Math 104B Homework #3 *

Instructor: Lihui Chai

General Instructions: Please write your homework papers neatly. You need to turn in both your codes and descriptions on the appropriate runs you made by following TA's instructions. Write your own code, individually. Do not copy codes!

1. (a) Implement (write a code) Gaussian Elimination with partial pivoting to solve a linear system $Ax = b$, given as output the coefficients of a nonsingular $n \times n$ matrix A and an n-vector b. Your code should produce the solution x or an error message (if A is singular) and the information needed for the LU factorization.

   (b) Test your Gaussian Elimination code for

$$A = \begin{pmatrix} 5 & 1 & 0 & 2 & 1 \\ 0 & 4 & 0 & 1 & 2 \\ 1 & 1 & 4 & 1 & 1 \\ 0 & 1 & 2 & 6 & 0 \\ 0 & 0 & 1 & 2 & 4 \end{pmatrix}, \qquad b = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{pmatrix}.$$

   (c) Test your Gaussian Elimination code for

$$A = \begin{pmatrix} 5 & 1 & 0 & 2 \\ 0 & 4 & 0 & 8 \\ 1 & 1 & 4 & 2 \\ 0 & 1 & 2 & 2 \end{pmatrix}, \qquad b = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}.$$

2. (a) Let A be an $n \times n$ upper or lower triangular matrix. Prove that the determinant of A is equal to the product of its diagonal entries, i.e. $\det(A) = a_{11}a_{22} \cdots a_{nn}$.

   (b) Prove that the product of the pivots in the Gaussian Elimination for $Ax = b$ is equal to the determinant of A up to a sign.

   (c) Prove that the product of two $n \times n$ lower (upper) triangular matrices is a lower (upper) triangular matrix.

(d) Let L_i be the unit upper triangular matrix that produces the i-th elimination step in the Gaussian Elimination algorithm, i.e.

$$L_i = \begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & -m_{i+1,i} & & & \\ & & -m_{i+2,i} & \ddots & & \\ & & \vdots & & \ddots & \\ & & -m_{n,i} & & & 1 \end{pmatrix}.$$

Prove that

$$L_i^{-1} = \begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & m_{i+1,i} & & & \\ & & m_{i+2,i} & \ddots & & \\ & & \vdots & & \ddots & \\ & & m_{n,i} & & & 1 \end{pmatrix}.$$

3. Find an LU factorization of the matrix

$$A = \begin{pmatrix} 5 & 1 & 0 \\ 0 & 4 & 0 \\ 1 & 1 & 4 \end{pmatrix}. \tag{1}$$

4. Find the Choleski factorization $A = LL^T$

$$A = \begin{pmatrix} 3 & -1 & 0 \\ -1 & 3 & -1 \\ 0 & -1 & 3 \end{pmatrix}. \tag{2}$$