1. Metropolis Simulation

1.a. MCMC transition matrix

The transition matrix W is written as

$$W(C \to C^{'}) = e^{-\beta(E(C^{'})-E(C))} \tag{1.1}$$

$$W(C^{'} \to C) = 1 \tag{1.2}$$

when $E(C^{'}) > E(C)$

1.a.1. Prove ergodicity

For ergodicity, it suffices to show that there exists one stationary probability.

$$P_{st}(C^{'}) = \sum_{C} P_{st}(C) W(C \to C^{'}) \tag{1.3}$$

Plugging in 1.1, one obtains

$$P_{st}(C^{'}) = e^{-\beta E(C^{'})} \sum_{C} P_{st}(C) e^{\beta E(C)} \tag{1.4}$$

$$P_{st}(C^{'}) = e^{-\beta E(C^{'})} \alpha \tag{1.5}$$

where α is just a constant.

1.a.2. Prove detailed balance w.r.t. equilibrium weight

$$\frac{W(C \to C^{'})}{W(C^{'} \to C)} = \frac{e^{-\beta(E(C^{'})-E(C))}}{1} \tag{1.6}$$

Rearranging 1.6, one obtains

$$e^{-\beta E(C)} W(C \to C^{'}) = e^{-\beta E(C^{'})} W(C^{'} \to C) \tag{1.7}$$

1.b. Metropolis algorithm implementation

The code, ising_MCMC.py, is appended. For system sizes, $4x4$, $8x8$, and $32x32$, 2500-sweep runs were run at 1.5, 2.0, 2.2, 2.3, 2.6 and 3.0 kT (coupling constant $J = 1$). Plots of $E(\tau)$ and $M(\tau)$ are attached. To run the program:

```
for size in 4 8 32; do
    for temperature in 1.5 2.0 2.2 2.3 2.6 3.0; do
        ./ising_MCMC.py $size $temperature --verbose \
            --out-pkl 2500_sweeps_${size}x${size}_${temperature}kT.pkl \
            | tee 2500_sweeps_${size}x${size}_${temperature}kT.log
    done
done
```

1.c. Simulation results

χ in the limit of high temperatures. Using Mathematica,

1.d. Critical exponents

1.e. Finite size scaling

2. Swendsen-Ma MCRG