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MA 521
Homework #1
12/16/2018

Problem B,Problem C, Problem D, Rudin 1.1, 1.2, 1.3a

Part A: Problem B We defined a rational number $\frac{m}{n}$ to be an equivalence class of pairs (m,n) under an equivalence relation. Check that the equivalence relation is transitive: if $(p,q) \sim (m,n)$ and $(m,n) \sim (a,b)$, then $(p,q) \sim (a,b)$.

By the definition of the relation, $\sim$, $(p,q) \sim (m,n)$ is equal to $pn = mq$. We then have two relations, $(p,q) \sim (m,n)$ and $(m,n) \sim (a,b)$. These have the following representations:

$$(pn = mq) \wedge (mb = an)$$

$$pnmb = mqan$$
$$pb = qa$$

which reduces to $(p,q) \sim (a,b)$. ∎

Part A: Problem C We defined addition of rational numbers in terms of representatives: $\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}$. Show that the addition of rational numbers is well-defined.

Let $(a,b) \sim (a_1,b_1)$ and $(c,d) \sim (c_1,d_1)$. Recall from the definition of the equivalence relation that it implies $ab_1 = a_1b$ and $cd_1 = c_1d$. To prove the property of addition holds, we need to show that, even if we choose different representatives, the equivalence relation is still valid. More explictly, $(a,b) + (c,d) \sim (a_1,b_1) + (c_1,d_1)$. After applying the definition of addition, we obtain:

$$(ad + bc, bd) \sim (a_1d_1 + b_1c_1, b_1d_1).$$

This implies that $(ad + bc)b_1d_1 = (a_1d_1 + b_1c_1)bd$. Let us focus on the left hand side of this equality:

$$(ad + bc)b_1d_1 = adb_1d_1 + bcb_1d_1$$
$$adb_1d_1 + bcb_1d_1 = d(ab_1)d_1 + b(cd_1)b_1.$$

Recall the initial equivalence we defined at the beginning with $ab_1 = a_1b$ and $cd_1 = c_1d$. Using these as necessary,

$$d(ab_1)d_1 + b(cd_1)b_1 = d(a_1b)d_1 + b(c_1d)b_1$$
$$d(a_1b)d_1 + b(c_1d)b_1 = bd(a_1d_1 + b_1c_1).$$

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Part B: Problem D Define a multiplication of rational numbers (corresponding to the one you are used to), and show this multiplication is well-defined.

Definition of multiplication in rationals: if $\frac{a}{b}$ and $\frac{c}{d}$ are rational numbers with $b, d \neq 0$,

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}.$$

Let $(a, b) \sim (a_1, b_1)$ and $(c, d) \sim (c_1, d_1)$. Recall from the definition of the equivalence relation that it implies $ab_1 = a_1 b$ and $cd_1 = c_1 d$. Again, we need to prove if different representatives will change the equivalence relations: $(a, b) \cdot (c, d) \sim (a_1, b_1) \cdot (c, d)$. Let us apply our definition of multiplication from above to get $(ac, bd) \sim (a_1 c_1, b_1 d_1)$.

This implies $acb_1 d_1 = a_1 c_1 bd$. Let's focus our attention on the left hand side. Recall the initial equivalence we defined at the beginning with $ab_1 = a_1 b$ and $cd_1 = c_1 d$. Using these as necessary, $(ab_1)(cd_1) = a_1 bc_1 d = a_1 c_1 bd$. Thus, the equivalence is still valid. ∎

Part B: Rudin 1.1 If r is rational $(r \neq 0)$ and x is irrational, prove that $r + x$ and rx are irrational.

Let us solve by proof of contridiction. Suppose $r + x$ and rx is rational. Since r is rational, $-r$ and $\frac{1}{r}$ are also rational. Thus, $(r + x) - r = x$ which implies x is rational by property of addition of rationals. Similarly, $rx \cdot (\frac{1}{r}) = x$ which suggests x is also rational by definition of multiplication of rationals. These are both clearly contridictions. Thus, $r + x$ and rx are irrational. ∎

Part C: Rudin 1.2 Prove that there is no rational number whose square is 12.

Let us solve by proof of contridiction. If there was a x such that $x^2 = 12$, we can write $x = \frac{m}{n}$ where m and n are not both multiples of 3. Then $x^2 = 12$ implies that

$$m^2 = 12n^2.$$

This shows that 3 divides m^2, and hence, that 3 divides m, so that 9 divides m^2. It then follows that n^2 is divisible by 3, so that n is a multiple of 3. This clearly shows a contridiction.

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Part D: Rudin 1.3a Prove: If $x \neq 0$ and $xy = xz$ then $y = z$.

If $xy = xz$ and $x \neq 0$, then $y = (1)y = \frac{1}{x}(xy) = \frac{1}{x}(xz) = (1)z = z$. ∎