Ravi Raju
MA 521
Homework #1
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Chapter 1: 1.7, 3.3, 3.4, 3.6, 4.5, 4.7, 4.17

Exercise 1.7. Let A and B be subsets of another X. Prove the following the following statements.

1. $A \cap B = A \setminus (A \setminus B)$
2. $A \subset B$ if and only if $X \setminus A \supset X \setminus B$.

Recall the definitions of $\cup$ and $\setminus$. $A \cup B = \{x : x \in A \text{ and } x \in B\}$. $A \setminus B = \{x \in A : x \notin B\}$.

1. Let $D = A \setminus B$. D is the set of elements in A that are strictly unique. Let $E = A \setminus D$. E is the relative complement of D in A, which only leaves elements common to both A and B.
2. (a) Let us prove this $\rightarrow$ direction first. Given $A \subset B$. This means that A will have a lesser or equal to number of elements in its set than B. It follows that $X \setminus A$ will contain all elements of the set $X \setminus B$. Thus, $X \setminus A \supset X \setminus B$.
   (b) Now the other direction, $\leftarrow$. Given $X \setminus A \supset X \setminus B$. Assume $\exists$ some $x \in A$ and $x \notin B$, which means that $A \not\subset B$. However, $x \notin X \setminus A$ and $x \in X \setminus B$ when we stated $X \setminus A \supset X \setminus B$. So, it must be $\forall x \in A$ must be also be $x \in B$ so $A \subset B$.

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Exercise 3.3. Let $f : A \rightarrow B$ be a function. Prove the following statements.

1. f is injective if and only if $f^{-1}(f(C)) = C$ for every subset C of A.
2. f is surjective if and only if $f^{-1}(f(D)) = D$ for every subset D of B.

First, let us list some useful definitions.
If $G \subset B$ then the inverse image, $f^{-1}(G)$ of G under f is $f^{-1}(G) = \{x \in A : f(x) \in G\}$.
If $f^{-1}(y)$ contains at most one element of A for each $y \in B$, then f is said to be injective.
If $f(A) = B$, we say that f maps A onto B, or that $f : A \rightarrow B$ is surjective.

1. (a) Let us prove this direction, $\rightarrow$. Given f is injective, let C_1 be some subset of A. f maps all elements of C_1 to some set $B_1 \subset B$. Applying the definition of the inverse image to this set B_1 under f yields $f^{-1}(B_1) = \{x \in A : f(x) \in B_1\}$. Since we know that f is injective, we know that the resulting set obtained from the inverse image has to be the original set, C_1.

(b) Now the other direction, $\leftarrow$. Now the other direction, $\leftarrow$. Given $f^{-1}(f(C)) = C$. Let us do proof by contradiction. Let x_1, x_2 be elements in C and assume that $f(x_1) = f(x_2)$ but $x_1 \neq x_2$ (this is another way to say f is not injective). Applying the given fact to a subset of C, $\{x_1\}$, yields $f^{-1}(f(\{x_1\})) = \{x \in C : f(x) \in f(C)\} = \{x_1, x_2\}$. Clearly, this is a contradiction since the set we put into the function and inverse image is not the same set that was returned. This proves that f has to be injective.

2. (a) Let us look at the $\rightarrow$ direction first. Given f is surjective. Let $C_1 = f^{-1}(D) = \{x \in A : f(x) \in D\}$. If we apply f to C_1, we will obtain our original set D since f is surjective.

   (b) Now for the other direction, $\leftarrow$. Given $f(f^{-1}(D)) = D$. Let us try to argue that f is not surjective. Let us call $C_2 = f^{-1}(D)$. What we mean when we call f not surjective is $f(C_2) \neq D$. But this goes against the given fact so it must be that f is surjective.

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Exercise 3.4. Let $f : A \to B$ and $g : B \to C$ be functions. Prove the following statements.

1. If f and g are both injective, then so is $g \circ f$.
2. If f and g are both surjective, then so is $g \circ f$.
3. If $g \circ f$ is surjective, then so is g.
4. Surjectivity of $g \circ f$ does not imply surjectivity of f.
5. If $g \circ f$ is injective, then so is f.
6. Injectivity of $g \circ f$ does not imply injectivity of g.

1. Let $h = g \circ f = g(f(a))$ for $a \in A$. To be injective, $h(a) = h(b) \implies a = b$. Substitute for h and use the fact that g and f are injective: $g(f(a)) = g(f(b)) \implies f(a) = f(b) \implies a = b$. So, $f \circ g$ is injective.
2. Let $h = g \circ f = g(f(a))$ for $a \in A$. To be surjective, $h(A) = C$ Substitute for h and use the fact that g and f are surjective: $g(f(A)) \implies g(B)$ since f is surjective $\implies g(B) = C$ since g is surjective. So, $f \circ g$ is surjective.
3. Let $h = g \circ f = g(f(a))$ for $a \in A$. Restate what $f(A)$ is and call it T: $f(A) = \{f(x) : x \in A\} = T$. So, $g(T) = \{g(y) : y \in T\}$. But since we know that h is surjective, it must span all $x \in C$. This is only possible if g is surjective.
4. Assume $\exists a_1 \in A : f(a_1) \notin B$. This states that f cannot be surjective. However, we know that if h is surjective, g must be surjective to map to all elements of C. Consider the example, $A = \{3\}, B = \{4,5\}, C = \{6\}$ where $f : A \to B$ by $f(3) = 4$ and $g : B \to C$ by $g(4) = g(5) = 6$. $f \circ g$ is surjective by $g(f(3)) = 6$ but $\nexists$ any $a \in A$ where $f(a) = 5$, implying f need not be surjective.

5. Given that $g \circ f$ is injective, that implies the following: for some $a_1, a_2 \in A, g(f(a_1)) = g(f(a_2)) \implies a_1 = a_2$. Assume f was not injective, then $f(a_1) = f(a_2)$ but $a_1 \neq a_2$. But this would violate that $g \circ f$ is injective since $a_1 = a_2$ so f must be injective.
6. For g to be injective, we need the condition for some $a_1, a_2 \in A, g(a_1) = g(a_2) \implies a_1 = a_2$. Consider the example, $A = \{3\}, B = \{4,5\}, C = \{6\}$ where $f : A \to B$ by $f(3) = 4$ and $g : B \to C$ by $g(4) = g(5) = 6. g \circ f(3) = 6$ is injective but $g(4) = g(5)$ leads to $4 \neq 5$, implying g need not be injective.

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Exercise 3.6. Let $f : X \to Y$ be a function. Prove the following statements.

1. If A and C are subsets of X, then $f(C \setminus A) \supset f(C) \setminus f(A)$.
2. f is injective if and only if $f(C \setminus A) = f(C) \setminus f(A)$ for any two subsets A and C of X.
3. If B and D are subsets of Y, then $f^{-1}(D \setminus B) = f^{-1}(D) \setminus f^{-1}(B)$.

1. Assume some $d \in f(C) \setminus f(A)$ but $d \notin f(C \setminus A)$. d needs to be in $f(C)$ then but not in $f(A)$. So that means that $\exists$ a particular $x : f(x) = d$ which cannot reside in A and must uniquely reside in C. But this contradicts our initial assumption that $d \notin f(C \setminus A)$ since this set contains elements unique to set C. Thus, $f(C \setminus A) \supset f(C) \setminus f(A)$.
2. (a) Let us look at the $\to$ direction first. Given f is injective. Let's try proof by contradiction. Assume that $f(x_1) \in f(C) \setminus f(A)$ but $f(x_1) \notin f(C \setminus A)$. $f(x_1) \in f(C) \setminus f(A) = \{w \in f(C) : w \notin f(A)\} \implies x_1 \in \{x \in C : x \notin A\}$ because f is injective. So then we apply f which then $f(x_1) \in f(C \setminus A)$, which contradicts our original statement. So, $f(C \setminus A) = f(C) \setminus f(A)$.
   (b) Now for the other direction, $\leftarrow$. Given $f(C \setminus A) = f(C) \setminus f(A)$. $f(C) \setminus f(A) = \{w \in f(C) : w \notin f(A)\}$, let us call this set D. $f(C \setminus A) = f(\{x \in C : x \notin A\})$. To show f is not injective, $f(x_1) = f(x_2) \implies x_1 \neq x_2$. Let's say that $\exists f(x_1) = f(x_2) \in D$ and that $x_1, x_2 \in C \setminus A$. So, $D = \{\ldots, f(x_1), \ldots\} = \{\ldots, f(x_2), \ldots\}$. This implies $D = f(\{\ldots, x_1, \ldots\})$ by definition of image but then this suggests that $x_2 \notin C \setminus A$, which is a contradiction. So, f must be injective.
3. Let us start with the LHS and work our way to the RHS. Recall $f^{-1}(D)\{w \in X : f(w) \in D\}$. So, $f^{-1}(D) \setminus f^{-1}(B) = \{w_1 \in \{w \in X : f(w) \in D\}, w_1 \notin \{w \in X : f(w) \in B\}\}$, this means to single out those elements in X that map to D uniquely. This can be rewritten as $\{w_1 \in X : f(w_1) \in \{w \in D : w \notin B\}\} = f^{-1}(\{w \in D : w \notin B\}) = f^{-1}(D \setminus B)$.

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Exercise 4.5 Assume that $card(A) \leq card(X)$ and $card(B) \leq card(Y)$. Prove that $card(B^A) \leq card(Y^X)$. Hint: Consider a function $\Phi(f) = h \circ f \circ k$, where $k : X \to A$ and $h : B \to Y$ are certain functions. Theorem 3.9 might be useful for the final step.

To prove $card(B^A) \leq card(Y^X)$ and $\Phi(f) = h \circ f \circ k$, where $k : X \to A$ and $h : B \to Y$ are certain functions, we need to show that Φ is injective by the definition of cardinality. $card(A) \leq card(X)$ tells us that $\exists$ some function that maps A to X that is injective. $card(B) \leq card(Y)$ tells us that $\exists$ some function that maps B to Y that is injective. By Theorem 3.9, for Φ to be injective, it needs to have a left inverse.

A left inverse for Φ is a function $\Phi_2 : Y^X \to B^A$ s.t. $\Phi_2 \circ \Phi = id_{B^A}.\Phi_2 \circ \Phi(f) = h_2 \circ \Phi(f) \circ k_2$, where $h_2 : Y \to B$ and $k_2 : A \to X$. From the given facts, we can select h to be injective and k to be surjective. This leads to $h_2 \circ \Phi(f) \circ k_2 = h_2 \circ h \circ f \circ k \circ k_2 = id_B \circ f \circ id_A = f$. So, Φ_2 is a left inverse so Φ is injective and $card(B^A) \leq card(Y^X)$. ∎

Exercise 4.7 Prove that for any set A, one has $\mathcal{P}(A) \sim \{0,1\}^A$.

$\mathcal{P}(A) \sim \{0,1\}^A \implies card(\mathcal{P}(A)) = card(\{0,1\}^A)$. So, $f : \mathcal{P}(A) \to \{0,1\}^A$ needs to be bijective for the previous statement to be true. Let us break it into two parts.

1. Prove f is injective. Let's do proof by contradiction. Assume $f(x_1) = f(x_2) \implies x_1 \neq x_2$. So, $f(x_1) = f(x_2) \in \{0,1\}^A$ and $x_1, x_2 \in \mathcal{P}(A)$. $\{0,1\}^A = \{\ldots, f(x_1), \ldots\} \to f(E)$, where $E = \mathcal{P}(A)$ by definition of image. But, $x_2 \notin E$ which establishes a condtradiction so f must be injective.
2. Prove f is surjective. Assume $a_1 \notin \mathcal{P}(A)$ where $f(a_1) \in \{0,1\}^A$. But $f^{-1}(\{0,1\}^A) = \{x \in \mathcal{P}(A) : f(x) \in \{0,1\}^A\}$ so $a_1 \in A$. This is a contradiction so f must be surjective.

If f is both injective and surjective, it must be bijective. Thus, $card(\mathcal{P}(A)) = card(\{0,1\}^A) \to \mathcal{P}(A) \sim \{0,1\}^A$. ∎

Exercise 4.17 Let A and B be sets, and assume $f : A \to B$ and $g : B \to A$ are injective functions.

1. Assume additionally that A is finite. Prove that f and g must actually be bijections.
2. Show by way of an example that both f and g may fail to be bijective if we do not assume that A is finite.

1. First, if A is finite, we know that it will be a countable set and that $\forall x \in A \to B$ because f is injective. Since f is injective, $card(A) \leq card(B)$. Similarly, $card(B) \leq card(A)$ because g is injective $\implies card(A) = card(B)$ by Thm. 4.2(5). So f and g must both be bijections.
2. Suppose $f : \mathbb{N}x\mathbb{N} \to \mathbb{N}$. It is clear to see that this function will be injective but there are elements in $\mathbb{N}$ where there is no mapping so it is not surjective. Define $g : \mathbb{N} \to \mathbb{N}x\mathbb{N}$ where $g(n) = (n, n)$, this is clearly injective but not surjective.

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