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Homework #1
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# Exercise 1.7, 3.3, 3.4, 3.6, 4.5, 4.7, 4.17

**Exercise 1.7.** Let $A$ and $B$ be subsets of another $X$. Prove the following the following statements.
(1) $A \cap B = A \setminus (A \setminus B)$
(2) $A \subset B$ if and only if $X \setminus A \supset X \setminus B$.

Recall the definitions of $\cup$ and $\setminus$. $A \cup B = \{x : x \in A \text{ and } x \in B\}$. $A \setminus B = \{x \in A : x \notin B\}$.
(1) Let $D = A \setminus B$. $D$ is the set of elements in $A$ that are strictly unique. Let $E = A \setminus D$. $E$ is the relative complement of $D$ in $A$, which only leaves elements common to both $A$ and $B$.

(2) Let us prove this $\rightarrow$ direction first.
Given $A \subset B$. This means that $A$ will have a lesser or equal to number of elements in its set than $B$. It follows that $X \setminus A$ will contain all elements of the set $X \setminus B$. Thus, $X \setminus A \supset X \setminus B$.

Now the other direction, $\leftarrow$. Given $X \setminus A \supset X \setminus B$. Assume $\exists$ some $x \in A$ and $x \notin B$, which means that $A \not\subset B$. However, $x \notin X \setminus A$ and $x \in X \setminus B$ when we stated $X \setminus A \supset X \setminus B$. So, it must be $\forall x \in A$ must be also be $x \in B$ so $A \subset B$. ∎

**Exercise 3.3.** Let $f : A \leftarrow B$ be a function. Prove the following statements.
(1) $f$ is injective if and only if $f^{-1}(f(C)) = C$ for every subset $C$ of $A$.
(2) $f$ is surjective if and only if $f^{-1}(f(D)) = D$ for every subset $D$ of $B$.

First, let us list some useful definitions.
If $G \subset B$ then the inverse image, $f^{-1}(G)$ of $G$ under $f$ is $f^{-1}(G) = \{x \in A : f(x) \in G\}$.
If $f^{-1}(y)$ contains at most one element of A for each $y \in B$, then $f$ is said to be injective.

(1) Let us prove this direction, $\rightarrow$.
Given $f$ is injective, let $C_1$ be some subset of $A$. $f$ maps all elements of $C_1$ to some set $B_1 \subset B$. Applying the definition of the inverse image to this set $B_1$ under f yields $f^{-1}(B_1) = \{x \in A : f(x) \in B_1\}$. Since we know that $f$ is injective, we know that the resulting set obtained from the inverse image has to be the original set, $C_1$.

Now the other direction, $\leftarrow$. Given $f^{-1}(f(C)) = C$. Let us do proof by contradiction. Let $x_1, x_2$ be elements in $C$ and assume that $f(x_1) = f(x_2)$ but $x_1 \neq x_2$ (this is another way to say $f$ is not injective). Applying the given fact to a subset of $C$, $\{x_1\}$, yields $f^{-1}(f(\{x_1\})) = \{x \in C : f(x) \in f(C)\} = \{x_1, x_2\}$. Clearly, this is a contradiction since the set we put into the function and inverse image is not the same set that was returned. This proves that $f$ has to be injective.

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