Wykład 1

wtorek, 5 października 2021 11:53

Slajd 5

Przykład

$z_1 = 4 + 3i \qquad Re z_1 = 4, \ Im z_1 = 3, \ \overline{z}_1 = 4 - 3i$

$z_3 = 18 + 0i \qquad Re z_3 = 18, \ Im z_3 = 0, \ \overline{z}_3 = 18 - 0i$

$z_4 = -2i = 0 + (-2)i \qquad Re z_4 = 0, \ Im z_4 = -2, \ \overline{z}_4 = 2i$

Slajd 6.

Mnożenie

$z_1 = a_1 + b_1 i, \quad z_2 = a_2 + b_2 i$

$z_1 \cdot z_2 = (a_1 + b_1 i) \cdot (a_2 + b_2 i) =$

$= a_1 a_2 + a_1 b_2 i + b_1 i \cdot a_2 + b_1 i \cdot b_2 i =$

$= a_1 a_2 + (a_1 b_2 + b_1 a_2) i + b_1 b_2 \underbrace{i^2}_{= -1} =$

$= a_1 a_2 + (a_1 b_2 + b_1 a_2) i - b_1 b_2 =$

$= (a_1 a_2 - b_1 b_2) + (a_1 b_2 + b_1 a_2) i$

$\frac{z_1}{z_2} = \frac{a_1 + b_1 i}{a_2 + b_2 i} \cdot \frac{a_2 - b_2 i}{a_2 - b_2 i} = \frac{(a_1 + b_1 i) \cdot (a_2 - b_2 i)}{(a_2 + b_2 i)(a_2 - b_2 i)} = \frac{(a_1 + b_1 i)(a_2 - b_2 i)}{a_2^2 - (b_2 i)^2} = \ldots$

$z_2 \neq 0 \qquad (a+b)(a-b) = a^2 - b^2$

$(b_2 i)^2 = b_2^2 \cdot \underbrace{i^2}_{-1} = -b_2^2$

Slajd 7. $z_1, z_2, z_3 \in \mathbb{C}$

1. $z_1 + z_2 = z_2 + z_1$
2. $(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$
3. $\underset{0 \in \mathbb{C}}{\exists} \quad 0 + z_1 = z_1, \quad 0 = 0 + 0i$
4. $\underset{(-z_1) \in \mathbb{C}}{\exists} \quad z_1 + (-z_1) = 0, \quad z = a + bi, \ -z = -a - bi$
5. $z_1 \cdot z_2 = z_2 \cdot z_1$

6. $(z_1 \cdot z_2) z_3 = z_1 \cdot (z_2 \cdot z_3)$

7. $\exists_{1 \in \mathbb{C}} \; 1 \cdot z_1 = z_1$ $\qquad 1 = 1 + 0i$

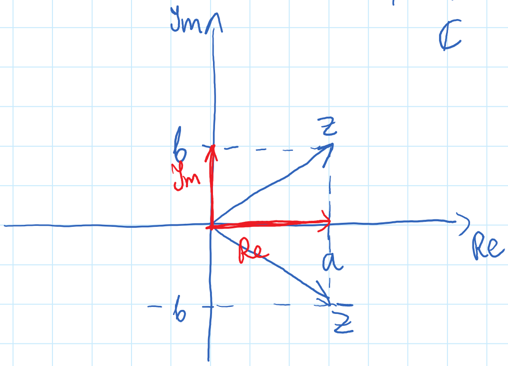
8. $\forall_{z_1 \neq 0} \exists_{z_1^{-1} \in \mathbb{C}} \; z_1 \cdot z_1^{-1} = 1$

9. $z_1(z_2 + z_3) = z_1 z_2 + z_1 z_3$

Slajd 9.

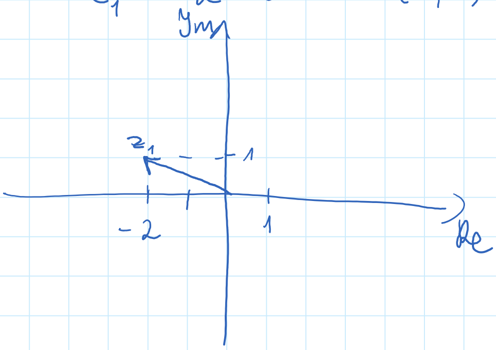
$z = a + bi$, $a, b \in \mathbb{R}$ $\qquad \bar{z} = a - bi$

$\mathbb{C}$ $\qquad (a, -b)$



(a, b)

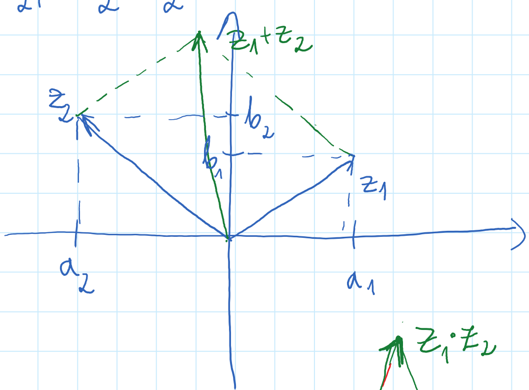
$z_1 = -2 + i$ $\qquad (-2, 1)$



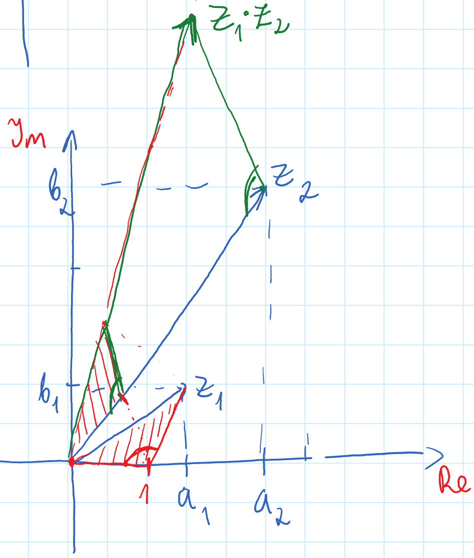
$z_1 = a_1 + b_1 i$

$a_1, a_2, b_1, b_2 \in \mathbb{R}$

$z_2 = a_2 + b_2 i$



Mnożenie



Przykład (slajd 11)

1. $z^2+4i=0$

Niech $z=a+bi$, $a,b\in\mathbb{R}$

$(a+bi)^2+4i=0$

$a^2+2abi+(bi)^2+4i=0$

$a^2+2abi-b^2+4i=0$

$(a^2-b^2)+(2ab+4)i=0+0i$

$\begin{cases} a^2-b^2=0 \\ 2ab+4=0 \end{cases}$

$2ab=-4 \quad /:2$

$ab=-2$

$\begin{cases} a^2-b^2=0 \\ 2ab=-4 \end{cases} \quad /:2a \quad (a\neq 0)$

$\begin{cases} a^2-b^2=0 \\ b=-\frac{2}{a} \end{cases}$

$a^2-\left(-\frac{2}{a}\right)^2=0$

$a^2-\frac{4}{a^2}=0 \quad /\cdot a^2$

$a^4-4=0$

$(a^2-2)(a^2+2)=0$

$a^2-2=0 \quad \vee \quad a^2+2=0$

$a^2=2 \qquad\qquad a^2=-2$

$a=\sqrt{2} \vee a=-\sqrt{2} \qquad$ sprz.

Stąd $b=-\frac{2}{\sqrt{2}}\cdot\frac{\sqrt{2}}{\sqrt{2}}$ lub $b=-\frac{2}{-\sqrt{2}}$

$b=-\sqrt{2} \qquad\qquad b=\sqrt{2}$

$\begin{cases} a=\sqrt{2} \\ b=-\sqrt{2} \end{cases} \vee \begin{cases} a=-\sqrt{2} \\ b=\sqrt{2} \end{cases} \qquad z=a+bi$

Więc $z\in\{\sqrt{2}-\sqrt{2}i, -\sqrt{2}+\sqrt{2}i\}$.

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2. $\operatorname{Re} z-3\operatorname{Im} z=2$, $z=a+bi$, $a,b\in\mathbb{R}$

$\operatorname{Re}(a+bi)-3\operatorname{Im}(a+bi)=2$

$a-3b=2$

$$\begin{cases} a = 2+3b \\ b \in \mathbb{R} \end{cases}$$

$$z \in \{ (2+3b)+bi \mid b \in \mathbb{R} \}$$

3. $$\frac{z+2}{i-1} = \frac{3z+i}{2+i}$$

$$(2+i)(z+2) = (i-1)(3z+i)$$

$$2z + 4 + iz + 2i = 3iz + \underset{=-1}{i^2} - 3z - i$$

Niech $z = a+bi \quad , a, b \in \mathbb{R}$

$$2(a+bi) + 4 + i(a+bi) + 2i = 3i(a+bi) - 1 - 3(a+bi) - i$$

$$2a + 2bi + 4 + ai - b + 2i = 3ai - 3b - 1 - 3a - 3bi - i$$

$$\begin{cases} 2a + 4 - b = -3b - 1 - 3a \\ 2b + a + 2 = 3a - 3b - 1 \end{cases}$$

$\operatorname{Im}(7+3i) = 3$

$$\begin{cases} 5a + 2b = -5 & / \cdot 2 \\ -2a + 5b = -3 & / \cdot 5 \end{cases}$$

$$+\begin{cases} 10a + 4b = -10 \\ -10a + 25b = -15 \end{cases}$$

$$29b = -25$$

$$b = -\frac{25}{29}$$

$$5a + 2 \cdot \left(-\frac{25}{29}\right) = -5 \quad / : 5$$

$$a - \frac{10}{29} = -1$$

$$a = -1 + \frac{10}{29}$$

$$a = \frac{-19}{29}$$

$$\begin{cases} a = -\frac{19}{29} \\ b = -\frac{25}{29} \end{cases}$$

$$\boxed{z = -\frac{19}{29} - \frac{25}{29} i}$$

Dowód ~~słazjot~~ $\overline{z_1 - z_2} = \overline{z_1} - \overline{z_2}$, $z_1 = a_1 + b_1 i$, $a_i, b_i \in \mathbb{R}$

Dowód $\forall_{z_1, z_2 \in \mathbb{C}} \quad \overline{z_1 - z_2} = \overline{z_1} - \overline{z_2}$, $z_1 = a_1 + b_1 i$, $a_i, b_i \in \mathbb{R}$

$z_2 = a_2 + b_2 i$ $\quad i = 1, 2$

Niech $L = \overline{z_1 - z_2}$, $P = \overline{z_1} - \overline{z_2}$.

$L = \overline{(a_1 + b_1 i) - (a_2 + b_2 i)} = \overline{a_1 + b_1 i - a_2 - b_2 i} =$

$= \overline{(a_1 - a_2) + (b_1 - b_2) i} = (a_1 - a_2) - (b_1 - b_2) i$

$P = \overline{a_1 + b_1 i} - \overline{a_2 + b_2 i} = a_1 - b_1 i - (a_2 - b_2 i) =$

$= a_1 - b_1 i - a_2 + b_2 i = (a_1 - a_2) - (b_1 - b_2) i$

$L = P$. Więc teza jest prawdziwa. $\square$

Dowód $\forall_{z_1, z_2 \in \mathbb{C}} \quad \overline{z_1 \cdot z_2} = \overline{z_1} \cdot \overline{z_2}$

Niech $z_1 = a_1 + b_1 i$, $z_2 = a_2 + b_2 i$, $a_i, b_i \in \mathbb{R}$, $i = 1, 2$

$z_1 \cdot z_2 = (a_1 + b_1 i)(a_2 + b_2 i) = a_1 a_2 + a_1 b_2 i + b_1 a_2 i + b_1 b_2 i^2 =$

$= a_1 a_2 + a_1 b_2 i + b_1 a_2 i - b_1 b_2 =$

$= (a_1 a_2 - b_1 b_2) + (a_1 b_2 + b_1 a_2) i$

$\overline{z_1 \cdot z_2} = (a_1 a_2 - b_1 b_2) - (a_1 b_2 + b_1 a_2) i$

$\overline{z_1} \cdot \overline{z_2} = (a_1 - b_1 i)(a_2 - b_2 i) = a_1 a_2 - a_1 b_2 i - b_1 a_2 i + b_1 b_2 i^2 =$

$= a_1 a_2 - a_1 b_2 i - b_1 a_2 i - b_1 b_2 =$

$= (a_1 a_2 - b_1 b_2) - (a_1 b_2 + b_1 a_2) i = \overline{z_1 \cdot z_2}$

$\square$

Dowód $z - \overline{z} = 2i \cdot \operatorname{Im} z$

Niech $z = a + bi$, $a, b \in \mathbb{R}$, $\operatorname{Im} z = b$

$z - \overline{z} = a + bi - \overline{(a + bi)} = a + bi - (a - bi) =$

$= \cancel{a} + bi - \cancel{a} + bi = 2bi = 2 (\operatorname{Im} z) \cdot i = 2i \operatorname{Im} z$

Przykład

Przykład

$$2z + (3-i)\bar{z} = 5+4i$$

Niech $z = a+bi$, $a, b \in \mathbb{R}$

$$2(a+bi) + (3-i)(a-bi) = 5+4i$$

$$2a+2bi+3a-3bi-ai+bi^2 = 5+4i$$

$$2a+2bi+3a-3bi-ai-b = 5+4i$$

// $2bi-3bi-ai =$
$= i\left(\frac{2b\cancel{i}}{\cancel{i}} - \frac{3b\cancel{i}}{\cancel{i}} - \frac{a\cancel{i}}{\cancel{i}}\right.$

$$(2a+3a-b)+(2b-3b-a)i = 5+4i$$

$$\begin{cases} 2a+3a-b=5 \\ 2b-3b-a=4 \end{cases}$$

$$\begin{cases} 5a-b=5 \\ -b-a=4 \end{cases}$$

$$\begin{cases} 5a-b=5 \\ -a-b=4 \quad /\cdot(-1) \end{cases}$$

$$+\begin{cases} 5a-b=5 \\ a+b=-4 \end{cases}$$

$$6a = 1 \quad /:6$$

$$a = \frac{1}{6}$$

$$\frac{1}{6}+b=-4$$

$$b = -4-\frac{1}{6}$$

$$b = -\frac{25}{6}$$

$$\begin{cases} a = \frac{1}{6} \\ b = -\frac{25}{6} \end{cases}$$

Rozwiązanie: $z = \frac{1}{6} - \frac{25}{6}i$