An Analysis of Simulated Stochastic Processes

A Computational and Theoretical Study

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Abstract

This report provides a detailed theoretical overview of twelve distinct stochastic processes, demonstrated through computational simulation. The processes are categorized into martingales (fair games), submartingales (favorable games), and supermartingales (unfavorable games). Each section provides the mathematical foundation for the process's classification, including rigorous derivations of its conditional expectation, and discusses the key insights and theorems associated with its behavior.

1 Martingales (Fair Games)

A stochastic process $\{M_n\}_{n\geq 0}$ is a martingale with respect to a filtration $\{\mathcal{F}_n\}_{n\geq 0}$ if it satisfies three conditions:

1. M_n is adapted to $\mathcal{F}_n$ for all n (i.e., M_n is knowable from the information at time n).
2. M_n is integrable, i.e., $\mathbb{E}[|M_n|] < \infty$ for all n.
3. $\mathbb{E}[M_{n+1} \mid \mathcal{F}_n] = M_n$.

This formalizes a "fair game," where the best prediction of the future value, given the entire past, is the current value.

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1.1 The Gambler's Ruin (Simple Symmetric Random Walk)

Model This process models a gambler's fortune, S_n, starting at $S_0 = c$. The fortune evolves as $S_n = c + \sum_{i=1}^{n} \xi_i$, where ξ_i are i.i.d. random variables representing a fair coin toss, with $\mathbb{P}(\xi_i = 1) = \mathbb{P}(\xi_i = -1) = 1/2$. The filtration $\mathcal{F}_n$ is the history of all coin tosses up to time n.

Martingale Proof The process S_n is a martingale because the expected gain at each step is zero.

$$\mathbb{E}[S_{n+1} \mid \mathcal{F}_n] = \mathbb{E}[S_n + \xi_{n+1} \mid \mathcal{F}_n] = S_n + \mathbb{E}[\xi_{n+1}] = S_n + 0 = S_n$$

Insight This process is a foundational application of the Optional Stopping Theorem. Let $\tau = \inf\{n : S_n = 0 \text{ or } S_n = N\}$ be the stopping time when the gambler is ruined or reaches a target N. The theorem states that $\mathbb{E}[S_\tau] = S_0 = c$. By letting $p_c = \mathbb{P}(\text{ruin starting from } c)$, we have $\mathbb{E}[S_\tau] = 0 \cdot p_c + N \cdot (1 - p_c)$. Solving $N(1 - p_c) = c$ gives the famous ruin probability $p_c = 1 - c/N$.

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1.2 De Moivre's Martingale

Model For a Simple Symmetric Random Walk S_n (as defined above, with $S_0 = 0$), the process $Y_n = S_n^2$ is *not* a martingale. The process $M_n = S_n^2 - n$ is De Moivre's Martingale.

Martingale Proof First, we show that S_n^2 has a deterministic drift.

$$\mathbb{E}[S_{n+1}^2 \mid \mathcal{F}_n] = \mathbb{E}[(S_n + \xi_{n+1})^2 \mid \mathcal{F}_n] = \mathbb{E}[S_n^2 + 2S_n\xi_{n+1} + \xi_{n+1}^2 \mid \mathcal{F}_n]$$

Using linearity of expectation and that S_n is known at time n, this becomes:

$$S_n^2 + 2S_n\mathbb{E}[\xi_{n+1}] + \mathbb{E}[\xi_{n+1}^2] = S_n^2 + 0 + 1 = S_n^2 + 1$$

The process increases by exactly 1 in expectation at each step. By subtracting the deterministic compensator n, we create a martingale $M_n = S_n^2 - n$.

$$\mathbb{E}[M_{n+1} \mid \mathcal{F}_n] = \mathbb{E}[S_{n+1}^2 - (n+1) \mid \mathcal{F}_n] = (S_n^2 + 1) - (n+1) = S_n^2 - n = M_n$$

Insight This is a fundamental example of the **Doob-Meyer Decomposition Theorem**, which states that any submartingale can be uniquely decomposed into a martingale and a predictable, increasing process (the compensator). Here, $S_n^2 = M_n + n$.

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1.3 Bernoulli Exponential Martingale

Example 1. *This construction creates a family of martingales from a sum of i.i.d. random variables. Let ξ_k be i.i.d. Bernoulli trials with $\mathbb{P}(\xi_k = 1) = p$. Let $S_n = \sum_{k=1}^n \xi_k$. The moment generating function of a single trial is:*

$$\varphi(\theta) = \mathbb{E}[e^{\theta\xi_1}] = pe^\theta + (1 - p)$$

The process $M_n = \frac{\exp(\theta S_n)}{\varphi(\theta)^n}$ is a martingale for any $\theta \in \mathbb{R}$.

Martingale Proof

$$\begin{aligned}\mathbb{E}[M_{n+1} \mid \mathcal{F}_n] &= \mathbb{E}\left[\frac{e^{\theta S_{n+1}}}{\varphi(\theta)^{n+1}} \mid \mathcal{F}_n\right] = \frac{1}{\varphi(\theta)^{n+1}}\mathbb{E}[e^{\theta(S_n+\xi_{n+1})} \mid \mathcal{F}_n] \\ &= \frac{e^{\theta S_n}}{\varphi(\theta)^{n+1}}\mathbb{E}[e^{\theta\xi_{n+1}}] \quad (\textit{since } \xi_{n+1} \textit{ is independent of } \mathcal{F}_n) \\ &= \frac{e^{\theta S_n}}{\varphi(\theta)^{n+1}} \cdot \varphi(\theta) = \frac{e^{\theta S_n}}{\varphi(\theta)^n} = M_n\end{aligned}$$

Insight This martingale is the Radon-Nikodym derivative process that allows for a change of probability measure. By choosing θ appropriately, we can re-weight probabilities to analyze the random walk as if it had a different success probability, a cornerstone of financial mathematics (e.g., switching to the risk-neutral measure).

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1.4 Pólya's Urn

Model An urn contains R_0 red and B_0 blue balls. At each step, a ball is drawn, its color is noted, and it is returned to the urn along with another ball of the same color. Let $X_n = \frac{R_n}{R_n+B_n}$ be the proportion of red balls at step n.

Martingale Proof The probability of drawing a red ball at step $n+1$ is X_n.

$$\begin{aligned}
\mathbb{E}[X_{n+1} \mid \mathcal{F}_n] &= \mathbb{P}(\text{draw red}) \cdot \left(\frac{R_n+1}{R_n+B_n+1}\right) + \mathbb{P}(\text{draw blue}) \cdot \left(\frac{R_n}{R_n+B_n+1}\right) \\
&= X_n \cdot \frac{R_n+1}{R_n+B_n+1} + (1-X_n) \cdot \frac{R_n}{R_n+B_n+1} \\
&= \frac{R_n(R_n+1)+B_nR_n}{(R_n+B_n)(R_n+B_n+1)} = \frac{R_n(R_n+B_n+1)}{(R_n+B_n)(R_n+B_n+1)} = X_n
\end{aligned}$$

Insight Although the expected proportion of red balls is always the current proportion, the variance of X_n increases with n. The process exhibits path dependence: early draws heavily influence the future. By the Martingale Convergence Theorem, X_n converges almost surely to a random variable X_∞ which follows a Beta distribution, $X_\infty \sim \text{Beta}(R_0, B_0)$.

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1.5 Critical Branching Process

Model A Galton-Watson process models a population size Z_n evolving via $Z_{n+1} = \sum_{i=1}^{Z_n} \xi_{i,n}$, where $\xi_{i,n}$ is the number of offspring of the i-th individual in generation n. The $\xi_{i,n}$ are i.i.d. with mean μ. If the mean offspring $\mu = 1$, the process is "critical."

Martingale Proof Using the tower property of conditional expectation:

$$\mathbb{E}[Z_{n+1} \mid \mathcal{F}_n] = \mathbb{E}\left[\sum_{i=1}^{Z_n} \xi_{i,n} \mid Z_n\right] = Z_n\mathbb{E}[\xi] = Z_n \cdot 1 = Z_n$$

Insight A critical branching process is a "fair game" on average, but its path behavior is dramatic. Unless every individual has exactly one offspring, the process is guaranteed to go extinct almost surely. This illustrates that expected value can be a poor summary of typical behavior.

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1.6 The Likelihood Ratio Martingale

Model Let $\mathbb{P}$ and $\mathbb{Q}$ be two probability measures. Consider a sequence of i.i.d. observations $X_1, X_2, \ldots$ which have density $p(x)$ under $\mathbb{P}$ and $q(x)$ under $\mathbb{Q}$. The likelihood ratio process is $L_n = \prod_{i=1}^{n} \frac{q(X_i)}{p(X_i)}$.

Martingale Proof We show L_n is a martingale under the measure $\mathbb{P}$.

$$\mathbb{E}_{\mathbb{P}}[L_{n+1} \mid \mathcal{F}_n] = \mathbb{E}_{\mathbb{P}}\left[L_n \cdot \frac{q(X_{n+1})}{p(X_{n+1})} \mid \mathcal{F}_n\right] = L_n \cdot \mathbb{E}_{\mathbb{P}}\left[\frac{q(X_{n+1})}{p(X_{n+1})}\right]$$

The expectation is an integral over the density $p(x)$:

$$\mathbb{E}_{\mathbb{P}}\left[\frac{q(X_{n+1})}{p(X_{n+1})}\right] = \int_{-\infty}^{\infty} \frac{q(x)}{p(x)} p(x) dx = \int_{-\infty}^{\infty} q(x) dx = 1$$

Thus, $\mathbb{E}_{\mathbb{P}}[L_{n+1} \mid \mathcal{F}_n] = L_n$.

Insight This process is the engine behind statistical hypothesis testing (e.g., the Sequential Probability Ratio Test) and is the discrete-time version of the process used in Girsanov's Theorem for changing measures in continuous time.

2 Submartingales (Favorable Games)

A process $\{X_n\}_{n\geq 0}$ is a **submartingale** if it is adapted and integrable, and satisfies $\mathbb{E}[X_{n+1} \mid \mathcal{F}_n] \geq X_n$. These processes model favorable games or quantities with a tendency to drift upwards.

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2.1 Biased Random Walk

Model Let $S_n = \sum_{i=1}^{n} \xi_i$, where $\mathbb{P}(\xi_i = 1) = p$ and $\mathbb{P}(\xi_i = -1) = 1 - p$. If the walk is biased upwards ($p > 1/2$), the expected step size $\mu = \mathbb{E}[\xi_i] = p - (1 - p) = 2p - 1$ is positive.

Submartingale Proof

$$\mathbb{E}[S_{n+1} \mid \mathcal{F}_n] = \mathbb{E}[S_n + \xi_{n+1} \mid \mathcal{F}_n] = S_n + \mathbb{E}[\xi_{n+1}] = S_n + \mu$$

Since $\mu > 0$, we have $\mathbb{E}[S_{n+1} \mid \mathcal{F}_n] > S_n$.

Insight This is the simplest model for a process with positive drift. By the Strong Law of Large Numbers, $S_n/n \to \mu$ almost surely, meaning the process grows linearly at a rate μ in the long run.

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2.2 Absolute Value of a Martingale

Model Let M_n be any non-trivial martingale (e.g., a fair random walk). The process $X_n = |M_n|$ is a submartingale.

Submartingale Proof The proof is a direct application of **Jensen's Inequality**. The function $f(x) = |x|$ is convex. For any convex function f and random variable Y, $\mathbb{E}[f(Y)] \geq f(\mathbb{E}[Y])$.

$$\mathbb{E}[|M_{n+1}| \mid \mathcal{F}_n] \geq |\mathbb{E}[M_{n+1} \mid \mathcal{F}_n]|$$

Since M_n is a martingale, $\mathbb{E}[M_{n+1} \mid \mathcal{F}_n] = M_n$. Therefore:

$$\mathbb{E}[|M_{n+1}| \mid \mathcal{F}_n] \geq |M_n|$$

Insight This demonstrates that applying a convex payoff function to a fair game results in a favorable game. It models situations where the magnitude of deviation from the mean is what matters, not the direction.

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2.3 Maximum of a Random Walk

Model Let S_n be a martingale. The running maximum process is $M_n = \max\{S_0, S_1, \ldots, S_n\}$.

Submartingale Proof By definition, the maximum at time $n+1$ is $M_{n+1} = \max(M_n, S_{n+1})$. This immediately implies that $M_{n+1} \geq M_n$ for all outcomes. Taking conditional expectations of both sides of this inequality:

$$\mathbb{E}[M_{n+1} \mid \mathcal{F}_n] \geq \mathbb{E}[M_n \mid \mathcal{F}_n]$$

Since M_n is known at time n (it is $\mathcal{F}_n$-measurable), $\mathbb{E}[M_n \mid \mathcal{F}_n] = M_n$. Thus, $\mathbb{E}[M_{n+1} \mid \mathcal{F}_n] \geq M_n$.

Insight This process models the "high-water mark" of a stock price or a gambler's fortune. It is central to probability theory and is the subject of **Doob's Maximal Inequality**, which bounds the probability that the maximum of a martingale exceeds a certain level.

3 Supermartingales (Unfavorable Games)

A process $\{X_n\}_{n\geq 0}$ is a **supermartingale** if it is adapted and integrable, and satisfies $\mathbb{E}[X_{n+1} \mid \mathcal{F}_n] \leq X_n$. These represent unfavorable games or quantities that tend to drift downwards.

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3.1 Wealth in a Sub-Fair Game

Model A gambler with wealth W_n employs a proportional betting strategy, wagering a fraction $\alpha \in (0, 1)$ of their wealth on a game with a negative expected net return $\mu < 0$. The return on a $1 bet is the random variable ξ with $\mathbb{E}[\xi] = \mu$. The wealth evolves as $W_{n+1} = W_n(1 - \alpha) + (W_n\alpha)(1 + \xi_{n+1}) = W_n(1 + \alpha\xi_{n+1})$.

Supermartingale Proof

$$\mathbb{E}[W_{n+1} \mid \mathcal{F}_n] = W_n\mathbb{E}[1 + \alpha\xi_{n+1}] = W_n(1 + \alpha\mu)$$

Since $\alpha > 0$ and $\mu < 0$, we have $(1 + \alpha\mu) < 1$. This implies $\mathbb{E}[W_{n+1} \mid \mathcal{F}_n] < W_n$.

Insight This mathematically demonstrates the principle of a "house edge." Even a tiny negative expected return (μ is slightly less than 0) ensures that, on average, wealth decays exponentially over time, drifting towards ruin.

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3.2 Coupon Collector's Problem

Model Let X_n be the number of unique coupons *remaining* to be collected from a set of N distinct coupons. At each step n, one coupon is drawn uniformly at random.

Supermartingale Proof Suppose at step n, we have $X_n = k$ coupons remaining. The probability of finding a new coupon on the next draw is $p_{new} = k/N$. If a new coupon is found, $X_{n+1} = k - 1$. Otherwise, $X_{n+1} = k$. The expected number of remaining coupons is:

$$\mathbb{E}[X_{n+1} \mid X_n = k] = (k-1) \cdot \frac{k}{N} + k \cdot \left(1 - \frac{k}{N}\right) = \frac{k^2 - k + Nk - k^2}{N} = k\left(1 - \frac{1}{N}\right)$$

Since $k \geq 1$ (if the game is not over), we have $k(1 - 1/N) < k$. Thus, $\mathbb{E}[X_{n+1} \mid \mathcal{F}_n] < X_n$.

Insight The process is a supermartingale that is guaranteed to stop at 0. The expected decrease per step, k/N, becomes smaller as k decreases. This explains why the last few coupons are disproportionately difficult to collect.

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3.3 Fair Game with a Transaction Cost

Model A player's wealth W_n evolves by playing a fair game with outcome ξ_{n+1} (where $\mathbb{E}[\xi_{n+1}] = 0$), but must pay a fixed transaction cost $c > 0$ for each play. The wealth evolves as $W_{n+1} = W_n + \xi_{n+1} - c$.

Supermartingale Proof

$$\mathbb{E}[W_{n+1} \mid \mathcal{F}_n] = \mathbb{E}[W_n + \xi_{n+1} - c \mid \mathcal{F}_n] = W_n + \mathbb{E}[\xi_{n+1}] - c = W_n - c$$

Since $c > 0$, we have $\mathbb{E}[W_{n+1} \mid \mathcal{F}_n] < W_n$.

Insight This models any business or trading strategy with fixed operating costs. Even if the core activity is perfectly balanced (a "fair game"), the constant drain of costs creates a supermartingale that drifts deterministically towards ruin unless offset by sufficiently large, positive-expectation events.