Wave Direction-based Recontruction of Stiffness in Magnetic Resonance Elastography

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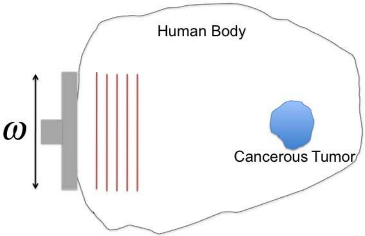
Background

Magnetic Resonance Elastography, MRE:

- uses vibrating plate to propagate waves into body
- utilizes MRI (motion-encoding gradient) to capture images of wave propagation
- used for detecting early stage cancer (tumors 3-5 mm in diameter)



Because cancerous tissue is stiffer than surrounding healthy tissue, a distortion of the wave is visible

We want to be able to reconstruct the stiffness of the inclusion and healthy tissue.

Modeling

The linear isotropic elasticity model:

$$\nabla(\lambda\nabla\cdot\vec{U})+\nabla\cdot[\mu(\nabla\vec{U}+(\nabla\vec{U})^T)]=\rho\vec{U}_{tt}$$

- $\rho>0$ is the mass density
- $\lambda>0$ is the compression stiffness
- $\mu>0$ is the shear stiffness

Utilizing

- piecewise constant (locally homogeneous) medium
- Shear wave dominates compression wave

our model reduces to

$$\mu\Delta\vec{U}=\rho\vec{U}_{tt}.$$

Modeling (con't)

$\vec{U} = (U_1, U_2, U_3)$ can be decoupled, i.e. if $U = U_3$, then

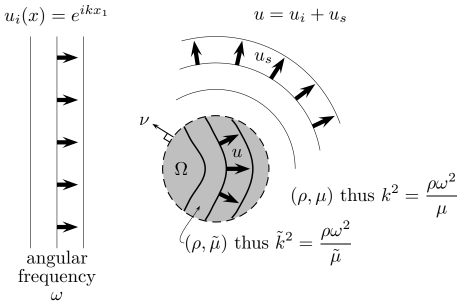
$$\mu \Delta U = \rho U_{tt}.$$

Then taking the Fourier Transform with respect to time, we get

$$\Delta u + k^2 u = 0, \text{ where } k^2 = \frac{\rho \omega^2}{\mu}$$

This is the Helmholtz equation with wavenumber, k.

Statement of Problem



- In surrounding healthy tissue

$$\Delta u + k^2 u = 0 \quad \text{in} \quad \bar{\Omega}^c \quad \text{where} \quad k^2 = \frac{\rho\omega^2}{\mu}$$

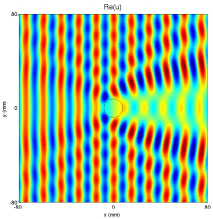
- In tumor which is penetrable object

$$\Delta u + \tilde{k}^2 u = 0 \quad \text{in} \quad \Omega \quad \text{where} \quad \tilde{k}^2 = \frac{\rho\omega^2}{\tilde{\mu}}$$

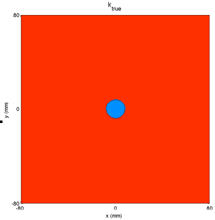
Motivation

The goal is to reconstruct the stiffness, μ.

- This is equivalent to reconstructing $k = \sqrt{\frac{\rho\omega^2}{\mu}}$.
- The strategy is to devise a completely local algorithm

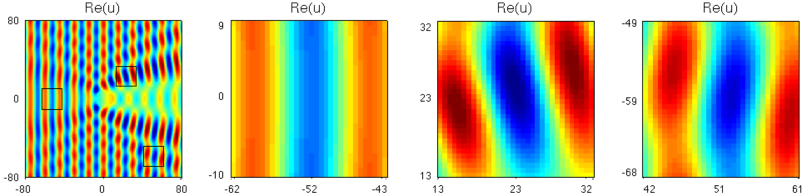


Input



Desired Output

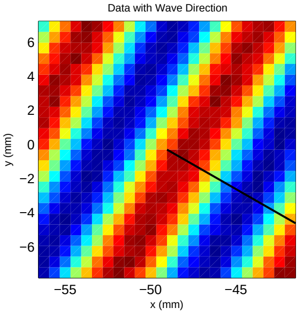
Plane Wave Assumption

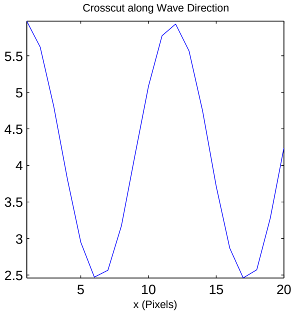


On a local window, the wave will look like a plane wave, i.e.,

$$u \approx Ae^{ik\vec{d}_0\cdot\vec{x}} \text{ where } A = |A|e^{i\phi}.$$

Overview of Algorithm Steps





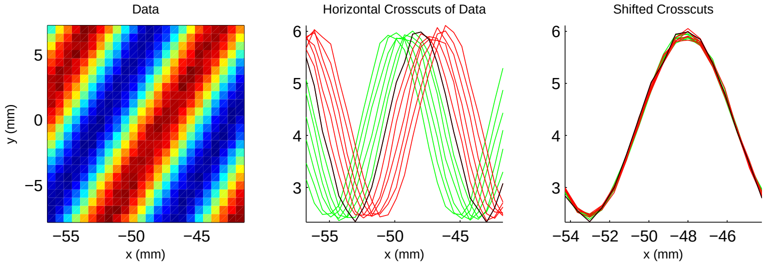
On a local window, the general idea is:

- Calculate propagation direction, $\vec{d}$.
- Along $\vec{d}$, the wavelength is $2\pi/k$.

Propagation Direction of $f(\vec{x})$

Let $f(\vec{x}) := Re(u) \approx |A| \cos\left(k\vec{d} \cdot \vec{x} + \phi\right)$.

- $f(x - \epsilon_0 y, y) = f(x, 0)$ if $\epsilon_0 = d_2/d_1$
- Take horizontal cuts of f at distinct heights y_i.
- Shift the cuts in the x-direction by ϵy_i.



If $\epsilon = \epsilon_0$, all of the shifted cuts must be exactly the same.

Propagation Direction: Variance

Our goal is to find this optimal ϵ_0.
Suppose our window is $S := [-L, L] \times [-L, L]$:

1. Take an average of all possible horizontal crosscuts:

$$C_\epsilon(x) := \frac{1}{2L} \int_{-L}^{L} f(x - \epsilon y', y')\, dy'$$

2. At each x, find the average variance of the crosscuts:

$$V_\epsilon(x) := \frac{1}{2L} \int_{-L}^{L} \left(f(x - \epsilon y, y) - C_\epsilon(x)\right)^2 \, dy$$

3. Average the variance:

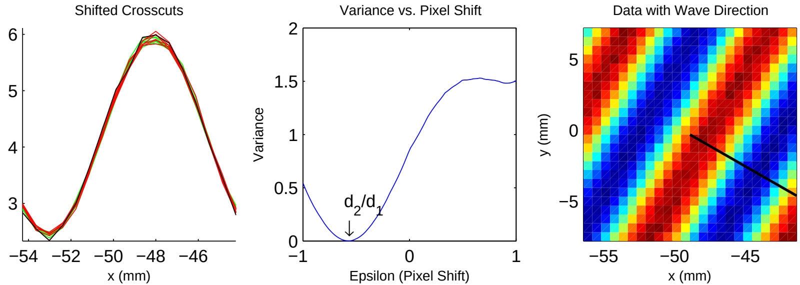
$$\mathrm{Var}(\epsilon) = \frac{1}{2L} \int_{-L}^{L} V_\epsilon(x)\, dx$$

Propagation Direction: Minimize Variance

We expect, for $\epsilon = \epsilon_0$:

- The ϵ_0-shifted cuts are perfectly stacked
- $\mathrm{Var}(\epsilon_0) = 0$.

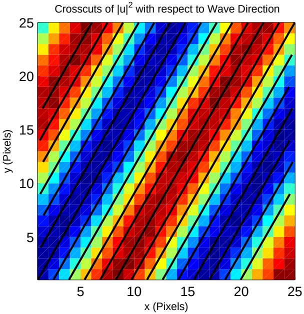
So we minimize $\mathrm{Var}(\epsilon_0) = 0$.

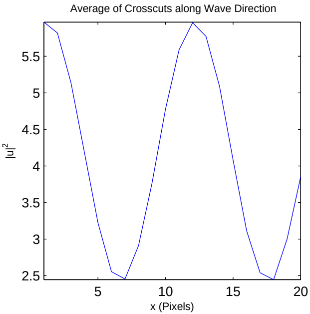


Crosscuts along $\vec{d}$

Once $\epsilon_0 = d_2/d_1$ is calculated,

- Take crosscuts parallel to $\vec{d}$.
- Averaged crosscuts will help reduce noise
- $h(s) := f(s\vec{d}) = |A| \cos(ks + \phi) \quad s \in [-L, L]$





Estimation of k: Fourier Transform

Taking the Fourier transform, we hope to recover k. Look at the Fourier transform of $h(s)$; for $n \in \mathbb{Z} \setminus \{0\}$,

$$\begin{aligned}\hat{h}(n) &= \int_{-L}^{L} h(s) e^{-i\frac{n\pi s}{L}}\, ds \\ &= |A|L\left(\operatorname{sinc}(kL - n\pi)e^{i\phi} + \operatorname{sinc}(kL + n\pi)e^{-i\phi}\right).\end{aligned}$$

- $\hat{h}(n) = 0 \iff kL = p\pi$ for some $p \in \mathbb{N}$ and $n \neq \pm p$.
- Thus, we can always find $n \in \mathbb{N}$ such that $\hat{h}(n) \neq 0$.

For $n \in \mathbb{N}$ such that $\hat{h}(n) \neq 0$, define

$$w(n) := (-1)^n \mathrm{Re}\left(\hat{h}(n)\right) = \frac{2|A|kL^2 \cos(\phi)\sin(kL)}{(kL)^2 - (n\pi)^2}$$

$$v(n) := (-1)^n \mathrm{Im}\left(\hat{h}(n)\right) = \frac{2n|A|\pi L \sin(\phi)\sin(kL)}{(kL)^2 - (n\pi)^2}$$

which gives

$$\left((kL)^2 - (n\pi)^2\right) w(n) = 2|A|kL^2 \cos(\phi)\sin(kL),$$
$$\left((kL)^2 - (n\pi)^2\right) \frac{v(n)}{n} = 2|A|\pi L \sin(\phi)\sin(kL).$$

Note that the RHS does not depend on n, so for $m \neq n$

$$(kL)^2(w(n) - w(m)) = \pi^2(n^2 w(n) - m^2 w(m))$$

$$(kL)^2\left(\frac{v(n)}{n} - \frac{v(m)}{m}\right) = \pi^2(nv(n) - mv(m))$$

Take a square sum:

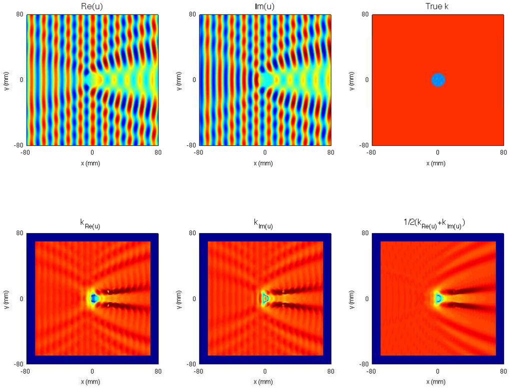
$$(kL)^4 \left[\left(\frac{v(n)}{n} - \frac{v(m)}{m} \right)^2 + (w(n) - w(m))^2 \right] \\ = \pi^4 \left[(nv(n) - mv(m))^2 + (n^2 w(n) - m^2 w(m))^2 \right]$$

And solve for k:

$$k = \frac{\pi}{L} \left(\frac{(nv(n) - mv(m))^2 + (n^2 w(n) - m^2 w(m))^2}{\left(\frac{v(n)}{n} - \frac{v(m)}{m} \right)^2 + (w(n) - w(m))^2} \right)^{1/4}$$

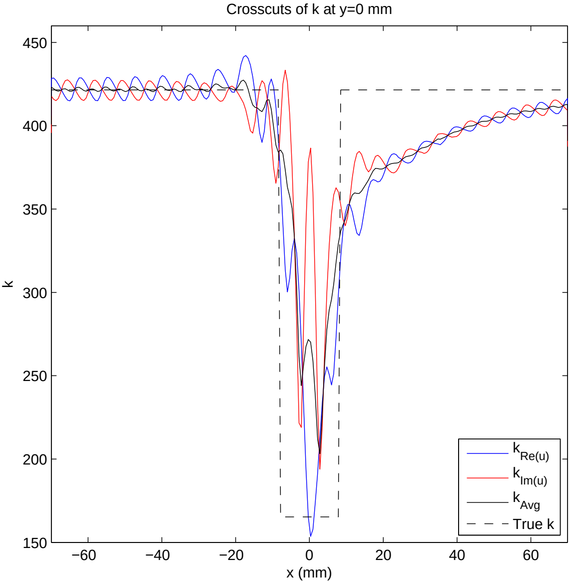
It can be shown that the denominator never vanishes.

$Re(u)$ versus $Im(u)$



- Reconstructions using just $Re(u)$ or $Im(u)$ have unwanted patterns.
- A simple algebraic average gives a better image.

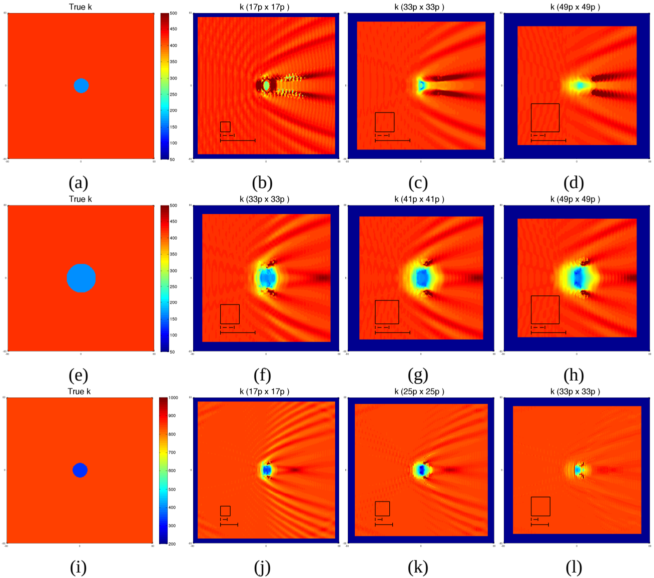
Center Crosscut



Different Window Sizes

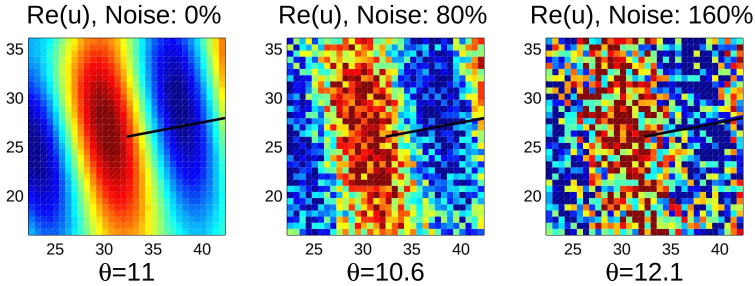
- Background
    - In homogenous case, box sizes larger than λ, guaranteed less than 3.4% error.
    - Expect similar results in background
    - Larger boxes will also reduce the effects of noise
- Inclusion
    - $\lambda_{inc} > \lambda_{bg}$, so larger box is needed.
    - Larger box will contain more points from background, harming resolution
    - Small inclusion with high stiffness (large λ_{inc}) is the most difficult case

Different Window Sizes



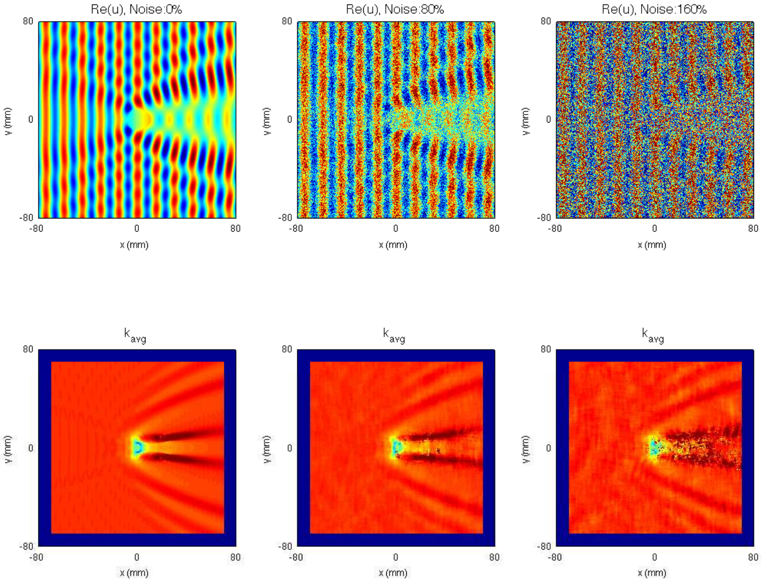
(a) (b) (c) (d)
(e) (f) (g) (h)
(i) (j) (k) (l)

Noise Effects on $\vec{d} = \tan\theta$



$\vec{d}$ reconstruction is extremely resilient to noise.

Overall Noise Effects



Direct Inversion Method

Recall the scalar Helmholtz equation, $\triangle u + k^2 u = 0$.

Let $u(x,y) = M(x,y)e^{i\phi(x,y)}$, where $M > 0$ and $\phi \in \mathbb{R}$. Plugging back into the differential equation, we get

$$[(\triangle M - M|\nabla\phi|^2) + i(M\triangle\phi + 2\nabla M \cdot \nabla\phi) + k^2 M]e^{i\phi} = 0.$$

Then solve for k:

$$k = \sqrt{\frac{M|\nabla\phi|^2 - \triangle M}{M}}$$

where $M = |u|$ and $|\nabla\phi|^2 = \left|\nabla\left(\frac{Re(u)}{|u|}\right)\right|^2 + \left|\nabla\left(\frac{Im(u)}{|u|}\right)\right|^2$.

Estimating the Derivatives

Assuming, $\phi, M \in C^2(\mathbb{R}^2)$, use centered difference for the gradient

$$\nabla\phi \approx \left(\frac{\phi(x+h,y) - \phi(x-h,y)}{2h}, \frac{\phi(x,y+h) - \phi(x,y-h)}{2h} \right)$$

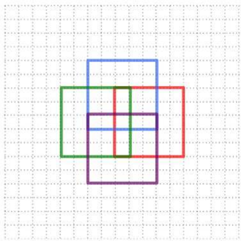
and five point stencil for laplacian,

$$\triangle M \approx \frac{M(x-h,y) + M(x+h,y) + M(x,y-h) + M(x,y+h) - 4M(x,y)}{h^2}$$

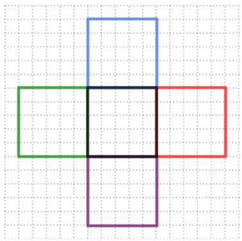
where both give $O(h^2)$ accuracy.

Anderssen's Derivative Method

Instead of using one point in each direction, use an average of values in each direction to approximate the derivatives [MOC, 1999].

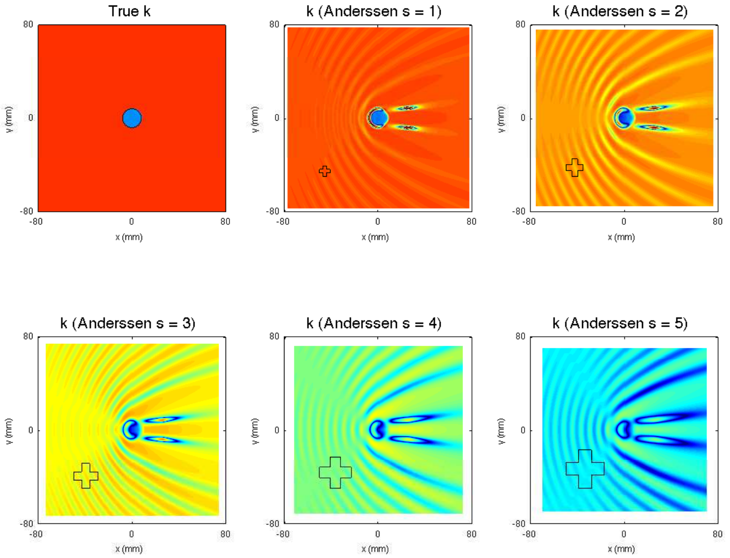


5×5 boxes for $\nabla\phi$

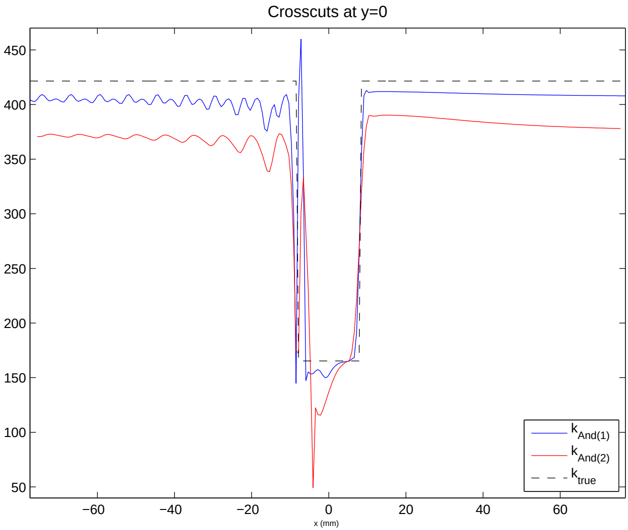


5×5 boxes for $\triangle M$

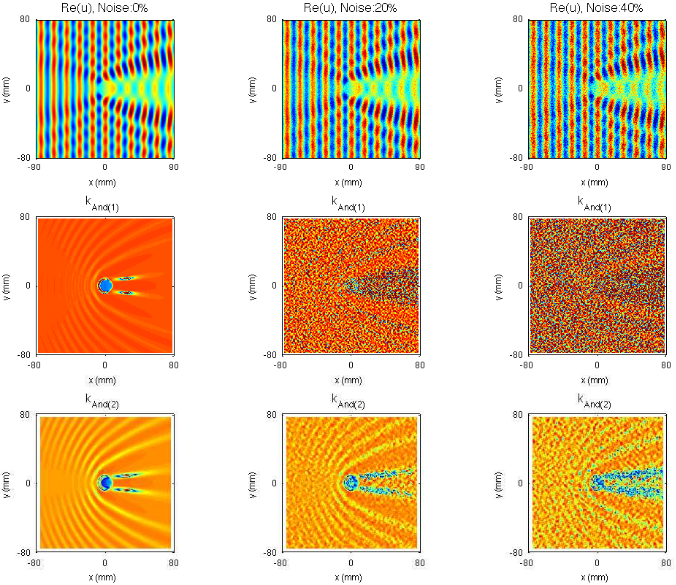
DIM with Anderssen Results



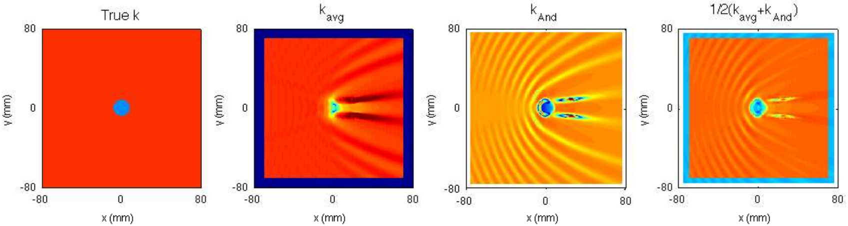
DIM Crosscuts



DIM with Anderssen Noise

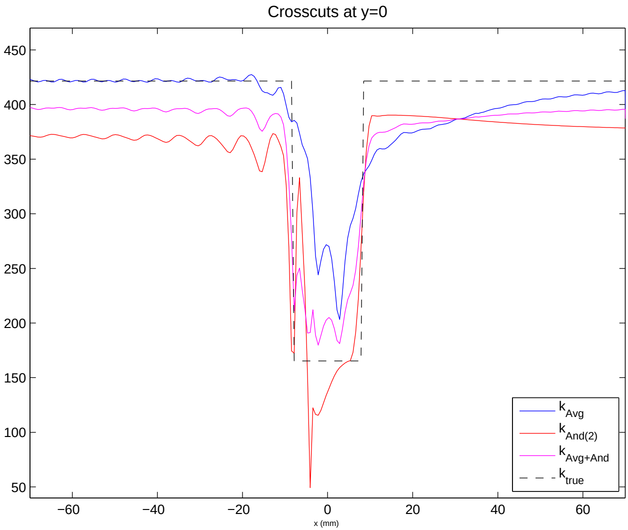


Combining Methods

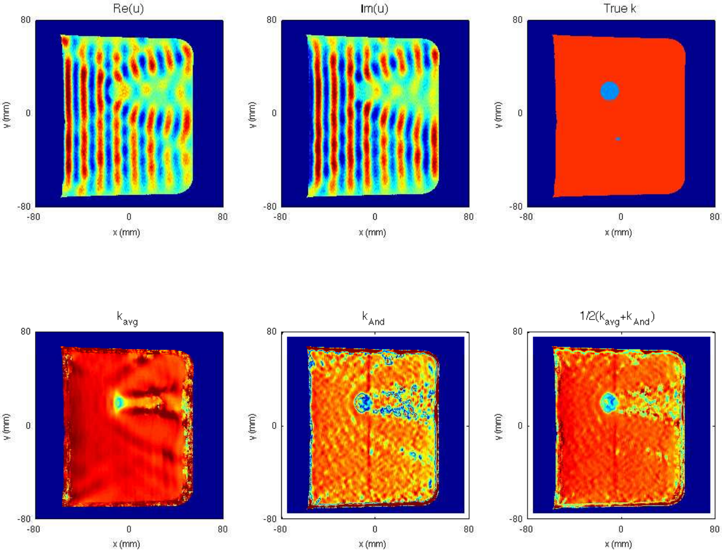


- The Wave-Direction based reconstruction has a better exterior reconstruction than the Direct Inversion Method.
- The Direct Inversion Method has a better inclusion reconstruction.
- An average of the two methods is better overall than each individual method.

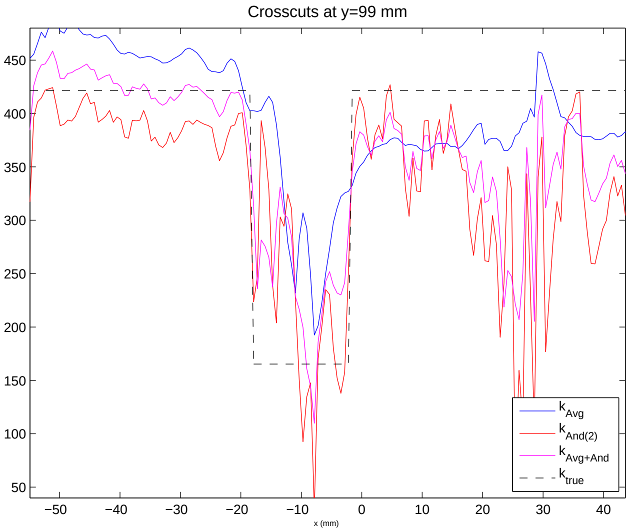
Combining Methods Crosscuts



Experimental Data



Combining Methods Crosscuts



Conclusion

- Wave Direction-base Reconstructions
    - Excellent in the background
    - Resilient to noise
    - The angle reconstruction is extremely resilent
    - Overestimates values in the inclusion
- Direct Inversion with Anderssen Reconstructions
    - Value-wise, it is good in the inclusion, but slightly underestimates
    - Maintains the shape of inclusion
    - Gives unwanted values behind inclusion, which could be misdiagnosed
    - Breaks down with noise
- Combination of Methods
    - Gives good reconstruction of inclusion
    - Sacrifices wave direction-base background reconstruction and resiliency to noise

Future Work

- Improve Combination of Methods
- Introduce viscoelastic medium, $k \in \mathbb{C}$
  - Both u_i and u_s will decay
  - Crosscuts will no longer be sinusoidal