# Billiards and Teichmüller Curves

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# Outline

Polygonal billiards

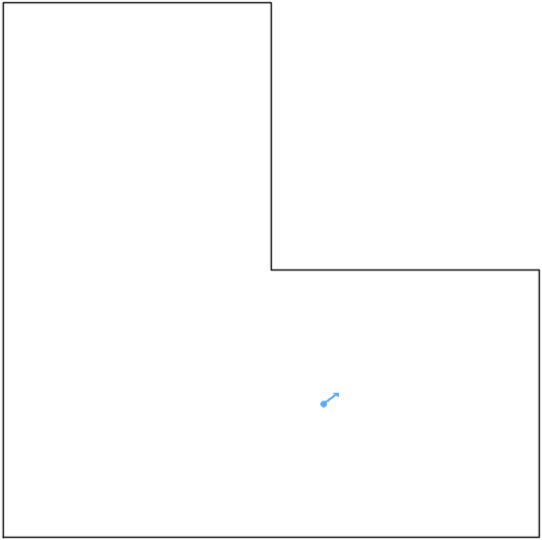
Some background

- Riemann surfaces
- Hyperbolic geometry

Moduli space of Riemann surfaces

Teichmüller Disks

# Polygonal billiards



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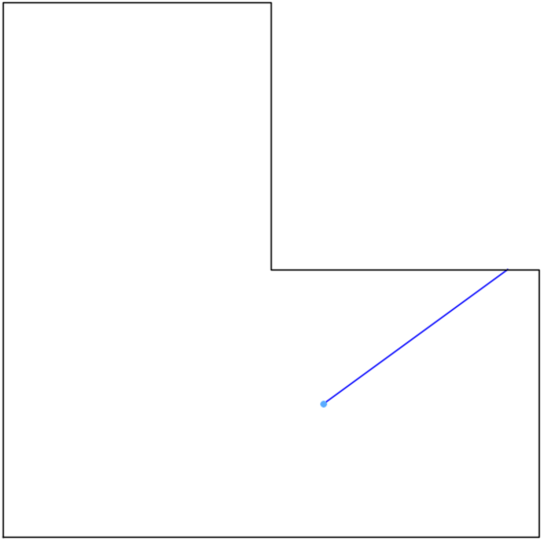

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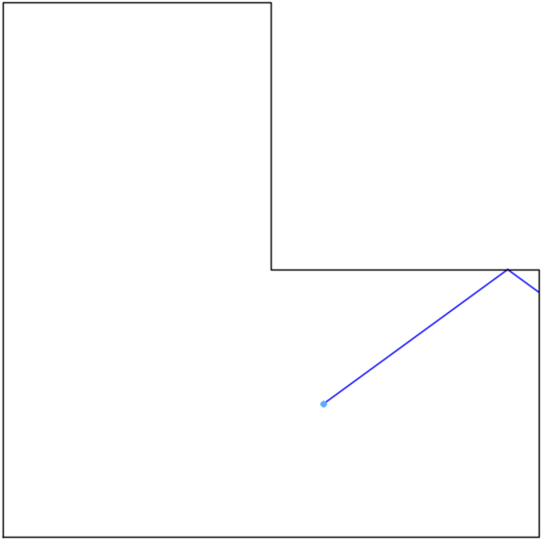

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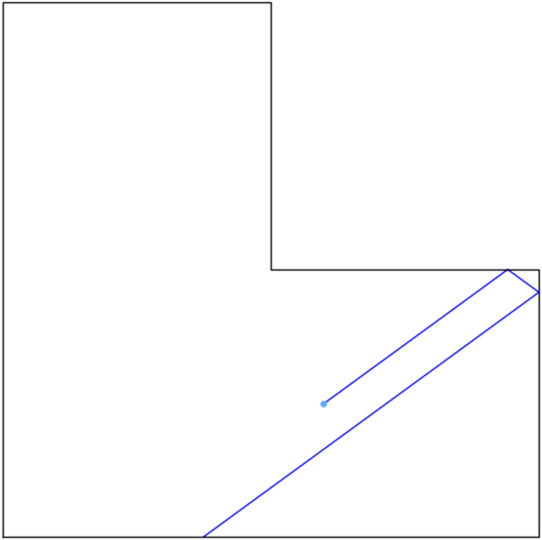

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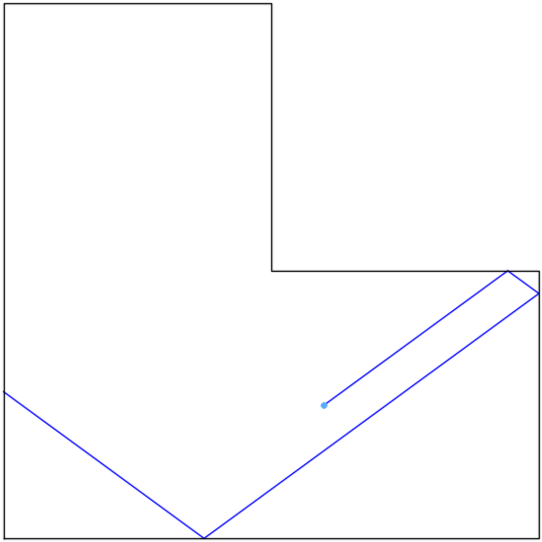

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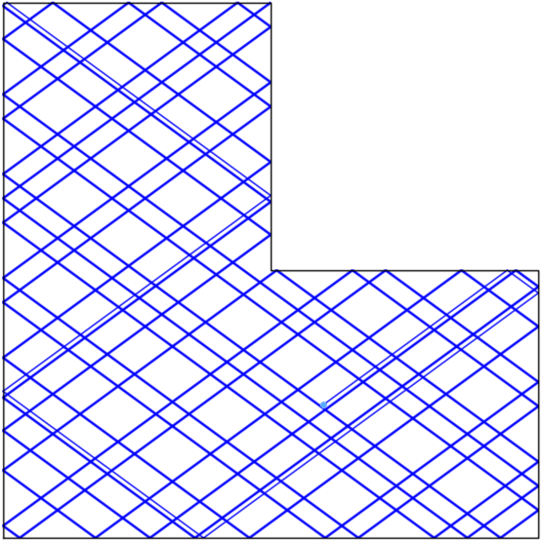

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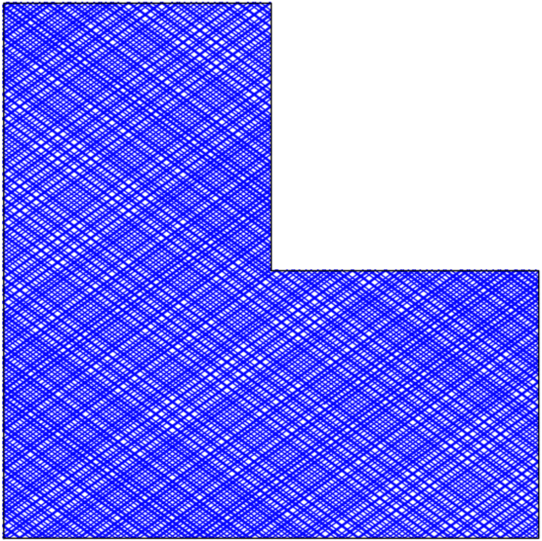

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We're interested in the trajectories of these billiards

1. Are there any periodic trajectories?
2. What other sorts of trajectories are there?
3. Can you easily classify when a trajectory behaves a particular way?

How do you even begin to answer these questions?

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# Unfolding a billiard

Following [ZK76] we can try to “straighten” the billiard trajectory to obtain a surface which has the following properties.

1. The surface covers the initial billiard table.
2. There is a natural metric on the surface.
3. Geodesics of the surface project to billiard trajectories in the original table.

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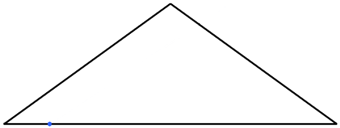
# Unfolding example

Isosceles triangle with angles $\left(\frac{\pi}{5}, \frac{\pi}{5}, \frac{3\pi}{5}\right)$.



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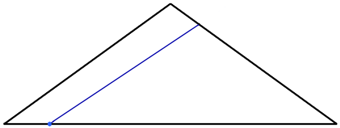

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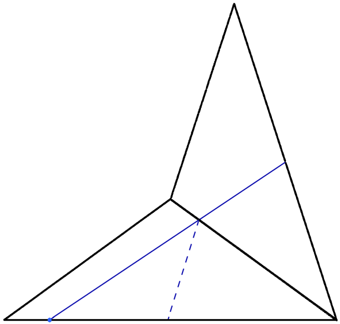

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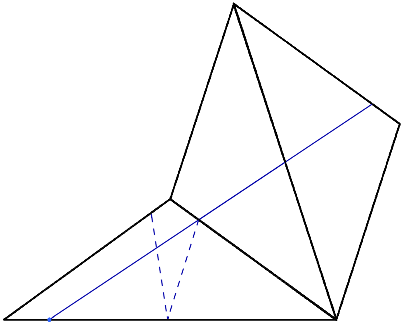

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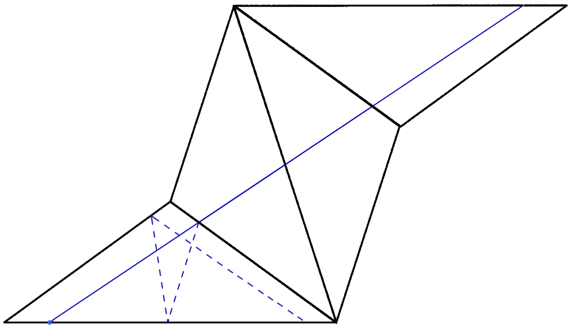

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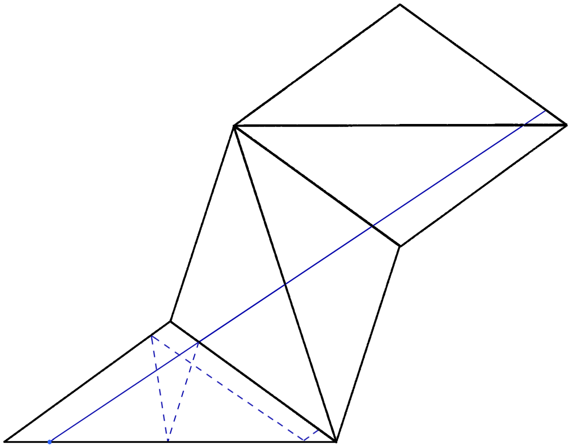

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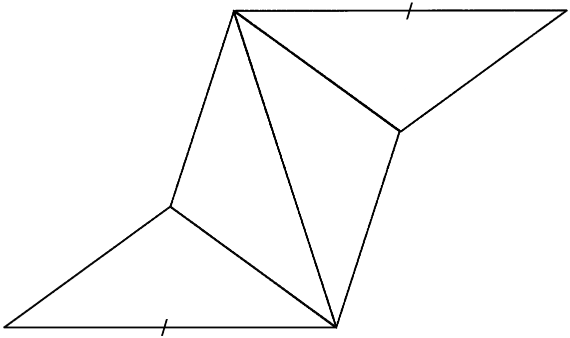

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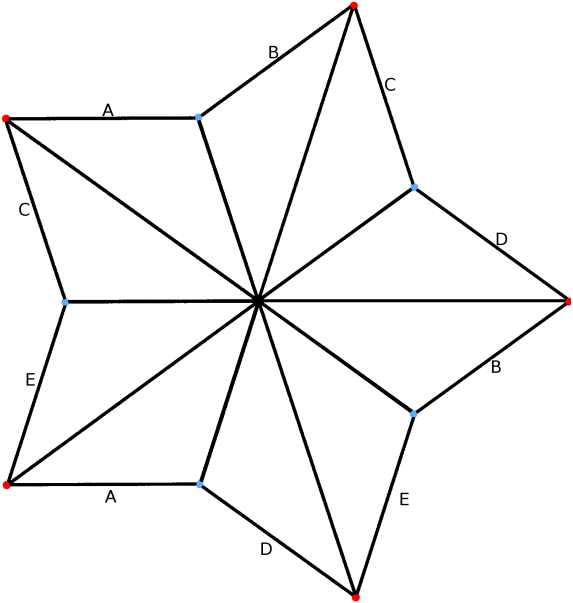
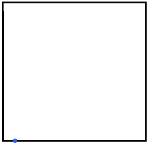


# Some observations

- These surfaces comes with a natural metric ($|dz|$ lifts from $\mathbb{C}$).
- With this metric the surface the surface is locally isometric to $\mathbb{C}$, away from points with cone angle $\theta > 2\pi$.
- Geodesics in this metric are just straight lines in the polygons.

# Unfolding example

Unfolding a square.



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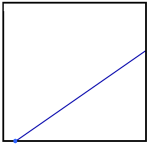

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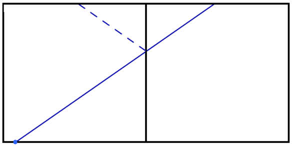

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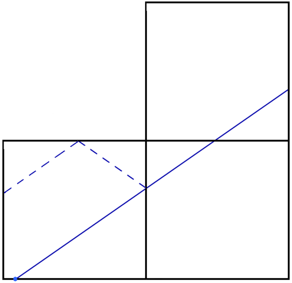

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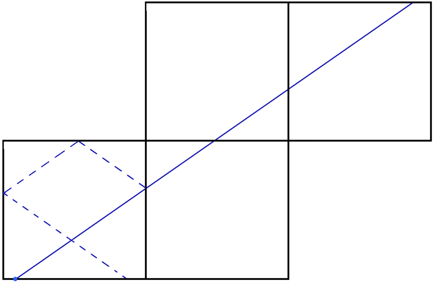

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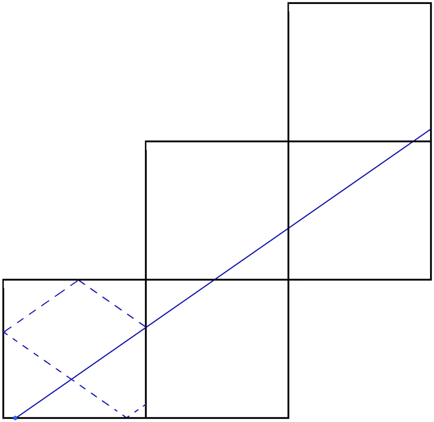

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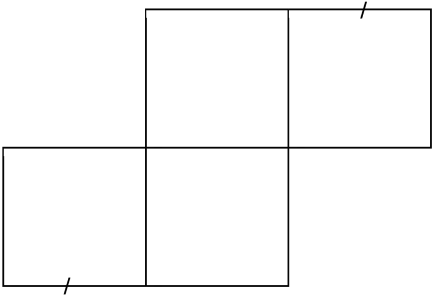

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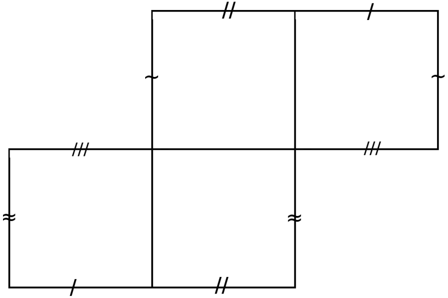

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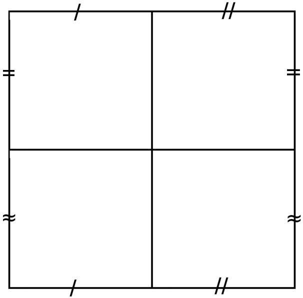

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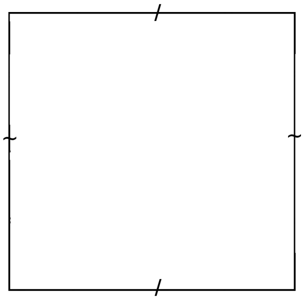

# Unfolding makes things easier

- It's a classical theorem of Weyl that the geodesics in a fixed direction on a flat torus are either periodic or uniformly distributed.
- Thus billiards in a square are either periodic or uniformly distributed, depending on the direction.

What other surfaces (and billiard tables) have this property?

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# Translation surfaces

To answer these questions we consider the sort of surfaces which arise from unfolding. These are called *translation surfaces* and there are several equivalent definitions.

1. A finite set of Euclidean polygons where each edge is glued by translation to another parallel edge of the same length.
2. A pair $(X, \omega)$ where $X$ is a compact Riemann surface and $\omega$ is a holomorphic 1-form on $X$.

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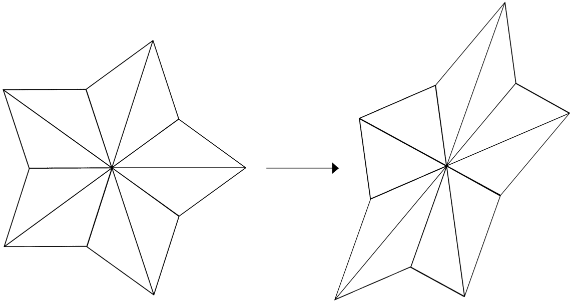
# Translation surfaces

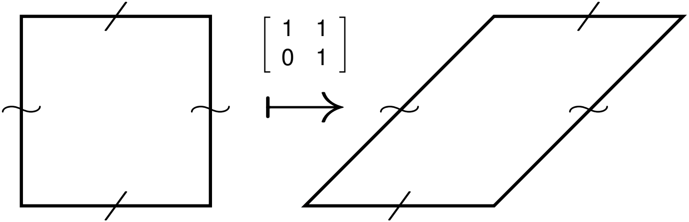
Thinking of the surface as polygons glued together, $\mathrm{SL}(2, \mathbb{R})$ acts on the set of translation surfaces by deforming the polygons.



(Note $\mathrm{SO}(2)$ just rotates the polygons; this is the same as changing the surface's notion of "vertical.")

# The Veech group

The stabilizer of a surface is called the *Veech group* of the surface, denoted $\mathrm{SL}(X, \omega)$.



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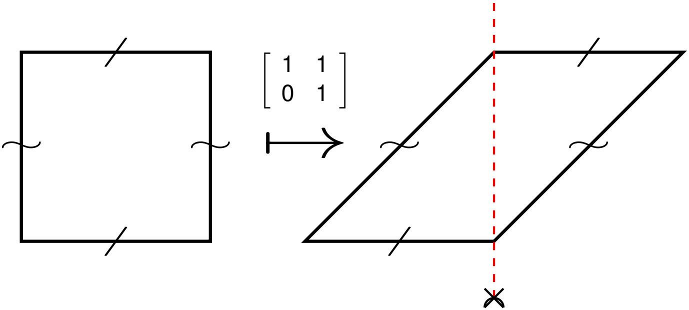

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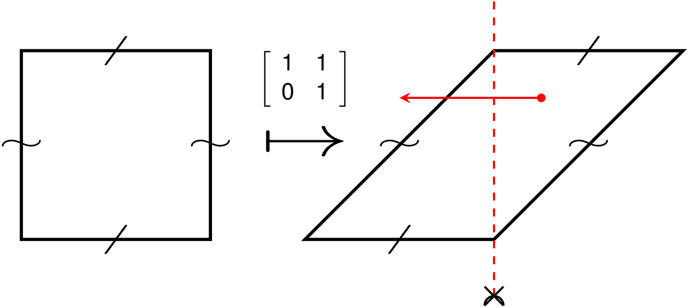

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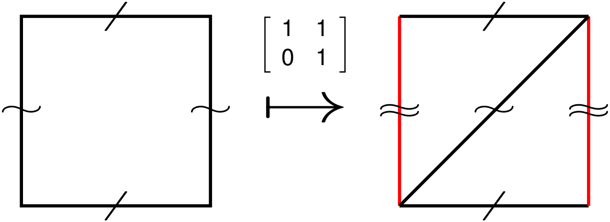

# The Veech dichotomy

- $\mathrm{SL}(X, \omega)$ is always a discrete subgroup of $\mathrm{SL}(2, \mathbb{R})$.
- When $\mathrm{SL}(X, \omega)$ is a lattice (has finite covolume with respect to Haar measure), then the surface satisfies the *Veech dichotomy*:
  - In a fixed direction, every regular trajectory is either periodic or uniformly distributed.

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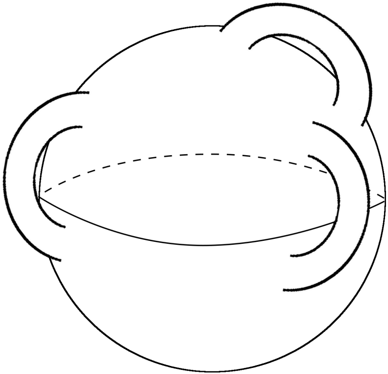
# Some topology

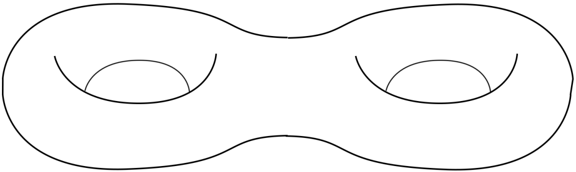
- Every compact orientable surface is homeomorphic to a sphere with a finite number of handles. The number of handles is the *genus* of the surface.



- The Euler characteristic of a surface of genus $g$ is $2 - 2g$.

# Riemann surfaces

A *Riemann surface* is a surface $X$ which locally “looks like” the complex plane.



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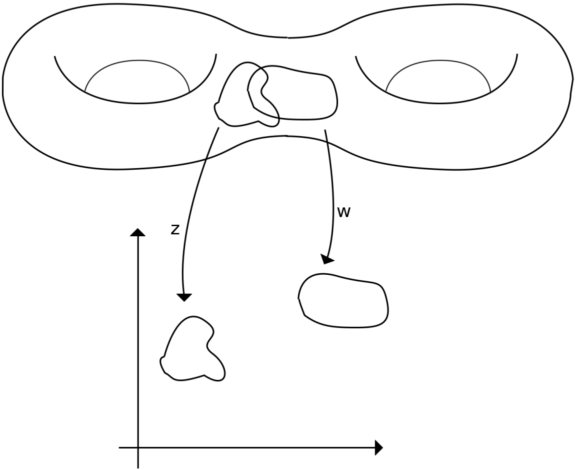

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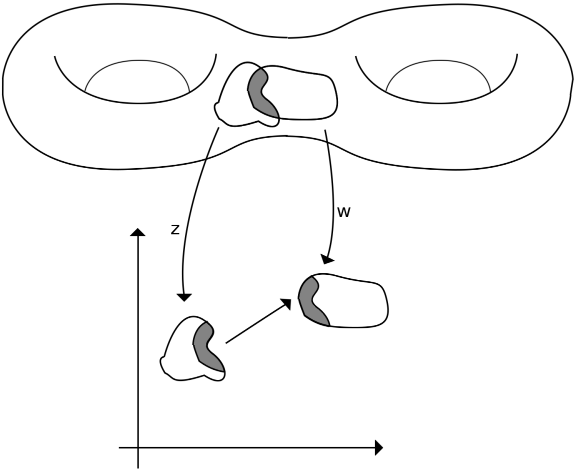

# Riemann surfaces

- An *algebraic curve* is the set of solutions to a polynomial equation.
- If the curve is non-singular, the implicit function theorem lets us turn the curve into a Riemann surface.
- Remarkably, you can go the other and turn a Riemann surface into an algebraic curve.
- (Particularly, compact Riemann surfaces are projective curves.)

# 1-forms

- To do integration on a Riemann surface, you need to pick coordinates.
- But there are lots of choices for coordinates, and your integral will change if you integrate naïvely.
- 1-forms are a way to get around this by taking coordinate changes into consideration.
- For each coordinate patch $(U, z)$ we associate a function $f_U(z)$
- If $(V, w)$ and $(U, z)$ overlap, then their functions should be related:
$$f_V(w) = f_U(z)\frac{\mathrm{d}z}{\mathrm{d}w}$$
More succinctly, $f_v(w)\mathrm{d}w = f_u(z)\mathrm{d}z$.

# Geometry of Riemann surfaces

We mostly care about the hyperbolic surfaces.

### Theorem (Gauss-Bonnet)

*Suppose $S$ is a surface of genus $g$. If the curvature at a point $P$ is given by $k(P)$, then*

$$\iint_S k \, \mathrm{d}A = 2\pi(2 - 2g)$$

If $g \geq 2$, then this integral is negative. If the curvature is constant, then it has to be a constant negative number (which we usually normalize to be -1).
For this reason, hyperbolic geometry is a useful tool.

# Hyperbolic geometry

One model of the hyperbolic plane is the upper half-plane $\mathbb{H}$ together with the metric

$$\mathrm{d}s = \frac{\sqrt{\mathrm{d}x^2 + \mathrm{d}y^2}}{y}.$$

This just means that the arclength of a curve $\gamma(t) = (x(t), y(t))$ for $t \in [a, b]$ is given by

$$\int_a^b \frac{\sqrt{x'(t)^2 + y'(t)^2}}{y(t)}\,\mathrm{d}t$$

# Hyperbolic geometry

- Orientation-preserving isometries of $\mathbb{H}$ with this metric have the form
$$z \mapsto \frac{az+b}{cz+d}$$
where $a, b, c, d \in \mathbb{R}$ and $ad - bc = 1$.
- $\mathrm{Isom}^+(\mathbb{H}) \cong \mathrm{PSL}(2, \mathbb{R})$
- $\mathrm{SL}(2, \mathbb{R})/\mathrm{SO}(2) \cong \mathbb{H}$.

# Hyperbolic geometry of Riemann surfaces

- Hyperbolic surfaces can be represented as $\mathbb{H}/\Gamma$ where $\Gamma$ is a discrete group of orientation-preserving isometries.
- (Compare to the purely topological statement about connected covers: $Y \cong \widetilde{X}/\pi_1(Y)$.)

# Moduli space

How many ways can you turn a topological surface of genus $g \geq 2$ into a Riemann surface?

- This is the same thing as asking how many different hyperbolic metrics can you give a surface.
- Consider two hyperbolic surfaces equivalent if they are isometric.
- The space of equivalence classes of hyperbolic surfaces of genus $g$ is called *moduli space* and denoted $\mathscr{M}_g$.

# Moduli space

- We can actually topologize $\mathscr{M}_g$ by giving it a metric.
- We need some way to of telling how far two non-isometric surfaces are from being isometric.
- For this we consider quasiconformal maps between the surfaces.

# Quasiconformal maps

A map is called *conformal* if it preserves angles. Two Riemann surfaces (resp., hyperbolic surfaces) are “the same” if there is a biholomorphic map (resp., isometry) between them. This happens only when there is a conformal map between the surfaces, so we see what the “closest to conformal map” is.

# Quasiconformal maps

- A map $f : X \to Y$ is called *quasiconformal* if the value

$$K(f) := \sup_{z \in X} \frac{|f_z| + |f_{\bar{z}}|}{|f_z| - |f_{\bar{z}}|}$$

  is finite. (This value measures the maximum amount of distortion by $f$.)
- The map $f$ is conformal if and only if $K(f) = 1$.
- We put a metric on $\mathscr{M}_g$ by

$$d(X, Y) = \inf_{f \in \mathrm{QC}(X,Y)} \log K(f)$$

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# The moduli space of translation surfaces

- A *translation surface* is a pair $(X, \omega)$ where $\omega$ is a holomorphic 1-form. (In coordinates, $\omega = f(z)\mathrm{d}z$.)
- The collection of all holomorphic 1-forms on a genus $g$ Riemann surface $X$ forms a $g$-dimensional vector space $\Omega^1 X$.
- Associating this vector space to each point of $\mathscr{M}_g$ we have a vector bundle denoted $\Omega\mathscr{M}_g$.
- (That is, each genus $g$ translation surface is a point in $\Omega\mathscr{M}_g$, so $\Omega\mathscr{M}_g$ is called the *moduli space of translation surfaces*.)

# Teichmüller disks

- There's a natural "forgetful" map $\pi : \Omega\mathscr{M}_g \to \mathscr{M}_g$ by dropping the 1-form; $(X, \omega) \mapsto X$.
- Recall that elements of $SO(2)$ just rotate the vertical direction on the surface. This is the same as multiplying the 1-form by some $e^{i\theta}$:

$$r_\theta \cdot (X, \omega) = (X, e^{i\theta}\omega)$$

# Teichmüller disks

- Given an $(X,\omega)$, we consider its orbit in $\mathcal{M}_g$:

$$\begin{aligned} f : \mathrm{SL}(2,\mathbb{R}) &\to \mathcal{M}_g \\ A &\mapsto \pi(A\cdot(X,\omega)) \end{aligned}$$

- This map is constant on $\mathrm{SO}(2)$-cosets:

$$A\cdot\mathrm{SO}(2) \mapsto \pi(A\cdot(X,\omega))$$

- Since $\mathbb{H} = \mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2)$, we get a map

$$\mathbb{H} \to \mathcal{M}_g$$

- The image of this map is called the *Teichmüller disk* of $(X,\omega)$.

# Teichmüller disks

### Theorem (Smillie's Theorem, [SW04])

*The Teichmüller disk of* $(X, \omega)$ *is closed if and only if* $\mathrm{SL}(X, \omega)$ *is a Veech surface.*

- $\mathbb{H}/\mathrm{SL}(X, \omega)$ is homeomorphic to the Teichmüller disk of $(X, \omega)$.
- $\mathbb{H}/\mathrm{SL}(X, \omega)$ is a Riemann surface – aka algebraic curve.
- In fact, the Teichmüller disk is isometrically embedded with respect to the hyperbolic metric of $\mathbb{H}/\mathrm{SL}(X, \omega)$ and the Teichmüller metric on $\mathscr{M}_g$.

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# The point of all this...

A problem of dynamical systems (finding optimal billiard tables) becomes a problem of complex algebraic geometry (finding certain algebraic curves in moduli space).
This opens the door to using algebraic geometry to study dynamical problems, and has been successfully used by people like McMullen to classify translation surfaces (and billiard tables) in genus 2.

# References

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