Compution of Interaction Amplitudes using Explicit Helicity Eigenstates

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Acknowledgements

I would like to express my sincere gratitude to Prof. Sreerup for his time and patience in instructing me through a number of classroom sessions at TIFR. He accompanied me along the rigor of the mathematical framework of the Dirac equation, often highlighting the physical significance of concepts like helicity and chirality. He clearly articulated the computational efficiency of determining Feynman amplitudes with explicit solutions of Dirac equation. I regret not seeing his task through to the end. I sincerely thank Prof. Sangita Bose for being extremely patient with me and giving me opportunities despite many disappointments. I have to acknowledge Prof. Arvind Ghosh for teaching a wonderful course on quantum field theory which gave me sufficient confidence to pursue this topic. I would like to wholeheartedly thank my parents for always being the biggest pillar of support in my life.

Abstract

In field theory, the interaction between colliding particles proceeds through N number of process which have their distinct Feynman diagrams and unique amplitude. The traditional method of computing the magnitude of Feynman amplitude involves multiplying the N amplitudes by their conjugate and applying summation over spin polarisation to simplify the expression using properties of trace and projection. This leads to $\vartheta(N^2)$ computation even when approached numerically. Instead, the explicit solutions of Dirac equation whose components are obtained in a given representation from momenta and helicity data, allow us to compute the amplitude numerically with $\vartheta(N)$. So, we put emphasis on a consistent and explicit derivation of these eigenket components that are complete and orthonormal.

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Chapter 1

Pauli's Fundamental Theorem

1.1 Statement

Any $N * N$ matrix can expressed as the linear superposition of N^2 independent matrices. In 2 dimensions, we are familiar with the Pauli matrices, which are traceless, hermitian and unitary. Similarly in Dirac's fermionic field theory in 4 dimensions, we are introduced to the Γ matrices, which will be used henceforth in the Dirac-Pauli (DP) representation.

$$\Gamma = \sum_{i=1}^{16} a_i \Gamma_i \tag{1.1}$$

1.2 Properties

The 16 Γ matrices are

$$I_4, \gamma^0 = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix}, \gamma^\alpha = \begin{pmatrix} 0 & \vec{\sigma^\alpha} \\ -\vec{\sigma^\alpha} & 0 \end{pmatrix} \gamma_5 = \begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix}, \gamma^\mu\gamma_5, \sigma^{\mu\nu}$$

They have interesting trace and anti-commutation properties given below:

$i.\ Tr(\gamma^\mu) = 0$ $\quad$ $ii.\ Tr(\gamma^\mu\gamma^\nu) = 4\eta^{\mu\nu}$

$iii.\ Tr(\gamma^\mu\gamma^\nu\gamma^\lambda) = 0$ $\quad$ $iv.\ Tr(\gamma_5) = 0$

$v.\ \{\gamma_5, \gamma^\mu\} = 0$ $\quad$ $vi.\ Tr(\gamma_5\gamma^\mu) = 0$

$vii.\ Tr(\gamma_5\gamma^\mu\gamma^\nu) = 0$ $\quad$ $viii.\ Tr(\gamma_5\gamma^\mu\gamma^\nu\gamma^\lambda) = 0$

$ix.\ \gamma_5.\gamma_5 = I_4$ $\quad$ $x.\ \{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$

$xi.\ \sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]$

1.3 Proof

The Γ_i's are independent if the equation

$$\sum_{i=1}^{16} a_i \Gamma_i = 0 \tag{1.2}$$

has only solution as $a_i = 0 \; \forall \; i$.
Let us rewrite the equation as

$$aI_4 + b_\mu \gamma^\mu + c\gamma_5 + d_\mu \gamma^\mu \gamma_5 + e_{\mu\nu}\sigma^{\mu\nu} = 0 \tag{1.3}$$

Pre-multiplying by each Γ_i and applying trace, we obtain the following results

$$\begin{aligned} i.\ aTr(I_4) &= 0 \\ \Rightarrow 4a &= 0 \\ \Rightarrow a &= 0 \end{aligned}$$

$$\begin{aligned} ii.\ b_\mu Tr(\gamma^\alpha \gamma^\mu) &= 0 \\ \Rightarrow b_\mu (4\eta^{\alpha\mu}) &= 0 \\ \Rightarrow 4\eta^{\alpha\alpha} b_\alpha &= 0 \\ \Rightarrow b_\alpha &= 0 \end{aligned}$$

$$\begin{aligned} iii.\ cTr(\gamma_5 . \gamma_5) &= 0 \\ \Rightarrow 4c &= 0 \\ \Rightarrow c &= 0 \end{aligned}$$

$$\begin{aligned} iv.\ d_\mu Tr(\gamma^\mu \gamma_5 . \gamma_5 \gamma^\alpha) &= 0 \\ \Rightarrow d_\mu (4\eta^{\mu\alpha}) &= 0 \\ \Rightarrow 4\eta^{\alpha\alpha} d_\alpha &= 0 \\ \Rightarrow d_\alpha &= 0 \end{aligned}$$

$$\begin{aligned} v.\ e_{\mu\nu} Tr(\sigma^{\alpha\beta}\sigma^{\mu\nu}) &= 0 \\ \Rightarrow e_{\mu\nu} Tr(i\gamma^\alpha \gamma^\beta i \gamma^\mu \gamma^\nu) &= 0 \;\because\; Sec\ 1.2.xi \end{aligned}$$

$$\begin{aligned} Now, Tr(\gamma^\alpha \gamma^\beta \gamma^\mu \gamma^\nu) &= Tr(\gamma^\alpha (2\eta^{\beta\mu} - \gamma^\mu \gamma^\beta)\gamma^\nu) \\ &= 8\eta^{\alpha\nu}\eta^{\beta\mu} - Tr(\gamma^\alpha \gamma^\mu \gamma^\beta \gamma^\nu) \\ &= 8\eta^{\alpha\nu}\eta^{\beta\mu} - 8\eta^{\beta\nu}\eta^{\alpha\mu} + Tr(\gamma^\alpha \gamma^\mu \gamma^\nu \gamma^\beta) \\ \Rightarrow Tr(\gamma^\alpha \gamma^\beta \gamma^\mu \gamma^\nu) &= 4\eta^{\alpha\nu}\eta^{\beta\mu} - 4\eta^{\beta\nu}\eta^{\alpha\mu} \end{aligned}$$

$$\begin{aligned} Therefore, \; -e_{\mu\nu}(4\eta^{\alpha\nu}\eta^{\beta\mu} - 4\eta^{\beta\nu}\eta^{\alpha\mu}) &= 0 \\ \Rightarrow -e_{\beta\alpha}(4\eta^{\alpha\alpha}\eta^{\beta\beta}) + e_{\alpha\beta}(4\eta^{\beta\beta}\eta^{\alpha\alpha}) &= 0 \\ \Rightarrow 8\eta^{\alpha\alpha}\eta^{\beta\beta} e_{\alpha\beta} &= 0 \;\because\; e_{\beta\alpha} = -e_{\alpha\beta} \\ \Rightarrow e_{\alpha\beta} &= 0 \end{aligned}$$

Thus, all the 16 Γ_i's are independent.

1.4 Expression for a_i's

The same method used in proving linear independence of Γ_i's in Sec 1.3 can be repeated with eqn. (1.1) to uniquely determine the set of coefficients a_i's.

$$4a = Tr(\Gamma) \Rightarrow a = \frac{1}{4}Tr(\Gamma) \tag{1.4}$$

$$4\eta^{\alpha\alpha}b_\alpha = Tr(\gamma^\alpha\Gamma) \Rightarrow b_\alpha = \frac{1}{4}Tr(\gamma_\alpha\Gamma) \tag{1.5}$$

$$8\eta^{\alpha\alpha}\eta^{\beta\beta}e_{\alpha\beta} = Tr(\sigma^{\alpha\beta}\Gamma) \Rightarrow e_{\alpha\beta} = \frac{1}{8}Tr(\sigma_{\alpha\beta}\Gamma) \tag{1.6}$$

$$4c = Tr(\gamma_5\Gamma) \Rightarrow c = \frac{1}{4}Tr(\gamma_5\Gamma) \tag{1.7}$$

$$4\eta^{\alpha\alpha}d_\alpha = Tr(\Gamma\gamma_5\gamma^\alpha) \Rightarrow d_\alpha = \frac{1}{4}Tr(\Gamma\gamma_5\gamma_\alpha) \tag{1.8}$$

1.5 Explicit Computation of a_i's

In Dirac field theory, the typical vertex factor in a Feynman diagram is of the form

$$A \sim g\bar{\psi}_1\Lambda\psi_2 \tag{1.9}$$

where ψ_1 and ψ_2 are the momentum and helicity eigenkets of external fermion lines at a vertex and Λ is some known product of the Γ_i matrices. This amplitude can be alternatively expressed as

$$A = g.Tr(\Lambda\psi_2\bar{\psi}_1) \tag{1.10}$$

where if we treat $\psi_2\bar{\psi}_1$ as the Γ matrix of Sec 1.4, then using eqn. (1.1)

$$A = g\sum_{i=1}^{16} a_i Tr(\Lambda\Gamma_i) \tag{1.11}$$

This form has the advantage that most of the $Tr(\Lambda\Gamma_i)$ terms vanish due the trace and anit-commutation relation between γ^μ matrices listed in Sec 1.2. For the non vanishing terms the co-efficients a_i's can be evaluated from the components of the eigenkets ψ_1 and ψ_2, which is demonstrated below.

$$Let \quad \psi_1 = \begin{pmatrix} a_1 \\ b_1 \\ c_1 \\ d_1 \end{pmatrix} \quad and \quad \psi_2 = \begin{pmatrix} a_2 \\ b_2 \\ c_2 \\ d_2 \end{pmatrix}$$

$$Then \quad \psi_1\bar{\psi_2} = \begin{pmatrix} a_1a_2^* & a_1b_2^* & -a_1c_2^* & -a_1d_2^* \\ b_1a_2^* & b_1b_2^* & -b_1c_2^* & -b_1d_2^* \\ c_1a_2^* & c_1b_2^* & -c_1c_2^* & -c_1d_2^* \\ d_1a_2^* & d_1b_2^* & -d_1c_2^* & -d_1d_2^* \end{pmatrix}$$

$$i. \quad a = \frac{1}{4}Tr(\psi_1\bar{\psi_2})$$
$$= \frac{1}{4}(a_1a_2^* + b_1b_2^* - c_1c_2^* - d_1d_2^*)$$

$$ii. \quad b_0 = \frac{1}{4}Tr(\gamma_0\psi_1\bar{\psi_2})$$
$$= \frac{1}{4}Tr\left(\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}\psi_1\bar{\psi_2}\right)$$
$$= \frac{1}{4}(a_1a_2^* + b_1b_2^* + c_1c_2^* + d_1d_2^*)$$

$$iii. \quad b_1 = \frac{1}{4}Tr(\gamma_1\psi_1\bar{\psi_2})$$
$$= \frac{1}{4}Tr\left(\begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}\psi_1\bar{\psi_2}\right)$$
$$= \frac{-1}{4}(d_1a_2^* + c_1b_2^* + b_1c_2^* + a_1d_2^*)$$

$$iv. \quad b_2 = \frac{1}{4}Tr(\gamma_2\psi_1\bar{\psi_2})$$
$$= \frac{1}{4}Tr\left(\begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}\psi_1\bar{\psi_2}\right)$$
$$= \frac{i}{4}(d_1a_2^* - c_1b_2^* + b_1c_2^* - a_1d_2^*)$$

$$v. \quad b_3 = \frac{1}{4}Tr(\gamma_3\psi_1\bar{\psi_2})$$
$$= \frac{1}{4}Tr\left(\begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}\psi_1\bar{\psi_2}\right)$$
$$= \frac{1}{4}(d_1b_2^* - a_1c_2^* - c_1a_2^* + b_1d_2^*)$$

$$vi. \quad c = \frac{1}{4}Tr(\gamma_5\psi_1\bar{\psi_2})$$
$$= \frac{1}{4}Tr\left(\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}\psi_1\bar{\psi_2}\right)$$
$$= \frac{1}{4}(c_1a_2^* + d_1b_2^* - a_1c_2^* - b_1d_2^*)$$

$$vii.\ \ d_0 = \frac{1}{4}Tr(\psi_1\bar{\psi}_2\gamma_5\gamma_0) = \frac{1}{4}Tr\left(\psi_1\bar{\psi}_2\begin{pmatrix}0 & 0 & -1 & 0\\ 0 & 0 & 0 & -1\\ 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\end{pmatrix}\right) = \frac{-1}{4}(a_1c_2^* + b_1d_2^* + c_1a_2^* + d_1b_2^*)$$

$$viii.\ \ d_1 = \frac{1}{4}Tr(\psi_1\bar{\psi}_2\gamma_5\gamma_1) = \frac{1}{4}Tr\left(\begin{pmatrix}0 & 1 & 0 & 0\\ 1 & 0 & 0 & 0\\ 0 & 0 & 0 & -1\\ 0 & 0 & -1 & 0\end{pmatrix}\psi_1\bar{\psi}_2\right) = \frac{i}{4}(d_1a_2^* - c_1b_2^* + b_1c_2^* - a_1d_2^*)$$

$$ix.\ \ d_2 = \frac{1}{4}Tr(\psi_1\bar{\psi}_2\gamma_5\gamma_2) = \frac{1}{4}Tr\left(\psi_1\bar{\psi}_2\begin{pmatrix}0 & -i & 0 & 0\\ i & 0 & 0 & 0\\ 0 & 0 & 0 & i\\ 0 & 0 & -i & 0\end{pmatrix}\right) = \frac{i}{4}(a_1b_2^* - b_1a_2^* + c_1d_2^* - d_1c_2^*)$$

$$x.\ \ d_3 = \frac{1}{4}Tr(\psi_1\bar{\psi}_2\gamma_5\gamma_3) = \frac{1}{4}Tr\left(\psi_1\bar{\psi}_2\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & -1 & 0 & 0\\ 0 & 0 & -1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}\right) = \frac{1}{4}(a_1a_2^* - b_1b_2^* + c_1c_2^* - d_1d_2^*)$$

$$xi.\ \ e_{01} = \frac{1}{8}Tr(\sigma_{01}\psi_1\bar{\psi}_2) = \frac{1}{8}Tr\left(\begin{pmatrix}0 & 0 & 0 & -i\\ 0 & 0 & -i & 0\\ 0 & -i & 0 & 0\\ -i & 0 & 0 & 0\end{pmatrix}\psi_1\bar{\psi}_2\right) = \frac{i}{8}(a_1d_2^* + b_1c_2^* - d_1a_2^* - c_1b_2^*)$$

$$xii.\ \ e_{02} = \frac{1}{8}Tr(\sigma_{02}\psi_1\bar{\psi}_2) = \frac{1}{8}Tr\left(\begin{pmatrix}0 & 0 & 0 & -1\\ 0 & 0 & 1 & 0\\ 0 & -1 & 0 & 0\\ 1 & 0 & 0 & 0\end{pmatrix}\psi_1\bar{\psi}_2\right) = \frac{1}{8}(c_1b_2^* + b_1c_2^* - a_1d_2^* - d_1a_2^*)$$

$$xiii.\ \ e_{03} = \frac{1}{8}Tr(\sigma_{03}\psi_1\bar{\psi}_2) = \frac{1}{8}Tr\left(\begin{pmatrix}0 & 0 & -i & 0\\ 0 & 0 & 0 & i\\ -i & 0 & 0 & 0\\ 0 & i & 0 & 0\end{pmatrix}\psi_1\bar{\psi}_2\right) = \frac{i}{8}(d_1b_2^* + a_1c_2^* - b_1d_2^* - c_1a_2^*)$$

$$xiv.\ \ e_{12} = \frac{1}{8}Tr(\sigma_{12}\psi_1\bar{\psi}_2) = \frac{1}{8}Tr\left(\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & -1 & 0 & 0\\ 0 & 0 & -1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}\psi_1\bar{\psi}_2\right) = \frac{1}{8}(a_1a_2^* - b_1b_2^* - c_1c_2^* + d_1d_2^*)$$

$$xv.\ \ e_{23} = \frac{1}{8}Tr(\sigma_{23}\psi_1\bar{\psi}_2) = \frac{1}{8}Tr\left(\begin{pmatrix}0 & 1 & 0 & 0\\ 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & 1 & 0\end{pmatrix}\psi_1\bar{\psi}_2\right) = \frac{1}{8}(b_1a_2^* + a_1b_2^* - d_1c_2^* - c_1d_2^*)$$

$$xvi.\ \ e_{13} = \frac{1}{8}Tr(\sigma_{13}\psi_1\bar{\psi}_2) = \frac{1}{8}Tr\left(\begin{pmatrix}0 & i & 0 & 0\\ -i & 0 & 0 & 0\\ 0 & 0 & 0 & i\\ 0 & 0 & -i & 0\end{pmatrix}\psi_1\bar{\psi}_2\right) = \frac{i}{8}(b_1a_2^* - a_1b_2^* - d_1c_2^* + c_1d_2^*)$$

Chapter 2

Helicitiy Eigenstates

2.1 Helicity Operator

$$\Pi = \frac{1}{p}\begin{pmatrix} \vec{\sigma}.\vec{p} & 0 \\ 0 & \vec{\sigma}.\vec{p} \end{pmatrix} \quad ; \quad H = \gamma_0\vec{\gamma}.\vec{p} + \gamma_0 m = \begin{pmatrix} mI_2 & \vec{\sigma}.\vec{p} \\ \vec{\sigma}.\vec{p} & -mI_2 \end{pmatrix} \tag{2.1}$$

It is observed that the Helicity operator commutes with the Dirac Hamiltonian i.e. $[\Pi, H] = 0$, therefore it is possible to simultaneously obtain eigenkets with definite helicity and four-momentum. The eigenstates of Π can be easily computed from the eigenstates of the $2 * 2$ matrix $\kappa = \vec{\sigma}.\vec{p}/p$

Now, $\kappa^2 = (\vec{\sigma}.\vec{p})^2/p^2 = I_2$, so the eigenvalues of κ are $\lambda = \pm 1$

2.2 Eigenstates

2.2.1 $\lambda = 1$ Eigenstate

$$\frac{\vec{\sigma}.\vec{p}}{p}e_+ = +e_+ \quad \Rightarrow \frac{1}{p}\begin{pmatrix} p_z & p_x - ip_y \\ p_x + ip_y & -p_z \end{pmatrix}\begin{pmatrix} x_+ \\ y_+ \end{pmatrix} = \begin{pmatrix} x_+ \\ y_+ \end{pmatrix}$$

$$\text{Let} \quad p_+ = p_x + ip_y, \quad p_s = p + p_z$$

$$p_- = p_x - ip_y, \quad p_d = p - p_z$$

$$\text{which gives } y_+ = \frac{p_d}{p_-}x_+ \text{ and } e_+ = N_+\begin{pmatrix} p_- \\ p_d \end{pmatrix}$$

where the normalisation constant is fixed by the requirement $e_+^\dagger e_+ = 1$ such that

$$\begin{aligned} N_+^2 \begin{pmatrix} p_+ & p_d \end{pmatrix} \begin{pmatrix} p_- \\ p_d \end{pmatrix} = 1 \quad \Rightarrow N_+^2 (2pp_d) = 1 \\ \Rightarrow e_+ = \frac{1}{\sqrt{2pp_d}} \begin{pmatrix} p_- \\ p_d \end{pmatrix} \end{aligned} \tag{2.2}$$

2.2.2 $\lambda = -1$ Eigenstate

$$\frac{\vec{\sigma}.\vec{p}}{p} e_- = -e_- \quad \Rightarrow \frac{1}{p} \begin{pmatrix} p_z & p_x - ip_y \\ p_x + ip_y & -p_z \end{pmatrix} \begin{pmatrix} x_- \\ y_- \end{pmatrix} = \begin{pmatrix} x_- \\ y_- \end{pmatrix}$$

$$\text{which gives } y_- = \frac{-p_s}{p_-} x_- \text{ and } e_- = N_- \begin{pmatrix} -p_- \\ p_s \end{pmatrix}$$

where the normalisation constant is fixed by the requirement $e_-^\dagger e_- = 1$ such that

$$\begin{aligned} N_-^2 \begin{pmatrix} -p_+ & p_s \end{pmatrix} \begin{pmatrix} -p_- \\ p_s \end{pmatrix} = 1 \quad \Rightarrow N_-^2 (2pp_s) = 1 \\ \Rightarrow e_- = \frac{1}{\sqrt{2pp_s}} \begin{pmatrix} -p_- \\ p_s \end{pmatrix} \end{aligned} \tag{2.3}$$

2.2.3 Orthonormality of eigenstates

$$\begin{aligned} e_+^\dagger e_- &= \frac{1}{\sqrt{2pp_d}} \frac{1}{2pp_s} \begin{pmatrix} p_+ & p_d \end{pmatrix} \begin{pmatrix} -p_- \\ p_s \end{pmatrix} \\ &= \frac{1}{2p\sqrt{p_d p_s}} (-p_+ p_- + p_d p_s) \\ &= 0 \end{aligned} \tag{2.4}$$

2.2.4 Completeness Relation

$$\begin{aligned} e_+^\dagger e_+ + e_-^\dagger e_- &= \frac{1}{2pp_d} \begin{pmatrix} p_+ & p_d \end{pmatrix} \begin{pmatrix} p_- \\ p_d \end{pmatrix} + \frac{1}{2pp_s} \begin{pmatrix} -p_+ & p_s \end{pmatrix} \begin{pmatrix} -p_- \\ p_s \end{pmatrix} \\ &= \frac{1}{2pp_d} \begin{pmatrix} p_s p_d & p_- p_d \\ p_d p_+ & p_d^2 \end{pmatrix} + \frac{1}{2pp_s} \begin{pmatrix} p_s p_d & -p_- p_s \\ -p_s p_+ & p_s^2 \end{pmatrix} \\ &= \frac{1}{2p} \begin{pmatrix} p_s + p_d & 0 \\ 0 & p_s + p_d \end{pmatrix} = I_2 \end{aligned} \tag{2.5}$$

2.3 Reconciliation with Collinear case

In particle accelerators, the collider axis is taken as the z-axis therefore the momentum of the colliding particles is only along the z-axis. In this case,

$$\vec{p} = (0, 0, p_z) \text{ and } p = |\vec{p}| = \pm p_z \tag{2.6}$$

and the Helicity operator has a simple form

$$\Pi = \frac{\sigma_z . p_z}{p} = \pm\sigma_z \text{ with } e_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ and } e_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{2.7}$$

Consider the case $p = p_z$, the eigenstates are simply defined as above but normalisation constant for the general form of the positive helicity eigenstate is ill-defined as

$$N_+ = \frac{1}{\sqrt{2pp_d}} \text{ but } p_d = 0 \tag{2.8}$$

In order to correct this singular nature, the general helicity eigenstates must agree with the eigenstates of σ_z in the limit $p_x, p_y \to 0$.
To construct this limitiing case, let us multiply e_+ by an overall phase factor such that

$$\begin{gathered} e_+ = \frac{e^{i\phi}}{\sqrt{2pp_d}} \begin{pmatrix} p_- \\ p_d \end{pmatrix}; \quad \phi = \arctan(\frac{p_y}{p_x}) \\ p_x = p_{xy}\cos\phi, \;\; p_y = p_{xy}\sin\phi; \quad p_{xy}^2 = p_x^2 + p_y^2 = p_s p_d = p_+ p_- \end{gathered} \tag{2.9}$$

Now, in the limit $p_{xy} \to 0$

$$e_+ = \frac{e^{i\phi}}{\sqrt{2pp_d}} \begin{pmatrix} p_x - ip_y \\ p_d \end{pmatrix} = \frac{e^{i\phi}}{\sqrt{2pp_d}} \begin{pmatrix} p_{xy}e^{-i\phi} \\ p_d \end{pmatrix} = \begin{pmatrix} \frac{p_{xy}}{\sqrt{p_s p_d}} \\ e^{i\phi}\sqrt{\frac{p_d}{p_s}} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{2.10}$$

$$e_- = \frac{1}{\sqrt{2pp_s}} \begin{pmatrix} -p_x + ip_y \\ p_s \end{pmatrix} = \begin{pmatrix} -e^{-i\phi}\frac{p_{xy}}{p_s} \\ \frac{p_s}{p_s} \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{2.11}$$

Chapter 3

Energy Eigenstates

3.1 Rest Frame

In the rest frame of the particle, the total momentum is zero. Since $p = 0$, at first glance it seems like the helicity eigenstates derived in previous chapter are no longer useful to us but since $\kappa = \vec{\sigma}.\hat{p}$, it only depends on the direction of momentum not the overall magnitude. Therefore, in the rest frame, any unit direction can be used to define the helicity eigenstates.

Furthermore, $E = \sqrt{\vec{p}^2 + m^2} = \pm m$. Thus, the energy eigenstates are given by

$$\begin{pmatrix} mI_2 & 0 \\ 0 & mI_2 \end{pmatrix} \omega(\vec{0}) = \pm m\omega(\vec{0}) \tag{3.1}$$

It is clear that the positive and negative energy eigenstates are given by

$$\omega_+(\vec{0}) = N_+ \begin{pmatrix} \phi \\ 0 \end{pmatrix} \quad , \quad \omega_-(\vec{0}) = N_- \begin{pmatrix} 0 \\ \chi \end{pmatrix} \tag{3.2}$$

The above solution further reaffirms the fact that any $\hat{p}$ can be used to construct spinors ϕ and χ.

3.2 Lab Frame

The Dirac Hamiltonian was given in Sec 1.1 as

$$H = \gamma_0\vec{\gamma}.\vec{p} + \gamma_0 m = \begin{pmatrix} mI_2 & \vec{\sigma}.\vec{p} \\ \vec{\sigma}.\vec{p} & -mI_2 \end{pmatrix} \tag{3.3}$$

Energy eigenvalues obtained from the above matrix are $E = \pm\sqrt{\vec{p}^2 + m^2} = \pm|E|$

3.2.1 Positive Energy Solutions

$$p^\mu = (|E|, \vec{p}), \quad \hat{H}\psi = |E|\psi, \quad \hat{\vec{P}}\psi = \vec{p}\psi \tag{3.4}$$

Let $\psi(x) = u(\vec{p})\exp(-ip.x)$ solve the Dirac equation. Then

$$(i\partial\!\!\!/ - m)\psi = 0 \Rightarrow (i(-ip\!\!\!/) - m)u(\vec{p}) = 0 \tag{3.5}$$

$$\Rightarrow (p\!\!\!/ - m)u(\vec{p}) = 0 \quad \because p\!\!\!/ = p_\mu\gamma^\mu \tag{3.6}$$

Possible solution for $u(\vec{p})$ is

$$u(\vec{p}) = f_+ \frac{(p\!\!\!/ + m)}{m}\begin{pmatrix}\phi\\0\end{pmatrix} \quad \because (p\!\!\!/ - m)(p\!\!\!/ + m) = p\!\!\!/^2 - m^2 = 0 \tag{3.7}$$

where ϕ is a normalised spinor i.e. $\phi^\dagger\phi = 1$. The normalisation constant f_+ is determined by the requirement that $\bar{\psi}\psi$ is a Lorentz scalar. Then

$$\begin{aligned}
&\bar{u}(\vec{p})u(\vec{p}) = 2m\\
\Rightarrow &\frac{f_+^2}{m^2}\begin{pmatrix}\phi^\dagger & 0\end{pmatrix}(p\!\!\!/ + m)^\dagger\gamma_0(p\!\!\!/ + m)\begin{pmatrix}\phi\\0\end{pmatrix} = 2m\\
\Rightarrow &2\frac{f_+^2}{m}\begin{pmatrix}\phi^\dagger & 0\end{pmatrix}\gamma_0(p\!\!\!/ + m)\begin{pmatrix}\phi\\0\end{pmatrix} = 2m\\
\Rightarrow &2\frac{f_+^2}{m}\begin{pmatrix}\phi^\dagger & 0\end{pmatrix}\begin{pmatrix}|E| + m & -\vec{\sigma}.\vec{p}\\ \vec{\sigma}.\vec{p} & -|E| + m\end{pmatrix}\begin{pmatrix}\phi\\0\end{pmatrix} = 2m\\
\Rightarrow &2\frac{f_+^2}{m}(|E| + m)(\phi^\dagger\phi) = 2m\\
\Rightarrow &f_+ = \frac{m}{\sqrt{|E| + m}}\\
\Rightarrow &u(\vec{p}) = \frac{p\!\!\!/ + m}{\sqrt{|E| + m}}\begin{pmatrix}\phi\\0\end{pmatrix}
\end{aligned}$$

The validity of the above solution can also be tested in the limiting case of the rest frame i.e. $\vec{p} = 0$

$$u(\vec{0}) = \frac{\gamma_0 m + m}{\sqrt{2m}}\begin{pmatrix}\phi\\0\end{pmatrix} = \sqrt{2m}\begin{pmatrix}\phi\\0\end{pmatrix}\ ,\ \bar{u}(\vec{0})u(\vec{0}) = 2m \tag{3.8}$$

3.2.2 Negative Energy Solutions

$$\hat{H}\psi = -|E|\psi, \quad \hat{\vec{P}}\psi = -\vec{p}\psi \tag{3.9}$$

Let $\psi(x) = v(\vec{p}) \exp(ip.x)$ solve the Dirac equation. Then

$$(i\partial\!\!\!/ - m)\psi = 0 \Rightarrow (i(ip\!\!\!/) - m)v(\vec{p}) = 0 \tag{3.10}$$
$$\Rightarrow (p\!\!\!/ + m)v(\vec{p}) = 0 \tag{3.11}$$

Possible solution for $v(\vec{p})$ is

$$v(\vec{p}) = f_- \frac{(-p\!\!\!/ + m)}{m} \begin{pmatrix} 0 \\ \chi \end{pmatrix} \tag{3.12}$$

where χ is a normalised spinor i.e. $\chi^\dagger \chi = 1$. Again, the normalisation constant f_- is determined by the requirement that $\bar{\psi}\psi$ is a Lorentz scalar. Then

$$\begin{aligned}
&\bar{v}(\vec{p})v(\vec{p}) = -2m \\
\Rightarrow &\frac{f_-^2}{m^2} \begin{pmatrix} 0 & \chi^\dagger \end{pmatrix} (p\!\!\!/ - m)^\dagger \gamma_0 (p\!\!\!/ - m) \begin{pmatrix} 0 \\ \chi \end{pmatrix} = -2m \\
\Rightarrow &\frac{f_-^2}{m^2} \begin{pmatrix} 0 & \chi^\dagger \end{pmatrix} (p\!\!\!/ - m) \begin{pmatrix} 0 \\ \chi \end{pmatrix} = -1 \\
\Rightarrow &\frac{f_-^2}{m^2} \begin{pmatrix} 0 & \chi^\dagger \end{pmatrix} \begin{pmatrix} |E| - m & -\vec{\sigma}.\vec{p} \\ \vec{\sigma}.\vec{p} & -|E| - m \end{pmatrix} \begin{pmatrix} 0 \\ \chi \end{pmatrix} = -1 \\
\Rightarrow &\frac{f_-^2}{m^2} (|E| + m)(\chi^\dagger \chi) = 1 \\
\Rightarrow &f_- = \frac{m}{\sqrt{|E| + m}} \\
\Rightarrow &v(\vec{p}) = \frac{-p\!\!\!/ + m}{\sqrt{|E| + m}} \begin{pmatrix} \phi \\ 0 \end{pmatrix}
\end{aligned}$$

Again, we may verify that this expressin gives correct results in the rest frame

$$v(\vec{0}) = \frac{-\gamma_0 m + m}{\sqrt{2m}} \begin{pmatrix} 0 \\ \chi \end{pmatrix} = \sqrt{2m} \begin{pmatrix} 0 \\ \chi \end{pmatrix} \tag{3.13}$$
$$\bar{v}(\vec{0})v(\vec{0}) = 2m \begin{pmatrix} 0 & \chi^\dagger \end{pmatrix} \gamma_0 \begin{pmatrix} 0 \\ \chi \end{pmatrix} = -2m \tag{3.14}$$

Chapter 4

Simultaneous Eigenket of Π and H

4.1 Solutions for $u(\vec{p})$

$$u(\vec{p}) = \frac{\not{p} + m}{\sqrt{|E| + m}} \begin{pmatrix} \phi \\ 0 \end{pmatrix} = \begin{pmatrix} \sqrt{|E| + m}\phi \\ \dfrac{\vec{\sigma}.\vec{p}}{\sqrt{|E| + m}}\phi \end{pmatrix} \tag{4.1}$$

The eigenket above is a general positive energy solution since the spinor ϕ is arbitrary. In order to extract positive energy eigenkets with definite helictiy, we consider the action of Π on $u(\vec{p})$.

$$\begin{aligned} \Pi u(\vec{p}) &= \frac{1}{p} \begin{pmatrix} \vec{\sigma}.\vec{p} & 0 \\ 0 & \vec{\sigma}.\vec{p} \end{pmatrix} u(\vec{p}) \\ &= \frac{1}{p} \begin{pmatrix} \sqrt{|E| + m}\vec{\sigma}.\vec{p}\phi \\ \dfrac{\vec{\sigma}.\vec{p}}{\sqrt{|E| + m}}\vec{\sigma}.\vec{p}\phi \end{pmatrix} \end{aligned}$$

Now, we observe that if we replace the arbitrary spinor ϕ by the eigenstates of κ, then

$$\Pi u(\vec{p}) = \frac{1}{p} \begin{pmatrix} \sqrt{|E| + m}\vec{\sigma}.\vec{p}e_{\pm} \\ \dfrac{\vec{\sigma}.\vec{p}}{\sqrt{|E| + m}}\vec{\sigma}.\vec{p}e_{\pm} \end{pmatrix} = \pm \begin{pmatrix} \sqrt{|E| + m}e_{\pm} \\ \dfrac{\vec{\sigma}.\vec{p}}{\sqrt{|E| + m}}e_{\pm} \end{pmatrix} = \pm u(\vec{p}) \tag{4.2}$$

4.2 Solutions for $v(\vec{p})$

$$v(\vec{p}) = \frac{-\not{p} + m}{\sqrt{|E| + m}} \begin{pmatrix} 0 \\ \chi \end{pmatrix} = \begin{pmatrix} \dfrac{\vec{\sigma}.\vec{p}}{\sqrt{|E| + m}}\chi \\ \sqrt{|E| + m}\chi \end{pmatrix} \tag{4.3}$$

However for negative energy solutions, we note that they are expressed with negative momentum i.e. $\hat{\vec{P}}\psi = -\vec{p}\psi$. Therefore

$$\begin{aligned}
\Pi v(\vec{p}) &= \frac{1}{p}\begin{pmatrix} \vec{\sigma}.-\vec{p} & 0 \\ 0 & \vec{\sigma}.-\vec{p} \end{pmatrix} v(\vec{p}) \\
&= \frac{-1}{p}\begin{pmatrix} \frac{\vec{\sigma}.\vec{p}}{\sqrt{|E|+m}}\vec{\sigma}.\vec{p}\chi \\ \sqrt{|E|+m}\vec{\sigma}.\vec{p}\chi \end{pmatrix}
\end{aligned}$$

Here, a choice of the arbitrary spinor $\chi = e_{\pm}$, gives

$$\Pi v(\vec{p}) = \frac{-1}{p}\begin{pmatrix} \frac{\vec{\sigma}.\vec{p}}{\sqrt{|E|+m}}\vec{\sigma}.\vec{p}e_{\pm} \\ \sqrt{|E|+m}\vec{\sigma}.\vec{p}e_{\pm} \end{pmatrix} = \mp \begin{pmatrix} \frac{\vec{\sigma}.\vec{p}}{\sqrt{|E|+m}}e_{\pm} \\ \sqrt{|E|+m}e_{\pm} \end{pmatrix} = \mp v(\vec{p}) \tag{4.4}$$

The final expressions for the simultaneous energy helicity eigenkets is therefore ultimately obtained to be

$$u_+(\vec{p}) = \begin{pmatrix} \sqrt{|E|+m}e_+ \\ \frac{\vec{\sigma}.\vec{p}}{\sqrt{|E|+m}}e_+ \end{pmatrix} = \begin{pmatrix} \sqrt{|E|+m}e_+ \\ \sqrt{|E|-m}e_+ \end{pmatrix} \qquad u_-(\vec{p}) = \begin{pmatrix} \sqrt{|E|+m}e_- \\ \frac{\vec{\sigma}.\vec{p}}{\sqrt{|E|+m}}e_- \end{pmatrix} = \begin{pmatrix} \sqrt{|E|+m}e_- \\ -\sqrt{|E|-m}e_- \end{pmatrix}$$

$$v_+(\vec{p}) = \begin{pmatrix} \frac{\vec{\sigma}.\vec{p}}{\sqrt{|E|+m}}e_- \\ \sqrt{|E|+m}e_- \end{pmatrix} = \begin{pmatrix} -\sqrt{|E|-m}e_- \\ \sqrt{|E|+m}e_- \end{pmatrix} \qquad v_-(\vec{p}) = \begin{pmatrix} \frac{\vec{\sigma}.\vec{p}}{\sqrt{|E|+m}}e_+ \\ \sqrt{|E|+m}e_+ \end{pmatrix} = \begin{pmatrix} \sqrt{|E|-m}e_+ \\ \sqrt{|E|+m}e_+ \end{pmatrix}$$

$$e_+ = \frac{e^{i\phi}}{\sqrt{2pp_d}}\begin{pmatrix} p_- \\ p_d \end{pmatrix} \quad e_- = \frac{1}{\sqrt{2pp_s}}\begin{pmatrix} -p_- \\ p_s \end{pmatrix}$$

Thus we have completely expressed the components of the eigenket in terms of the momenta and energy.

Chapter 5

Properties of Eigenkets

5.1 Orthonormality

$$
\begin{aligned}
i.\ & \bar{u}_\pm(\vec{p})u_\pm(\vec{p}) \\
&= \begin{pmatrix} \sqrt{|E|+m}e_\pm^\dagger & \pm\sqrt{|E|-m}e_\pm^\dagger \end{pmatrix} \gamma_0 \begin{pmatrix} \sqrt{|E|+m}e_\pm \\ \pm\sqrt{|E|-m}e_\pm \end{pmatrix} \\
&= (|E|+m)e_\pm^\dagger e_\pm - (|E|-m) + e_\pm^\dagger e_\pm \\
&= 2m \\
ii.\ & \bar{v}_\pm(\vec{p})v_\pm(\vec{p}) \\
&= \begin{pmatrix} \mp\sqrt{|E|-m}e_\pm^\dagger & \sqrt{|E|+m}e_\pm^\dagger \end{pmatrix} \gamma_0 \begin{pmatrix} \mp\sqrt{|E|-m}e_\pm \\ \sqrt{|E|+m}e_\pm \end{pmatrix} \\
&= (|E|-m)e_\pm^\dagger e_\pm - (|E|+m)e_\pm^\dagger e_\pm \\
&= -2m \\
iii.\ & \bar{u}_+(\vec{p})u_-(\vec{p}) \\
&= \begin{pmatrix} \sqrt{|E|+m}e_+^\dagger & \sqrt{|E|-m}e_+^\dagger \end{pmatrix} \gamma_0 \begin{pmatrix} \sqrt{|E|+m}e_- \\ -\sqrt{|E|-m}e_- \end{pmatrix} \\
&= (|E|+m)e_+^\dagger e_- + (|E|-m)e_+^\dagger e_- \\
&= 0 \\
iv.\ & \bar{u}_\pm(\vec{p})v_\mp(\vec{p}) \\
&= \begin{pmatrix} \sqrt{|E|+m}e_\pm^\dagger & \pm\sqrt{|E|-m}e_\pm^\dagger \end{pmatrix} \gamma_0 \begin{pmatrix} \pm\sqrt{|E|-m}e_\pm \\ \sqrt{|E|+m}e_\pm \end{pmatrix} \\
&= \pm\sqrt{|E|^2-m^2}e_\pm^\dagger e_\pm - \pm\sqrt{|E|^2-m^2}e_\pm^\dagger e_\pm \\
&= 0
\end{aligned}
$$

5.2 Projection Operators

Let $\Lambda_{\pm}$ be the positive and negative energy projection operators respectively. From the above orthonormal relations

$$\bar{u}_r(\vec{p})u_s(\vec{p}) = +2m\delta_{rs} \tag{5.1}$$

$$\bar{v}_r(\vec{p})v_s(\vec{p}) = -2m\delta_{rs}\ ,\ \ r,s = +,- \tag{5.2}$$

We also have the Dirac equations

$$(\not{p} - m)u_r(\vec{p}) = 0 \tag{5.3}$$

$$(\not{p} + m)v_r(\vec{p}) = 0\ ,\ \ r = +,- \tag{5.4}$$

Thus, the projection operators can be equivalently written as

$$\Lambda_+ = \frac{\not{p} + m}{2m} = \frac{1}{2m}\sum_r u_r(\vec{p})\bar{u}_r(\vec{p}) \tag{5.5}$$

$$\Lambda_- = \frac{\not{p} - m}{-2m} = \frac{-1}{2m}\sum_r v_r(\vec{p})\bar{v}_r(\vec{p}) \tag{5.6}$$

$$\Lambda_+ + \Lambda_- = \frac{\not{p} + m}{2m} - \frac{\not{p} - m}{2m} = \frac{1}{2m}\sum_r u_r(\vec{p})\bar{u}_r(\vec{p}) - v_r(\vec{p})\bar{v}_r(\vec{p}) = I_4 \tag{5.7}$$

We obtained the completeness relation of the eigenkets from orthonormality constraints and form of the Dirac equation but we must test our solutions explicitly for completeness.

5.3 Completeness Relation

$$\sum_r u_r(\vec{p})\bar{u}_r(\vec{p})$$

$$= u_+(\vec{p})\bar{u}_+(\vec{p}) + u_-(\vec{p})\bar{u}_-(\vec{p})$$

$$= \begin{pmatrix} \sqrt{|E|+m}e_+ \\ \sqrt{|E|-m}e_+ \end{pmatrix} \begin{pmatrix} \sqrt{|E|+m}e_+^\dagger & -\sqrt{|E|-m}e_+^\dagger \end{pmatrix} \begin{pmatrix} \sqrt{|E|+m}e_- \\ -\sqrt{|E|-m}e_- \end{pmatrix} \left(\begin{pmatrix} \sqrt{|E|+m}e_- \\ -\sqrt{|E|-m}e_- \end{pmatrix}\right)^\dagger \gamma_0$$

$$= \begin{pmatrix} (|E|+m)e_+e_+^\dagger & -pe_+e_+^\dagger \\ pe_+e_+^\dagger & -(|E|-m)e_+e_+^\dagger \end{pmatrix} + \begin{pmatrix} (|E|+m)e_-e_-^\dagger & pe_-e_-^\dagger \\ -pe_-e_-^\dagger & -(|E|-m)e_-e_-^\dagger \end{pmatrix}$$

$$= \begin{pmatrix} (|E|+m)(e_+e_+^\dagger + e_-e_-^\dagger) & -p(e_+e_+^\dagger - e_-e_-^\dagger) \\ p(e_+e_+^\dagger - e_-e_-^\dagger) & -(|E|-m)(e_+e_+^\dagger + e_-e_-^\dagger) \end{pmatrix}$$

$$= \begin{pmatrix} (|E|+m)I_2 & -\vec{\sigma}.\vec{p} \\ \vec{\sigma}.\vec{p} & (-|E|+m)I_2 \end{pmatrix}$$

$$= \not{p} + m$$

$$
\begin{aligned}
&\sum_r v_r(\vec{p})\bar{v}_r(\vec{p})\\
&= v_+(\vec{p})\bar{v}_+(\vec{p}) + v_-(\vec{p})\bar{v}_-(\vec{p})\\
&= \begin{pmatrix} -\sqrt{|E|-m}e_- \\ \sqrt{|E|+m}e_- \end{pmatrix} \begin{pmatrix} -\sqrt{|E|-m}e_-^\dagger & -\sqrt{|E|+m}e_-^\dagger \end{pmatrix} \begin{pmatrix} \sqrt{|E|-m}e_+ \\ \sqrt{|E|+m}e_+ \end{pmatrix} \begin{pmatrix} \sqrt{|E|-m}e_+ \\ \sqrt{|E|+m}e_+ \end{pmatrix}^\dagger \gamma_0\\
&= \begin{pmatrix} (|E|-m)e_-e_-^\dagger & pe_-e_-^\dagger \\ -pe_-e_-^\dagger & -(|E|+m)e_-e_-^\dagger \end{pmatrix} + \begin{pmatrix} (|E|-m)e_+e_+^\dagger & -pe_+e_+^\dagger \\ pe_+e_+^\dagger & -(|E|+m)e_+e_+^\dagger \end{pmatrix}\\
&= \begin{pmatrix} (|E|-m)(e_-e_-^\dagger + e_+e_+^\dagger) & -p(-e_-e_-^\dagger + e_+e_+^\dagger) \\ p(-e_-e_-^\dagger + e_+e_+^\dagger) & -(|E|+m)(e_-e_-^\dagger + e_+e_+^\dagger) \end{pmatrix}\\
&= \begin{pmatrix} (|E|-m)I_2 & -\vec{\sigma}.\vec{p} \\ \vec{\sigma}.\vec{p} & (-|E|-m)I_2 \end{pmatrix}\\
&= \not{p} - m
\end{aligned}
$$

Thus, we have ascertained that our explicit eigenket solutions are complete as

$$\sum_r u_r(\vec{p})\bar{u}_r(\vec{p}) - v_r(\vec{p})\bar{v}_r(\vec{p}) = 2mI_4 \tag{5.8}$$

Chapter 6

Massless Limit

In Standard Model, neutrinos are massless fermions; also in accelerators, the colliding particles are most often ultra-relativistic and therefore have a simple spectrum $E = \pm p$. Since $m = 0$, the Dirac equation admits only 2 independent eigenkets instead of 4 for a given momenta. This is illustrated below

$$\begin{aligned} \psi^+ &= u(\vec{p})\exp(-ip.x) & \psi^- &= v(\vec{p})\exp(+ip.x) \\ i\partial\!\!\!/\psi^+ &= 0 & i\partial\!\!\!/\psi^- &= 0 \\ i(-ip\!\!\!/)u(\vec{p}) &= 0 & i(ip\!\!\!/)v(\vec{p}) &= 0 \\ p\!\!\!/u(\vec{p}) &= 0 & -p\!\!\!/v(\vec{p}) &= 0 \end{aligned}$$

Possible massless solutions are

$$\begin{aligned} u(\vec{p}) &= f_+ p\!\!\!/ \begin{pmatrix} \phi \\ 0 \end{pmatrix} \;,\quad \because p\!\!\!/^2 = 0 \\ \Rightarrow u^\dagger u &= |f_+|^2 \begin{pmatrix} \phi^\dagger & 0 \end{pmatrix} p\!\!\!/^\dagger p\!\!\!/ \begin{pmatrix} \phi \\ 0 \end{pmatrix} \\ &= |f_+|^2 \begin{pmatrix} \phi^\dagger & 0 \end{pmatrix} \begin{pmatrix} 2p^2 & -2p\vec{\sigma}.\vec{p} \\ -2p\vec{\sigma}.\vec{p} & 2p^2 \end{pmatrix} \begin{pmatrix} \phi \\ 0 \end{pmatrix} \\ &= |f_+|^2 2p^2(\phi^\dagger\phi) \end{aligned}$$

The normalisation constant is determined from the requirement that $u^\dagger u = \bar{u}\gamma_0 u$ is a Lorentz vector and $u^\dagger u = 2E$. Thus,

$$f_+ = \frac{1}{\sqrt{p}} \tag{6.1}$$

$$u(\vec{p}) = \frac{p\!\!\!/}{\sqrt{p}} \begin{pmatrix} \phi \\ 0 \end{pmatrix} = \begin{pmatrix} \sqrt{p}\phi \\ \frac{\vec{\sigma}.\vec{p}}{\sqrt{p}}\phi \end{pmatrix} \tag{6.2}$$

Since $m = 0$, $\bar{u}u$ is a zero scalar.

$$\begin{aligned}
\bar{u}u &= \begin{pmatrix} \phi^\dagger & 0 \end{pmatrix} \not{p}^\dagger \gamma_0 \not{p} \begin{pmatrix} \phi \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} \phi^\dagger & 0 \end{pmatrix} \gamma_0 \not{p}^2 \begin{pmatrix} \phi \\ 0 \end{pmatrix} \\
&= 0
\end{aligned}$$

$v(\vec{p})$ satisfies the same Dirac eqn. but let us consider solutions of the same form as massive case, though inevitably we must arrive at same result as $u(\vec{p})$.

$$\begin{aligned}
v(\vec{p}) &= -f_- \not{p} \begin{pmatrix} 0 \\ \chi \end{pmatrix} \\
\Rightarrow v^\dagger v &= |f_-|^2 \begin{pmatrix} 0 & \chi^\dagger \end{pmatrix} \not{p}^\dagger \not{p} \begin{pmatrix} 0 \\ \chi \end{pmatrix} \\
&= |f_-|^2 \begin{pmatrix} 0 & -\chi^\dagger \end{pmatrix} \begin{pmatrix} 2p^2 & -2p\vec{\sigma}.\vec{p} \\ -2p\vec{\sigma}.\vec{p} & 2p^2 \end{pmatrix} \begin{pmatrix} 0 \\ \chi \end{pmatrix} \\
&= -|f_-|^2 2p^2 (\chi^\dagger \chi)
\end{aligned}$$

Again, $v^\dagger v = 2E = -2p$, which gives

$$f_- = \frac{1}{\sqrt{p}} \tag{6.3}$$

$$v(\vec{p}) = \frac{\not{p}}{\sqrt{p}} \begin{pmatrix} 0 \\ \chi \end{pmatrix} = \begin{pmatrix} \dfrac{\vec{\sigma}.\vec{p}}{\sqrt{p}} \chi \\ \sqrt{p}\chi \end{pmatrix} \tag{6.4}$$

The simultaneous energy helicity eigenkets in the massless limit are given by

$$i.\ u_+(\vec{p}) = \begin{pmatrix} \sqrt{p} e_+ \\ \dfrac{\vec{\sigma}.\vec{p}}{\sqrt{p}} e_+ \end{pmatrix} = \sqrt{p} \begin{pmatrix} e_+ \\ e_+ \end{pmatrix} \qquad ii.\ u_-(\vec{p}) = \begin{pmatrix} \sqrt{p} e_- \\ \dfrac{\vec{\sigma}.\vec{p}}{\sqrt{p}} e_- \end{pmatrix} = \sqrt{p} \begin{pmatrix} e_- \\ -e_- \end{pmatrix}$$

$$iii.\ v_+(\vec{p}) = \begin{pmatrix} \dfrac{\vec{\sigma}.\vec{p}}{\sqrt{p}} e_- \\ \sqrt{p} e_- \end{pmatrix} = \sqrt{p} \begin{pmatrix} -e_- \\ e_- \end{pmatrix} \qquad iv.\ v_-(\vec{p}) = \begin{pmatrix} \dfrac{\vec{\sigma}.\vec{p}}{\sqrt{p}} e_+ \\ \sqrt{p} e_+ \end{pmatrix} = \sqrt{p} \begin{pmatrix} e_+ \\ e_+ \end{pmatrix}$$

It can be seen that $u_+(\vec{p})$ is identical to $v_-(\vec{p})$ and $u_-(\vec{p})$ is identical to $v_+(\vec{p})$ i.e. there are only 2 independent eigenkets as was evident from the form of solutions of Dirac eqn.

Concluding Remarks

The statement and proof of Pauli's fundamental theorem allowed us to uniquely express the vertex factor of a Feynman diagram in a linear sum of Γ matrices with useful trace and anti-commutation properties. Given a particular process, this method allowed us to consider contributions from only very small number of coefficients. In order to accurately determine these a_i's, the energy helicity eigenkets were derived rigorously and made consistent with limiting cases of massless/ultra-relativistic particles, rest frame and collinear beams. Furthermore, the orthonormality and completeness properties of these eigenstates were verified with the explicitly obtained solutions.