MATH 1060-014 Name: Key

2.4 - 2.6 Homework Table: ______

1. (SECTION 2.4)

Use the $\varepsilon - \delta$ definition of a limit to prove the following:

a. $\lim_{x \to -3} (1 - 4x) = 13$

Given $\varepsilon > 0$, we need $\delta > 0$ such that if $0 < |x - (-3)| < \delta$, then $|(1 - 4x) - 13| < \varepsilon$. But,

$$|(1 - 4x) - 13| < \varepsilon \iff \cdots |-4x-12| < \varepsilon \iff |-4(x+3)| < \varepsilon$$
$$\iff 4 \cdot |x+3| < \varepsilon \iff |x+3| < \frac{\varepsilon}{4}$$

So if we choose $\delta = \varepsilon/4$, then

$$0 < |x - (-3)| < \delta \implies |(1-4x)-13| < \varepsilon$$

Thus, $\lim_{x \to -3} (1 - 4x) = 13$ by the definition of a limit.

b. $\lim_{x \to 2} (x^2 - 4x + 5) = 1$.

Given $\varepsilon > 0$, we need $\delta > 0$ such that if $0 < |x-2| < \delta$, then $|(x^2-4x+5)-1| < \varepsilon$.

Now,

$$|(x^2-4x+5)-1| < \varepsilon \iff |x^2-4x+4| < \varepsilon$$
$$\iff |(x-2)^2| < \varepsilon$$
$$\iff (x-2)^2 < \varepsilon$$
$$\iff (x-2) < \sqrt{\varepsilon}$$

So let $\delta = \sqrt{\varepsilon}$.

Then, $0 < |x-2| < \delta \iff |x-2| < \sqrt{\varepsilon}$
$$\iff |(x-2)^2| < \varepsilon$$
$$\iff |(x^2-4x+5)-1| < \varepsilon$$

Thus, $\lim_{x \to 2} (x^2-4x+5) = 1$ by the definition of a limit.

2. (SECTION 2.5)

a. Show that f is continuous on $(-\infty, \infty)$. $f(x) = \begin{cases} x^2 & \text{if } x < 1 \\ \sqrt{x} & \text{if } x \geq 1. \end{cases}$

(Hint: For $x = 1$, explore both one-sided limits and $f(1)$. Also be sure to describe the continuity of f for all $x \neq 1$.)

Since $f(x) = x^2$ on $(-\infty, 1)$ which is a polynomial, f is continuous on $(-\infty, 1)$.
Since $f(x) = \sqrt{x}$ on $(1, \infty)$ is a root function f is continuous on $(1, \infty)$.

At $x=1$,
$$\left.\begin{array}{l} \lim_{x\to 1^-} f(x) = \lim_{x \to 1^-} (x^2) = 1 \\ \lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} (\sqrt{x}) = 1 \end{array}\right\} \Rightarrow \lim_{x\to 1} f(x) = 1 = f(1)$$

Thus, f is continuous at $x=1$. So f is continuous on $(-\infty, \infty)$.

b. Find the numbers at which f is discontinuous. At each of these numbers specify whether f is continuous from the right, from the left, or neither?

$$f(x) = \begin{cases} 1 + x^2 & \text{if } x \leq 0 \\ 2 - x & \text{if } 0 < x \leq 2 \\ (x-2)^2 & \text{if } x > 2. \end{cases}$$

$x=0$: $f(0) = 1$

* continuous from the left.

$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} (1+x^2) = 1 = f(0)$.

$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} (2-x) = 2$

Note: $f(x)$ is continuous at $x=2$.
$x=0$ is the only discontinuity of f.

c. Use the Intermediate Value Theorem to show that there is a root of the given equation in the specified interval: $e^x = 3 - 2x$, $(0, 1)$

$e^x = 3-2x \Leftrightarrow e^x + 2x - 3 = 0$ Let $f(x) = e^x + 2x - 3$.

$a=0$
$b=1$
$N=0$.

$f(x)$ is continuous on $[0,1]$.

$f(0) = 1 + 0 - 3 = -2 < 0$

$f(1) = e + 2 - 3 = e - 1 > 0$

Since $-2 < 0 < e-1$, by the IVT there is a number c in $(0,1)$ such that $f(c) = 0$. Thus, there is a root of the equation $e^x = 3-2x$ in the interval $(0,1)$.

3. (Section 2.6)

a. Sketch the graph of a function f that satisfies all of the given conditions:

$f(0) = 3,$

$\lim_{x\to 0^-} f(x) = 4,$

$\lim_{x\to 0^+} f(x) = 2,$

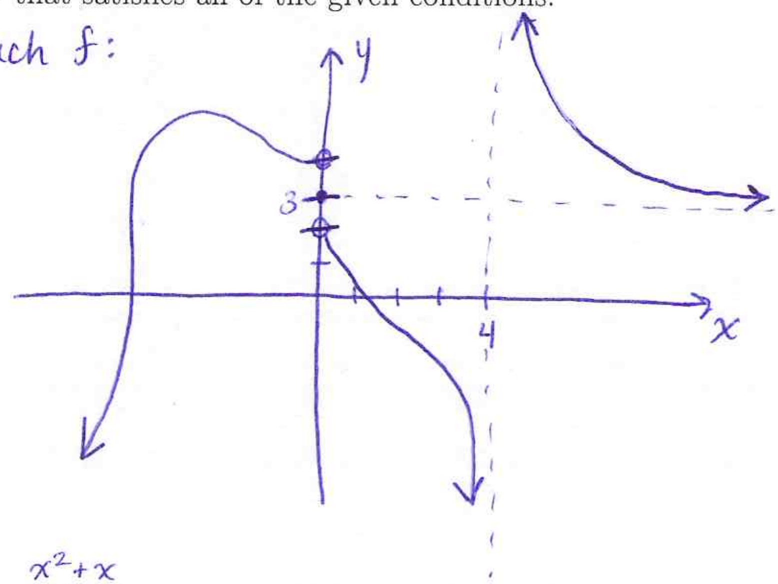
$\lim_{x\to -\infty} f(x) = -\infty,$

$\lim_{x\to 4^-} f(x) = -\infty,$

$\lim_{x\to 4^+} f(x) = \infty,$

$\lim_{x\to \infty} f(x) = 3.$

One such f:



b. Find $\lim_{x\to\infty} \frac{x^2+x}{3-x}$.

$$\lim_{x\to\infty} \frac{x^2+x}{3-x} = \lim_{x\to\infty} \frac{\frac{x^2+x}{x}}{\frac{3-x}{x}} = \lim_{x\to\infty} \frac{x+1}{\frac{3}{x}-1} = -\infty$$

since $x+1 \to \infty$ as $x\to\infty$

and $\frac{3}{x}-1 \to -1$ as $x\to\infty$.

c. Find $\lim_{x\to-\infty} (x+\sqrt{x^2+2x})$.

$$\lim_{x\to-\infty} (x+\sqrt{x^2+2x}) = \lim_{x\to-\infty} (x+\sqrt{x^2+2x}) \cdot \frac{x-\sqrt{x^2+2x}}{x-\sqrt{x^2+2x}}$$

$$= \lim_{x\to-\infty} \frac{x^2-(x^2+2x)}{x-\sqrt{x^2+2x}}$$

$$= \lim_{x\to-\infty} \frac{-2x}{x-\sqrt{x^2+2x}}$$

$$= \lim_{x\to-\infty} \frac{-2}{1-\frac{\sqrt{x^2+2x}}{-\sqrt{x^2}}}$$

[Note: since $x<0$, $x = -\sqrt{x^2}$]

$$= \lim_{x\to-\infty} \frac{-2}{1+\sqrt{1+\frac{2}{x}}}$$

$$= \frac{-2}{2} = -1.$$

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Quiz 2.3 Table: ______

Directions: For #1-3, find the limit if it exists. If the limit does not exist, explain why. Remember laws of limits and show all work!

Note: Though not required, if you do use direct substitution from the start on any problem you MUST explain why you can use direct substitution.

1. $\lim_{x\to 1}(x^4 - 3x)(x^2+3) = \left[\lim_{x\to 1}(x^4-3x)\right]\cdot\left[\lim_{x\to 1}(x^2+3)\right]$

$= \left[\lim_{x\to 1}x^4 - 3\cdot\lim_{x\to 1}x\right]\cdot\left[\lim_{x\to 1}x^2 + \lim_{x\to 1}3\right]$

$= \left((1)^4 - 3\cdot 1\right)\cdot\left((1)^2+3\right)$

$= (-2)(4) = \boxed{-8}$

*Could use direct substitution here also b/c the function is polynomial and x=1 is in the domain.

2. $\lim_{x\to -6}\frac{2x+12}{|x+6|}$

$\lim_{x\to -6^+}\frac{2x+12}{|x+6|} = \lim_{x\to -6^+}\frac{2x+12}{x+6} = \lim_{x\to -6^+}\frac{2(x+6)}{x+6} = \lim_{x\to -6^+} 2 = 2$

$\lim_{x\to -6^-}\frac{2x+12}{|x+6|} = \lim_{x\to -6^-}\frac{2(x+6)}{-(x+6)} = \lim_{x\to -6^-}(-2) = -2$

Since $\lim_{x\to -6^+}\frac{2x+12}{|x+6|} \neq \lim_{x\to -6^-}\frac{2x+12}{|x+6|}$, $\boxed{\lim_{x\to -6}\frac{2x+12}{|x+6|} \text{ does not exist.}}$

3. $\lim_{x\to -4}\frac{\sqrt{x^2+9}-5}{x+4} = \lim_{x\to -4}\frac{\sqrt{x^2+9}-5}{x+4}\cdot\frac{\sqrt{x^2+9}+5}{\sqrt{x^2+9}+5} = \lim_{x\to -4}\frac{x^2+9-25}{(x+4)(\sqrt{x^2+9}+5)}$

$= \lim_{x\to -4}\frac{x^2-16}{(x+4)(\sqrt{x^2+9}+5)} = \lim_{x\to -4}\frac{(x+4)(x-4)}{(x+4)(\sqrt{x^2+9}+5)}$

$= \lim_{x\to -4}\frac{x-4}{\sqrt{x^2+9}+5} = \frac{-4-4}{\sqrt{(-4)^2+9}+5} = \frac{-8}{\sqrt{25}+5} = \frac{-8}{10} = \boxed{-\frac{4}{5}}$

4. **True or False**: (circle one)

a) $\frac{x^2+x-6}{x-2} = x+3.$ True **False**

b) $\lim_{x\to 5}\frac{x^2-25}{x+5} = \lim_{x\to 5}(x-5).$ **True** False

Bonus: What is the first name of the Teaching Assistant for this class? Tianhui (or Tia)

What is the first name of the SI for this class? Brad