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MA 521
Homework #3
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Chapter 2: 5.4, 5.5, 6.4
Chapter 3: 1.7, 2.3, 2.4, 2.6, 2.8, 2.11, 2.12, 2.16, 2.20

Exercise 5.4. Let a and b be real numbers. Show that the following three equalities hold:

$$\bigcap_{x>b}(a,x) = (a,b], \qquad \bigcup_{n=1}^{\infty}[a+\frac{1}{n}, b-\frac{1}{n}] = (a,b), \qquad \bigcap_{n=1}^{\infty}(a+n,+\infty) = \emptyset.$$

1. Let A_1 be the set $(a, b+\epsilon_1)$ for some $\epsilon_1 > 0$. This translates to $\{x \in \overline{\mathbb{R}} : a < x < b+\epsilon_1\}$. Since this is a general intersection, another set in the collection will some subset, $A_2 \subset A_1, A_2 = \{x \in \overline{\mathbb{R}} : a < x < b+\epsilon_2\}$ where $\epsilon_1 > \epsilon_2$. There are many more subsets of A_2 and will continue for $\epsilon_{n+1} < \epsilon_n, n \in \mathbb{N}$. The key insight to note is that b is a common element between all subsets. Clearly, the upper bounds of these subsets are tending to b so that will be the upper bound of general intersection. So, $A_1 \cap A_2 \cap \cdots = \{x \in \overline{\mathbb{R}} : a < x \leq b\} = (a, b]$.
2.
3.

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Exercise 5.5. Let $a_1, a_2, \ldots$ be any enumeration of the negative rational numbers; let $b_1, b_2, \ldots$ be any enumeration of the positive rational numbers. Show that the following two equalities hold:

$$\bigcap_{j=1}(a_j, b_j) = \{0\}, \qquad \bigcup_{j=1}^{\infty}(a_j, b_j) = \mathbb{R}\,.$$

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Exercise 6.4. Prove there exists no order $\leq$ that makes $(\mathbb{C}, +, \cdot, \leq)$ into an ordered field.

Let us argue by contradiction. Assume that $\mathbb{C}$ is a ordered field. Let us analyze the case $(0,1) = i$.

- If $i > 0, x, y \in \mathbb{C}$ and $x = i, y = i, i \cdot i > 0 \implies -1 > 0$, which is a contradiction.
- If $i < 0$, $x, y \in \mathbb{C}$ and $x = -i, y = -i$ (since $x, y > 0$ for the second condition to hold), $-i \cdot -i < 0 \rightarrow i^2 > 0 \implies -1 > 0$, which is a contradiction.

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Exercise 1.7. Let $\|\cdot\|$ be a norm on a real vector space V. Prove the *reverse triangle inequality*:

$$|\|x\| - \|y\|| \leq \|x - y\|$$

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Exercise 2.3. Let X be any set. Prove that the discrete metric $d : X \times X \to \mathbb{R}$(defined by $d(x, y) = 1$ if $x \neq y$ and $d(x, x) = 0$ for $x \in X$) satisfies the triangle inequality and is therefore a metric on X.

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Exercise 2.4. Determine which of the following functions are metrics on $\mathbb{R}$. Prove your answer in each case.

- $d_1(x, y) = \sqrt{|x - y|}$.
- $d_2(x, y) = |x - 2y|$.
- $d_3(x, y) = \frac{|x-y|}{1+|x-y|}$.

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Exercise 2.6. Let $\|\cdot\|$ denote the Euclidean norm $\mathbb{R}^2$, i.e. $\|(x_1, x_2)\| = \sqrt{x_1^2 + x_2^2}$. Consider the function from $\mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$, defined by

$$d(x, y) = \|x_1 - y_1\| + \|x_2 - y_2\|, \qquad (x = (x_1, x_2), y = (y_1, y_2)).$$

1. Prove that d is a metric on $\mathbb{R}^2$.
2. On a sheet of graph paper, draw the set $B_d((5, 1), 3)$. Use dotted lines to indicate the boundary, which is not included in the set you are drawing. (Hint: it may be easier to figure out what the set looks like if you first consider $B_d((0, 0), 3)$.
3. On the same graph as in the previous part, draw $B_{d_u}((-3, 2), 1)$, where d_u denotes the square metric.

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Exercise 2.8. Let (X, d) be a metric space, and let E be a subset of X. The *diameter* of E in (X, d) is defined by the formula

$$\mathrm{diam}_d(E) = \sup\{d(x, y) : x, y \in E\}.$$

1. Prove that for any $r > 0$ and $x \in X$, we have $\mathrm{diam}(B(x, r)) \leq 2r$.
2. If X is any set and d is the discrete metric, show $\mathrm{diam}(B(x, r)) = 0$.
3. If $X = \mathbb{R}^n$ for some $n \in \mathbb{N}$ and d is the Euclidean metric, prove that $\mathrm{diam}(B(x, r)) = 2r$.

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Exercise 2.11. As in Example 2.7, let $X = \mathbb{R}^2, Y = [-1, 3] \times [2, 4]$, and let d denote the Euclidean metric on $X = \mathbb{R}^2$. Let $p = (3, 4)$ and let $q = (2, 4)$. Arguing *directly from the definition of an interior point* (i.e., without using Exercise 2.12), show that q is an interior point $B_Y(p, 2)$ with respect to Y, but q is not an interior point $B_Y(p, 2)$ with respect to X. In addition, draw a picture on a piece of graph paper that illustrates the idea of your proof.

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Exercise 2.12. Let (X, d) be a metric space, and let Y be a subset of X. Prove that

$$(*)\, \mathrm{Int}_X(U) = \mathrm{Int}_Y(U) \cap \mathrm{Int}_X(Y).$$

In the notation of Exercise 2.11, the equality $(*)$ gives an alternate explanation of why q is not an interior point of $B_Y(p, 2)$ with respect to X: It is because $q \notin \mathrm{Int}_X(Y)$, as can be seen from the picture you drew in that Exercise.

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Exercise 2.16. Let (X, d) be a metric space, and let U be a subset of X. Use Proposition 2.15 to prove that $\mathrm{Int}_X(U)$ is open in X.

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Exercise 2.20. Let (X, d) be a metric space. Assume that $U \subset Y \subset X$, and additionally that Y is open X. Prove that U is open in Y if and only if U is open in X. (Note: There at least two possible solutions; one uses Theorem 2.19, the other uses Exercise 2.12.)

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