

Exercise 1.7, 3.3, 3.4, 3.6,  
4.5, 4.7, 4.17

**Exercise 1.7.** Let  $A$  and  $B$  be subsets of another  $X$ . Prove the following the following statements.

1.  $A \cap B = A \setminus (A \setminus B)$
2.  $A \subset B$  if and only if  $X \setminus A \supset X \setminus B$ .

Recall the definitions of  $\cup$  and  $\setminus$ .  $A \cup B = \{x : x \in A \text{ and } x \in B\}$ .  $A \setminus B = \{x \in A : x \notin B\}$ .

1. Let  $D = A \setminus B$ .  $D$  is the set of elements in  $A$  that are strictly unique. Let  $E = A \setminus D$ .  $E$  is the relative complement of  $D$  in  $A$ , which only leaves elements common to both  $A$  and  $B$ .
2. (a) Let us prove this  $\rightarrow$  direction first. Given  $A \subset B$ . This means that  $A$  will have a lesser or equal to number of elements in its set than  $B$ . It follows that  $X \setminus A$  will contain all elements of the set  $X \setminus B$ . Thus,  $X \setminus A \supset X \setminus B$ .  
(b) Now the other direction,  $\leftarrow$ . Given  $X \setminus A \supset X \setminus B$ . Assume  $\exists$  some  $x \in A$  and  $x \notin B$ , which means that  $A \not\subset B$ . However,  $x \notin X \setminus A$  and  $x \in X \setminus B$  when we stated  $X \setminus A \supset X \setminus B$ . So, it must be  $\forall x \in A$  must be also be  $x \in B$  so  $A \subset B$ .

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**Exercise 3.3.** Let  $f : A \rightarrow B$  be a function. Prove the following statements.

1.  $f$  is injective if and only if  $f^{-1}(f(C)) = C$  for every subset  $C$  of  $A$ .
2.  $f$  is surjective if and only if  $f^{-1}(f(D)) = D$  for every subset  $D$  of  $B$ .

First, let us list some useful definitions.

If  $G \subset B$  then the inverse image,  $f^{-1}(G)$  of  $G$  under  $f$  is  $f^{-1}(G) = \{x \in A : f(x) \in G\}$ .

If  $f^{-1}(y)$  contains at most one element of  $A$  for each  $y \in B$ , then  $f$  is said to be injective.

If  $f(A) = B$ , we say that  $f$  maps  $A$  onto  $B$ , or that  $f : A \rightarrow B$  is surjective.

1. (a) Let us prove this direction,  $\rightarrow$ . Given  $f$  is injective, let  $C_1$  be some subset of  $A$ .  $f$  maps all elements of  $C_1$  to some set  $B_1 \subset B$ . Applying the definition of the inverse image to this set  $B_1$  under  $f$  yields  $f^{-1}(B_1) = \{x \in A : f(x) \in B_1\}$ . Since we know that  $f$  is injective, we know that the resulting set obtained from the inverse image has to be the original set,  $C_1$ .

- (b) Now the other direction,  $\leftarrow$ . Now the other direction,  $\leftarrow$ . Given  $f^{-1}(f(C)) = C$ . Let us do proof by contradiction. Let  $x_1, x_2$  be elements in  $C$  and assume that  $f(x_1) = f(x_2)$  but  $x_1 \neq x_2$  (this is another way to say  $f$  is not injective). Applying the given fact to a subset of  $C$ ,  $\{x_1\}$ , yields  $f^{-1}(f(\{x_1\})) = \{x \in C : f(x) \in f(C)\} = \{x_1, x_2\}$ . Clearly, this is a contradiction since the set we put into the function and inverse image is not the same set that was returned. This proves that  $f$  has to be injective.
2. (a) Let us look at the  $\rightarrow$  direction first. Given  $f$  is surjective. Let  $C_1 = f^{-1}(D) = \{x \in A : f(x) \in D\}$ . If we apply  $f$  to  $C_1$ , we will obtain our original set  $D$  since  $f$  is surjective.
- (b) Now for the other direction,  $\leftarrow$ . Given  $f(f^{-1}(D)) = D$ . Let us try to argue that  $f$  is not surjective. Let us call  $C_2 = f^{-1}(D)$ . What we mean when we call  $f$  not surjective is  $f(C_2) \neq D$ . But this goes against the given fact so it must be that  $f$  is surjective.

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**Exercise 3.4.** Let  $f : A \rightarrow B$  and  $g : B \rightarrow C$  be functions. Prove the following statements.

1. If  $f$  and  $g$  are both injective, then so is  $g \circ f$ .
2. If  $f$  and  $g$  are both surjective, then so is  $g \circ f$ .
3. If  $g \circ f$  is surjective, then so is  $g$ .
4. Surjectivity of  $g \circ f$  does not imply surjectivity of  $f$ .
5. If  $g \circ f$  is injective, then so is  $f$ .
6. Injectivity of  $g \circ f$  does not imply injectivity of  $g$ .

1. Let  $h = g \circ f = g(f(a))$  for  $a \in A$ . To be injective,  $h(a) = h(b) \implies a = b$ . Substitute for  $h$  and use the fact that  $g$  and  $f$  are injective:  $g(f(a)) = g(f(b)) \implies f(a) = f(b) \implies a = b$ . So,  $f \circ g$  is injective.
2. Let  $h = g \circ f = g(f(a))$  for  $a \in A$ . To be surjective,  $h(A) = C$ . Substitute for  $h$  and use the fact that  $g$  and  $f$  are surjective:  $g(f(A)) \implies g(B)$  since  $f$  is surjective  $\implies g(B) = C$  since  $g$  is surjective. So,  $f \circ g$  is surjective.
3. Let  $h = g \circ f = g(f(a))$  for  $a \in A$ . Restate what  $f(A)$  is and call it  $T$ :  $f(A) = \{f(x) : x \in A\} = T$ . So,  $g(T) = \{g(y) : y \in T\}$ . But since we know that  $h$  is surjective, it must span all  $x \in C$ . This is only possible if  $g$  is surjective.
4. Surjectivity of  $g \circ f$  does not imply surjectivity of  $f$ .
5. If  $g \circ f$  is injective, then so is  $f$ .
6. Injectivity of  $g \circ f$  does not imply injectivity of  $g$ .

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**Exercise 3.6.** Let  $f : X \rightarrow Y$  be a function. Prove the following statements.

1. If  $A$  and  $C$  are subsets of  $X$ , then  $f(C \setminus A) \supset f(C) \setminus f(A)$ .
2.  $f$  is injective if and only if  $f(C \setminus A) = f(C) \setminus f(A)$  for any two subsets  $A$  and  $C$  of  $X$ .
3. If  $B$  and  $D$  are subsets of  $Y$ , then  $f^{-1}(D \setminus B) = f^{-1}(D) \setminus f^{-1}(B)$ .

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**Exercise 4.5** Assume that  $\text{card}(A) \leq \text{card}(X)$  and  $\text{card}(B) \leq \text{card}(Y)$ . Prove that  $\text{card}(B^A) \leq \text{card}(Y^X)$ . Hint: Consider a function  $\Phi(f) = h \circ f \circ k$ , where  $k : X \rightarrow A$  and  $h : B \rightarrow Y$  are certain functions. Theorem 3.9 might be useful for the final step.

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**Exercise 4.7** Prove that for any set  $A$ , one has  $\mathcal{P}(A) \sim \{0, 1\}^A$ .

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**Exercise 4.17** Let  $A$  and  $B$  be sets, and assume  $f : A \rightarrow B$  and  $g : B \rightarrow A$  are injective functions.

1. Assume additionally that  $A$  is finite. Prove that  $f$  and  $g$  must actually be bijections.
2. Show by way of an example that both  $f$  and  $g$  may fail to be bijective if we do not assume that  $A$  is finite.