

Chapter 1: 4.18, 4.19, 4.22;
Chapter 2: 1.6, 1.7, 2.3, 2.4, 2.5, 3.3, 4.6, 4.8, 4.9, 4.10

Exercise 4.18. Let A and B be sets. Assume A is infinite, B is countable, and A and B are disjoint. Prove $A \sim A \cup B$. Hint: The strategy of Theorem 4.16 may be useful.

If A is infinite, we have $C \subset A$, a countably infinite set. By Proposition 4.21, the union of two countable sets is still countable, $B \cup C$, which is countably infinite. Since $((A \cup B) \setminus B \cup C) \cap C$ and $((A \cup B) \setminus B \cup C) \cap (B \cup C)$ are both empty, $A \cup B = ((A \cup B) \setminus B \cup C) \cup (B \cup C) \sim ((A \cup B) \setminus B \cup C) \cup C = A$. ■

Exercise 4.19. Let X and Y be sets. Assume Y is countable and $X \setminus Y$ is infinite. Prove that $X \sim X \cup Y \sim X \setminus Y$. Hint: Each of the equivalences can be done extremely quickly if you use the previous exercise and some set manipulations.

If $X \setminus Y$ is infinite, $X \setminus Y$ must have a countably infinite subset. This means that X must be infinite. We can use Exercise 4.18 but we need to prove that X and Y are disjoint sets. Let's solve by contradiction.

Assume $a_1 \in X, Y$ s.t. $X \cap Y = \{a_1\}$. This means that $X \setminus Y$ will be a proper subset of X . We can apply Theorem 4.16 to say $X \sim X \setminus Y$. But then, $X \cap Y \sim (X \cap Y) \cap Y = \emptyset$, which is a contradiction. This suggests that X and Y are disjoint and apply Exercise 4.18 directly to say $X \sim X \cup Y \sim X \setminus Y$. ■

Exercise 4.22. Let X be a countable set.

1. Prove inductively that $X^n \sim X^{n-1} \times X$ for any $n \in \mathbb{N}$.
2. Prove inductively that X^n is countable for any $n \in \mathbb{N}$.

1. WLOG, let $n = 2$. For the base case, by definition of n -tuples, $X^2 = X \times X \sim X^1 \times X$. For the inductive step, assume statement is true for n , $X^{n+1} = (X \times X \times \dots) \times X = X^n \times X \sim X^{(n+1)-1} \times X$.
2. WLOG, let $n = 2$. $X^2 = X \times X = \{(a, b) : a \in X \text{ and } b \in X\}$. If $X \cup X$ is countable by Proposition 4.21, then $X \times X$ should also be countable. For the inductive step, let $n = k + 1$ assume X^k is countable. $X^{k+1} = X \times X^k \implies X$ is countable and X^k is countable by assumption so by Proposition 4.21, X^{k+1} should be countable. ■

Exercise 1.6. Let E, F , and G be nonempty subsets of an ordered set (S, \leq) . Prove the following statements.

1. If $\alpha \in S$ is a lower bound for E and $\beta \in S$ is an upper bound for E , then $\alpha \leq \beta$.
2. $\sup E \leq \inf F$ if and only if $x \leq y$ for any $x \in E, y \in F$.
3. If $E \subset G$, then $\sup E \leq \sup G$.

1. By definition of upper bound, $\forall x \in E : x \leq \beta$. By definition of lower bound, $\forall x \in E : x \geq \alpha$. So, $\alpha \leq x \leq \beta \implies \alpha \leq E \leq \beta \implies \alpha \leq \beta$.
2. (a) Let us prove this \rightarrow direction first. Given $\sup E \leq \inf F$. Let's solve by contradiction. Assume $x > y$ for any $x \in E, y \in F$. Say $\beta_1 = \sup E$, implying β_1 is an upper bound for E . So by definition, $x < \beta_1 \forall x \in E$. Say $\alpha_1 = \inf F$, implying α_1 is a lower bound for F . So by definition, $\alpha_1 \leq y \forall y \in F$. By the given statement, $\beta_1 \leq \alpha_1 \implies x \leq \beta_1 \leq \alpha_1 \leq y$. This establishes a contradiction so $x \leq y$.
 (b) Now the other direction, \leftarrow . Given $x \leq y$ for any $x \in E, y \in F$. Let β_2 be the upper bound for E . Let α_2 be the upper bound for F . $x \leq \beta_2 \leq \alpha_2 \leq y$; the tightest bounds for this expression would be if $\beta_2 = \sup E$ and $\alpha_2 = \inf F$. $x \leq \sup E \leq \inf F \leq y \implies \sup E \leq \inf F$.
3. Let $a = \sup G$ and $b = \sup E$. Assume $b > a$. If b is larger than a , a could not be the upper bound of G since $E \subset G$. So, this establishes a contradiction and $\sup E \leq \sup G$. ■

Exercise 1.7. Let (S, \leq) be an ordered set, let f and g be functions from X to S and let A be a subset of X . Assume that $f(x) \leq g(x)$ for all $x \in A$, and that furthermore $\sup_A f$ and $\sup_A g$ exist in S . Prove that $\sup_A f \leq \sup_A g$.

Given $\sup_A f = \sup\{f(x) : x \in A\} = \beta, \sup_A g = \sup\{g(x) : x \in A\} = \alpha$, and $f(x) \leq g(x) \forall x \in A$. Clearly, since β is an upper bound for f , $f(x) \leq \beta \leq g(x) \forall x \in A$. Since α is an upper bound for g , $f(x) \leq \beta \leq g(x) \leq \alpha \forall x \in A \implies \beta \leq \alpha = \sup_A g \leq \sup_A f$. ■

Exercise 2.3. Let A be a nonempty subset of an ordered field $(F, +, \cdot, \leq)$. Assume that $\sup A$ and $\inf A$ exist in F , and let c be any element of F . Define the set $cA := \{ca : a \in A\}$.

1. Prove that $c \geq 0$, then $\sup(cA) = c \sup A$.
2. What is $\sup(cA)$ if $c \leq 0$? Prove your answer is correct.

1. WLOG, let $c > 0$. Let B_1 be an upper bound for A . $\sup cA = \sup(\{ca : a \in A\}) = C_1 = cB_1 = c \sup A$.
2. Prove $\sup(cA) = c \inf(A)$. Let $\inf A = C_2$ and $cC_2 = B_2$. So, $\{B_2 \geq ca : a \in A\}$ since A is an ordered field. $\{ca : a \in A\} \leq B_2 \implies$ tightest upper bound is $\sup(cA)$. ■

Exercise 2.4 Let A be a nonempty subset of an ordered field $(F, +, \cdot, \leq)$. Assume that $\sup A$ and $\inf A$ exist in F . Define $A + B := \{a + b : a \in A, b \in B\}$. Prove that $\sup(A + B) = \sup A + \sup B$ by filling in the details of the following outline:

- Denote $s = \sup A, t = \sup B$. Then $s + t$ is an upper bound for $A + B$.
- Let u be any upper bound for $A + B$, and let a be any element of A . Then $t \leq u - a$.
- We have $s + t \leq u$. Consequently, $\sup(A + B)$ exists in F and is equal to $s + t = \sup A + \sup B$.

Let $s = \sup A, t = \sup B$. By definition of supremum, no element in $A + B$ is greater than $s + t$ so it must be an upper bound. Let u be any upper bound for $A + B$, and let a be any element of A . Then $t \leq u - a$. Let's choose $u = s + t + 1$ and plugging that into the later expression yields $t \leq s + t + 1 - a \implies a - 1 \leq s$, which will always be true since s is an upper bound on A . If u is an upper bound on $A + B$, $\sup(A + B)$ is the tightest bound which is $s + t$ so $\sup(A + B) = \sup A + \sup B$. ■

Exercise 2.5. Let f and g be functions from a set X to an ordered field $(F, +, \cdot, \leq)$. Let A be a subset of X .

- Prove that the following inequality holds, assuming the relevant suprema all exist.

$$(*) \sup_{x \in A} (f(x) + g(x)) \leq \sup_{x \in A} f(x) + \sup_{x \in A} g(x).$$

- Show by way of an example that equality might not hold in $(*)$, even if the suprema all exist. (Hint: This is probably easiest if you choose X to be a set with two elements, and $F = \mathbb{Q}$.)

- $\forall x_0 \in A, f(x_0) + g(x_0) \leq f(x_0) + g(x_0). \forall x_0, \exists x_1, x_2 \in A : f(x_0) + g(x_0) \leq f(x_1) + g(x_2)$. Let $f(x_1) = \sup_{x \in A} f(x)$ and $g(x_2) = \sup_{x \in A} g(x)$. $\sup_X (f(x) + g(x)) \leq \sup_X f(x) + \sup_X g(x)$.
- Let $X = \{a, b\}$, $f : a \rightarrow 4, b \rightarrow 5$, and $g : a \rightarrow -1, b \rightarrow -2$. Clearly, $\sup f = 5$ and $\sup g = -1$ but $\sup f(x) + g(x) = \sup\{3, 3\} = 3$. This proves that equality doesn't hold.

Exercise 3.3. Using the strategies similar to those proofs in this section, prove the following statements.

1. There is no rational whose square is 20.
 2. The set $A := \{r \in \mathbb{Q} : r^2 < 20\}$ has no least upper bound in \mathbb{Q} .
1. Assume p is a rational number s.t. $p^2 = 20$. Since $p \in \mathbb{Q}$, we can write $p = \frac{m}{n}$, where m and n are integers with no common factors. So, $p^2 = 20 \rightarrow m^2 = 20n^2$. This shows that 5 divides m^2 , and hence, that 5 divides m , so that 25 divides m^2 . It then follows that n^2 is divisible by 5, so that n is a multiple of 5. This is clearly a contradiction.

2. First, we want to break the proof into two steps:

- (a) $p \in \mathbb{Q}$ is an upper bound for A if and only if $p^2 > 20$ and $p \geq 0$.
- (b) If $p^2 > 20$ and $p > 0$, then there exists $q \in \mathbb{Q}$ such that $0 \leq q \leq p$ and $q^2 > 20$.

If p is not an upper bound for A , then $\exists r \in A$ s.t. $r > p$. But then $20 > r^2 > rp > p^2$, which contradicts initial definition of p in (a). So now we prove $p^2 > 20$. So if $0 \leq p^2 \leq 20$, then $p^2 < 20$. This implies $q = p + \frac{20-p^2}{p+20}$. So, $q \in A > p$ because $20 - p^2$ is positive. To see this, we need to prove that $20 - q^2 > 0$.

$$q = p \frac{(p+20)}{(p+20)} + \frac{20-p^2}{p+20} = \frac{20p+20}{p+20},$$

so

$$\begin{aligned} 20 - q^2 &= 20 \left(\frac{(p+20)^2}{(p+20)^2} \right) - \left(\frac{(20p+20)^2}{(p+20)^2} \right) = \frac{20(p^2 + 40p + 400) - 400p^2 + 800p + 400}{(p+20)^2} \\ &= \frac{-380p^2 + 7600}{(p+20)^2} = \frac{380(20 - p^2)}{(p+20)^2} > 0. \end{aligned}$$

So, $q^2 < 20$ meaning that p is not an upper bound for A . This leaves that the upper bounds of A in \mathbb{Q} are the numbers $p \in \mathbb{Q}$ such that $p^2 > 20$ and $p \geq 0$.

Now, for last part, we need to prove that p is not the least upper bound for A in \mathbb{Q} .

So, recall that $q = p - \frac{p^2-20}{p+20}$ is an upper bound for A which is less than p . This is because of the positivity of $p^2 - 20$. $q \geq 0$ follows from

$$\frac{p^2 - 20}{p + 20} \leq \frac{p^2 + 20p}{p + 20} = p.$$

This leads us to the conclusion

$$q^2 - 20 = \frac{380(p^2 - 20)}{(p+20)^2} > 0.$$

So $q^2 > 20$. Thus, q is an upper bound for A which is less than p ; that is, p is not the least upper bound for A in \mathbb{Q} . But, p is just an arbitrary upper bound so there is no least upper bound for A in \mathbb{Q} . So it follows that A doesn't have the least upper bound property in \mathbb{Q} and it follows that \mathbb{Q} doesn't have the least upper bound property. ■

Exercise 4.6. Elements of $\mathbb{R} \setminus \mathbb{Q}$ are called *irrational numbers*.

1. Assume r is rational and x is irrational. Show that $r + x$ is irrational. Show that rx is irrational unless $r = 0$.
2. Use the Archimedean property of \mathbb{R} to prove that the set of irrational numbers is dense in \mathbb{R} . (Hint: First prove if x and y are real numbers with $y - x > \sqrt{2}$, then there exists an integer m such that $x < m\sqrt{2} < y$.)

1. Let us solve by proof of contradiction. Suppose $r + x$ and rx is rational. Since r is rational, $-r$ and $\frac{1}{r}$ are also rational for $r \neq 0$. Thus, $(r + x) - r = x$ which implies x is rational. Similarly, $rx \cdot (\frac{1}{r}) = x$ which suggests x is also rational. These are both clearly contradictions. Thus, $r + x$ and rx are irrational.
2. We need to prove that if $y - x > \sqrt{2}$, \exists an integer m s.t. $x < m\sqrt{2} < y$. Let m be the smallest positive integer such that $m\sqrt{2} > nx$. $x < \frac{m\sqrt{2}}{n} < y$; since $nx < m\sqrt{2}$ by definition of m , $(m - 1)\sqrt{2} < nx$. On the other hand, $nx < ny - \sqrt{2}$ so $(m - 1)\sqrt{2} < nx < ny - \sqrt{2}$. This reduces to $\sqrt{2}m < nx < ny \implies \sqrt{2}m < ny$. This finishes the proof. ■

Exercise 4.8. Assume $a, b \in \mathbb{R}$. Prove that $a \leq b$ if and only if $a \leq b + \epsilon$ for every $\epsilon > 0$. ■

Exercise 4.9. Let E be a set of real numbers, let s be an upper bound for E . Prove that $s = \sup E$ if and only if for every $\epsilon > 0$ there exists $x \in E$ such that $x > s - \epsilon$. ■

Exercise 4.10. Let A and B be nonempty sets of real numbers. Decide whether the following statements are true or false. If true, give a proof; if false, give a counterexample.

1. If $\sup A < \inf B$, then there exists a $c \in \mathbb{R}$ satisfying $a < c < b$ for all $a \in A$ and $b \in B$.
2. If there exists a $c \in \mathbb{R}$ satisfying $a < c < b$ for all $a \in A$ and $b \in B$, then $\sup A < \inf B$. ■