Math 104B: Homework 1

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1. Hermite's interpolation formula is given as follows:

$$H_{n-1}(x) = f[x_0] + \sum_{j=1}^{n-1} f[x_0, ..., x_j](x - x_0)...(x - x_{j-1}) \tag{1}$$

Since, we are interpolating through the points $x = -1, 0, 0, 1$, $n = 4$ and we solve for $H_3(x)$.

$H_3(x) = f[x_0] + f[x_0, x_1](x - x_0) + f[x_0, x_1, x_2](x - x_0)(x - x_1) + f[x_0, x_1, x_2, x_3](x - x_0)(x - x_1)(x - x_2)$

We must now construct a table of first, second, and third, divided differences:

x_j	$f(x)$	FDD	SDD	TDD
$x_0 = -1$	$f(-1)$			
		$f[x_0, x_1] = f(0) - f(-1)$		
$x_1 = 0$	$f(0)$		$f[x_0, x_1, x_2] = f'(0) - f(0) + f(-1)$	
		$f[x_1, x_2] = f'(0)$		$f[x_0, x_1, x_2, x_3] = -f'(0)+$ $(f(1) - f(-1))/2$
$x_2 = 0$	$f(0)$		$f[x_1, x_2, x_3] = -f'(0) - f(0) + f(1)$	
		$f[x_2, x_3] = f(1) - f(0)$		
$x_3 = 1$	$f(1)$			

Plugging in our divided difference values, we obtain:

$$\begin{aligned} H_3(x) = f(-1) + [f(0) - f(-1)](x + 1) + [f'(0) - f(0) + f(-1)][x(x + 1)] + \\ [-f'(0) + (f(1) - f(-1))/2][x^2(x + 1)] \end{aligned}$$

We then approximate the integral of $f(x)$ over $[-1, 1]$ by integrating $H_3(x)$ over [-1,1]. We treat the divided difference values as constants, and integrate with respect to x.

$$\begin{aligned} \int_{-1}^{1} f(x)dx &\approx \int_{-1}^{1} H_3(x)dx \\ &= 2f(-1) + 2[f(0) - f(-1)] + \frac{2}{3}[f'(0) - f(0) + f(-1)] + \frac{2}{3}[-f'(0) + \frac{f(1) - f(-1)}{2}] \\ &= \frac{1}{3}f(-1) + \frac{4}{3}f(0) + \frac{1}{3}f(1) \\ &= \frac{1}{3}[f(-1) + 4f(0) + f(-1)] \end{aligned}$$

We can see that we arrived at the elementary Simpson's rule for approximation of an integral.

2. Given

$$f(x) = \begin{cases} 1 + x, -1 \leq x \leq 0 \\ 1 - x, 0 \leq x \leq 1 \end{cases}$$

we approximate $\int_{-1}^{1} f(x)dx$ using the

(a) elementary Trapezoidal Rule:

$$\begin{aligned}\int_{-1}^{1} f(x)dx &\approx (b-a)(\frac{f(a)+f(b)}{2}) \\ &= (1-(-1))(\frac{f(-1)+f(1)}{2} \\ &= 2(\frac{0+0}{2}) = 0\end{aligned}$$

(b) elementary Simpson Rule:

$$\begin{aligned}\int_{-1}^{1} f(x)dx &\approx \frac{b-a}{6}(f(a)+4f(\frac{a+b}{2})+f(b)) \\ &= \frac{1-(-1)}{6}(f(-1)+4f(0)+f(1)) \\ &= \frac{1}{3}(4) = \frac{4}{3}\end{aligned}$$

(c) elementary Trapezoidal Rule over [-1,0] and then [0,1]:

$$\begin{aligned}\int_{-1}^{1} f(x)dx &= \int_{-1}^{0} f(x)dx + \int_{0}^{1} f(x)dx \\ &\approx (0-(-1))(\frac{f(0)+f(-1)}{2}) + (1-0)(\frac{f(0)+f(1)}{2}) \\ &= \frac{1}{2}(1+0) + \frac{1}{2}(1+0) = \frac{1}{2} + \frac{1}{2} = 1\end{aligned}$$

The actual value of $\int_{-1}^{1} f(x)dx$ can be computed as follows:

$$\begin{aligned}\int_{-1}^{1} f(x)dx &= \int_{-1}^{0} f(x)dx + \int_{0}^{1} f(x)dx \\ &= \int_{-1}^{0} (1+x)dx + \int_{0}^{1} (1-x)dx = 1\end{aligned}$$

The error term of the elementary Trapezoidal Rule becomes: $|1-0| = 1$. This is because the Trapezoidal rule converges to the x-axis in this scenario. So we obtain an integral value of 0.

The error term of the elementary Simpson Rule becomes: $|1-\frac{4}{3}| = \frac{1}{3}$. This is because the Simpson's rule uses a parabola and over-approximates the given linear function. So we end up with a value slightly above our desired value.

The error term of the elementary Trapezoidal Rule over [-1,0] and [0,1] becomes: $|1-1| = 0$. This is because when we divide the integral over the two intervals, we obtain the same exact formula as the exact value of the integral.

3. (a) The corresponding Legendre Polynomials are $1, x, x^2 - \frac{1}{2}, x^3 - \frac{3}{5}x$. Take $n = 2$. The roots of ψ_3 are $x_0 = -\sqrt{3/5}, x_1 = 0, x_2 = \sqrt{3/5}$. Therefore, the corresponding Gaussian quadrature is

$$\int_{-1}^{1} f(x) \approx A_0 f(-\sqrt{\frac{3}{5}}) + A_1 f(0) + A_2 f(\sqrt{\frac{3}{5}}) \tag{2}$$

Now we must solve for A_0, A_1, A_2.

$$\int_{-1}^{1} 1dx = 2 = A_0 + A_1 + A_2$$
$$\int_{-1}^{1} xdx = 0 = A_0x_0 + A_1x_1 + A_2x_2$$
$$\int_{-1}^{1} x^2dx = \frac{2}{3} = A_0x_0^2 + A_1x_1^2 + A_2x_2^2$$

Plugging in values of x_0, x_1, x_2 into our system of equations, we obtain:

$$2 = A_0 + A_1 + A_2$$
$$0 = -\sqrt{\frac{3}{5}}A_0 + \sqrt{\frac{3}{5}}A_2$$
$$\frac{2}{3} = \frac{3}{5}A_0 + \frac{3}{5}A_2$$

Solving this system of equations, $A_0 = \frac{5}{9}, A_1 = \frac{8}{9}, A_2 = \frac{5}{9}$. Plugging these values back into equation (2), we obtain:

$$\int_{-1}^{1} f(x) \approx \frac{5}{9}f(-\sqrt{\frac{3}{5}}) + \frac{8}{9}f(0) + \frac{5}{9}f(\sqrt{\frac{3}{5}}) \tag{3}$$

(b) Since the equation above is an interpolatory Gaussian quadrature it has a degree of $k = 2n + 1$. We use ψ_3 as the orthogonal polynomial so $n = 2$. Therefore, the degree of precision for this Gaussian quadrature is $2(2) + 1 = 5$.

(c) Actual value of $\int_{-1}^{1} e^x = 2.3504$.

Using 3-point Gaussian quadrature:

$$\int_{-1}^{1} e^x \approx \frac{5}{9}e^{-\sqrt{3/5}} + \frac{8}{9}e^0 + \frac{5}{9}e^{\sqrt{3/5}}$$
$$= 2.3503$$

Using elementary Simpson rule:

$$\int_{-1}^{1} e^x \approx \frac{1}{3}(e^{-\sqrt{3/5}} + 4e^0 + e^{\sqrt{3/5}})$$
$$= 2.2102$$

We can see that the accuracy of the 3-point Gaussian quadrature is much higher than that of the elementary Simpson rule.

(d) We are attempting to approximate

$$\int_{0}^{4} \frac{\sin x}{x}dx$$

We can apply a coordinate transform and adjust the variables, calculating

$$\int_{a}^{b} \frac{\sin t}{t}dt \tag{4}$$

where

$$t = \frac{b-a}{2}x + \frac{b+a}{2}$$
$$dt = \frac{b-a}{2}dx$$

We can see that when $x = -1, t = a$ and when $x = 1, t = b$ So equation (4) becomes:

$$\int_{-1}^{1} (\frac{\sin{(2x+2)}}{2x+2})2dx$$

Using our 3-point Gaussian quadrature formula (equation (3)) we can approximate this integral using $f(x) = \frac{sin(2x+2)}{2x+2}$, and multiply the overall result by 2 to account for the 2 in the front of the dx term. We obtain 1.77, which is very close to the actual value of 1.76.