Wykład 5

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13:04

Przykład 1 (slajd 9)

$W(x) = x^2 - 2x + 2$

$\Delta = (-2)^2 - 4\cdot 1\cdot 2 = 4 - 8 = -4 = (2\cdot i)^2$

$\delta = 2i \quad \vee \delta = -2i$

$x_1 = \frac{2-2i}{2} = 1 - i$

$x_2 = \frac{2+2i}{2} = 1 + i$

Przykład 2 (slajd 9)

$W(z) = 6z^2 + (5i-3)z - 1 - i$

$\Delta = (5i-3)^2 - 4\cdot 6\cdot(-1-i) =$

$= -25 - 30i + 9 + 24 + 24i = -6i + 8$

Szukamy $\sqrt{-6i+8}$, $\delta^2 = -6i + 8$, $\delta = a + bi$, $a, b \in \mathbb{R}$

$(a+bi)^2 = -6i + 8$

$a^2 + 2abi - b^2 = -6i + 8$

$\begin{cases} a^2 - b^2 = 8 \\ 2ab = -6 \quad /:2b, \quad b \neq 0 \end{cases}$

$\begin{cases} a^2 - b^2 = 8 \\ a = -\frac{3}{b} \end{cases}$

$\left(-\frac{3}{b}\right)^2 - b^2 = 8$

$\frac{9}{b^2} - b^2 = 8 \quad /\cdot b^2$

$9 - b^4 = 8b^2$

$-b^4 - 8b^2 + 9 = 0$

$t = b^2, \quad t \geq 0$

$\Delta_t = 64 + 36 = 100$

$\sqrt{\Delta_t} = 10$

$t_1 = \frac{8-10}{-2} = 1$

$t_2 = \frac{8+10}{-2} = -9 < 0$ sprzeczne

$1 = b^2$

$b = \pm 1$

$$b=1 \qquad\qquad b=-1$$
$$a=-\frac{3}{1} \quad \vee \quad a=-\frac{3}{-1}$$
$$a=-3 \qquad\qquad a=3$$
$$\delta \in \{-3+i,\ 3-i\}$$

$$z_1=\frac{-(5i-3)-(-3+i)}{2\cdot 6}=\frac{-5i+3+3-i}{12}=\frac{6-6i}{12}=\frac{1-i}{2}$$

$$z_2=\frac{-(5i-3)+(-3+i)}{2\cdot 6}=\frac{-5i+3-3+i}{12}=\frac{-4i}{12}=-\frac{i}{3}$$

Przykład.

$$W(z)=1z^3-9z^2+21z-5$$

$$W_0(z)=az^3+bz^2+cz+d$$
$$z=y-\frac{b}{3a}$$

$$a=1,\ b=-9,\ c=21,\ d=-5$$

Podstawienie $z=y-\frac{-9}{3\cdot 1}=y+3$

$$W(y+3)=(y+3)^3-9(y+3)^2+21\cdot(y+3)-5=$$
$$W_1(y) \quad =y^3+3y^2\cdot 3+3\cdot y\cdot 3^2+3^3-9(y^2+6y+9)+$$
$$+21y+63-5=$$
$$=y^3+9y^2+27y+27-9y^2-54y-81+$$
$$+21y+63-5=$$
$$=y^3-6y+4$$

tw. Cardana $\rightarrow$ y_1, y_2, y_3

$$z=y+3$$
$$\begin{cases} z_1=y_1+3 \\ z_2=y_2+3 \\ z_3=y_3+3 \end{cases}$$
miejsca zerowe wielomianu W

Przykład.

$$W(z) = z^3 + 3z - 4 \qquad p = 3, \; q = -4$$

$$\Delta = q^2 + \frac{4}{27} \cdot p^3 = 16 + \frac{4}{\cancel{27}} \cdot \cancel{27} = 16 + 4 = 20 > 0$$

Przypadek ①

$$\delta = \sqrt{20} = \sqrt{4 \cdot 5} = 2\sqrt{5}$$

$$u^3 = \frac{1}{2} \cdot (-q - \delta) = \frac{1}{2} \cdot (4 - 2\sqrt{5}) = 2 - \sqrt{5}$$

$$v^3 = \frac{1}{2}(-q + \delta) = \frac{1}{2}(4 + 2\sqrt{5}) = 2 + \sqrt{5}$$

$$\varepsilon = -\frac{1}{2} + \frac{\sqrt{3}}{2} i$$

$$\varepsilon^2 = \left(-\frac{1}{2} + \frac{\sqrt{3}}{2} i\right)^2 = \frac{1}{4} - \frac{\sqrt{3}}{2} i - \frac{3}{4} = -\frac{1}{2} - \frac{\sqrt{3}}{2} i$$

$$z_1 = u + v = \sqrt[3]{2 - \sqrt{5}} + \sqrt[3]{2 + \sqrt{5}} = 1 \quad !!$$

$$z_2 = \varepsilon u + \varepsilon^2 v = \left(-\frac{1}{2} + \frac{\sqrt{3}}{2} i\right) \cdot \sqrt[3]{2 - \sqrt{5}} + \left(-\frac{1}{2} - \frac{\sqrt{3}}{2} i\right) \sqrt[3]{2 + \sqrt{5}} = -\frac{1}{2} \cdot (1 + i\sqrt{15}) \Leftarrow \text{spr.}$$

$$z_3 = \overline{z_2} = -\frac{1}{2}(1 - i\sqrt{15})$$

Przykład.

$$W(z) = z^3 - 3z + \sqrt{3}, \quad p = -3, \quad q = \sqrt{3}$$

$$\Delta = 3 + \frac{4}{\cancel{27}} \cdot (-\cancel{27}) = 3 - 4 = -1 < 0$$

Przypadek 3.

$$\delta^2 = -1 \iff \delta = i \;\vee\; \delta = -i$$

$$u^3 = \frac{1}{2}(-\sqrt{3} - i), \quad v^3 = \frac{1}{2}(-\sqrt{3} + i)$$

$$u = \overline{v}$$

Postać trygonometryczna u^3:

$$w_u = -\frac{\sqrt{3}}{2} - \frac{1}{2} i \qquad |w_u| = \sqrt{\frac{3}{4} + \frac{1}{4}} = 1$$

$$\begin{cases} \cos\varphi_u = -\frac{\sqrt{3}}{2} \\ \sin\varphi_u = -\frac{1}{2} \end{cases}$$

$\underline{\text{III}}$ ćw.

$$\varphi_u = \pi + \frac{\pi}{6} = \frac{7}{6}\pi$$

$\sin\varphi_u = -\frac{1}{2}$ $\qquad \varphi_u = \pi + \frac{\pi}{6} = \frac{7}{6}\pi$

$\omega_u = \cos\frac{7}{6}\pi + i\sin\frac{7}{6}\pi$

Pierwiastki trzeciego stopnia:

$$u_0 = \cos\frac{\frac{7}{6}\pi + 0}{3} + i\sin\frac{\frac{7}{6}\pi+0}{3} =$$

$$= \cos\frac{7}{18}\pi + i\sin\frac{7}{18}\pi$$

$$u_1 = \cos\left(\frac{7}{18}\pi + \frac{2}{3}\pi\right) + i\sin\left(\frac{7}{18}\pi + \frac{2}{3}\pi\right) =$$

$$= \cos\left(\frac{7}{18}\pi + \frac{12}{18}\pi\right) + i\sin\left(\frac{7}{18}\pi + \frac{12}{18}\pi\right) =$$

$$= \cos\frac{19}{18}\pi + i\sin\frac{19}{18}\pi = \leftarrow$$

$$= \cos\left(\pi + \frac{\pi}{18}\right) + i\sin\left(\pi + \frac{\pi}{18}\right) =$$

$$= -\cos\frac{\pi}{18} - i\sin\frac{\pi}{18}$$

$$u_2 = \cos\left(\frac{19}{18}\pi + \frac{2}{3}\pi\right) + i\sin\left(\frac{19}{18}\pi + \frac{2}{3}\pi\right) =$$

$$= \cos\left(\frac{19}{18}\pi + \frac{12}{18}\pi\right) + i\sin\left(\frac{19}{18}\pi + \frac{12}{18}\pi\right) =$$

$$= \cos\left(\frac{31}{18}\pi\right) + i\sin\left(\frac{31}{18}\pi\right) =$$

$$= \cos\left(2\pi - \frac{5}{18}\pi\right) + i\sin\left(2\pi - \frac{5}{18}\pi\right) =$$

$$= \cos\frac{5}{18}\pi - i\sin\frac{5}{18}\pi$$

Szukamy v takiego, że $u = \bar{v}$, $v = \bar{u}$

oraz $3uv = 3$, $uv = 1$

$$u \cdot \bar{u} = 1$$

$$|u|^2 = 1$$

Postać trygonometryczna v^3.

Niech $\omega_v = \frac{1}{2}(-\sqrt{3} + i)$. Wówczas:

$$|\omega_v| = \sqrt{\frac{3}{4} + \frac{1}{4}} = \sqrt{1} = 1$$

$$\begin{cases} \cos\varphi_v = -\frac{\sqrt{3}}{2} \\ \sin\varphi_v = \frac{1}{2} \end{cases} \qquad \text{II ćw.} \qquad \varphi_v = \pi - \frac{\pi}{6} = \frac{5}{6}\pi$$

$\omega_v = \cos\frac{5}{6}\pi + i\sin\frac{5}{6}\pi$

Pierwiastki trzeciego stopnia:

$$v_0 = \cos\frac{\frac{5}{6}\pi + 0}{3} + i\sin\frac{\frac{5}{6}\pi + 0}{3} =$$

$$= \cos\frac{5}{18}\pi + i\sin\frac{5}{18}\pi$$

$$v_1 = \cos\left(\frac{5}{18}\pi + \frac{2}{3}\pi\right) + i\sin\left(\frac{5}{18}\pi + \frac{2}{3}\pi\right) =$$

$$\cos\frac{5}{18}\pi + i\sin\frac{5}{18}\pi$$

$$v_1 = \cos\left(\frac{5}{18}\pi + \frac{2}{3}\pi\right) + i\sin\left(\frac{5}{18}\pi + \frac{2}{3}\pi\right) =$$
$$= \cos\frac{17}{18}\pi + i\sin\frac{17}{18}\pi$$

$$v_2 = \cos\left(\frac{17}{18}\pi + \frac{2}{3}\pi\right) + i\sin\left(\frac{17}{18}\pi + \frac{2}{3}\pi\right) =$$
$$= \cos\frac{29}{18}\pi + i\sin\frac{29}{18}\pi$$

Mamy, że $v_2 = \overline{u_0}$, ponieważ

$$u_0 = \cos\frac{7}{18}\pi + i\sin\frac{7}{18}\pi$$

oraz $v_2 = \cos\frac{29}{18}\pi + i\sin\frac{29}{18}\pi =$

$$= \cos\left(2\pi - \frac{7}{18}\pi\right) + i\sin\left(2\pi - \frac{7}{18}\pi\right) = \overline{u_0}$$

Postać trygonometryczna ε oraz ε^2:

$\varepsilon = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$ $\quad |\varepsilon| = 1$

$$\begin{cases} \cos\varphi_\varepsilon = -\frac{1}{2} \\ \sin\varphi_\varepsilon = \frac{\sqrt{3}}{2} \end{cases}$$

II ćw.

$\varphi_\varepsilon = \pi - \frac{\pi}{3} = \frac{2}{3}\pi$

$$\varepsilon = \cos\frac{2}{3}\pi + i\sin\frac{2}{3}\pi$$
$$\varepsilon^2 = \cos\frac{4}{3}\pi + i\sin\frac{4}{3}\pi \quad \text{, bo } 2\cdot\frac{2}{3}\pi = \frac{4}{3}\pi$$

$$u\cdot\varepsilon = \cos\left(\frac{7}{18}\pi + \frac{2}{3}\pi\right) + i\sin\left(\frac{7}{18}\pi + \frac{2}{3}\pi\right) =$$
$$= \cos\left(\frac{7}{18}\pi + \frac{12}{18}\pi\right) + i\sin\left(\frac{7}{18}\pi + \frac{12}{18}\pi\right) =$$
$$= \cos\frac{19}{18}\pi + i\sin\frac{19}{18}\pi =$$
$$= \cos\left(\frac{19}{18}\pi - 2\pi\right) + i\sin\left(\frac{19}{18}\pi - 2\pi\right) =$$
$$= \cos\left(-\frac{17}{18}\pi\right) + i\sin\left(-\frac{17}{18}\pi\right) =$$
$$= \underline{\underline{\cos\frac{17}{18}\pi - i\sin\frac{17}{18}\pi}}$$

$$u\cdot\varepsilon^2 = \cos\left(\frac{7}{18}\pi + \frac{4}{3}\pi\right) + i\sin\left(\frac{7}{18}\pi + \frac{4}{3}\pi\right) =$$
$$= \cos\left(\frac{7}{18}\pi + \frac{24}{18}\pi\right) + i\sin\left(\frac{7}{18}\pi + \frac{24}{18}\pi\right) =$$
$$= \cos\frac{31}{18}\pi + i\sin\frac{31}{18}\pi =$$
$$= \cos\left(2\pi - \frac{5}{18}\pi\right) + i\sin\left(2\pi - \frac{5\pi}{18}\right) =$$
$$= \underline{\underline{\cos\frac{5}{18}\pi - i\sin\frac{5}{18}\pi}}$$

Pierwiastki $w(z)$:

$$z_1 = 2\cdot \mathrm{Re}(u) = 2\cdot\cos\frac{7}{18}\pi = 2\cdot\cos\left(\pi - \frac{11\pi}{18}\right) = -2\cos\frac{11}{18}\pi$$

$$z_2 = 2\cdot \mathrm{Re}(\varepsilon\cdot u) = 2\cdot\cos\frac{17}{18}\pi$$

$$z_3 = 2\cdot \mathrm{Re}(\varepsilon^2\cdot u) = 2\cdot\cos\frac{5}{18}\pi$$

Przykład:

$$W(z) = z^3 - 3z + 2 \quad , \quad p = -3, \quad q = 2$$

$$\Delta = q^2 + \frac{4}{27}p^3 = 4 + \frac{4}{\cancel{27}}\cdot(-\cancel{27}) = 4 - 4 = 0$$

Przypadek 2.

$$z_1 = -2\sqrt[3]{\frac{q}{2}} = -2\cdot\sqrt[3]{\frac{2}{2}} = -2$$

$$z_2 = z_3 = \sqrt[3]{\frac{q}{2}} = \sqrt[3]{\frac{2}{2}} = 1$$

Przykład.

$$W(z) = 2\cdot(z+5)^1(z-2i+1)^2\cdot(z+3-i)^5 z^1$$

$$z + 5 = 0 \ \vee\ z - 2i + 1 = 0 \ \vee\ z + 3 - i = 0 \ \vee\ z = 0$$

krotność 1

Stopień wielomianu: 1 + 2 + 5 + 1 krotność 5 krotność 1

Przykład.

$$W(z) = z^3 + 1$$

$$z^3 + 1 = 0$$

$$\omega_0 \quad z^3 = -1 \quad -1 \qquad n = 3$$

obrót o $\frac{2}{3}\pi$: $\cos\frac{2}{3}\pi + i\sin\frac{2}{3}\pi = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$

$$\omega_1 = -1\cdot\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) = \frac{1}{2} - \frac{\sqrt{3}}{2}i$$

$$\omega_2 = \omega_1\cdot\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) = \left(\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)\cdot\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) =$$

$$= -\left(\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)^2 = -\left(\frac{1}{4} - \frac{\sqrt{3}}{2}i - \frac{3}{4}\right) =$$

$$= -\left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right) = \frac{1}{2} + \frac{\sqrt{3}}{2}i$$

$$W(z) = 1 \cdot \left(z - (-1)\right) \cdot \left(z - \left(\frac{1}{2} - \frac{\sqrt{3}}{2} i\right)\right) \cdot \left(z - \left(\frac{1}{2} + \frac{\sqrt{3}}{2} i\right)\right)$$