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3.3 LA - Derivatives of Trigonometric Functions Table: ______

1. Differentiate $g(t) = \dfrac{t\sin t}{1+t}$.

$$g'(t) = \frac{(1+t)[t\cdot\cos t + \sin t] - t\sin t}{(1+t)^2}$$

$$= \boxed{\frac{t\cos t}{1+t} + \frac{\sin t}{(1+t)^2}}$$

OR $\boxed{\dfrac{(1+t)t\cos t + \sin t}{(1+t)^2}}$

2. Differentiate $f(p) = e^4(p^2 - \cos p)$.

$$f'(p) = 2e^4 p + e^4 \sin p$$

$$= \boxed{e^4(2p + \sin p)}$$

3. If $f(t) = \csc t$, find $f''(\frac{\pi}{6})$.

$$f'(t) = -\csc t\cot t$$

$$f''(t) = -(\csc t(-\csc^2 t) + \cot t(-\csc t\cot t))$$

$$= \csc^3 t + \csc t\cot^2 t$$

$$f''(\tfrac{\pi}{6}) = \frac{1}{(\sin(\frac{\pi}{6}))^3} + \frac{\cos^2(\frac{\pi}{6})}{\sin^3(\frac{\pi}{6})}$$

$$= 8 + \tfrac{3}{4}(8) = 8 + 6 = \boxed{14}$$

4. Differentiate $f(t) = \dfrac{\cot t}{e^t}$.

$$f'(t) = \frac{e^t(-\csc^2 t) - \cot t\cdot e^t}{e^{2t}}$$

$$= \frac{e^t(-\csc^2 t - \cot t)}{e^{2t}}$$

$$= \boxed{\frac{-\csc^2 t - \cot t}{e^t}} \text{ OR } \boxed{-\frac{\csc^2 t + \cot t}{e^t}}$$

5. Find an equation of the the tangent line to the curve $f(x) = x + \tan x$ at the point (π, π).

$$f'(x) = 1 + \sec^2 x$$

$$m_{tan} = f'(\pi) = 1 + \sec^2\pi = 1 + (-1)^2 = 2$$

$$y - \pi = 2(x - \pi)$$

$\Rightarrow \boxed{y = 2x - \pi}$ is an equation of the tangent line.

6. For what values of x is the graph of $f(x) = e^x\cos x$ at a relative minimum or maximum?

$$f'(x) = e^x(-\sin x) + \cos x\cdot e^x = e^x(\cos x - \sin x)$$

Relative max or min $\Rightarrow$ horizontal tangent line $\Rightarrow$ slope of zero!

$f'(x) = 0 \iff e^x(\cos x - \sin x) = 0$

$\iff \cos x = \sin x$

$\iff \tan x = 1$

$\iff \boxed{x = \frac{\pi}{4} \pm n\pi \text{ for } n = 0,1,2,\ldots}$

7. Evaluate $\lim_{m \to 0} \dfrac{\sin 3m}{5m^3 - 4m}$.

$$\lim_{m \to 0} \frac{\sin 3m}{5m^3-4m} = \lim_{m \to 0} \left(\frac{\sin 3m}{3m} \cdot \frac{3}{5m^2-4} \right)$$
$$= \lim_{m \to 0} \frac{\sin 3m}{3m} \cdot \lim_{m \to 0} \frac{3}{5m^2-4}$$
$$= 1 \cdot \left(-\frac{3}{4}\right) = \boxed{-\frac{3}{4}}$$

8. Find the limit: $\lim_{x \to 0} \dfrac{\sin 3x \sin 5x}{x^2}$.

$$\lim_{x \to 0} \frac{\sin 3x \cdot \sin 5x}{x^2} = \lim_{x \to 0} \left(3 \cdot \frac{\sin 3x}{3x} \cdot 5 \cdot \frac{\sin 5x}{5x} \right)$$
$$= 3 \cdot \lim_{x \to 0} \frac{\sin 3x}{3x} \cdot 5 \cdot \lim_{x \to 0} \frac{\sin 5x}{5x}$$
$$= 3 \cdot (1) \cdot 5 \cdot (1)$$
$$= \boxed{15}$$

9. Let $y = \sin\theta \cos\theta$. Find $\left.\dfrac{dy}{d\theta}\right|_{\theta=\pi}$.

$$\frac{dy}{d\theta} = \sin\theta(-\sin\theta) + \cos\theta\cos\theta = \cos^2\theta - \sin^2\theta$$

$$\left.\frac{dy}{d\theta}\right|_{\theta=\pi} = \cos^2(\pi) - \sin^2(\pi) = (-1)^2 - 0^2 = \boxed{1}$$

10. Define $f(x) = \sin x$. Find $f^{99}(x)$ by finding the first few derivatives and observing the pattern that occurs.

$99 = 4(24) + 3$

$f'(x) = \cos x$

$f''(x) = -\sin x$

$f'''(x) = -\cos x$

$f^{(4)}(x) = \sin x \Rightarrow$ every 4^{th} derivative is $\sin x$, so $f^{(100)}(x) = \sin x$.

$$\boxed{f^{(99)}(x) = -\cos x}$$

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Learning Activity 3.4 – The Chain Rule

1. Use Version 1 of the Chain Rule to find the derivative of $y = f(x) = (3x+7)^{10}$.

Inside Function: $u = g(x) = 3x + 7$ $\qquad \dfrac{du}{dx} = 3$

Outside Function: $y = f(u) = u^{10}$ $\qquad \dfrac{dy}{du} = 10u^9$

$$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx} = 10u^9 \cdot 3 = 30u^9 = 30(3x+7)^9$$

2. Use Version 1 of the Chain Rule to find the derivative of $y = f(x) = \sqrt{\cos x}$.

Inside Function: $u = g(x) = \cos x$ $\qquad \dfrac{du}{dx} = -\sin x$

Outside Function: $y = f(u) = \sqrt{u} = u^{1/2}$ $\qquad \dfrac{dy}{du} = \dfrac{1}{2}u^{-1/2} = \dfrac{1}{2\sqrt{u}}$

$$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx} = \frac{1}{2\sqrt{u}}\cdot\left(-\sin x\right) = \frac{-\sin x}{2\sqrt{\cos x}}$$

3. Find the derivative of the function by any means you choose. You do not need to simplify.

a. $y = \sqrt{x^2 + 9x - 1}$

$$y' = \frac{1}{2}\left(x^2 + 9x - 1\right)^{-1/2}(2x+9) = \frac{2x+9}{2\sqrt{x^2+9x-1}}$$

b. $y = \sin(x \cos x)$

$$y' = \cos(x\cos x)\{[1]\cos x + x[-\sin x]\}$$
$$= \cos(x\cos x)(\cos x - x\sin x)$$

c. $y = (\sec x + \tan x)^5$

$$\frac{dy}{dx} = 5(\sec x + \tan x)^4\left(\sec x\tan x + \sec^2 x\right) = 5\sec x(\sec x + \tan x)^5$$

d. $y = \sqrt{\cos(\sin^2 x)}$

$$y' = \frac{1}{2}\left(\cos\left(\sin^2 x\right)\right)^{\frac{-1}{2}}[-\sin(\sin^2 x)(2\sin x\cos x)]$$

$$= -\sin(\sin^2 x)\sin x\cos x\left(\cos\left(\sin^2 x\right)\right)^{\frac{-1}{2}}$$

$$= \frac{-\sin(\sin^2 x)\sin x\cos x}{\sqrt{\cos(\sin^2 x)}}$$

e. $y = \left(\dfrac{3x}{4x+2}\right)^5$

$$\frac{dy}{dx} = 5\left(\frac{3x}{4x+2}\right)^4\left(\frac{[3](4x+2)-3x[4]}{(4x+2)^2}\right)$$

$$= 5\frac{3^4x^4}{(4x+2)^4}\left(\frac{\cancel{12x}+6-\cancel{12x}}{(4x+2)^2}\right)$$

$$= \frac{5\cdot 81x^4\cdot 6}{(4x+2)^6} = \frac{2430x^4}{(4x+2)^6}$$

2. The function h is the composite function defined as $h(x) = f(g(x))$. The following values for the functions f, g, f', and g' are given:

   $g(3) = 6$, $g'(3) = 4$, $f'(3) = 2$, and $f'(6) = 7$

   Find the value of $h'(3)$.

$$\begin{aligned} h(x) &= f(g(x)) \\ h'(x) &= f'(g(x))\cdot g'(x) \\ h'(3) &= f'(g(3))\cdot g'(3) \\ &= f'(6)\cdot 4 \\ &= 7\cdot 4 \\ &= 28 \end{aligned}$$

3. Assume f is a differentiable function whose graph passes through the point $(1, 4)$. If $g(x) = f\left(x^2\right)$ and the line tangent to the graph of f at $(1, 4)$ is $y = 3x - 1$, determine each of the following.

   a. $g(1) = f\left(1^2\right) = f(1) = 4$

   b. $g'(x) = f'\left(x^2\right)\cdot 2x$

   c. $g'(1) = f'\left(1^2\right)\cdot 2(1) = 2f'\left(1\right) = 2\cdot 3 = 6$

   d. Find an equation of the line tangent to the graph of g when $x = 1$.

   $m_{\tan} = g'(1) = 6$
   Point of tangency: $(1,\ g(1)) = (1, 4)$
   Equation of tangent: $y - 4 = 6(x - 1) \quad \rightarrow \quad y = 6x - 2$

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Learning Activity 3.5 – Implicit Differentiation

Table: ______

1 – 2. Find $\frac{dy}{dx}$ for each of the functions.

1. $x + \sin y = xy$

$$\frac{d}{dx}(x + \sin y) = \frac{d}{dx}(xy)$$
$$1 + \left(\cos y\right) \cdot \frac{dy}{dx} = [1]y + x\left[\frac{dy}{dx}\right]$$
$$(\cos y - x)\frac{dy}{dx} = y - 1$$
$$\frac{dy}{dx} = \frac{y-1}{\cos y - x}$$

2. Use implicit differentiation to find $\frac{dy}{dx}$.

$\cos(xy) + x^4 = y^4$

$$\frac{d}{dx}\left[\cos(xy) + x^4\right] = \frac{d}{dx}\left[y^4\right]$$
$$\left(-\sin(xy)\right) \cdot \left([1]y + x\left[\frac{dy}{dx}\right]\right) + 4x^3 = 4y^3\frac{dy}{dx}$$
$$-y\sin(xy) - x\sin(xy)\frac{dy}{dx} - 4y^3\frac{dy}{dx} = -4x^3$$
$$\frac{dy}{dx} = \frac{-4x^3 + y\sin(xy)}{-4y^3 - x\sin(xy)} \quad \text{OR} \quad \frac{4x^3 - y\sin(xy)}{4y^3 + x\sin(xy)}$$

3. Find the slope of the curve at (0, 1).

$y^5 + x^3 = y^2 + 9x$

$$5y^4y' + 3x^2 = 2yy' + 9$$
$$y' = \frac{9 - 3x^2}{5y^4 - 2y} \qquad y'\Big|_{(0,1)} = \frac{9 - 3(0)^2}{5(1)^4 - 2(1)} = 3$$

4. Find the equation of the line tangent to the curve $x^2y^2 + xy = 2$ at the point $(-1, -1)$.

$$\frac{d}{dx}\left(x^2y^2 + xy\right) = \frac{d}{dx}(2)$$
$$[2x]y^2 + x^2\left[2y\frac{dy}{dx}\right] + [1]y + x\left[\frac{dy}{dx}\right] = 0$$
$$(2x^2y + x)\frac{dy}{dx} = -y - 2xy^2$$
$$\frac{dy}{dx} = \frac{-y - 2xy^2}{2x^2y + x}$$
$$m_{\tan} = \left.\frac{dy}{dx}\right|_{(-1,-1)} = \frac{-(-1) - 2(-1)(-1)^2}{2(-1)^2(-1) + (-1)} = \frac{1+2}{-2-1} = -1$$

Equation of tangent line is $y - (-1) = -1(x - (-1))$ or $y = -x - 2$.

5. Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ using implicit differentiation. Evaluate both at the given point.

$xy + 3 = y$ at $(4, -1)$

$$[1]y + x\left[\frac{dy}{dx}\right] + 0 = \frac{dy}{dx}$$
$$\frac{dy}{dx} = \frac{-y}{x-1}$$
$$\left.\frac{dy}{dx}\right|_{(4,-1)} = \frac{-(-1)}{4-1} = \frac{1}{3}$$
$$\frac{d^2y}{dx^2} = \frac{\left[-\frac{dy}{dx}\right](x-1) - (-y)[1]}{(x-1)^2} = \frac{\left[-\left(\frac{-y}{\cancel{x-1}}\right)\right]\cancel{(x-1)} + y}{(x-1)^2} = \frac{2y}{(x-1)^2}$$
$$\left.\frac{d^2y}{dx^2}\right|_{(4,-1)} = \frac{2(-1)}{(4-1)^2} = \frac{-2}{9}$$