Math 104B: Homework 3

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1. (a)

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function [x,L,U] = gauss(A,b)
% Computer code for solving a linear system using Gaussian elimination
% and partial pivoting (interchanging rows)
% Input:  A  --- nxn matrix of coefficients
%         b  --- n-vector
% Output  x  --- solution vector to the system
%         L  --- lower triangular matrix from LU factorization
%         U  --- upper triangular matrix from LU factorization
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% Find LU factorization of A, then use back-substitution for efficiency
[L,U,P] = lu_factorization(A);
try
    x = U\(L\(P*b));
catch
    disp('A is singular, no solution')
end

    function [L,U,P] = lu_factorization(A)
    % Nested function to calculate LU factorization of matrix A using
    % partial pivoting
    % Input:  A  --- nxn matrix of coefficients
    % Output: L  --- lower triangular matrix from LU factorization
    %         U  --- upper triangular matrix from LU factorization
    %         P  --- permutation matrix such that PA = LU
    % Set up vector sizes
    n = size(A,1);
    I = eye(n);
    O = zeros(n);
    L = I;
    U = O;
    P = I;
    % For loop for finding new pivot, uses switch_rows to switch rows if
    % necessary
    for k = 1:n
        if k == 1
            v(k:n) = A(k:n,k);
        else
            z = L(1:k-1,1:k -1)\ A(1:k-1,k);
            U(1:k-1,k) = z;
            v(k:n) = A(k:n,k)-L(k:n,1:k-1)*z;
        end
        if k<n
            x = v(k:n); p = (k-1)+find(abs(x) == max(abs(x)));
            switch_rows(k,p);
            L(k+1:n,k) = v(k+1:n)/v(k);
            if k > 1, adjust_L(k,p); end
        end
        U(k,k) = v(k);
    end
            function switch_rows(k,p)
            % Nested function to place element (p,k) in the (k,k) position
            % Input:  rows k and p
            % Output: No output, adjusts matrix in outer function
                x = P(k,:); P(k,:) = P(p,:); P(p,:) = x;
                x = A(k,:); A(k,:) = A(p,:); A(p,:) = x;
                x = v(k); v(k) = v(p); v(p) = x;
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                end

                function adjust_L(k,p)
                % Nested function to adjust the lower triangular matrix once
                % rows have been switched
                % Input:  rows k and p
                % Output: No output, adjusts matrix in outer function
                    x = L(k,1:k-1); L(k,1:k-1) = L(p,1:k-1);
                    L(p,1:k-1) = x;
                end
        end % End of lu_factorization
    end % End of gauss.m
```

(b) The solution for the given matrix and vector is

$$x = \begin{bmatrix} -0.1708 \\ -0.0675 \\ 0.4603 \\ 0.5245 \\ 0.8727 \end{bmatrix}$$

where

$$L = \begin{bmatrix} 1.0000 & 0 & 0 & 0 & 0 \\ 0 & 1.0000 & 0 & 0 & 0 \\ 0.2000 & 0.2000 & 1.0000 & 0 & 0 \\ 0 & 0.2500 & 0.5000 & 1.0000 & 0 \\ 0 & 0 & 0.2500 & 0.3423 & 1.0000 \end{bmatrix} U = \begin{bmatrix} 5.0000 & 1.0000 & 0 & 2.0000 & 1.0000 \\ 0 & 4.0000 & 0 & 1.0000 & 2.0000 \\ 0 & 0 & 4.0000 & 0.4000 & 0.4000 \\ 0 & 0 & 0 & 5.5500 & -0.7000 \\ 0 & 0 & 0 & 0 & 4.1396 \end{bmatrix}$$

(c) There are no solutions (A is singular). However, we do obtain the following LU factorization:

$$L = \begin{bmatrix} 1.0000 & 0 & 0 & 0 \\ 0 & 1.0000 & 0 & 0 \\ 0.2000 & 0.2000 & 1.0000 & 0 \\ 0 & 0.2500 & 0.5000 & 1.0000 \end{bmatrix} U = \begin{bmatrix} 5 & 1 & 0 & 2 \\ 0 & 4 & 0 & 8 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

2. (a) We will use proof by induction. We will start with a 2x2 matrix: Let

$$A = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$$

Then

$$|A| = ac$$

Now we will prove for an nxn upper triangular matrix. Let

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ 0 & a_{22} & a_{23} & \dots & a_{2n} \\ 0 & 0 & a_{33} & \dots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & a_{nn} \end{bmatrix}$$

Using the cofactor expansion of A, we obtain

$$
\begin{aligned}
|A| &= (-1)^{1+1} a_{11} \begin{bmatrix} a_{22} & a_{23} & \dots & a_{2n} \\ 0 & a_{33} & \dots & a_{3n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_{nn} \end{bmatrix} \\
&= a_{11} \begin{bmatrix} a_{22} & a_{23} & \dots & a_{2n} \\ 0 & a_{33} & \dots & a_{3n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_{nn} \end{bmatrix}
\end{aligned}
$$

If we continue cofactor expansion for the remaining entries we can see that the determinant is the product of the diagonal entries. Hence,

$$|A| = a_{11}a_{22}a_{33}...a_{nn}$$

(b) When we perform Gaussian elimination to solve a system $Ax = b$, we reduce the coefficient matrix A into an upper triangular matrix. So we get

$$
\begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ 0 & a_{22} & a_{23} & \dots & a_{2n} \\ 0 & 0 & a_{33} & \dots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_n \end{bmatrix}
$$

We know from part (a) that the determinant of an upper triangular matrix is the product of the diagonals. Also here the diagonals serve as the pivot elements. Therefore, the product of the pivots is equal to the determinant of A.

(c) We will first prove that the product of two upper triangular matrices is an upper triangular matrix. Let A and B denote these two matrices. We must show that $(AB)_{ij} = 0$
We will use the formula for matrix multiplication.

$$(AB)_{ij} = \sum_{k=i}^{n} A_{ik}B_{kj} + \sum_{k=1}^{i-1} A_{ik}B_{kj}$$

Note that for the term on the left in our sum above, $k \geq i > j$, so $B_{kj} = 0$ since B is an upper triangular matrix.
Note that for the term on the right in our sum above, $i > k$, so $A_{ik} = 0$ since A is an upper triangular matrix.
Hence $(AB)_{ij} = 0$.

Now suppose that A and B are lower triangular matrices and $i < j$. Using our formula for matrix multiplication, we obtain:

$$(AB)_{ij} = \sum_{k=1}^{n} A_{ik}B_{kj} + \sum_{k=j}^{n} A_{ik}B_{kj}$$

Using the same logic as for upper triangular matrices, we obtain 0 for both of the terms of the sum and $(AB)_{ij} = 0$

(d) We will use proof by induction again. Suppose that

$$L_2 = \begin{bmatrix} 1 & 0 \\ -m_{21} & 1 \end{bmatrix}$$

Then

$$L_2^{-1} = \frac{1}{1-0} \begin{bmatrix} 1 & 0 \\ m_{21} & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ m_{21} & 1 \end{bmatrix}$$

We will now perform cofactor expansion of L_3:

$$cofactorsofL_3 = \begin{bmatrix} 1 & m_{21} & m_{21}m_{32} + m_{31} \\ 0 & 1 & m_{32} \\ 0 & 0 & 1 \end{bmatrix}$$

We know that the inverse of L_3 is the adjoint of L_3 divided by the determinant of L_3. So we obtain

$$L_3^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ m_{21} & 1 & 0 \\ m_{21}m_{32} + m_{31} & m_{32} & 1 \end{bmatrix}$$

which can be reduced to:

$$L_3^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ m_{21} & 1 & 1 \\ m_{31} & m_{32} & 1 \end{bmatrix}$$

Using our induction claim:

$$L_i^{-1} = \begin{bmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & m_{i+1,i} & & & \\ & & m_{i+2,i} & \ddots & & \\ & & \vdots & & \ddots & \\ & & m_{n,i} & & & 1 \end{bmatrix}$$

3. I will use a shortcut here to find the LU factorization of the given matrix

$$\begin{aligned} A &= \begin{bmatrix} 5 & 1 & 0 \\ 0 & 4 & 0 \\ 1 & 1 & 4 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ ... & 1 & 0 \\ ... & ... & 1 \end{bmatrix} \begin{bmatrix} 5 & 1 & 0 \\ 0 & 4 & 0 \\ 1 & 1 & 4 \end{bmatrix} \end{aligned}$$

Ensure row 2, column 1 of matrix on the right is 0 using row operations. It already is 0, so we know row 2, column 1 of matrix on the left is 0.

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ ... & ... & 1 \end{bmatrix} \begin{bmatrix} 5 & 1 & 0 \\ 0 & 4 & 0 \\ 1 & 1 & 4 \end{bmatrix}$$

Ensure row 3, column 1 of matrix on the right is 0 using row operations. Use $-\frac{1}{5}R_1 + R_3$ to obtain this. That means row 3, column 1 of matrix on the left is $\frac{1}{5}$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1/5 & ... & 1 \end{bmatrix} \begin{bmatrix} 5 & 1 & 0 \\ 0 & 4 & 0 \\ 0 & 4/5 & 4 \end{bmatrix}$$

Ensure row 3, column 2 of matrix on the right is 0 using row operations. Use $-\frac{1}{5}R_2 + R_3$ to obtain this. That means row 3, column 2 of matrix on the left is $\frac{1}{5}$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1/5 & 1/5 & 1 \end{bmatrix} \begin{bmatrix} 5 & 1 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

where

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1/5 & 1/5 & 1 \end{bmatrix} U = \begin{bmatrix} 5 & 1 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

4. I will find the Choleski factorization using the following method:

$$A = \begin{bmatrix} 3 & -1 & 0 \\ -1 & 3 & -1 \\ 0 & -1 & 3 \end{bmatrix} = LL^T$$

where

$$\begin{aligned} \begin{bmatrix} 3 & -1 & 0 \\ -1 & 3 & -1 \\ 0 & -1 & 3 \end{bmatrix} &= \begin{bmatrix} a & 0 & 0 \\ b & d & 0 \\ c & e & f \end{bmatrix} \begin{bmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{bmatrix} \\ &= \begin{bmatrix} a^2 & ab & ac \\ ab & b^2+d^2 & bc+de \\ ac & bc+de & c^2+e^2+f^2 \end{bmatrix} \end{aligned}$$

Solving for the variables one by one, we get $a = \sqrt{3}, b = -\sqrt{1/3}, c = 0, d = \sqrt{8/3}, e = -\sqrt{3/8}, f = \sqrt{21/8}$.

Therefore,

$$\begin{bmatrix} 3 & -1 & 0 \\ -1 & 3 & -1 \\ 0 & -1 & 3 \end{bmatrix} = \begin{bmatrix} \sqrt{3} & 0 & 0 \\ -\sqrt{1/3} & \sqrt{8/3} & 0 \\ 0 & -\sqrt{3/8} & \sqrt{21/8} \end{bmatrix} \begin{bmatrix} \sqrt{3} & -\sqrt{1/3} & 0 \\ 0 & \sqrt{8/3} & -\sqrt{3/8} \\ 0 & 0 & \sqrt{21/8} \end{bmatrix}$$

where

$$L = \begin{bmatrix} \sqrt{3} & 0 & 0 \\ -\sqrt{1/3} & \sqrt{8/3} & 0 \\ 0 & -\sqrt{3/8} & \sqrt{21/8} \end{bmatrix} L^T = \begin{bmatrix} \sqrt{3} & -\sqrt{1/3} & 0 \\ 0 & \sqrt{8/3} & -\sqrt{3/8} \\ 0 & 0 & \sqrt{21/8} \end{bmatrix}$$