In-class Quiz #3 **Name:** KEY

*Please **circle** your answers (:*

1. Consider the parabola $y = 5x - x^2$.

(a) Find the slope of the tangent line to the parabola at the point (1, 4).

$f'(x) = 5 - 2x$

$f'(1) = 5 - 2(1)$

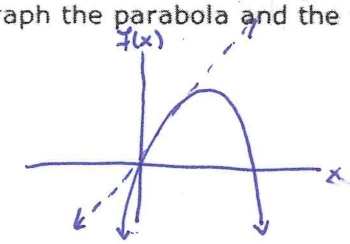
$\boxed{= 3}$

(b) Find an equation of the tangent line in part (a).

$y - 4 = 3(x - 1)$

$y = 3x + 1$

(c) Graph the parabola and the tangent line.



2. Find the derivative of the function using the definition of derivative.

$f(t) = 7t - 2t^2$

$$f'(t) = \lim_{h\to 0} \frac{f(t+h) - f(t)}{h} = \lim_{h\to 0} \frac{[7(t+h) - 2(t+h)^2] - [7t - 2t^2]}{h}$$

$$= \lim_{h\to 0} \frac{7t + 7h - 2t^2 - 4th - 2h^2 - 7t + 2t^2}{h}$$

$$= \lim_{h\to 0} \frac{7h - 4th - 2h^2}{h} = \lim_{h\to 0} 7 - 4t - 2h \boxed{= 7 - 4t}$$

State the domain of the function. (Enter your answer using interval notation.)

$(-\infty, \infty)$

State the domain of its derivative. (Enter your answer using interval notation.)

$(-\infty, \infty)$

3. Find the derivative of the function using the definition of derivative.

$g(x) = \sqrt{7 - x}$

$$g'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{\sqrt{7-(x+h)} - \sqrt{7-x}}{h} \cdot \frac{\sqrt{7-x-h} + \sqrt{7-x}}{\sqrt{7-x-h} + \sqrt{7-x}}$$

$$= \lim_{h\to 0} \left(\frac{(7-x-h) + -(7+x)}{h(\sqrt{7-x-h} + \sqrt{7-x})} \right) = \lim_{h\to 0} \frac{-1}{\sqrt{7-x-h} + \sqrt{7-x}} = \frac{-1}{\sqrt{7-x} + \sqrt{7-x}} \boxed{= \frac{-1}{2\sqrt{7-x}}}$$

State the domain of the function. (Enter your answer using interval notation.)

$(-\infty, 7]$

State the domain of its derivative. (Enter your answer using interval notation.)

$(-\infty, 7)$

4. (a) Differentiate the function.

$$y = \frac{6x^2 + 6x + 2}{\sqrt{x}} = 6x^{3/2} + 6x^{1/2} + 2x^{-1/2}$$

$$y' = 6 \cdot \frac{3}{2} x^{1/2} + 6 \cdot \frac{1}{2} x^{-1/2} - x^{-3/2}$$

$$\boxed{= 9x^{1/2} + 3x^{-1/2} - x^{-3/2}}$$

(b) Differentiate the function.

$$u = \sqrt[7]{t} + 2\sqrt{t^7}$$

$$u' = \left(\frac{1}{7} t^{\frac{1}{7} - \frac{7}{7}}\right) + \left(\frac{7}{2} \cdot 2t^{7/2 - 2/2}\right)$$

$$\boxed{= \frac{1}{7} t^{-6/7} + 7t^{5/2}}$$

(c) Differentiate the function.

$$z = \frac{A}{y^{12}} + Be^y$$

$$z' = A(-12y^{-13}) + B(e^y)$$

$$\boxed{= \frac{-12A}{y^{13}} + Be^y}$$

(d) Find equations of the tangent line and normal line to the curve at the given point.

$y = x^4 + 2e^x, \quad (0, 2)$

$f(x) = x^4 + 2e^x$

$f'(x) = 4x^3 + 2e^x$

$f'(0) = 0 + 2 \boxed{= 2}$

tangent line $y = 2x + 2$

normal line $y = -\frac{1}{2}x + 2$

$m_{tan} = 2$

$y - 2 = 2(x - 0)$

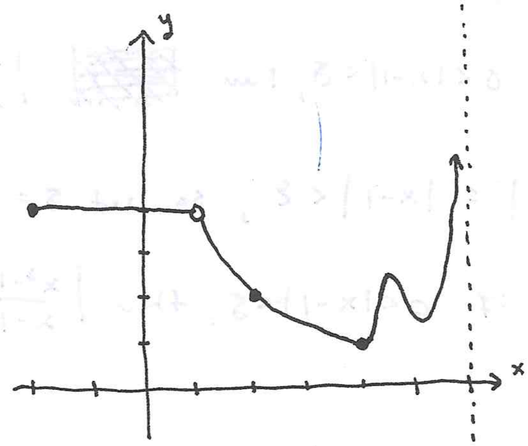
$\Rightarrow y = 2x + 2$

$m_{norm} = \frac{-1}{m_{tan}} = -\frac{1}{2}$

$y - 2 = -\frac{1}{2}(x - 0)$

$\Rightarrow y = -\frac{1}{2}x + 2$

(5.) Use the graph to answer the following questions:



a) $\lim\limits_{h \to 0} \dfrac{f(4+h) - f(4)}{h} = \boxed{0}$

b) $\lim\limits_{h \to 0} \dfrac{f(1+h) - f(1)}{h} = \boxed{DNE}$

c) $\lim\limits_{x \to -1^+} \dfrac{df}{dx} = \boxed{0}$

d) $f(2) = \boxed{2}$

(6.) Find the following limits:

a) $\lim\limits_{x \to 1} \cos^{-1}\left(\ln e^{x^2-4}\right) = \lim\limits_{x \to 1} \cos^{-1}(x^2-4) = \cos^{-1}(-3)$ so $\boxed{DNE}$

b) $\lim\limits_{x \to -\infty} \dfrac{\sqrt{x^4+x^2} + x^2}{2x^2 - x} = \lim\limits_{x \to -\infty} \dfrac{\sqrt{\frac{x^4}{x^4} + \frac{x^2}{x^4}} + \frac{x^2}{x^2}}{\frac{2x^2}{x^2} - \frac{x}{x^2}}$

$= \lim\limits_{x \to -\infty} \dfrac{\sqrt{1 + \frac{1}{x^2}} + 1}{2 - \frac{1}{x}}$

$\boxed{= 1}$

(7.) Let $f(x) = \dfrac{x^2-1}{x-1}$. Use the epsilon-delta definition of a limit to prove $\lim\limits_{x \to 1} f(x) = 2$.

Given $\varepsilon > 0$, find $\delta > 0$ s.t. if $0 < |x-1| < \delta$, then $\left| \dfrac{x^2-1}{x-1} - 2 \right| < \varepsilon$.

But $\left| \dfrac{(x-1)(x+1)}{(x-1)} - 2 \right| = |x+1-2| = |x-1| < \varepsilon$, so set $\delta = \varepsilon$.

Given $\varepsilon > 0$, we choose $\delta = \varepsilon$. So if $0 < |x-1| < \delta$, then

$$\left| \frac{x^2-1}{x-1} - 2 \right| = \left| \frac{\cancel{(x-1)}(x+1)}{\cancel{(x-1)}} - 2 \right| = |x+1-2| = |x-1| < \varepsilon = \delta$$

therefore $\lim\limits_{x \to 1} f(x) = 2$ by the definition of a limit.

(8.) Use the limit definition of the derivative to find $f'(x)$ if $f(x) = \dfrac{x}{3x-1}$.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\left[\frac{x+h}{3(x+h)-1}\right] - \frac{x}{3x-1}}{h}$$

$$= \lim_{h \to 0} \frac{\frac{(3x-1)(x+h) - (3x+3h-1)(x)}{(3x-1)(3x+3h-1)}}{h}$$

$$= \lim_{h \to 0} \left[\frac{3x^2 + 3xh - x - h - 3x^2 - 3xh + x}{(3x-1)(3x+3h-1)} \right] \cdot \frac{1}{h}$$

$$= \lim_{h \to 0} \frac{-1}{(3x-1)(3x+3h-1)}$$

$$= \frac{-1}{(3x-1)(3x-1)}$$

$$= \frac{-1}{(3x-1)^2}$$

9. Let $f(x) = 2x^2 - 4$

(a) Find all values of x s.t. the tangent line of $f(x)$ is parallel to the line $y = 2x$.

$f(x)$ is parallel to the line $y=2x$ for all values of x that make $f'(x) = 2$

$f(x) = 2x^2 - 4$

$f'(x) = 4x \iff f(x) = 2$

$\Rightarrow 4x = 2$

$\Rightarrow x = 1/2$

$f(x)$ is parallel to $y = 2x$ when $\boxed{x = 1/2}$

(b) Find the equation of the line normal to $f(x)$ at $x = 1$. Put your answer in slope-intercept form ($y = mx + b$).

$y + 2 = -1/4\,(x-1)$

$\Rightarrow \boxed{y = -\frac{1}{4}x - \frac{7}{4}}$

(c) Let $g(x) = -9x + 10$. Show there exists a solution to the equation $f(x) = g(x)$.

$f(x) = g(x) \iff 0 = f(x) - g(x)$

$0 = 2x^2 - 4 + 9x - 10$

$\iff 0 = 2x^2 + 9x - 14$

$$\left.\begin{aligned} x=0 &: 2(0)^2 - 4 + 9(0) = -4 \\ x=2 &: 2(4) - 4 + 9(2) = 22 \end{aligned}\right\} \text{sign change on } [0,2]$$

Since polynomials are continuous, $f(x) - g(x)$ continuous on $[0,2]$, so by IVT, there is a solution since there is a sign change on $[0,2]$.

(10.) Let $f(x)$ be a function s.t.

$$1 - |x-2| \le 3f(x) \le 1 + |x-2|$$

is true for all real numbers. Use the Squeeze Theorem to find $\lim_{x \to 2} f(x)$.

$$1 - |x-2| \le 3f(x) \le 1 + |x-2|$$

$$\Leftrightarrow \quad \lim_{x \to 2} \frac{1 - |x-2|}{3} \le \lim_{x \to 2} f(x) \le \lim_{x \to 2} \frac{1 + |x-2|}{3}$$

$$\Leftrightarrow \quad \frac{1}{3} \le \underset{\text{at } x=2}{f(2)} \le \frac{1}{3}$$

$$\Rightarrow \quad \lim_{x \to 2} f(x) = \frac{1}{3}$$