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MA 521
Homework #11
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Chapter 8: 2.12, 2.15, 2.16, 2.17, 2.18, 2.19, 2.20, 3.3

Exercise 2.12 Which $n \in \mathbb{N}$ have the property that $f^n \in \mathcal{R}([a,b])$ implies $f \in \mathcal{R}([a,b])$? Give proofs(s) and counterexamples(s) to show your answer is correct and complete.

We knwo that if f is differentiable on $[a,b]$ that is must be continuous on $[a,b]$. By contradition, we will show that f is bounded. If f is unbounded at $[a,b]$ then $\exists x \in [a,b]$ s.t. $\nexists M$ s.t. $|x| < M$. Let's analyze the limit as $\lim_{h \to 0} \frac{f(x+h)-f(x)}{h} = f'(x)$. Without loss of generality, assume $f(x+h)$ is finite. By our previous statement, we claimed that $f(x)$ was unbounded which implies this limit can't exist. This means that $f^1 \notin \mathcal{R}([a,b])$ which is a contradition. Since f is continuous and bounded, we can say that by Theorem 2.8 $f \in \mathcal{R}([a,b])$. If this holds for f', then it will hold for all $n > 1$ as well. ∎

Exercise 2.15 Show that if $f : [a,b] \to \mathbb{R}$ is continuous and $F(x) := \int_a^x f(t)dt = 0$ for all $x \in [a,b]$, then $f(x) = 0$ for all $x \in [a,b]$. Provide an example to show that the statement is false if f is not continuous.

We want $\lim_{t \to x} \frac{F(t)-F(x)}{t-x} = F'(x) = f(x) = 0 \, \forall x \in [a,b]$. We know that $F(x) := \int_a^x f(t)dt = 0$. So this means that $\forall x \in [a,b], F(x) = F(a)$ by the Fundamental Theorem of Calculus. So then, substitue $F(a)$ for $F(x)$ and $\lim_{t \to x} \frac{F(t)-F(a)}{t-x} = F'(x) = f(x) = 0 \forall x \in [a,b]$. Let's pick the same function $f(x) = 0$ but at some point $x \in [a,b]$, there exists a removable discontinuity where $f(x) = 1$. If we choose the partitions of the function correctly, we can obtain the original expression with $f(x) \neq 0 \forall x \in [a,b]. F(x) := \int_a^x f(t)dt = 0$. ∎

Exercise 2.16 Assume f and g are differentiable functions on $[a,b]$ and assume $f', g' \in \mathcal{R}([a,b])$. Show that the integration by parts formula is valid:

$$\int_a^b fg'dx = f(b)g(b) - f(a)g(a) - \int_a^b f'gdx.$$

Make sure you show the relevant functions are Riemann integrable when you do the proof!

Let's analyze the product of the functions, fg. Apply the product rule so $\frac{d}{dx}fg = f'g + fg'$. Apply the Fundamental Theorem of Calculus, $\int_a^b \frac{d}{dx}(fg)dx = \int_a^b fg'dx + \int_a^b fg'dx \rightarrow \int_a^b fg'dx = \int_a^b \frac{d}{dx}(fg)dx - \int_a^b fg'dx$. Evaluate the expression so $\int_a^b fg'dx = f(b)g(b) - f(a)g(a) - \int_a^b f'gdx$. By Theorem 2.10(b), $\frac{d}{dx}(fg) \in \mathcal{R}$ so both functions in integrals are Reimann integrable. ∎

Exercise 2.17 Assume $g : [a,b] \to \mathbb{R}$ is differentiable, that g' is continuous, and M and m are upper and lower bounds, respectively, for the function g. Assume $f : [m,M] \to \mathbb{R}$ is continuous. SHow that the change of variables formula is valid:

$$\int_a^b f(g(x))g'(x)dx = \int_{g(a)}^{g(b)} f(t)dt.$$

Again, part of the exercise is to check that the relevant functions are Riemann integrable when you do the proof!

The chain rule is defined as $\frac{d}{dx}F(g(x)) = F'(g(x))g'(x)$, where $f = F'$. Apply the Fundamental Theorem of Calculus $\int_a^b \frac{d}{dx}F(g(x))dx = F(g(b)) - F(g(a)) = \int_{g(a)}^{g(b)} f(t)dt$. $g(x) \in \mathcal{R}$ since g is continuous and bounded. By Thm. 2.9, $f \in \mathcal{R}$ so $f(g) \in \mathcal{R}$. g' is continuous on a compact set so it is bounded so $g' \in \mathcal{R}$. For the other function, since f is continuous and $g(a), g(b)$ are bounded on a compact so $f \in \mathcal{R}$. ∎

Exercise 2.18 Assume $f \in \mathcal{R}([a,b])$, but that f has a jump discontinuity at $c \in (a,b)$, i.e. $f(c-) \neq f(c+)$. Show that $F(x) := \int_a^x f(t)dt$ is not differentiable at $x = c$.

$|\frac{F(x)-F(c)}{x-c}| < \epsilon \rightarrow |\frac{\int_a^x g(t)dt - \int_a^c f(t)dt}{x-c} - \frac{\int_c^x f(c)dx}{x-c}| = |\frac{\int_c^x f(t)dt}{x-c} - \frac{\int_c^x f(c)dt}{x-c}| \rightarrow \frac{1}{|x-c|}|\int_c^x f(t) - f(c)dt| \leq \frac{1}{|x-c|}\int_c^x f(t) - f(c)dt$. Since f is continuous at c, $\exists\delta > 0$ s.t. for all $x \in [a,b]$ if $|x-c| < \delta$ then $|g(x) - (c)| < \epsilon$. So take any $\epsilon > 0$, we then have that $\exists\delta > 0$ s.t. $\forall x \in [a,b]$ if $(c-\delta, c)$. $|\frac{F(x)-F(c)}{x-c} - f(c)| \leq \frac{1}{|x-c|}\int_c^x |f(t) - f(c)|dt < \frac{1}{|x-c|}\int_c^x \epsilon dt = \epsilon$. So, the left derivative of F at c is $f(c)$. Right side is similar derivation. But since we know that a jump discontinuity exists at c, the limit cannot exist so F is not differentiable at $x = c$. ∎

Exercise 2.19 Given a function $f : [a,b] \to \mathbb{R}$, define its total variation Tf by

$$Tf = \sup\{\sum_{k=1}^{n} |f(x_k) - f(x_{k-1})|\},$$

where the supremum is taken over all partitions P of $[a,b]$. Show that if f' is continuous, then

$$Tf = \int_b^a |f'(x)|dx.$$

(Hint: Use the FTC for one inequality, and use the MVT for the other direction.)

$Tf = \sup\{\sum_{k=1}^{n} |f(x_k) - f(x_{k-1})|\} \frac{x_k - x_{k-1}}{x_k - x_{k-1}} \to \sup\{\sum_{k=1}^{n} |f'(x)|(x_k - x_{k-1})\} \geq lim_{n\to\infty} \sup\{\sum_{k=1}^{n} |f'(x)|(x_k - x_{k-1})\}$ by MVT. So, $\int_a^b |f'(x)|dx \leq \sup\{\sum_{k=1}^{n} |f'(x)|(x_k - x_{k-1})\}$. ■

Exercise 2.20 Assume g is bounded, $g \in \mathcal{R}([0,1])$ and g is continuous at 0. Show that

$$\lim_{n\to\infty} \int_0^1 g(x^n)dx = g(0).$$

Hint: Consider the difference $\int_0^1 g(x^n)dx - g(0)$; add and subtract $\int_0^c g(x^n)dx$ for a carefully chosen c, and then that $\int_0^c dx$ is close to $cg(0)$ for large enough n.

Define $k = \max_{[0,1]} |g|$. So for $\epsilon > 0$, $c = 1 - \frac{\epsilon}{2k} \to |\int_0^1| < k(1-c) < \frac{\epsilon}{2}$. So $\exists \delta$ s.t. $0 < y < \delta \to |g(y) - g(0)| < \frac{\epsilon}{2}$. Take N, so $n > N$ $c^{\frac{1}{n}} < \delta$. So, $\int \max_{0<x<c} |g(x^n - g(0)|dx = \max_{0<y<c^{\frac{1}{n}}} |g(y) - g(0)| < \frac{\epsilon}{2}$. ■

Exercise 3.3 Let (f_n) be a sequence of real-valued, Riemann integrable functions on the interval $[a,b]$. Assume that $f_n(x) \to 0$ as $n \to \infty$ for each $x \in [a,b]$, and additionally,

$$\sum_{n=1}^{\infty} |f_{n+1}(x) - f_n(x)|$$

converges uniformly on $[a,b]$.

1. Show that $\lim_{n\to\infty} \int_a^b f_n dx \to 0$.
2. Is it necessarily the case that the series $\sum_{n=1}^{\infty} f_n$ converges uniformly? Give a proof or counterexample to support your answer.

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