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MA 521
Homework #11
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Chapter 8: 2.12, 2.15, 2.16, 2.17, 2.18, 2.19, 2.20, 3.3

Exercise 2.12 Which $n \in \mathbb{N}$ have the property that $f^n \in \mathcal{R}([a,b])$ implies $f \in \mathcal{R}([a,b])$? Give proofs(s) and counterexamples(s) to show your answer is correct and complete.

We knwo that if f is differentiable on $[a,b]$ that is must be continuous on $[a,b]$. By contradition, we will show that f is bounded. If f is unbounded at $[a,b]$ then $\exists x \in [a,b]$ s.t. $\nexists M$ s.t. $|x| < M$. Let's analyze the limit as $\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = f'(x)$. Without loss of generality, assume $f(x+h)$ is finite. By our previous statement, we claimed that $f(x)$ was unbounded which implies this limit can't exist. This means that $f^1 \notin \mathcal{R}([a,b])$ which is a contradition. Since f is continuous and bounded, we can say that by Theorem 2.8 $f \in \mathcal{R}([a,b])$. If this holds for f', then it will hold for all $n > 1$ as well. ∎

Exercise 2.15 Show that if $f : [a,b] \to \mathbb{R}$ is continuous and $F(x) := \int_a^x f(t)dt = 0$ for all $x \in [a,b]$, then $f(x) = 0$ for all $x \in [a,b]$. Provide an example to show that the statement is false if f is not continuous.

We want $\lim_{t\to x} \frac{F(t)-F(x)}{t-x} = F'(x) = f(x) = 0\,\forall x \in [a,b]$. We know that $F(x) := \int_a^x f(t)dt = 0$. So this means that $\forall x \in [a,b], F(x) = F(a)$ by the Fundamental Theorem of Calculus. So then, substitue $F(a)$ for $F(x)$ and $\lim_{t\to x} \frac{F(t)-F(a)}{t-x} = F'(x) = f(x) = 0 \forall x \in [a,b]$. Let's pick the same function $f(x) = 0$ but at some point $x \in [a,b]$, there exists a removable discontinuity where $f(x) = 1$. If we choose the partitions of the function correctly, we can obtain the original expression with $f(x) \neq 0 \forall x \in [a,b]. F(x) := \int_a^x f(t)dt = 0$. ∎

Exercise 2.16 Assume f and g are differentiable functions on $[a,b]$ and assume $f', g' \in \mathcal{R}([a,b])$. Show that the integration by parts formula is valid:

$$\int_a^b fg'dx = f(b)g(b) - f(a)g(a) = \int_a^b f'gdx.$$

Make sure you show the relevant functions are Riemann integrable when you do the proof!

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Exercise 2.17 Assume $g : [a,b] \to \mathbb{R}$ is differentiable, that g' is continuous, and M and m are upper and loewr bounds, respectively, for the function g. Assume $f : [m, M] \to \mathbb{R}$ is continuous. SHow that the change of variables formula is valid:

$$\int_a^b f(g(x))g'(x)dx = \int_{g(a)}^{g(b)} f(t)dt.$$

Again, part of the exercise is to check that the relevant functions are Riemann integrable when you do the proof!

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Exercise 2.18 Assume $f \in \mathcal{R}([a,b])$, but that f has a jump discontinuity at $c \in (a,b)$, i.e. $f(c-) \neq f(c+)$. Show that $F(x) := \int_a^x f(t)dt$ is not differentiable at $x = c$.

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Exercise 2.20 Assume g is bounded, $g \in \mathcal{R}([0,1])$ and g is continuous at 0. Show that

$$\lim_{n\to\infty} \int_0^1 g(x^n)dx = g(0).$$

Hint: Consider the difference $\int_0^1 g(x^n)dx - g(0)$; add and subtract $\int_0^c g(x^n)dx$ for a carefully chosen c, and then that $\int_0^c dx$ is close to $cg(0)$ for large enough n.

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Exercise 3.3 Let (f_n) be a sequence of real-valued, Riemann integrable functions on the interval $[a,b]$. Assume that $f_n(x) \to 0$ as $n \to \infty$ for each $x \in [a,b]$, and additionally,

$$\sum_{n=1}^{\infty} |f_{n+1}(x) - f_n(x)|$$

converges uniformly on $[a,b]$.

1. Show that $\lim_{n\to\infty} \int_a^b f_n dx \to 0$.
2. Is it necessarily the case that the series $\sum_{n=1}^{\infty} f_n$ converges uniformly? Give a proof or counterexample to support your answer.

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