + c + |z-y| \to -c \leq 0 \leq c$. So $d_1$ is a metric on $\mathbb{R}$ .
2. Assume $d_2(x,y) = d_2(y,x) \to |x-2y| = |y-2x| \forall\, x, y \in \mathbb{R}$ . Choose $x = 0$ and $y = 1$ so $d_2(x,y) = 1 = d_2(y,x) = 2$. So, $d_2$ is clearly not a metric on $\mathbb{R}$ .
3. For any $x, y \in \mathbb{R}$, $|x-y| \geq 0$ so $\frac{|x-y|}{1+|x-y|} \geq 0$. $d_3(x,y) = \frac{|x-y|}{1+|x-y|} = \frac{|(-1)y-x|}{1+|(-1)y-x|} = \frac{|y-x|}{1+|y-x|} = d_3(y,x)$. Assume $\frac{|x-y|}{1+|x-y|} > \frac{|x-y|}{1+|x-y|} + \frac{|x-y|}{1+|x-y|}$. Choose $x, y = 5$ and $z = 0$. So, $0 > \frac{|5|}{6} + \frac{|5|}{6}$ and clearly this is a contradiction so $d_3$ follows triangle inequality. So, $d_3$ is a metric on $\mathbb{R}$.

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**Exercise 2.6.** Consider the function $d : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$, defined by

$$d(x,y) = |x_1 - y_1| + |x_2 - y_2|, \qquad (x = (x_1, x_2), y = (y_1, y_2)).$$

1. Prove that $d$ is a metric on $\mathbb{R}^2$.
2. On a sheet of graph paper, draw the set $B_d((5,1),3)$. Use dotted lines to indicate the boundary, which is not included in the set you are drawing. (Hint: it may be easier to figure out what the set looks like if you first consider $B_d((0,0),3)$.)
3. On the same graph as in the previous part, draw $B_{d_u}((-3,2),1)$, where $d_u$ denotes the square metric.

For nonnegativity, analyze two cases where $x, y \in \mathbb{R}^2$.

1. Case 1: Let $(x_1, x_2) = (y_1, y_2)$. So, $d(x,y) = |x_1 - y_1| + |x_2 - y_2| = |x_1 - x_1| + |x_2 - x_2| = 0$.
2. Case 2: Let $x \neq y \, \forall x, y \in \mathbb{R}^2$. Claim that $d(x,y) < 0$ so $d$ is not a metric. $d(x,y) = |x_1 - y_1| + |x_2 - y_2| < 0$. $|x_1 - y_1| < -|x_2 - y_2|$. Choose $x = (0,0)$ and $y = (1,2)$. So, $|0-1| < -|0-2| \to 1 < -2$. Clearly, this is false so $d(x,y) \geq 0$.

For symmetry, $d(x,y) = |x_1 - y_1| + |x_2 - y_2| = |(-1)y_1 - x_1| + |(-1)y_2 - x_2| = |y_1 - x_1| + |y_2 - x_2| = d(y,x)$.
For triangle inequality, $d(x,y) = |x_1 - y_1| + |x_2 - y_2| \leq |x_1 - z_1| + |z_1 - y_1| + |x_2 - z_2| + |z_2 - y_2| = d(x,z) + d(z,y)$. Drawings attached on back ■

**Exercise 2.8.** Let $(X, d)$ be a metric space, and let $E$ be a subset of $X$. The *diameter* of $E$ in $(X, d)$ is defined by the formula

$$\text{diam}_d(E) = \sup\{d(x,y) : x, y \in E\}.$$

1. Prove that for any $r > 0$ and $x \in X$, we have $\text{diam}(B(x,r)) \leq 2r$.
2. If $X$ is any set and $d$ is the discrete metric, show $\text{diam}(B(x,r)) = 0$ for any $r \leq 1$, while $\text{diam}(B(x,r)) = 1$ for any $r > 1$.
3. If $X = \mathbb{R}^n$ for some $n \in \mathbb{N}$ and $d$ is the Euclidean metric, prove that $\text{diam}(B(x,r)) = 2r$.

1. $B(x,r) = \{y \in X : d(x,y) < r\}$. $\text{diam}_d(\{y \in X : d(x,y) : x, y \in B(x,r)\}) = \sup\{d(x,y) : x, y \in B(x,r)\} \leq \sup\{d(x,z) : z, y \in B(x,r)\} + \sup\{d(z,y) : y, z \in B(x,r)\}$. So, $\text{diam}_d(\{y \in X : d(x,y) : x, y \in B(x,r)\}) \leq r + r = 2r$.
2. Recall discrete metric is defined as $d(x,y) = 1$ if $x \neq y$ and $d(x,y) = 0$ if $x = y$.
   (a) For $r \leq 1$, $B_{(X,d)}(x,r) = \{y \in X : d(x,y) < r\}$. The cases where $x \neq y$ will yield an empty set since 1 is not greater 1. So, we will only have the set,$D$, where $x = y$. So, $\sup\{d(x,y) : x, y \in D\} = 0$.

(b) For $r > 1$, $B_{(X,d)}(x,r) = \{y \in X : d(x,y) < r\} = \{y \in X : x \neq y\} \cap \{y \in X : x \in y\} = B$. $\sup\{d(x,y) : x,y \in B\} = 1$. Any point, $y$ in the metric space will fulfill the condition from the ball since $d$ is the discrete metric. The maximum can only be 1 so it must be the sup of the set.

3. Choose two points $a_1, a_2 \in B(x,r)$ s.t. $a_1 = [x_1 + r - \epsilon, x_2, x_3, \dots], a_2 = [x_1 - (r - \epsilon), x_2, x_3, \dots]$. $\operatorname{diam} B(x,r) = \sup\{d(a_1,a_2) : a_1, a_2 \in B(x,r)\}$. Apply triangle inequality to $d(a_1,a_2) \to d(a_1,a_2) \leq d(a_1,x) + d(x,a_2)$. So, $d(a_1,a_2) \leq \|r - \epsilon\| + \|(-1)(r-\epsilon)\|$. $d(a_1,a_2) \leq 2r - 2\epsilon \forall \epsilon > 0$. Use Exer. 4.8, $d(a_1,a_2) + 2\epsilon \leq 2r \to d(a_1,a_2) \leq 2r$. So, $\operatorname{diam} B(x,r) = \sup\{d(a_1,a_2) \leq 2r\} = 2r$.

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**Exercise 2.11.** As in Example 2.7, let $X = \mathbb{R}^2, Y = [-1,3] \times [2,4]$, and let $d$ denote the Euclidean metric on $X = \mathbb{R}^2$. Let $p = (3,4)$ and let $q = (2,4)$. Arguing *directly from the definition of an interior point* (i.e., without using Exercise 2.12), show that $q$ is an interior point of $B_Y(p,2)$ with respect to $Y$, but $q$ is not an interior point of $B_Y(p,2)$ with respect to $X$. In addition, draw a picture on a piece of graph paper that illustrates the idea of your proof.

Let $D = B_Y(p,2)$ with respect to $Y$. To show that $q$ is an interior point of $D$, $\exists r > 0$ s.t. $B_Y(q,r) \subset D$. Set $r = 1$ so for some $x \in B_Y(q,1)$ s.t. $d(x,q) < 1$ and $x \in Y$. By triangle inequality, $d(x,p) \leq d(x,q) + d(p,q)$. $d(x,q) < 1$ and $d(p,q) = 1$ so $d(x,q) + d(p,q) < 2 \to x \in D$. So, $B_Y(q,1) \subset D \to q \in \operatorname{Int}_Y(D)$.
Let $E = B_Y(p,2)$. To show that $q$ is not an interior point of $E$, $x \in B_X(q,r), x \notin E$, for some $r > 0$. Take $x = (2, 4 + \frac{r}{2})$. $d(x,q) = \frac{r}{2} \to x \in B_X(q,r)$. But, $x \notin Y$ since $(4 + \frac{r}{2} > 4)$. So, $q$ is not an interior point of $E$ with respect to $X$.
Drawings attached on back ■

**Exercise 2.12.** Let $(X,d)$ be a metric space, and let $Y$ be a subset of $X$. Prove that for any subset $U$ of $Y$, we have

$$(*)\ \operatorname{Int}_X(U) = \operatorname{Int}_Y(U) \cap \operatorname{Int}_X(Y).$$

In the notation of Exercise 2.11, the equality $(*)$ gives an alternate explanation of why q is not an interior point of $B_Y(p,2)$ with respect to $X$: It is because $q \notin \operatorname{Int}_X(Y)$, as can be seen from the picture you drew in that Exercise.

$\operatorname{Int}_X(U) = \{x \in U : x$ is an interior point of $U$ with respect to $X\}$. That means there is some $B_X(x,r) \subset U$, s.t. $r > 0$. Call this ball, $D$. Since $D \subset U \to D \subset \operatorname{Int}_Y(U)$. Also, since $U \subset Y$, $D \subset Y$. So, $D \subset \operatorname{Int}_X(Y)$. So, $\operatorname{Int}_X(U) \subset \operatorname{Int}_Y(U) \cap \operatorname{Int}_X(Y)$.
Now for the other direction, $\operatorname{Int}_Y(U) = \{x \in U : x$ is an interior point of $U$ with respect to $Y\}$. That means there is some $B_Y(x,r) \subset U$, s.t. $r > 0$. Call this ball, $E$. Since $E \subset U$, $E \subset Y \to E \subset \operatorname{Int}_X(Y)$. So, $E \subset \operatorname{Int}_Y(U) \cap \operatorname{Int}_X(Y) \subset U \subset \operatorname{Int}_X(U)$. So, $\operatorname{Int}_X(U) = \operatorname{Int}_Y(U) \cap \operatorname{Int}_X(Y)$. ■

**Exercise 2.16.** Let $(X, d)$ be a metric space, and let $U$ be a subset of $X$. Use Proposition 2.15 to prove that $\mathrm{Int}_X(U)$ is open in $X$.

For $x \in U$, since $U \subset X$ and $(X, d)$ is a metric space, $B_X(x, r)$ is open in $X$. Choose $y \in U$ so $y$ is the interior point of $B_X(y, r)$ using Proposition 2.15. $B_X(y, r) \cap U = B_U(y, r) \subset U$. So, $y$ is an interior point of $U$ wrt $X$ so $\forall y \in U$ are interior points so $\mathrm{Int}_X U = U$. ∎

**Exercise 2.20.** Let $(X, d)$ be a metric space. Assume that $U \subset Y \subset X$, and additionally that $Y$ is open $X$. Prove that $U$ is open in $Y$ if and only if $U$ is open in $X$. (Note: There at least two possible solutions; one uses Theorem 2.19, the other uses Exercise 2.12.)

($\rightarrow$)Given $U$ is open in $Y$, applying Exer 2.12, $\mathrm{Int}_X(U) = \mathrm{Int}_Y(U) \cap \mathrm{Int}_X(Y) = U \cap Y = U$.
($\leftarrow$)Given $U$ is open in $X$, applying Exer 2.12, $\mathrm{Int}_X(U) = U = \mathrm{Int}_Y(U) \cap \mathrm{Int}_X(Y) = \mathrm{Int}_Y(U) \cap Y = \mathrm{Int}_Y(U)$.

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