Chapter 2: 5.4, 5.5, 6.4
Chapter 3: 1.7, 2.3, 2.4, 2.6, 2.8, 2.11, 2.12, 2.16, 2.20

Exercise 5.4. Let a and b be real numbers. Show that the following three equalities hold:

$$\bigcap_{x>b}(a,x) = (a,b], \qquad \bigcup_{n=1}^{\infty}[a+\frac{1}{n}, b-\frac{1}{n}] = (a,b), \qquad \bigcap_{n=1}^{\infty}(a+n, +\infty) = \emptyset.$$

∎

Exercise 5.5. Let $a_1, a_2, \ldots$ be any enumeration of the negative rational numbers; let $b_1, b_2, \ldots$ be any enumeration of the positive rational numbers. Show that the following two equalities hold:

$$\bigcap_{j=1}(a_j, b_j) = \{0\}, \qquad \bigcup_{j=1}^{\infty}(a_j, b_j) = \mathbb{R},$$

.

∎

Exercise 6.4. Prove there exists no order $\leq$ that makes $(\mathbb{C}, +, \cdot, \leq)$ into an ordered field.

Let us argue by contradiction. Assume that $\mathbb{C}$ is a ordered field. Let us analyze the case $(0,1) = i$.

- If $i > 0$, $x, y \in \mathbb{C}$ and $x = i, y = i, i \cdot i > 0 \implies -1 > 0$, which is a contradiction.

- If $i < 0$, $x, y \in \mathbb{C}$ and $x = -i, y = -i$ (since $x, y > 0$ for the second condition to hold), $-i \cdot -i < 0 \to i^2 > 0 \implies -1 > 0$, which is a contradiction.

∎

Exercise 1.7. Let $\|\cdot\|$ be a norm on a real vector space V. Prove the *reverse triangle inequality*:

$$|\|x\| - \|y\|| \leq \|x - y\|$$

∎

Exercise 2.3. Let X be any set. Prove that the discrete metric $d : X \times X \to \mathbb{R}$(defined by $d(x,y) = 1$ if $x \neq y$ and $d(x,x) = 0$ for $x \in X$) satisfies the triangle inequality and is therefore a metric on X.

Exercise 2.4. Determine which of the following functions are metrics on $\mathbb{R}$. Prove your answer in each case.

- $d_1(x,y) = \sqrt{|x-y|}$.
- $d_2(x,y) = |x-2y|$.
- $d_3(x,y) = \frac{|x-y|}{1+|x-y|}$.

■

Exercise 2.6. Let $\|\cdot\|$ denote the Euclidean norm $\mathbb{R}^2$, i.e. $\|(x_1,x_2)\| = \sqrt{x_1^2 + x_2^2}$. Consider the function from $\mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$, defined by

$$d(x,y) = \|x_1 - y_1\| + \|x_2 - y_2\|, \qquad (x = (x_1,x_2), y = (y_1,y_2)).$$

1. Prove that d is a metric on $\mathbb{R}^2$.

2. On a sheet of graph paper, draw the set $B_d((5,1),3)$. Use dotted lines to indicate the boundary, which is not included in the set you are drawing. (Hint: it may be easier to figure out what the set looks like if you first consider $B_d((0,0),3)$.)

3. On the same graph as in the previous part, draw $B_{d_u}((-3,2),1)$, where d_u denotes the square metric.

■

Exercise 2.8. Let (X,d) be a metric space, and let E be a subset of X. The *diameter* of E in (X,d) is defined by the formula

$$\mathrm{diam}_d(E) = \sup\{d(x,y) : x,y \in E\}.$$

1. Prove that for any $r > 0$ and $x \in X$, we have $\mathrm{diam}(B(x,r)) \leq 2r$.

2. If X is any set and d is the discrete metric, show $\mathrm{diam}(B(x,r)) = 0$.

3. If $X = \mathbb{R}^n$ for some $n \in \mathbb{N}$ and d is the Euclidean metric, prove that $\mathrm{diam}(B(x,r)) = 2r$.

■

Exercise 2.11. As in Example 2.7, let $X = \mathbb{R}^2, Y = [-1,3] \times [2,4]$, and let d denote the Euclidean metric on $X = \mathbb{R}^2$. Let $p = (3,4)$ and let $q = (2,4)$. Arguing *directly from the definition of an interior point* (i.e., without using Exercise 2.12), show that q is an interior point $B_Y(p,2)$ with respect to Y, but q is not an interior point $B_Y(p,2)$ with respect to X. In addition, draw a picture on a piece of graph paper that illustrates the idea of your proof.

■

Exercise 2.12. Let (X, d) be a metric space, and let Y be a subset of X. Prove that

$$(*) \operatorname{Int}_X(U) = \operatorname{Int}_Y(U) \cap \operatorname{Int}_X(Y).$$

In the notation of Exercise 2.11, the equality $(*)$ gives an alternate explanation of why q is not an interior point of $B_Y(p, 2)$ with respect to X: It is because $q \notin \operatorname{Int}_X(Y)$, as can be seen from the picture you drew in that Exercise.

■

Exercise 2.16. Let (X, d) be a metric space, and let U be a subset of X. Use Proposition 2.15 to prove that $\operatorname{Int}_X(U)$ is open in X.

■

Exercise 2.20. Let (X, d) be a metric space. Assume that $U \subset Y \subset X$, and additionally that Y is open X. Prove that U is open in Y if and only if U is open in X. (Note: There at least two possible solutions; one uses Theorem 2.19, the other uses Exercise 2.12.)

■