Test 2 – Answer Key
Version A

Student's Printed Name: ___________________________ **CUID:** ___________

Instructor: ____ **Section:** ___________

Instructions: You are **not** permitted to use a calculator on any portion of this test. You are **not** allowed to use a textbook, notes, or any technology on any portion of this test. All devices must be turned off and stored out of sight while you are in the testing room.

During this test, any communication with any person (other than the instructor or a designated proctor) in any form, including written, signed, verbal, or digital, is understood to be a violation of academic integrity.

No part of this test may be removed from the examination room.

Read each question very carefully. To receive full credit for your work, you must:

1. Show legible, logical, and relevant justification which supports your final answer.
2. Use complete and correct mathematical notation.
3. Include proper units, if necessary.
4. Give exact numerical values whenever possible.

You have **90 minutes** to complete the entire test.

On my honor, I have neither given nor received inappropriate or unauthorized information at any time before or during this test.

Student's Signature: ____

Do not write below this line.

Free Response Problem	Possible Points	Points Earned	Free Response Problem	Possible Points	Points Earned
1.a.	8		5.a.	6	
1.b.	8		5.b.	6	
1.c.	8		6.a.	7	
1.d.	8		6.b.	7	
2.a.	8		**Test Total**	100	
2.b.	6				
3.	12				
4.a.	4				
4.b.	7				
4.c.	5				

Read each question carefully. In order to receive full credit you must show legible, logical, and relevant justification which supports your final answer. Give numerical answers as exact values. You are NOT permitted to use a calculator on any portion of this test.

1. **(8 pts. each)** Find the indicated derivatives. Assume *f(x)* and *g(x)* are differentiable functions wherever they appear.

a) Find $\frac{dy}{dx}$ if $y=\sqrt{[f(x)]^2\,g(2x)}$.

$$y=\sqrt{[f(x)]^2\,g(2x)}$$
$$y=\left([f(x)]^2\,g(2x)\right)^{1/2}$$
$$\frac{dy}{dx}=\frac{1}{2}\left([f(x)]^2\,g(2x)\right)^{-1/2}\frac{d}{dx}\left[[f(x)]^2\,g(2x)\right]$$
$$\frac{dy}{dx}=\frac{1}{2}\left([f(x)]^2\,g(2x)\right)^{-1/2}\left[[f(x)]^2\,g'(2x)(2)+g(2x)(2)f(x)f'(x)\right]$$

Work on Problem:	Points
Derivative of square root function	2 points
Derivative of $[f(x)]^2\,g(2x)$	6 points

Notes:
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b) Find $\frac{dy}{dx}$ if $\sin(x)\cos(y)=\sin(x)+\cos(y)$.

$$\sin(x)\cos(y)=\sin(x)+\cos(y)$$
$$\sin(x)\left[-\sin(y)\frac{dy}{dx}\right]+\cos(y)\cos(x)=\cos(x)-\sin(y)\frac{dy}{dx}$$
$$-\sin(x)\sin(y)\frac{dy}{dx}+\sin(y)\frac{dy}{dx}=\cos(x)-\cos(y)\cos(x)$$
$$\frac{dy}{dx}\left[-\sin(x)\sin(y)+\sin(y)\right]=\cos(x)-\cos(y)\cos(x)$$
$$\frac{dy}{dx}=\frac{\cos(x)-\cos(y)\cos(x)}{-\sin(x)\sin(y)+\sin(y)}$$

Work on Problem:	Points
Derivative of left side	4 points
Derivative of right side	2 points
Solves for $\frac{dy}{dx}$	2 points

Notes:
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c) Find $h'(x)$ if $h(x)=\tan^{-1}\left(\dfrac{e^{x^2}}{2^x}\right)$. Do not try to simplify $h'(x)$.

$$h(x)=\tan^{-1}\left(\frac{e^{x^2}}{2^x}\right)$$

$$h'(x)=\frac{1}{1+\left(\frac{e^{x^2}}{2^x}\right)^2}\frac{d}{dx}\left(\frac{e^{x^2}}{2^x}\right)$$

$$h'(x)=\frac{1}{1+\left(\frac{e^{x^2}}{2^x}\right)^2}\left(\frac{2^x e^{x^2}(2x)-e^{x^2}2^x\ln 2}{(2^x)^2}\right)$$

Work on Problem:	Points
Derivative of the inverse tangent function	3 points
Derivative of the argument for the inverse tangent function	5 points
Notes: •	

d) Find $\dfrac{dy}{dx}$ if $y=x^{\sin(x^2)}$.

$$y=x^{\sin(x^2)}$$

(logaritmic differentiation)

$$\ln(y)=\ln(x^{\sin(x^2)})$$

$$\ln(y)=\sin(x^2)\ln x$$

$$\frac{1}{y}\frac{dy}{dx}=\sin(x^2)\left(\frac{1}{x}\right)+\ln x\cos(x^2)(2x)$$

$$\frac{dy}{dx}=\left[\sin(x^2)\left(\frac{1}{x}\right)+\ln x\cos(x^2)(2x)\right]y$$

$$\frac{dy}{dx}=\left[\sin(x^2)\left(\frac{1}{x}\right)+\ln x\cos(x^2)(2x)\right]x^{\sin(x^2)}$$

Work on Problem:	Points
Natural log of both sides	1 point
Simplifies right side with log properties	1 point
Derivative of left side	1 point
Derivative of right side	3 points
Solves for $\dfrac{dy}{dx}$	1 point
Replaces y	1 point
Notes: • Subtract ½ for missing derivative notation	

2. **(14 pts.)** A stone is thrown upward from the edge of a bridge over a large river. The height of the stone over the river t seconds after it is thrown is $f(t) = -16t^2 + 32t + 48$ feet.

a) (**8 pts.**) Determine the maximum height of the stone.

$s(t) = -16t^2 + 32t + 48$
$v(t) = -32t + 32$
Solve
$v(t) = 0$
$-32t + 32 = 0$
$t = 1$
Evaluate $s(t)$ at $t = 1$
$s(1) = -16(1)^2 + 32(1) + 48$
$s(1) = 64$ feet

Work on Problem:	**Points**
Finds velocity function	2 points
Sets velocity equal to zero	2 points
Solves for t	2 points
Calculates height	2 points

Notes:
- Subtract ½ point for omitting units from final answer

b) (6 **pts.**) Find the speed **and** velocity of the stone at the moment it reaches the surface of the river.

Solve
$s(t) = 0$
$-16t^2 + 32t + 48 = 0$
$-16(t^2 - 2t - 3) = 0$
$-16(t-3)(t+1) = 0$
$t = 3$ or $\cancel{t = -1}$

Evaulate $v(t)$ at $t = 3$
$v(3) = -32(3) + 32$
$v(3) = -64$ feet per second
speed(3)= $|v(3)| = |-64| = 64$ feet per second

Work on Problem:	**Points**
Sets height to zero	2 points
Solves for t	2 points
Evaluates velocity at $t=3$	1 point
Calculates speed at $t=3$	1 point

Notes:
- Subtract ½ point for omitting units from final answers

3. (**12 pts.**) Two cylindrical swimming pools are being filled simultaneously at the exact **same** constant rate (in cubic meters per minute). The smaller pool has a radius of 5 meters, and the water level in it rises at a rate of 0.5 meters per minute. The larger pool has a radius of 8 meters.

Note: The volume *V* of a circular cylinder of radius *r* and height *h* is $V = \pi r^2 h$

a) (**7 pts.**) At what rate is water entering the smaller pool?

$V = \pi(5)^2 h \quad (r \text{ is constant})$

$V = 25\pi h$

$\frac{dV}{dt} = 25\pi \frac{dh}{dt}$

substitute for $\frac{dh}{dt}$

$\frac{dV}{dt} = 25\pi(0.5)$

$\frac{dV}{dt} = 12.5\pi = \frac{25\pi}{2} \text{ m}^3/\text{min}$

Work on Problem:	**Points**
Recognizes that radius is constant	2 points
Derivative of left side	1 points
Derivative of right side	2 points
Substitutes value for $\frac{dh}{dt}$	1 point
Answer with units	1 point

Notes:
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b) (**5 pts.**) At what rate is the water level rising in the larger pool?

$V = \pi(8)^2 h \quad (r \text{ is constant})$

$V = 64\pi h$

$\frac{dV}{dt} = 64\pi \frac{dh}{dt}$

substitute for $\frac{dV}{dt}$

$\frac{25\pi}{2} = 64\pi \frac{dh}{dt}$

$\frac{dh}{dt} = \frac{25\pi}{2(64\pi)}$

$\frac{dh}{dt} = \frac{25}{128} \text{ m}/\text{min}$

Work on Problem:	**Points**
Recognizes that radius is constant	1 point
Derivative of left side	1 point
Derivative of right side	1 point
Substitutes value for $\frac{dV}{dt}$	1 point
Answer with units	1 point

Notes:
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4. **(16 pts.)** Let f and g be everywhere differentiable functions. Let $h(x)=\frac{1}{27}\left[f(g(x))\right]^2$. Use the values in the table below to answer questions a-c below.

	***x*=0**	***x*=1**	***x*=2**	***x*=3**
$f(x)$	0	1	8	27
$f'(x)$	0	3	12	27
$g(x)$	1	3	9	19
$g'(x)$	0	2	4	12

a) (**4 pts.**) Find $h(1)$.

$$h(x)=\frac{1}{27}\left[f(g(x))\right]^2$$
$$h(1)=\frac{1}{27}\left[f(g(1))\right]^2$$
$$=\frac{1}{27}\left[f(3)\right]^2$$
$$=\frac{1}{27}\left[27\right]^2$$
$$=27$$

Work on Problem:	**Points**
Substitutes *x=1*	1 point
Calculates answer	3 points

Notes:
- Subtract one point for missing work

b) (**7 pts.**) Find $h'(1)$.

$$h(x)=\frac{1}{27}\left[f(g(x))\right]^2$$
$$h'(x)=\frac{2}{27}\left[f(g(x))\right]f'(g(x))\,g'(x)$$
$$h'(1)=\frac{2}{27}\left[f(g(1))\right]f'(g(1))\,g'(1)$$
$$=\frac{2}{27}\left[f(g(1))\right]f'(g(1))\,g'(1)$$
$$=\frac{2}{27}\left[f(3)\right]f'(3)\,g'(1)$$
$$=\frac{2}{27}(27)(27)(2)$$
$$=108$$

Work on Problem:	**Points**
Finds $h'(x)$	4 points
Calculates answer	3 points

Notes:
- Subtract 2 points if $h'(x)$ not shown

c) (**5 pts.**) Find the equation of the line tangent to the curve $y=h(x)$ at $x=1$.

$$y-27=108(x-1)$$

Work on Problem:	**Points**
Substitutes values into a line equation	5 points

Notes:

5. **(6 pts. each)** Find the following limits. Show all work. Do **NOT** use L'Hopital's Rule.

a) $\lim\limits_{x\to 0}\dfrac{A\tan(3x)}{Bx}$ (*A* and *B* are real number constants)

$$\lim_{x\to 0}\frac{A\tan(3x)}{Bx}\ \frac{0}{0},\ \text{Indeterminate Form}$$
$$=\frac{A}{B}\lim_{x\to 0}\frac{\tan(3x)}{x}$$
$$=\frac{A}{B}\lim_{x\to 0}\frac{\frac{\sin(3x)}{\cos(3x)}}{x}$$
$$=\frac{A}{B}\lim_{x\to 0}\frac{\sin(3x)}{x\cos(3x)}$$
$$=\frac{3A}{B}\lim_{x\to 0}\left[\frac{\sin(3x)}{3x}\cdot\frac{1}{\cos(3x)}\right]$$
$$=\frac{3A}{B}\lim_{x\to 0}\frac{\sin(3x)}{3x}\lim_{x\to 0}\frac{1}{\cos(3x)}$$
$$=\frac{3A}{B}(1)(1)$$
$$=\frac{3A}{B}$$

Work on Problem:	**Points**
Recognizes the indeterminate form 0/0	1 point
Rewrites tangent in terms of sine and cosine	2 points
Multiplies by 3/3	1 point
Separates into two limits	1 point
Calculates final answer	1 point
Notes: •	

b) $\lim\limits_{x\to 0}\dfrac{3\sec^5 x}{\sin^{-1}x+4}$

$$\lim_{x\to 0}\frac{3\sec^5 x}{\sin^{-1}x+4}=\frac{3(\sec 0)^5}{\sin^{-1}0+4}=\frac{3(1)^5}{0+4}=\frac{3}{4}$$

Work on Problem:	**Points**
Substitutes	2 points
Value for sec(0)	1.5 points
Value for arcsin(0)	2 points
Calculates final answer	0.5 points
Notes: •	

6. **(7 pts. each)** Suppose the population P (billions of cells) of the bacteria *calculosium adnauseam* exhibits exponential growth when placed in a nutrient broth in a laboratory.

a) If an initial population P_0 doubles in size every 20 **minutes**, find an expression for the population P after t **hours**.

20 minutes = $\frac{1}{3}$ hour

$P = P_0 e^{kt}$

know $P(1/3) = 2P_0$

$2P_0 = P_0 e^{k(1/3)}$

$2P_0 = P_0 e^{k(1/3)}$

$2 = e^{k(1/3)}$

$\ln 2 = \ln e^{k(1/3)}$

$\ln 2 = \frac{k}{3}$

$3\ln 2 = k$

$P = P_0 e^{(3\ln 2)t}$

Work on Problem:	**Points**
Solves for k	5 point
Finds equation for P	2 points

Notes:
- Subtract two for wrong time units

b) How long (in hours) will it take the population to reach exactly 13 times its original amount? Give your answer in terms of natural logarithms.

$13P_0 = P_0 e^{(3\ln 2)t}$

$13 = e^{(3\ln 2)t}$

$\ln(13) = \ln(e^{(3\ln 2)t})$

$\ln(13) = (3\ln 2)t\ln(e)$

$t = \frac{\ln(13)}{3\ln 2}$

Work on Problem:	**Points**
Sets $P = 13P_0$	2 points
Solves for t	5 points

Notes:
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