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Homework #8
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Chapter 5: 3.9, 3.10
Chapter 6: 1.9, 4.2, 4.5

Exercise 3.9 A collection $\mathcal{A}$ of real-valued functions on a set E is said to be *uniformly bounded* on E if there exists $M > 0$ such that $|f(x)| \leq M$ for all $x \in E$, for all $f \in \mathcal{A}$. (So each function is bounded, and the same bound works for all functions in $\mathcal{A}$.) Let (f_n) be a sequence of bounded functions which converges uniformly to a limit function f. Prove that $\{f_n\}$ is uniformly bounded.

(f_n) contains sequence of all bounded functions. By Prop 3.8, $f_n \to f$ uniformly iff $d_u(f, f_n) = \sup|f_n(x) - f(x)|$ as $n \to \infty$. So, choose $n \in \mathbb{N}$ s.t. $\max(|f_1(x) - f(x)|, \ldots, |f_n(x) - f(x)|, \ldots) \forall\, x \in E$. Take this value s.t. $M = |f_n(x) - f(x)| + 1$. By Prop 3.8, this is the largest deviation possible and all other functions will lie in $B(E)$ so they will also be bounded by M. So, $\{f_n\}$ is uniformly bounded. ∎

Exercise 3.10 Let (f_n) and (g_n) be sequences of real-valued functions on a set E, which converge uniformly on E to limit functions f and g, respectively.

1. Prove that $(f_n + g_n)$ converges to $f + g$, uniformly on E.
2. If each f_n and each g_n is bounded, show that $(f_n g_n)$ converges uniformly to fg on E.

1. So, for $(f_n + g_n)$ to converge uniformly, we need to show that $|f_n(x) + g_n(x) - f(x) + g(x)| < \epsilon\, \forall \epsilon > 0$. Apply the trianle inequality so $|f_n(x) + g_n(x) - f(x) + g(x)| \leq |f_n(x) - f(x)| + |g_n(x) - g(x)| < \epsilon_1 + \epsilon_2$, where ϵ_1 is the $\sup|f_n - f|$ and ϵ_2 is the $\sup|g_n - g|$. Since f, g both converge uniformly on E, $f_n + g_n$ is also uniformly converges on E.
2. $(f_n) \leq M$,$(g_n) \leq L$,where $L, M \in \mathbb{R}$, $|g_n(x) - g(x)| < \epsilon_1$, and $|f_n(x) - f(x)| < \epsilon_2$. We need to prove that $|f_n(x)g_n(x) - f(x)g(x)| < \epsilon \forall\, \epsilon > 0$. So, $|f_n(x)g_n(x) - f(x)g(x)| \leq |f_n(x)||g_n(x) - g(x)| + |g(x)||f_n(x) - f(x)| = M\epsilon_1 + L\epsilon_2$. So, $f_n + g_n$ uniformly converges to fg on E.

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Exercise 1.9 Prove the second and third points in Prop 1.8.

1. For $\to$, by definition of limits, for all nbd V of q, $\exists$ a nbd U of $+\infty$ s.t. $x \in U \cap B \setminus \{+\infty\} \neq \emptyset \to g(x) \in V$. So pick M s.t. $(M, +\infty) \subset B$. So, pick $V = B_{\overline{\mathbb{R}}}(q, \epsilon)$ for some ϵ in V. So, $g(x) \in B_{\overline{\mathbb{R}}}(q, \epsilon) \to |g(x) - q| < \epsilon$. For the other direction, for every $\epsilon > 0$, $\exists M \in \mathbb{R}$ s.t. $x > M$ and $x \in B$ together imply that $|g(x) - q| < \epsilon$. $|g(x) - q| < \epsilon \to g(x) \in B_{\overline{\mathbb{R}}}(q, \epsilon)$ for every $\epsilon > 0$. Pick an M in $\mathbb{R}$ and set U in $\mathbb{R}$ as $(M, +\infty)$. So $x \in (M, +\infty)$. So $x \in (M, +\infty)$ and $x \in B$ and $(M, +\infty) \cap B \setminus \{+\infty\} \neq \emptyset$.

2. For every neighborhood V of $+\infty$, $\exists$ a nbd U of $+\infty$ s.t. $x \in U \cap C \setminus \{+\infty\} \neq \emptyset$. Let U be the neighborhood of $+\infty$ for some $P \in \mathbb{R}$, $(P, +\infty)$. So, let $x \in C$ and $x > P$ to be in $(P, +\infty)$. So, let $h(x) \in V$ around $+\infty$. So, for any choice of N, $h(x)$ will always be in V. For $\leftarrow$, let U be a neighborhood of $+\infty$ s.t. $x \in (P, +\infty)$ and $x \in C$. $h(x) > N$ implies that V can be chosen as $(N, +\infty)$. To see if $+\infty$ is limit point, $U \cap C \setminus \{+\infty\} \neq \emptyset$.

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Exercise 4.2 Let $(s_n)_{n=1}^{\infty}$ and $(t_n)_{n=1}^{\infty}$ be sequences in $\overline{\mathbb{R}}$ and let $(u_n)_{n=1}^{\infty}$ be a sequence in $\mathbb{R}$. Prove the following statements.

1. If $s_n \leq t_n$ for each $n \in \mathbb{N}$ and $\lim_{n\to\infty} s_n = +\infty$, then $\lim_{n\to\infty} t_n = +\infty$ as well.
2. If (s_n) and (t_n) converge in $\overline{\mathbb{R}}$ to s and t, respectively, and if $s_n \leq t_n$ for each $n \in \mathbb{N}$, then $s \leq t$.

1. If $\lim_{n\to\infty} s_n = +\infty$, for every $N \in \mathbb{R}$, $\exists P \in \mathbb{R}$ s.t. $x > P$ and $x \in \mathbb{N}$ together impu $(s_n) > N$. This says that for every neighborhood V of $+\infty, \exists$ neighorhood U of $+\infty$ s.t. $U \cap \mathbb{N} \setminus \{+\infty\}$. So, pick V of $+\infty$ s.t. $s_n \in V$ and $\exists U$ of $+\infty$ s.t. $U \cap \mathbb{N} \setminus \{+\infty\} \neq \emptyset$. Since $(t_n) > (s_n) \forall n \in \mathbb{N}, t_n \in V$ and the same neighborhood U will still satisify the limit point condition since t_n maps $\mathbb{N} \to \overline{\mathbb{R}}$.

2. The upper limit of $s(n)$ is defined $\lim_{n\to\infty} \sup s_n = \inf_{n \in \mathbb{N}} (\sup_{k \geq n} s_k)$ and the lower limit of $t(n)$ is defined as $\lim_{n\to\infty} \inf t_n = \sup_{n \in \mathbb{N}} (\inf_{k \geq n} t_k)$. For any choice of n, the upper limit of s_n will always be smaller or equal than the lower limit of t_n since $s_k \leq t_k$. Since both sequences conver to s and t respectively, we can claim $s \leq t$.

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Exercise 4.5 Let $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ be sequences in of real numbers. Prove that

$$\lim_{n\to} \sup(a_n + b_n) \leq \lim_{n\to\infty} \sup(a_n) + \lim_{n\to\infty} \sup(b_n),$$

provided that the RHS isn't of the form $\infty - \infty$.

For all $k \in \mathbb{N}$, set $A_k = \sup\{a_n : n \geq k\}$, $B_k = \sup\{b_n : n \geq k\}$, and $C_k = \sup\{a_n + b_n : n \geq k\}$. Now for a particular K, for all $n \geq k$, we have $a_n + b_n \leq A_k + B_k$. $C_k = \sup(a_n + b_n) \leq A_k + B_k$. So, $sup\{a_n + b_n : n \geq k\} = \sup\{a_n : n \geq k\} + \sup\{b_n : n \geq k\}$. Take limit of both sides of inequality, $\lim_{n\to\infty} sup\{a_n + b_n : n \geq k\} = \lim_{n\to\infty} \sup\{a_n : n \geq k\} + \lim_{n\to\infty} \sup\{b_n : n \geq k\}$. ■