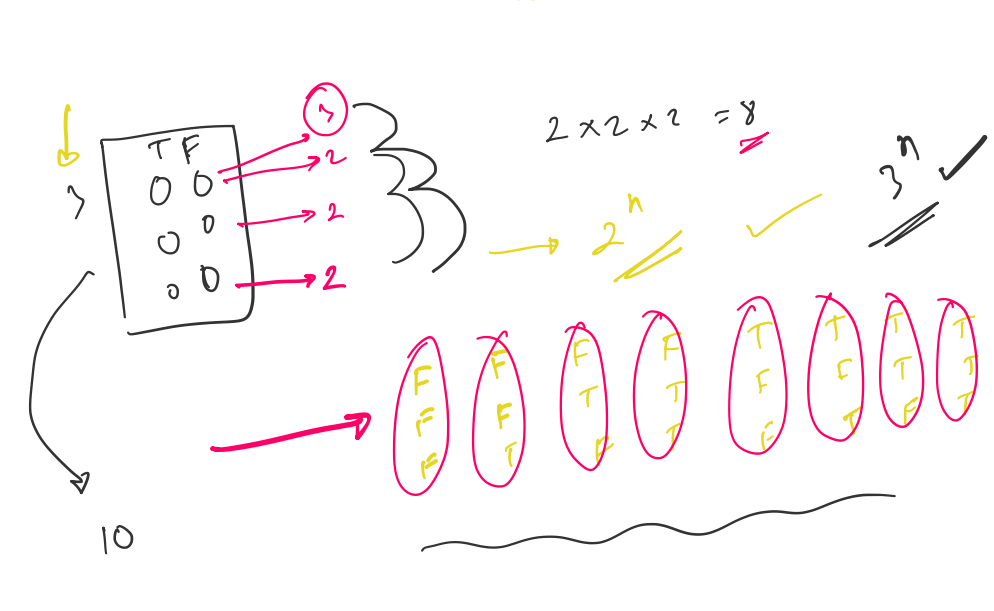
Maths Combinatorics September

Monday, 27 September 2021 9:05 PM

Combinatorics

- Combinatorics!
  It deals with selecting / arranging elements in particular order!
  → Count!



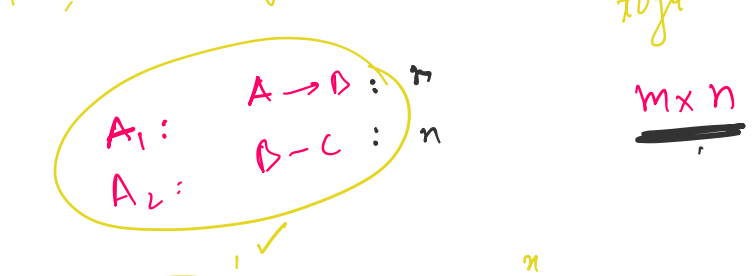
$2\times2\times2 = 8$ → 2^n ✓ 3^n ✓

- Product Rule

If 2 events A_1 & A_2 can occur in m and n ways.
Then, the no of ways of doing A_1 & A_2 together is:

A_1: A → B : m
A_2: B – C : n

m × n

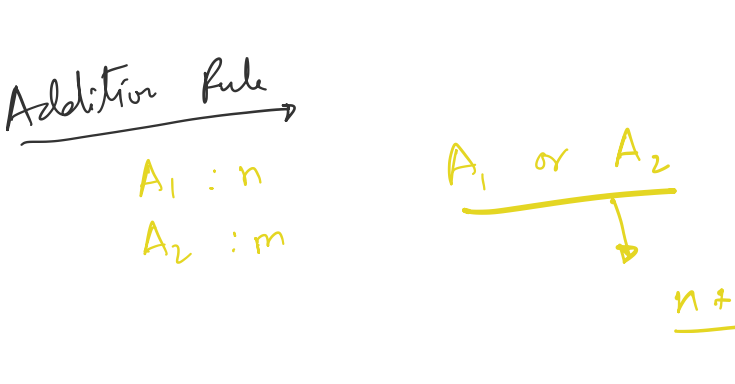


1x	2x	3x
1y	2y	3y

3 × 2 = 6 !

- Addition Rule
  A_1 : n
  A_2 : m

A_1 or A_2 → n + m !


A
B
1
2
3
x
y
5
3 + 2
1, 2, 3, x, y

- Permutation
  It is the act of arranging elements in a sequence or order!

(1,2) , (2,1) → 2

a, b, c →
abc
acb
bac
bca
cab
cba
6

~~abc~~ 2 1

$\underset{1}{3}\ \underset{2}{2}\ \underset{3}{1}$ → $3\times2\times1 = 3! = 6$

$n! = \lfloor n = n\times(n-1)\times(n-2)\ \dots \times2\times1$

$\boxed{\#\text{ ways of permuting n elements} : n!}$

- N elements. Select R elements from it & then arrange them.
  distinct
  # ways : ?

N = 5
abcde 4
r = 2

$\underline{5}\ \underline{4} = 5\times4 = 20$

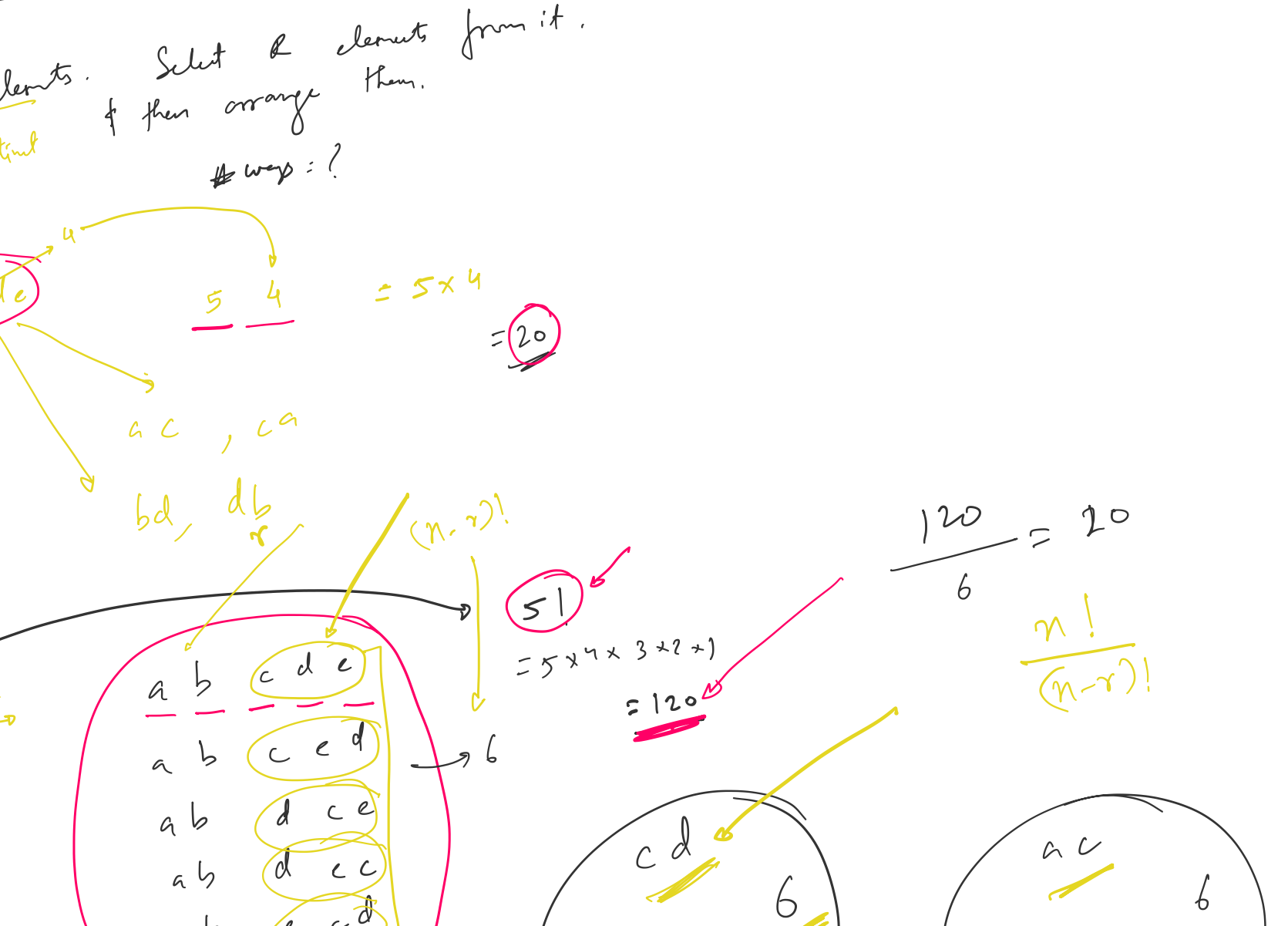
ac , ca
bd, db

$(n-r)!$

$5! = 5\times4\times3\times2\times1 = 120$

$\frac{120}{6} = 20$ $\frac{n!}{(n-r)!}$

abcde → 20


ab cde
ab ced
ab dce
ab dec
ab ecd
ab edc
6
cd 6
ac 6
ba 6
dc 6
co 6

$$\boxed{{}^nP_r = \frac{n!}{(n-r)!}}$$

$${}^5P_2 = \frac{5!}{(5-2)!} = \frac{5!}{3!} = \frac{5\times4\times3!}{3!} = 20$$

- S = "a b b c" → 4!
  4
  $a\ b_1\ c\ b_2$
  $a\ b_2\ c\ b_1$

$\frac{4!}{2}$ → 12 !

- 5 balls.
  3W, 1R, 1B
  W_1, W_2, W_3

→ 5! $\frac{5!}{3!}$

$W_1\ B\ W_2\ R\ W_3$
$W_1\ B\ W_3\ R\ W_2$
$W_2\ B\ W_1\ R\ W_3$
$W_2\ B\ W_3\ R\ W_1$
$W_3\ B\ W_1\ R\ W_2$
$W_3\ B\ W_2\ R\ W_1$

→ W B W R W
3! → 1

- N items : R are repeating
  # permutations = $\frac{N!}{R!}$

- N items : R are same, S are same
  6 : 3W , 2R 1B
  $\frac{6!}{3!\times2!}$

- N items, $r_1, r_2, r_3 \dots r_k$ are same

$\boxed{\#\text{ Ways} = \frac{N!}{r_1!\times r_2!\times r_3}}$

- Combination:
  → deals with selecting the objects.

(1,2) = (2,1)

52 Cards:
Select 3 cards : = $\frac{52\times51\times50}{3\times2\times1}$

- N items:
  Select r items
  $= \boxed{{}^nC_r = \frac{n!}{(n-r)!\,r!}}$

S = "abcde" → ${}^5P_3 = \frac{5!}{(5-3)!} = \frac{5!}{2!}$

${}^5C_3 = \boxed{{}^nC_r = \frac{{}^nP_r}{r!}}$

r!
6
abc
acb
bac
bca
cab
cba
→ 1

$\boxed{{}^nC_r = \frac{{}^nP_r}{r!}}$ → $\frac{n!}{(n-r)!\,r!}$

$$\boxed{{}^nC_r = \frac{n!}{(n-r)!\times r!}}$$

${}^nC_r$ → binomial coefficients !

$(x+y)^n = {}^nC_0 x^0 y^n + {}^nC_1 x^1 y^{n-1} + \dots + {}^nC_n x^n y^0$

- properties of Combination

① $\boxed{{}^nC_r = {}^nC_{n-r}}$ ${}^nC$

${}^5C_2$ abcde n = 5 ${}^5C_{5-2} = {}^5C_3$

ab	cde
ac	bde
ad	bce
ae	bcd
bc	ade
bd	ace
be	acd
cd	abe
ce	abd
de	abc

${}^5C_2 = {}^5C_3$

$\boxed{{}^nC_r = {}^nC_{n-r}}$

$\frac{n!}{(n-r)!\,r!}$ $\frac{n!}{(n-(n-r))!\,(n-r)!}$ $\frac{n!}{r!\,(n-r)!}$

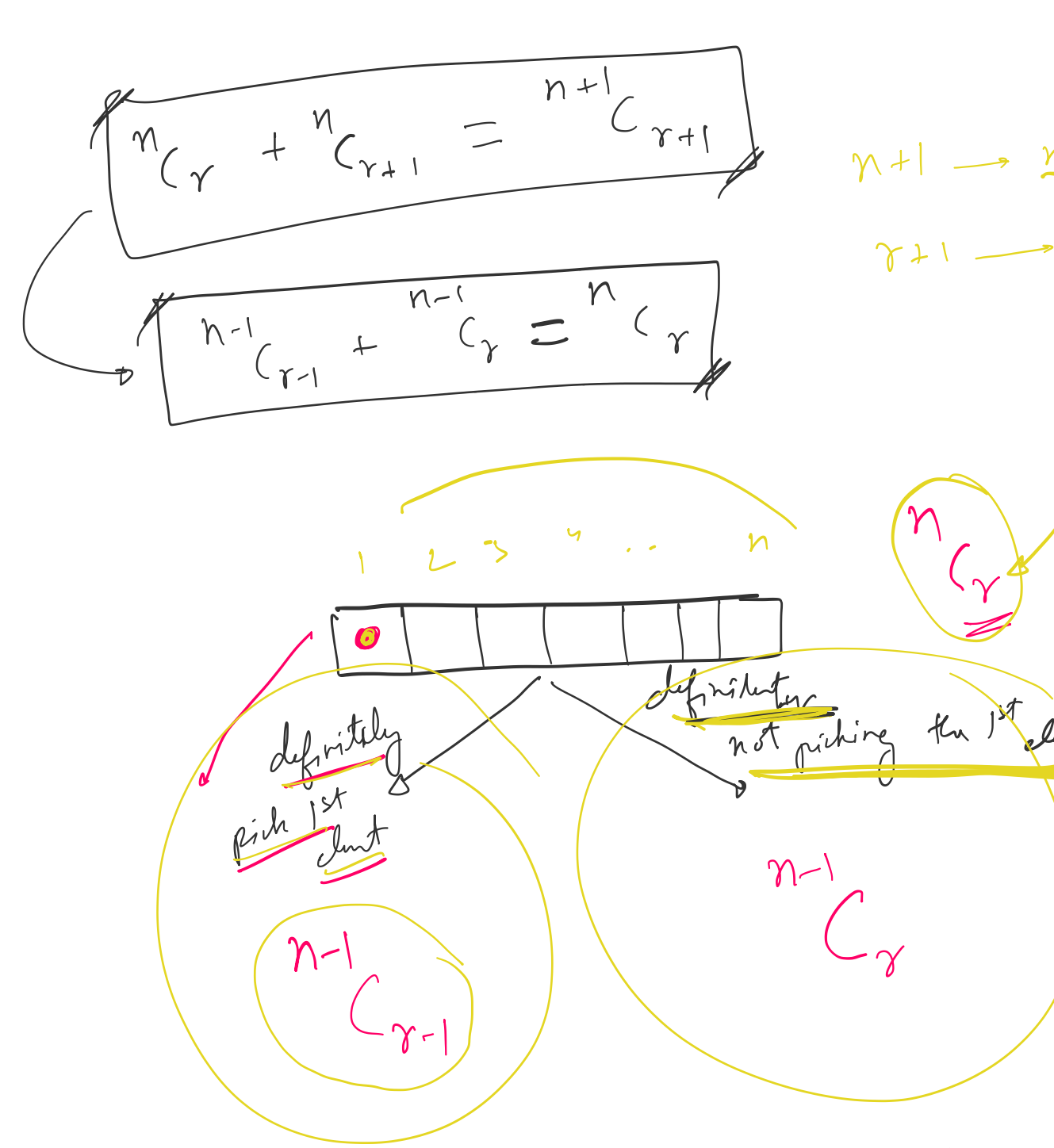
- $\boxed{{}^nC_0 = 1}$ $\frac{\cancel{n!}}{0!\,\cancel{(n)!}} = 1$ $0! = 1$

- $\boxed{{}^nC_1 = n}$ $\frac{n!}{1!\,(n-1)!} \rightarrow \frac{n\times\cancel{(n-1)!}}{\cancel{(n-1)!}} = n$

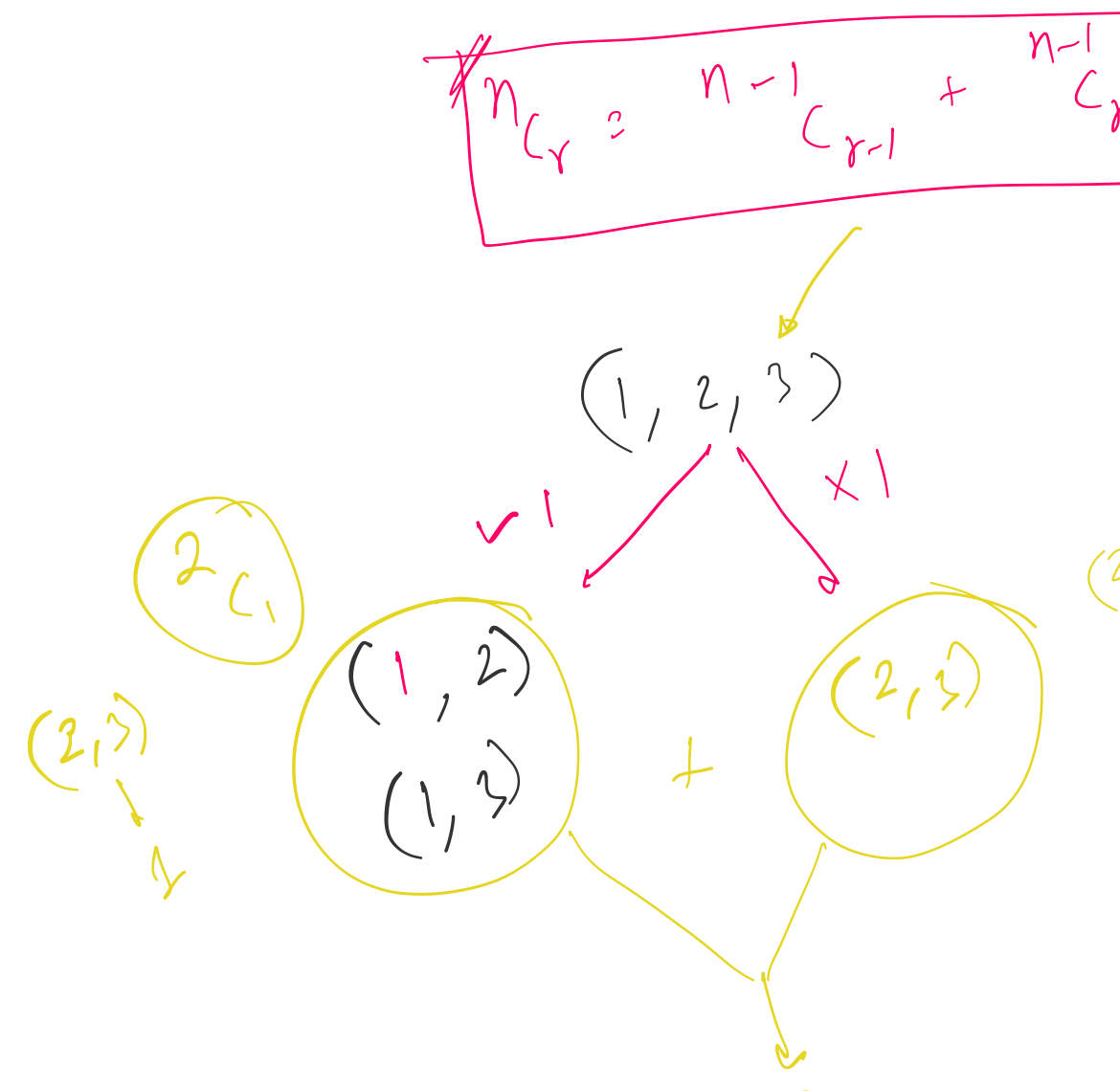
- $\boxed{{}^nC_n = 1}$

- $\boxed{{}^nC_r + {}^nC_{r+1} = {}^{n+1}C_{r+1}}$ $n+1 \rightarrow n$ $r+1 \rightarrow r$

$\boxed{{}^{n-1}C_{r-1} + {}^{n-1}C_r = {}^nC_r}$

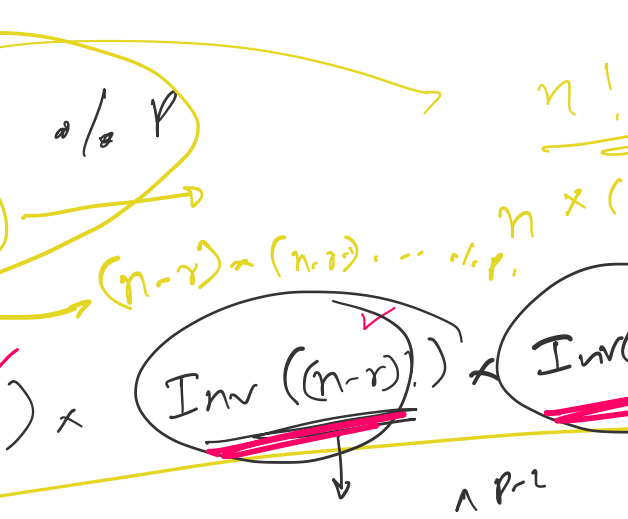

1 2 3 4 .. n
nCr
definitely pick 1st element
n-1 C r-1
definitely not picking the 1st element
n-1 C r

$$\boxed{{}^nC_r = {}^{n-1}C_{r-1} + {}^{n-1}C_r}$$


(1, 2, 3)
✓1
×1
2C1
(1, 2)
(1, 3)
+
(2, 3)
(2,3)
1
3

${}^3C_2 = 3$ ✓
(2,3) ${}^2C_2 = 1$

- $\boxed{{}^nC_0 + {}^nC_1 + {}^nC_2 + {}^nC_3 + \dots + {}^nC_n = 2^n}$
  Select × 1 2 3 ... n



2^n ← 2×2×2
(1, 2, 3)
()
(1) (2) (3)
(1,2) (1,3)
(2,3)
(1,2,3)

- ${}^nC_r \% P$ ${}^{100}C_{45} \% P$

$\frac{n!}{(n-r)!\,r!} \% P$ $n!\%P$! $n\times(n-1)\times(n-2)\dots$ $(n-r)\times(n-r-1)\dots\%P$

O(N)

$\left((n!\%P)\times \text{Inv}((n-r)!)\times\text{Inv}(r!)\right)\%P$

$\times (n-r)!^{\wedge P-2} \times (r!)^{\wedge P-2} \%P$

$\left(\frac{1}{b}\right)\%P = (b^{-1})\%P = \text{Inv}(b)$
$= b^{P-2}\%P$ ✓ → O(lg P)

$\boxed{TC : O(N + \lg P)}$

$\%\ 10^9+7$
${}^3C_2$
${}^{300}C_{200}$ → ${}^{350}C_{250}$ ${}^{300}C_3$

f
0 1 2. — – — 10^5
1 1 . . . . $\%P$
O(N)

$f(i: 2 \rightarrow 10^5)$
$f(i) = (f(i-1)\times i)\%P$

${}^nC_r \rightarrow \frac{n!}{r!\,(n-r)!}$ → Inv O(lg P)
1 Q : O(lg P)

Q $\boxed{TC : O(N + Q.\lg P)}$
$10^5 + 10^5 \times 30$

$N \le 10^5$
$Q \le 10^5$
$P \le 10^9$

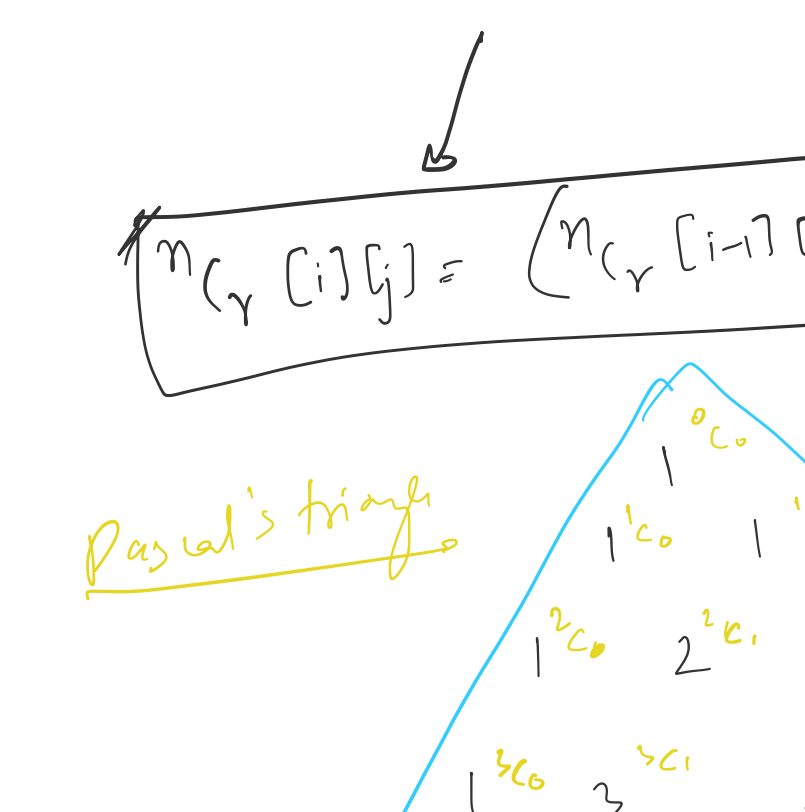
31×10^5
3.1×10^6 ✓

$${}^nC_r = \prod_{i=1}^{r} \frac{n-i-1}{i}$$

- ${}^nC_r \% m$

${}^{300}C_{150} \%\ 10^{16}$

${}^nC_r \% m = \left({}^{n-1}C_{r-1} + {}^{n-1}C_r\right)\%m$

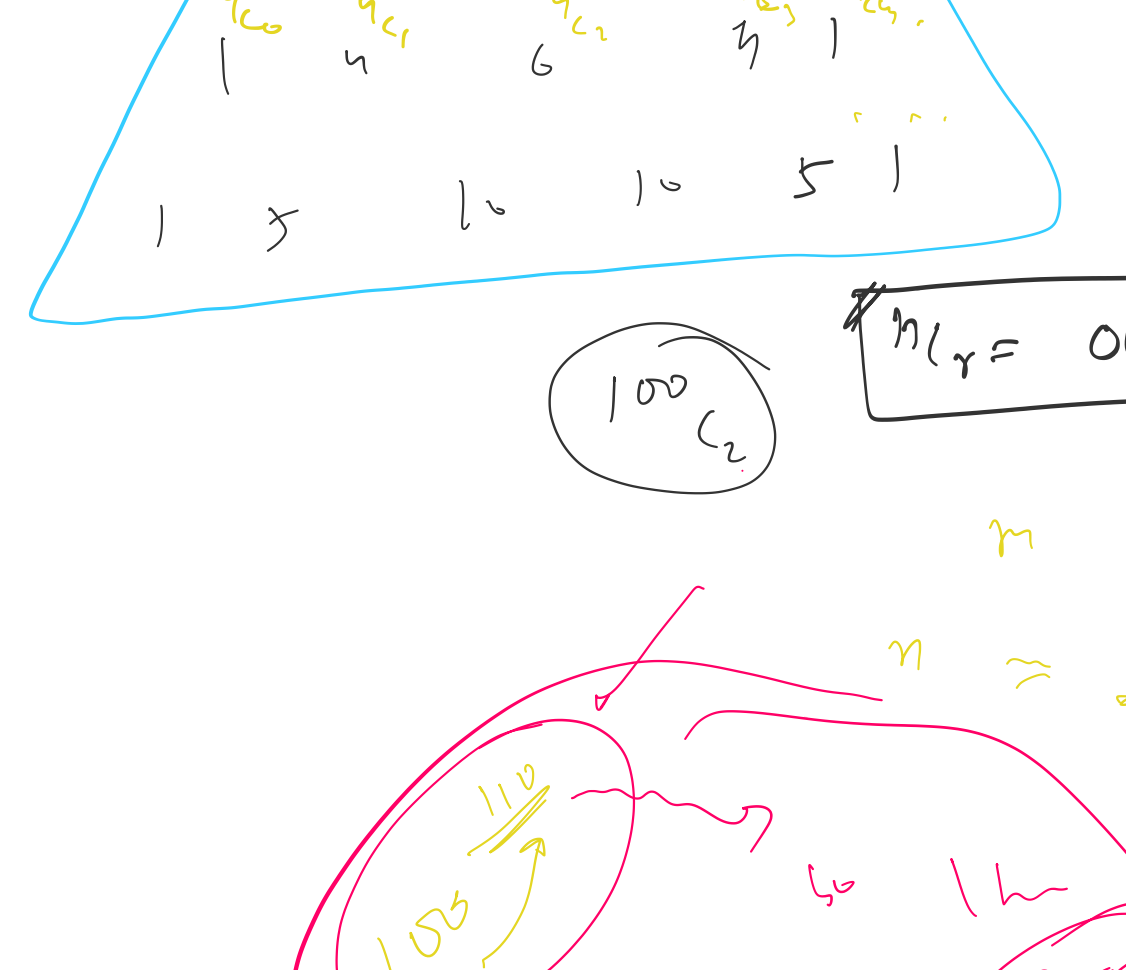

r
% m
0 1 2 3 4
0
1
2
3
4
n
nCr
1
1 1
1 2 1
1 3 3 1
1 4 6 4 1

${}^nC_r$ n ≥ r

${}^2C_1 = {}^{2-1}C_{1-1} + {}^{2-1}C_1$
$= {}^1C_0 + {}^1C_1$
(n-1, r-1) (n-1, r)
(n, r)

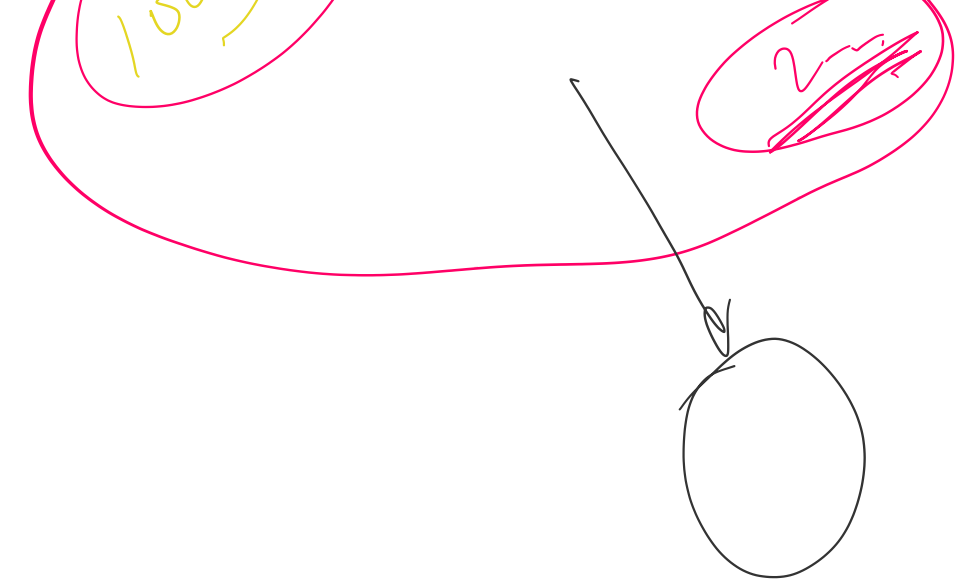
$$\boxed{{}^nC_r[i][j] = \left({}^nC_r[i-1][j-1] + {}^nC_r[i-1][j]\right)\%m}$$

Pascal's triangle


1 0C0
1 1C0 1 1C1
1 2C0 2 2C1 1 2C2
1 3C0 3 3C1 3 3C2 1 3C3
4C0 1 4C1 4 4C2 6 4C3 4 4C4 1
1 5 10 10 5 1

${}^{100}C_2$ $\boxed{{}^nC_r = O(n\times r)}$

m
n ≈ 1000


100
110
50
1 hr
2 min