

1.25, 1.26, 1.28, 1.30, 1.31, 2.7, 2.9, 2.16, 2.18.

Exercise 1.25 Let (X, d) be a metric space. Let E and Y be subsets of X such that $E \subset Y$. Prove that

$$\text{Cl}_Y(E) = \text{Cl}_X(E) \cap Y.$$

For \subset , let $x \in \text{Cl}_Y(E)$. $\text{Cl}_Y(E) = E \cup \text{Lim}_Y(E) = E \cup (\text{Lim}_X(E) \cap Y)$ (by Exercise 1.10) $= (E \cup \text{Lim}_X(E)) \cap Y = \text{Cl}_X(E) \cap Y$. So, $x \in \text{Cl}_X(E) \cap Y$. For \supset , let $x \in \text{Cl}_X(E) \cap Y$. So, $\text{Cl}_X(E) = (E \cup \text{Lim}_X(E)) \cap Y = E \cup (\text{Lim}_X(E) \cap Y) = E \cup \text{Lim}_Y(E) = \text{Cl}_Y(E)$. ■

Exercise 1.26 Let (X, d) be a metric space.

1. Prove that for any collection \mathbb{E} of subsets of X , we have

$$\bigcup_{E \in \mathbb{E}} \bar{E} \subset \overline{\bigcup_{E \in \mathbb{E}} E}$$

and equality holds if \mathbb{E} is finite.

2. Prove that for any collection \mathbb{E} of subsets of X , we have

$$\bigcap_{E \in \mathbb{E}} \bar{E} \supset \overline{\bigcap_{E \in \mathbb{E}} E}$$

and equality holds if \mathbb{E} is finite.

3. Give examples that demonstrate that equality might fail in part (1) if \mathbb{E} is not finite, and equality might fail in part (2) even if \mathbb{E} is finite.

1. Let $x \in \bigcup_{E \in \mathbb{E}} \bar{E}$. So, for some E , $x \in E \cup \text{Lim}_X(E)$. $\overline{\bigcup_{E \in \mathbb{E}} E} = [\bigcup_{E \in \mathbb{E}} E] \cup [\text{Lim}_X(\bigcup_{E \in \mathbb{E}} E)]$. So, $E \subset \bigcup_{E \in \mathbb{E}} E$ and $\text{Lim}_X(E) \subset \text{Lim}_X(\bigcup_{E \in \mathbb{E}} E)$ by Exercise 1.9. So, $x \in \overline{\bigcup_{E \in \mathbb{E}} E}$. For \supset , let $K = \bigcup_{E \in \mathbb{E}} \bar{E}$. Since K is union of finite number of closed sets, K is a closed set. All $x \in \bigcap_{E \in \mathbb{E}} \bar{E} \rightarrow x \in \text{Cl}_X(E)$ and so $x \in K$. Thus, $\bigcap_{E \in \mathbb{E}} \bar{E} \subset K$.
- 2.
- 3.

■

Exercise 1.28 Let (X, d) be a metric space.

1. Prove that $x \in X$ and $r > 0$, we have $\overline{B_X(x, r)} \subset \{y \in X : d(x, y) \leq r\}$. (Hint: Take complements and draw a picture.) Note the inclusion $\overline{B_X(x, r)} \subset B_X(x, r + \epsilon)$ follows for any $\epsilon > 0$.
2. Give an example using the discrete metric that demonstrates that equality need not hold in the inclusion $\overline{B_X(x, r)} \subset \{y \in X : d(x, y) \leq r\}$ that you proved in (1).
3. Prove that in \mathbb{R}^n under the Euclidean metric $d(x, y) = \|x - y\|$, we have $\overline{B_{\mathbb{R}^n}(x, r)} = \{y \in \mathbb{R}^n : \|x - y\| \leq r\}$. (Again a picture might be useful)
4. Using part (1), prove if A is bounded in (X, d) then \overline{A} is also bounded in (X, d) .

1. $\overline{B_X(x, r)} = B_X(x, r) \cup \text{Lim}_X(B_X(x, r))$. Let's analyze some $z \in B_X(x, r)$. For any such z , $d(x, z) < r \rightarrow z \in \{y \in X : d(x, y) \leq r\}$. Now assume that $y \in \text{Lim}_X(B_X(x, r))$. $\text{Lim}_X(B_X(x, r)) = \{x : U \cap \{B_X(x, r) \setminus \{x\}\} \neq \emptyset\}$ where U is any neighborhood of x . The set of limit points will contain those points with $d(x, y) \leq r$. Any y that has $d(x, y) \geq r + \epsilon$ will violate definition of limit point. So, $\text{Lim}_X(B_X(x, r)) \subset \{y \in X : d(x, y) \leq r\}$. This completes the inclusion, \subset .
2. Let $r = 1$ and $y \in \{y \in X : d(x, y) \leq r\}$. $y \notin B_X(x, 1) = \{x\}$ and we need to show $y \notin \text{Lim}_X(B_X(x, 1)) = U \cap (\{x\} \setminus y)$. Choose U to be $B_X(y, 1)$ so $y \notin \text{Lim}_X(B_X(x, 1))$.
3. Let $D = \{y \in \mathbb{R}^n : \|x - y\| \leq r\}$. For \subset , $\overline{B_{\mathbb{R}^n}(x, r)} = B_{\mathbb{R}^n}(x, r) \cup \text{Lim}_X(B_{\mathbb{R}^n}(x, r))$. If $y \in B_{\mathbb{R}^n}(x, r)$, $d(x, y) < r$ so $y \in D$. If $y \in \text{Lim}_X(B_{\mathbb{R}^n}(x, r)) = U \cap (B_{\mathbb{R}^n}(x, r) \setminus \{y\})$ for any neighborhood around y . This includes y when $d(x, y) = r$ so $y \in D$. For \supset , choose $y \in D$ s.t. $d(x, y) < r$ so $y \in B_{\mathbb{R}^n}(x, r)$. Let $z \in D$ be s.t. $d(x, z) = r$. So, for any $\epsilon > 0$, $\nexists B_X(z, \epsilon)$ s.t. $B_X(z, \epsilon) \cap (B_{\mathbb{R}^n}(x, r) \setminus \{z\}) \neq \emptyset$. Thus, $z \in \text{Lim}_X(B_{\mathbb{R}^n}(x, r))$.
4. Let $a \in \text{Lim}_X(A) = B_X(x, R) \cup A \subset \{a\} = \{\dots, b, \dots\}$. $d(a, x) \leq d(a, b) + d(b, x) \leq R$. So, $\text{Lim}_X(A) \in B_X(x, R)$ so \overline{A} is bounded. ■

Exercise 1.30 Let (X, d) be a metric space, and let E be a subset of X .

1. Show that E is dense in X if and only if any nonempty open subset of X contains a point of E .
2. Suppose $E \subset Y \subset X$. Prove that E is dense in Y if and only if $\text{Cl}_X(E) \supset Y$.

1. For \rightarrow , let $x \in \text{Lim}_X(E)$. Clearly, for any open set U , $U \cap E \setminus \{x\} \neq \emptyset$ so U contains another point than x . For \leftarrow , let's argue via proof by contrapositive. Assume that no nonempty open subsets contain a point of E . So, $x \in U$ but $x \notin E$ and $x \notin \text{Lim}_X(E)$ since $U \cap (E \setminus \{x\}) = \emptyset$. So, $x \in X$ but $x \notin \text{Cl}_X(E)$. So, E is not dense in X for this case.
2. E is dense in Y if and only if $\text{Cl}_Y(E) = Y$. So, $\text{Cl}_X(E) \cap Y$ (apply properties from Exer 1.10) $= Y$ if and only if $\text{Cl}_X(E) = Y$. ■

Exercise 1.31 Previously, we said that a subset E of \mathbb{R} was dense in \mathbb{R} if for any real numbers a and b , there exists a number $c \in E$ which lies between a and b . Show that in \mathbb{R} , the new, more general definition of dense agrees with the old one. That is, show that a subset E of \mathbb{R} is dense in \mathbb{R} according to the new definition if and only if it is dense according to the old one. (Hint: Use Exercise 1.30(1).)

Applying the first part of 1.30 yields $\text{Cl}_{\mathbb{R}}(E) = \mathbb{R}$ if and only if any nonempty open subset of \mathbb{R} contains a point of E . If $a, b \in \mathbb{R}$ are taken as end points of an open interval, there will be another point $c \in E$ which directly leads to the old definition. This proves both directions. ■

Exercise 2.7 Let $S = (p_n)_{n=1}^{\infty}$ be a sequence in \mathbb{R} whose image is $(\mathbb{Q} \cap (0, 1)) \cup \{5\}$. What are the two possibilities for S^* ? Justify your answers.

Set $\text{Im } S = (\mathbb{Q} \cap (0, 1)) \cup \{5\}$. So, $(\text{Im } S)' = [0, 1]$ and $S_{\infty} = \{5\}$. $S^* = \{[0, 1] \cup \{5\}, [0, 1]\}$ because 5 might appear a finite number of times in the sequence so $5 \notin S_{\infty}$. ■

Exercise 2.9 Prove Proposition 2.8.

Prove (1) \rightarrow (2). $p_n \rightarrow p$ in X means $p_n \in B_X(p, r)$ for $r > 0$ s.t. $N \in \mathbb{N}$ and $n \geq N$. Suppose \exists some subsequence p_{n_k} for $n_k \geq N$ s.t. $p_{n_k} \notin B_X(p, r)$. Then, p_n does not converge to p as it violates the definition of $p_n \rightarrow p$ in X as $n \rightarrow \infty$. Prove (2) \rightarrow (3). $S^* = (\text{Im } S)' \cup S_{\infty}$. This is true by definition since S^* is the set of all subsequential limits and all subsequences converges to p in X . So, $S^* = p$. Since all subsequences converge to a point p in X , they converge in X . Prove (3) \rightarrow (1). If $p \in S_{\infty}$ then $\exists p \in B_X(x, r)$ for all but finitely many points of p_n for $n \geq N, N \in \mathbb{N}$. If $p \in (\text{Im } S)'$, then $\text{Lim}_X(\text{Im}(p_n)) = \{p\}$. Thus, \exists a neighbourhood, U , of p s.t. $(U) \cap \text{Im}(S) \setminus \{p\} \neq \emptyset$. Also, every subsequence of $p_n \rightarrow p$ as $n \rightarrow \infty$. Thus, $\exists n \geq N$ s.t. $p_n \in U \forall n$, which shows that $p_n \rightarrow p$ as $n \rightarrow \infty$. ■

Exercise 2.16 Let (X, d) be a metric space, and let $(x_n)_{n=1}^{\infty}$ be a sequence in X . Prove the following statements.

1. If $(x_n)_{n=1}^{\infty}$ converges in X , then it is Cauchy in X .
2. If $(x_n)_{n=1}^{\infty}$ is Cauchy in X , then it is bounded in X .

1. By definition of convergence, we have any neighbourhood U s.t. $n \geq N$ implies $x_n \in U$. Let $U = B_X(x, r)$. By the triangle inequality. $d(x_m, x_n) \leq d(x_m, x) + d(x, x_n) \leq 2r$ so Cauchy in X .
2. If (x_n) is Cauchy then $m \geq n \geq N$ implies $d(x_m, x_n) < \epsilon$. Thus, all but finitely many points are within $B_X(x, \epsilon)$. Let $r = \max(d(x_j, x))$ when $j \leq N$. Construct $B_X(x, r + \epsilon)$ where $\epsilon > 0$. Thus all points of $\text{Im}(x_n)$ are contained within $B_X(x, r + \epsilon)$ so $\text{Im}(x_n)$ is bounded so (x_n) is bounded. ■

Exercise 2.18 Let (X, d) be a metric space, and let Y be a subset of X . Prove the following statements.

1. If Y is complete, then Y is closed in X .
2. If X is complete and Y is closed in X , then Y is complete.

1. Every Cauchy sequence in Y converges in Y . So, \exists a sequence $(x_n)_{n=1}^{\infty}$ in Y that converges to x in X . So $x \in \text{Lim}_X(Y)$ and $\text{Lim}_X(Y) \subset Y$ so Y is closed.
2. Since Y is closed, $\text{Lim}_X(Y) \subset Y$. From Exer 1.13, every Cauchy sequence in X s.t. $x_n \rightarrow x$ and $x \in Y$ means every Cauchy sequence also converges to x in Y . This means it is a limit point of Y which are all contained in Y . Thus. Y is complete. ■