Student's Printed Name: ___________________________ **CUID:** ___________

Instructor: _______________________________________ **Section:** ___________

Instructions: You are **not** permitted to use a calculator on any portion of this test. You are **not** allowed to use any textbook, notes, cell phone, laptop, PDA, or any technology on either portion of this test. All devices must be turned off while you are in the testing room.

During this test, any communication with any person (other than the instructor or a designated proctor) in any form, including written, signed, verbal, or digital, is understood to be a violation of academic integrity.

No part of this test may be removed from the examination room.

Read each question very carefully. In order to receive full credit for the free response portion of the test, you must:

1. Show legible and logical (relevant) justification which supports your final answer.
2. Use complete and correct mathematical notation.
3. Include proper units, if necessary.
4. Give exact numerical values whenever possible.

You have **90 minutes** to complete the entire test.

On my honor, I have neither given nor received inappropriate or unauthorized information at any time before or during this test.

Student's Signature: ____

Do not write below this line.

Free Response Problem	Possible Points	Points Earned	Free Response Problem	Possible Points	Points Earned
1.a.	7		5.a.	4	
1.b.	7		5.b.	8	
1.c.	7		6.a.	6	
1.d.	7		6.b.	6	
2.a.	11		7.	8	
2.b.	3		**Test Total**	100	
3.	14				
4.a.	6				
4.b.	6				

Read each question carefully. In order to receive full credit you must show legible and logical (relevant) justification which supports your final answer. Give answers as exact answers. You are NOT permitted to use a calculator on any portion of this test.

1. **(7 pts. each)** Find the indicated derivatives. Assume *g(x)* is a differentiable function wherever it appears. Cancel common factors wherever appropriate.

a) Find $f'(x)$ if $f(x) = x^2 g(\sin(\cos(x)))$.

$$f(x) = x^2 g(\sin(\cos x))$$
$$f'(x) = x^2 g'(\sin(\cos x))\cos(\cos x)(-\sin x) + 2xg(\sin(\cos x))$$

Work on Problem:	Points
Uses Product Rule	2 points
$x^2 g'(x)$	3 points
$2xg(x)$	2 points

Notes:

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b) Find $\frac{dy}{dx}$ if $\sin(5y) = ye^{x^2}$.

$$\sin(5y) = ye^{x^2}$$
$$\cos(5y)(5)\frac{dy}{dx} = y(2xe^{x^2}) + e^{x^2}\frac{dy}{dx}$$
$$5\cos(5y)\frac{dy}{dx} - e^{x^2}\frac{dy}{dx} = 2xye^{x^2}$$
$$\frac{dy}{dx}\left(5\cos(5y) - e^{x^2}\right) = 2xye^{x^2}$$
$$\frac{dy}{dx} = \frac{2xye^{x^2}}{5\cos(5y) - e^{x^2}}$$

Work on Problem:	Points
Derivative of left side	2 points
Derivative of right side	3 points
Solves for $\frac{dy}{dx}$	2 points

Notes:

- Other equivalent forms of the answer obtained with logarithmic differentiation can get full credit.

c) Find $h'(x)$ if $h(x) = \ln\left(g\left(\frac{3^x}{e^{x-1}}\right)\right)$.

$$h(x) = \ln\left(g\left(\frac{3^x}{e^{x-1}}\right)\right)$$

$$h'(x) = \frac{1}{g\left(\frac{3^x}{e^{x-1}}\right)} g'\left(\frac{3^x}{e^{x-1}}\right) \frac{(e^{x-1})(3^x)\ln 3 - (3^x)(e^{x-1})}{(e^{x-1})^2}$$

$$h'(x) = \frac{1}{g\left(\frac{3^x}{e^{x-1}}\right)} g'\left(\frac{3^x}{e^{x-1}}\right) \frac{(e^{x-1})(3^x)\left[\ln 3 - 1\right]}{(e^{x-1})^2}$$

$$h'(x) = \frac{1}{g\left(\frac{3^x}{e^{x-1}}\right)} g'\left(\frac{3^x}{e^{x-1}}\right) \frac{(3^x)\left[\ln 3 - 1\right]}{e^{x-1}}$$

Work on Problem:	**Points**
Derivative natural log function	1 point
Derivative of function *g*	1 point
Derivative of argument for function g	4 points
Factor and reduce	1 point

Notes:
- Subtract ½ for missing derivative notation

d) Find $\frac{dy}{dx}$ if $y = x^{(x^3)}$.

$$y = x^{(x^3)}$$

(logaritmic differentiation)

$$\ln(y) = \ln(x^{(x^3)})$$

$$\ln(y) = x^3 \ln x$$

$$\frac{1}{y}\frac{dy}{dx} = x^3\left(\frac{1}{x}\right) + (\ln x)(3x^2)$$

$$\frac{dy}{dx} = \left[x^2 + 3x^2 \ln x\right] y$$

$$\frac{dy}{dx} = \left[x^2 + 3x^2 \ln x\right] x^{(x^3)}$$

Work on Problem:	**Points**
Natural log of both sides	1 point
Simplifies right side with log properties	1 point
Derivative of left side	1 point
Derivative of right side	2 points
Solves for $\frac{dy}{dx}$	1 point
Replaces *y*	1 point

Notes:
- Subtract ½ for missing derivative notation

2. You are adding a layer of special insulating material to the outer surface of the **hemispherical** dome of a large building. The radius of the dome is 50 feet and the insulating material will have a thickness of 0.10 feet.

Note: The volume V of a **sphere** of radius r is $V = \frac{4\pi}{3} r^3$.

a) (**11 pts.**) Use differentials to estimate how much of the insulating material (in cubic feet) you will have to purchase for the project.

$V = \frac{1}{2}\left(\frac{4\pi}{3} r^3\right)$ (volume of hemisphere)

$V = \frac{2\pi}{3} r^3$

$\frac{dV}{dr} = \frac{2\pi}{3}(3r^2)$

$dV = 2\pi r^2 dr$

when $r = 50$ and $dr = 0.10$

$dV = 2\pi(50)^2(0.10)$

$dV = 2\pi(2500)(0.10)$

$dV = 500\pi \ ft^3$

Work on Problem:	Points
Volume of hemisphere	1 point
Differentiates both sides	4 points
Solves for differential dV	1 point
Substitutes values for r and dr	2 points
Calculates dV	2 points
Units	1 point

Notes:
- Subtract 2 points for calculating dV for a sphere

b) (**3 pts.**) Structural engineers estimate the dome of the building can safely support an additional 90,000 pounds. The insulating material you are using weighs 60 pounds per cubic foot. Use your result from part (a) to determine if your insulating project puts the building in jeopardy.

Weight $\approx (500\pi)(60) = 30{,}000\pi$ pounds

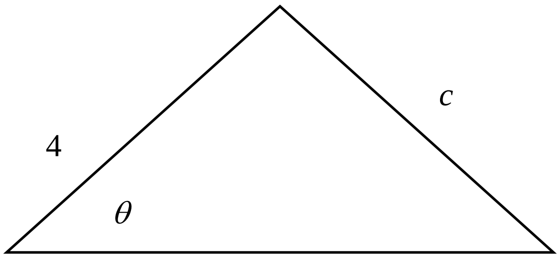
$30{,}000\pi > 90{,}000$

so the project puts the building in jeopardy.

Work on Problem:	Points
Multiples result from (a) by 60	1 point
Conclusion	2 points

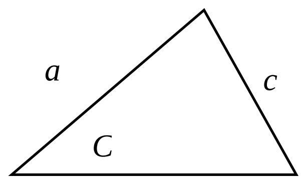
Notes:
- Maximum of 2 points for correctly using a wrong result from part (a)

3. (**14 pts.**) Consider the triangle shown below, with sides of lengths 4 meters, 5 meters, and angle θ.



Find the rate at which the length of side c is changing when angle $\theta = \frac{\pi}{3}$ and θ is **increasing** at 2 radians per second.

Hint: Law of Cosines: $c^2 = a^2 + b^2 - 2ab\cos C$



$c^2 = 4^2 + 5^2 - 2(4)(5)\cos\theta$

$c^2 = 41 - 40\cos\theta$

$2c\frac{dc}{dt} = 0 - 40\left(-\sin\theta\frac{d\theta}{dt}\right)$

$\frac{dc}{dt} = \frac{40\sin\theta\frac{d\theta}{dt}}{2c}$

when $\theta = \frac{\pi}{3}$

$\frac{dc}{dt} = \frac{40\sin\left(\frac{\pi}{3}\right)(2)}{2\sqrt{21}}$

$\frac{dc}{dt} = \frac{40\left(\frac{\sqrt{3}}{2}\right)(2)}{2\sqrt{21}}$

$\frac{dc}{dt} = \frac{20\sqrt{3}}{\sqrt{21}}$ meters/second

solve for c when $\theta = \frac{\pi}{3}$

$c = \sqrt{41 - 40\cos\frac{\pi}{3}}$

$c = \sqrt{41 - 40\left(\frac{1}{2}\right)}$

$c = \sqrt{41 - 20}$

$c = \sqrt{21}$

Work on Problem:	**Points**
Equation for c^2 with values for a and b	2 points
Derivative of left side	2 points
Derivative of right side	3 points
Calculates c when $\theta = \frac{\pi}{3}$	3 points
Substitutes values for θ and c	1 point
Solves for $\frac{dc}{dt}$	2 points
units	1 point

Notes:
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4. Let f and g be differentiable functions. Use the values in the table below to answer the questions that follow.

	***x*=0**	***x*=1**	***x*=2**	***x*=3**
$f(x)$	0	1	8	27
$f'(x)$	0	3	12	27
$g(x)$	1	3	9	19
$g'(x)$	0	2	4	12

a) **(6 pts.)** Find $h'(1)$ if $h(x) = \left[f(x)g(x)\right]^2$.

$$h(x) = \left[f(x)g(x)\right]^2$$
$$h'(x) = 2f(x)g(x)\left[f(x)g'(x) + f'(x)g(x)\right]$$
$$h'(1) = 2f(1)g(1)\left[f(1)g'(1) + f'(1)g(1)\right]$$
$$h'(1) = 2(1)(3)\left[(1)(2) + (3)(3)\right]$$
$$h'(1) = 6\left[11\right]$$
$$h'(1) = 66$$

Work on Problem:	**Points**
Finds $h'(x)$	4 points
Substitutes values and solves	2 points
Notes: •	

b) **(6 pts.)** Find $P'(3)$ if $P(x) = f\left(g\left(\frac{x}{3}\right)\right)$.

$$P(x) = f\left(g\left(\frac{x}{3}\right)\right)$$
$$P'(x) = f'\left(g\left(\frac{x}{3}\right)\right)g'\left(\frac{x}{3}\right)\left(\frac{1}{3}\right)$$
$$P'(3) = f'\left(g\left(\frac{3}{3}\right)\right)g'\left(\frac{3}{3}\right)\left(\frac{1}{3}\right)$$
$$P'(3) = f'\left(g(1)\right)g'(1)\left(\frac{1}{3}\right)$$
$$P'(3) = f'(3)g'(1)\left(\frac{1}{3}\right)$$
$$P'(3) = (27)(2)\left(\frac{1}{3}\right)$$
$$P'(3) = 18$$

Work on Problem:	**Points**
Finds $P'(x)$	4 points
Substitutes values and solves	2 points
Notes: •	

5. A tank holds 5000 gallons of water which drains from a hole in the bottom. Torricelli's Law gives the volume V of water remaining in the tank t minutes after it starts draining:

$$V(t) = 5000\left(1 - \frac{t}{40}\right)^2$$

a) (**4 pts.**) At what time will the tank be empty?

Solve $V(t) = 0$

$$5000\left(1 - \frac{t}{40}\right)^2 = 0$$

$$\left(1 - \frac{t}{40}\right)^2 = 0$$

$$1 - \frac{t}{40} = 0$$

$$\frac{t}{40} = 1$$

$t = 40$ minutes

Work on Problem:	**Points**
Sets $V(t) = 0$	1 point
Solves for t	3 points

Notes:
- Subtract 4 points for only solving $V'(t) = 0$
- Subtract 3 points for major algebraic errors

b) (8 **pts.**) At what rate is the volume changing when $t = 20$?

$$V(t) = 5000\left(1 - \frac{t}{40}\right)^2$$

$$V'(t) = 5000(2)\left(1 - \frac{t}{40}\right)\left(-\frac{1}{40}\right)$$

$$V'(20) = 5000(2)\left(1 - \frac{20}{40}\right)\left(-\frac{1}{40}\right)$$

$$V'(20) = 5000(2)\left(\frac{1}{2}\right)\left(-\frac{1}{40}\right)$$

$$V'(20) = -\frac{5000}{40}$$

$V'(20) = -125$ gallons/minute

Work on Problem:	**Points**
Finds $V'(t)$	5 point
Substitutes $t = 20$	1 points
Finds $V'(20)$	1 point
units	1 point

Notes:
- No points awarded for finding $V(20)$

6. A sample of Strontium-90 will decay to **25%** of its original amount in 56 days.

a) (**6 pts.**) A sample has a mass of 60 mg initially. Find the amount remaining after 70 days. Give your final answer in terms of e and natural logarithms.

letting $m =$ mass of Strontium-90

$m(t) = 60e^{kt}$

know $m(56) = 0.25(60) = 15$

apply $m(56) = 15$

$15 = 60e^{k(56)}$

$\frac{1}{4} = e^{k(56)}$

$\ln\left(\frac{1}{4}\right) = \ln e^{k(56)}$

$\ln\left(\frac{1}{4}\right) = 56k$

$k = \frac{\ln\left(\frac{1}{4}\right)}{56}$

$m(t) = 60e^{\frac{\ln(1/4)}{56}t}$

$m(70) = 60e^{\frac{\ln(1/4)}{56}70}$ mg

Work on Problem:	Points
Solves for k	5 point
Substitutes $t = 70$	1 points

Notes:
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b) (**6 pts.**) How long will it take the initial amount of 60 mg to decay to 6 mg? Give your final answer in terms of natural logarithms.

$m(t) = 60e^{\frac{\ln(1/4)}{56}t}$

solve

$m(t) = 6$

$6 = 60e^{\frac{\ln(1/4)}{56}t}$

$\frac{6}{60} = e^{\frac{\ln(1/4)}{56}t}$

$\ln(1/10) = \ln e^{\frac{\ln(1/4)}{56}t}$

$\ln(1/10) = \frac{\ln(1/4)}{56}t$

$t = \frac{56\ln(1/10)}{\ln(1/4)}$ days

Work on Problem:	Points
Sets $m(t) = 6$	1 point
Solves for t	5 points

Notes:
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7. (**8 pts.**) Prove $\dfrac{1+\tanh x}{1-\tanh x}=e^{2x}$

$$\frac{1+\tanh x}{1-\tanh x}=\frac{1+\dfrac{\sinh x}{\cosh x}}{1-\dfrac{\sinh x}{\cosh x}}$$

$$=\frac{\cosh x+\sinh x}{\cosh x-\sinh x}$$

$$=\frac{\dfrac{e^x+e^{-x}}{2}+\dfrac{e^x-e^{-x}}{2}}{\dfrac{e^x+e^{-x}}{2}-\dfrac{e^x-e^{-x}}{2}}$$

$$=\frac{e^x+e^{-x}+e^x-e^{-x}}{e^x+e^{-x}-e^x+e^{-x}}$$

$$=\frac{2e^x}{2e^{-x}}$$

$$=\frac{e^x}{e^{-x}}$$

$$=e^{x-(-x)}$$

$$=e^{2x}$$

Work on Problem:	**Points**
Converts to sinh$x$ and cosh$x$	2 point
Definitions of sinh$x$ and cosh$x$ in terms of e	2 points
Simplification	4 points

Notes:

- Subtract 1 point for improper use of equal signs