Math 104A Homework #5 Least Square Approximation and Orthogonal Polynomials*

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1. The solution $P_n(x)$ to the Least Squares Approximation problem of f by a polynomial of degree at most n is given explicitly in terms of orthogonal polynomials $\psi_0(x)$, $\psi_1(x)$, ..., $\psi_n(x)$, where ψ_j is a polynomial of degree j, by

$$P_n(x) = \sum_{j=0}^{n} a_j \psi_j(x), \quad a_j = \frac{\langle f, \psi_j \rangle}{\langle \psi_j, \psi_j \rangle}.$$

   (a) Let $\mathcal{P}_n$ be the space of polynomials of degree at most n. Prove that the error $f - P_n$ is orthogonal to this space, i.e. $\langle f - P_n, q \rangle = 0$ for any $q \in \mathcal{P}_n$.

   (b) Using the analogy of vectors interpret this result geometrically (recall the concept of orthogonal projection).

2. (a) Obtain the first 4 Legendre polynomials in $[-1, 1]$.

   (b) Find the least squares polynomial approximations of degrees 1, 2, and 3 for the function $f(x) = \mathrm{e}^x$ on $[-1, 1]$.

   (c) What is the polynomial least squares approximation of degree 4 for $f(x) = x^3$ on $[-1, 1]$? Explain.

3. Plot the monic Chebyshev polynomials $\tilde{T}_0(x)$, $\tilde{T}_1(x)$, $\tilde{T}_2(x)$, $\tilde{T}_3(x)$, and $\tilde{T}_4(x)$.

4. The concentration c of a radioactive material decays according to the law $c(t) = b\mathrm{e}^{-at}$ where t represents time in seconds, $a = 0.1\,\mathrm{sec}^{-1}$, and b is the initial concentration.

   (a) Using the Least Squares method and the data table (Table 1) below find b.

t_j (sec)	1	2	3	4
c_j	0.91	0.80	0.76	0.65

   (b) Find the error in the least squares approximation.