Network-Flow Descriptions of Base-2 Expansions of Integer Variables

Frank M. Muldoon and Warren P. Adams

Clemson University

February 8, 2012

Outline

- Review a standard integer variable formulation using base-2 expansions.
- Give a network representation of a binary lexicographic ordering and related it to the base-2 expansion of an integer variable.
- Provide lexicographic extensions to be used on knapsack problems.
- Provide computational experience on a set of difficult integer knapsack programs.

Base-2 Integer Variable Representation

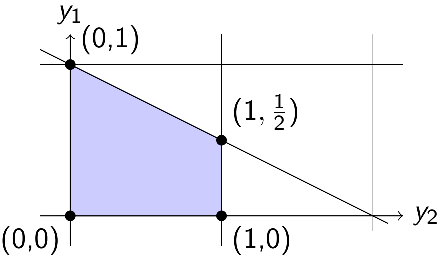
- Consider an integer variable x having $\ell \leq x \leq u$ realizing $n = u - \ell + 1$ possible values.
- It is well known in the literature that an integer variable x can be represented with $\lceil \log_2 n \rceil = m$ binary variables $\mathbf{y}$ as

$$P \equiv \{(x, \mathbf{y}) \in \mathbb{R} \times \mathbb{R}^m : \quad x = \ell + \sum_{j=1}^{m} 2^{m-j} y_j, \quad x \leq u, \ \mathbf{y} \text{ binary}\}.$$

- It is known that P is not *ideal* in that $\text{conv}(P) \subset \overline{P}$ when $\log_2(n) < \lceil \log_2 n \rceil$.

Example with $\text{conv}(P) \subset \overline{P}$

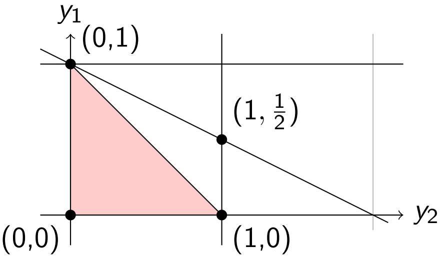
Consider an integer variable x that can realize the values $\{0, 1, 2\}$. Then the set
$\overline{P} = \{(x, y_1, y_2) \in \mathbb{R} \times \mathbb{R}^2 : x = 2y_1 + y_2 \leq 2,\ 0 \leq y_1, y_2 \leq 1\}$
projected onto the **y** space is



which has a nonbinary extreme point at $(x, y_1, y_2) = (2, \frac{1}{2}, 1)$.

Example with $\text{conv}(P) \subset \overline{P}$

The $\text{conv}(P)$ projected onto the $\mathbf{y}$ space shown in red is



Base-2 Integer Variable Representation

- Notice that the feasible binary $\mathbf{y} \in \mathbb{R}^m$ to P are of the form $\{(0, \ldots, 0, 0), (0, \ldots, 0, 1), (0, \ldots, 1, 0), \ldots, \boldsymbol{\alpha}^T\}$ for some unique binary $\boldsymbol{\alpha}$ satisfying

$$\sum_{j=1}^{m} 2^{m-j} \alpha_j = u - \ell$$

- Thus, the set of feasible binary $\mathbf{y}$ solutions to P are lexicographical less than or equal to $\boldsymbol{\alpha}$.

Review and Notation

- Recall that a nonzero binary vector $\mathbf{y}$ is lexicographically nonpositive, denoted $\mathbf{y} \preceq \mathbf{0}$, if the first nonzero entry is negative.
- Also, given two vectors, $\mathbf{y}^1$ and $\mathbf{y}^2$, the vector $\mathbf{y}^1$ is lexicographically less than $\mathbf{y}^2$, denoted $\mathbf{y}^1 \preceq \mathbf{y}^2$, if $\mathbf{y}^1 - \mathbf{y}^2 \preceq \mathbf{0}$.
- Given a binary vector $\boldsymbol{\alpha} \in \mathbb{R}^m$ with $m \geq 2$, partition the set $M \equiv \{1, \dots, m\}$ into the two subsets $M_0 = \{i \in M : \alpha_i = 0\}$ and $M_1 = \{i \in M : \alpha_i = 1\}$.

Lexicographic Example

- A binary vector $\mathbf{y}$ satisfies $\mathbf{y} \preceq \boldsymbol{\alpha}$ if and only if, for each $i \in M_0$ such that $y_i = 1$, there exists some $j \in M_1,\ j < i$ with $y_j = 0$.
- **Example 1**
  Consider the binary vector $\boldsymbol{\alpha}^T = (1,0,1,1,0,1,0)$ with $M_0 = \{2,5,7\}$ and $M_1 = \{1,3,4,6\}$.

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \\ y_7 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \quad ? \quad \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \\ \alpha_6 \\ \alpha_7 \end{bmatrix}$$

Lexicographic Example

- A binary vector $\mathbf{y}$ satisfies $\mathbf{y} \preceq \boldsymbol{\alpha}$ if and only if, for each $i \in M_0$ such that $y_i = 1$, there exists some $j \in M_1,\ j < i$ with $y_j = 0$.
- **Example 1**
  Consider the binary vector $\boldsymbol{\alpha}^T = (1, 0, 1, 1, 0, 1, 0)$ with $M_0 = \{2, 5, 7\}$ and $M_1 = \{1, 3, 4, 6\}$.

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \\ y_7 \end{bmatrix} = \begin{bmatrix} 1 \\ \color{blue}{0} \\ 0 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \quad ? \quad \begin{bmatrix} 1 \\ \color{blue}{0} \\ 1 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \\ \alpha_6 \\ \alpha_7 \end{bmatrix}$$

Lexicographic Example

- A binary vector $\mathbf{y}$ satisfies $\mathbf{y} \preceq \boldsymbol{\alpha}$ if and only if, for each $i \in M_0$ such that $y_i = 1$, there exists some $j \in M_1,\ j < i$ with $y_j = 0$.
- **Example 1**
  Consider the binary vector $\boldsymbol{\alpha}^T = (1,0,1,1,0,1,0)$ with $M_0 = \{2,5,7\}$ and $M_1 = \{1,3,4,6\}$.

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \\ y_7 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ \color{red}{1} \\ 1 \\ 1 \end{bmatrix} \quad ? \quad \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \\ \color{red}{0} \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \\ \alpha_6 \\ \alpha_7 \end{bmatrix}$$

Lexicographic Example

- A binary vector $\mathbf{y}$ satisfies $\mathbf{y} \preceq \boldsymbol{\alpha}$ if and only if, for each $i \in M_0$ such that $y_i = 1$, there exists some $j \in M_1$, $j < i$ with $y_j = 0$.
- **Example 1**
  Consider the binary vector $\boldsymbol{\alpha}^T = (1, 0, 1, 1, 0, 1, 0)$ with $M_0 = \{2, 5, 7\}$ and $M_1 = \{1, 3, 4, 6\}$.

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \\ y_7 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ {\color{blue}0} \\ 1 \\ {\color{red}1} \\ 1 \\ 1 \end{bmatrix} \quad ? \quad \begin{bmatrix} 1 \\ 0 \\ {\color{blue}1} \\ 1 \\ {\color{red}0} \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \\ \alpha_6 \\ \alpha_7 \end{bmatrix}$$

Lexicographic Example

- A binary vector $\mathbf{y}$ satisfies $\mathbf{y} \preceq \boldsymbol{\alpha}$ if and only if, for each $i \in M_0$ such that $y_i = 1$, there exists some $j \in M_1,\ j < i$ with $y_j = 0$.
- **Example 1**
  Consider the binary vector $\boldsymbol{\alpha}^T = (1, 0, 1, 1, 0, 1, 0)$ with $M_0 = \{2, 5, 7\}$ and $M_1 = \{1, 3, 4, 6\}$.

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \\ y_7 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \preceq \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \\ \alpha_6 \\ \alpha_7 \end{bmatrix}$$

Lexicographic Example

- A binary vector $\mathbf{y}$ satisfies $\mathbf{y} \preceq \boldsymbol{\alpha}$ if and only if, for each $i \in M_0$ such that $y_i = 1$, there exists some $j \in M_1,\ j < i$ with $y_j = 0$.
- **Example 2**
  Consider the binary vector $\boldsymbol{\alpha}^T = (1, 0, 1, 1, 0, 1, 0)$ with $M_0 = \{2, 5, 7\}$ and $M_1 = \{1, 3, 4, 6\}$.

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \\ y_7 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \quad ? \quad \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \\ \alpha_6 \\ \alpha_7 \end{bmatrix}$$

Lexicographic Example

- A binary vector $\mathbf{y}$ satisfies $\mathbf{y} \preceq \boldsymbol{\alpha}$ if and only if, for each $i \in M_0$ such that $y_i = 1$, there exists some $j \in M_1,\ j < i$ with $y_j = 0$.
- **Example 2**
  Consider the binary vector $\boldsymbol{\alpha}^T = (1,0,1,1,0,1,0)$ with $M_0 = \{2,5,7\}$ and $M_1 = \{1,3,4,6\}$.

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \\ y_7 \end{bmatrix} = \begin{bmatrix} 1 \\ {\color{red}1} \\ 0 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \quad ? \quad \begin{bmatrix} 1 \\ {\color{red}0} \\ 1 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \\ \alpha_6 \\ \alpha_7 \end{bmatrix}$$

Lexicographic Example

- A binary vector $\mathbf{y}$ satisfies $\mathbf{y} \preceq \boldsymbol{\alpha}$ if and only if, for each $i \in M_0$ such that $y_i = 1$, there exists some $j \in M_1$, $j < i$ with $y_j = 0$.
- **Example 2**
  Consider the binary vector $\boldsymbol{\alpha}^T = (1, 0, 1, 1, 0, 1, 0)$ with $M_0 = \{2, 5, 7\}$ and $M_1 = \{1, 3, 4, 6\}$.

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \\ y_7 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \not\preceq \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \\ \alpha_6 \\ \alpha_7 \end{bmatrix}$$

Observation 1

Given the binary vectors $\mathbf{y}, \boldsymbol{\alpha} \in \mathbb{R}^m$ and the sets M, M_0, and M_1, we have $\mathbf{y} \preceq \boldsymbol{\alpha}$ if and only the following inequalities are satisfied.

$$y_i \leq \sum_{\substack{j \in M_1 \\ j < i}} (1 - y_j) \ \forall \ i \in M_0$$

Based on this observation, the set

$$S \equiv \left\{ \mathbf{y} \in \mathbb{R}^m : y_i \leq \sum_{\substack{j \in M_1 \\ j < i}} (1 - y_j), \ \forall \ i \in M_0, \mathbf{y} \text{ binary} \right\}$$

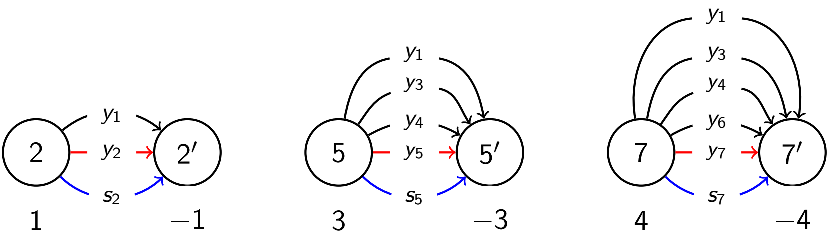
characterizes those binary vectors $\mathbf{y}$ having $\mathbf{y} \preceq \boldsymbol{\alpha}$.

Network Representation

- We wish to show that $\text{conv}(S) = \overline{S}$.
- The network representation is motivated by the inequalities found in S modified by adding slack variables $\mathbf{s}$ to form equality constraints.
- Each constraint, represented by some $i \in M_0$, is enforced using an individual network by creating two nodes, i and i', with directed arcs y_i, y_j for $j < i$, $j \in M_1$, and s_i originating at node i and terminating at node i'.
- Each of the y_i variables are bounded between 0 and 1, while the slack variables s_i are only restricted to be nonnegative.

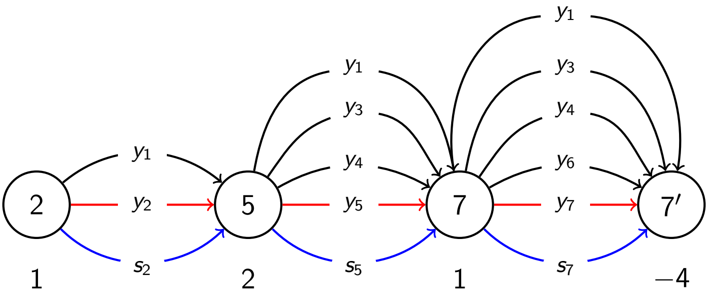
Individual Network Example

- Let $\alpha^T = (1, 0, 1, 1, 0, 1, 0)$ with $M_0 = \{2, 5, 7\}$ and $M_1 = \{1, 3, 4, 6\}$.
- The constrains of S with slacks can be written as $y_1 + y_2 + s_2 = 1$, $y_1 + y_3 + y_4 + y_5 + s_5 = 3$, and $y_1 + y_3 + y_4 + y_6 + y_7 + s_7 = 4$,.
- The three networks are given below.



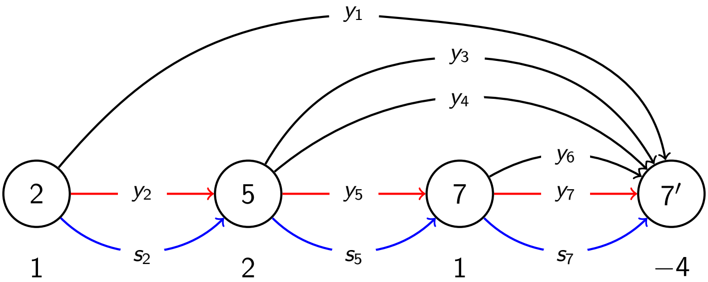
Network Representation

- The networks are not separable because once an arc appears in an individual network, it appears in every subsequent network creating equal flow constraints.
- The structure of these individual networks allows them to be combined by merging adjacent nodes and adding the supplies into the network denoted $\mathcal{N}(\mathbf{y}, \mathbf{s})$.



Network Representation

- The networks are not separable because once an arc appears in an individual network, it appears in every subsequent network creating equal flow constraints.
- The structure of these individual networks allows them to be combined by merging adjacent nodes and adding the supplies into the network denoted $\mathcal{N}(\mathbf{y}, \mathbf{s})$.



Convex Hull Result for S

- The network construction gives that every extreme point of $\mathcal{N}(\mathbf{y}, \mathbf{s})$ has $\mathbf{y}$ binary.
- As a result, the projection of the network onto the $\mathbf{y}$ space gives the following convex hull result on the set S.

$$\text{Proj}_{\mathbf{y}}(\mathcal{N}(\mathbf{y}, \mathbf{s})) = \overline{S} = \text{conv}(S),$$

where

$$\text{Proj}_{\mathbf{y}}(\bullet) \equiv \{\mathbf{y} : \text{there exists an } \mathbf{s} \text{ so that } (\mathbf{y}, \mathbf{s}) \in \bullet\}$$

Network Results

- Observe that the network representation of the lexicographic ordering can be applied to find the conv(P).
- For two binary vectors $\mathbf{y}, \boldsymbol{\alpha} \in \mathbb{R}^m$ we have

$$\mathbf{y} \preceq \boldsymbol{\alpha} \iff \sum_{j=1}^{m} 2^{m-j} y_j \leq \sum_{j=1}^{m} 2^{m-j} \alpha_j = u - \ell.$$

- As a result

$$\mathbf{y} \preceq \boldsymbol{\alpha} \iff \ell + \sum_{j=1}^{m} 2^{m-j} y_j \leq \ell + \sum_{j=1}^{m} 2^{m-j} \alpha_j = u.$$

New Formulation of P

The set P can be rewritten as

$$P = \left\{ (x, \mathbf{y}) \in \mathbb{R} \times \mathbb{R}^m : x = \ell + \sum_{j=1}^{m} 2^{m-j} y_j,\ \mathbf{y} \preceq \boldsymbol{\alpha},\ \mathbf{y} \text{ binary} \right\}.$$

Where the lexicographic ordering $\mathbf{y} \preceq \boldsymbol{\alpha}$ can be represented as a network in a higher dimensional space as

$$\left\{ (x, \mathbf{y}, \mathbf{s}) \in \mathbb{R} \times \mathbb{R}^m \times \mathbb{R}^{m_0} : x = \ell + \sum_{j=1}^{m} 2^{m-j} y_j,\ (\mathbf{y}, \mathbf{s}) \in \mathcal{N}(\mathbf{y}, \mathbf{s}) \right\}.$$

A projection onto the $(x, \mathbf{y})$ spaces gives the convex hull result

$$\operatorname{conv}(P) = \left\{ (x, \mathbf{y}) \in \mathbb{R} \times \mathbb{R}^m : x = \ell + \sum_{j=1}^{m} 2^{m-j} y_j,\ \mathbf{y} \in \overline{S} \right\}.$$

More General Lexicographic Restrictions

- Everything discussed so far has dealt with a binary $\mathbf{y}$ lexicographically less than or equal to some binary $\boldsymbol{\alpha}$.
- Similar results hold for some binary $\mathbf{y}$ lexicographically bigger than or equal to a binary $\boldsymbol{\beta}$ denoted $\mathbf{y} \succeq \boldsymbol{\beta}$.
- Given a binary vector $\boldsymbol{\beta} \in \mathbb{R}^m$ with $m \geq 2$, partition the set $M \equiv \{1, \ldots, m\}$ into the two subsets $N_0 = \{i \in M : \beta_i = 0\}$ and $N_1 = \{i \in M : \beta_i = 1\}$.
- A binary vector $\mathbf{y}$ satisfies $\mathbf{y} \succeq \boldsymbol{\beta}$ if and only if, for each $i \in N_1$ such that $y_i = 0$, there exists some $j \in N_0,\ j < i$ with $y_j = 1$.

Observation 2

Given the binary vectors $\mathbf{y}, \boldsymbol{\beta} \in \mathbb{R}^m$ and the sets M, N_0, and N_1, we have $\mathbf{y} \succeq \boldsymbol{\beta}$ if and only the following inequalities are satisfied.

$$1 - y_i \leq \sum_{\substack{j \in N_0 \\ j < i}} y_j \ \forall\ i \in N_1$$

Based on this observation, the set

$$Q \equiv \left\{ \mathbf{y} \in \mathbb{R}^m : 1 - y_i \leq \sum_{\substack{j \in N_0 \\ j < i}} y_j,\ \forall\ i \in N_1, \mathbf{y} \text{ binary} \right\}$$

characterizes those binary vectors $\mathbf{y}$ having $\mathbf{y} \succeq \boldsymbol{\beta}$.

Network Representation of Q

- Making a variable substitution of $y_i' = 1 - y_i$ for all $i \in M$ in the set Q' gives a similar structure found in P.

$$Q' \equiv \left\{ \mathbf{y}' \in \mathbb{R}^m : y_i' \leq \sum_{\substack{j \in N_0 \\ j < i}} (1 - y_j'),\ \forall\, i \in N_1, \mathbf{y}' \text{ binary} \right\}$$

- This is apparent because given two binary vectors $\mathbf{y}, \boldsymbol{\beta} \in \mathbb{R}^m$ we have

$$\mathbf{y} \succeq \boldsymbol{\beta} \Longleftrightarrow \mathbf{1} - \mathbf{y} \preceq \mathbf{1} - \boldsymbol{\beta}.$$

- Thus, the network representation is valid for Q and shows that $\text{conv}(Q) = \overline{Q}$.

Lexicographic Extensions

Let $\boldsymbol{\beta}, \boldsymbol{\alpha} \in \mathbb{R}^m$ with $\boldsymbol{\beta} \preceq \boldsymbol{\alpha}$. An integer variable x bounded between ℓ and u can be represented using the relationship

$$\boldsymbol{\beta} \preceq \mathbf{y} \preceq \boldsymbol{\alpha} \iff \ell = \sum_{j=1}^{m} 2^{m-j}\beta_j \leq \sum_{j=1}^{m} 2^{m-j} y_j \leq \sum_{j=1}^{m} 2^{m-j}\alpha_j = u,$$

as

$$P' \equiv \left\{ (x, \mathbf{y}) \in \mathbb{R} \times \mathbb{R}^m : x = \sum_{j=1}^{m} 2^{m-j} y_j,\ \boldsymbol{\beta} \preceq \mathbf{y} \preceq \boldsymbol{\alpha},\ \mathbf{y} \text{ binary} \right\}.$$

Lexicographic Extensions

The lexicographic relationship

$$\{\mathbf{y} \in \mathbb{R}^m : \boldsymbol{\beta} \preceq \mathbf{y} \preceq \boldsymbol{\alpha},\ \mathbf{y} \text{ binary}\}$$

can be represented as

$$T \equiv \{\mathbf{y} \in \mathbb{R}^m : \quad 1 - y_i \leq \sum_{\substack{j \in N_0 \\ j < i}} y_j \ \forall\ i \in N_1, \quad y_i \leq \sum_{\substack{j \in M_1 \\ j < i}} (1 - y_j) \ \forall\ i \in M_0,\ \mathbf{y} \text{ binary}\}.$$

Question. Is $\overline{T} = \text{conv}(T)$?

Lexicographic Extensions

Theorem

$conv(T) = \overline{T}$.

The proof shows every extreme point consisting of m linearly independent constraint of T is binary.
It gives the following convex hull result.

$$\text{conv}(P') \equiv \left\{ (x, \mathbf{y}) \in \mathbb{R} \times \mathbb{R}^m : x = \sum_{j=1}^{m} 2^{m-j} y_j,\ \mathbf{y} \in \overline{T} \right\}.$$

Coefficients

- The formulations P and P' represent integer variables using coefficients that are powers of 2.
- These coefficients can be relaxed to any such nonnegative coefficients that are "weakly super-decreasing" (equivalently, weakly super-increasing).
- That is, nonnegative coefficients γ_j of the form

$$\gamma_j \geq \sum_{i=j+1}^{m} \gamma_i, \ \forall j = 1, 2, \ldots, m-1.$$

- Everything proven using base-2 coefficients still holds for weakly super-decreasing coefficients.

0-1 Knapsack Polytopes

Consider a 0-1 knapsack polytope of the form

$$KP(\mathbf{y}) \equiv \left\{ \mathbf{y} \in \mathbb{R}^m : \kappa_1 \leq \sum_{j=1}^{m} \gamma_j y_j \leq \kappa_2, \ \mathbf{y} \text{ binary} \right\},$$

where κ_1 and κ_2 are scalars and the γ_j's are weakly super-decreasing.
Find the smallest κ_1' and lexicographically smallest $\boldsymbol{\beta}$ such that

$$\kappa_1 \leq \kappa_1' = \sum_{j=1}^{m} \gamma_j \boldsymbol{\beta}_j.$$

0-1 Knapsack Polytopes

Also, find the largest κ_2' and lexicographically largest $\boldsymbol{\alpha}$ such that

$$\kappa_2 \geq \kappa_2' = \sum_{j=1}^{m} \gamma_j \boldsymbol{\alpha}_j.$$

The knapsack polytope can be written as

$$KP(\mathbf{y}) = \{\mathbf{y} \in \mathbb{R}^m : \boldsymbol{\beta} \preceq \mathbf{y} \preceq \boldsymbol{\alpha}, \mathbf{y} \text{ binary}\},$$

Using the Theorem we know the convex hull is given as

$$\text{conv}(KP(\mathbf{y})) = \{\mathbf{y} \in \mathbb{R}^m : \mathbf{y} \in \overline{T}.\}$$

Convex Hull

- The convex hull result for $KP(\mathbf{y})$ for the special case when $\kappa_1 = 0$ was shown by Laurent and Sassano (1992) by forming the minimal cover inequalities of S and then using a result from Seymour (1977).
- Our approach considered a network perspective motivated by a lexicographic ordering of binary vectors that was independent of the previous work.

Computational Experience

We tested whether the formulation given by $\overline{P}$ or conv(P) generated from $\overline{P}$ and the set $\overline{S}$ improved computational efficiency on the following optimization problem

$$
\begin{aligned}
\text{MIKP :minimize} \quad & \sum_{i=1}^{p} x_i + (pU)r \\
\text{subject to:} \quad & \sum_{i=1}^{p} (2x_i) + r = pU - 1 \\
& 0 \le x_i \le U, \;\; i = 1, \ldots, p \\
& x_i \text{ integer}, \quad i = 1, \ldots, p \\
& r \ge 0.
\end{aligned}
$$

Computational Experience

The first formulation replaces the integer variables with a base-2 expansion.

$$
\begin{aligned}
\text{BKP1 :minimize} \quad & \sum_{i=1}^{p} x_i + (pU)r \\
\text{subject to:} \quad & \sum_{i=1}^{p} (2x_i) + r = pU - 1 \\
& x_i = \sum_{j=1}^{\lceil \log_2(U+1) \rceil} 2^{\lceil \log_2(U+1) \rceil - j} y_j^i, && \forall i \\
& x_i \leq U, && \forall i \\
& y_j^i \text{ integer}, && \forall i, j \\
& r \geq 0.
\end{aligned}
$$

Computational Experience

The second formulation replaces the upper bound restrictions

$$x_i \leq U, \ \forall i$$

with the minimal cover inequalities of S

$$y_j^i \leq \sum_{\substack{k \in M_1 \\ k < j}} (1 - y_k), \ \forall i, \ \forall j \in M_0$$

Computational Experience

51 trials were solved using ILOG CPLEX 11.0 (with *presolve* disabled) for various parameters p and U.

- 46 of the 51 problems showed computational improvement in CPU time and nodes explored.
- Inequalities of the form

$$y_j^i \leq 1 - y_1^i,\ i = 1, \ldots, p,\ j = 2, \ldots \lceil \log_2(U+1) \rceil$$

  tended to be more effective than equalities of the form

$$y_j^i \leq \sum_{k=1}^{m-1} (1 - y_j^i),\ i = 1, \ldots, p,\ j = 2, \ldots \lceil \log_2(U+1) \rceil$$

Future Research

- Continue computational testing to see which type of problems and structures the cuts added are most effective.
- We only considered a base-2 expansion of the integer variable x, but do the same results hold for base-n expansions of x?
    1. The network structure is no longer valid because minimal cover inequalities are not used.
    2. We believe there exists a variation on the minimal cover inequalities by adding coefficients and modifying the right-hand sides to form the convex hull.
    3. Given two vectors (using any base or a mix of bases), we should be able to determine the convex hull of all integer vectors lexicographically between some β and α.

Thank You !

References

Laurent, M., and Sassano, A. "A Characterization of Knapsacks with the Max-Flow-Min-Cut Property," *Operations Research letters*, Vol. 11, No. 2, pp 105-110, 1992.

Seymour, P.D., "The Matroids with the Max-Flow-Min-Cut Property," *Journal of Combinatorial Theory Series B*, Vol. 23, pp 189-222, 1977.