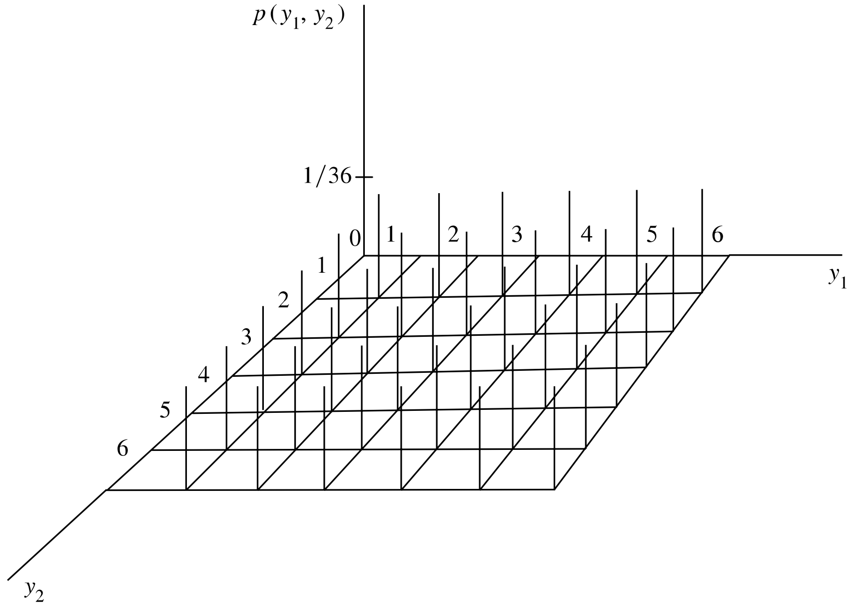

FIGURE 5.1 Bivariate probability function; y_1 = number of dots on die 1, y_2 = number of dots on die 2

DEFINITION 5.1 Let Y_1 and Y_2 be discrete random variables. The *joint* (or bivariate) *probability function* for Y_1 and Y_2 is given by

$$p(y_1, y_2) = P(Y_1 = y_1, Y_2 = y_2), \qquad -\infty < y_1 < \infty, -\infty < y_2 < \infty.$$

In the single-variable case discussed in Chapter 3, we saw that the probability function for a discrete random variable Y assigns nonzero probabilities to a finite or countable number of distinct values of Y in such a way that the sum of the probabilities is equal to 1. Similarly, in the bivariate case the joint probability function $p(y_1, y_2)$ assigns nonzero probabilities to only a finite or countable number of pairs of values (y_1, y_2). Further, the nonzero probabilities must sum to 1.

THEOREM 5.1 If Y_1 and Y_2 are discrete random variables with joint probability function $p(y_1, y_2)$, then

1. $p(y_1, y_2) \geq 0$ for all y_1, y_2.
2. $\sum_{y_1, y_2} p(y_1, y_2) = 1$, where the sum is over all values (y_1, y_2) that are assigned nonzero probabilities.

As in the univariate discrete case, the joint probability function for discrete random variables is sometimes called the *joint probability mass function* because it specifies the probability (mass) associated with each of the possible pairs of values for the random variables. Once the joint probability function has been determined for discrete random variables Y_1 and Y_2, calculating joint probabilities involving Y_1 and Y_2 is straightforward. For the die-tossing experiment, $P(2 \leq Y_1 \leq 3, 1 \leq Y_2 \leq 2)$ is

$$\begin{aligned} P(2 \leq Y_1 \leq 3, 1 \leq Y_2 \leq 2) &= p(2,1) + p(2,2) + p(3,1) + p(3,2) \\ &= 4/36 = 1/9. \end{aligned}$$

EXAMPLE **5.1** A local supermarket has three checkout counters. Two customers arrive at the counters at different times when the counters are serving no other customers. Each customer chooses a counter at random, independently of the other. Let Y_1 denote the number of customers who choose counter 1 and Y_2, the number who select counter 2. Find the joint probability function of Y_1 and Y_2.

Solution We might proceed with the derivation in many ways. The most direct is to consider the sample space associated with the experiment. Let the pair $\{i, j\}$ denote the simple event that the first customer chose counter i and the second customer chose counter j, where $i, j = 1, 2,$ and 3. Using the mn rule, the sample space consists of $3 \times 3 = 9$ sample points. Under the assumptions given earlier, each sample point is equally likely and has probability 1/9. The sample space associated with the experiment is

$$S = [\{1, 1\}, \{1, 2\}, \{1, 3\}, \{2, 1\}, \{2, 2\}, \{2, 3\}, \{3, 1\}, \{3, 2\}, \{3, 3\}].$$

Notice that sample point $\{1, 1\}$ is the only sample point corresponding to $(Y_1 = 2, Y_2 = 0)$ and hence $P(Y_1 = 2, Y_2 = 0) = 1/9$. Similarly, $P(Y_1 = 1, Y_2 = 1) = P(\{1, 2\} \text{ or } \{2, 1\}) = 2/9$. Table 5.1 contains the probabilities associated with each possible pair of values for Y_1 and Y_2—that is, the joint probability function for Y_1 and Y_2. As always, the results of Theorem 5.1 hold for this example.

Table 5.1 Probability function for Y_1 and Y_2, Example 5.1

	y_1		
y_2	0	1	2
0	1/9	2/9	1/9
1	2/9	2/9	0
2	1/9	0	0

■

As in the case of univariate random variables, the distinction between jointly discrete and jointly continuous random variables may be characterized in terms of their (joint) distribution functions.

DEFINITION **5.2** For any random variables Y_1 and Y_2, the joint (bivariate) distribution function $F(y_1, y_2)$ is

$$F(y_1, y_2) = P(Y_1 \leq y_1, Y_2 \leq y_2), \qquad -\infty < y_1 < \infty, -\infty < y_2 < \infty.$$