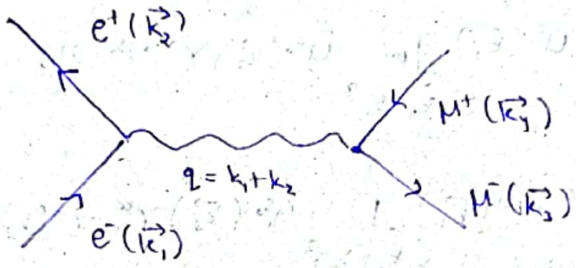
1. Interaction :-

$$e^- + e^+ \longrightarrow \mu^+ + \mu^-$$

$k_1, s_1 \quad k_2, s_2 \quad k_4, s_4 \quad k_3, s_3 \qquad s_i = \frac{1}{2}, \; i = 1, 4.$

$$|i\rangle = d^{\dagger}_{s_2}(\vec{k}_2)\, c^{\dagger}_{s_1}(\vec{k}_1)\, |0\rangle$$

$$|f\rangle = g^{\dagger}_{s_4}(\vec{k}_4)\, b^{\dagger}_{s_3}(\vec{k}_3)\, |0\rangle$$



$$T_{fi} = 2 \times \frac{(-ie)^2}{2!} \int d^4x_1 \int d^4x_2 \, \langle f | \bar{\psi}^{(-)}_{\alpha}(x_1)\, \gamma^{\mu}_{\alpha\beta}\, \psi^{(-)}_{\beta}(x_1)\, \bar{\Lambda}^{(+)}_{\lambda}(x_2)\, \gamma^{\nu}_{\lambda\sigma}\, \Lambda^{(+)}_{\sigma}(x_2) | i \rangle \, \underbrace{A_{\mu}(x_1) A_{\nu}(x_2)}$$

$$= -e^2 \int d^4x_1 \int d^4x_2 \, \langle f | b^{\dagger}_{s_3}(\vec{k}_3)\, g^{\dagger}_{s_4}(\vec{k}_4)\, d_{s_2}(\vec{k}_2)\, c_{s_1}(\vec{k}_1) | i \rangle \, i D^F_{\mu\nu}(x_1 - x_2)$$

$$\times \left(\frac{m_e}{V E_{\vec{k}_1}}\right)^{1/2} \left(\frac{m_e}{V E_{\vec{k}_2}}\right)^{1/2} \left(\frac{m_\mu}{V E_{\vec{k}_3}}\right)^{1/2} \left(\frac{m_\mu}{V E_{\vec{k}_4}}\right)^{1/2} \bar{u}^{s_3}_{\alpha}(\vec{k}_3)\, \gamma^{\mu}_{\alpha\beta}\, v^{s_4}_{\beta}(\vec{k}_4)$$

$$\times \bar{v}^{s_2}_{\lambda}(\vec{k}_2)\, \gamma^{\nu}_{\lambda\delta}\, u^{s_1}_{\delta}(\vec{k}_1) \times e^{ix_1 k_3}\, e^{ix_1 k_4}\, e^{-ix_2 k_1}\, e^{-ix_2 k_2}$$

Now,

$\langle f |$ creation/ annih. op. $| i \rangle$

$$= \langle 0 | b_{s_3}(\vec{k}_3)\, g_{s_4}(\vec{k}_4)\, b^{\dagger}_{s_3}(\vec{k}_3)\, g^{\dagger}_{s_4}(\vec{k}_4)\, d_{s_2}(\vec{k}_2)\, c_{s_1}(\vec{k}_1)\, d^{\dagger}_{s_2}(\vec{k}_2)\, c^{\dagger}_{s_1}(\vec{k}_1) | 0 \rangle$$

$$= 1 \qquad \because \; \{c_s(\vec{k}), d_{s'}(\vec{k}')\} = 0$$

and,

$$\int d^4x_1 \int d^4x_2 \, i D^F_{\mu\nu}(x_1 - x_2)\, e^{ix_1(k_3 + k_4)}\, e^{-ix_2(k_1 + k_2)}$$

$$= \int d^4x_1 \int d^4x_2 \int \frac{d^4q}{(2\pi)^4} \, i D_{F_{\mu\nu}}(q)\, e^{-ix_1(q - k_3 - k_4)}\, e^{ix_2(q - k_1 - k_2)}$$

$$= \int \frac{d^4 q}{(2\pi)^4} \left(i\, D^F_{\mu\nu}(q = k_1+k_2)\right) \delta^4\left(\Sigma k_i - \Sigma k_f\right)(2\pi)^4$$

$$= \int d^4 q \;\; i\, D_{F_{\mu\nu}}(q = k_1+k_2)\, \delta^4(K_i - K_f)$$

and, $D_{F_{\mu\nu}}(q) = -\dfrac{g_{\mu\nu}}{q^2}$

So, the invariant Feynman amplitude for the interaction is,

$$\mathcal{M} = ie^2\, \bar{v}^{s_2}(\vec{k_2})\gamma^\mu u^{s_1}(\vec{k_1}) \frac{g_{\mu\nu}}{(k_1+k_2)^2} \bar{u}^{s_3}(\vec{k_3})\gamma^\nu v^{s_4}(\vec{k_4})$$

$$= i\, \bar{v}^{s_2}(\vec{k_2})\gamma^\mu u^{s_1}(\vec{k_1}) \frac{1}{(k_1+k_2)^2} \bar{u}^{s_3}(\vec{k_3})\gamma_\mu v^{s_4}(\vec{k_4})$$

$\because$ let coupling $e^2 = 1$

2. Spinor formalism :-

In high energy colliders, most fermions are ultra relativistic, so we treat them as massless.

Also, for massless fermions, both $[\eta, H] = [\gamma_5, H] = 0$ i.e both helicity and chirality commute with Dirac Hamiltonian.

Now, in Weyl representation;

$$\gamma^0 = \begin{pmatrix} 0 & \mathbb{I}_2 \\ \mathbb{I}_2 & 0 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \quad \gamma_5 = \begin{pmatrix} -\mathbb{I}_2 & 0 \\ 0 & \mathbb{I}_2 \end{pmatrix}$$

$$\eta = \begin{pmatrix} \vec{\sigma}\cdot\hat{p} & 0 \\ 0 & \vec{\sigma}\cdot\hat{p} \end{pmatrix}$$

Now, the eigenstates of $\bar{\sigma}.\hat{p}$ are given by,

$\bar{\sigma}.\hat{p}\, e_{\pm} = \pm\, e_{\pm}$, where $e_+ = \frac{e^{i\varphi}}{\sqrt{2pp_d}}\begin{pmatrix} p_- \\ p_d \end{pmatrix}$, $e_- = \frac{e^{-i\varphi}}{\sqrt{2pp_d}}\begin{pmatrix} -p_d \\ p_+ \end{pmatrix}$

$\therefore\ p_{\pm} = p_x \pm ip_y$, $\tan\varphi = \frac{p_y}{p_x}$

$p = (p_x^2 + p_y^2 + p_z^2)^{1/2}$, $p_d = p - p_z$

Positive helicity eigenstates are given by $\begin{pmatrix} e_+ \\ 0 \end{pmatrix}$, $\begin{pmatrix} 0 \\ e_+ \end{pmatrix}$,

whereas right handed i.e positive chirality states are given by $\begin{pmatrix} 0 \\ \varphi \end{pmatrix}$

Hence, $\begin{pmatrix} 0 \\ \alpha e_+ \end{pmatrix}$ is the right handed / positive helicity fermion eigenstate.

It is easily verified that

$$\not{p}\, u_+(p) = \begin{bmatrix} 0 & p_{a\dot{b}} \\ p^{\dot{a}b} & 0 \end{bmatrix} u_+(p)$$

$$= 0$$

$\therefore\ p_{a\dot{b}} = p_\mu \sigma^\mu{}_{a\dot{b}}$

$= p\,\mathbb{I} - \vec{p}.\bar{\sigma}$

$p^{\dot{a}b} = p_\mu(\bar{\sigma}^\mu)^{\dot{a}b}$

$= p\,\mathbb{I} + \vec{p}.\bar{\sigma}$

Negative helicity eigenstates are given by $\begin{pmatrix} e_- \\ 0 \end{pmatrix}$, $\begin{pmatrix} 0 \\ e_- \end{pmatrix}$,

whereas left handed i.e negative chirality states are given by $\begin{pmatrix} \chi \\ 0 \end{pmatrix}$.

Hence, $\begin{pmatrix} \beta e_- \\ 0 \end{pmatrix}$ is the left handed / negative helicity fermion eigenstate.

This is also easily verified that,

$$\not{p}\, u_-(p) = \begin{bmatrix} 0 & p_{a\dot{b}} \\ p^{\dot{a}b} & 0 \end{bmatrix} u_-(p)$$

$$= 0$$

$\therefore\ p\,(\mathbb{I} + \bar{\sigma}.\hat{p})\, e_- = 0$

$p\,(\mathbb{I} - \bar{\sigma}.\hat{p})\, e_+ = 0$

* We also notice that

$$(e_+^\dagger) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}(e_-) \quad \& \quad (e_-^\dagger) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}(e_+)$$

Finally, the helicity eigenstates are represented in Weyl formalism as

$$u_+(p) = \sqrt{2p}\begin{bmatrix} 0 \\ e_+ \end{bmatrix} = \begin{bmatrix} 0 \\ |p\rangle^{\dot{a}} \end{bmatrix}$$

$\therefore\ \alpha = \beta = \sqrt{2p}$ is fixed by the requirement

$$u_-(p) = \sqrt{2p}\begin{bmatrix} e_- \\ 0 \end{bmatrix} = \begin{bmatrix} |p]_a \\ 0 \end{bmatrix}$$

$$\not{p} = u_+(p)\bar{u}_+(p) + u_-(p)\bar{u}_-(p)$$

and
$$\bar{u}_+(p) = u_+^\dagger(p)\gamma^0 = \sqrt{2p}\,[e_+^\dagger \;\; 0] \equiv [\,[p|^a \;\; 0\,],$$
$$\bar{u}_-(p) = u_-^\dagger(p)\gamma^0 = \sqrt{2p}\,[0 \;\; e_-^\dagger] \equiv [\,0 \;\; \langle p|_{\dot{a}}\,]$$

Additionally via crossing symmetry, anti-fermion states are given as,

$$v_\pm(p) = u_\mp(p) \quad \& \quad \bar{v}_\pm(p) = \bar{u}_\mp(p)$$

and, we have the following properties for the two component square and angle spinors,

(i) $|p]_a^* = \langle p|_{\dot{a}}$ (ii) $[p|^{a*} = |p\rangle^{\dot{a}}$

(iii) $|p]_a = \varepsilon_{ab}[p|^b$, $\langle p|_{\dot{a}} = \varepsilon_{\dot{a}\dot{b}}|p\rangle^{\dot{b}}$, $\varepsilon_{ab} = \varepsilon_{\dot{a}\dot{b}}$ are anti-symmetric with $\varepsilon_{12} = -1$

(iv) $[p|^a = \varepsilon^{ab}|p]_b$, $|p\rangle^{\dot{a}} = \varepsilon^{\dot{a}\dot{b}}\langle p|_{\dot{b}}$, $\varepsilon^{ab} = \varepsilon^{\dot{a}\dot{b}}$ are anti-symmetric with $\varepsilon^{12} = 1$

(3) Computing the Feynman amplitude.

$$\underset{k_1,s_1}{e^-} + \underset{k_2,s_2}{e^+} \longrightarrow \underset{k_3,s_3}{\mu^-} + \underset{k_4,s_4}{\mu^+}$$

$$k_1 + k_2 = k_3 + k_4.$$

Now, crossing symmetry for massless fermions requires that $s_1 = \bar{s}_2$ & $s_3 = \bar{s}_4$, so first we consider the helicity amplitude.

$$A(e^-_L e^+_R \mu^-_R \mu^+_L) = i\, \bar{v}_+(2)\gamma^\mu u_-(1) \frac{1}{s_{12}} \bar{u}_+(3)\gamma_\mu v_-(4) \quad \because s_{ij} = (p_i+p_j)^2$$

$$= \frac{i}{s_{12}} \bar{u}_-(2)\gamma^\mu u_-(1)\, \bar{u}_+(3)\gamma_\mu u_+(4) \quad \because \text{crossing symmetry}$$

$$= \frac{i}{s_{12}} \langle 2^- | \gamma^\mu | 1^- \rangle \langle 3^+ | \gamma_\mu | 4^+ \rangle$$

$$= \frac{i}{s_{12}} \langle 1^+ | \gamma^\mu | 2^+ \rangle \langle 3^+ | \gamma_\mu | 4^+ \rangle \quad \because \text{charge conjugation of current gives } \langle i^+|\gamma^\mu|j^+\rangle = \langle j^-|\gamma^\mu|i^-\rangle$$

$$= \frac{i}{s_{12}}\, 2\,[13]\langle 42\rangle \quad \because \text{Fierz rearrangement. } \langle i^+|\gamma^\mu|j^+\rangle\langle k^+|\gamma_\mu|l^+\rangle = 2\langle lj\rangle[ik]$$

$\#$ $[ij]\langle ji\rangle = 2k_i^\mu k_{j\mu} = s_{ij}$ & $[ij]^* = \langle ji\rangle$

$\Rightarrow$ $[ij] = \sqrt{s_{ij}}\, e^{i\varphi}$ & $\langle ji\rangle = \sqrt{s_{ij}}\, e^{-i\varphi}$

Thus,
$$A(e^-_L e^+_R \mu^-_R \mu^+_L) = \frac{2i\sqrt{s_{13}}\sqrt{s_{42}}\, e^{i\delta}}{s_{12}} \quad \because \delta \text{ is some phase!}$$

$$= 2i \frac{s_{24}}{s_{12}} e^{i\delta}$$

$$= -i e^{i\delta} \frac{t}{s} \times 2$$

and, for massless fermions
$$s_{13} = (p_1+p_3)^2 = -(p_1-p_3)^2 = -(p_4-p_2)^2 = (p_2+p_4)^2 = s_{24}$$

where, t and s are the usual Mandelstam variables given by $s = (p_A+p_B)^2 = (p_1+p_2)^2$ & $t = (p_A-p_C)^2 = (p_1-p_3)^2$

In the center of momentum frame, $\vec{p}_1 = -\vec{p}_2 = \vec{p}_i$ & $\vec{p}_3 = -\vec{p}_4 = \vec{p}_f$
& $|\vec{p}_i| = |\vec{p}_f| = p$

$t = -2p^2(1-\cos\theta)$, where θ is the scattering ~~opt~~ angle
$s = 4p^2$ b/w $\vec{p}_3$ and $\vec{p}_1$

So, for this particular helicity configuration, we have.

$$A(e^-_L e^+_R \mu^-_R \mu^+_L) = -2ie^{i\delta} \times \frac{-2p^2(1-\cos\theta)}{4p^2}$$
$$= ie^{i\delta}(1-\cos\theta)$$

Alternatively, the result can be expressed entirely in terms of $\langle\rangle$ spinors as,

$$A(e^-_L e^+_R \mu^-_R \mu^+_L) = 2i\frac{[13]\langle 42\rangle}{\langle 12\rangle[21]} \times \frac{\langle 13\rangle}{\langle 13\rangle}$$

$$= 2i\frac{\langle 42\rangle([13]\langle 13\rangle)}{\langle 12\rangle([21]\langle 13\rangle)}$$

$$= 2i\frac{\langle 42\rangle[\cancel{24}]\langle 24\rangle}{\langle 12\rangle[\cancel{24}]\langle 43\rangle}$$

$$= 2i\frac{\langle 24\rangle^3}{\langle 12\rangle\langle 34\rangle}$$

$\because [13]\langle 13\rangle$
$= -s_{13}$
$= -s_{24}$
$= \langle 24\rangle[24]$

$\because [21]\langle 13\rangle$
$= [21\cancel{1}13\rangle$
$= [2|\cancel{2}+\cancel{1}-\cancel{4}|3\rangle$
$= [2|\cancel{4}|3\rangle$
$= [2|4]\langle 43\rangle$

and, $|A(e^-_L e^+_R \mu^-_R \mu^+_L)|^2 = 4\frac{s_{24}^2}{s_{12}^2}$

(4) Different Helicity configurations.

Since, the electron & positron helicities must be opposite, & the same goes for the muon/anti-muon helicities, we have four different helicity configurations to be summed over,

(a) $e^-_L e^+_R \mu^-_R \mu^+_L$ (b) $e^-_R e^+_L \mu^-_L \mu^+_R$ ~~(c)~~ ~~$e^-_R e^+_L \mu^-_L \mu^+_R$~~ ~~(d)~~ ~~$e^-_L e^+_R \mu^-_R \mu^+_L$~~

(c) $e^-_R e^+_L \mu^-_R \mu^+_L$ (d) $e^-_L e^+_R \mu^-_L \mu^+_R$

For (b); parity(P) operation flips all the momenta in opposite direction, but not the spin, so helicities get reversed, which interchanges all the angle and square spinors $\langle\,\rangle \leftrightarrow [\,]$

Hence, $A(e^-_R e^+_L \mu^-_L \mu^+_R) = 2i \dfrac{\langle 13\rangle [42]}{s_{12}}$

$= -2i e^{i\theta'} \dfrac{t}{s}$ (or) $2i \dfrac{[24]^2}{[12][34]}$

which is same as (a) up to a phase factor.

For (c); charge conjugation (C) operation on the electron line, interchanges the electron and positron charges,

$$\underset{k_1}{e^-}\ \underset{k_2}{e^+} \xrightarrow{C} \underset{k_2}{e^-}\ \underset{k_1}{e^+}$$

and, $\underset{k_2}{e^-_L}\ \underset{k_1}{e^+_R}$ gives the term $\bar{u}_-(1)\gamma^\mu u_-(2)$ i.e $\langle 1^-|\gamma^\mu|2^-\rangle$

i.e it interchanges the electron & positron momenta of (a)

but, $\langle 1^-|\gamma^\mu|2^-\rangle = \langle 2^+|\gamma^\mu|1^+\rangle$ (charge conjugation of current)

thus, C operation has flipped the electron/positron helicities and gives the amplitude for the $e^-_R e^+_L$ configuration

Hence, $A(e^-_R e^+_L \mu^-_R \mu^+_L) = A(e^-_L e^+_R \mu^-_R \mu^+_L)_{1 \leftrightarrow 2}$

$$= \frac{2i}{S_{12}} [23]\langle 41\rangle$$

$$= 2i \frac{\sqrt{S_{23}}\sqrt{S_{14}}}{S_{12}} e^{i\delta}, \quad \delta \text{ is some phase.}$$

$$= 2i \frac{S_{14}}{S_{12}} e^{i\delta}$$

$$= -2i \frac{u}{s} e^{i\delta}$$

$$= -2i \times \frac{-2p^2(1+\cos\theta)}{4p^2} e^{i\delta}$$

$$= i e^{i\delta} (1+\cos\theta) \quad (\text{or}) \quad -2i \frac{\langle 14\rangle^2}{\langle 12\rangle\langle 34\rangle}$$

For (d); parity operation (P) on (c), interchanges the square and angle spinors $\langle\rangle \leftrightarrow []$ and the momenta $1 \leftrightarrow 2$

$$A(e^-_L e^+_R \mu^-_L \mu^+_R) = A(e^-_L e^+_R \mu^-_R \mu^+_L)_{\langle\rangle \leftrightarrow [], 1\leftrightarrow 2}$$

$$= 2i \frac{\langle 23\rangle[41]}{S_{12}}$$

$$= -2i \frac{u}{s} e^{i\delta'}$$

$$= i e^{i\delta'}(1+\cos\theta) \quad \text{or} \quad -2i \frac{[14]^2}{[12][34]}$$

Now, since the 4 helicity configuration do not interfere, the total amplitude is given by,

$$|A|^2 = |A_{(a)}|^2 + |A_{(b)}|^2 + |A_{(c)}|^2 + |A_{(d)}|^2$$

$$= 2[(1+\cos\theta)^2 + (1-\cos\theta)^2] \quad (\text{or}) \quad 2\left[\frac{S_{14}^2}{S_{12}^2} + \frac{S_{24}^2}{S_{12}^2}\right]$$

$$= 4(1+\cos^2\theta) \quad (\text{or}) \quad \frac{2(S_{14}^2 + S_{24}^2)}{S_{12}^2}$$