Math 104A: Homework 5

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1. (a) Assuming that $q \in P_n$, we can say that q is a polynomial of degree at most n. In other words, we can write

$$q = \sum_{j=0}^{n} b_j \psi_j$$

Using properties of inner products and the given equations, we can write the following:

$$\begin{aligned} < f - P_n, q > &=< f, q > - < P_n, q > && (1) \\ &=< f, \sum_{j=0}^{n} b_j \psi_j > - < \sum_{j=0}^{n} a_j \psi_j, \sum_{j=0}^{n} b_j \psi_j > && (2) \\ &= \sum_{j=0}^{n} b_j < f, \psi_j > - \sum_{j=0}^{n} a_j b_j < \psi_j, \psi_j > && (3) \end{aligned}$$

Rewriting the given equation, we obtain:

$$< f, \psi_j >= a_j < \psi_j, \psi_j >$$

So equation (3) simplifies to

$$\begin{aligned} < f - P_n, q > &= \sum_{j=0}^{n} a_j b_j < \psi_j, \psi_j > - \sum_{j=0}^{n} a_j b_j < \psi_j, \psi_j > \\ &= 0 \end{aligned}$$

Hence the error term $f - P_n$ is orthogonal to q or any polynomial of degree n.

(b) We know from part (a) that $f - P_n$ is orthogonal to any polynomial from P_n, so we can write

$$P_n + (f - P_n) = f$$

We can then deduce that f is the sum of two orthogonal vectors. Let U and V denote the subspaces, containing each of these orthogonal vectors, respectively, as follows:

$$\begin{aligned} U &= P_n \\ V &= f - P_n \end{aligned}$$

We can say there is an orthogonal projection from the space P_n onto the two subspaces containing U and V. The equations are shown as follows:

$$\begin{aligned} proj_U(f - P_n) &= proj_U(f) - proj_U(P_n) \\ &= P_n - P_n = 0 \end{aligned}$$

We then can write:

$$\begin{aligned} proj_U(P_n) + proj_U(f - P_n) &= proj_U(P_n) + 0 \\ &= proj_U(P_n) \end{aligned}$$

2. (a) We find the Legendre polynomials using the following formula:

$$\begin{aligned} P_0(x) &= 1 \\ P_n(x) &= \frac{1}{n!2^n}\frac{d^n}{dx^n}(x^2-1)^n \end{aligned}$$

Therefore, the first four Legendre polynomials are:

$$\begin{aligned} P_0(x) &= 1 \\ P_1(x) &= x \\ P_2(x) &= \frac{1}{2}(3x^2-1) \\ P_3(x) &= \frac{1}{2}(5x^3-3x) \end{aligned}$$

(b) We use the following formula to find the least square approximation:

$$l_n(x) = \sum_{j=0}^{n} \frac{<f,P_j>}{<P_j,P_j>} P_j(x)$$

where each $P(x)$ is a Legendre polynomial. Therefore,

$$l_n(x) = \sum_{j=0}^{n} \beta_j P_j(x) \text{ where } \beta_j = \frac{<f,P_j>}{<P_j,P_j>}$$

So

$$\begin{aligned}
\beta_0 &= \frac{< f, P_0 >}{< P_0, P_0 >} \\
&= \frac{\int_{-1}^{1}(1)(e^x)dx}{\int_{-1}^{1}(1)(1)dx} \\
&= \frac{1}{2}(e - 1/e) \\
&= 1.1752
\end{aligned}$$

$$\begin{aligned}
\beta_1 &= \frac{< f, P_1 >}{< P_1, P_1 >} \\
&= \frac{\int_{-1}^{1}(x)(e^x)dx}{\int_{-1}^{1}(x)(x)dx} \\
&= 1.1037
\end{aligned}$$

$$\begin{aligned}
\beta_2 &= \frac{< f, P_2 >}{< P_2, P_2 >} \\
&= \frac{\frac{1}{2}\int_{-1}^{1}(3x^2 - 1)(e^x)dx}{\frac{1}{4}\int_{-1}^{1}(3x^2 - 1)^2 dx} \\
&= 0.3564
\end{aligned}$$

$$\begin{aligned}
\beta_3 &= \frac{< f, P_3 >}{< P_3, P_3 >} \\
&= \frac{\frac{1}{2}\int_{-1}^{1}(5x^3 - 3x)(e^x)dx}{\frac{1}{4}\int_{-1}^{1}(5x^3 - 3x)^2 dx} \\
&= 0.0803
\end{aligned}$$

So

$$\begin{aligned}
l_1(x) &= \beta_0 P_0(x) + \beta_1 P_1(x) \\
&= 1.1752(1) + 1.1037(x) \\
&= 1.1752 + 1.1037x
\end{aligned}$$

$$\begin{aligned}
l_2(x) &= \beta_0 P_0(x) + \beta_1 P_1(x) + \beta_2 P_2(x) \\
&= 1.1752(1) + 1.1037(x) + 0.3664(0.5)(3x^2 - 1)
\end{aligned}$$

$$\begin{aligned}
l_3(x) &= \beta_0 P_0(x) + \beta_1 P_1(x) + \beta_2 P_2(x) + \beta_3 P_3(x) \\
&= 1.1752(1) + 1.1037(x) + 0.3664(0.5)(3x^2 - 1) + 0.0803(0.5)(5x^3 - 3x) \\
&= 0.9968 + 0.9832x + 0.5340x^2 + 0.2008x^3
\end{aligned}$$

(c) Let $a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4$ be the least square approximation for $f(x) = x^3$ on [-1,1].

Using the least square approximation method, we obtain the following system of equations (not shown because the scratch work is very long):

$$30a_0 + 10a_2 + 6a_4 = 0 \tag{4}$$
$$10a_1 + 6a_3 = 6 \tag{5}$$
$$70a_0 + 42a_2 + 30a_4 = 0 \tag{6}$$
$$14a_1 + 10a_3 = 10 \tag{7}$$
$$126a_0 + 90a_2 + 70a_4 = 0 \tag{8}$$

Combining equations (5) and (7) we obtain $a_3 = 1$. Solving for the other constants we get $a_0 = a_1 = a_2 = a_4 = 0$. Therefore, the least square approximation is x^3. This is because we are attempting to approximate a polynomial of degree 3 with a 4 degree approximation. So we get an accurate result.

3. The first 5 Chebyshev polynomials are given as follows:

$$\begin{aligned} T_0(x) &= 1 \\ T_1(x) &= x \\ T_2(x) &= 2x^2 - 1 \\ T_3(x) &= 4x^3 - 3x \\ T_4(x) &= 8x^4 - 8x^2 + 1 \end{aligned}$$

Using the following formulas:

$$\begin{aligned} \widetilde{T}_0(x) &= 1 \\ \widetilde{T}_n(x) &= \frac{1}{2^{n-1}} T_n(x) \end{aligned}$$

we obtain the monic Chebyshev polynomials as follows:

$$\begin{aligned} \widetilde{T}_0(x) &= 1 \\ \widetilde{T}_1(x) &= x \\ \widetilde{T}_2(x) &= x^2 - \frac{1}{2} \\ \widetilde{T}_3(x) &= x^3 - \frac{3}{4}x \\ \widetilde{T}_4(x) &= x^4 - x^2 + \frac{1}{8} \end{aligned}$$

Here is the code for plotting the functions above, along with the figure:

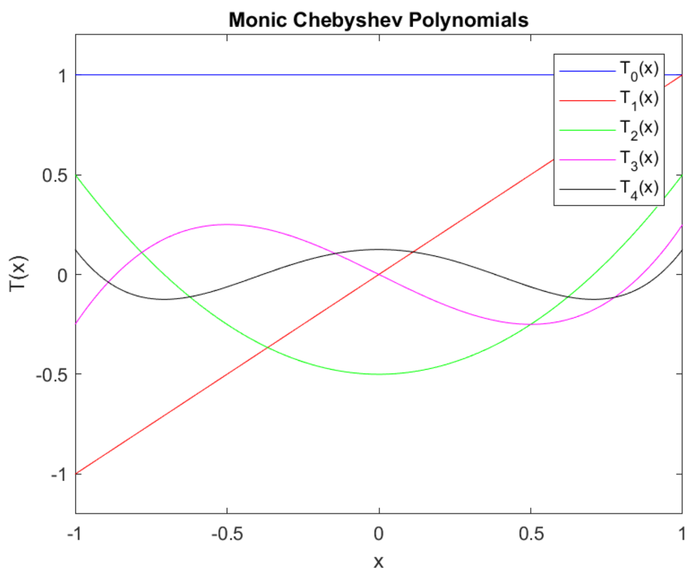
```
% Computer code for plotting Monic Chebyshev polynomials
%
% Input: No arguments are necessary for running this function
% Output: A plot of T0(x), T1(x), T2(x), T3(x), T4(x)
%
% Author: Raghav Thirumulu, Perm 3499720
% Date:    07/24/2018

x=-1:.01:1;

T0 = x.^0;
T1 = x;
```

```
T2 = (x.^2) - 0.5;
T3 = (x.^3) - (0.75.*x);
T4 = (x.^4) - (x.^2) + 1/8;

plot(x,T0,'b'); hold on
plot(x,T1,'r');
plot(x,T2,'g');
plot(x,T3,'m');
plot(x,T4,'k');
title('Monic Chebyshev Polynomials');
axis([-1 1 -1.2 1.2]);
legend('T_0(x)','T_1(x)','T_2(x)','T_3(x)','T_4(x)');
xlabel('x');
ylabel('T(x)'); hold off
```



4. (a)

$$\begin{aligned} c(t) &= be^{-at} \\ ln(c(t)) &= ln(be^{-at}) \\ &= lnb + ln(e^{-at}) \\ &= lnb - at \\ &= lnb - 0.1t \end{aligned}$$

We can now set up the equation $y = A + Bt$ when $y = lnc, A = lnb, B = -0.1$. The normal equations are of the form:

$$\sum y_i = 4A + (\sum t_i)B \tag{9}$$

$$\sum (t_i y_i) = (\sum t_i)A + (\sum t_i{}^2)B \tag{10}$$

We must now tabulate the data so we can solve for A and B:

t_i	c(t)	t_i^2	y_i	$t_i y_i$
1	0.91	1	-0.0943	-0.0943
2	0.80	4	-0.2231	-0.4462
3	0.76	9	-0.2744	-0.8232
4	0.05	16	-0.4307	-1.7228

So now we obtain:

$$\sum t_i = 10$$
$$\sum t_i^2 = 30$$
$$\sum y_i = -1.0225$$
$$\sum (t_i y_i) = -3.0865$$

Plugging in these values back into the normal equations (9) and (10) gives us $A = 0.0095, B = -0.1$ So now our equation becomes $c(t) = (e^{0.0095})(e^{-0.1t}) = (1.0095)e^{-0.1t}$. So our initial concentration for $t = 0$ is 1.0095

(b) Our error term takes the form:

$$\begin{aligned} S^2 &= \sum (y_i - f(t_i))^2 \\ &= (0.91 - 0.91343)^2 + (0.8 - 0.8265)^2 + (0.76 - 0.7478)^2 + (0.65 - 0.6766)^2 \\ &= 0.0015704 \end{aligned}$$