In-class quiz #8, 9

Name: KEY

(1.) Find $\frac{dy}{dx}$ if $y = x^{\cos^{-1} x}$

$$\ln y = \cos^{-1}x \ln x$$

So $\frac{1}{y}\frac{dy}{dx} = \cos^{-1}x\left(\frac{1}{x}\right) + \ln\left(-\frac{1}{\sqrt{1-x^2}}\right)$

$\Rightarrow \frac{dy}{dx} = y\left(\frac{\cos^{-1}x}{x} - \frac{\ln x}{\sqrt{1-x^2}}\right)$

$\Rightarrow \boxed{\frac{dy}{dx} = x^{\cos^{-1}x}\left(\frac{\cos^{-1}x}{x} - \frac{\ln x}{\sqrt{1-x^2}}\right)}$

(2.) Find $\frac{dy}{dx}$ if $\sin(2x+y) = \tan(3^x)$

$$\cos(2x+y)\left[2 + \frac{dy}{dx}\right] = \sec^2(3^x)(3^x \ln 3)$$

$\Rightarrow 2\cos(2x+y) + \cos(2x+y)\frac{dy}{dx} = \sec^2(3^x)(3^x \ln 3)$

$\Rightarrow \frac{dy}{dx}(\cos(2x+y)) = \sec^2(3^x)(3^x \ln 3) - 2\cos(2x+y)$

$\Rightarrow \frac{dy}{dx} = \frac{\sec^2(3^x)(3^x \ln 3) - 2\cos(2x+y)}{\cos(2x+y)}$

$\Rightarrow \boxed{\frac{dy}{dx} = \frac{\sec^2(3^x)(3^x \ln 3)}{\cos(2x+y)} - 2}$

(3.) Find $f'(x)$ if $f(x) = \ln\left(\dfrac{\tan(e^{x^2})}{\sec(x^3)}\right)$

$$f(x) = \ln(\tan(e^{x^2})) - \ln(\sec(x^3))$$

$$f'(x) = \left[\frac{1}{\tan(e^{x^2})} \cdot \sec^2(e^{x^2})\, e^{x^2}(2x)\right] - \left[\frac{1}{\sec(x^3)} \sec(x^3)\tan(x^3)\,(3x^2)\right]$$

$$\boxed{f'(x) = \frac{2x e^{x^2} \sec^2(e^{x^2})}{\tan(e^{x^2})} - 3x^2 \tan(x^3)}$$

(4.) Find $g'(x)$ if $g(x) = h\left(\sin\left(f\left(\frac{3^x}{x^2}\right)\right)\right)$

$$g'(x) = h'\left(\sin\left(f\left(\frac{3^x}{x^2}\right)\right)\right)\cos\left(f\left(\frac{3^x}{x^2}\right)\right) f'\left(\frac{3^x}{x^2}\right)\left[\frac{x^2 3^x \ln 3 - 3^x(2x)}{(x^2)^2}\right]$$

$$\boxed{g'(x) = h'\left(\sin\left(f\left(\frac{3^x}{x^2}\right)\right)\right)\cos\left(f\left(\frac{3^x}{x^2}\right)\right) f'\left(\frac{3^x}{x^2}\right)\left[\frac{3^x(x\ln 3 - 2)}{x^3}\right]}$$

5. Your friend John Wayne has a sphere in his house with radius of 4 ft.

i) Use a differential to estimate the increase in radius that you would get if the volume of the sphere increases by 10 cubic ft.

Hint: The volume of a sphere is given by $V = \frac{4\pi}{3} r^3$.

$$V = \frac{4\pi}{3} r^3$$

$$\Rightarrow \frac{dV}{dr} = 4\pi r^2$$

$$\Rightarrow dV = 4\pi r^2 dr$$

$$\Rightarrow dr = \frac{dV}{4\pi r^2} \qquad \Rightarrow dr = \frac{(10)}{4\pi (4)^2} = \frac{10}{64\pi}$$

$$\Rightarrow \boxed{dr = \frac{5}{32\pi} \text{ ft}}$$

$\Rightarrow$ The radius would increase by $\frac{5}{32\pi}$ ft.

ii) Use a differential to estimate the change in surface area of the sphere if the radius increases from 4 ft to 4.5 ft.

Hint: The surface area of sphere is given by $S = 4\pi r^2$.

$$S = 4\pi r^2$$

$$\Rightarrow \frac{dS}{dr} = 8\pi r$$

$$\Rightarrow dS = 8\pi r\, dr \qquad \Rightarrow dS = 8\pi (4)(0.5)$$

$$\Rightarrow \boxed{dS = 16\pi \text{ ft}^2}$$

$\Rightarrow$ The change in surface area would be 16π ft^2

6. Two cylindrical swimming pools are being filled simultaneously at the exact same constant rate (in cubic meters per minute). The smaller pool has a radius of 5 meters, and the water level in it rises at a rate of 0.5 meters per minute. The larger pool has a radius of 8 meters.

Note: The Volume V of a circular cylinder of radius r and height h is $V = \pi r^2 h$.

i) At what rate is water entering the smaller pool?
↳ means find $\frac{dV}{dt}$

$V = \pi r^2 h$

$\Rightarrow V = \pi(5)^2 h$ (r is constant)

$\Rightarrow V = 25\pi h$

$\Rightarrow \frac{dV}{dt} = 25\pi \frac{dh}{dt}$

$\Rightarrow \frac{dV}{dt} = 25\pi(.5)$

$= 12.5\pi$

$\Rightarrow \boxed{\frac{dV}{dt} = \frac{25\pi}{2}\ m^3/min}$ $\Rightarrow$ Water is entering the smaller pool at a rate of $\frac{25\pi}{2}\ \frac{m^3}{min}$

ii) At what rate is the water level rising in the larger pool?
↳ means find $\frac{dh}{dt}$

Note that $\frac{dV}{dt} = \frac{25\pi}{2}\ \frac{m^3}{min}$ is the same as the value found in part i) because the pools are being filled at the same constant rate.

$V = \pi r^2 h$

$\Rightarrow V = \pi(8)^2 h$ (r is constant)

$\Rightarrow V = 64\pi h$

$\Rightarrow \frac{dV}{dt} = 64\pi \frac{dh}{dt}$

$\Rightarrow \frac{dh}{dt} = \frac{dV}{dt} \cdot \frac{1}{64\pi}$

$\Rightarrow \frac{dh}{dt} = \frac{25\pi}{2} \cdot \frac{1}{64\pi}$

$\Rightarrow \boxed{\frac{dh}{dt} = \frac{25}{128}\ \frac{m}{min}}$ $\Rightarrow$ The water level is rising at a rate of $\frac{25}{128}\ \frac{m}{min}$

7. Let f and g be everywhere differentiable functions.

Let $h(x) = \frac{1}{27}[f(g(x))]^2$.

	$x=0$	$x=1$	$x=2$	$x=3$
$f(x)$	0	1	8	27
$f'(x)$	0	3	12	27
$g(x)$	1	3	9	19
$g'(x)$	0	2	4	12

i) Is h increasing or decreasing at $x=1$? Why?

h is $\boxed{\text{increasing at } x=1}$ because the slope is positive since $h'(1) > 0$ (see part ii) for details)

ii) Find $h'(1)$.

$$h'(x) = \frac{2}{27}[f(g(x))]\, f'(g(x))\, g'(x)$$

$$h'(1) = \frac{2}{27}[f(g(1))]\, f'(g(1))\, g(1)$$

$$= \frac{2}{27}[f(3)]f'(3)(2)$$

$$= \frac{2}{27}(27)(27)(2)$$

$$\boxed{=108}$$

iii) Find the equation of the line tangent to the curve $y = h(x)$ at $x=1$. Put your answer in the form $y = mx+b$.

Note $h(1) = \frac{1}{27}[f(g(1))]^2$

$= \frac{1}{27}[f(3)]^2$

$= \frac{1}{27}[27]^2$

$= 27$

$y - h(1) = h'(1)(x-1)$

$\Rightarrow y - 27 = 108(x-1)$

$\Rightarrow y - 27 = 108x - 108$

$\Rightarrow \boxed{y = 108x - 81}$

(8.) The half-life of Strontium-91 is 56 days.

i) Assuming an initial amount of 60 mg, how long will it take the initial amount of 60mg to decay to 6 mg? Give your answer in terms of natural logarithms.

Let m = amount of Strontium-91

$m(56) = \frac{1}{2} m_0$

$m(t) = m_0 e^{kt}$

$\Rightarrow m(56) = m_0 e^{k(56)}$

$\Rightarrow \frac{1}{2} m_0 = m_0 e^{k(56)}$

$\Rightarrow \frac{1}{2} = e^{k(56)}$

$\Rightarrow \ln\left(\frac{1}{2}\right) = k(56)$

$\Rightarrow k = \dfrac{\ln\left(\frac{1}{2}\right)}{56}$

Therefore, $m(t) = 60\, e^{\left(\frac{\ln\left(\frac{1}{2}\right)}{56}\right)t}$

So, $6 = 60\, e^{\left(\frac{\ln\left(\frac{1}{2}\right)}{56}\right)t}$ and we want to solve for t

$\Rightarrow \frac{1}{10} = e^{\left(\frac{\ln\left(\frac{1}{2}\right)}{56}\right)t}$

$\Rightarrow \ln\left(\frac{1}{10}\right) = \left(\frac{\ln\left(\frac{1}{2}\right)}{56}\right)t$

$\Rightarrow \boxed{t = \dfrac{\ln 1/10}{\left(\frac{\ln 1/2}{56}\right)}}$ $\Rightarrow$ For 60 mg to decay to 6 mg it will take $t = \dfrac{56 \ln 1/10}{\ln 1/2}$ days

ii) At what rate is the amount of Strontium-91 decaying 4 hours after ~~its~~ initial amount of 60 mg.

Note that $\dfrac{4\,hr}{24\,hr} = \dfrac{1}{6}$ days and let $y = m(t)$

$y = 60\, e^{\left(\frac{\ln\left(\frac{1}{2}\right)}{56}\right)t}$

$\Rightarrow y' = 60\, e^{\left(\frac{\ln\left(\frac{1}{2}\right)}{56}\right)t} \cdot \left(\frac{\ln\left(\frac{1}{2}\right)}{56}\right)$

Therefore, $\boxed{y'\left(\tfrac{1}{6}\right) = 60\, e^{\left(\frac{\ln\left(\frac{1}{2}\right)}{56}\right)\left(\frac{1}{6}\right)} \cdot \left(\frac{\ln\left(\frac{1}{2}\right)}{56}\right)}$

(9.) Find the absolute maximum and absolute minimum values of the functions on the given interval and the x-values where they occur.

i) $f(x) = x^{3/5}(x^{1/5}+1)$ $\quad x \in [-1, 1]$

$f(x) = x^{4/5} + x^{3/5}$

$\Rightarrow f'(x) = \frac{4}{5}x^{-1/5} + \frac{3}{5}x^{-2/5}$

$f'(x) = 0 \Leftrightarrow \frac{4}{5}x^{-1/5} + \frac{3}{5}x^{-2/5} = 0$ which has no real solutions

$f'(x)$ DNE at $x = 0$.

$f(0) = 0$

$f(-1) = 0$

$f(1) = 2$

$$\boxed{\begin{array}{l}\text{Absolute min of } 0 \text{ at } x = 0, -1 \\ \text{Absolute max of } 2 \text{ at } x = 1\end{array}}$$

ii) $g(x) = 3x^2 + 4x + 7$ $\quad x \in [-8, 8]$

$g'(x) = 6x + 4$

$g'(x) = 0 \Leftrightarrow 6x + 4 = 0 \Leftrightarrow x = -2/3$

$g'(x)$ is continuous everywhere thus exists everywhere

$g(-2/3) = \frac{4}{3} - \frac{8}{3} + 7 = \frac{17}{3}$

$g(-8) = 192 - 32 + 7 = 167$

$g(8) = 192 + 32 + 7 = 231$

$$\boxed{\begin{array}{l}\text{Absolute minimum of } \frac{17}{3} \text{ at } x = -2/3 \\ \text{Absolute maximum of } 231 \text{ at } x = 8\end{array}}$$

(10) Prove the following identities;

i) $\dfrac{1+\tanh x}{1-\tanh x} = e^{2x}$

$$\frac{1+\tanh x}{1-\tanh x} = \frac{1+\left(\frac{\sinh x}{\cosh x}\right)}{1-\left(\frac{\sinh x}{\cosh x}\right)} = \frac{\left(\frac{\cosh x}{\cosh x}\right)+\left(\frac{\sinh x}{\cosh x}\right)}{\left(\frac{\cosh x}{\cosh x}\right)-\left(\frac{\sinh x}{\cosh x}\right)}$$

$$= \frac{\cosh x+\sinh x}{\cosh x-\sinh x}$$

$$= \frac{\left(\frac{e^x+e^{-x}}{2}+\frac{e^x-e^{-x}}{2}\right)}{\left(\frac{e^x+e^{-x}}{2}-\frac{e^x-e^{-x}}{2}\right)}$$

$$= \frac{2e^x}{2e^{-x}}$$

$$= e^x \cdot e^{-(-x)} = e^{2x}, \quad \boxed{\text{thus } \frac{1+\tanh x}{1-\tanh x} = e^{2x}}$$

ii) $\sinh(2x) - 2\sinh x\cosh x = 0$

Want to show $\left(\dfrac{e^{2x}-e^{-2x}}{2}\right) - 2\left(\dfrac{e^x-e^{-x}}{2}\right)\left(\dfrac{e^x+e^{-x}}{2}\right) = 0$

So let's show $2\left(\dfrac{e^x-e^{-x}}{2}\right)\left(\dfrac{e^x+e^{-x}}{2}\right) = \dfrac{e^{2x}-e^{-2x}}{2}$

$$\cancel{2}\left(\frac{e^x-e^{-x}}{\cancel{2}}\right)\left(\frac{e^x+e^{-x}}{2}\right) = \frac{e^{2x}+\cancel{1}-\cancel{1}-e^{-2x}}{2}$$

$$= \frac{e^{2x}-e^{-2x}}{2}$$

$$\boxed{\text{Therefore } \sinh(2x) - 2\sinh x\cosh x = 0}$$