Ravi Raju
MA 521
Homework #6
3/14/2019

2.24, 3.12, 3.26, 5.3, 5.7, 5.8, 5.12, 5.13.

Exercise 2.24 Let (X, d) be a metric space. Show that if X is totally bounded, then X is bounded.

If X is totally bounded, then it can be covered by finitely many balls of radius ϵ if $\{x_1, x_2, \ldots, x_n\} \in X$ s.t. $\bigcup_{i=1}^{n} B_{(X,d)}(x_j, \epsilon)$. So, simply choose the x_n which has the ball with the maximum radius, r_m and construct another ball with this radius $+r$ s.t. all balls are contained within $B_X(x, r_m + r)$. So, X is bounded. ∎

Exercise 3.12 Let (X, d) be a metric space. Assume F and K are subsets of X, with F closed and K compact. Then $F \cap K$ is compact.

If K is a compact subset of X, then K is closed and bounded in X. The intersection of closed sets is closed so $F \cap K$ is closed. By Thm 3.10, $F \cap K \subset K$ and K is compact so $F \cap K$ is also compact. ∎

Exercise 3.26 Give an example of a collection $\mathcal{A}$ of bounded subsets of $\mathbb{R}$ such that $\mathcal{A}$ has the finite intersection property, but $\bigcap_{A \in \mathcal{A}} A = \emptyset$. Hint: If $A \subset \mathbb{R}$ is bounded in $\mathbb{R}$, what else can prevent it from being compact?

The subset $K = (0, \frac{1}{n})$ is bounded because for any n chosen, $\exists$ another $n_2 \in \mathbb{R}$ s.t. $n_2 < n$ so $K \subset B_{\mathbb{R}}(0, \frac{1}{n_2})$. Let $\mathcal{B}$ be a finite subcollection of $\mathcal{A}$ so then $G = \bigcap_{A \in \mathcal{B}} (0, \frac{1}{n})$. Since the metric space is $\mathbb{R}$ and $\mathcal{B}$ is a finite collection, $\exists \sup B = L$ s.t. $\frac{1}{L+1}$ is in all subsets $(0, \frac{1}{n}) \forall n \in \mathcal{B}$ so $\bigcap_{A \in \mathcal{B}} A \neq \emptyset$. So $\mathcal{A}$ has FIP. But for $\bigcap_{A \in \mathcal{A}} (0, \frac{1}{n}) \to (0, 0)$ because the $\sup \frac{1}{n}$ as $n \to \infty = 0$. $(0, 0)$ is empty $\to \bigcap_{A \in \mathcal{A}} (0, \frac{1}{n}) = \emptyset$. ∎

Exercise 5.3 Let $\mathcal{A}$ be a collection of convex subsets of $\mathbb{R}^k$. Show that $B := \bigcap_{A \in \mathcal{A}} A$ is convex.

Let's do proof by contradiction. Let $B = \bigcap_{A \in \mathcal{A}} A$. Assume B is not convex. Let $a, b \in B$ so then $\exists\, t \in [0, 1]$ s.t. $z \in (1 - t)a + tb \notin B$. But $z \in A \forall A \in \mathcal{A} \to z \notin B$ so $B \neq \bigcap_{A \in \mathcal{A}} A$. This is clearly a contradiction so B is convex. ∎

Exercise 5.7 Let (X, d) be a metric space and let A and B be disjoint subsets of X. Prove that if A and B are both open in X, then A and B are seperated.

We need to show that $A \cap \overline{B} = B \cap \overline{A} = \emptyset$. So, let's analyze the first statement: $\overline{A} \cap B = (A \cup \text{Lim}_X(A)) \cap B = (A \cap B) \cup (\text{Lim}_X(A) \cap B)$. A and B are disjoint so the only set we need to be concerned with is $\text{Lim}_X(A) \cap B$. Consider the intersection of $\text{Lim}_X(A) \cap \text{Lim}_X(B) = C$. Without loss of generality, choose $x \in C \rightarrow x \in \text{Lim}_X(A)$ and $\text{Lim}_X(B) \not\subset B$ since B is open. So, $x \notin \text{Lim}_X(A) \cap B$. So, $\overline{A} \cap B = \emptyset$. This holds true for the other case as well and so A and B are both seperated. ∎

Exercise 5.8 Let E be a connected subset of a metric space (X, d). Show that $\overline{E}$ is connected.

If E is connected, then $E \subset \text{Lim}_X(E)$. If E is connected, then E has no isolated points. If E had isolated points, then $\exists$ some $x \in E$ s.t. $x \notin \text{Lim}_X(E)$. Thus, $\exists$ some neighbourhood U of x s.t. $U \cap \backslash E\{x\} = \emptyset$. Then, E can be written as the union of two seperated sets $E = E \setminus \{x\} \cup \{x\}$, implying E is not connected which is false. Thus, $\overline{E}$ is connected. ∎

Exercise 5.12 Let (X, d) be a metric space, and let $\mathcal{C}$ be a collection of connected subsets of X. Assume $A = \bigcap_{C \in \mathcal{C}} C$ is nonempty. Show that $B = \bigcup_{C \in \mathcal{C}} C$ is connected.

Let's solve this problem via proof by contrapositive. Let B not be connected so this implies that $B = Z \cup Y$ s.t. $Z \cap \overline{Y} = Y \cap \overline{Z} = \emptyset$. Take connected subset $C_1 \in \mathcal{C}$ s.t. $C_1 \subset B$. By Thm. 5.11, $C_1 \subset Z$ or $C_1 \subset Y$. So $Z \cap Y = \emptyset \rightarrow \bigcap_{C \in \mathcal{C}} C \neq \emptyset$. So B is connected. ∎

Exercise 5.13 Let $X = \mathbb{R}^2$. Give an example of a connected subset E of X, such that $\text{Int}_X(E)$ is not connected. Prove both that your set E is connected and that its interior is not. ((Hint: Consider the union of two convex sets joined at a point. You may assume without proof the fact that convexity implies connectedness in $\mathbb{R}^2$.)

Let A and B be convex sets in $\mathbb{R}^2$ s.t. A is a closed ball with radius 1 centered at $(1, 0)$ and B is a ball with radius 1 centered at $(-1, 0)$. Because we can assume that convexity implies connectedness, we can claim that both A and B are connected. If $\mathcal{C}$ is the collection of all connected subsets of $\mathbb{R}^2$ then by Exer. 5.12 and assuming that $\bigcap_{C \in \mathcal{C}} C \neq \emptyset$, $A \cup B$ is also connected. However, consider the point at $(0, 0)$ and call it x. For any $\epsilon > 0$, $\exists y \in B_{\mathbb{R}^2}(0, \epsilon)$ ($(0, -\epsilon)$ for e.g.) s.t. $y \notin A, B$. So, $x \notin \text{Int}_{\mathbb{R}^2}(A \cup B)$. This leads to $\text{Int}_{\mathbb{R}^2}((A \cup B) \setminus x) = \text{Int}_{\mathbb{R}^2}((A \setminus x \cup B \setminus x)$ which reduces into two seperated sets expressed as a union. Thus, the interior is not connected. ∎