Math 104A Homework #2 *

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General Instructions: Please write your homework papers neatly. You need to turn in both full printouts of your codes and the appropriate runs you made. Write your own code, individually. Do not copy codes!

1. (a) Write the Lagrangian form of the interpolating polynomial $P_2(x)$ corresponding to the data in the table below:

x_j	$f(x_j)$
0	1
1	1
3	-5

   (b) Use $P_2(x)$ you obtained in (a) to approximate $f(2)$.

2. **(optional)** We proved in class that

$$\|f - P_n\|_\infty \leq (1 + \Lambda_n) \|f - P_n^*\|_\infty , \tag{1}$$

where P_n is the interpolating polynomial of f at the nodes x_0, ..., x_n, P_n^* is the best approximation of f, in the maximum (infinity) norm, by a polynomial of degree at most n, and

$$\Lambda_n = \left\| \sum_{j=0}^{n} \left| l_j^{(n)} \right| \right\|_\infty , \tag{2}$$

is the Lebesgue constant (here the $l_j^{(n)}$ are the elementary Lagrange polynomials).

   (a) Write a computer code to evaluate the Lebesgue function

$$L^{(n)}(x) = \sum_{j=0}^{n} \left| l_j^{(n)}(x) \right| , \tag{3}$$

   associated to a given set of pairwise distinct nodes x_0, ..., x_n.

(b) Consider the equidistributed points $x_j = -1 + j(2/n)$ for $j = 0, ..., n$. Write a computer code that uses (a) to evaluate and plot $L^{(n)}(x)$ (evaluate $L^{(n)}(x)$ at a large number of points $\bar{x}$ to have a good plotting resolution, e.g. $\bar{x}_k = -1 + k(2/n_e)$, $k = 0, ..., n_e$ with $n_e = 1000$) for $n = 4$, 10, and 20. Estimate Λ_n for these three values of n.

(c) Repeat (b) with the nodes given by $x_j = \cos(\frac{j\pi}{n})$, $j = 0, ..., n$. Contrast the behavior of $L^{(n)}(x)$ and Λ_n with those corresponding to the equidistributed points in (b).

3. (a) Implement the Barycentric Formula for evaluating the interpolating polynomial for arbitrarily distributed nodes $x_0, ..., x_n$; you need to write a function or script that computes the barycentric weights $\lambda_j^{(n)} = 1/\Pi_{k\neq j}(x_j - x_k)$ first and another code to use these values in the Barycentric Formula. Make sure to test your implementation.

(b) Consider the following table of data

x_j	$f(x_j)$
0.00	0.0000
0.25	0.7071
0.50	1.0000
0.75	0.7071
1.25	-0.7071
1.50	-1.0000

Use your code in (a) to find $P_5(2)$ as an approximation of $f(2)$.

4. *The Runge Example.* Let

$$f(x) = \frac{1}{1+x^2}, \quad x \in [-5, 5]. \tag{4}$$

Using your Barycentric Formula code (Prob. 3) and (5) and (6) below, evaluate and plot the interpolating polynomial of $f(x)$ corresponding to

(a) the equidistributed nodes $x_j = -5 + j(10/n)$, $j = 0, ..., n$ for $n = 4$, 8, and 12.

(b) the nodes $x_j = 5\cos(\frac{j\pi}{n})$, $j = 0, ..., n$ for $n = 4$, 8, 12, and 100.

(c) Repeat (a) for $f(x) = e^{-x^2/5}$ for $x \in [-5, 5]$ and comment on the result.

Remark 1. *It can be shown that for equidistributed nodes one can use the barycentric weights*

$$\lambda_j^{(n)} = (-1)^j \binom{n}{j}, \quad j = 0, ..., n, \tag{5}$$

where $\binom{n}{j}$ is the binomial coefficient (nchoosek(n,j) in Matlab). It can be shown that for the nodes $x_j = \frac{a+b}{2} + \frac{b-a}{2}\cos(\frac{j\pi}{n})$, $j = 0, ..., n$, in $[a, b]$, one can use

$$\lambda_j^{(n)} = \begin{cases} \frac{1}{2}(-1)^j & \textit{for } j = 0 \textit{ or } j = n \\ (-1)^j & \textit{for } j = 1, ..., n-1. \end{cases} \tag{6}$$

Make sure to employ (5) and (6) in your Barycentric Formula code for this problem. To plot the corresponding $P_n(x)$ evaluate $P_n(x)$ at a large number of points $\bar{x}$ to have a good plotting *resolution, e.g.* $\bar{x}_k = -5 + k(10/n_e)$, $k = 0, ..., n_e$ *with* $n_e = 5000$. *Note that your Barycentric Formula cannot be used to evaluate* $P_n(x)$ *when* x *coincides with an interpolating node! Plot also* f *for comparison. Compare (a) and (b) and comment on the result in view of what you observed in Prob. 2.*