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MA 521
Homework #1
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Exercise 1.7, 3.3, 3.4, 3.6, 4.5, 4.7, 4.17

Exercise 1.7. Let A and B be subsets of another X. Prove the following the following statements.

1. $A \cap B = A \setminus (A \setminus B)$
2. $A \subset B$ if and only if $X \setminus A \supset X \setminus B$.

Recall the definitions of $\cup$ and $\setminus$. $A \cup B = \{x : x \in A \text{ and } x \in B\}$. $A \setminus B = \{x \in A : x \notin B\}$.

1. Let $D = A \setminus B$. D is the set of elements in A that are strictly unique. Let $E = A \setminus D$. E is the relative complement of D in A, which only leaves elements common to both A and B.
2. (a) Let us prove this $\rightarrow$ direction first. Given $A \subset B$. This means that A will have a lesser or equal to number of elements in its set than B. It follows that $X \setminus A$ will contain all elements of the set $X \setminus B$. Thus, $X \setminus A \supset X \setminus B$.
   (b) Now the other direction, $\leftarrow$. Given $X \setminus A \supset X \setminus B$. Assume $\exists$ some $x \in A$ and $x \notin B$, which means that $A \not\subset B$. However, $x \notin X \setminus A$ and $x \in X \setminus B$ when we stated $X \setminus A \supset X \setminus B$. So, it must be $\forall x \in A$ must be also be $x \in B$ so $A \subset B$.

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Exercise 3.3. Let $f : A \leftarrow B$ be a function. Prove the following statements.

1. f is injective if and only if $f^{-1}(f(C)) = C$ for every subset C of A.
2. f is surjective if and only if $f^{-1}(f(D)) = D$ for every subset D of B.

First, let us list some useful definitions.
If $G \subset B$ then the inverse image, $f^{-1}(G)$ of G under f is $f^{-1}(G) = \{x \in A : f(x) \in G\}$.
If $f^{-1}(y)$ contains at most one element of A for each $y \in B$, then f is said to be injective.
If $f(A) = B$, we say that f maps A onto B, or that $f : A \rightarrow B$ is surjective.

1. (a) Let us prove this direction, $\rightarrow$. Given f is injective, let C_1 be some subset of A. f maps all elements of C_1 to some set $B_1 \subset B$. Applying the definition of the inverse image to this set B_1 under f yields $f^{-1}(B_1) = \{x \in A : f(x) \in B_1\}$. Since we know that f is injective, we know that the resulting set obtained from the inverse image has to be the original set, C_1.

(b) Now the other direction, $\leftarrow$. Now the other direction, $\leftarrow$. Given $f^{-1}(f(C)) = C$. Let us do proof by contradiction. Let x_1, x_2 be elements in C and assume that $f(x_1) = f(x_2)$ but $x_1 \neq x_2$ (this is another way to say f is not injective). Applying the given fact to a subset of C, $\{x_1\}$, yields $f^{-1}(f(\{x_1\})) = \{x \in C : f(x) \in f(C)\} = \{x_1, x_2\}$. Clearly, this is a contradiction since the set we put into the function and inverse image is not the same set that was returned. This proves that f has to be injective.

2. (a) Let us look at the $\rightarrow$ direction first. Given f is surjective. Let $C_1 = f^{-1}(D) = \{x \in A : f(x) \in D\}$. If we apply f to C_1, we will obtain our original set D since f is surjective.

   (b) Now for the other direction, $\leftarrow$. Given $f(f^{-1}(D)) = D$. Let us try to argue that f is not surjective. Let us call $C_2 = f^{-1}(D)$. What we mean when we call f not surjective is $f(C_2) \neq D$. But this goes against the given fact so it must be that f is surjective.

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Exercise 3.4. Let $f : A \leftarrow B$ be a function. Prove the following statements.

1. f is injective if and only if $f^{-1}(f(C)) = C$ for every subset C of A.
2. f is surjective if and only if $f^{-1}(f(D)) = D$ for every subset D of B.