LECTURE 7

DATE: 8 NOVEMBER 2021

7. Measurable sets and the multiple integral

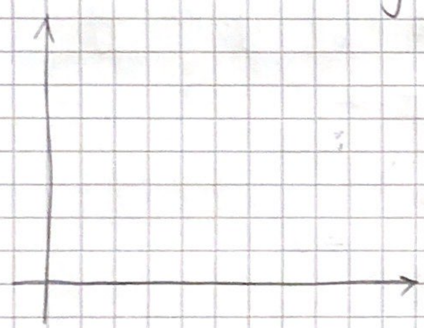
Short Recap: The Riemann Integral

- simple functions don't have simple antiderivatives

$$\int e^{-x^2} dx \quad , \quad \int \frac{\sin x}{x} dx \quad , \quad \int \frac{1}{\ln x} dx$$

- RIEMANN : "Concentrate on the Riemann Integral (not antiderivatives)!"

- Continuous functions are Riemann integrable
($\exists$ integrable functions which are discontinuous)



area under graph $f = ?$

Answer $= \int_a^b f(x)\, dx$

Riemann $\int$

Idea : cut $[a,b]$ in slices ("partition" $a = x_0, x_1, \dots, x_n = b$)

observe ξ_m intermediate points ξ ($\xi_k \in [x_{k-1}, x_k]$)

area of one slice $= f(\xi_k)(x_k - x_{k-1})$

Riemann sum = add up all slices

$$\sigma = (f; \Delta; \xi) = \sum_{k=1}^{n} f(\xi_k)(x_k - x_{k-1}) \longrightarrow \int_a^b f(x)\, dx$$

$$\max_k (x_k - x_{k-1}) \to 0$$

§7.1. A simple example

$f:[a,b]\times[c,d]\to\mathbb{R}$ continuous

Then $\int_a^b \left(\int_c^d f(x,y)\,dy\right)dx = \int_c^d\left(\int_a^b f(x,y)\,dx\right)dy$

Idea of Proof:

Define $H(t) = \int_a^t \left(\int_c^d f(x,y)\,dy\right)dx - \int_c^d\left(\int_a^t f(x,y)\,dx\right)dy$

$H:[a,b]\to\mathbb{R}$

Aim: Show that $H(b)=0$

We compute the derivative

$$\underbrace{H'(t)}_{\frac{d}{dt}H(t)} = \int_c^d f(t,y)\,dy - \int_c^d \frac{d}{dt}\left(\int_a^t f(x,y)\,dx\right)dy$$

$= 0 \implies H = \text{constant}$

$H(b) = H(a) = 0$

because $\int_a^a \ldots = 0$

Geometry

$[a,b]\times[c,d]$

fix $x\in[a,b]$ then integrate w.r.t. y
then integrate w.r.t. x

§7.2. Jordan Measurability:

The point is: we first have to establish which domains $D \subset \mathbb{R}^d$ are "good" for integrating f over them

Want: for $f : D \to \mathbb{R}$ "$\int_D f(x_1, \ldots, x_d)$ "dx"

integral for function of several variables

We need a framework in which:

LENGTH — AREA — VOLUME — ?

$d=1$ $d=2$ $d=3$ d arbitrary

$$B = [a_1, b_1] \times [a_2, b_2] \times \ldots \times [a_d, b_d]$$ "box" in dimension d

$\operatorname{int} B = (a_1, b_1) \times \ldots \times (a_d, b_d)$ interior of B

$v(B) = (b_1 - a_1)(b_2 - a_2) \ldots (b_d - a_d)$ "volume" of the box

$v(\emptyset) = 0 \to$ the empty set has zero volume

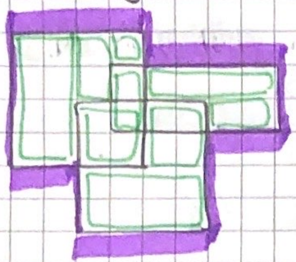
Def: We call a set $A \in \mathbb{R}^d$ elementary if it is a finite version of (nonoverlapping) boxes

$$A = \bigcup_{j=1}^{N} B_i, \quad B_i \text{ box}, \quad \operatorname{int} B_i \cap \operatorname{int} B_j = \emptyset \quad \forall i \neq j$$

For such A: $v(A) = \sum_{i=1}^{N} v(B_i)$

you can define / compute volume

Overlapping vs nonoverlapping :



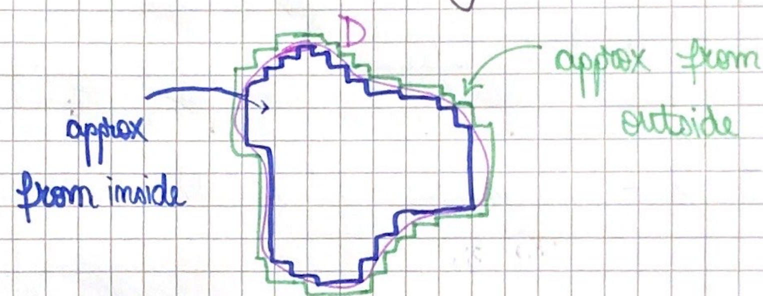
$A = \bigcup B_i = \bigcup \tilde{B}_j$

overlapping nonoverlapping

Now consider a bounded set $D \subset \mathbb{R}^n$

$$m_i(D) = \sup \{ v(A) : A \text{ elementary and } A \subseteq D \}$$

$$m_o(D) = \inf \{ v(A) : A \text{ elementary and } D \subseteq A \}$$



<u>Def</u>: A bounded set $D \subset \mathbb{R}^d$ is <u>Jordan measurable</u> if $m_i(D) = m_o(D)$ (inner and outer approx coincide). The common value will be denoted by $m(D)$ and is called the Jordan <u>measure</u> of D.

LENGTH – AREA – VOLUME – MEASURE

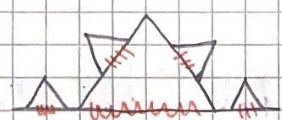
$d=1$ $d=2$ $d=3$ d arbitrary

Examples and counterexamples: (dimension = 2)

- Any set D with piecewise smooth boundary is Jordan measurable (in $\mathbb{R}^2$).

etc.

- A famous counterexample: the von Koch snowflake



etc.

Fractal and NOT Jordan measurable

Remark: An important extension of Jordan measurability is due to LEBESGUE (1901, 1902, 1904 book)

The only difference is that L. considers elementary sets which are not finite but countable reunions of boxes.

§7.3. The multiple integral (in the sense of Riemann)

- $D \subset \mathbb{R}^d$ bounded and Jordan measurable

$\Delta = \{D_1, \dots, D_n\}$ a partition of D

Jordan measurable

$\text{int } D_i \cap \text{int } D_j = \emptyset \qquad i \neq j$

$D = D_1 \cup \dots \cup D_m$

$(\|\Delta\| :=) \max\limits_i \delta(D_i) = \max\limits_i \left(\sup\{\|x-y\| : x, y \in D_i\} \right)$

diameter of D_i

$\xi = \{\xi_i\}, \ \xi_i \in D_i$ intermediate points

- $f : D \to \mathbb{R}$

Jordan measure

$\sigma(f; \Delta; \xi) = \sum\limits_{i=1}^{m} f(\xi_i)\, m(D_i)$

Riemann sum

<u>Def:</u> f is Riemann integrable if the Riemann sum converges to $\mathcal{I} \in \mathbb{R}$ (some value)

(as $n \to \infty$ and $\max \delta(D_i) \to 0$)

The limit $\mathcal{I} := \int_D f(x)\, dx$ or $= \iiint \dots \int f(x_1, \dots, x_d)\, dx_1 \dots dx_d$

notations

8. Computation of multiple integrals

IDEA : reduce the multiple integral to the computation of (several) simple integrals

Theorem 1 (FUBINI) : $f : A \times B \to \mathbb{R}$ integrable,

A, B bounded and Jordan measurable

Then $\iint\limits_{A\times B} f(x,y)\,dx\,dy = \int\limits_A \left(\int\limits_B f(x,y)\,dy \right) dx = \int\limits_B \left(\int\limits_A f(x,y)\,dx \right) dy$

Example : $D = \underbrace{[0,1]}_{A} \times \underbrace{[1,2]}_{B}$

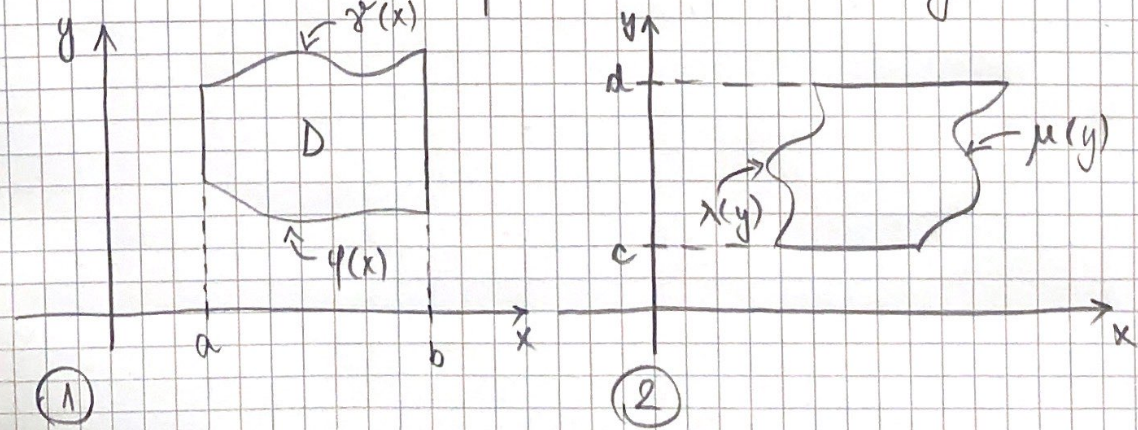
$$\iint\limits_D xy\,dx\,dy = \int_0^1 \left(\int_1^2 xy\,dy \right) dx =$$

$$= \left(\int_0^1 x\,dx \right) \left(\int_1^2 y\,dy \right) =$$

$$= \frac{x^2}{2}\Big|_0^1 \cdot \frac{y^2}{2}\Big|_1^2 = \frac{1}{2} \cdot \left(2 - \frac{1}{2}\right) = 1 - \frac{1}{4} = \frac{3}{4}$$

§8.1. Double integrals

Domain D is simple w.r.t. Ox or Oy



① $\iint_{D} f(x,y)\,dx\,dy = \int_{a}^{b} \left(\int_{\varphi(x)}^{\gamma(x)} f(x,y)\,dy \right) dx$

② $\iint_{D} f(x,y)\,dx\,dy = \int_{c}^{d} \left(\int_{\lambda(x)}^{\mu(x)} f(x,y)\,dx \right) dy$

Ex: D is by $y = x$ and $y = x^2$

$\varphi(x) = x^2, \quad \gamma(x) = x$

$$y = \iint_{D} (x+y)\,dx\,dy = \int_{0}^{1} \left(\int_{x^2}^{x} (x+y)\,dy \right) dx$$

$$= \int_{0}^{1} \left[xy + \frac{y^2}{2} \right] \Bigg|_{y=x^2}^{y=x} dx = \ldots = \frac{3}{20}$$