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MA 521
Homework #9
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Chapter 7: 2.16, 2.17, 3.6, 3.7

Exercise 2.16 For each of the following sequences $(a_n)_{n=1}^{\infty}$, prove whether the series $\sum_{n=1}^{\infty} a_n$ converges or diverges. (If it converges, you do not need to find the limit.)

1. $a_n = \sqrt{n+1} - \sqrt{n}$.
2. $a_n = \frac{\sqrt{n+1}-\sqrt{n}}{n}$.
3. $a_n = (\sqrt[n]{n} - 1)^n$.
4. $a_n = \frac{(-1)^n}{\log n}$ for $n \geq 2$ (and $a_1 = 0$).

1. Enumerate partial sums of a_n. $s_n = (\sqrt{2} - \sqrt{1}) + (\sqrt{3} - \sqrt{2}) \cdots + (\sqrt{k+1} - \sqrt{k}) = -1 + (\sqrt{2} - \sqrt{2}) \cdots + (\sqrt{k} - \sqrt{k}) + \sqrt{k+1}$. So, $\sqrt{k+1} - 1$ as $n \to \infty$ diverges so series diverges.
2. Multiply numerator and denominator by $\frac{\sqrt{n+1}+\sqrt{n}}{\sqrt{n+1}+\sqrt{n}}$. So, $\frac{\sqrt{n+1}-\sqrt{n}}{n} = \frac{1}{n\sqrt{n+1}+\sqrt{n}} < n^{\frac{-3}{2}}$. According to Theorem 2.4, $\frac{\sqrt{n+1}-\sqrt{n}}{n}$ diverges.
3. Use Root Test so $\lim_{n\to\infty} \sup(\sqrt[n]{n} - 1) < 1$. $\lim_{n\to\infty} \sup \sqrt[n]{n} < 2$. $\lim_{n\to\infty} \sup \frac{\log n}{n} < \log 2$. $\lim_{n\to\infty} \sup n < 2^n \to \frac{n}{2^n} < 1$, which is true for all n. So series converges.
4. Use Alternating Series Test to show that $\frac{1}{\log n}$ is monotonically decreasing. For all $n \geq 2$, $\frac{1}{\log n+1} < \frac{1}{\log n}$. So by AST, this series converges.

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Exercise 2.17 Consider the series

$$\sum_{n=1}^{\infty} \frac{1}{1+z^n}.$$

Determine which values of $z \in \mathbb{R}(z \neq -1)$ make the series convergent and which make it divergent. Prove your answers are correct.

Let's use the Ratio Test. $|\frac{a_{n+1}}{a_n}| = |(\frac{1+z}{1+z^2}, \frac{1+z^2}{1+z^3}, \frac{1+z^3}{1+z^4}, \ldots, \frac{1+z^n}{1+z^{n+1}})|$. So, $\lim_{n\to\infty} \sup|\frac{1+z^n}{1+z^{n+1}}| = |\frac{z^n}{z^{n+1}}| = \frac{1}{|z|} < 1$. If $|z| > 1$, then $\sum_n^{\infty} \frac{1}{1+z^n}$ converges. Now, $\lim_{n\to\infty} \inf|\frac{1+z^n}{1+z^{n+1}}| = |\frac{z^n}{z^{n+1}}| = \frac{1}{|z|} > 1$. If $|z| < 1, \sum_{n=1}^{\infty} \frac{1}{1+z^n}$ diverges. If $z = 1$, $\sum_{n=1}^{\infty} \frac{1}{1+1^n}$. We know that $\sum_{n=1}^{\infty} \frac{1}{2}$ diverges so by comparison test, $\sum_{n=1}^{\infty} \frac{1}{1+z^n}$ diverges for $z = 1$. ■

Exercise 3.6 Assume that $\sum_{n=1}^{\infty} a_n$ converges absolutely. Prove that $\sum_{n=1}^{\infty} \frac{\sqrt{|a_n|}}{n}$ converges. (Hint: Use the inequality $2AB \leq A^2 + B^2$, valid for any real numbers A, B).

If $\sum_{n=1}^{\infty} a_n$ converges absolutely, then $\sum_{n=1}^{\infty} |a_n|$ converges. Consider any term in $\sum_{n=1}^{\infty} \frac{\sqrt{|a_n|}}{n}$ and set it equal to AB such that $A = \sqrt{|a_n|}, B = \frac{1}{n}$. So, $2\frac{\sqrt{|a_n|}}{n} \leq |a_n| + \frac{1}{n^2} \forall n \in \mathbb{N}$. By comparison test, $2\frac{\sqrt{|a_n|}}{n}$ converges since $|a_n| + \frac{1}{n^2}$ converges. So, $\frac{\sqrt{|a_n|}}{n}$ also must converge.

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Exercise 3.7

1. Assume that $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ converge absolutely. Prove that $\sum_{n=1}^{\infty} (a_n + b_n)$ absolutely as well.
2. Assume that $\sum_{n=1}^{\infty} a_n$ converges. Does it follow that $\sum_{n=1}^{\infty} a_{2n}$ converges? Give a proof or counterexample.
3. Assume that $\sum_{n=1}^{\infty} a_n$ converges absolutely. Does it follow that $\sum_{n=1}^{\infty} a_{2n}$ converges absolutely? Give a proof or counterexample.

1. If $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ absolutely converge, then $\sum_{n=1}^{\infty} |a_n|$ and $\sum_{n=1}^{\infty} |b_n|$ converge. By triangle inequality, $\sum_{n=1}^{\infty} |a_n + b_n| \leq \sum_{n=1}^{\infty} |a_n| + \sum_{n=1}^{\infty} |b_n|$. By comparison test, $\sum_{n=1}^{\infty} |a_n + b_n|$ converges. So, $\sum_{n=1}^{\infty} (a_n + b_n)$ absolutely converges.
2. No. Consider the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$, which converges by AST. But, $\sum_{n=1}^{\infty} \frac{(-1)^{2n}}{2n} = \sum_{n=1}^{\infty} \frac{(1)}{2n}$ is a divergent series.
3. Yes. If $\sum_{n=1}^{\infty} a_n$ converges absolutely, then $\sum_{n=1}^{\infty} |a_n|$ converges. For any $k \in \mathbb{N}, \sum_{n=1}^{k} |a_{2n}| \leq \sum_{n=1}^{k} |a_n| \leq \sum_{n=1}^{\infty} |a_n|$. So, $\sum_{n=1}^{\infty} absa_{2n}$ converges which implies $\sum_{n=1}^{\infty} a_{2n}$ converges absolutely.

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