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Homework #7
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1.2, 2.2, 2.9, 2.10, 2.11, 2.23, 2.32, 2.40, 2.41.

Exercise 1.2 Let (X, d_X) and (Y, d_Y) be metric spaces, and let E be a subset of X. Let $f : E \to Y$ be a function, and let p be a limit point of E in X. Prove that $f(x) \to q$ as $x \to p$ if and only if for every $\epsilon > 0$, there exists $\delta > 0$ such that $x \in E$ and $0 < d_X(x, p) < \delta$ imply together that $d_Y(f(x), q) < \epsilon$.

For $\to$, let $x \in E$ and for some $\delta > 0$. Since p is a lim point of E, $\exists$ some δ s.t. $x \in B_X(p, \delta)$. $x \to p$ means for every $\delta > 0 \exists N \in \mathbb{N}$ s.t. $n \geq N$ that $x_n \in B_X(p, \delta)$. $f(x) \to q$ means for every $\epsilon > 0 \exists N \in \mathbb{N}$ s.t. for $n \geq \mathbb{N}$, $f(x_n) \in B_Y(q, \epsilon)$. This says that $d(f(x), q) < \epsilon$. For $\leftarrow$, since p is limit point, $\exists$ a sequence $(x_n)_{n=1}^{\infty}$ in $E \setminus \{p\}$ which converges to p in X so $x \to p$. This satisfies $x \in E$ and $0 < d_X(x, p) < \delta$ for some $\delta > 0$. For every $\epsilon > 0$ if $d_Y(f(x), q) < \epsilon$ implies $f(x_n) \in B_Y(q, \epsilon)$ s.t. $\exists N \in \mathbb{N}$ s.t. for $n \geq \mathbb{N}$. ∎

Exercise 2.2 Let (X, d_X) and (Y, d_Y) be metric spaces; let $f : X \to Y$ be a function. Prove that f is continuous at $p \in X$ if and only if for every $\epsilon > 0$, there exists $\delta > 0$ such that $x \in B_X(p, \delta)$ implies $f(x) \in B_Y(f(p), \epsilon)$.

Let $x \in B_X(p, \delta)$ for some $\delta > 0$. By def. of f being continuous, $x \in B_X(p, \delta)$ implies for every neighborhood V of $f(p)$, $f(x) \in V$. This includes neighborhoods with every $\epsilon > 0$ so $f(x) \in B_Y(f(p), \epsilon)$. For $\to$, for every $\epsilon > 0$, $\exists \delta > 0$ s.t. $x \in B_X(p, \delta) \to f(x) \in B_Y(f(p), \epsilon)$. Let $B_Y(f(p), \epsilon)$ be a neighborhood V of $f(p)$. If $f(x) \in B_Y(f(p), \epsilon), x \in B_X(p, \delta)$ for some δ which is a neighborhood of p in X so f is continuous. ∎

Exercise 2.9 Assume $f : \mathbb{R} \to \mathbb{R}$ is a function satisfying $\lim_{h \to 0}[f(x + h) - f(x - h)] = 0$, for all $x \in \mathbb{R}$. Does it follow that f must be continuous? If so, give a proof; if not, give a counterexample.

No, define $f(x) \in \mathbb{R}$ to be x if $x < x_0, -x_0 if x = x_0, x if x > x_0$. So essentially, there is a hole in the linear function such that it is defined at another point. x_0 is a limit point of $f(x)$ but $\lim_{x \to x_0} f(x) \neq f(x_0)$. ∎

Exercise 2.10 Let (X, d_X) and (Y, d_Y) be metric spaces and $f : X \to Y$ a function.

1. Show that f is continuous if and only if $f^{-1}(C)$ is closed on X whenever C is closed in Y.
2. Show that $f : X \to Y$ is continuous if and only if $f(\overline{A}) \subset \overline{f(A)}$ for every subset A of X.
3. Consider the (continuous) function $g : \mathbb{R} \to \mathbb{R}$ given by $g(x) = \frac{1}{1+x^2}$. Give an example of a subset A of $\mathbb{R}$ such that $g(\overline{A}) \neq \overline{g(A)}$.

1. Let $C \subset Y$ and C is closed. For $\rightarrow$, because f is continuous, $B = f^{-1}(A)$ is open in X. So, $X \setminus B$ is closed in X. So, $f^{-1}(C) = X \setminus B$. For $\leftarrow$, $Y \setminus C$ is open in Y. $f^{-1}(Y \setminus C) = X \setminus f^{-1}(C)$ is open so f is continuous by Prop 2.6.
2. For $\rightarrow$, $f^{-1}(\overline{f(A)})$ is closed in X. Let $x \in \overline{A} \rightarrow f(x) \subset f(\overline{A})$. Since $\overline{A}$ is the smallest closed set containing A, $\overline{A} \subset f^{-1}(\overline{f(A)})$. So $f(\overline{A}) \subset \overline{f(A)}$. For $\leftarrow$, C is closed in Y so $D = f^{-1}(C)$ needs to be closed in X for f to be continuous. $f(\overline{D}) \subset \overline{f(D)} = f(f^{-1}(C)) \subset \overline{C} = C$. This means that $\overline{D} \subset f^{-1}(C) = D$, making D closed. So, then f is continuous.
3. Let $A = [0, \infty)$. $g(\overline{A}) = g([0, \infty]) = (0, 1]$ and $\overline{g(A)} = [0, 1]$.

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Exercise 2.11 Let (X, d_X) and (Y, d_Y) be metric spaces and let f and g be continuous functions from X to Y. Assume E is a dense subset of X.

1. Prove that $f(E)$ is dense in $f(X)$. (Hint: Use Exercise 1.30) in Chapter 4 and Exercise 2.10 above.)
2. Prove that if $f(x) = g(x)$ for all $x \in E$, then $f(x) = g(x)$ for all $x \in X$.

1. By 2.10, $f(\overline{E}) \subset \overline{f(E)}$. We want to prove that for an open subset $f(C)$ of Y that $f(E) \cap f(C) \neq \emptyset$. So, $Y \setminus f(C)$ is a closed set in Y then also $f^{-1}(Y \setminus f(C))$ is a closed set in X. $f^{-1}(Y \setminus f(C)) = X \setminus f^{-1}(f(C))$ is closed so $f^{-1}(f(C))$ is open in X. So, $E \cap f^{-1}(f(C)) \neq \emptyset$. So there exists an element$x \in E$ s.t. $x \in f^{-1}(f(C))$ so $f(x) \in f(C)$. So, $f(x) \in f(E) \rightarrow f(C) \cap f(E) \neq \emptyset$. So $f(E)$ is dense in $f(X)$.
2. Assume by contradiction that $f(a) \neq g(a), a \in X$. Let $d(f(a), g(a)) = r > 0$. Since f is continuous at a so $\exists \delta_1 > 0$ s.t. $f(B(a, \delta_1)) \subset B(f(a), \frac{r}{4})$. g is continuous at a so $\exists \delta_2 > 0$ s.t. $g(B(a, \delta_2)) \subset B(g(a), \frac{r}{4})$. Take $\delta = \min(\delta_1, \delta_2)$. Then, $f(B(a, \delta)) \subset B(f(a), \frac{r}{4})$ and $g(B(a, \delta)) \subset B(g(a), \frac{r}{4})$. Since E is dense in X so $B(a, \delta) \cap E \neq \emptyset$. Take $k \in B(a, \delta) \cap E$. Then, $f(k) = g(k)$. So, $f(k) \in B(f(a), \frac{r}{4})$ and $g(k) \in B(g(a), \frac{r}{4})$. Hence, by triangle inequality, we have $d(f(a), g(a)) \leq d(f(a), f(k)) + d(f(k), g(k)) + d(g(k), g(a)) < \frac{2r}{4}$, which is false since $d(f(a), g(a)) = r$. So, $f(x) = g(x)$ for all $x \in X$

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Exercise 2.23

1. Find a closed subset of E of $\mathbb{R}$ and a continuous function $f : \mathbb{R} \rightarrow \mathbb{R}$ is continuous such that $f(E)$ is not closed.
2. Find a bounded subset E of $\mathbb{R}$ and a continuous function $f : E \rightarrow \mathbb{R}$ such that $f(E)$ is not bounded.
3. Show that if E is a bounded subset of $\mathbb{R}$ and $f : \mathbb{R} \rightarrow \mathbb{R}$ is continuous, then $f(E)$ is bounded.

1. Consider the function $f = \arctan(x)$ on the closed interval $[0, \infty)$. f is continuous but does not yield a closed set as $f(0) = 0$ and $f(\infty) = \frac{\pi}{2}$ so the interval was $[0, \frac{\pi}{2})$.
2. Let E be the set $[0, 1]$ for $\frac{1}{x}$ defined on E. $\nexists$ any r such that any ball in $\mathbb{R}$ can contain $\frac{1}{0}$.
3. For some $\delta > 0$, $E \subset B_X(p, \delta)$ since E is bounded for some $p \in X$. So by definition of continuity of f, $f(B_X(p, \delta)) \subset B_Y f(p), \epsilon)$ so $f(E)$ is bounded.

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Exercise 2.32 Prove that the set $R^2 \setminus \{0, 0\}$ is path-connected, and therefore connected. Then, use the function $\frac{x}{|x|}$ to show that $S = \{x \in \mathbb{R}^2 : |x| = 1\}$ is connected.

Let $a_0, a_1 \in \mathbb{R}^2 \setminus \{0, 0\}$. So, use polar coordinates for a curve: $a_i = r_i(\cos\theta_i, \sin\theta_i)$ with $r_i > 0$ and let $f(t) = r(t)(\cos\theta(t), \sin\theta(t))$ where $r(t) = (1-t)r_0 + tr_1$ and $\theta(t) = (1-t)\theta_0 + t\theta_1$. Then $\theta(0) = \theta_0, \theta(1) = \theta_1, r(0) = r_0, r(1) = r_1 \to f(0) = a_0$ and $f(1) = a_1$. $\forall\, t \in [0, 1], r(t) > 0$ since $t, 1-t, r_0, r_1 > 0$. So, $f(t) \neq 0 \forall t$ and therefore f defines path from a_0 to a_1. So, $\mathbb{R}^2 \setminus \{0, 0\}$ is path-connected and is connected.
Set $f(t) = \frac{a(1-t)+tb}{|a(1-t)+tb|}$. Since $|a(1-t) + tb| \in S$, this will always be constrained to 1. So then $f(0) = a, f(1) = b$ and S is path-connected and thus connected. ■

Exercise 2.40 Assume $f : X \to Y$ and $g : Y \to Z$ are uniformly continuous functions, where $(X, d_X), (Y, d_Y)$, and (Z, d_Z) are metric spaces. Prove that $g \circ f$ is uniformly continuous.

Take (x, y) in X s.t. $d_X(x, y) < \delta(\delta > 0)$ then $d_Y(f(x), f(y)) < \epsilon$. Set $\delta_1 = \epsilon$ so that $d_Y(f(x), f(y)) < \delta_1$ then $d_Z(g(f(x)), g(f(y))) < \epsilon_2$. So for any choice of arbitrary ϵ_2 so $g \circ f$ is uniformly continuous. ■

Exercise 2.41 Let E be a bounded subset of $\mathbb{R}^k$ and let $f : E \to \mathbb{R}$ be a uniformly continuous function. Show that f is bounded. (Hint: You will need to use compactness of $\overline{E}$ at some point.)

$\overline{E}$ is closed and bounded so it is compact. Because $\overline{E}$ is compact, it can be written as $\bigcup_{i=1}^{n} B_{R^k}(x_i, \delta)$ which is a finite open cover. Uniform continuity of f states that for $d(x, y) < \delta$ in $x, y \in \mathbb{R}^k$ then $d(f(x), f(y)) < \epsilon$. Get the max distance of all $f(x_i)$ with each other and add ϵ and set this quantity to r. Then choose any $f(x_i)$ as center as $B_X(f(x_i), r)$ so f is bounded. ■