Twierdzenie: $z_1, z_2 \in \mathbb{C}$

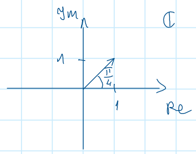
$$z_1 = |z_1| (\cos\varphi_1 + i\sin\varphi_1), \quad // \varphi_1 = \arg z_1$$
$$z_2 = |z_2| (\cos\varphi_2 + i\sin\varphi_2), \quad // \varphi_2 = \arg z_2$$

$z_1 = z_2 \iff$

$$|z_1| = |z_2| = 0 \quad \text{lub} \quad (|z_1| = |z_2| > 0 \ \wedge \ \varphi_1 = \varphi_2 + 2k\pi \text{ dla pewnego } k \in \mathbb{Z})$$

$$z_1 \cdot z_2 = |z_1| \cdot |z_2| \cdot (\cos(\varphi_1 + \varphi_2) + i\sin(\varphi_1 + \varphi_2))$$
$$\frac{z_1}{z_2} = \frac{|z_1|}{|z_2|} (\cos(\varphi_1 - \varphi_2) + i\sin(\varphi_1 - \varphi_2)), \quad z_2 \neq 0$$

Przykład



$$(1+i)(\sqrt{3}+i)$$

$z_1 = 1+i$ $\qquad 1+i = \sqrt{2} \cdot (\cos\frac{\pi}{4} + i\sin\frac{\pi}{4})$

$z_2 = \sqrt{3}+i$

$$|z_2| = \sqrt{(\sqrt{3})^2 + 1^2} = \sqrt{3+1} = \sqrt{4} = 2$$

$$\begin{cases} \cos\varphi = \frac{\sqrt{3}}{2} & + \\ \sin\varphi = \frac{1}{2} & + \end{cases} \Big\} \text{ I ćw.} \qquad \varphi = \frac{\pi}{6}$$

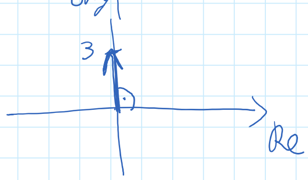
$$\sqrt{3} + i = 2 \cdot (\cos\frac{\pi}{6} + i\sin\frac{\pi}{6})$$

$$z_1 \cdot z_2 = \sqrt{2} \cdot 2 \left(\cos\left(\frac{\pi}{4} + \frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{4} + \frac{\pi}{6}\right)\right) =$$
$$= 2\sqrt{2} \left(\cos\left(\frac{3}{12}\pi + \frac{2}{12}\pi\right) + i\sin\left(\frac{3}{12}\pi + \frac{2}{12}\pi\right)\right) =$$
$$= 2\sqrt{2} \left(\cos\frac{5}{12}\pi + i\sin\frac{5}{12}\pi\right) =$$
$$= 2\sqrt{2} \left(\cos\left(\frac{\pi}{2} - \frac{\pi}{12}\right) + i\sin\left(\frac{\pi}{2} - \frac{\pi}{12}\right)\right) =$$
$$= 2\sqrt{2} \left(\sin\frac{\pi}{12} + i\cos\frac{\pi}{12}\right) = 2\sqrt{2} \cdot \left(\frac{\sqrt{6} - \sqrt{2}}{4} + i \cdot \frac{\sqrt{6} + \sqrt{2}}{4}\right) =$$
$$= \frac{\sqrt{2} \cdot \sqrt{6} - 2}{2} + i\,\frac{\sqrt{2} \cdot \sqrt{6} + 2}{2} = \frac{2\sqrt{3} - 2}{2} + i\left(\frac{2\sqrt{3} + 2}{2}\right) =$$
$$= (\sqrt{3} - 1) + i(\sqrt{3} + 1)$$

$// \sin\frac{\pi}{12} = \frac{\sqrt{6} - \sqrt{2}}{4}$

$// \cos\frac{\pi}{12} = \frac{\sqrt{6} + \sqrt{2}}{4}$

Przykład



$$\frac{3i}{1+i} = \frac{z_1}{z_2}, \quad z_2 \neq 0$$

$$z_1 = 3i = 3\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$$
$$z_2 = 1+i = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$$
$$\frac{z_1}{z_2} = \frac{3}{\sqrt{2}} \cdot \left(\cos\left(\frac{\pi}{2} - \frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{2} - \frac{\pi}{4}\right)\right) =$$
$$= \frac{3\sqrt{2}}{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) = \frac{3\sqrt{2}}{2}\left(\frac{\sqrt{2}}{2} + i\,\frac{\sqrt{2}}{2}\right) =$$

$= \frac{3}{2} + \frac{3}{2} i$

Przykład.

(2) $z^2 = (\bar{z})^2$

$\arg \bar{z} = 2\pi - \arg z$ (**)

Moduły

$|z| = |\bar{z}|$

$|z^2| = \underbrace{|z|^2}$

$|(\bar{z})^2| = |(\bar{z})|^2 = \underbrace{|z|^2}$

$|z^2| = |(\bar{z})^2| \Rightarrow |z| \in \mathbb{R}_+$

Argumenty

(3*) $\arg(z^2) = 2 \cdot \arg z + 2k\pi \quad , k \in \mathbb{Z}$

(4*) $\arg((\bar{z})^2) = 2 \cdot \arg \bar{z} + 2k_1\pi, \quad k_1 \in \mathbb{Z}$

$\overset{(**)}{=} 2 \cdot (2\pi - \arg z) + 2k_1\pi$

Równość argumentów:

$\arg(z^2) = \arg((\bar{z})^2) + 2k_2\pi, \quad k_2 \in \mathbb{Z} \iff$

z (3*) i (4*):

$2 \arg z + 2k\pi = 2 \cdot (2\pi - \arg z) + 2k_1\pi + 2k_2\pi$

$2\arg z + 2\arg z = 4\pi - 2k\pi + 2k_1\pi + 2k_2\pi$

Niech $l = k_1 + k_2 - k \in \mathbb{Z}$.

$4 \arg z = 4\pi + 2l\pi \quad /:4$

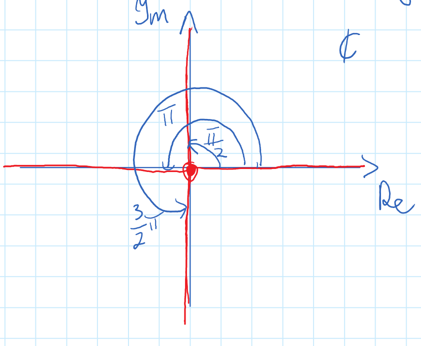
$\arg z = \pi + \frac{1}{2} l\pi \quad , \quad l \in \mathbb{Z}$

niech $l = 0 \Rightarrow \arg z = \pi + 0 = \pi$

niech $l = 1 \Rightarrow \arg z = \pi + \frac{\pi}{2} = \frac{3}{2}\pi$

niech $l = -1 \Rightarrow \arg z = \pi - \frac{\pi}{2} = \frac{\pi}{2}$

niech $l = -2 \Rightarrow \arg z = \pi - \pi = 0$



$z = 0$ jest rozwiązaniem?

$|z|^2 = |\bar{z}|^2 = 0$

$\arg z = 0$

Przykład:

$\frac{\pi}{4} \le \arg \frac{i}{z} \le \frac{\pi}{2} \quad , \quad z \neq 0$

$$\frac{\pi}{4} \le \arg\frac{i}{z} \le \frac{\pi}{2} \quad , \; z \ne 0$$

$$\frac{\pi}{4} \le \arg i - \arg z + 2k\pi \le \frac{\pi}{2} \quad , \; k \in \{0,1\}$$

$\arg i = \frac{\pi}{2}$

$$\frac{\pi}{4} \le \frac{\pi}{2} \underbrace{- \arg z} + 2k\pi \le \frac{\pi}{2}$$

$$\frac{\pi}{4} - \frac{\pi}{2} - 2k\pi \le -\arg z \le \frac{\pi}{2} - \frac{\pi}{2} - 2k\pi$$

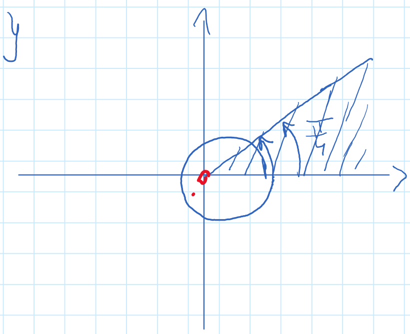
$$-\frac{\pi}{4} - 2k\pi \le -\arg z \le -2k\pi \quad / \cdot (-1)$$

$$\frac{\pi}{4} + 2k\pi \ge \arg z \ge 2k\pi$$

$$2k\pi \le \arg z \le \frac{\pi}{4} + 2k\pi \quad , \; k \in \{0,1\}$$

$k = 0$

$$0 \le \arg z \le \frac{\pi}{4}$$



$k = 1$

$$\begin{cases} 2\pi \le \arg z \le \frac{\pi}{4} + 2\pi \\ 0 \le \arg z < 2\pi \end{cases}$$ sprzeczne

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Przykład. (slajd 22)

$(\sqrt{3} - i)^{60}$, $z = \sqrt{3} - i$, $n = 60$

$$|z| = \sqrt{3+1} = 2$$

$$\begin{cases} \cos\varphi = \frac{\sqrt{3}}{2} \\ \sin\varphi = -\frac{1}{2} \end{cases}$$

$\underline{IV}$ ćw.

$$\varphi = 2\pi - \frac{\pi}{6} = \frac{11}{6}\pi$$

$$z = 2\cdot\left(\cos\frac{11}{6}\pi + i\sin\frac{11}{6}\pi\right)$$

$$z^{60} = 2^{60}\cdot\left(\cos 60\cdot\frac{11}{6}\pi + i\sin 60\cdot\frac{11}{6}\pi\right) =$$

$$= 2^{60}\cdot\left(\cos 110\pi + i\sin 110\pi\right) =$$

$$= 2^{60}\left(\cos 0 + i\sin 0\right) = 2^{60}(1 + i\cdot 0) = 2^{60}$$

Przykład. (slajd 22)

Wyrazić $\cos 3\varphi$ za pomocą $\cos\varphi$ i $\sin\varphi$.

Weźmy $(\cos\varphi + i\sin\varphi)^3 = \boxed{\cos 3\varphi + i\sin 3\varphi}$

$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$

$$(\cos\varphi + i\sin\varphi)^3 = \cos^3\varphi + 3\cdot\cos^2\varphi\cdot i\sin\varphi + 3\cos\varphi\cdot(i\sin\varphi)^2 +$$

$i^2 = -1$

$$(\cos\varphi + i\sin\varphi)^3 = \cos^3\varphi + 3\cdot\cos^2\varphi\cdot i\sin\varphi + 3\cos\varphi\cdot(i\sin\varphi)^2 + (i\sin\varphi)^3 =$$

$a = \cos\varphi$, $b = i\sin\varphi$

$$= \cos^3\varphi + 3i\cos^2\varphi\cdot\sin\varphi + 3\cos\varphi\cdot \underbrace{i^2}_{=-1}\cdot\sin^2\varphi + \underbrace{i^3}_{=-i}\sin^3\varphi =$$

$$= \cos^3\varphi + 3i\cos^2\varphi\cdot\sin\varphi - 3\cos\varphi\cdot\sin^2\varphi - i\sin^3\varphi =$$

$$= \left(\cos^3\varphi - 3\cos\varphi\cdot\sin^2\varphi\right) + i\left(3\cos^2\varphi\cdot\sin\varphi - \sin^3\varphi\right)$$

$i^2 = -1$

$i^3 = i^2\cdot i = -1i$

Wzór:

$$\cos 3\varphi = \cos^3\varphi - 3\cos\varphi\cdot\sin^2\varphi =$$
$$= \cos^3\varphi - 3\cos\varphi\cdot(1-\cos^2\varphi) =$$
$$= \cos^3\varphi - 3\cos\varphi + 3\cos^3\varphi =$$
$$= 4\cos^3\varphi - 3\cos\varphi$$

$$\sin 3\varphi = \ldots$$

Przykład (slajd 24)

$$\sin^2 x = \left(\frac{e^{ix}-e^{-ix}}{2i}\right)^2 = \frac{1}{(2i)^2}\cdot\left(e^{ix}-e^{-ix}\right)^2 =$$

$$= \frac{1}{(2i)^2}\cdot\left(e^{ix}\cdot e^{ix} - 2e^{ix}\cdot e^{-ix} + e^{-ix}\cdot e^{-ix}\right) =$$

$$= \frac{1}{(2i)^2}\cdot\left(e^{2ix} - 2e^{ix-ix} + e^{-2ix}\right) =$$

$$= \frac{1}{(2i)^2}\cdot\left(e^{2ix} - 2 + e^{-2ix}\right) =$$

$$= \frac{1}{(2i)^2}\left(\left(e^{2ix} + e^{-2ix}\right) - 2\right) =$$

$$= \frac{e^{2ix}+e^{-2ix}}{(2i)^2} - \frac{2}{(2i)^2} =$$

$$\frac{1}{(2i)^2} = \frac{1}{2}\cdot\frac{1}{2\cdot i^2} = \frac{1}{2}\cdot\left(-\frac{1}{2}\right)$$

$$\cos x = \frac{e^{ix}+e^{-ix}}{2}$$

$2x$

$$\cos 2x = \frac{e^{2ix}+e^{-2ix}}{2}$$

$$= \frac{e^{2ix}+e^{-2ix}}{2}\cdot\left(-\frac{1}{2}\right) - \frac{2}{-4} =$$

$$= -\frac{1}{2}\cos 2x + \frac{1}{2} = \frac{1}{2}(1-\cos 2x)$$

$$\boxed{\sin^2 x = \frac{1}{2}(1-\cos 2x)} \quad !!$$

Przykład (slajd 24)

$$\sin\alpha\cdot\cos\beta = \frac{e^{i\alpha}-e^{-i\alpha}}{2i}\cdot\frac{e^{i\beta}+e^{-i\beta}}{2} =$$

$$= \frac{1}{4i}\left(e^{i\alpha}\cdot e^{i\beta}+e^{i\alpha}\cdot e^{-i\beta}-e^{-i\alpha}e^{i\beta}-e^{-i\alpha}\cdot e^{-i\beta}\right)=$$

$$= \frac{1}{4i}\left(e^{i(\alpha+\beta)}+e^{i(\alpha-\beta)}-e^{-i(\alpha-\beta)}-e^{-i(\alpha+\beta)}\right)=$$

$$= \frac{1}{4i}\left(\left(e^{i(\alpha+\beta)}-e^{-i(\alpha+\beta)}\right)+\left(e^{i(\alpha-\beta)}-e^{-i(\alpha-\beta)}\right)\right)=$$

$$= \frac{e^{i(\alpha+\beta)}-e^{-i(\alpha+\beta)}}{2\cdot 2i}+\frac{e^{i(\alpha-\beta)}-e^{-i(\alpha-\beta)}}{2\cdot 2i}=$$

$$= \frac{1}{2}\sin(\alpha+\beta)+\frac{1}{2}\cdot\sin(\alpha-\beta)$$

Przykład (slajd 25)

$$z^2=\bar{z}$$

$z_1=z_2 \iff \left(|z_1|=|z_2| \wedge \varphi_1=\varphi_2+2k\pi,\ k\in\mathbb{Z}\right)$ lub $|z_1|=|z_2|=0$

1° Czy $z=0$ jest rozwiązaniem?

$0^2=\bar{0}$

$0=0$ TAK

2° Niech $z\neq 0$.

$z=|z|\cdot e^{i\varphi}$ $\bar{z}=|z|e^{-i\varphi}$

$z^2=|z|^2\cdot e^{2i\varphi}$

$$z^2=\bar{z}$$

$$|z|^2\cdot e^{2i\varphi}=|z|\cdot e^{-i\varphi}$$

$$\begin{cases}|z|^2=|z|\\ e^{2i\varphi}=e^{-i\varphi}\end{cases}$$

$$\begin{cases}|z|^2=|z|\\ 2\varphi=-\varphi+2k\pi, & k\in\mathbb{Z}\end{cases}$$

$|z|^2=|z|$, $|z|>0$ $2\varphi=-\varphi+2k\pi$

$|z|^2-|z|=0$ $3\varphi=2k\pi$

$|z|(|z|-1)=0$ $\varphi=\frac{2}{3}k\pi$

$|z|=0 \vee |z|-1=0$ $\varphi=\{0, \frac{2}{3}\pi, \frac{4}{3}\pi\}$ ($k=0$, $k=1$, $k=2$)

$|z|=1$

// $\varphi=\frac{2}{3}\cdot 2\pi=\frac{4}{3}\pi$

// $k=3$

// $\varphi=\frac{2}{3}\cdot 3\pi=2\pi$

0

Rozwiązanie:

$$z\in\{0, 1\cdot e^{0i}, 1\cdot e^{\frac{2}{3}\pi i}, 1\cdot e^{\frac{4}{3}\pi i}\}$$