Supplementary Material for State Logging in Chemical Reaction Networks

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1 Robustness Properties

Many CRNs are designed to complete a specific task. In many cases, they are even designed to only work for a specific set of parameters. Any changes in the parameters can disrupt the functionality of the CRN. We seek to define a CRN to monitor the concentration of a signal and a CRN to compute the AND of many CRN signals robustly correct, i.e., the correct output even with some perturbation on the parameters and the inputs. Robustness of CRNs has been formally defined in [2]. Here we give a brief overview of the robustness parameters used.

We use the following parameters for robustness. A vector $\boldsymbol{\delta} = (\delta_1, \delta_2, \delta_3, \delta_4)$ defines four different types of parameters on the construction of the CRN. δ_1 defines the perturbation allowed on the input signal. This means, if an input signal X has two valid states, α and β, then X is a valid input when

$$X > \alpha - \delta_1 \text{ or } X < \beta + \delta_1,$$

meaning X is high or low respectively.

δ_2 is the accuracy of the output function. That is, when the CRN computes its output, up to δ_2 error is allowed.

δ_3 is the perturbation allowed in the initial state. If a CRN is constructed with an inital value of Y_0 in species Y, the CRN is robust in terms of δ_3 if it outputs correctly for any initial condition $\hat{Y}_0$, where

$$Y_0 - \delta_3 < \hat{Y}_0 < Y_0 + \delta_3.$$

δ_4 is the perturbation allowed in the rate constants. A CRN is considered robust in terms of δ_4 if for every rate constant k, it outputs correctly for any rate constant $\hat{k}$ where

$$k - \delta_4 < \hat{k} < k + \delta_4.$$

We use ϵ to describe the perturbation allowed in the output. If an output signal X has two valid states, 0 and 1, then X is a valid output when

$$X > 1 - \epsilon \; or \; X < \epsilon,$$

meaning X is 0 or 1 respectively.

2 Construction of a Concentration Monitor

We define a device that monitors the concentration of an input signal and provides an output based on its amount. Given an input signal X, the device monitors for certain concentration thresholds. If the concentration of X is above the target threshold α, then within τ time the concentrations of the output species, $\hat{X}$ and $\hat{Y}$ will have concentrations of the form

$$\hat{X} > 1 - \epsilon \tag{1}$$
$$\hat{Y} < \epsilon \tag{2}$$

Similarly, if the concentration of X is below the target threshold β then within τ time the concentrations of the output species, $\hat{X}$ and $\hat{Y}$ will have concentrations of the form

$$\hat{X} < \epsilon \tag{3}$$
$$\hat{Y} > 1 - \epsilon \tag{4}$$

This device extends previous work to define a signal restoration device [1]. The signal restorer monitored for inputs of above $1 - \delta_1$ and below δ_1. Our work extends this construction for arbitrary inputs $\alpha - \delta_1$ and $\beta + \delta_1$.

2.1 Construction 1

Given the real numbers $\tau > 0$, $\epsilon \in \left(0, \frac{1}{2}\right)$, $\alpha > 0$, $0 \leq \beta < \alpha$, and $\boldsymbol{\delta} = (\delta_1, \delta_2, \delta_3, \delta_4)$ where $0 < \delta_1 < \frac{\alpha - \beta}{2 + \alpha + \beta}$, $\delta_2 \in [0, \epsilon)$, $\delta_3 \in (0, \frac{1}{2})$ and $\delta_4 > 0$, let

$b = \frac{(\alpha-\delta_1)(1+\beta)}{(1+\alpha)(\beta+\delta_1)}$ and $n = \lceil 2log_b(\frac{8}{\epsilon-\delta_2})\rceil$. Define the I/O CRN [1] $N(\tau, \epsilon, \boldsymbol{\delta}) = (U, R, S)$ where

$$U = \{X\}, \qquad S = \{L_i | 0 \leq i \leq n\} \cup \{\hat{X}, \hat{Y}\} \tag{5}$$

and where R contains the reactions

$$X + L_i \xrightarrow{k_1} X + L_{i+1} \qquad (\forall 0 \leq i < n) \tag{6}$$

$$L_i \xrightarrow{k_1} L_0 \qquad (\forall 0 < i \leq n) \tag{7}$$

$$\hat{Y} + L_n \xrightarrow{k_2} \hat{X} + L_n \tag{8}$$

$$\hat{X} \xrightarrow{k_2} \hat{Y}, \tag{9}$$

and where the rate constants k_1 and k_2 are defined by

$$k_1 = 2\delta_4 + \frac{2n\log(2n)}{\tau(\alpha-\delta_1)} + \frac{2}{\tau}\log\left(10\left(\frac{8}{\epsilon-\delta_2}\right)^2\left(\frac{1+\alpha}{\alpha-\delta_1}\right)^n\right) + \frac{\delta_4(2+\delta_1)}{\delta_1} \tag{10}$$

$$k_2 = \frac{2}{\tau}\log\left(\frac{3}{\epsilon-\delta_2}\right) + 4\delta_4 \tag{11}$$

We define the initial concentrations of the species as

$$\hat{X} = 0 \tag{12}$$

$$\hat{Y} = q \tag{13}$$

$$L_0 = p \tag{14}$$

$$L_i = 0 \qquad (\forall 0 < i \leq n) \tag{15}$$

where $p = \frac{10}{\epsilon-\delta_2}\left(\frac{1+\alpha}{\alpha-\delta_1}\right)^n + \delta_3$ and $q = 1 + \delta_3$.

2.2 Proof of Correctness

The device consists of two components, a cascade of species $L_0, \ldots, L_n$, and an output component of species $\hat{X}$ and $\hat{Y}$. The cascade moves in two directions, it climbs forward one step at a time, and it falls backwards all the way to the first species. The output component moves the concentration of $\hat{X}$ and $\hat{Y}$ based on the concentration of L_n.

The ODEs for the species $L_0, \dots, L_n$ can be defined from their reactions.

$$\frac{dl_0}{dt} = \sum_{i=1}^{n} k_1 l_i - (k_1 x)(l_0), \tag{16}$$

$$\frac{dl_i}{dt} = (k_1 x) l_{i-1} - (k_1 x + k_1) l_i, \quad \text{for } 0 < i < n, \tag{17}$$

$$\frac{l_n}{dt} = (k_1 x) l_{n-1} - k_1 l_n. \tag{18}$$

The ODEs for $\hat{X}$ and $\hat{Y}$ are

$$\frac{d\hat{x}}{dt} = (k_2 l_n)\hat{y} - k_2 \hat{x} \tag{19}$$

$$\frac{d\hat{y}}{dt} = k_2 \hat{x} - (k_2 l_n)\hat{y} \tag{20}$$

Analysis on these ODEs was performed in [1], and it provides a number of useful lemmas to prove the correctness of our CRN.

Our goal is to show that the device has correct output for a given input. It suffices to show the correctness of the device on an input greater than α and an input lower than β over a time interval $\boldsymbol{I} = [t_1, t_2]$

We first analyze the value of $l_n(t)$, the concentration of L_n at time t. Using this, we will show the value of the output component species.

Lemma 2.1 *Given a CRN defined as in Construction 1, with a time interval $\boldsymbol{I} = [t_1, t_2]$. If the concentration of the input signal X is above $\alpha - \delta_1$ for all $t \in \boldsymbol{I}$, then by time $t_1 + \tau$,*

$$\hat{x}(t) > 1 - \epsilon. \tag{21}$$

First consider the behavior for an input above α. This means that for all $t \in \boldsymbol{I}$, the input X is above α, so $x(t) > \alpha - \delta_1$. By this assumption, the input X has a constant concentration during the interval $\boldsymbol{I}$. Therefore, we wrap the concentration of X and the rate constant k_1 into the forward rate variable f. Since the backward rate is always constant we abstract it to the backward rate variable b.

We minimize the forward rate of the cascade by assuming all the concentration of $L_0, \dots, L_n$ is in L_0 at time t_1, that is $l_0(t) = p$. Since the rate constants k_1 can be perturbed by at most δ_4, we minimize the forward rate f and maximize the backwards rate b, giving $f = (k_1 - \delta_4)(\alpha - \delta_1)$ and $b = (k_1 + \delta_4)$. By Lemma 6.6 from [1], for all time $t \in \boldsymbol{I}$,

$$l_n(t) > p\left(\frac{f}{f+b}\right)^n \sum_{i=n}^{\infty} \frac{t^i (f+b)^i}{i!} e^{-(f+b)(t-t_1)}, \tag{22}$$

Since l_n is monotonically increasing, for all $t \in [t_1 + \frac{\tau}{2}, t_2]$, $l_n(t) \geq l_n(\frac{\tau}{2})$ and therefore

$$l_n(t) > p\left(\frac{f}{f+b}\right)^n \sum_{i=n}^{\infty} \frac{t^i (f+b)^i}{i!} e^{-(f+b)\frac{\tau}{2}}, \tag{23}$$

Applying Lemma 6.7 from [1], for all $t \in [t_1 + \frac{\tau}{2}, t_2]$,

$$l_n(t) > p\left(\frac{f}{f+b}\right)^n \left(1 - ne^{-\frac{1}{n}(f+b)\frac{\tau}{2}}\right). \tag{24}$$

Since $k_1 > \delta_4 + \frac{2n\log(2n)}{\tau(\alpha-\delta_1)}$, Corrollary 6.8 from [1] tells us

$$l_n(t) > p\left(\frac{f}{f+b}\right)^n \left(\frac{1}{2}\right) = \frac{p}{2}\left(\frac{(k_1-\delta_4)(\alpha-\delta_1)}{(k_1-\delta_4)(\alpha-\delta_1)+(k_1+\delta_4)}\right)^n \tag{25}$$

$$= \frac{p}{2}\left(\frac{(\alpha-\delta_1)}{(\alpha-\delta_1)+u}\right)^n \tag{26}$$

where $u = \frac{k_1+\delta_4}{k_1-\delta_4}$. Since $k_1 > \frac{\delta_4(2+\delta_1)}{\delta_1}$, we know $u < 1+\delta_1$ and therefore

$$l_n(t) > \frac{p}{2}\left(\frac{(\alpha-\delta_1)}{(1+\alpha)}\right)^n \tag{27}$$

Since the initial concentration can be perturbed by at most δ_3, $p > \frac{10}{\epsilon-\delta_2}\left(\frac{1+\alpha}{\alpha-\delta_1}\right)^n$ and therefore

$$l_n(t) > \frac{5}{\epsilon-\delta_2} \tag{28}$$

Now consider the output component. The cascade component performed its operations in the first $\frac{\tau}{2}$ time of the interval $\boldsymbol{I}$. Let f be the rate of converting $\hat{Y}$ into $\hat{X}$ and b be the rate of converting $\hat{X}$ back into $\hat{Y}$. By Equation 28, we know that by $t_1 + \frac{\tau}{2}$,

$$l_n(t) > \frac{5}{\epsilon-\delta_2}. \tag{29}$$

Therefore, $f = (k_2 - \delta_4)\frac{5}{\epsilon-\delta_2}$ and $b = k_2 + \delta_4$. Lemmas 6.12 and 6.13 from [1] show that for f,b,δ_4, and k_2 as defined above, $\hat{x}(t) > q - \epsilon + \delta_2$. Since the output decision can be perturbed by at most δ_2,

$$\hat{x}(t) > q - \epsilon. \tag{30}$$

Since q can be perturbed by at most δ_3,

$$\hat{x}(t) > 1 - \epsilon. \tag{31}$$

Lemma 2.2 *Given a CRN defined as in Construction 1, with a time interval* $\boldsymbol{I} = [t_1, t_2]$. *If the concentration of the input signal* X *is below* $\beta + \delta_1$ *for all* $t \in \boldsymbol{I}$, *then by time* $t_1 + \tau$,

$$\hat{x}(t) < \epsilon. \tag{32}$$

Now consider an input below β. This means that for all $t \in \boldsymbol{I}$, the input X is below β, so $x(t) < \beta + \delta_1$. Since the rate constants k_1 can be perturbed by at most δ_4, we maximize the forward rate f and minimize the backwards rate b, giving $f = (k_1 + \delta_4)(\beta + \delta_1)$ and $b = (k_1 - \delta_4)$. By Lemma 6.10 from [1], for all $t \in \boldsymbol{I}$,

$$l_n(t) < pe^{-b(t-t_1)} + p\left(\frac{f}{f+b}\right)^n \left(1 - e^{-b(t-t_1)}\right) \tag{33}$$

Since this function is monotonically decreasing, for all $t \in [t_1 + \frac{\tau}{2}, t_2]$,

$$\begin{aligned} l_n(t) &< p\left(\frac{f}{f+b}\right)^n + pe^{-b\frac{\tau}{2}} && (34)\\ &= p\left(\frac{(k_1+\delta_4)(\beta+\delta_1)}{(k_1+\delta_4)(\beta+\delta_1) + (k_1-\delta_4)}\right)^n + pe^{-b\frac{\tau}{2}} && (35)\\ &= p\left(\frac{(\beta+\delta_1)}{(\beta+\delta_1)+u}\right)^n + pe^{-b\frac{\tau}{2}}, && (36) \end{aligned}$$

where $u = \frac{k_1 - \delta_4}{k_1 + \delta_4}$. Since $k_1 > \frac{\delta_4(2-\delta_1)}{\delta_1}$, we know that $u > 1 - \delta_1$ and therefore for all $t \in [t_1 + \frac{\tau}{2}, t_2]$,

$$l_n(t) < p\left(\frac{(\beta+\delta_1)}{(1+\beta)}\right)^n + pe^{-b\frac{\tau}{2}}. \tag{37}$$

Since $p < \frac{10}{\epsilon - \delta_2}\left(\frac{1+\alpha}{\alpha-\delta_1}\right)^n + 2\delta_3$,

$$\begin{aligned} l_n(t) &< \frac{10}{\epsilon-\delta_2}\left(\frac{1+\alpha}{\alpha-\delta_1}\right)^n\left(\frac{(\beta+\delta_1)}{(1+\beta)}\right)^n + 2\delta_3\left(\frac{(\beta+\delta_1)}{(1+\beta)}\right)^n + pe^{-b\frac{\tau}{2}} && (38)\\ &< \frac{10}{\epsilon-\delta_2}\left(\frac{1+\alpha}{\alpha-\delta_1}\right)^n\left(\frac{(\beta+\delta_1)}{(1+\beta)}\right)^n + \left(\frac{(\beta+\delta_1)}{(1+\beta)}\right)^n + pe^{-b\frac{\tau}{2}} && (39)\\ &< \frac{10+\epsilon-\delta_2}{\epsilon-\delta_2}\left(\frac{(1+\alpha)(\beta+\delta_1)}{(\alpha-\delta_1)(1+\beta)}\right)^n + pe^{-b\frac{\tau}{2}} && (40)\\ &< \frac{32}{3(\epsilon-\delta_2)}\left(\frac{(\alpha-\delta_1)(1+\beta)}{(1+\alpha)(\beta+\delta_1)}\right)^{-n} + pe^{-b\frac{\tau}{2}}. && (41) \end{aligned}$$

Since $n \geq \log_{\left(\frac{(\alpha-\delta_1)(1+\beta)}{(1+\alpha)(\beta+\delta_1)}\right)} \left(\frac{64}{(\epsilon-\delta_2)^2}\right)$,

$$l_n(t) < \frac{32}{3(\epsilon - \delta_2)} \left(\frac{(\epsilon - \delta_2)^2}{64}\right) + pe^{-b\frac{\tau}{2}} \tag{42}$$

$$= \frac{(\epsilon - \delta_2)}{6} + pe^{-b\frac{\tau}{2}}. \tag{43}$$

As we showed before $p < \frac{32}{3(\epsilon-\delta_2)} \left(\frac{1+\alpha}{\alpha-\delta_1}\right)^n + 2\delta_3$,

$$l_n(t) < \frac{(\epsilon - \delta_2)}{6} + \frac{32}{3(\epsilon - \delta_2)} \left(\frac{1+\alpha}{\alpha - \delta_1}\right)^n e^{-b\frac{\tau}{2}} + 2\delta_3 e^{-b\frac{\tau}{2}}. \tag{44}$$

Since $b = k_1 - \delta_4 > \frac{2}{\tau} \log \left(\left(\frac{640}{(\epsilon-\delta_2)^2}\right)\left(\frac{1+\alpha}{\alpha-\delta_1}\right)^n\right)$,

$$l_n(t) < \frac{(\epsilon - \delta_2)}{6} + \frac{(\epsilon - \delta_2)}{60} + 2\delta_3 e^{-b\frac{\tau}{2}} \tag{45}$$

$$< \frac{(\epsilon - \delta_2)}{6} + \frac{(\epsilon - \delta_2)}{30}, \tag{46}$$

whence for all $t \in [t_1 + \frac{\tau}{2}, t_2]$

$$l_n(t) < \frac{(\epsilon - \delta_2)}{5} \tag{47}$$

Now consider the output component. The cascade component performed its operations in the first $\frac{\tau}{2}$ time of the interval $\boldsymbol{I}$. Let f be the rate of converting $\hat{Y}$ into $\hat{X}$ and b be the rate of converting $\hat{X}$ back into $\hat{Y}$. By Equation 47, we know that by $t_1 + \frac{\tau}{2}$,

$$l_n(t) < \frac{(\epsilon - \delta_2)}{5} \tag{48}$$

Therefore, $f = (k_2 + \delta_4)\frac{(\epsilon-\delta_2)}{5}$ and $b = k_2 - \delta_4$. Lemmas 6.12 and 6.13 from [1] show that for f,b,δ_4, and k_2 as defined above, $\hat{y}(t) > q - \epsilon + \delta_2$. Since $\hat{x}(t) + \hat{y}(t) = q$ for all $t \in [0, \infty)$,

$$\hat{x}(t) < \epsilon - \delta_2 \tag{49}$$

$$\hat{y}(t) > q - \epsilon + \delta_2 \tag{50}$$

for all $t \in [t_1 + \tau, t_2]$. Since q can be perturbed by at most δ_3, and the output can be perturbed by at most δ_2

$$\hat{x}(t) < \epsilon \tag{51}$$

$$\hat{y}(t) > 1 - \epsilon \tag{52}$$

3 K-Input AND Gate

We construct an AND gate that takes k inputs, X_j for all $0 \leq j < k$. The AND Gate has two output species, $\hat{X}$ and $\hat{Y}$. We consider all inputs to be binary, that is they are considered to be 1 when their concentration is above some $1 - \hat{\delta_1}$ and they are considered to be 0 if their concentration is below $\hat{\delta_1}$. If all inputs $X_0, \ldots, X_{k-1}$ are considered to be 1 over a time interval $\boldsymbol{I} = [t_1, t_2]$, then by $t_1 + \tau$ time the concentrations of the output species, $\hat{X}$ and $\hat{Y}$ will have concentrations of the form

$$\hat{X} > 1 - \epsilon \tag{53}$$

$$\hat{Y} < \epsilon \tag{54}$$

Similarly, if the concentration of any X_j is below $\hat{\delta_1}$ then within τ time the concentrations of the output species, $\hat{X}$ and $\hat{Y}$ will have concentrations of the form

$$\hat{X} < \epsilon \tag{55}$$

$$\hat{Y} > 1 - \epsilon \tag{56}$$

3.1 Construction 2

We construct the AND gate as above in Construction 1 with a minor modification. We replace reaction 6 with the following reaction

$$X_j + L_i \xrightarrow{k_1} X_j + L_{i+1} \qquad (\forall 0 \leq i < n)(\forall 0 \leq j < k). \tag{57}$$

The ODEs for this cascade change to

$$\frac{dl_0}{d_t} = \sum_{i=1}^{n} k_1 l_i - k_1 \sum_{j=0}^{k-1} (x_j)(l_0), \tag{58}$$

$$\frac{dl_i}{d_t} = k_1 \sum_{j=0}^{k-1} (x_j) l_{i-1} - (k_1 \sum_{j=0}^{k-1} (x_j) + k_1) l_i, \quad \text{for } 0 < i < n, \tag{59}$$

$$\frac{l_n}{dt} = (k_1 \sum_{j=0}^{k-1} (x_j)) l_{n-1} - k_1 l_n. \tag{60}$$

That is, all input species equally push the cascade forward.

We also add some constraints to the values of α, β, and δ_1. We define α as

$$\alpha = k, \tag{61}$$

since we want the input to be above α when all input species X_j are considered to be 1. We define β as

$$0 \leq \beta < \alpha, \tag{62}$$

however, the closer β approaches α the more accurate the AND gate is. Ideally, we define β as at least

$$\beta = \alpha - 1. \tag{63}$$

Each input species is allowed to be perturbed by at most $\hat{\delta_1}$. Therefore we add the constraint

$$k\hat{\delta_1} < \delta_1 \tag{64}$$

which ensures that, no matter how far any input species is perturbed within its bounds, the perturbation will never exceed δ_1.

3.2 Proof of Correctness

Lemma 3.1 *Given a CRN defined as in Construction 2, with a time interval $\boldsymbol{I} = [t_1, t_2]$. If the concentration of all input signals X_j are above $1 - \hat{\delta_1}$ for all $t \in \boldsymbol{I}$, then by time $t_1 + \tau$,*

$$\hat{x}(t) > 1 - \epsilon. \tag{65}$$

If the concentration of at least one species X_j is below $\hat{\delta_1}$ for all $t \in \boldsymbol{I}$, then by time $t_1 + \tau$,

$$\hat{x}(t) < \epsilon. \tag{66}$$

Since all X_js are pushing the same cascade forward, it is useful to define the function $x(t) = \sum_{j=0}^{k-1} x_j(t)$. Observe that if we substitute this input $x = \sum_{j=0}^{k-1}(x_j)$ into Construction 2, the ODEs are the same as in Construction 1.

It suffices to show then that when all inputs $X_0, \dots X_{k-1}$ are above $1 - \hat{\delta_1}$ for all $t \in \boldsymbol{I}$, that the concentration of X will be above $\alpha - \delta_1$, and if the concentration of at least one species is below $\hat{\delta_1}$ within the interval $\boldsymbol{I}$, then the concentration of X will be below $\beta + \delta_1$.

Consider the case where all inputs $X_0, \dots X_{k-1}$ are considered to be 1. By their perturbation bounds, this means that each input

$$x_j(t) > 1 - \hat{\delta_1} \tag{67}$$

By the definition of X,

$$x(t) = \sum_{j=0}^{k-1}(x_j(t)). \tag{68}$$

Since all the inputs are considered to be 1,

$$\begin{aligned} x(t) &> \sum_{j=0}^{k-1}(1-\hat{\delta_1}) && (69)\\ &= k(1-\hat{\delta_1}) && (70)\\ &= k - k\hat{\delta_1} && (71)\\ &= \alpha - k\hat{\delta_1} && (72)\\ x(t) &> \alpha - \delta_1. && (73) \end{aligned}$$

By the proof 2.2, since $x(t) > \alpha - \delta_1$ for the time interval $\boldsymbol{I}$, by time $t_1 + \tau$,

$$\hat{x}(t) > 1 - \epsilon \tag{74}$$

For an output of 0, the worst case is when only 1 input species X_l has a value below $\hat{\delta_1}$ and all other species have their highest value,

$$x_j(t) < 1 + \hat{\delta_1} \tag{75}$$

By the definition of X,

$$x(t) = \sum_{j=0}^{k-1}(x_j(t)). \tag{76}$$

Assume that input X_l has a value below $\hat{\delta_1}$. Since all of the inputs other than X_l are considered to be 1,

$$\begin{aligned} x(t) &< (\sum_{j=0}^{k-2}(1+\hat{\delta_1})) + x_l(t) && (77)\\ &= (k-1)(1+\hat{\delta_1}) + x_l(t) && (78)\\ &= k + k\hat{\delta_1} - 1 - \hat{\delta_1} + x_l(t) && (79)\\ &< k + k\hat{\delta_1} - 1 - \hat{\delta_1} + \hat{\delta_1} && (80)\\ &= k + k\hat{\delta_1} - 1 && (81) \end{aligned}$$

Since $\alpha = k$,

$$x(t) < \alpha - 1 + k\hat{\delta_1}. \tag{82}$$

Since $\beta = \alpha - 1$,

$$x(t) < \beta + k\hat{\delta_1}. \tag{83}$$

Whence by the definition of $\hat{\delta_1}$,

$$x(t) < \beta + \delta_1. \tag{84}$$

By the proof 2.2, since $x(t) < \beta + \delta_1$ for the time interval $\boldsymbol{I}$, by time $t_1 + \tau$,

$$\hat{x}(t) < \epsilon. \tag{85}$$

4 Follower CRNs

We prove the correctness of the Follower CRN design. The Follower CRN has the reactions

$$log + X \xrightarrow{k} log + X + C, \tag{86}$$

$$log + C \xrightarrow{k} log. \tag{87}$$

where X is an input signal, C is a species to hold the value of X, log is a catalyst, and k is a rate constant.

The theory of Time Dilation on CRNs [1], states that a species used catalytically in all reactions of a CRN does not affect the behavior of the CRN, only its speed. Therefore, the behavior of the above reactions is the same as

$$X \xrightarrow{k} X + C, \tag{88}$$

$$C \xrightarrow{k} \emptyset. \tag{89}$$

A benefit of time dilation is that by controlling the amount of log, we can speed up or slow down the rates of reactions (86) and (87). Therefore, it is sufficient to show that the behavior of reactions (88) and (89) is correct. Since we are only trying to show the behavior of the reactions and not the speed, and X is only used catalytically, we can assume X to be constant. The behavior of the species C is determined by the first-order ODE

$$\frac{dc}{dt} = kx - kc \tag{90}$$

which has the solution

$$c(t) = (c_0 - x)e^{-kt} + x \tag{91}$$

where c_0 is the initial concentration of c. Therefore, as time increases, the concentration of C will approach X at an exponential rate.

We can control the speed of these reactions by increasing or decreasing the rate constant k along with the concentration of log. Therefore, we can construct a Follower CRN to track the value of another concentration with any desired speed.

References

[1] Titus H Klinge. *Modular and Robust Computation with Chemical Reaction Networks.* PhD thesis, Iowa State University, 2016.

[2] Titus H. Klinge, James I. Lathrop, and Jack H. Lutz. Robust biomolecular finite automata. Technical Report 1505.03931, arXiv.org e-Print archive, 2015.