

2.24, 3.12, 3.26, 5.3, 5.7, 5.8, 5.12, 5.13.

Exercise 2.24 Let (X, d) be a metric space. Show that if X is totally bounded, then X is bounded.

Let $X = \bigcup_{B_X} B_X(x, \epsilon) = B_X(x, r)$. Choose $\zeta > 0$ so $B_X(x, r) \subset B_X(x, r + \zeta)$ so X is bounded. ■

Exercise 3.12 Let (X, d) be a metric space. Assume F and K are subsets of X , with F closed and K compact. Then $F \cap K$ is compact.

If K is a compact subset of X , then K is closed and bounded in X . The intersection of closed sets is closed so $F \cap K$ is closed. By Thm 3.10, $F \cap K \subset K$ and K is compact so $F \cap K$ is also compact. ■

Exercise 3.26 Give an example of a collection \mathcal{A} of bounded subsets of \mathbb{R} such that \mathcal{A} has the finite intersection property, but $\bigcap_{A \in \mathcal{A}} A = \emptyset$. Hint: If $A \subset \mathbb{R}$ is bounded in \mathbb{R} , what else can prevent it from being compact?

Exercise 5.3 Let \mathcal{A} be a collection of convex subsets of \mathbb{R}^k . Show that $B := \bigcap_{A \in \mathcal{A}} A$ is convex.

Let's do proof by contradiction. Let $B = \bigcap_{A \in \mathcal{A}} A$. Assume B is not convex. Let $a, b \in B$ so then $\exists t \in [0, 1]$ s.t. $z \in (1 - t)a + tb \notin B$. But $z \in A \forall A \in \mathcal{A} \rightarrow z \notin B$ so $B \neq \bigcap_{A \in \mathcal{A}} A$. This is clearly a contradiction so B is convex. ■

Exercise 5.7 Let (X, d) be a metric space and let A and B be disjoint subsets of X . Prove that if A and B are both open in X , then A and B are separated.

We need to show that $A \cap \overline{B} = B \cap \overline{A} = \emptyset$. So, let's analyze the first statement: $\overline{A} \cap B = (A \cup \text{Lim}_X(A)) \cap B = (A \cap B) \cup (\text{Lim}_X(A) \cap B)$. A and B are disjoint so the only set we need to be concerned with is $\text{Lim}_X(A) \cap B$. Consider the intersection of $\text{Lim}_X(A) \cap \text{Lim}_X(B) = C$. Without loss of generality, choose $x \in C \rightarrow x \in \text{Lim}_X(A)$ and $\text{Lim}_X(B) \not\subset B$ since B is open. So, $x \notin \text{Lim}_X(A) \cap B$. So, $\overline{A} \cap B = \emptyset$. This holds true for the other case as well and so A and B are both separated. ■

Exercise 5.8 Let E be a connected subset of a metric space (X, d) . Show that \bar{E} is connected.

If E is connected, then $E \subset \text{Lim}_X(E)$. If E is connected, then E has no isolated points. If E had isolated points, then \exists some $x \in E$ s.t. $x \notin \text{Lim}_X(E)$. Thus, \exists some neighbourhood U of x s.t. $U \cap \bar{E} \setminus \{x\} = \emptyset$. Then, E can be written as the union of two separated sets $E = E \setminus \{x\} \cup \{x\}$, implying E is not connected which is false. Thus, \bar{E} is connected. ■

Exercise 5.12 Let (X, d) be a metric space, and let \mathcal{C} be a collection of connected subsets of X . Assume $A = \bigcap_{C \in \mathcal{C}} C$ is nonempty. Show that $B = \bigcup_{C \in \mathcal{C}} C$ is connected.

Let's solve this problem via proof by contrapositive. Let B not be connected so this implies that $B = Z \cup Y$ s.t. $Z \cap \bar{Y} = Y \cap \bar{Z} = \emptyset$. Take connected subset $C_1 \in \mathcal{C}$ s.t. $C_1 \subset B$. By Thm. 5.11, $C_1 \subset Z$ or $C_1 \subset Y$. So $Z \cap Y = \emptyset \rightarrow \bigcap_{C \in \mathcal{C}} C \neq \emptyset$. So B is connected. ■

Exercise 5.13 Let $X = \mathbb{R}^2$. Give an example of a connected subset E of X , such that $\text{Int}_X(E)$ is not connected. Prove both that your set E is connected and that its interior is not. ((Hint: Consider the union of two convex sets joined at a point. You may assume without proof the fact that convexity implies connectedness in \mathbb{R}^2 .)

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