Ravi Raju
MA 521
Homework #4
2/28/2018

1.9, 1.10, 1.13, 1.14, 1.15, 1.21, 1.24

Exercise 1.9 Let E_1 and E_2 be subsets of a metric space (X, d). Prove that

$$\mathrm{Lim}_X(E_1 \cup E_2) = \mathrm{Lim}_X(E_1) \cup \mathrm{Lim}_X(E_2).$$

For $\subset$, assume $x \in \mathrm{Lim}_X(E_1 \cup E_2)$ and U is a neighborhood of x in X. So, $\emptyset \neq U \cap (E_1 \cup E_2) \setminus \{x\}$. $U \cap [(E_1 \setminus \{x\}) \cup (E_2 \setminus \{x\})] = [U \cap (E_1 \setminus \{x\}) \cup U \cap (E_2 \setminus \{x\})] = \mathrm{Lim}_X(E_1) \cup \mathrm{Lim}_X(E_2)$. For $\supset$, using Proposition 1.8, $E_1 \subset E_1 \cup E_2 \rightarrow \mathrm{Lim}_X(E_1) \subset \mathrm{Lim}_X(E_1 \cup E_2)$. Similarly, $E_2 \subset E_1 \cup E_2 \rightarrow \mathrm{Lim}_X(E_2) \subset \mathrm{Lim}_X(E_1 \cup E_2)$. So, $\mathrm{Lim}_X(E_1) \cup \mathrm{Lim}_X(E_2) \subset \mathrm{Lim}_X(E_1 \cup E_2)$. So, $\mathrm{Lim}_X(E_1 \cup E_2) = \mathrm{Lim}_X(E_1) \cup \mathrm{Lim}_X(E_2)$. ■

Exercise 1.10 Let (X, d) be a metric space, and assume $E \subset Y \subset X$. Prove that

$$\mathrm{Lim}_Y(E) = \mathrm{Lim}_X(E) \cap Y.$$

For $\subset$, $\mathrm{Lim}_Y(E) = V \cap (E \setminus \{x\}) = (U \cap Y) \cap (E \setminus \{x\})$, where V is open in Y and U is open in X (applied Theorem 2.19). So, $(U \cap Y) \cap (E \setminus \{x\}) \rightarrow [U \cap E \setminus \{x\}] \cap Y = \mathrm{Lim}_X(E) \cap Y$. For $\supset$, $\mathrm{Lim}_X(E) \cap Y \rightarrow [V \cap (E \setminus \{x\})] \cap Y$, where V is an open set neighborhood of x with respect to X. So, $[(V \cap Y) \cap [E \setminus \{x\} \cap Y]]$. Applying Theorem 2.19 to $V \cap Y = U$ where U is open in Y. Then, $U \cap [E \setminus \{x\} \cap Y] = U \cap [E \setminus \{x\}]$, since $E \subset Y$. So, $U \cap [E \setminus \{x\}] \rightarrow \mathrm{Lim}_Y(E)$. So, $Lim_Y(E) = \mathrm{Lim}_X(E) \cap Y$. ■

Exercise 1.13 Let (X, d) be a metric space, and assume $Y \subset X$. Let $(x_n)_{n=1}^{\infty}$ be a sequence in Y and let x be a point of X. Prove that the following two statements are equivalent:

1. $x_n \rightarrow x$ in X, and $x \in Y$.
2. $x_n \rightarrow x$ in Y.

For $\subset$, if $x_n \rightarrow x$ in X, and $x \in Y$, then for some neighborhood V of x and $\exists$ some $n \in \mathbb{N}$ s.t. $x_n \in V$. By Theorem 2.19, $U = V \cap Y \implies U$ is open in Y so $x_n \in U$. This implies that $x_n \rightarrow x$ and $x \in Y$. For $\supset$, $\exists$ a neighborhood V around x with respect to Y s.t. $n \in \mathbb{N}$ that $x_n \in V$. $V \subset Y \subset X$ so $x \in Y$. If $x_n \in V$, then $V = U \cap Y$ where $x_n \in U$. So, $x_n \rightarrow x$ in X. ■

Exercise 1.14 Let (X, d) be a metric space, and let $(x_n)_{n=1}^{\infty}$ be a sequence in Y and let x be a point of X. Prove that the following statements are equivalent:

1. $x_n \rightarrow x$ in X
2. For every $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $n \geq N$ implies $x_n \in B_X(x, \epsilon)$(i.e. $d(x, x_n) < \epsilon$).
3. $d(x, x_n) \rightarrow 0$ as $n \rightarrow \infty$.

Let's consider the first two points first. For $x_n \to x$ in X, $\exists$ a neighborhood U around x in X s.t. $n \in \mathbb{N}$ that $x_n \in U$ and $n \geq N$, where $N \in \mathbb{N}$. For $\subset$, set U to the ball $B_X(x, \epsilon)$ for $\epsilon > 0$ so $x_n \in B_X(x, \epsilon)$. For $\supset$, use the same reasoning that $n \geq N$ implies $B_X(x, \epsilon)$. $B_X(x, \epsilon)$ is a neighborhood of x which is open in X. Since $x_n \in B_X(x, \epsilon)$ for $n \geq N$, so $x_n \to x$ in X. For the last equality, let us look at points 2 and 3. To prove the direction (2) $\to$ (3), let's argue via proof by contrapositive. Assume that $d(x, x_n) \to \zeta$ as $n \to \infty$ for some $n \in \mathbb{N}$. If this is true, then $\exists$ some $\epsilon > 0$ s.t. $\zeta > \epsilon$ so then $x_n \notin B_X(x, \epsilon)$. So, for x_n to be in the set $B_X(x, \epsilon)$, $d(x, x_n)$ must tend to 0 as $n \to \infty$. For the other inclusion $\supset$, for any ϵ selected, there will always be another $n \in \mathbb{N}$ such that $\epsilon > d(x, x_n)$ so $x_n \in B_X(x, \epsilon)$. ■

Exercise 1.15 Let $(s_n)_{n=1}^{\infty}$ and $(t_n)_{n=1}^{\infty}$ be sequences of real numbers, with $t_n > 0$ for each $n \in \mathbb{N}$. Assume that $t_n \to 0$ as $n \to \infty$.

- Prove that if $|s_n - s| < t_n$ for all $n \in \mathbb{N}$, then $s_n \to s$ as $n \to \infty$.
- Prove that if $\frac{1}{n} \to 0$ as $n \to \infty$.

- Given $|s_n - s| < t_n \forall n$ is true, for any $t_n > 0$, $\exists N \in \mathbb{N}$ s.t. $n \geq N$ implies $s_n \in B_X(s, t_n)$ as $d(s_n, s) < t_n \forall n \in \mathbb{N}$ so $s_n \to s$ as $n \to \infty$.
- Let $a_n = (\frac{1}{n})_{n=1}^{\infty}$. Clearly, $\forall n \in \mathbb{N}, a_{n+1} < a_n$ (this is the Archimedean property). If $a_N < \epsilon$ for $N \in \mathbb{N}$ for any $\epsilon > 0$, $\exists n \geq N$ s.t. $a_n < a_N < \epsilon$ so $a_n \in B_X(0, \epsilon)$.

■

Exercise 1.21 Let (X, d) be a metric space, and let E be a subset of X. Prove that $\text{Lim}_X(E)$ is a closed set of X.

Let $x \in \text{Lim}_X(E')$ where $E' = \text{Lim}_X(E)$. For any neighborhood U around x, $\exists x' \in U \cap (E' \setminus \{x\})$. So, $x' \in \text{Lim}_X(E)$. For any neighborhood V around x', $\exists y \in V \cap (E \setminus \{x'\})$ such that $y \in E$. For any neighborhood N around x, $y \in N \cap (E \setminus \{x\})$ so $x \in \text{Lim}_X(E)$. ■

Exercise 1.24 Let (X, d) be a metric space, and let E be a subset of X. Prove that

$$X \setminus \text{Cl}_X(E) = \text{Int}_X(X \setminus E)$$

.

For $\subset$, $X \setminus \text{Cl}_X(E) = X \setminus (E \cup \text{Lim}_X(E)) = (X \setminus E) \cap (X \setminus \text{Lim}_X(E))$. Recall from Exercise 1.21 that $\text{Lim}_X(E)$ is a closed set so $X \setminus \text{Lim}_X(E)$ is open. Let $x \in X \setminus E$ and $x \in X \setminus \text{Lim}_X(E)$. $x \in \text{Int}_X(X \setminus \text{Lim}_X(E)) \to \exists \epsilon > 0$ s.t. $B_X(x, \epsilon) \subset \text{Int}_X(X \setminus \text{Lim}_X(E))$. Since $x \in X \setminus E$, for $\epsilon > 0$, $B_X(x, \epsilon) \subset X \setminus E$ so $x \in \text{Int}_X(X \setminus E)$. For $\supset$, let $x \in \text{Int}_X(E) \subset X \setminus E$ so $x \in X \setminus E$. $x \subset \text{Int}_X(X \setminus E) \subset \text{Int}(X)$. This says that since x is in this open set, $B_X(x, r)$ is open. $x \notin \text{Lim}_X(E)$ as $\text{Lim}_X(E)$ is closed; that is, $B_X(x, r) \not\subset \text{Lim}_X(E)$. So, $x \in X \setminus \text{Lim}_X(E)$. Thus, $X \setminus \text{Cl}_X(E) = \text{Int}_X(X \setminus E)$. ■