Ravi Raju
MA 521
Homework #2
2/6/2019

Chapter 1: 4.18, 4.19, 4.22;
Chapter 2: 1.6, 1.7, 2.3, 2.4, 2.5, 3.3, 4.6, 4.8, 4.9, 4.10

Exercise 4.18. Let A and B be sets. Assume A is infinite, B is countable, and A and B are disjoint. Prove $A \sim A \cup B$. Hint: The strategy of Theorem 4.16 may be useful.

If A is infinite, we have $C \subset A$, a countably infinite set. By Proposition 4.21, the union of two countable sets is still countable, $B \cup C$, which is countably infinite. Since $((A \cup B) \setminus B \cup C) \cap C$ and $((A \cup B) \setminus B \cup C) \cap (B \cup C)$ are both empty, $A \cup B = ((A \cup B) \setminus B \cup C) \cup (B \cup C) \sim ((A \cup B) \setminus B \cup C) \cup C = A$. ■

Exercise 4.19. Let X and Y be sets. Assume Y is countable and $X \setminus Y$ is infinite. Prove that $X \sim X \cup Y \sim X \setminus Y$. Hint: Each of the equivalences can be done extremely quickly if you use the previous exercise and some set manipulations.

If $X \setminus Y$ is infinite, $X \setminus Y$ must have a countably infinite subset. This means that X must be infinite. We can use Exercise 4.18 but we need to prove that X and Y are disjoint sets. Let's solve by contradiction.

Assume $a_1 \in X, Y$ s.t. $X \cap Y = \{a_1\}$. This means that $X \setminus Y$ will be a proper subset of X. We can apply Theorem 4.16 to say $X \sim X \setminus Y$. But then, $X \cap Y \sim (X \cap Y) \cap Y = \emptyset$, which is a contradicts our assumption. This suggest that X and Y are disjoint and apply Exercise 4.18 directly to say $X \sim X \cup Y \sim X \setminus Y$. ■

Exercise 4.22. Let X be a countable set.

1. Prove inductively that $X^n \sim X^{n-1} \times X$ for any $n \in \mathbb{N}$.
2. Prove inductively that X^n is countable for any $n \in \mathbb{N}$.

1. WLOG, let $n = 2$. For the base case, by definition of n-tuples, $X^2 = X \times X \sim X^1 \times X$. For the inductive step, assume statement is true for n, $X^{n+1} = (X \times X \times \dots) \times X = X^n \times X = X^{(n+1)-1} \times X$.
2. WLOG, let $n = 2$. $X^2 = X \times X = \{(a, b) : a \in X \text{ and } b \in X\}$. If $X \cup X$ is countable by Proposition 4.21, then $X \times X$ should also be countable. For the inductive step, let $n = k + 1$ assume X^k is countable. $X^{k+1} = X \times X^k \implies X$ is countable and X^k is countable by assumption so by Proposition 4.21, X^{k+1} should be countable.

■

Exercise 1.6. Let E, F, and G be nonempty subsets of an ordered set $(S, \leq)$. Prove the following statements.

1. If $\alpha \in S$ is a lower bound for E and $\beta \in S$ is an upper bound for E, then $\alpha \leq \beta$.
2. $\sup E \leq \inf F$ if and only if $x \leq y$ for any $x \in E, y \in F$.
3. If $E \subset G$, then $\sup E \leq \sup G$.

1. By definition of upper bound, $\forall x \in E : x \leq \beta$. By definition of lower bound, $\forall x \in E : x \geq \alpha$. So, $\alpha \leq x \leq \beta \implies \alpha \leq E \leq \beta \implies \alpha \leq \beta$.
2. (a) Let us prove this $\rightarrow$ direction first. Given $\sup E \leq \inf F$. Let's solve by contradiciton. Assume $x > y$ for any $x \in E, y \in F$. Say $\beta_1 = \sup E$, implying β_1 is an upper bound for E. So by definition, $x < \beta_1 \, \forall x \in E$. Say $\alpha_1 = \inf F$, implying α_1 is a lower bound for F. So by definition, $\alpha_1 \leq y \, \forall y \in F$. By the given statement, $\beta_1 \leq \alpha_1 \implies x \leq \beta_1 \leq \alpha_1 \leq y$. This establishes a contradiction so $x \leq y$.
   (b) Now the other direction, $\leftarrow$. Given $x \leq y$ for any $x \in E, y \in F$. Let β_2 be the upper bound for E. Let α_2 be the upper bound for F. $x \leq \beta_2 \leq \alpha_2 \leq y$; the tightest bounds for this expression would be if $\beta_2 = \sup E$ and $\alpha_2 = \inf F$. $x \leq \sup E \leq \inf F \leq y \implies \sup E \leq \inf F$.
3. Let $a = \sup G$ and $b = \sup E$. Assume $b > a$. If b is larger than a, a could not be the upper bound of G since $E \subset G$. So, this establishes a contradiction and $\sup E \leq \sup G$.

■

Exercise 1.7. Let $(S, \leq)$ be an ordered set, let f and g be functions from X to S and let A be a subset of X. Assume that $f(x) \leq g(x)$ for all $x \in A$, and that furthermore $\sup_A f$ and $\sup_A g$ exist in S. Prove that $\sup_A f \leq \sup_A g$.

Given $\sup_A f = \sup\{f(x) : x \in A\} = \beta, \sup_A g = \sup\{g(x) : x \in A\} = \alpha$, and $f(x) \leq g(x) \, \forall x \in A$. Clearly, since β is an upper bound for f, $f(x) \leq \beta \leq g(x) \forall x \in A$. Since α is an upper bound for g, $f(x) \leq \beta \leq g(x) \leq \alpha \, \forall x \in A \implies \beta \leq \alpha = \sup_A f \leq \sup_A g$. ■

Exercise 2.3. Let A be a nonempty subset of an ordered field $(F, +, \cdot, \leq)$. Assume that $\sup A$ and $\inf A$ exist in F, and let c be any element of F. Define the set $cA := \{ca : a \in A\}$.

1. Prove that $c \geq 0$, then $\sup(cA) = c \sup A$.
2. What is $\sup(cA)$ if $c \leq 0$? Prove your answer is correct.

1. WLOG, let $c > 0$. Let B_1 be an upper bound for A. $\sup cA = \sup(\{ca : a \in A\}) = C_1 = cB_1 = c \sup A$.
2. Prove $\sup(cA) = c \inf(A)$. Let $\inf A = C_2$ and $cC_2 = B_2$. So, $\{B_2 \geq ca : a \in A\}$ since A is an ordered field. $\{ca : a \in A\} \leq B_2 \implies$ tightest upper bound is $\sup(cA)$.

■

Exercise 2.4 Let A be a nonempty subset of an ordered field $(F, +, \cdot, \leq)$. Assume that $\sup A$ and $\inf A$ exist in F. Define $A + B := \{a + b : a \in A, b \in B\}$. Prove that $\sup(A + B) = \sup A + \sup B$ by filling in the details of the following outline:

- Denote $s = \sup A, t = \sup B$. Then $s + t$ is an upper bound for $A + B$.
- Let u be any upper bound for $A + B$, and let a be any element of A. Then $t \leq u - a$.
- We have $s + t \leq u$. Consequently, $\sup(A + B)$ exists in F and is equal to $s + t = \sup A + \sup B$.

Let $s = \sup A, t = \sup B$. By definition of supremum, no element in $A + B$ is greater than $s + t$ so it must be an upper bound. Let u be any upper bound for $A + B$, and let a be any element of A. Then $t \leq u - a$. Let's choose $u = s + t + 1$ and plugging that into the later expression yields $t \leq s + t + 1 - a \implies a - 1 \leq s$, which will always be true since s is an upper bound on A. If u is an upper bound on $A + B$, $\sup(A + B)$ is the tightest bound which is $s + t$ so $\sup(A + B) = \sup A + \sup B$. ∎

Exercise 2.5. Let f and g be functions from a set X to an ordered field $(F, +, \cdot, \leq)$. Let A be a subset of X.

- Prove that the following inequality holds, assuming the relevant suprema all exist.

$$(*) \sup_{x \in A}(f(x) + g(x)) \leq \sup_{x \in A} f(x) + \sup_{x \in A} g(x).$$

- Show by way of an example that equality might not hold in $(*)$, even if the suprema all exist. (Hint: This is probably easiest if you choose X to be a set with two elements, and $F = \mathbb{Q}$.)

- $\forall x_0 \in A, f(x_0) + g(x_0) \leq f(x_0) + g(x_0). \forall x_0, \exists x_1, x_2 \in A : f(x_0) + g(x_0) \leq f(x_1) + g(x_2)$. Let $f(x_1) = \sup_{x \in A} f(x)$ and $g(x_2) = \sup_{x \in A} g(x)$. $\sup_X(f(x) + g(x) \leq \sup_X f(x) + \sup_X g(x)$.
- Let $X = \{a, b\}$, $f : a \to 4, b \to 5$, and $g : a \to -1, b \to -2$. Clearly, $\sup f = 5$ and $\sup g = -1$ but $\sup f(x) + g(x) = \sup\{3, 3\} = 3$. This proves that equality doesn't hold.

∎

Exercise 3.3. Using the strategies similar to those proofs in this section, prove the following statements.

1. There is no rational whose square is 20.
2. The set $A := \{r \in \mathbb{Q} : r^2 < 20\}$ has no least upper bound in $\mathbb{Q}$.

1. Assume p is a rational number s.t. $p^2 = 20$. Since $p \in \mathbb{Q}$, we can write $p = \frac{m}{n}$, where m and n are integers with no common factors. So, $p^2 = 20 \to m^2 = 20n^2$. This shows that 5 divides m^2, and hence, that 5 divides m, so that 25 divides m^2. It then follows that n^2 is divisible by 5, so that n is a multiple of 5. This is clearly a contradiction.

2. First, we want to break the proof into two steps:

   (a) $p \in \mathbb{Q}$ is an upper bound for A if and only if $p^2 > 20$ and $p \geq 0$.
   (b) If $p^2 > 20$ and $p > 0$, then there exists $q \in \mathbb{Q}$ such that $0 \leq q \leq p$ and $q^2 > 20$.

   If p is not an upper bound for A, then $\exists r \in A$ s.t. $r > p$. But then $20 > r^2 > rp > p^2$, which contradicts initial definition of p in (a). So now we prove $p^2 > 20$. So if $0 \leq p^2 \leq 20$, then $p^2 < 20$. This implies $q = p + \frac{20-p^2}{p+20}$. So, $q \in A > p$ because $20 - p^2$ is positive. To see this, we need to prove that $20 - q^2 > 0$.

$$q = p\frac{(p+20)}{(p+20)} + \frac{20-p^2}{p+20} = \frac{20p+20}{p+20},$$

   so

$$20 - q^2 = 20(\frac{(p+20)^2}{(p+20)^2}) - (\frac{(20p+20)^2}{(p+20)^2}) = \frac{20(p^2+40p+400) - 400p^2 + 800p + 400}{(p+20)^2}$$

$$= \frac{-380p^2 + 7600}{(p+20)^2} = \frac{380(20-p^2)}{(p+20)^2} > 0.$$

   So, $q^2 < 20$ meaning that p is not an upper bound for A. This leaves that the upper bounds of A in $\mathbb{Q}$ are the numbers $p \in \mathbb{Q}$ such that $p^2 > 20$ and $p \geq 0$.
   Now, for last part, we need to prove that p is not the least upper bound for A in $\mathbb{Q}$. So, recall that $q = p - \frac{p^2-20}{p+20}$ is an upper bound for A which is less than p. This is becuase of the positivity of $p^2 - 20$. $q \geq 0$ follows from

$$\frac{p^2-20}{p+20} \leq \frac{p^2+20p}{p+20} = p.$$

   This leads us to the conclusion

$$q^2 - 2 = \frac{380(p^2-20)}{(p+20)^2} > 0.$$

   So $q^2 > 20$. Thus, q is an upper bound for A which is less than p; that is, p is not the least upper bound for A in $\mathbb{Q}$. But, p is just an arbitrary upper bound so there is no least upper bound for A in $\mathbb{Q}$. So it follows that A doesn't have the least upper bound property in $\mathbb{Q}$ and it follows that $\mathbb{Q}$ doesn't have the least upper bound property.

■

Exercise 4.6. Elements of $\mathbb{R} \setminus \mathbb{Q}$ are called *irrational numbers*.

1. Assume r is rational and x is irrational. Show that $r + x$ is irrational. Show that rx is irrational unless $r = 0$.
2. Use the Archimedean property of $\mathbb{R}$ to prove that the set of irrational numbers is dense in $\mathbb{R}$. (Hint: First prove if x and y are real numbers with $y - x > \sqrt{2}$, then there exists an integer m such that $x < m\sqrt{2} < y$.)

1. Let us solve by proof of contridiction. Suppose $r + x$ and rx is rational. Since r is rational, $-r$ and $\frac{1}{r}$ are also rational for $r \neq 0$. Thus, $(r + x) - r = x$ which implies x is rational. Similarly, $rx \cdot (\frac{1}{r}) = x$ which suggests x is also rational. These are both clearly contridictions. Thus, $r + x$ and rx are irrational.
2. We need to prove that if $y - x > \sqrt{2}, \exists$ an integer m s.t. $x < m\sqrt{2} < y$. Let m be the smallest positive integer such that $m\sqrt{2} > nx, n \in \mathbb{N}$. $x < \frac{m\sqrt{2}}{n} < y$; since $nx < m\sqrt{2}$ by definition of m, $(m-1)\sqrt{2} < nx$. On the other hand, $nx < ny - \sqrt{2}$ so $(m-1)\sqrt{2} < nx < ny - \sqrt{2}$. This reduces to $\sqrt{2}m < nx < ny \implies \sqrt{2}m < ny$. This finishes the proof.

■

Exercise 4.8. Assume $a, b \in \mathbb{R}$. Prove that $a \leq b$ if and only if $a \leq b + \epsilon$ for every $\epsilon > 0$.

1. Let us prove this $\rightarrow$ direction first. Given $a \leq b$. By the definition of a well-ordered field, it's obvious to see that for any $\epsilon > 0$, $a + 0 \leq b + \epsilon$.
2. Now the other direction, $\leftarrow$.

■

Exercise 4.9. Let E be a set of real numbers, let s be an upper bound for E. Prove that $s = \sup E$ if and only if for every $\epsilon > 0$ there exists $x \in E$ such that $x > s - \epsilon$.

$E \subset \mathbb{R}$ and s is an upper bound for E. Because E is a nonempty set in $\mathbb{R}$ and it is bounded above, it has the least upper bound property, implying that $\sup E$ exists.

1. Let us prove this $\rightarrow$ direction first. Given $s = \sup E$. It is easiest to do this via contradiction. This means that $\exists x \in E$ s.t. $x < s - \epsilon$. This would suggest that this particular $x + \epsilon$ would be a tighter bound for E than s. We know that is cannot be the case since $s = \sup E$, which is the least upper bound by definition.
2. Now the other direction, $\leftarrow$. The main property we need to demonstrate is that s is actually the least upper bound. Given for every $\epsilon > 0$ there exists $x \in E$ such that $x > s - \epsilon$.. Assume that there is an upper bound for E called d s.t. $s > d$. If $x > s - \epsilon$, then $d > x > s - \epsilon$. Let's look at $x > s - \epsilon$ which is the same as $x + \epsilon > s$. Using our definition of d from above, $x + \epsilon > d$ so $x > d - \epsilon$. So our inequality expression becomes $d > x > d - \epsilon > s - \epsilon$. Looking at the last part, $d - \epsilon > s - \epsilon \implies d > s$, which violates the above definition of d. This shows there is no d s.t. $s > d$ so s is the least upper bound for E. It necessarily follows then $s = \sup E$.

■

Exercise 4.10. Let A and B be nonempty sets of real numbers. Decide whether the following statements are true or false. If true, give a proof; if false, give a counterexample.

1. If $\sup A < \inf B$, then there exists a $c \in \mathbb{R}$ satisfying $a < c < b$ for all $a \in A$ and $b \in B$.
2. If there exists a $c \in \mathbb{R}$ satisfying $a < c < b$ for all $a \in A$ and $b \in B$, then $\sup A < \inf B$.

1. True. Let $x = \sup A$ and $y = \inf B$. Clearly, $a < x < y < b \, \forall a \in A, b \in B$. We need to prove that if $y - x > \epsilon, \epsilon \in \mathbb{R}, \exists$ an integer m s.t. $x < m\epsilon < y$. Let m be the smallest positive integer such that $m\epsilon > nx, n \in \mathbb{N}$. $x < \frac{m\epsilon}{n} < y$; since $nx < m\epsilon$ by definition of m, $(m-1)\epsilon < nx$. On the other hand, $nx < ny - \epsilon$ so $(m-1)\epsilon < nx < ny - \epsilon$. This reduces to $m\epsilon < nx < ny \implies m\epsilon < ny$. This quantity is the c we are looking for since $\frac{m}{n}$ is a rational number and ϵ is real. This concludes the proof.
2. False. Let A be the set of all negative $\mathbb{R}$ and B be the set of all positive $\mathbb{R}$. Clearly, $c = 0$ so $\sup A = \inf B$, disproving the claim.

∎