1. ① 5 is $O((\log n)^5)$ because 5 is a constant

② $(\log n)^5$ is $O(n^{0.1})$ because $\lim_{n\to\infty} \frac{(\log n)^5}{n^{0.1}}$

$= \lim_{n\to\infty} \frac{5(\log n)^4 \frac{1}{n}}{0.1 n^{0.9}}$

$= \lim_{n\to\infty} \frac{5(\log n)^4 \frac{1}{n}}{\frac{1}{0.1 n^{0.9}}}$

$= \lim_{n\to\infty} \frac{0.5(\log n)^4}{n^{0.1}} \cdots, = \lim_{n\to\infty} \frac{c(1)}{n^{0.1}} = 0$

③ $n^{0.1}$ is $O(5n)$ because $\lim_{n\to\infty} \frac{5n}{n^{0.1}} = \lim_{n\to\infty} 5n^{0.9} = \infty$

④ $5n$ is $O(2n\log\log n)$ because $\lim_{n\to\infty} \frac{5n}{2n\log\log n} = \lim_{n\to\infty} \frac{5}{2\log\log n} = 0$

⑤ $2n\log\log n$ is $O(4^{\log n})$ because $4^{\log n} = 2^{2\log n} = (2^{\log n})^2 = n^2$

$\because \log\log n \le \log n \le n$ for $n \ge 1$

$\therefore \lim_{n\to\infty} \frac{2n\log\log n}{n^2} = \lim_{n\to\infty} \frac{2\log\log n}{n} = 0$

⑥ $4^{\log n}$ is $O(n^5)$, $n^2 < n^5$

⑦ n^5 is $O(5^n)$, if both side take log, $\log n^5 = 5\log n$, $\log 5^n = n\log 5$

$\because n > \log n$, $\therefore n\log 5 > 5\log n$ $\therefore 5^n > n^5$

⑧ 5^n is $O(n!)$, 5^n is every time multiply by 5,

if $n\to\infty$, $n!$ every time multiply numbers are bigger than 5 $\therefore n! > 5^n$

⑨ $n!$ is $O(2^{2^n})$, if both side take log, $\log(n!)$ is $\Theta\log(n\log n)$,

$\log 2^{2^n} = 2^n$ $\therefore n\log n$ is $O(2^n)$

$\therefore n!$ is $O(2^{2^n})$

2. $S(n) = \sum_{i=1}^{n} \log i = \log(n!)$
$f(n) = n\log n$

Proof: ① claim $S(n)$ is $O(n\log n)$

$$\begin{aligned} S(n) &= \log(n!) \\ &= \log 1 + \log 2 + \dots + \log(n-1) + \log n \\ &\le \log n + \log n + \dots + \log n + \log n \quad \text{for } n \ge 1 \\ &= n\log n \end{aligned}$$

$\therefore$ we found $c=1$ and $n_0 = 1$ such that $S(n) \le n f(n)$ for $n \ge n_0$
$\therefore S(n)$ is $O(n\log n)$

② claim $S(n)$ is $\Omega(n\log n)$

$$\begin{aligned} S(n) &= \log(n!) \\ &= \log 1 + \log 2 + \dots + \log(\tfrac{n}{2}) + \dots + \log n \\ &\ge \log(\tfrac{n}{2}) + \log(\tfrac{n}{2}) + \dots + \log(\tfrac{n}{2}) \quad \text{for } n \ge 1 \\ &= \tfrac{n}{2}\log(\tfrac{n}{2}) \\ &= \tfrac{n}{2}(\log n - 1) \\ &= \tfrac{1}{2}n\log n - \tfrac{1}{2}n \\ &\ge \tfrac{1}{2}n\log n \end{aligned}$$

$\therefore$ we found $c = \frac{1}{2}$ and $n_0 = 1$ such that $S(n) \ge n f(n)$ for $n \ge n_0$
$\therefore S(n)$ is $\Omega(n\log n)$

3. ⓐ $T(n) = n + \frac{n}{2} + \frac{n}{4} + \dots + 1 = 2n - 1$
$T(n)$ is $O(n)$

ⓑ Outer loop has $\log n$ times
Inner loop has $1+2+4+8+\dots+n = \sum_{i=0}^{n} 2^i = 2^{n+1} - 1$
$\therefore T(n) = 2^{(\log n + 1)} - 1 = 2^{\log n} 2^1 - 1 = 2n - 1$
$\therefore T(n)$ is $O(n)$.

ⓒ outer loop runs $\log n$ times
Inner loop runs n times every time as long as it executed
$\therefore T(n)$ is $O(n\log n)$

4. Proof $\sum_{i=1}^{n}(2i-1) = n^2$ for all $n \geq 1$

Base case: for $n=1$, LHS $= 2(1)-1 = 1$

$= 1^2 = 1 =$ RHS

Induction Hypothesis: Assume that $\sum_{i=1}^{n}(2i-1) = n^2$ for some $n \geq 1$

Induction Step: Consider $n+1$

$$\sum_{i=1}^{n+1}(2i-1) = \sum_{i=1}^{n}(2i-1) + [2(n+1)-1]$$

$$= n^2 + [2(n+1)-1] \quad \text{by I.H.}$$

$$= n^2 + 2n + 1$$

$$= (n+1)^2$$

$\therefore$ the identity holds for $n+1$, and by induction, the identity holds for all $n \geq 0$.

5. Step 1: get the sum1 of $0+1+2+\ldots+n-1$ using $\frac{(n-1)(n-1+1)}{2} = \frac{n(n-1)}{2}$

Step 2: set the value $i=0$, sum2 $=0$, missing $=0$

Step 3: while the value of i is less than n, repeat 4 through 5

Step 4: sum2 = array[i] + sum2

Step 5: Add 1 to i to move to the next integer

Step 6: missing = sum1 - sum2

Step 7: Print missing number, missing

Step 8: Stop