Numerical schemes for Classical Chemotaxis Equations PC-239

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Introduction

Chemotaxis (movement of biological cells in response to chemical signals) was modeled by Keller-Segel in 1970. Although there are several models, we focus on the classical one, given by the following equations in $\Omega \subset \mathbb{R}^n$:

$$\begin{cases} u_t = \alpha_0 \Delta u - \alpha_1 \nabla \cdot (u \nabla v), \quad x \in \Omega,\ t > 0, & \text{(1a)} \\ v_t = \alpha_2 \Delta v - \alpha_3 v + \alpha_4 u, \quad x \in \Omega,\ t > 0, & \text{(1b)} \\ \nabla u \cdot \mathbf{n} = \nabla v \cdot \mathbf{n} = 0, \quad x \in \partial\Omega,\ t > 0, & \text{(1c)} \\ u(x,0) = u_0(x),\ \ v(x,0) = v_0(x), \quad x \in \Omega, & \text{(1d)} \end{cases}$$

where u and v represent density of cells and chemical-signal, respectively.

From an analytical point of view, a lot of research has been recently done (see e.g. [2] and references therein) and interesting results about global in time existence, mass conservation, energy, blow-up and positivity of solution have been published. However, there is not a large literature on *numerical analysis* for (1), and reproducing former properties is an interesting challenge. This work is mainly focused on development of *positivity preserving numerical schemes*, related to discontinous *Galerkin methods*, which *decouple* calculus of u and v.

Energy-Stable Semi-Discretization in Time

Given a partition of time interval $(0,T)$ into subintervals of size $k > 0$, we approximate u and v at each time step t^{m+1} by an implicit Euler scheme[a] as follows:

$$\begin{cases} \delta_t u^{m+1} - \nabla u^{m+1} + \alpha_1 \nabla \cdot (u^{m+1} \nabla v^{m+1}) = 0, & \text{(2a)} \\ \delta_t v^{m+1} - \alpha_2 \nabla v^{m+1} + \alpha_3 v^{m+1} - \alpha_4 u^m = 0, & \text{(2b)} \end{cases}$$

where δ_t is the backward difference operator. For this scheme, we can show the **energy-stability**: **(COMPLETAR A PARTIR DE TFG DE ALBA?)**

MPP Space Discretization for (2a)

Let $\mathcal{T}_h$ a mesh of Ω and let U_h be a space of $\mathbb{P}_k^d$ (discontinous) polynomials in elements $K \in \mathcal{T}_h$ **(mejorar esta definición, como en [1], sección 2.1)**. We fix[b] $k = 1$. Let V_h be an space of (conforming or not) FE. For each $m \geq 0$ let ${v_h}^{m+1} \in V_h$ computed from (2b), where ${u_h}^{m+1}$ is replaced by $P_{V_h}({u_h}^{m+1})$ (its $L^2(\Omega)$–projection on W_h) and let ${\mathbf{w}_h}^{m+1} = P_{U_h^2}(\nabla {v_h}^{m+1})$.

Let us consider the following discrete problem: find ${u_h}^{m+1} \in U_h$,

$$\int_\Omega \delta_t {u_h}^{m+1} \phi + a_h^{\mathrm{sip}}({u_h}^{m+1}, \phi) + a_h^{\mathrm{god}}({u_h}^{m+1}, \phi) = 0 \quad \forall \phi \in U_h, \tag{3}$$

where

$$\begin{aligned} a_h^{\mathrm{sip}}(u,\phi) &= \sum_{K \in \mathcal{T}_h} \int_K \nabla_h u \nabla_h \phi - \sum_{e \in \mathcal{E}_h} \int_e \left(\{\!\{ \nabla_h u \cdot \mathbf{n}_e \}\!\} [\![\phi]\!] + \{\!\{ \nabla_h \phi \cdot \mathbf{n}_e \}\!\} [\![u]\!] \right) \\ &\quad + \sigma \sum_{e \in \mathcal{E}_h} \int_e \frac{1}{h_e} [\![u]\!] [\![\phi]\!], \\ a_h^{\mathrm{god}}(u,\phi) &= -\sum_K \int_K u \left({\mathbf{w}_h}^{m+1} \cdot \nabla \phi \right) + \sum_{e \in \mathcal{E}_h} \{ u\, {\mathbf{w}_h}^{m+1} \cdot \mathbf{n}_e \}_\star \end{aligned}$$

and

$$\{ u\, {\mathbf{w}_h}^{m+1} \cdot \mathbf{n}_e \}_\star = \textbf{...completar...}$$

is the Godunov (upwind) flux [1].

[a] However, this work can be applied to higher order implicit methods or, following [1] and references therein, to parallel explicit high-order approximations via Strong Stability Preserving (SSP) methods

[b] Results presented here might be improved to high order space approximations by using Bernstein polynomials, see[1]

Test 1. Time Schemes Convergence Order

Exact solution: In order to compute error orders, we introduce the exact solution $u = v = e^{-2t} \cos x \cos y + 2$, adding the following term to the right hand side of (1.a):

$$f = -\alpha_1 (4 \cos x \cos y (e^{2t} + \cos x \cos y) - \cos^2 x - \cos^2 y) e^{-4t}.$$

Experiment data: $\Omega = [0,\pi]^2 \subset \mathbb{R}^2$, $\alpha_i = 1$ for $i = 1, \dots, 4$ and $u_0 = v_0 = 2 + \cos x \cos y$. Discretization 50×50 P2-Lagrange, $k_i = 4 \cdot 10^{-1}/2^i$ for $i = 0, \dots, 5$. Here we show errors orders in norm $L^2(0,T; H^1(\Omega))$.

Test 2. Blow-up and Positivity

References

[1] R. Anderson, V. Dobrev, T. Kolev, D. Kuzmin, M. Quezada de Luna, R. Rieben, and V. Tomov. High-order local maximum principle preserving (MPP) discontinuous Galerkin finite element method for the transport equation. *Journal of Computational Physics*, 334:102–124, 2017.

[2] N. Bellomo, A. Belloquid, Y. Tao, and M. Winkler. Towards a mathematical theory of keller-segel models of pattern formation in biologicl tissues. *Math. Models Meth. Appl. Sci.*, 25(9):1663–1763, 2015.

[3] M. Farina, M. Marras, and G. Viglialoro. On explicit lower bounds and blow-up times in a model of chemotaxis. *Discret. Contin. Dyn. Syst. Suppl.*, pages 409–417, 2015.

[4] F. Hecht. New development in freefem++. *J. Numer. Math.*, 20(3-4):251–265, 2012.

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