Math 104A: Homework 6

Raghav Thirumulu, Perm 3499720
rrajuthirumulu@umail.ucsb.edu

August 8, 2018

1. Given

$$c_k = \sum_{j=0}^{N-1} f_j e^{-i2\pi kj/N} \tag{1}$$

For c_0, the equation simplifies to:

$$\begin{aligned} c_0 &= \sum_{j=0}^{N-1} f_j e^0 \\ &= \sum_{j=0}^{N-1} f_j = f_0 + f_1 + ... + f_{N-1} \end{aligned}$$

Since we know that each f_j for $j = 0, 1, ..., N-1$ is real, the sum of real numbers is also a real number by the closure property of a field. Hence, c_0 is real.

We can rewrite the right side of equation (1) in trigonometric form complex:

$$c_k = \sum_{j=0}^{N-1} f_j[cos(\frac{-2\pi kj}{N}) + isin(\frac{-2\pi kj}{N})]$$

If we substitute $N-k$ for k, then we obtain the following equation:

$$\begin{aligned} c_{N-k} &= \sum_{j=0}^{N-1} f_j[cos(\frac{-2\pi(N-k)j}{N}) + isin(\frac{-2\pi(N-k)j}{N})] \\ &= \sum_{j=0}^{N-1} f_j[cos(\frac{-2\pi(N)j}{N} - \frac{-2\pi(k)j}{N}) + isin(\frac{-2\pi(N)j}{N} - \frac{-2\pi(k)j}{N})] \\ &= \sum_{j=0}^{N-1} f_j[cos(-2\pi j - \frac{-2\pi(k)j}{N}) + isin(-2\pi j - \frac{-2\pi(k)j}{N})] \end{aligned}$$

We know that $cos(\theta - 2\pi j) = cos(\theta)$ and $sin(\theta - 2\pi j) = sin(\theta)$ for $j = 0, 1, ...N-1$. Therefore, using trigonometric properties, the equation above simplifies to:

$$\begin{aligned} c_{N-k} &= \sum_{j=0}^{N-1} f_j[cos(-\frac{-2\pi kj}{N}) + isin(-\frac{-2\pi kj}{N})] \\ &= \sum_{j=0}^{N-1} f_j[cos(\frac{-2\pi kj}{N}) - isin(-\frac{2\pi kj}{N})] \\ &= \overline{c_k} \end{aligned}$$

We know the equation above holds true because the *cos* portion of the trigonometric form of a complex number represents the real part of a complex number and the *isin* portion of the equation represents the imaginary part of the same complex number. To find the complex conjugate, we simply negate the imaginary part, as shown in the steps above.

2. I am using the fft and ifft package in MATLAB. Here are the formulas for the packages:

$$Y(k) = \sum_{j=1}^{n} X(j) W_n^{(j-1)(k-1)}$$
$$X(j) = \frac{1}{n} \sum_{k=1}^{n} Y(k) W_n^{-(j-1)(k-1)}$$

where $W_n = e^{(-2\pi i)/n}$ is one of n roots of unity.

3. We have the following relation:

$$\begin{aligned}
\sum_{j=0}^{N-1} (cos(kx_j))^2 &= \sum_{j=0}^{N-1} (cos(-\pi\frac{N}{2} + N/2\frac{j\pi}{N/2}))^2 \\
&= \sum_{j=0}^{N-1} (cos(j - N/2)\pi)^2 \\
&= \sum_{j=0}^{N-1} (-1)^{2(j-N/2)} \\
&= \sum_{j=0}^{N-1} (1) = N
\end{aligned}$$

Therefore, we can now consider:

$$c_k = \frac{N}{2}(-1)^k (a_k + ib_k)$$

We now have:

$$P_N(x) = \frac{1}{N} \sum_{k=0}^{N-1} c_k e^{ikx}$$
$$c_k = \sum_{j=0}^{N-1} f_j e^{2i\pi kj/N}$$

We get this from the fact that for any $n \in \mathbb{Z}$

$$e^{in\pi} = cos(n\pi) + isin(n\pi) = (-1)^n$$

Using the formula from the fft package in MATLAB, we now have:

$$P_N(x) = \frac{1}{N} \sum_{k=1}^{N} Y(k) e^{i(k-1)x}$$

4. Plugging in the values of f_j for each x_j into an array X and running Y = fft(X,8) produces the vector Y(k) necessary for $\text{P}_8(x)$.

$$P_8(x) = \frac{1}{8}\sum_{k=1}^{8} Y(k)e^{i(k-1)x}$$

where

$$Y(k) = \begin{bmatrix} 10.1286 \\ -4.5212i \\ -1.0862 \\ 0.1796i \\ 0.0438 \\ -0.1796i \\ -1.0862 \\ 4.5212i \end{bmatrix}$$

Hence

$$P_8'(x) = \frac{1}{8}\sum_{k=1}^{8} i(k-1)Y(k)e^{i(k-1)x}$$

The error is computed using

$$f' = cos(x)e^{sin(x)}$$

And the error term becomes:

$$error = |f' - P_8'(x)|$$

5. (a)

```
function P = fft_deriv(n,xbar)
% Computer code for evaluating derivative of PN(x) using FFT and IFFT
% Input:  n    --- number of points to interpolate
%         xbar --- point we want to approximate
% Output  P    --- evaluation of derivative of PN(xbar)
% Author: Raghav Thirumulu, Perm 3499720
% Date:   08/08/2018

% Create vector of length n to store our interpolation points
x = zeros(1,n);
y = zeros(1,n);

% Iterate through and solve for xj and fj
for j=1:n
    x(j) = 2*pi*(j-1)/n;
    y(j) = exp(sin(x(j)));
end

% Find Fourier coefficients related to fj
Y = fft(y,n);
X = ifft(Y,n);

% Solve for the derivative
P = 0;
for k=1:n
    P = P + (1i*(k-1)*Y(k)*exp(1i*(k-1)*xbar));
end

P = P/n;
```

(b) We can see that we obtain a similar result from the previous question.

(c) The error term reduced as we increase the value of N.